C^*-equivalences of k-graph and $\mathbb{N}$-graph algebras through graph transformations

by

Benjamin Listhartke

B.A., Truman State University, 2014

B.S., Truman State University, 2014

M.S., Kansas State University, 2016

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2024

Abstract

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Approved by:

Major Professor
Dr. Sarah Reznikoff

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Ben Listhartke will invent some kind of new math." I was honored and touched, and his inspiration continues even today.

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Dedication

To my wife, to my parents, and to my God.

Introduction

Directed graphs can be used to study dynamical systems as a subject transitions between various states. The properties of the underlying state graph reveals information about the subject. The subject of symbolic dynamics (see [Lind and Marcus, 1995]) is an aspect of this study, where a mathematical space (known as a shift space) inherits properties from the associated graph. On a different note, the study of C^*-algebras was developed to provide a framework for quantum mechanics in physics. It is also useful in studying non-commutative geometry. One can construct a C^*-algebra for a directed graph by studying partial isometries on a Hilbert space which satisfy relations known as the Cuntz-Krieger relations. Thus, studying C^*-algebras for directed graphs (known as "graph algebras") is useful because these graph algebras are similar to shift spaces in that their properties arise from the graph, but include much more structure (for example, an ideal lattice) that can be studied and used.

Several important theorems regarding classification of C^*-algebras have been found using graph algebras. For example, Drinen and Tyler proved that any approximately finite C^*-algebra is Morita equivalent to a graph algebra. Important theorems like the gauge-invariant uniqueness theorem and Cuntz-Krieger uniqueness theorem are also widely used and rely on graph algebra techniques. Recently, the study of graph algebras has expanded to include higher-rank graphs, which are k-colored directed graphs whose edges satisfy certain edge relations. Many of the important theorems for classification of directed graph algebras has been lifted to the study of k-graph algebras.

One major question is whether the structure of the graph algebra changes when the underlying graph is modified through a transformation such as an insplit or an outsplit. For example, the Willams Conjecture (studied in the Lind/Marcus text mentioned above) asks whether two directed graphs are isomorphic if their shift spaces are equivalent. In this paper, we add on to others' previous results regarding certain graph transformations which preserve the ideal structure in the graph algebra.

In Chapter 1, an overview of directed graphs is given, as well as some results in symbolic dynamics which mirror the results later studied in the rest of the paper. One important result in symbolic dynamics is related to knowing how the shift space of a graph changes when the underlying graph is transformed. In addition, we introduce the notion of a k-colored directed graph, known as a k-graph. We also showcase some results between how a k-graph can be transformed and what happens to a related directed graph.

In Chapter 2, some related topics from functional analysis are described. The mathematical structure of a C^*-algebra is defined and some basic examples are given. One major example of a C^*-algebra is done by building functions on a groupoid. Some other structural elements of C^*-algebra (namely, the multiplier algebra and the diagonal subalgebra) are mentioned.

In Chapter 3, we outline the two constructions of a graph C^*-algebra. The first is by using the groupoid method mentioned in the previous chapter, using the boundary path space of the graph. The second method is by using the well-known Cuntz-Krieger relations. We also describe several equivalences put on graph algebras and their mirrored concepts in symbolic dynamics. The primary equivalence is Morita equivalence, which preserves the ideal structure of the graph algebra. These notions of graph algebras and Morita equivalence are then lifted to k-graphs. Lastly a few propositions are given to related graphs of different values of k.

In [Eckhardt et al., 2020], several results are proven which show that Morita equivalence of the C^*-algebra is preserved when a k-graph is transformed via insplitting, delay, sink deletion, and reduction. The outsplit case for k-graphs is proven in Chapter 4, following a result shown in [Eilers and Ruiz, 2019].

In Chapter 5, the topics are extended further to consider C^*-algebras formed with $\mathbb{N}$-graphs, which are introduced by Schenkel in [Schenkel, 2022] as the limiting case of k-graph algebras as $k \to \infty$. Several results from [Eckhardt et al., 2020] are lifted to $\mathbb{N}$-graph algebras, namely that the sink deletion, delay, and reduction transformations of an $\mathbb{N}$-graph preserve Morita equivalence.

Chapter 1

Graph Theory, Symbolic Dynamics, and Higher-Rank Graphs

1.1 Directed Graphs

The notation $\mathbb{N} = \{0, 1, 2, \dots\}$ is used in this paper. Also, $\mathbb{N}^k$ carries a binary operator of componentwise addition. We use $e^i \in \mathbb{N}^k$ to denote the k-tuple which has 1 in the ith position and zeros elsewhere.

A directed graph is a pair $E = (\mathcal{V}, \mathcal{E})$ of vertices and edges between these vertices. That is, $\mathcal{V}$ is a set and an edge $e \in \mathcal{E}$ is an ordered pair $(v_i, v_f) \in \mathcal{V} \times \mathcal{V}$ marking an arrow from v_i to v_f. There are also maps $r, s\colon \mathcal{E} \to \mathcal{V}$ with $r(e) = v_f$ and $s(e) = v_i$. We assume the vertex and edge sets are countable in size (by which we mean either finite or countably infinite).

Edges can be joined together to form a path $\mu = e_0 e_1 \dots e_n$ if $s(e_i) = r(e_{i+1})$ for $0 \leq i \leq n-1$. In addition, paths can be concatenated: $\mu = e_0 e_1 \dots e_{n-1}$ and $\nu = e_n e_{n+1} \dots e_m$ can form $\mu\nu = e_1 \dots e_m$. (The reader will note that paths are written "backwards" from usual.) Let E^* be the set of finite paths of E. We construct the length function $d\colon E^* \to \mathbb{N}$ by saying $d(e_0 \dots e_{n-1}) = n$. It is clear that d is additive: $d(\mu\nu) = d(\mu) + d(\nu)$. We extend the range and source maps so that $r(\mu) = r(e_0)$ and $s(\mu) = s(e_n)$. For a graph E we denote

$E^n := \{\mu\colon d(\mu) = n\}$. Extending the notion, $E^0 := \mathcal{V}$. With this in mind, the set of all finite paths in E can also be denoted $E^* = \cup_{n=0}^{\infty} E^n$. We use E^∞ to represent the set of infinite paths on E; that is, $\mu \in E^\infty$ iff $\exists (e_i)_{i=0}^{\infty}\colon \mu = e_0 e_1 e_2 e_3 \ldots$ (where again $s(e_i) = r(e_{i+1})$ holds for all i, so $\forall i \in \mathbb{N} : e_0 \ldots e_{i-1} \in E^i$). In addition, we denote $vE^n := \{\mu \in E^n\colon r(\mu) = v\}$ and $E^n v := \{\mu \in E^n\colon s(\mu) = v\}$. Lastly $vE^n w := \{\mu \in E^n\colon s(\mu) = w, r(\mu) = v\} = vE^n \cap E^n w$. The notion of $E^\infty v$ does not make sense in this context.

Using matrices can assist in the understanding of graphs. Let E be a graph, and enumerate the vertices $\mathcal{V} = \{v_0, v_1, \ldots\}$. Note that here $\mathcal{V}$ is not necessarily finite. Construct a matrix $M_E \in M_{|\mathcal{V}|\times|\mathcal{V}|}(\mathbb{N} \cup \{\infty\})$ where $(M_E)_{ij}$ is the number of edges from v_j to v_i. We call M_E the adjacency matrix associated with E. We say E is *row-finite* if $\sum_j M_{ij} < \infty$ for each i in the adjacency matrix. (Or put another way, the set $vE^1 = \{e \in \mathcal{E}\colon r(e) = v\}$ is finite for every $v \in \mathcal{V}$.) In addition, given a square matrix M with $M_{ij} \in \mathbb{N}$ for all i, j, the directed graph E_M can be constructed. Because of this, often describing a graph via its adjacency matrix (e.g. $E = E_M$) is common, with vertex and edge sets $\mathcal{V}_M, \mathcal{E}_M$ respectively.

We say a vertex $v \in E^0$ is a sink if $vE^1 = \emptyset$. Similarly we say v is a source if $E^1 v = \emptyset$. A vertex is an infinite receiver if $|vE^1| = \infty$.

1.2 Boundary Path Spaces

The boundary path space will be useful in Chapter 3 when defining a path groupoid. See [Carlsen and Winger, 2018] for more details. For a directed graph E, a vertex $v \in E^0$ is called *singular* if $r^{-1}(v) = \emptyset$ (i.e., v is a source) or $|r^{-1}(v)| = \infty$ (i.e., v is an infinite receiver). Then we say a vertex w is *regular* if $w \in E^0 \backslash E^0_{\text{sing}}$. (Note that if a directed graph is row-finite and source free, then all vertices are regular.) We then define the boundary path space ∂E to be the set of infinite paths on E or paths which end at a singular vertex. (Again if E is row-finite and source-free then $\partial E = E^\infty$.)

We can equip the boundary path space with a topology, using a basis of cylinder sets.

Assume E is row-finite and source-free and choose $\mu \in E^*$. Define

$$Z(\mu) := \{\mu x \in E^\infty : s(\mu) = r(x), x \in E^\infty\}$$

The collection $\{Z(\mu)\}$ forms a basis for a topology on E^∞. Observe that each $Z(\mu)$ is closed, open, and compact. This makes the boundary path space locally compact and Hausdorff. As such, a groupoid can be constructed from this boundary path space using the construction outlined in Chapter 2.

1.3 Symbolic Dynamics with Directed Graphs

The topic of symbolic dynamics uses directed graphs to illustrate the concept of shift spaces. Much work has been done on this topic, the beginnings of which are condensed into the book "Symbolic Dynamics and Coding," by Lind and Marcus [Lind and Marcus, 1995]. A brief overview follows.

Definition 1.3.1. A *shift space* is a set X of bi-infinite sequences $x = \dots x_{-2}x_{-1}x_0.x_1x_2\dots = \{x_i\}_{i\in\mathbb{Z}}$ paired with a shift map $\sigma\colon X \to X$ via $\sigma(\{x_i\}) = \{x_{i+1}\}$. In particular, a shift space can be formed from a directed graph $E = (E^0, E^1, r, s)$ by constructing bi-infinite paths on E, i.e. $x \in X_E$ iff $x = \{e_i\}_{i\in\mathbb{Z}}$ where $e_i \in E^1\ \forall i$ and $s(e_i) = r(e_{i+1})$.

We say two shift spaces (X and Y) are conjugate if there exists a map $\Phi\colon X \to Y$ such that $\sigma_Y(\Phi(x)) = \Phi(\sigma_X(x)) \forall x \in X$.

Clearly if two graphs are isomorphic then their corresponding shift spaces will be conjugate. However it should be noted that shift spaces can be conjugate without their corresponding graphs being isomorphic.

1.4 k-graphs

Definition 1.4.1. A graph of rank k is a pair (Λ, d) which is defined by Kumjian and Pask in [Kumjian and Pask, 2000] as a countable category Λ with a functor $d\colon \Lambda \to \mathbb{N}^k$ satisfying the factorisation property: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m, d(\nu) = n$. (These graphs of higher rank are more colloquially called k-graphs.) In the same paper, the authors remark that if E is a directed graph, then E^* defines a 1-graph. Conversely, if Λ is a 1-graph, then a directed graph E_Λ can be constructed with $E^* = \Lambda$.

Using this idea, we present an alternative view of k-graphs. Let a *k-graphical object* G be a multiple $G = (\mathcal{V}, \mathcal{E}_1, \ldots, \mathcal{E}_k)$ where $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ if $i \neq j$ and where each pair $(\mathcal{V}, \mathcal{E}_i)$ is a directed graph. (Think of a graph where the edges are colored using k different colors.) Let $G^* := \left(\cup_{i=1}^k \mathcal{E}_i\right)^*$ be the set of all finite paths in G. The length function is extended to $d\colon G^* \to \mathbb{N}^k$ where $d(\mu)$ is defined so that d is functorial and with $d(e) = e^i$ if $e \in \mathcal{E}_i$ (For such an edge, we say that it "has color i.")

Note that if G is a k-graphical object then (G, d) is a countable category. So (G, d) admits a k-graph if and only if a unique factorization can be defined on its paths. In this case, $\forall n\colon G^n = \cup_{i=1}^k \mathcal{E}_i^n$.

It is worth mentioning that the choices of color when sketching a k-graphical object are arbitrary. (i.e. permuting the colors in the k-graphical object doesn't modify the k-graph structure in any way.) The reader can include a map $c_\Lambda\colon \{1, \ldots, k\} \to \{k \text{ colors}\}$ so that the statement "e has color i" to mean "e has color $c_\Lambda(i)$." For all examples of Λ in this paper, red edges will be color i_1, blue edges will be color i_2, etc.

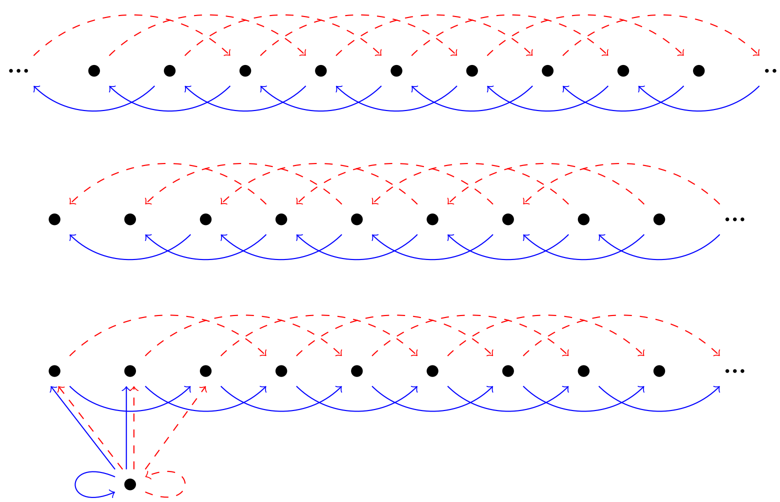
Example 1.4.1. The following are examples of k-graphs, each built from graphs which have (countably) infinite vertices. These examples stem from the desire to generalize the description of types of infinite paths in [Carlsen and Winger, 2018]. For each example, a path groupoid can be built which will later be used to build a C^*-algebra. See chapters 2 and 3 for more details.

- Choose $n \in \mathbb{Z}^k$. Let the vertices of H_n be enumerated via $\mathbb{Z}$, and let the edges be defined via $H_n^{e_j} = \{v_i \to v_{i+n_j}\}$. If $n = (a, b) \in \mathbb{Z}^2$ then $\mathcal{G}(H_{a,b})$ is congruent to the groupoid $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with interesting multiplication depending on the choice of n.

- Choose $n \in \mathbb{N}^k$. Let the vertices of $H_n^\dagger$ be enumerated via $\mathbb{N}$ and let the edges be defined via $H_n^{\dagger e_i} = \{v_i \to v_{i+n_j}\}$. If $n = (a, b) \in \mathbb{N}^2$ then $\mathcal{G}(H_{a,b}^\dagger)$ is congruent to the groupoid $\mathbb{N} \times \mathbb{N} \times \mathbb{Z}$ with interesting multiplication depending on the choice of n.

- Choose $n \in \mathbb{N}^k$. Consider the k-graph drawn by the previous example but with arrows reversed and a "total source" vertex with additional necessary paths. This forms $H^{\uparrow n}$. A theorem of Carlsen and Rout from [Carlsen and Rout, 2019] proves that for any $m, n \in \mathbb{N}^k \backslash \{0\}$, the groupoids for $H^{\uparrow m}$ and $H^{\uparrow n}$ are isomorphic.

Below are depictions of $H_{(3,-2)}$, $H_{(3,2)}^\dagger$ and $H^{\uparrow(3,2)}$ respectively.



Definition 1.4.2. For a k-graph Λ, the function $d \colon \Lambda \to \mathbb{N}^k$ describes the length function d of a path. Then for $m \in \mathbb{N}^k$, we denote $\Lambda^m := \{\lambda \in \Lambda \colon d(\lambda) = m\}$. Let $d_i \colon \Lambda \to \mathbb{N}$ be the number of edges in μ of color i. Then let $d_{\text{tot}}(\mu) := \sum_i d_i(\mu)$. Finally for $n \in \mathbb{N}$ let $\Lambda^n := \{\lambda \in \Lambda \colon d_{\text{tot}}(\lambda) = n\}$. In particular, Λ^0 is the collection of vertices in the k-graph and $\Lambda^1 := \bigcup_{i=1}^k \Lambda^{e_i}$ is the collection of single edges (any color) in the k-graph.

We denote $\mathbb{1} := (\underbrace{1, \cdots, 1}_{k})$ so that $\Lambda^{\mathbb{1}}$ is the collection of "rainbow paths" of length k.

1.5 Outsplits of k-graphs

Definition 1.5.1. We say that edges f, g are *outsplit conjugate at* w if $f, g \in s^{-1}(w)$ and $\exists a, b \in \Lambda^1$ such that $af = bg$. Similarly we say that paths $\lambda_1, \lambda_2 \in \Lambda^{\mathbb{1}}$ are *outsplit conjugate at* w if $\lambda_1, \lambda_2 \in s^{-1}(w)$ and $\exists a, b \in \Lambda^1$ such that $a\lambda_1 = b\lambda_2$.

We construct the outsplit of a higher rank graph by selecting a vertex w and partitioning $s^{-1}(w)$ among n disjoint sets $\mathcal{E}_i$. (Note some $\mathcal{E}_i$ may be empty.) We also require that edges which are outsplit conjugate at w to share the same edge set. (We call this the *pairing condition.*) Then Λ_{out} is built as follows:

- $\Lambda^0_{\text{out}} \coloneqq \{v^1 \colon v \in \Lambda^0\} \cup \{w^i \colon 2 \leq i \leq n\}$
- $\Lambda^{e_i}_{\text{out}} \coloneqq \{f^1 \colon f \in \Lambda^{e_i}, r_\Lambda(f) \neq w\} \cup \{f^1, \dots, f^n \colon e \in \Lambda^{e_i}, r_\Lambda(f) = w\}$
- $s_{\Lambda_{\text{out}}}(f^j) \coloneqq \begin{cases} s_\Lambda(f)^1 & \text{if } f \in \Lambda^{e_i}, s_\Lambda(f) \neq w \\ w^k & \text{if } f \in \Lambda^{e_i}, s_\Lambda(f) = w, f \in \mathcal{E}_k \end{cases}$
- $r_{\Lambda_{\text{out}}}(f^j) \coloneqq \begin{cases} r_\Lambda(f)^1 & \text{if } f \in \Lambda^{e_i}, r_\Lambda(f) \neq w \\ w^j & \text{if } f \in \Lambda^{e_i}, r_\Lambda(f) = w \end{cases}$

This builds Λ_{out} as a k-graphical object. Furthermore, because of the pairing condition, we can set Λ_{out} as a k-graph using the following edge relations. First suppose $af = bg$ in Λ. No matter which vertex is $s(f) = s(g)$, then we can find squares which emulate $af = bg$ in the outsplit. Depending on where a, b are sourced will affect the relations which follow.

- If $s(a), s(b) \neq w$ then $a^1 f^1 = b^1 g^1$ in Λ_{out}.
- If $s(a) \neq w, b \in \mathcal{E}_n$ then $a^1 f^1 = b^1 g^n$ in Λ_{out}.
- If $a \in \mathcal{E}_m, s(b) \neq w$ then $a^1 f^m = b^1 g^1$ in Λ_{out}.
- If $a \in \mathcal{E}_m, b \in \mathcal{E}_n$ then $a^1 f^m = b^1 g^n$ in Λ_{out}.
- If $r(a) = r(b) = w$ then for any $1 \leq i \leq n$ we have $a^i x^m = b^i y^p$ where $r(x^m) = s(a^i), r(y^p) = s(b^i)$.

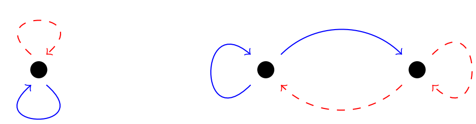
Claim 1.5.1. *Let Λ be a row-finite, source-free k-graph and let Λ_{out} be an outsplit of Λ at a regular vertex w. Then Λ_{out} is also a row-finite, source-free k-graph.*

Proof. We wish to show that Λ_{out} still satisfies the factorization property. To do so, we use the lemma here (proven later as Claim 4.0.2):

Lemma 1.5.0.1. *For all $\lambda \in \Lambda_{out}$, there exists a path $\tilde{\lambda} \in \Lambda$ with $d(\tilde{\lambda}) = d(\lambda)$ where $\tilde{\lambda}$ is obtained by removing the superscripts of λ.*

Suppose $\lambda \in \Lambda_{\text{out}}$ with $d(\lambda) = d_1 + d_2$. We wish to show that there exists $\mu, \nu \in \Lambda_{\text{out}}$ with $d(\mu) = d_1, d(\nu) = d_2$, and $\mu\nu = \lambda$. By the lemma, there exists a path $\tilde{\lambda} \in \Lambda$ with $d(\tilde{\lambda}) = d(\lambda) = d_1 + d_2$. By the factorization property of Λ, there exist $\tilde{\mu}, \tilde{\nu}$ with $d(\tilde{\mu}) = d_1, d(\tilde{\nu}) = d_2$ and $\tilde{\mu}\tilde{\nu} = \tilde{\lambda}$. Lift $\tilde{\mu}, \tilde{\nu}$ to unique paths $\mu, \nu \in \Lambda_{\text{out}}$ and note that $\mu\nu = \lambda$. □

Remark 1.5.1. The pairing condition is necessary to ensure that the outsplit still satisfies the factorization property. Indeed, if $af = bg$ in Λ, then the factorization for Λ_{out} given earlier works. In addition, consider the counterexample of an outsplit given below where the pairing condition is not considered. Note that the factorization property is not satisfied in the outsplit:



Definition 1.5.2. For a higher-rank graph Λ, we define $G(\Lambda)$ to be the "rainbow monochromization" of Λ, which is the directed graph with $G^0 := \Lambda^0$ and $G^1 := \Lambda^{\mathbb{1}}$.

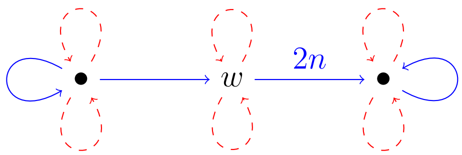
Theorem 1.5.1. *Suppose Λ is a k-graph. Let Λ_{out} be an outsplit of Λ at a regular vertex w. Also let $G(\Lambda)$ be the rainbow monochromization of Λ. Then $G(\Lambda_{out})$ is an outsplit of $G(\Lambda)$ at w.*

Proof. Consider the outsplit of Λ at a regular vertex w. Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be the partition of $s^{-1}(w)$. The goal is to find a sufficient partition of $s^{-1}(w) \cap \Lambda^{\mathbb{1}}$ so that $G(\Lambda)$ is outsplit.

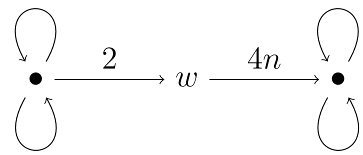
Recall that if $f, g \in s^{-1}(w)$ and $af = bg$ then $f, g \in \mathcal{E}_i$ for some i. Let $\mathscr{E}_1, \ldots, \mathscr{E}_n$ be a partition of $s^{-1}(w)$ in $G(\Lambda)$ by $\lambda = \mu f \in \mathscr{E}_i \Leftrightarrow f \in \mathcal{E}_i$. Put another way, if f, g are outsplit conjugate in Λ and $\lambda_1 = \mu f$ and $\lambda_2 = \nu g$ then λ_1, λ_2 are outsplit conjugate in $G(\Lambda)$. We partition $s^{-1}(w)$ in $G(\Lambda)$ following a similar pairing condition as partitioning $s^{-1}(w)$ in Λ. □

Example 1.5.1. On the next page is a k-graph, with $s^{-1}(w) = \{d_1, d_2, e_1, \ldots, e_{2n}\}$ and two possible outsplits, depending on the partition of $s^{-1}(w)$ and the edge relations defined for the picture. Observe that $s^{-1}(w)$ includes two red (dashed) loops and n blue (solid) edges. So the partition can take one of two forms: either both red loops are in the same $\mathcal{E}_i$ set, or one loop is in $\mathcal{E}_1$ and the other loop is in $\mathcal{E}_2$. Whichever blue edges are outsplit conjugate to each loop will follow due to the pairing condition. Beside each k-graph is its associated rainbow monochromization. Observe that a similar phenomenon occurs for the partition of $s^{-1}(w)$ within $G(\Lambda)$. Either all of the paths are filtered into one of the vertices, or the paths are exactly split halfway.

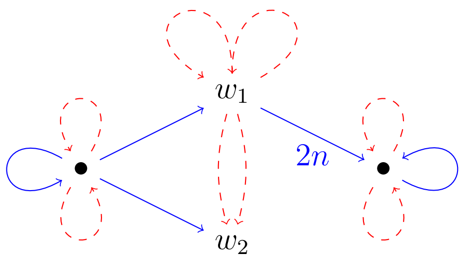
Λ



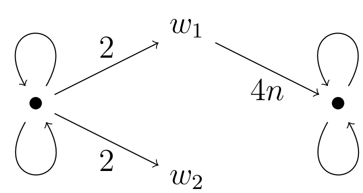
$G(\Lambda)$



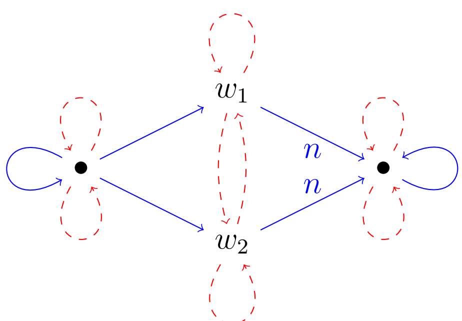
Option 1 for Λ_{out}



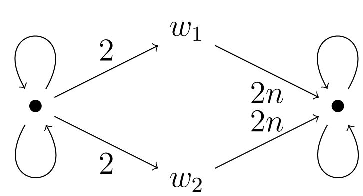
Option 1 for $G(\Lambda_{\text{out}})$



Option 2 for Λ_{out}



Option 2 for $G(\Lambda_{\text{out}})$



Chapter 2

Functional Analysis Basics

Remark 2.0.1. For this chapter, $\mathbb{F}$ will refer to a field (usually $\mathbb{R}$ or $\mathbb{C}$).

Definition 2.0.1. Let X be a vector space over a field $\mathbb{F}$. An inner product on X is a function $\langle \cdot, \cdot \rangle \colon X \times X \to \mathbb{F}$ which satisfies the following properties for all $a, b \in X$ and all $x, y, z \in \mathbb{F}$:

(a) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$;

(b) $\langle x, ay + bz \rangle = \overline{a}\langle x, y \rangle + \overline{b}\langle x, z \rangle$;

(c) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$;

(d) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

Definition 2.0.2. Let X be a vector space over a field $\mathbb{F}$. A norm on X is a function $\|\cdot\| \colon X \to \mathbb{R}$ which satisfies the following properties for all $a, b \in X$ and for all $\lambda \in \mathbb{F}$:

(a) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$;

(b) $\|\lambda x\| = |\lambda| \|x\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Remark 2.0.2. An inner product induces a norm by describing $\|x\| := \langle x, x \rangle^{1/2}$. Similarly, a norm induces a metric by describing $d(x, y) := \|x - y\|$, and in turn a metric induces a topology known as the metric topology.

Definition 2.0.3. A Hilbert space is a vector space over a field $\mathbb{F}$ that is also equipped with an inner product such that the space is complete with respect to the metric induced by the inner product.

Definition 2.0.4. A Banach space is a vector space over a field $\mathbb{F}$ that is also equipped with a norm such that the space is complete with respect to the distance induced by this norm.

Example 2.0.1. Vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces when $\mathbb{R}^n$ is equipped with the dot product and when $\mathbb{C}^n$ is equipped with the standard Hermitian inner product. Note also that all Hilbert spaces are Banach spaces due to the norm and metric induced by the inner product.

Definition 2.0.5. An algebra is a vector space over a field $\mathbb{F}$, which is further equipped with a bilinear product. (This also gives the algebra a ring structure.) We say an algebra is associative if the multiplication operation is associative. We say the algebra is unital if it contains an identity element with respect to the multiplication. (Often the field $\mathbb{F}$ is known and doesn't need to be specified. If it does, the shorthand "$\mathbb{F}$-algebra" is commonly used.)

Example 2.0.2. Let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices with elements from a field $\mathbb{F}$. This forms an associative unital $\mathbb{F}$-algebra.

Example 2.0.3. Let X be a set with the discrete topology. Define $l^\infty(X)$ to be the set of bounded functions $f\colon X \to \mathbb{C}$ and equip the norm $\|f\| := \sup_{i \in I} \{|f(i)|\}$. This is a Banach space. Furthermore, define $c_0(I)$ to be the set of functions $f\colon I \to \mathbb{C}$ such that $\forall \epsilon > 0$, the set $\{i \in I\colon |f(i) \geq \epsilon\}$ is finite. Then $c_0(I)$ is a Banach space as well when equipped with the same norm. If $I = \mathbb{N}$ then $l^\infty(\mathbb{N})$ consists of all bounded sequences of complex numbers, and $c_0(\mathbb{N})$ consises of sequences which converge to 0.

Example 2.0.4. More generally, let X be a set and let $1 \leq p < \infty$. Define $l^p(X)$ to be the set of functions $f\colon X \to \mathbb{C}$ such that $S(f) := \sum_{x \in X} |f(x)|^p$ is finite, then define the norm on $l^p(X)$ to be $\|f\|_p := S(f)^{1/p}$. Then $l^p(X)$ is a Banach space.

Definition 2.0.6. A Banach algebra is an associative algebra (over either $\mathbb{R}$ or $\mathbb{C}$) equpped with a norm, such that the algebra is complete with respect to its induced metric. (Put anotherway, a Banach algebra is an algebra and a Banach space.)

Example 2.0.5. Both $\mathbb{R}$ and $\mathbb{C}$ are Banach algebras, since multiplication is well-defined. But $\mathbb{R}^n$ and $\mathbb{C}^n$ are not algebras if $n > 1$ using the typical norm, because an appropriate multiplication is not defined between vectors. However, let the norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ be defined by $\|x\| \coloneqq \max_{1\leq i\leq n} |x_i|$ and then define multiplication component-wise betwen the vectors. This construction builds a Banach algebra.

Example 2.0.6. The algebra $M_n(\mathbb{F})$ from Example 2.0.2 can be equipped with a matrix norm to make it a Banach algebra.

Example 2.0.7. Let X be a locally compact Hausdorff space and consider $C_0(X)$ the space of continuous function $f\colon X \to \mathbb{C}$ which vanish at infinity (i.e. $\forall\epsilon > 0$, the set $\{x \in X\colon |f(x)| > \epsilon\}$ is compact) is a vector space over $\mathbb{C}$ using function addition and scalar multiplication. Furthermore, equipping $C_0(X)$ with the norm $\|f\| \coloneqq \sup_{x\in X} |f(x)|$ makes it a Banach space. Lastly define function multiplication in the usual fashion to make $C_0(X)$ a Banach algebra.

Definition 2.0.7. An involution map on a Banach $\mathbb{F}$-algebra $\mathscr{B}$ is a function $*\colon \mathscr{B} \to \mathscr{B}$ which satisfies the following properties for all $a, b \in \mathscr{B}$ and $\lambda \in \mathbb{F}$:

(a) $(a^*)^* = a$;

(b) $(ab)^* = b^*a^*$;

(c) $(\lambda a + b)^* = \overline{\lambda}a^* + b^*$.

Definition 2.0.8. A C^*-algebra is a Banach algebra with is further equipped with an involution map which satisfies the adjoint property $\|a^*a\| = \|a\|^2$.

Example 2.0.8. The Banach algebra $M_n(\mathbb{C})$ from Example 2.0.6 has an involution map given by the conjugate transpose. This involution map satisfies the adjoint property. So $M_n(\mathbb{C})$ is a C^*-algebra.

Theorem 2.0.1. *More generally, let H be a Hilbert space. Then $B(H) \coloneqq \{f \colon H \to H \colon \|f\| < \infty\}$ is a C^*-algebra.*

Definition 2.0.9. Let H be a Hilbert space, and let $\mathcal{K}(H)$ represent the functions of H with "compact support," i.e. $K(H) \coloneqq \{f \colon H \to \mathbb{C} \mid H \backslash f^{-1}(0) \text{ is compact}\}$. Note that $K(H)$ is a $*$-subalgebra of $B(H)$ (and thus a C^*-algebra also).

Definition 2.0.10. Let $T \in B(H)$. We say T is a compact operator if the image of each bounded set under T is relatively compact. Denote $\mathcal{K}$ to be the set of bounded operators of H.

Definition 2.0.11. Let A be a C^*-algebra, and let $p \in A$ be a projection. Then pAp is a C^*-subalgebra of A known as a corner. A C^*-subalgebra is full if it is not contained in any proper (two-sided) closed ideal.

Theorem 2.0.2. *Let A be a finite-dimensional C^*-algebra. Then there exists a partition $dim(A) = n_1 + ...n_k$ such that $A \cong \oplus_i M_{n_i}(\mathbb{C})$.*

Theorem 2.0.3. *Let A be a C^*-algebra. Then there exists a Hilbert space H for which $A \subseteq B(H)$ as a subalgebra.*

Proof. This is known as the Gelfand-Naimark-Segal (GNS) construction. □

Definition 2.0.12. A representation of a C^*-algebra A is a pair (H, π) where H is a Hilbert space and π is a $*$-homomorphism $\pi \colon A \to B(H)$. If A has an identity, it is assumed that $\pi(1) = 1$. We say the representation is faithful if π is injective.

To prove a space is a C^*-algebra, it will often be the goal to construct a suitable Hilbert space H and find an appropriate representation into $B(H)$. Observe that there is a choice of representation π into $B(H)$. The "best" choice is the universal representation.

Definition 2.0.13. Let A be a C^*-algebra. The universal representation of A is a $*$-isomorphism $A \hookrightarrow B(H)$. Notice that as a $*$-isomorphism, this representation is faithful.

2.1 Groupoid C^*-algebras

We can build C^*-algebras from topological groups and groupoids. We will focus on the groupoid construction. (Note that via Example 2.1.1 below that the group construction is a specific example of the groupoid construction.)

The groupoid is a generalization of the group. Think of a groupoid as a group whose binary operation is not fully defined for all elements.

Definition 2.1.1. A groupoid is a set $\mathcal{G}$ equipped with an inverse map and a partial multiplication map with the following multiplication properties defined:

- Multiplication is associative when possible. (i.e. for all $a, b \in \mathcal{G}$, if ab and ac are defined, then so are $(ab)c$ and $a(bc)$. Furthermore, these two are equal.)
- aa^{-1} and $a^{-1}a$ are always defined.
- If ab is defined, then $aa^{-1}b = b$ and $a^{-1}ab = b$.

Note that according to the previous definition, we get $(a^{-1})^{-1} = a$ and also $(ab)^{-1} = b^{-1}a^{-1}$. Some further definitions which help us study groupoids are given below.

Definition 2.1.2. Let $\mathcal{G}$ be a groupoid. The set $\mathcal{G}^0$ consists of the "identity elements" of $\mathcal{G}$. That is, $\mathcal{G}^0 \coloneqq \{g^{-1}g | g \in \mathcal{G}\}$. Furthermore, for the sake of convenience denote $\mathcal{G}^2 \coloneqq \{(f, g) | f, g, fg \in \mathcal{G}\}$. (Think of $\mathcal{G}^2$ as the collection of possible products.) It can be helpful to visualize $\mathcal{G}$ as a graph, where $\mathcal{G}^2$ works as the edges of the graph between the elements of $\mathcal{G}$. You can define source and range maps $s, r \colon \mathcal{G} \to \mathcal{G}^0$ using $s(g) = g^{-1}g$ and $r(g) = gg^{-1}$.

Example 2.1.1. A group G is a groupoid where $\mathcal{G}^{(0)} = \{e\}$.

Example 2.1.2. Let X be a set with an equivalence relation $\equiv$. Note that the equivalence relation can be described as a set $R \coloneqq \{(x, y) \in R \times R \colon x \equiv y\}$. Then define a groupoid with $\mathcal{G} = R$ and define multiplication as $(x, y)(y, z) = (x, z)$ only when the inner parts match.

Often, groupoids will be given topologies. The following definitions will be helpful.

Definition 2.1.3. A topological groupoid $\mathcal{G}$ is étale if the maps r, s are local homeomorphisms, i.e. for every $g \in \mathcal{G}$ there exist open sets $U, V \ni g$ such that $r(U), s(V)$ are open in $\mathcal{G}$ and both $r|_U \colon U \to r(U)$ and $s|_V \colon V \to s(V)$ are homeomorphisms. For an étale groupoid $\mathcal{G}$, a subset $B \subseteq \mathcal{G}$ is a bisection if there is an open set $B \subseteq U \subseteq \mathcal{G}$ such that $r|_U, s|_U$ are both homeomorphisms onto open subsets of $\mathcal{G}^0$.

Definition 2.1.4. Let $\mathcal{G}$ be an étale groupoid which is locally compact and Hausdorff. Denote $C_c(\mathcal{G})$ to be the set of compactly supported continuous complex-valued functions on $\mathcal{G}$. Also, denote $\mathcal{C}(\mathcal{G})$ to be the set of functions in $C_c(\mathcal{G})$ which are supported in some $\mathcal{G}$-set.

For an étale groupoid $\mathcal{G}$ which is locally compact and Hausdorff, we can append $C_c(\mathcal{G})$ with the following additional structure so that it is a $*$-algebra. For $\lambda \in \mathbb{F}$ and $x, y \in C_c(\mathcal{G})$, we define:

(a) $(\lambda \cdot x)(g) = \lambda x(g)$;

(b) $(x + y)(g) = x(g) + y(g)$;

(c) $x^*(g) = \overline{x(g^{-1})}$;

(d) $(x * y)(g) = \sum_{r(h)=r(g)} x(h)y(h^{-1}g)$

Then further equip with the norm $\|a\| := \sup\{\|\pi(a)\| \mid \pi \colon C_c(\mathcal{G}) \to B(H)\}$ and complete the space with respect to its norm to form a C^*-algebra, denoted by $C^*(\mathcal{G})$. Alternatively, if we equip with the norm $\|a\|_r := \sup\{\|\pi_\lambda^u(a)\| \mid u \in \mathcal{G}^0\}$ (where π_λ^u is a specific representation) and complete the space with respect to this norm, we form the reduced C^*-algebra denoted by $C_r^*(\mathcal{G})$.

Lemma 2.1.0.1. *Let $\mathcal{G}$ be a groupoid for which $C_r^*(\mathcal{G})$ exists. There is a groupoid $\mathcal{H}$ such that $C_r^*(\mathcal{H}) \cong C_r^*(\mathcal{G}) \otimes \mathsf{M}_n(\mathbb{C})$.*

Proof. Construct $\mathcal{H} = \mathcal{G} \times (\{1, \ldots, n\} \times \{1, \ldots, n\})$ with operations and topology defined as follows:

- $(g,(i,j))\,(h,(k,l)) = (gh,(i,l))$, only defined if $j = k$ and if gh is defined in $\mathcal{G}$.
- $s\left((g,(i,j))\right) = (s(g),(j,j))$
- $r\left((g,(i,j))\right) = (r(g),(i,i))$
- The basis for the topology of $\mathcal{H}$ is $\{\mathcal{U}_g = (U_g, \{i\}, \{j\}) \colon g \in U_g \in \tau(\mathcal{G})\}$.

Then let $\phi\colon C_r^*(\mathcal{G}) \otimes \mathsf{M}_n(\mathbb{C}) \to C_r^*(\mathcal{H})$ via $\phi(f) = \sum_{i,j=1}^{n} f_{ij} \otimes e_{ij})$ where $f_{ij}(g) = f(g,(i,j)$. This homomorphism is invertible, with $\phi^{-1}(h \otimes e_{ij}) = h_{ij}$ where

$$h_{ij}(g,(k,l)) = \begin{cases} h(g) & \text{if } (k,l) = (i,j) \\ 0 & \text{else} \end{cases}.$$

Furthermore, the map indeed acts as a homomorphism. It preserves properties (a) and (b) above by definition of a sum. To show the preservation of property (c), observe that $f^*(g,(i,j)) = \overline{f(g^{-1},(j,i))}$. So then for all $g \in \mathcal{G}$, we get

$$\phi(f^*)(g) = \sum_{i,j=1}^{n} \overline{f_{ji}(g^{-1})} \otimes e_{ij} = \overline{\sum_{i,j=1}^{n} f_{ji}(g^{-1}) \otimes e_{ji}} = \overline{\sum_{i,j=1}^{n} f_{ij}(g^{-1}) \otimes e_{ij}} = \overline{\phi(f)(g^{-1})} = (\phi(f))^*(g)$$

We need to show that property (d) persists. Let $y = f_1 * f_2 \in C_c(\mathcal{H})$ and note that

$$\begin{aligned} y(g,(i,j)) = (f_1 * f_2)(g,(i,j)) &= \sum_{h \in \mathcal{H}, r(h) = (g,(i,i))} f_1(h) f_2(h^{-1}(g,(i,j))) \\ &= \sum_{a \in \mathcal{G}, r(a) = r(g), k \in \{1,\cdots,n\}} f_1(a,(i,k)) f_2(a^{-1}g,(k,j)) \end{aligned}$$

Thus for all $g \in \mathcal{G}$ we get

$$\begin{aligned}
\phi(f_1 * f_2)(g) = \phi(y)(g) &= \sum_{i,j=1}^{n} y_{ij}(g) \otimes e_{ij} \\
&= \sum_{a \in \mathcal{G}, r(a)=r(g)} \left(\sum_{i,j=1}^{n} \overbrace{\sum_{k=1}^{n} (f_1)_{ik}(a)(f_2)_{kj}(a^{-1}g) \otimes e_{ij}}^{\text{matrix multiplication}} \right) \\
&= \sum_{a \in \mathcal{G}, r(a)=r(g)} \left[\left(\sum_{i,k=1}^{n} (f_1)_{ik}(a) \otimes e_{ik} \right) \left(\sum_{k,j=1}^{n} (f_2)_{kj}(a^{-1}g) \otimes e_{kj} \right) \right] \\
&= \sum_{a \in \mathcal{G}, r(a)=r(g)} \phi(f_1)(a)\phi(f_2)(a^{-1}g) \\
&= (\phi(f_1) * \phi(f_2))(g)
\end{aligned}$$

□

2.2 The Diagonal Subalgebra

Definition 2.2.1. For a groupoid $\mathcal{G}$, the diagonal subalgebra $D(\mathcal{G}) \subseteq C_r^*(\mathcal{G})$ is defined as $D(\mathcal{G}) \coloneqq c_0(\mathcal{G}^0)$.

The above definition coincides with a diagonal subalgebra of a Cuntz-Krieger algebra, via $S_\alpha S_\alpha^* \leftrightarrow \chi_{Z(\alpha,\alpha)}$.

2.3 The Multiplier Algebra

Definition 2.3.1. For any C^*-algebra A, the multiplier algebra $M(A)$ is the largest unital C^*-algebra containing A as an (non-degenerate) ideal. Define the set $M(A)$ to be pairs of maps $L, R\colon A \to A$ such that for all $a, b \in A$ we get $aL(b) = R(a)b$. Note that L, R are both bounded linear maps, with $\|L\| = \|R\|$. Thus we can equip $M(A)$ with a norm and an adjoint map to make it a C^*-algebra. Let $\|(L,R)\| \coloneqq \|L\|$ and $(L,R)^* \coloneqq (R^\sharp, L^\sharp)$, where

$R^\sharp(a) = R(a^*)^*$. Addition is component-wise and multiplication is given by composition: $(L_1, R_1) + (L_2, R_2) = (L_1 + R_1, L_2 + R_2)$ and $(L_1, R_1)(L_2, R_2) = (L_1 \circ L_2, R_2 \circ R_1)$.

Example 2.3.1. If A is a unital C^*-algebra, then $M(A) = A$.

Example 2.3.2. Let H be a Hilbert space, and let A be the set of compact operators on H. Then $M(A) = B(H)$.

Chapter 3

Graph Algebras

3.1 Cuntz-Krieger relations

Definition 3.1.1. For a row-finite directed graph $E = (E^0, E^1, r, s)$, we assign a family of projections and isometries $\{P_v \colon v \in E^0\} \cup \{S_e \colon e \in E^1\}$ which satisfy the following properties:

(CK1) The projections P_v are mutually orthogonal.

(CK2) For $e \in E^1$, $S_e^* S_e = P_{s(e)}$.

(CK3) For $v \in E^0$, $P_v = \sum_{r(e)=v} S_e S_e^*$.

This is known as a Cuntz-Krieger family for E. We can take the C^*-closure of the family $\{P_v, S_e\}$ to form a C^*-algebra, often denoted $C^*(P_v, S_e)$.

For any applicable graph E, there may be several Cuntz-Krieger families which can be defined. Each family generates a related C^*-algebra. Kumjian and Pask determined the existence of a *universal* graph algebra (denoted $C^*(E)$), such that for any graph algebra generated by a Cuntz-Krieger family (for example, $A = C^*(Q_v, T_e)$), there exists a $*$-homomorphism $\phi\colon C^*(E) \to A$ such that $\phi(P_v) = Q_v$ and $\phi(S_e) = T_e$ for all $v \in E^0$ and $e \in E^1$. This universal algebra is unique up to $*$-isomorphism.

Remark 3.1.1. For a path $\mu = e_n \cdots e_1 \in E^*$, a partial isometry is formed by applying each respective isometry in turn: $s_\mu \coloneqq s_{e_n} \cdots s_{e_1}$. If E is source-free, then the universal graph algebra $C^*(E) = C^*(p_v, s_e)$ can also be generated using $\mathcal{A} = \{s_\mu s_\nu^* \colon \mu, \nu \in G^*\}$. To see this, note that $p_v \in \operatorname{span}(A)$ via (CK3) and $s_e = s_e p_{s(e)} = s_e \sum s_f s_f^* = \sum s_{ef} s_f^* \in \operatorname{span}(\mathcal{A})$. (The other inclusion is obvious.)

The universal graph algebra admits a gauge action, which is a family of maps $\beta \colon \mathbb{T} \to \operatorname{Aut}(C^*(E))$ with $\beta_z(P_v) = P_v$ and $\beta_z(S_e) = zS_e$. Note that the gauge action extends to finite paths on E. For $\mu = e_n e_{n-1} \cdots e_1 \in E^*$, we have $\beta_z(\mu) = z^{d(\mu)} S_\mu$. Furthermore, $\beta_z(S_e^*) = z^{-1} S_e$ so that $\beta_z(S_e^* S_e) = \beta_z(P_{r(e)}) = P_{r(e)}$. As seen in Theorem 3.4.6, we are often interested in when a homomorphism between two graph algebras intertwines the gauge action.

3.2 The path groupoid

A groupoid can of a graph can be constructed by utillizing relationships between infinite paths. This construction is written in [Carlsen and Winger, 2018]. Consider the set

$$\mathcal{G}_E \coloneqq \{(x, m-n, y) \mid x, y \in \partial E,\ m, n \in \mathbb{N},\ \text{and } \sigma_E^m(x) = \sigma_E^n(y)\},$$

and define multiplication via $(x, k, y)(w, l, z) = (x, k+l, z)$ when $y = w$ and undefined otherwise. Note that the inverse operation is $(x, k, y)^{-1} = (y, -k, x)$ and $\mathcal{G}_E^0 = \{(x, 0, x) : x \in E^0\}$. The groupoid is further endowed with a topology given by a basis which is carried over from the boundary path space:

$$Z(U, m, n, V) \coloneqq \{(x, k, y) \in \mathcal{G}_E : x \in U, y \in V, k = m - n, \sigma_E^m(x) = \sigma_E^n(y)\}$$

Each cylinder set is closed, open, and compact.

As seen in Chapter 2, two C^*-algebras can be constructed from this groupoid, depending on choice of norm. These are labeled $C^*(\mathcal{G}_E)$ and $C_r^*(\mathcal{G}_E)$ respectively.

Theorem 3.2.1. *Let E be a directed graph, and let $\mathcal{G}_E$ be the associated path groupoid. Define $C^*(E)$ to be the graph's universal C^*-algebra. Then $C^*(E) \cong C^*_r(\mathcal{G}_E) \cong C^*(\mathcal{G}_E)$.*

Proof. This is proven in the higher-rank case as Corollary 3.5 in [Kumjian and Pask, 2000]. □

3.3 Types of shift space equivalence

In this section, we give a brief overview of different kinds of equivalence for shift spaces. As mentioned in Chapter 1, if the underlying directed graphs E and F are isomorphic, then $X_E \cong X_F$. However there are some weaker conditions of equivalence between shift spaces which are also worth studying.

Some definitions of equivalence are based on the notion of a one-sided shift space. The collection of one-sided shift spaces has a one-to-one correspondence with the collection of (two-sided) shift spaces, and so both can be studied. The definition of orbit equivalence and the related theorem below are attributed to [Carlsen and Winger, 2018].

Definition 3.3.1. Let E and F be directed graphs. A homeomorphism $h\colon \partial E \to \partial F$ is called an orbit equivalence if there exist continuous functions $k, l\colon \partial E^{\geq 1} \to \mathbb{N}$ and $k', l'\colon \partial F^{\geq 1} \to \mathbb{N}$ such that

$$\sigma_F^{k(x)}(h(\sigma_E(x))) = \sigma_F^{l(x)}(h(x)) \text{ and } \sigma_E^{k'(y)}(h^{-1}(\sigma_F(y))) = \sigma_E^{l'}(h^{-1}(y))$$

for all $x \in \partial E^{\geq 1}$ and for all $y \in \partial F^{\geq 1}$. We say the graphs are orbit equivalent if there exists such a map.

Theorem 3.3.1. *Let E and F be directed graphs and let $\mathcal{G}_E, \mathcal{G}_F$ be their corresponding path groupoids. Then $\mathcal{G}_E, \mathcal{G}_F$ are isomorphic if and only if there is an orbit equivalence $h\colon \partial E \to \partial F$ which preserves isolated evantually periodic points.*

Definition 3.3.2. Let E, F be row-finite directed graphs. We say that E and F are *flow* *equivalent* if there exists a sequence of standard transformations from E to F. (A standard graph transformation is one of six types: an insplit, outsplit, in-amalgamation, out-amalgamation, expansion, or contraction.) We say there is a flow equivalence between adjacency matrices A and B if their corresponding graphs (E_A and E_B) are flow equivalent.

Shift spaces have their own definition of flow equivalence. As the naming convention implies, if two graphs are flow equivalent then their corresponding shift spaces are also flow equivalent.

Definition 3.3.3. For a shift space X, define an equivalence relation $\sim$ on $X \times \mathbb{R}$ via $(\sigma_X^n(x), t) \sim (x, t+n)$. Apply the product topology to $X \times \mathbb{R}$, then define the suspension of X (denoted SX) via the quotient of $X \times \mathbb{R}/\sim$ equipped with the quotient topology.

Definition 3.3.4. We say two shift spaces X and Y are flow equivalent if there exists a homeomorphism $\psi\colon SX \to SY$ such that whenever $a, b > 0$ and $\psi([(x,t)]) = [(y,r)]$, then there exists $v > 0$ such that $\psi([(x, t+a)]) = [(y, r+v)]$ and there exists $w > 0$ such that $\psi^{-1}([(y, r+b)]) = [(x, t+w)]$.

Theorem 3.3.2. *Suppose that A and B are non-negative irreducible square integer matrices, and neither matrix is in the trivial flow equivalence class. Assume A is $n \times n$ and B is $m \times m$. Then A and B are flow equivalent if and only if the following conditions are both satisfied:*

- $\det(I_n - A) = \det(I_m - B)$
- $\mathbb{Z}^n/(I_n - A)\mathbb{Z}^n \cong \mathbb{Z}^m/(I_m - B)\mathbb{Z}^m$

The values defined as $PS(A) = \det(I_n - A)$ and $BF(A) = \mathbb{Z}^n/(I_n - A)\mathbb{Z}^n$ are called the Parry-Sullivan number and Bowen-Franks group of A, respectively. For a graph E, the invariants are defined using their adjacency matrix: $PS(E) \coloneqq PS(A_E)$ and $BF(E) \coloneqq BF(A_E)$.

(This theorem is known as Franks' Theorem, as seen in [Abrams et al., 2011].)

Definition 3.3.5. Let A and B be adjacency matrices for graphs. We say that A is *shift equivalent with lag n* to B (or likewise their graphs are shift equivalent with lag n) if there exist matrices R and S such that $AR = RB$, $SA = BS$, $A^n = RS$ and $B^n = SR$. Often the value of n is unimportant and is not mentioned. We use $A \sim_{SE} B$ to denote shift equivalence.

Definition 3.3.6. Let A and B be adjacenty matrices for graphs. We say that A is *strong shift equivalent* to B (or likewise their graphs are strong shift equivalent) if there exists a finite sequence of matrices $\{A_i\}_{i=0}^m$ where $A = A_0$, $B = A_m$, and for each adjacent pair A_i and A_{i+1}, there exist matrices R_i and S_i such that $A_i = R_iS_i$ and $A_{i+1} = S_iR_i$. We use $A \sim_{SSE} B$ to denote strong shift equivalence.

Remark 3.3.1. It is worth mentioning that strong shift equivalence is a stronger condition than flow equivalence. Furthermore, flow equivalence can be determined by studying the graphs of the respective shift spaces and forming a sequence of constructions. The generalized question of whether such a classification can be determined for graph algebras is studied (and the reason for the main theorem, in the next chapter).

3.4 Morita Equivalence for graph algebras

In the language of graph algebras, the (strong) Morita equivalence is the most frequently used correspondence. It is weaker than an isomorphism between C^*-algebras but many important properties are still preserved between Morita equivalence. (There are two types of Morita equivalence defined between C^*-algebras. Both will be defined here. It is understood that the (strong) Morita equivalence is the desired property, and hence in all future chapters, the (strong) will be dropped.)

Definition 3.4.1. Two unital rings R and S are said to be *(weakly) Morita equivalent* if their respective categories of modules are equivalent.

Note the requirement of the rings to be unital, which C^*-algebras often are not.

Definition 3.4.2. Two C^*-algebras A and B are *(strongly) Morita equivalent* if there exists an $A-B$-imprivity bimodule X (put another way, X is both a full left Hilbert A-module and a full Hilbert right B-module which also satisfies the inner product properties below).

(a) For all $x, y \in X, a \in A, b \in B$, we have that $\langle a \cdot x, y\rangle_B = \langle x, a^* \cdot y\rangle_B$ and also ${}_A\langle x \cdot b, y\rangle = {}_A\langle x, y \cdot b^*\rangle$.

(b) For all $x, y, z \in X$, we have that ${}_A\langle x, y\rangle \cdot z = x \cdot \langle y, z\rangle_B$.

For an in-depth look on properties which are preserved under the (strong) Morita equivalence of C^*-algebras, see [an Huef et al., 2005].

Definition 3.4.3. Two C^*-algebras A and B are *stably isomorphic* if there exists an isomorphism between $A \otimes \mathcal{K}(\mathcal{H})$ and $B \otimes \mathcal{K}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space.

The following important theorems regarding Morita equivalence between C^*-algebras come from [Raeburn and Williams].

Theorem 3.4.1. *Stable isomorphism implies Morita equivalence.*

Theorem 3.4.2. *Two C^*-algebras A and B are Morita equivalent if and only if there exists a C^*-algebra C with complementary full corners isomorphic to A and B respectively. (That is, there exist projections $p, q \in M(C)$ such that $A = pCp, B = qCq$ and $p + q = 1_{M(C)}$.) We call C the linking algebra.*

Theorem 3.4.3. *If A and B are Morita equivalent, then there exists a lattice isomorphism between the sets of closed (two-sided) ideals $\mathcal{I}(A) \to \mathcal{I}(B)$. This is known as the Rieffel correspondence.*

Proposition 3.4.1. *Let A,B be Morita equivalent C^*-algebras with $J \in \mathcal{I}(A), K \in \mathcal{I}(B)$. Then A/J and B/K are C^*-algebras, and $A/J, B/K$ are Morita equivalent.*

Theorem 3.4.4. *(Brown-Green-Rieffel Theorem) Morita equivalence implies stable isomorphism.*

In the case of directed graphs, Morita equivalence of the graph algebra coincides nicely with the shift equivalences learned for shift spaces. For example, Bates and Pask proved in 2004 that for row-finite graphs E and F, we have $E \sim_{SSE} F \Rightarrow C^*(E) \sim_{ME} C^*(F)$.

(See [Bates and Pask, 2004].) Furthermore, the algebras formed via the below definition also gives rise to Morita equivalence between the two graph algebras.

Definition 3.4.4. Let E be a row-finite graph, and let $\mathbb{K}$ be a field. The *Leavitt path* $\mathbb{K}$-*algebra* is the $\mathbb{K}$-algebra generated by a set of pairwise orthogonal idempotents $\{P_v : v \in E^0\}$ and a set of variables $\{S_e, S_e^* : e \in E^1\}$, satisfying the following properties:

- $P_{s(e)}S_e = S_e P_{r(e)} = S_e$ for all $e \in E^1$.
- $P_{r(e)}S_e^* = S_e^* P_{s(e)} = S_e^*$ for all $e \in E^1$.
- $S_e^* S_e' = \delta_{e,e'} P_{r(e)}$ for all $e, e' \in E^1$.
- $P_v = \sum_{\{e \in E^1 : s(e)=v\}} S_e S_e^*$ for every $v \in E^0$ for which $s^{-1}(v) \neq \emptyset$.

The Leavitt-path $\mathbb{K}$-algebra is denoted $L_{\mathbb{K}}(E)$. Often if the field is not important to the discussion, the $\mathbb{K}$ is dropped and the algebra is labeled $L(E)$.

The following theorem comes from [Abrams and Tomforde, 2009].

Theorem 3.4.5. *Let E and F be row-finite graphs. If $L_{\mathbb{C}}(E)$ is Morita equivalent to $L_{\mathbb{C}}(F)$ then $C^*(E)$ is (strongly) Morita equivalent to $C^*(F)$.*

We finish this section with a few important uniqueness theorems for directed graphs, quoted from [Raeburn, 2005]. These uniqueness theorems have been generalized to work for k-graphs as well. (For example, see [Kumjian and Pask, 2000] and [Brown et al., 2014].)

Theorem 3.4.6. *(Gauge Invariant Uniqueness Theorem)Let G be a row-finite directed graph, and suppose that $\{T, Q\}$ is a Cuntz-Krieger G-family in a C^*-algebra B where each Q_v is nonzero. Suppose there is a continuous action $\beta : \mathbb{T} \to Aut\ B$ such that $\beta_z(T_e) = zT_e$ for every $e \in G^1$ and $\beta_z(Q_v) = Q_v$ for all $v \in G^0$. Then the representation $\pi_{T,Q}$ is an isomorphism of $C^*(G)$ onto $C^*(T, Q)$.*

Theorem 3.4.7. *(Cuntz-Krieger Uniqueness Theorem (1)) Let G be a row-finite directed graph in which every cycle has an entry, and suppose that $\{T, Q\}$ is a Cuntz-Krieger G-family in a C^*-algebra B where each Q_v is nonzero. Then the homomorphism $C^*(G) \to B$ is an isomorphism of $C^*(G)$ onto $C^*(T, Q)$.*

Corollary 3.4.1. *(Cuntz-Krieger Uniqueness Theorem (2))Let G be a row-finite directed graph in which every cycle has an entry. Suppose that $\{S, P\}$ and $\{T, Q\}$ are both Cuntz-Krieger G-families on some Hilbert space where each P_v and Q_v are nonzero. Then there exists an isomorphism ϕ from $C^*(S, P)$ onto $C^*(T, Q)$ such that $\phi(S_e) = T_E$ for every edge and $\phi(P_v) = Q_v$ for every vertex.*

As mentioned in 3.3.1, a motivating question for graph algebras is whether a classifcation exists between directed graphs which preserve Morita equivalence. In addition, the question remains whether such a classification extends to k-graphs.

3.5 k-graphs

Definition 3.5.1. For a k-graph $\Lambda = (\Lambda^0, \Lambda^{e_1}, \dots, \Lambda^{e_k})$, we assign a family of projections and isometries $\{P_v \colon v \in \Lambda^0\} \cup \{S_e \colon e \in \Lambda^1\}$ which satisfy the following properties:

(CK1) The projections P_v are mutually orthogonal.

(CK2) For $e \in \Lambda^1$, $S_e^* S_e = P_{s(e)}$.

(CK3) For $v \in \Lambda^0$, $P_v = \sum_{r^{-1}(v) \cap \Lambda^{e_i}} S_e S_e^*$ for $1 \le i \le k$.

(CK4) If f, g are paired via $af = bg$, then $S_a S_f = S_b S_g$.

This is a Cuntz-Krieger family which generates a C^*-algebra. Note that in the $k = 1$ case, a 1-graph is equivalent to a directed graph. Similar to the graph algebra experience, there exists a unique (up to $*$-isomorphism) universal C^*-algebra, typically denoted $C^*(\Lambda)$.

Much work has been done on finding transformations of k-graphs which preserve Morita equivalence (see [Eckhardt et al., 2020]). Their work involves utilizing a result from [Allen, 2005]. The core of the argument is copied below.

Definition 3.5.2. For a k-graph Λ, let $X \subseteq \Lambda^0$. Define the *saturation of* X (denoted $\Sigma(X)$) to be the smallest set S such that $X \subseteq S$ and which satisfies the following properties:

(Heredity) If $v \in S$ and $\lambda \in v\Lambda$, then $s(\lambda) \in S$.

(Saturation) If $\{s(\lambda) \colon \lambda \in v\Lambda^n\} \subseteq S$ for some $n \in \mathbb{N}^k$, then $v \in S$.

Remark 3.5.1. Saturated and hereditary sets are useful helpful in finding ideals of C^*-algebras formed from directed graphs. See Theorem 4.9 from [Raeburn, 2005].

Theorem 3.5.1. *Let Λ be an row-finite k-graph and let $X \subseteq \Lambda^0$. Define $P_X \coloneqq \sum_{v \in X} p_v$ (note that $P_X \in M(C^*(\Lambda))$). Then $P_X C^*(\Lambda) P_X$ is Morita equivalent to $C^*(\Lambda)$ if $\Sigma(X) = \Lambda^0$.*

We end the chapter with two propositions which allow comparison between graph algebras of varying values of k. The proofs require building a Cuntz-Krieger family and applying the gauge invariant uniqueness theorem.

Proposition 3.5.1. *Let Λ be a row-finite, source-free $(k+1)$-graph, and let ${}^k\Lambda$ be the k-graph which is obtained by deleting all edges of color $(k+1)$. Then ${}^k\Lambda$ is also row-finite and source-free, and $C^*({}^k\Lambda) \subseteq C^*(\Lambda)$ as a $*$-subalgebra.*

Proof. Let $\iota \colon {}^k\Lambda \to \Lambda$ be the standard inclusion map. Let $\{p_v, s_e\}$ generate $C^*(\Lambda)$ and construct $\{Q_v, T_e\}$ a Cuntz-Krieger ${}^k\Lambda$-family in $C^*(\Lambda)$ via $Q_v = p_{\iota(v)}$ and $T_e = s_{\iota(e)}$. □

Proposition 3.5.2. *Let Λ be a k-graph and let $G(\Lambda)$ be the rainbow monochromization of Λ as in Definition 1.5.2. Then $C^*(G(\Lambda)) \subseteq C^*(\Lambda)$ as a $*$-subalgebra.*

Proof. Every $e = e_\lambda \in G(\Lambda)$ exists as a path $\lambda \in \Lambda^{\mathbb{1}}$. Let $\{p_v, s_e\}$ generate $C^*(\Lambda)$ and construct $\{Q_v, T_e\}$ a Cuntz-Krieger $G(\Lambda)$-family in $C^*(\Lambda)$ via $Q_v = p_v$ and $T_e = s_\lambda$. □

Chapter 4

Outsplit Proof

We now prove the main theorems of the paper.

Theorem 4.0.1. *Let Λ be a (row-finite, source-free) k-graph and let $w \in \Lambda^0$ be a regular vertex. Partition $s^{-1}(w)$ as a finite disjoint union of (possibly empty) subsets $\mathcal{E}_i$, asserting $\mathcal{E}_1 \neq \emptyset$ if $s^{-1}(w) \neq \emptyset$. Define $\psi \colon C^*(\Lambda) \to C^*(\Lambda_{out})$ by $\psi(p_v) = p_{v^1}$ for $v \in \Lambda^0$,and for all $e \in \Lambda^1$ set*

$$\psi(s_e) = \begin{cases} s_{e^1} & \text{if } s_\Lambda(e) \neq w \\ \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^1} s_{e^1} s_{f^j} s_{f^1}^* & \text{if } s_\Lambda(e) = w \text{ and } e \in \mathcal{E}_j \end{cases}.$$

Then ψ is an injective $*$-homomorphism such that $\gamma_z^{\Lambda_{out}} \circ \psi = \psi \circ \gamma_z^{\Lambda}$, $\psi(C^*(\Lambda)) = qC^*(\Lambda_{out})q$, and $\psi(\mathcal{D}_\Lambda) = q\mathcal{D}_{\Lambda_{out}}$ where $q = \sum_{v \in \Lambda^0} p_{v^1}$.

Proof. Set $P_v = p_{v^1}$ for all $v \in \Lambda^0$, and for all $e \in \Lambda^1$ set

$$S_e = \begin{cases} s_{e^1} & \text{if } s_\Lambda(e) \neq w \\ \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^1} s_{e^1} s_{f^j} s_{f^1}^* & \text{if } s_\Lambda(e) = w \text{ and } e \in \mathcal{E}_j \end{cases}$$

We will first show that $\{P_v, S_e\}$ is a Cuntz-Krieger Λ-family in $C^*(\Lambda_{\text{out}})$. Note that for any $e, f \in \Lambda^1$,

$$S_e^* S_f = \begin{cases} s_{e^1}^* s_{f^1} & \text{if } e, f \notin s_\Lambda^{-1}(w) \\ \sum_{g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{g^1} s_{g^j}^* s_{e^1}^* s_{f^1} & \text{if } e \in \mathcal{E}_j \text{ and } f \notin s_\Lambda^{-1}(w) \\ \sum_{g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{e^1}^* s_{f^1} s_{g^j} s_{g^1}^* & \text{if } f \in \mathcal{E}_j \text{ and } e \notin s_\Lambda^{-1}(w) \\ \sum_{g,h \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{g^1} s_{g^j}^* s_{e^1}^* s_{f^1} s_{h^m} s_{h^1}^* & \text{if } e \in \mathcal{E}_j \text{ and } f \in \mathcal{E}_m \end{cases}$$

$$= \begin{cases} \delta_{e,f} P_{s_\Lambda(e)} & \text{if } e, f \notin s_\Lambda^{-1}(w) \\ \delta_{e,f} \sum_{g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{g^1} s_{g^1}^* & \text{if } e, f \in \mathcal{E}_j \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta_{e,f} P_{s_\Lambda(e)}.$$

Furthermore, for any $e \in \Lambda^1$,

$$S_e S_e^* = \begin{cases} s_{e^1} s_{e^1}^* & \text{if } s_\Lambda(e) \neq w \\ \sum_{f,g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{e^1} s_{f^j} s_{f^1}^* s_{g^1} s_{g^j}^* s_{e^1}^* & \text{if } s_\Lambda(e) = w \text{ and } e \in \mathcal{E}_j \end{cases}$$

$$= \begin{cases} s_{e^1} s_{e^1}^* & \text{if } s_\Lambda(e) \neq w \\ \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{e^1} s_{f^j} s_{f^j}^* s_{e^1}^* & \text{if } s_\Lambda(e) = w \text{ and } e \in \mathcal{E}_j \end{cases}$$

$$= \begin{cases} s_{e^1} s_{e^1}^* & \text{if } s_\Lambda(e) \neq w \\ s_{e^1} p_{w^j} s_{e^1}^* & \text{if } s_\Lambda(e) = w \text{ and } e \in \mathcal{E}_j \end{cases}$$

$$= s_{e^1} s_{e^1}^* \leq p_{r_{\Lambda_{\text{out}}}(e^1)} = P_{r_\Lambda(e)}.$$

Let v be a regular vertex in Λ. Then v^1 is a regular vertex in Λ_{out} and $r_{\Lambda_{\text{out}}}^{-1}(v^1) = \{e^1 \colon e \in r_\Lambda^{-1}(v)\}$. Therefore, for any $1 \leq i \leq k$ we have

$$P_v = p_{v^1} = \sum_{e \in r_\Lambda^{-1}(v) \cap \Lambda^{e_i}} s_{e^1} s_{e^1}^* = \sum_{e \in r_\Lambda^{-1}(v) \cap \Lambda^{e_i}} S_e S_e^*.$$

These observations show that CK1, CK2, and CK3 hold for this Λ-family within Λ_{out}. To show that CK4 holds, we introduce a lemma:

Lemma 4.0.1.1. *Let $x \in r^{-1}(w) \cap \Lambda^{e_i}$ and let $1 \leq j \leq n$. Then*

$$\sum_{f \in r^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^* s_{x^1} = s_{x^j}.$$

To prove the lemma, first rewrite any $f \in r^{-1}(w) \cap \Lambda^{\mathbb{1}}$ so that the last edge (f_k) is color e_i. Observe that the Cuntz-Krieger properties of Λ_{out} require that $s_{f^k}^* s_{x^1} = 0$ unless $f_k = x^1$. So without loss of generality, we can assume $f = x\mu$ in the sum (i.e., $f^i = \mu' x^i$ for the equivalent $\mu' \in \Lambda_{\text{out}}$). Therefore $s_{f^j} s_{f^1}^* s_{x^1} = s_{x^j} s_{\mu'} s_{\mu'}^* s_{x^1}^* s_{x^1} = s_{x^j} s_{\mu'} s_{\mu'}^*$. By CK3, $\sum s_{\mu'} s_{\mu'}^* = p_{s(x^i)}$ and the proof is complete.

Suppose that $ax = by$ in Λ. We split into cases. First, assume that $s(x) = s(y) \neq w$. There are four cases depending on $s(a)$ and $s(b)$.

- Case 1: If $s_\Lambda(a), s_\Lambda(b) \neq w$, then $a^1 x^1 = b^1 y^1$ in Λ_{out}. Furthermore, we have

$$S_a S_x = s_{a^1} s_{x^1} = s_{b^1} s_{y^1} = S_b S_y.$$

- Case 2: If $s_\Lambda(a) \neq w$ and $b \in \mathcal{E}_i$, then $a^1 x^1 = b^1 y^i$ in Λ_{out}. Furthermore, we have

$$S_a S_x = s_{a^1} s_{x^1} = s_{b^1} s_{y^i} = \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{b^1} s_{f^i} s_{f^1}^* s_{y^1} = S_b S_y.$$

- Case 3: If $a \in \mathcal{E}_j$ and $s_\Lambda(b) \neq w$, then $a^1 x^j = b^1 y^1$ in Λ_{out}. This case is equivalent to Case 2.

- Case 4: If $a \in \mathcal{E}_j$ and $b \in \mathcal{E}_i$, then $a^1x^j = b^1y^i$ in Λ_{out}. Furthermore, we have

$$\begin{aligned} S_aS_x = \sum_{f \in r_\Lambda^{-1}(w)\cap\Lambda^{\mathbb{1}}} s_{a^1}s_{f^j}s_{f^1}^*s_{x^1} &= s_{a^1}s_{x^j} \\ &= s_{b^1}s_{y^i} \\ &= \sum_{f \in r_\Lambda^{-1}(w)\cap\Lambda^{\mathbb{1}}} s_{b^1}s_{f^i}s_{f^1}^*s_{y^1} = S_bS_y. \end{aligned}$$

Now assume that $s(x) = s(y) = w$. Then x, y are outsplit conjugate and belong in the same partition $\mathcal{E}_k$. Once again there are four cases depending on $s(a)$ and $s(b)$.

- Case 1: If $s_\Lambda(a), s_\Lambda(b) \neq w$, then $a^1x^1 = b^1y^1$ in Λ_{out}. Furthermore, we have

$$S_aS_x = s_{a^1}s_{x^1} \sum_{f \in r_\Lambda^{-1}(w)\cap\Lambda^{\mathbb{1}}} s_{f^k}s_{f^1}^* = s_{b^1}s_{y^1} \sum_{f \in r_\Lambda^{-1}(w)\cap\Lambda^{\mathbb{1}}} s_{f^k}s_{f^1}^* = S_bS_y.$$

- Case 2: If $s_\Lambda(a) \neq w$ and $b \in \mathcal{E}_i$, then $a^1x^1 = b^1y^i$ in Λ_{out}. Furthermore, we have

$$\begin{aligned} S_aS_x &= s_{a^1}s_{x^1} \sum_{f \in r_\Lambda^{-1}(w)\cap\Lambda^{\mathbb{1}}} s_{f^k}s_{f^1}^* \\ &= s_{b^1}s_{y^i} \sum_{f \in r_\Lambda^{-1}(w)\cap\Lambda^{\mathbb{1}}} s_{f^k}s_{f^1}^* \\ &= \sum_{f,g \in r_\Lambda^{-1}(w)\cap\Lambda^{\mathbb{1}}} s_{b^1}s_{g^i}s_{g^1}^*s_{y^1}s_{f^k}s_{f^1}^* = S_bS_y. \end{aligned}$$

- Case 3: If $a \in \mathcal{E}_j$ and $s_\Lambda(b) \neq w$, then $a^1x^j = b^1y^1$ in Λ_{out}. This case is equivalent to Case 2.

- Case 4: If $a \in \mathcal{E}_j$ and $b \in \mathcal{E}_i$, then $a^1 x^j = b^1 y^i$ in Λ_{out}. Furthermore, we have

$$\begin{aligned} S_a S_x = \sum_{f,g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{a^1} s_{g^j} s_{g^1}^* s_{x^1} s_{f^k} s_{f^1}^* &= s_{a^1} s_{x^j} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^k} s_{f^1}^* \\ &= s_{b^1} s_{y^i} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^k} s_{f^1}^* \\ &= \sum_{f,h \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{b^1} s_{h^i} s_{h^1}^* s_{y^1} s_{f^k} s_{f^1}^* = S_b S_y. \end{aligned}$$

This proves that $\{P_v, S_e\}$ is a Cuntz-Krieger Λ-family in $C^*(\Lambda_{\text{out}})$. By the universal property of $C^*(\Lambda)$, there exists a $*$-homomorphism $\psi\colon C^*(\Lambda) \to C^*(\Lambda_{\text{out}})$ such that $\psi(p_v) = P_v$ and $\psi(s_e) = S_e$. It's clear that $\gamma_z^{\Lambda_{\text{out}}} \circ \psi = \psi \circ \gamma_z^{\Lambda}$ on $\{p_v, s_e\}$ and thus the two are equivalent on $C^*(\Lambda)$. Since $\psi(p_v) \neq 0$ for all $v \in \Lambda^0$, the gauge-invariant uniqueness theorem for k-graphs gives that ψ is injective.

It remains to be shown that $\psi(C^*(\Lambda)) = qC^*(\Lambda_{\text{out}})q$ and $\psi(D_\Lambda) = qD_{\Lambda_{\text{out}}}$.

Claim 4.0.1. *For any finite path $\nu = e_m e_{m-1} \dots e_2 e_1 \in \Lambda^{\geq 1}$, there exists a finite path $\mu \in \Lambda_{out}$ such that $r_{\Lambda_{out}}(\mu) \in \{v^1 \colon v \in \Lambda^0\}$,*

$$s_{\Lambda_{out}}(\mu) = \begin{cases} s_\Lambda(e_1)^1 & \text{if } e_1 \notin s_\Lambda^{-1}(w) \\ w^j & \text{if } e_1 \in \mathcal{E}_j \end{cases},$$

and

$$\psi(s_\nu) = \begin{cases} s_\mu & \text{if } e_1 \notin s_\Lambda^{-1}(w) \\ s_\mu \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^* & \text{if } e_1 \in \mathcal{E}_j \end{cases}.$$

We will prove the claim by induction on the length of the paths. Suppose $\nu = e \in \Lambda^1$. Then

$$\psi(s_e) = \begin{cases} s_{e^1} & \text{if } e_1 \notin s_\Lambda^{-1}(w) \\ s_{e^1} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^* & \text{if } e_1 \in \mathcal{E}_j \end{cases}.$$

Set $\mu = e^1$. Then $r_{\Lambda_{\text{out}}}(\mu) = r_\Lambda(e)^1$,

$$s_{\Lambda_{\text{out}}}(e^1) = \begin{cases} s_\Lambda(e)^1 & \text{if } s_\Lambda(e) \neq w \\ w^j & \text{if } e \in \mathcal{E}_j \end{cases},$$

and

$$\psi(s_e) = \begin{cases} s_\mu & \text{if } e \notin s_\Lambda^{-1}(w) \\ s_\mu \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^* & \text{if } e \in \mathcal{E}_j \end{cases}.$$

Now suppose the claim is true for any path in Λ with length m. Let $\nu = e_m e_{m-1} \dots e_1 e_0 \in \Lambda^{m+1}$. We consider the following cases:

- Suppose $s_\Lambda(e_1) \neq w$ and $s_\Lambda(e_0) \neq w$. Then $s(e_m \dots e_1) = r(e_0) = v \neq w$. By induction hypothesis, there exists $\mu' \in \Lambda_{\text{out}}$ such that $r_{\Lambda_{\text{out}}}(\mu') \in \{v^1 \colon v \in \Lambda^0\}$, $s_{\Lambda_{\text{out}}}(\mu') = v^1$ and $\psi(s_{e_m \dots e_1}) = s_{\mu'}$. It follows that $s_{\Lambda_{\text{out}}}(\mu') = v^1 = r_{\Lambda_{\text{out}}}(e_0^1)$ which implies that $\mu = \mu' e_0^1$ with the desired conditions.

- Suppose $s_\Lambda(e_1) \neq w$ and $e_0 \in \mathcal{E}_j$. By induction hypothesis, there exists $\mu' \in \Lambda_{\text{out}}$ such that $r_{\Lambda_{\text{out}}}(\mu') \in \{v^1 \colon v \in \Lambda^0\}$, $s_{\Lambda_{\text{out}}}(\mu') = s_\Lambda(e_1)^1$ and $\psi(s_{e_m \dots e_1}) = s_{\mu'}$. Set $\mu = \mu' e_0^1$ which will satisfy the conditions.

- Suppose $e_1 \in \mathcal{E}_j$ and $e_0 \notin s_\Lambda^{-1}(w)$. Again by induction on $e_m \dots e_1$ There exists $\mu' \in \Lambda_{\text{out}}$ with $r_{\Lambda_{\text{out}}}(\mu') \in \{v^1 \colon v \in \Lambda^0\}$, $s_{\Lambda_{\text{out}}}(\mu') = w^j$, and $\psi(s_{e_m \dots e_1}) = s_{\mu'} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^*$. Note that $s_{\Lambda_{\text{out}}}(\mu') = w^j = r_\Lambda(e_0)^1$ which allows $\mu = \mu' e_0^j$ to satisfy the conditions.

- Suppose $e_1 \in \mathcal{E}_j$ and $e_0 \in \mathcal{E}_n$. Inducting on $e_m \dots e_1$ gives $\mu' \in \Lambda_{\text{out}}$ with $r_{\Lambda_{\text{out}}}(\mu') \in \{v^1 : v \in \Lambda^0\}$, $s_{\Lambda_{\text{out}}}(\mu') = w^j$, and $\psi(s_{e_m \dots e_1}) = s_{\mu'} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^*$. Note that

$r_{\Lambda_{\text{out}}}(e_0) = w^j = s_{\Lambda_{\text{out}}}(\mu')$ thus setting $\mu = \mu' e_0^j$ will satisfy the conditions. We have

$$\psi(s_\nu) = \psi(s_\mu)\psi(e_0) = \left(s_{\mu'} \sum_{g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{g^j} s_{g^1}^* \right) \left(\sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{e_0^1} s_{f^n} s_{f^1}^* \right)$$

$$= s_{\mu'} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{e_0^j} s_{f^j} s_{f^1}^* = s_\mu \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^*$$

The above bullets prove Claim 1.

Claim 4.0.2. *For all $\mu \in \Lambda_{out}$, there exists a path $\nu_\mu \in \Lambda$ with $d(\nu_\mu) = d(\mu)$ where ν_μ is obtained by removing the superscripts of μ.*

To prove the claim, first suppose $\mu = v^i$ for some $v \in \Lambda^0$. Then $\nu_\mu = v$ satisfies the claim. Now let $\mu = e_m^{i_m} \dots e_2^{i_2} e_1^{i_1} \in \Lambda_{\text{out}}$. We want to show $\nu_\mu = e_m \dots e_1$ is a path in Λ (i.e. that $r_\Lambda(e_j) = s_\Lambda(e_{j+1})$ for $1 \leq j \leq m-1$). Again this breaks into cases.

- Suppose $r_\Lambda(e_j) \neq w$. Then $i_j = 1$ and $s_{\Lambda_{\text{out}}}(e_{j+1}^{i_{j+1}}) = s_\Lambda(e_{j+1})^1$. But $r_{\Lambda_{\text{out}}}(e_j^{i_j}) = s_{\Lambda_{\text{out}}}(e_{j+1}^{i_{j+1}}) = s_\Lambda(e_{j+1})^1$, so we conclude $s_\Lambda(e_{j+1}) \neq w$ and thus $s_\Lambda(e_{j+1})^1 = s_{\Lambda_{\text{out}}}(e_{j+1}^{i_{j+1}}) = r_{\Lambda_{\text{out}}}(e_j^{i_j}) = r_\Lambda(e_j)^1$.
- Suppose $r_\Lambda(e_j) = w$. Then $s_{\Lambda_{\text{out}}}(e_{j+1}^{i_{j+1}}) = r_{\Lambda_{\text{out}}}(e_j^{i_j}) = w^{i_j}$. But this implies $e_{j+1} \in \mathcal{E}_{i_j}$ and therefore $s_\Lambda(e_{j+1}) = w$.

The above bullets prove Claim 2.

Claim 4.0.3. *Suppose $\mu \in \Lambda_{out}$ and $\nu_\mu \in \Lambda$ as defined in the previous claim. Then*

$$\psi(s_{\nu_\mu}) = \begin{cases} s_\mu & \text{if } s_{\Lambda_{out}}(\mu) = v^1 \text{ for } v \neq w \\ s_\mu \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^* & \text{if } s_{\Lambda_{out}}(\mu) = w^j \end{cases}.$$

Once again we use induction to prove, this time on the length of μ. First, if $\text{len}(\mu) = 0$ then

WLOG $\mu = v^1$ for some vertex $v \in \Lambda$. So $\nu_\mu = v$ and

$$\psi(p_{\nu_\mu}) = p_{v^1} = \begin{cases} p_{v^1} & \text{if } v \neq w \\ p_{w^1} & \text{if } v = w \end{cases}$$
$$= \begin{cases} p_{v^1} & \text{if } v \neq w \\ p_{w^1} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^1}^* & \text{if } v = w \end{cases}$$
$$= \begin{cases} s_\mu & \text{if } s_{\Lambda_{\text{out}}}(\mu) = v^1 \text{ for } v \neq w \\ s_\mu \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^1}^* & \text{if } s_{\Lambda_{\text{out}}}(\mu) = w^1 \end{cases}$$

Now suppose that the claim is true for any $\mu \in \Lambda_{\text{out}}^{\text{len}(\mu)=m}$ and let $\mu \in \Lambda_{\text{out}}^{m+1}$. (Say, $\mu = \mu' e^i$ where $\mu' \in \Lambda_{\text{out}}^m$.) Let ν_μ and $\nu_{\mu'}$ be the corresponding paths in Λ for μ and μ' respectively. Note that by construction, $\nu_\mu = \nu_{\mu'} e$. There are four cases:

- Suppose $s_{\Lambda_{\text{out}}}(\mu) = v^1$ with $v \neq w$ and $s_{\Lambda_{\text{out}}}(\mu') \neq w^j$. Then $s_\Lambda(\nu_{\mu'}) \neq w$ and $\psi(s_{\nu_{\mu'}}) = s_{\mu'}$. Note $r_\Lambda(e) = s_\Lambda(\nu_{\mu'}) \neq w$, therefore $e^i = e^1$ and

$$\psi(s_{\nu_\mu}) = \psi(s_{\nu_{\mu'}})\psi(s_e) = s_{\mu'} s_{e^1} = s_{\mu'} s_{e^i} = s_\mu.$$

- Suppose $s_{\Lambda_{\text{out}}}(\mu) = v^1$ with $v \neq w$ and $s_{\Lambda_{\text{out}}}(\mu') = w^j$. Then $\psi(s_{\nu_{\mu'}}) = s_{\mu'} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^*$ and $r_\Lambda(e)^i = r_{\Lambda_{\text{out}}}(e^i) = s_{\Lambda_{\text{out}}}(\mu') = w^j$. This implies $r_\Lambda(e) = w$ and $i = j$.Thus, we get

$$\psi(s_{\nu_\mu}) = \psi(s_{\nu_{\mu'}})\psi(s_e) = s_{\mu'} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^* s_{e^1} = s_{\mu'} s_{e^i} = s_\mu.$$

- Suppose $s_{\Lambda_{\text{out}}}(\mu) = w^j$ and $s_{\Lambda_{\text{out}}}(\mu') = v^1$ with $v \neq w$. Then $s_\Lambda(\nu_{\mu'}) =\neq w$, $\psi(s_{\nu_{\mu'}}) = s_{\mu'}$, and $s_{\Lambda_{\text{out}}}(e^i) = s_{\Lambda_{\text{out}}}(\mu) = w^j$. Note that this implies $e \in \mathcal{E}_j$. Observe also that

$r_\Lambda(e) = s_\Lambda(\nu_{\mu'}) \neq w$, so $e^i = e^1$ and

$$\psi(s_{\nu_\mu}) = \psi(s_{\nu_{\mu'}})\psi(s_e) = s_{\mu'} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{e^1} s_{f^j} s_{f^1}^* = s_\mu \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^*.$$

- Suppose $s_{\Lambda_{\text{out}}}(\mu) = w^j$ and $s_{\Lambda_{\text{out}}}(\mu') = w^n$. Then

$$\psi(s_{\nu_{mu'}}) = s_{\mu'} \sum_{g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{g^n} s_{g^1}^* \qquad \text{and} \qquad \psi(s_e) = \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_e s_{f^j} s_{f^1}^*$$

since $s_{\Lambda_{\text{out}}}(\mu) = s_{\Lambda_{\text{out}}}(\mu) = w^j$. So we get

$$\psi(s_{\nu_\mu}) = \psi(s_{\nu_{\mu'}})\psi(s_e) = s_{\mu'} \left(\sum_{g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{g^n} s_{g^1}^* \right) \left(\sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{e^1} s_{f^j} s_{f^1}^* \right).$$

Observe that $w^n = s_{\Lambda_{\text{out}}}(\nu') = r_{\Lambda_{\text{out}}}(e^i) = r_\Lambda(e)^i$ thus we get $s_\Lambda(e) = w$ and $i = n$. In addition,

$$\begin{aligned}
\psi(s_{\nu_\mu}) &= s_{\mu'} \left(\sum_{g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{g^n} s_{g^1}^* \right) \left(\sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_e s_{f^j} s_{f^1}^* \right) \\
&= s_{\mu'} s_{e^n} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^n} s_{f^1}^* \\
&= s_{\mu'} s_{e^i} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^* \\
&= s_\mu \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^1}^*.
\end{aligned}$$

These bullets prove Claim 3. Using these claims, we can finish the proof. First, note that

$\psi(C^*(\Lambda)) \subseteq qC^*(\Lambda_{\text{out}})q$ by observing

$$\begin{aligned}\psi(s_\nu s_\nu^*) &= \psi(p_{r_\Lambda(\nu)})\psi(s_\nu s_\nu^*)\psi(p_{r_\Lambda(\nu)}) \\ &= p_{r_\Lambda(\nu)^1}\psi(s_\nu)\psi(s_\nu)^* p_{r_\Lambda(\nu)^1} \\ &= q\psi(s_\nu)\psi(s_\nu)^* q.\end{aligned}$$

We next show that $q\mathcal{D}_{\Lambda_{\text{out}}} = \psi(\mathcal{D}_\Lambda)$. To prove $\psi(\mathcal{D}_\Lambda) \subseteq q\mathcal{D}_{\Lambda_{\text{out}}}$, assume $\nu \in \Lambda$. If ν is a vertex, then $\psi(p_\nu) = p_{\nu^1} \in q\mathcal{D}_{\Lambda_{\text{out}}}$. Otherwise assume $\nu = e_m \ldots e_1$. From Claim 1, there exists a path $\mu \in \Lambda_{\text{out}}$ such that $r_{\Lambda_{\text{out}}}(\mu) \in \{v^1 \colon v \in \Lambda^0\}$,

$$s_{\Lambda_{\text{out}}}(\mu) = \begin{cases} s_\Lambda(e_1)^1 & \text{if } e_1 \notin s_\Lambda^{-1}(w) \\ w^j & \text{if } e_1 \in \mathcal{E}_j \end{cases},$$

and

$$\psi(s_\nu) = \begin{cases} s_\mu & \text{if } e_1 \notin s_\Lambda^{-1}(w) \\ s_\mu \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^1} s_{f^j} s_{f^1}^* & \text{if } e_1 \in \mathcal{E}_j \end{cases}.$$

Then observe that

$$\begin{aligned}\psi(s_\nu s_\nu^*) &= \begin{cases} s_\mu s_\mu^* & \text{if } e_1 \notin r_\Lambda^{-1}(w) \\ \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^1} s_\mu s_{f^j} s_{f^j}^* s_\mu^* & \text{if } e_1 \in \mathcal{E}_j \end{cases} \\ &= \begin{cases} s_\mu s_\mu^* & \text{if } e_1 \notin s_\Lambda^{-1}(w) \\ \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^1} s_{\mu f^j} s_{\mu f^j}^* & \text{if } e_1 \in \mathcal{E}_j \end{cases} \\ &= \begin{cases} q s_\mu s_\mu^* & \text{if } e_1 \notin s_\Lambda^{-1}(w) \\ q \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^1} s_{\mu f^j} s_{\mu f^j}^* & \text{if } e_1 \in \mathcal{E}_j \end{cases}\end{aligned}$$

(since $r_{\Lambda_{\text{out}}}(\mu) \in \{v^1 \colon v \in \Lambda^0\}$). This shows that $\psi(\mathcal{D}_\Lambda) \subseteq q\mathcal{D}_{\Lambda_{\text{out}}}$.

We still need to show $qC^*(\Lambda_{\text{out}})q \subseteq \psi(C^*(\Lambda))$ and $q\mathcal{D}_{\Lambda_{\text{out}}} \subseteq \psi(\mathcal{D}_\Lambda)$. Let $\mu_1, \mu_2 \in \Lambda_{\text{out}}$ such that $s_{\Lambda_{\text{out}}}(\mu_1) = s_{\Lambda_{\text{out}}}(\mu_2)$ and $r_{\Lambda_{\text{out}}}(\mu_1), r_{\Lambda_{\text{out}}}(\mu_2) \in \{v^1 \colon v \in \Lambda^0\}$. By Claim 2, there exist corresponding paths $\nu_1 \coloneqq \nu_{\mu_1}$ and $\nu_2 \coloneqq \nu_{\mu_2}$ in Λ. Then by Claim 3,

$$\begin{aligned}
\psi(s_{\nu_1} s_{\nu_2}^*) &= \begin{cases} s_{\mu_1} s_{\mu_2}^* & \text{if } s_{\Lambda_{\text{out}}}(\mu_1) = s_{\Lambda_{\text{out}}}(\mu_2) = v^1 \text{ for } v \neq w \\ \sum_{f,g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{\mu_1} s_{f^j} s_{f^1}^* s_{g^1} s_{g^j}^* s_{\mu_2}^* & \text{if } s_{\Lambda_{\text{out}}}(\mu_1) = s_{\Lambda_{\text{out}}}(\mu_2) = w^j \end{cases} \\
&= \begin{cases} s_{\mu_1} s_{\mu_2}^* & \text{if } s_{\Lambda_{\text{out}}}(\mu_1) = s_{\Lambda_{\text{out}}}(\mu_2) = v^1 \text{ for } v \neq w \\ \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{\mu_1} s_{f^j} s_{f^j}^* s_{\mu_2}^* & \text{if } s_{\Lambda_{\text{out}}}(\mu_1) = s_{\Lambda_{\text{out}}}(\mu_2) = w^j \end{cases} \\
&= s_{\mu_1} s_{\mu_2}^*
\end{aligned}$$

where the last equality occurs since $s_{\Lambda_{\text{out}}}(\mu_1) = w^j$ and $p_{w^j} = \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^j} s_{f^j}^*$. This proves both that $qC^*(\Lambda)q \subseteq \psi(C^*(\Lambda_{\text{out}}))$ and $q\mathcal{D}_{\Lambda_{\text{out}}} \subseteq \psi(\mathcal{D}_\Lambda)$ as desired. □

Theorem 4.0.2. *Let Λ be a (row-finite, source-free) k-graph and let $w \in \Lambda^0$ be a regular vertex. Partition $s^{-1}(w)$ as a finite disjoint union of (possibly empty) subsets,*

$$s^{-1}(w) = \mathcal{E}_1 \sqcup \mathcal{E}_2 \sqcup \cdots \sqcup \mathcal{E}_n$$

with $\mathcal{E}_1 \neq \emptyset$ if $s^{-1}(w) \neq \emptyset$. Let $\{e_{i,j}\}$ be a system of matrix units in $\mathsf{M}_n(\mathbb{C})$. Set $p = \sum_{v \in \Lambda^0 \setminus \{w\}} p_v \otimes e_{1,1} + p_w \otimes 1_{\mathsf{M}_n(\mathbb{C})}$. Then there exists a $*$-isomorphism $\Phi \colon C^*(\Lambda_{out}) \to p(C^*(\Lambda) \otimes \mathsf{M}_n(\mathbb{C}))p$ such that $\Phi(\mathcal{D}_{\Lambda_{out}}) = p(\mathcal{D}_\Lambda \otimes c_0(\{1, \dots, n\}))$, and $(\gamma_z \otimes id_{\mathsf{M}_n}) \circ \Phi = \Phi \circ \gamma_z^{\Lambda_{out}}$ for all $z \in \mathbb{T}$.

Proof. Let $\psi \colon C^*(\Lambda) \to C^*(\Lambda_{\text{out}})$ be the injective $*$-homomorphism and q the projection given in the previous theorem. Set $\Psi = \psi \otimes \text{id}_{\mathsf{M}_n(\mathbb{C})}$ and set

$$V = \sum_{v \in \Lambda^0 \setminus \{w\}} p_{v^1} \otimes e_{1,1} + \sum_{i=1}^{n} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^i}^* \otimes e_{i,1}.$$

Note that

$$\begin{aligned}
V^*V &= \sum_{v\in\Lambda^0\setminus\{w\}} p_{v^1}\otimes e_{1,1} + \left(\sum_{i,j=1}^{n} \sum_{f,g\in r_\Lambda^{-1}(w)\cap\Lambda^{\mathbb{1}}} (s_{f^i}s_{f^1}^*\otimes e_{1,i})(s_{g^1}s_{g^j}^*\otimes e_{j,1})\right)\\
&= \sum_{v\in\Lambda^0\setminus\{w\}} p_{v^1}\otimes e_{1,1} + \sum_{i=1}^{n} \sum_{f\in r_\Lambda^{-1}(w)\cap\Lambda^{\mathbb{1}}} s_{f^1}s_{f^i}^*\otimes e_{1,1}\\
&= \sum_{v\in\Lambda^0\setminus\{w\}} p_{v^1}\otimes e_{1,1} + \sum_{i=1}^{n} p_{w^i}\otimes e_{1,1}\\
&= 1_{C^*(\Lambda_{\text{out}})}\otimes e_{1,1}
\end{aligned}$$

and

$$\begin{aligned}
VV^* &= \sum_{v\in\Lambda^0\setminus\{w\}} p_{v^1}\otimes e_{1,1} + \left(\sum_{i,j=1}^{n} \sum_{f,g\in r_\Lambda^{-1}(w)\cap\Lambda^{\mathbb{1}}} (s_{f^1}s_{f^i}^*\otimes e_{i,1})(s_{g^j}s_{g^1}^*\otimes e_{1,j})\right)\\
&= \sum_{v\in\Lambda^0\setminus\{w\}} p_{v^1}\otimes e_{1,1} + \sum_{i=1}^{n} s_{f^1}s_{f^i}^*\otimes e_{i,i}\\
&= \sum_{v\in\Lambda^0\setminus\{w\}} p_{v^1}\otimes e_{1,1} + \sum_{i=1}^{n} p_w^1\otimes e_{i,i}\\
&= \sum_{v\in\Lambda^0\setminus\{w\}} p_{v^1}\otimes e_{1,1} + p_{w^1}\otimes 1_{\mathsf{M}_n(\mathbb{C})}.
\end{aligned}$$

This shows that V is a partial isometry in $C^*(\Lambda_{\text{out}})\otimes\mathsf{M}_n(\mathbb{C})$ such that $V^*V = 1_{C^*(\Lambda_{\text{out}})}\otimes e_{1,1}$, $VV^* = \sum_{v\in\Lambda^0\setminus\{w\}} p_{v^1}\otimes e_{1,1} + p_{w^1}\otimes 1_{\mathsf{M}_n(\mathbb{C})}$ and $\left(\gamma_z^{\Lambda_{\text{out}}}\otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})}\right)(V) = V$ for all $z\in\mathbb{T}$.

Define $\beta\colon C^*(\Lambda_{\text{out}})\to Q\left(C^*(\Lambda_{\text{out}})\otimes\mathsf{M}_n(\mathbb{C})\right)Q$ by $\beta(x) = VxV^*$, where

$$Q = VV^* = \sum_{v\in\Lambda^0\setminus\{w\}} p_{v^1}\otimes e_{1,1} + p_{w^1}\otimes 1_{\mathsf{M}_n(\mathbb{C})},$$

Then β is a $*$-isomorphism such that

$$\begin{aligned}((\gamma_z^{\Lambda_{\mathrm{out}}} \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})}) \circ \beta)(x) &= (\gamma_z^{\Lambda_{\mathrm{out}}} \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})(VxV^*) \\ &= V(\gamma_z^{\Lambda_{\mathrm{out}}} \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})(x)V^* \\ &= (\beta \circ (\gamma_z^{\Lambda_{\mathrm{out}}} \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})}))(x)\end{aligned}$$

for all $x \in C^*(\Lambda_{\mathrm{out}}) \otimes e_{1,1}$. Note also that $(\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})(p) = Q$ and $Q = q \otimes e_{1,1} + \sum_{i=2}^n p_{w^1} \otimes e_{i,i} \leq q \otimes 1_{\mathsf{M}_n(\mathbb{C})}$. Thus

$$\begin{aligned}(\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})(p(C^*(\Lambda) \otimes \mathsf{M}_n(\mathbb{C}))p) &= Q(\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})(C^*(E) \otimes \mathsf{M}_n(\mathbb{C}))Q \\ &= Q(qC^*(\Lambda_{\mathrm{out}})q \otimes \mathsf{M}_n(\mathbb{C}))Q \\ &= Q(q \otimes 1_{\mathsf{M}_n(\mathbb{C})})(C^*(\Lambda_{\mathrm{out}}) \otimes \mathsf{M}_n(\mathbb{C}))(q \otimes 1_{\mathsf{M}_n(\mathbb{C})})Q \\ &= Q(C^*(\Lambda_{\mathrm{out}}) \otimes \mathsf{M}_n(\mathbb{C}))Q\end{aligned}$$

and

$$\begin{aligned}(\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})(p(\mathcal{D}_\Lambda \otimes c_0(\{1,2,\dots,n\})) &= Q(\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})(\mathcal{D}_\Lambda \otimes c_0(\{1,2,\dots,n\})) \\ &= Q(q\mathcal{D}_{\Lambda_{\mathrm{out}}} \otimes c_0(\{1,2,\dots,n\})) \\ &= Q(q \otimes 1_{\mathsf{M}_n(\mathbb{C})})(\mathcal{D}_{\Lambda_{\mathrm{out}}} \otimes c_0(\{1,2,\dots,n\})) \\ &= Q(\mathcal{D}_{\Lambda_{\mathrm{out}}} \otimes c_0(\{1,2,\dots,n\})).\end{aligned}$$

Define $\Phi\colon C^*(\Lambda_{\mathrm{out}}) \to p(C^*(\Lambda) \otimes \mathsf{M}_n(\mathbb{C}))p$ by $\Phi(a) = ((\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})^{-1} \circ \beta)(a \otimes e_{1,1})$. Then

Φ is a $*$-isomorphism and

$$\begin{aligned}(\gamma_z^{\Lambda} \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})}) \circ \Phi(a) &= \left((\gamma_z^{\Lambda} \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})}) \circ (\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})^{-1} \circ \beta\right)(a \otimes e_{1,1}) \\ &= \left((\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})^{-1} \circ (\gamma_z^{\Lambda_{\mathrm{out}}} \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})}) \circ \beta\right)(a \otimes e_{1,1}) \\ &= \left((\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})^{-1} \circ \beta \circ (\gamma_z^{\Lambda_{\mathrm{out}}} \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})\right)(a \otimes e_{1,1}) \\ &= \left((\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})^{-1} \circ \beta\right)(\gamma_z^{\Lambda_{\mathrm{out}}}(a) \otimes e_{1,1}) \\ &= \Phi \circ \gamma_z^{\Lambda_{\mathrm{out}}}(a)\end{aligned}$$

for all $a \in C^*(\Lambda_{\mathrm{out}})$.

We are left to show that $\Phi(\mathcal{D}_{\Lambda_{\mathrm{out}}}) = p(\mathcal{D}_{\Lambda} \otimes c_0(\{1, 2, \ldots, n\}))$. We prove the stronger result below.

Claim 4.0.4. $\beta(\mathcal{D}_{\Lambda_{out}} \otimes e_{1,1}) = V(\mathcal{D}_{\Lambda_{out}} \otimes e_{1,1})V^* = Q(\mathcal{D}_{\Lambda_{out}} \otimes c_0(\{1, 2, \ldots, n\}))$.

(Note the first equality is the definition of the map β. The second equality is the claim.) For the first inclusion, assume $\mu \in \Lambda_{\mathrm{out}}$ and note that

$$\begin{aligned}V s_\mu s_\mu^* \otimes e_{1,1} &= \left(\sum_{v \in \Lambda^0 \setminus \{w\}} p_{v^1} \otimes e_{1,1} + \sum_{i=1}^{n} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^i}^* \otimes e_{i,1}\right)(s_\mu s_\mu^* \otimes e_{1,1}) \\ &= \begin{cases} s_\mu s_\mu^* \otimes e_{1,1} & \text{if } r_{\Lambda_{\mathrm{out}}}(\mu) = v^1 \text{ with } v \neq w \\ \displaystyle\sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^j}^* s_\mu s_\mu^* \otimes e_{j,1} & \text{if } r_{\Lambda_{\mathrm{out}}}(\mu) = w^j \end{cases},\end{aligned}$$

therefore

$$V(s_\mu s_\mu^* \otimes e_{1,1})V^* = \begin{cases} s_\mu s_\mu^* \otimes e_{1,1} & \text{if } r_{\Lambda_{\text{out}}}(\mu) = v^1 \text{ with } v \neq w \\ \displaystyle\sum_{f,g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^j}^* s_\mu s_\mu^* s_{g^j} s_{g^1}^* \otimes e_{j,j} & \text{if } r_{\Lambda_{\text{out}}}(\mu) = w^j \end{cases}$$

$$= \begin{cases} s_\mu s_\mu^* \otimes e_{1,1} & \text{if } r_{\Lambda_{\text{out}}}(\mu) = v^1 \text{ with } v \neq w \\ \displaystyle\sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^j}^* s_\mu s_\mu^* s_{f^j} s_{f^1}^* \otimes e_{j,j} & \text{if } r_{\Lambda_{\text{out}}}(\mu) = w^j \end{cases}.$$

From here we break into two cases.

- Suppose $r_{\Lambda_{\text{out}}}(\mu) = v^1$ where $v \neq w$. Then $s_\mu s_\mu^* \otimes e_{1,1} \in (p_{v^1} \otimes e_{1,1})(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\})) \subseteq Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\}))$.
- Suppose $r_{\Lambda_{\text{out}}}(\mu) = w^j$. Then

$$\sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^j}^* s_\mu s_\mu^* s_{f^j} s_{f^1}^* \otimes e_{j,j} = s_{f^1} s_{\mu'} s_{\mu'}^* s_{f^1}^* \otimes e_{j,j}$$

$$\in (p_{w^1} \otimes e_{j,j})(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\})).$$

  Then observe that $(p_{w^1} \otimes e_{j,j})(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\})) \subseteq Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\}))$.

In either case, $V(s_\mu s_\mu^* \otimes e_{1,1})V^* \in Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\}))$. In addition, observe that for all $v \in \Lambda^0 \backslash \{w\}$:

$$V(p_{v^1} \otimes e_{1,1})V^* = p_{v^1} \otimes e_{1,1} \in Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\}))$$

and also, if $v = w^j$:

$$
\begin{aligned}
V(p_{w^j} \otimes e_{j,j})V^* &= \left(\sum_{i=1}^{n} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^i}^* \otimes e_{i,1} \right) (p_{w^j} \otimes e_{j,j}) \left(\sum_{i=1}^{n} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^i}^* \otimes e_{i,1} \right) \\
&= \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} (s_{f^1} s_{f^j} \otimes e_{j,1})(p_{w^j} \otimes e_{1,1})(s_{f^j} s_{f^1}^* \otimes e_{1,j}) \\
&= \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^1}^* \otimes e_{j,j} \\
&= (p_{w^1} \otimes 1_{\mathsf{M}_n(\mathbb{C})}) \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^1}^* \otimes e_{j,j} \\
&\in Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \dots, n\})).
\end{aligned}
$$

This proves the forward inclusion. To prove the reverse inclusion, consider $s_\mu s_\mu^* \otimes e_{l,l} \in Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \dots, n\}))$. Then we get

$$
\begin{aligned}
(s_\mu s_\mu^* \otimes e_{l,l})V &= (s_\mu s_\mu^* \otimes e_{l,l}) \left(\sum_{v \in \Lambda^0 \setminus \{w\}} p_{v^1} \otimes e_{1,1} + \sum_{j=1}^{n} \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^j}^* \otimes e_{j,1} \right) \\
&= \begin{cases} (s_\mu s_\mu^* \otimes e_{1,1}) \left(\sum_{v \in \Lambda^0 \setminus \{w\}} p_{v^1} \otimes e_{1,1} + \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^1}^* \otimes e_{1,1} \right) & \text{if } l = 1 \\ (s_\mu s_\mu^* \otimes e_{l,l}) \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^1} s_{f^l}^* \otimes e_{l,1} & \text{if } l \neq 1 \end{cases} \\
&= \begin{cases} (s_\mu s_\mu^* \otimes e_{1,1})(q \otimes e_{1,1}) & \text{if } l = 1 \\ \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_\mu s_\mu^* s_{f^1} s_{f^l}^* \otimes e_{l,1} & \text{if } l \neq 1 \end{cases}
\end{aligned}
$$

which then gives

$$
V^*(s_\mu s_\mu^* \otimes e_{l,l})V = \begin{cases} q s_\mu s_\mu^* q \otimes e_{1,1} & \text{if } l = 1 \\ \sum_{f,g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{g^l} s_{g^1}^* s_\mu s_\mu^* s_{f^1} s_{f^l}^* \otimes e_{1,1} & \text{if } l \neq 1 \end{cases}.
$$

We split into two cases.

- Suppose $l = 1$. Then clearly $qs_\mu s_\mu^* q \otimes e_{1,1} \in q\mathcal{D}_{\Lambda_{\text{out}}} \otimes e_{1,1} \subseteq \mathcal{D}_{\Lambda_{\text{out}}} \otimes e_{1,1}$.

- On the other hand, suppose $l \neq 1$. Then

$$\begin{aligned}\sum_{f,g \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{g^l} s_{g^1}^* s_\mu s_\mu^* s_{f^1} s_{f^l}^* \otimes e_{1,1} &= \sum_{f \in r_\Lambda^{-1}(w) \cap \Lambda^{\mathbb{1}}} s_{f^l} s_{\mu'} s_{\mu'}^* s_{f^l}^* \otimes e_{1,1} \\ &\in \mathcal{D}_{\Lambda_{\text{out}}} \otimes e_{1,1}\end{aligned}$$

.

In either case, $V^*(Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\})))V \subseteq D_{\Lambda_{\text{out}}} \otimes e_{1,1}$. This implies

$$\begin{aligned}Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\})) &= VV^*(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\}))VV^* \\ &\subseteq V(\mathcal{D}_{\Lambda_{\text{out}}} \otimes e_{1,1})V^* \\ &\subseteq Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\})).\end{aligned}$$

Thus $\beta(\mathcal{D}_{\Lambda_{\text{out}}} \otimes e_{1,1}) = V(\mathcal{D}_{\Lambda_{\text{out}}} \otimes e_{1,1})V^* = Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\}))$ as claimed.
Using the claim, we get

$$\begin{aligned}\Phi(\mathcal{D}_{\Lambda_{\text{out}}}) &= ((\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})^{-1} \circ \beta)(\mathcal{D}_{\Lambda_{\text{out}}} \otimes e_{1,1}) \\ &= (\psi \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})})^{-1}(Q(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\}))) \\ &= p(\mathcal{D}_{\Lambda_{\text{out}}} \otimes c_0(\{1, 2, \ldots, n\})).\end{aligned}$$

□

The next theorem will show that the outsplit process preserves Morita equivalence.

Theorem 4.0.3. *There exists a $*$-isomorphism $\Theta \colon C^*(\Lambda_{out}) \otimes \mathcal{K} \to C^*(\Lambda) \otimes \mathcal{K}$ such that $\Theta(\mathcal{D}_{\Lambda_{out}} \otimes c_0) = \mathcal{D}_\Lambda \otimes c_0$ and $(\gamma_z^\Lambda \otimes id_\mathcal{K}) \circ \Theta = \Theta \circ (\gamma_z^{\Lambda_{out}} \otimes id_\mathcal{K})$.*

Proof. This proof relies on (9) $\Rightarrow$ (8) of Corollary 11.3 from [Carlsen et al., 2021], which for reference is quoted as a lemma here.

Lemma 4.0.3.1. *Let Γ be a discrete group, and let $(G_1, c_1), (G_2, c_2)$ be second-countable Γ-graded ample Hausdorff groupoids such that each $Iso(c_i^{-1}(id_\Gamma))^\circ$ is torsion-free and abelian. Then the following are equivalent:*

(9) There are $C_r^*(c_i^{-1}(e))$-full projections $p_i \in M(C_0(G_i^{(0)}))$ and an isomorphism $\phi\colon p_1 C_r^*(G_1)p_1 \to p_2 C_r^*(G_2)p_2$ such that $\phi(p_1 C_0(G_1^{(0)})) = p_2 C_0(G_2^{(0)})$ and $\delta_{c_2} \circ \phi = (\phi \otimes id_{C_r^*(\Gamma)}) \circ \delta_{c_1}$ on $p_1 C_r^*(G_1)p_1$.

(8) There is an isomorphism $\Theta\colon C_r^*(G_1) \otimes \mathcal{K} \to C_r^*(G_1) \otimes \mathcal{K}$ satisfying $\Theta(C_0(G_1^{(0)}) \otimes \mathbb{C}) = C_0(G_2^{(0)}) \otimes \mathbb{C}$ and $(\delta_{c_2} \otimes id_\mathcal{K}) \circ \Theta = (\Theta \otimes id_{C_r^*(\Gamma)}) \circ (\delta_{c_1} \otimes id_\mathcal{K})$.

To use their notation in the corollary, let $\Gamma = \mathbb{Z}^k$ be the discrete group. Let $G_1 = \mathcal{G}(\Lambda_{\text{out}})$ be the path groupoid with $c_1 = d$, the degree map the groupoid. Note that $C_r^*(G_1) = C^*(\Lambda_{\text{out}})$ via the groupoid construction of the space. Similarly, let $G_2 = \mathcal{H}$ be the groupoid constructed to extend $\mathcal{G}(\Lambda)$ using Lemma 2.1.0.1 so that $C_r^*(G_2) = C^*(\Lambda) \otimes \mathsf{M}_n(\mathbb{C})$. Let $c_2 =$. Let p_1 be the identity map on $C^*(\Lambda_{\text{out}})$ so that $p_1 C_r^*(G_1)p_1 = p_1 C^*(\Lambda_{\text{out}})p_1 = C^*(\Lambda_{\text{out}})$. Let $p_2 = p$, the projection defined in Theorem 4.0.2. We make several claims in order to satisfy the requirements of (9) in the lemma:

1. p_1 and p_2 are indeed full projections in their respective space.

2. There exists an isomorphism $\phi\colon C^*(\Lambda_{\text{out}}) \to p(C^*(\Lambda) \otimes \mathsf{M}_n(\mathbb{C}))p$.

3. $\phi(p_1 C_0(G_1^{(0)})) = p_2 C_0(G_2^{(0)})$.

4. $\delta_{c_2} \circ \phi = (\phi \otimes \mathrm{id}_{C_r^*(\mathbb{Z}^k)}) \otimes \delta_{c_1}$ on $p_1 C_r^*(G_1)p_1$.

(Observe that Claims 2, 3, and 4 are satisfied via Theorem 4.0.2 since $p_1 C_0(G_1^{(0)}) = \mathcal{D}_{\Lambda_{\text{out}}}$, $p_2 C_0(G_2^{(0)}) = \mathcal{D}_\Lambda \otimes c_0(\{1, \ldots, n\})$, $\delta_{c_1} = \gamma_z^{\Lambda_{\text{out}}}$, and $\delta_{c_2} = \gamma_z \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})}$.)

We prove Claim 1. First, clearly the identity projection is full in $C^*(\Lambda_{\text{out}})$. Let A be the closed (two-sided) ideal in $C^*(\Lambda) \otimes \mathsf{M}_n(\mathbb{C})$ generated by $p(S_\gamma S_\delta^* \otimes e_{m,n})p$. It suffices to show that $(S_\alpha S_\beta^* \otimes e_{i,j}) \in A$ for all $\alpha, \beta \in \Lambda$ and for all i, j. Denote v_m be the vertex such that the vertex projection $p_{v_m} = p_{m,m}$, and let $v_m = r(\alpha), v_n = r(\beta)$. We get

$$\left(S_\alpha S_\gamma^* \otimes e_{i,m}\right)\left(p_{m,m} S_\gamma S_\delta^* p_{n,n} \otimes e_{m,n}\right)\left(S_\delta S_\beta^* \otimes e_{n,j}\right) = S_\alpha S_\gamma^* p_{m,m} S_\gamma S_\delta^* p_{n,n} S_\delta S_\beta^* \otimes e_{i,j} = S_\alpha S_\beta^* \otimes e_{i,j}.$$

Then by Lemma 4.0.3.1 there is an isomorphism $\Theta\colon C_r^*(G_1) \otimes \mathcal{K} \to C_r^*(G_2) \otimes \mathcal{K}$ (which itself is isomorphic to $C^*(\Lambda) \otimes \mathcal{K}$). Furthermore, the isomorphism satisfies $\Theta(C_0(G_1^{(0)}) \otimes \mathbb{C}) = C_0(G_2^{(0)}) \otimes \mathbb{C}$ and $(\delta_{c_2} \otimes \mathrm{id}_{\mathcal{K}}) \circ \Theta = (\Theta \otimes \mathrm{id}_{C_r^*(\mathbb{Z}^k)}) \circ (\delta_{c_1} \otimes \mathrm{id}_{\mathcal{K}})$. Retranslating, this means that $\Theta(\mathcal{D}_{\Lambda_{\text{out}}} \otimes \mathbb{C}) = \mathcal{D}_\Lambda \otimes c_0(\{1, \ldots, n\})$ and $((\gamma_z \otimes \mathrm{id}_{\mathsf{M}_n(\mathbb{C})}) \otimes \mathrm{id}_{\mathcal{K}}) \circ \Theta = (\Theta \otimes \mathrm{id}_{C_r^*(\mathbb{Z}^k)}) \circ (\gamma_z^{\Lambda_{\text{out}}} \otimes \mathrm{id}_{\mathcal{K}})$. These equations simpilfy to the desired ones, which completes the proof. $\square$

Chapter 5

$\mathbb{N}$-graphs

5.1 $\mathbb{N}$-graphs and $\mathbb{N}$-graph algebras

Definition 5.1.1. An $\mathbb{N}$-graph Λ is an infinite sequence of sets, $(\Lambda^0, \Lambda^{e_1}, \Lambda^{e_2}, \dots)$, equipped with range and source maps $r, s\colon \bigcup_{i=1}^{\infty} \Lambda^{e_i} \to \Lambda^0$ satisfying the factorization property. (Often the notation $\Lambda^1 \coloneqq \bigcup_{i=1}^{\infty} \Lambda^{e_i}$ is used instead for brevity.)

Observe that like k-graphs, the factorization property of an $\mathbb{N}$-graph can be checked sufficiently on tricolor paths.

Lemma 5.1.0.1. *Let Λ be an $\mathbb{N}$-colored graph with a complete collection of edge relations ("squares"). Then Λ satisfies the factorization property (and thus Λ is an $\mathbb{N}$-graph) if and only if it satisfies the factorization property for the set $\Lambda_3 = \{\lambda \in G : d(\lambda) = e_i + e_j + e_k, i \neq j \neq k\}$.*

Proof. Clearly Λ_3 is satisfied if Λ is an $\mathbb{N}$-graph. We focus on the other case. Assume Λ_3 satisfies the factorization property and consider $\lambda \in \Lambda$. If λ is formed with only one color, then the factorization property is automatically satisfied. Similarly, if λ is formed with only two colors, then the complete collection of edge relations satisfies the factorization for λ. So assume λ is formed with 3 or more colors. Let $d_1 + d_2 = d(\lambda)$. We wish to construct $\mu, \nu \in \Lambda$ such that $d(\mu) = d_1, d(\nu) = d_2$, and $\lambda = \mu\nu$. Let $\lambda = \lambda_n \lambda_{n-1} \cdots \lambda_2 \lambda_1$ and consider

$\nu = \lambda_{|d_2|} \cdots \lambda_1$. If $d(\nu) = d_2$ then we are finished. Otherwise there exists $\lambda_i \in \nu$ which does not belong. Consequently there also must also exist $\lambda_j \in \mu$ which does not belong. If $i \in \{|d_2|, |d_2| - 1, |d_2|\}$ then using the factorization of the subpath $\lambda_{|d_2|}\lambda_{|d_2|-1}\Lambda_{|d_2|-2}$ we can rewrite so that the unwanted edge is at $\tilde{\lambda}_{|d_2|}$. Otherwise if $i \in \{|d_2| - 3, |d_2| - 4\}$ then using the factorization of $\lambda_{|d_2|-2}\lambda_{|d_2|-3}\lambda_{|d_2|-4}$ we can write so that the unwanted edge is at $\tilde{\lambda}_{|d_2|-2}$ and then apply factorization to the leading triple to get the unwanted edge to $\hat{\lambda}_{|d_2|}$. Follow the same process inductively to see that ν can be rewritten with the unwanted edge at the end. A similar process can be done to move the unwanted edge of μ to the front. Suppose this new version is written $\lambda = \tilde{\mu}\tilde{\nu}$ (note that $\tilde{\mu} = \mu$ and $\tilde{\nu} = \nu$ but with different orders of edge colors). Apply an edge relation to $\tilde{\mu}_1\tilde{\nu}_{|d_2|}$ to form $\hat{\mu}$ and $\hat{\nu}$. Observe that $\lambda = \tilde{\mu}\tilde{\nu} = \hat{\mu}\hat{\nu}$ but now μ, ν are each one color closer to being correct. Repeat this process for each unwanted edge pair, rewriting μ, ν so that the colors are properly sorted. When $d(\nu) = d_2$, then the end result is a factorization of λ. □

Definition 5.1.2. An $\mathbb{N}$-graph Λ is row-finite if $|r^{-1}(v) \cap \Lambda^{e_i}| < \infty$ for all $v \in \Lambda^0$ and $i \in \mathbb{N}$.

Definition 5.1.3. A vertex $v \in \Lambda^0$ is a sink if $s^{-1}(v) \cap \Lambda^{e_i} = \emptyset$ for some $i \in \mathbb{N}$.

Definition 5.1.4. For an $\mathbb{N}$-graph $\Lambda = (\Lambda^0, \Lambda^{e_1}, \dots)$, we assign a family of projections and isometries $\{P_v : v \in \Lambda^0\} \cup \{S_e : e \in \Lambda^1\}$ which satisfy the following properties:

(CK1) The projections P_v are mutually orthogonal.

(CK2) For $e \in \Lambda^1$, $S_e^* S_e = P_{s(e)}$.

(CK3) For $v \in \Lambda^0$, $P_v = \sum_{r^{-1}(v) \cap \Lambda^{e_i}} S_e S_e^*$ for $i \in \mathbb{N}$.

(CK4) If f, g are paired via $af = bg$, then $S_a S_f = S_b S_g$.

This is a Cuntz-Krieger family which generates $C^*(\Lambda)$.

Much work has already been done in generalizing theorems for k-graphs to $\mathbb{N}$-graphs. For example, the gauge invariant uniqueness theorem and Cuntz-Krieger uniqueness theorem are both generalized. (See [Schenkel, 2022]). One interesting result from Schenkel's work is written below.

Theorem 5.1.1. *Let* Λ *be a row-finite* $\mathbb{N}$*-graph with no sources. Define* ${}^k\Lambda := \{\lambda \in \Lambda \colon d_{e^n}(\lambda) = 0$ *when* $n > k\}$*. Then* ${}^k\Lambda$ *is a* k*-graph, and furthermore,* $C^*(\Lambda) = \overline{\bigcup_{k\in\mathbb{N}} C^*({}^k\Lambda)}$*.*

In this section we focus on lifting some of the graph transformations known for k-graphs which preserve Morita equivalence to $\mathbb{N}$-graph algebras. These results will largely follow the work done in [Eckhardt et al., 2020] which also uses results from [Allen, 2005]. These necessary results will first need to be lifted to $\mathbb{N}$-graphs, as shown below.

Definition 5.1.5. For $\lambda, \mu \in \Lambda$, we define the set of minimal common extensions as $\Lambda^{\min}(\lambda, \mu) := \{(\alpha, \beta) \colon \lambda\alpha = \mu\beta, d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}$. We say that Λ is finitely aligned if $\Lambda^{\min}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$.

Definition 5.1.6. A set $E \subseteq v\Lambda$ is called exhaustive if for every $\mu \in v\Lambda$ there exists $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$.

Remark 5.1.1. Note that if Λ is row-finite and source-free then any possible $v\Lambda^n$ is finite and exhaustive.

Definition 5.1.7. Let Λ be a row-finite $\mathbb{N}$-graph and let $X \subseteq \Lambda^0$ be non-empty. A Cuntz-Krieger (Λ, X) family is a collection of partial isometries $\{T_{\alpha,\beta} \colon \alpha, \beta \in X\Lambda$ and $s(\alpha) = s(\beta)\}$ subject to the following relations (for any $\alpha, \beta, \lambda, \mu \in X\Lambda$ with $s(\alpha) = s(\beta)$ and $s(\lambda) = s(\mu)$):

(i) $T^*_{\alpha,\beta} = T_{\beta,\alpha}$;

(ii) $T_{\alpha,\beta}T_{\lambda,\mu} = \sum_{(\beta',\lambda')\in\Lambda^{\min}(\beta,\lambda)} T_{\alpha\beta',\mu\lambda'}$; and

(iii) $\prod_{\lambda\in E} (T_v - T_{\lambda,\lambda}) = 0$ for all $v \in X$ and finite exhaustive $E \subseteq v\Lambda$.

Remark 5.1.2. Because of (i) and (ii) above, the set $\{T_{\lambda,\lambda} \colon \lambda \in X\Lambda\}$ is a set of commuting projections, and in particular, the set $\{T_v \colon v \in X\}$ is a set of mutually orthogonal projections. Furthermore, when $X = \Lambda^0$, the above definition reduces to a Cuntz-Krieger Λ-family.

As a collection of partial isometries, completing the space with respect to the norm gives rise to a C^*-algebra. We define the universal C^*-algebra as usual:

$$C^*(\Lambda, X) := \overline{\operatorname{span}}\{T_{\alpha,\beta} \colon \alpha, \beta \in X\Lambda, s(\alpha) = s(\beta)\}.$$

Because of the universality of $C^*(\Lambda, X)$, there is a strongly continuous gauge action (labeled γ) of $\mathbb{T}^{\mathbb{N}}$ via $\gamma_z(T_{\alpha,\beta}) = z^{d(\alpha)-d(\beta)}T_{\alpha,\beta}$. Allen produces a gauge-invariant uniqueness theorem for these C^*-algebras, which can be lifted as seen below.

Theorem 5.1.2. *Let Λ be a row-finite $\mathbb{N}$-graph with $X \subseteq \Lambda^0$. Let $\{t_{\alpha,\beta}\}$ be a Cuntz-Krieger (Λ, X)-family and let π be a representation of $C^*(\Lambda, X)$ such that $\pi(T_{\alpha,\beta}) = t_{\alpha,\beta}$. Suppose that for each $v \in \Lambda^0$ with $X\Lambda v \neq \emptyset$ there exists a path $\mu_v \in x\Lambda v$ with $\pi(T_{\mu_v}) \neq 0$, and suppose that there is a strongly continuous action δ of $\mathbb{T}^{\mathbb{N}}$ on $C^*(t_{\alpha,\beta})$ such that $\delta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}^{\mathbb{N}}$. Then π is faithful.*

Remark 5.1.3. The definition used in [Allen, 2005] for both 5.1.7 and 5.1.2 above only requires Λ to be finitely aligned, which is less restrictive than requiring it to be row-finite. Perhaps in later work the proofs which follow in this chapter can be generalized further?

The focus for this section is on the following result and its consequences:

Corollary 5.1.1. *Let Λ be a row-finite $\mathbb{N}$-graph and $\{s_\lambda\}$ be a Cuntz-Krieger Λ-family. Let $X \subseteq \Lambda^0$ and $\{T_{\alpha,\beta}\}$ be a Cuntz-Krieger (Λ, X)-family. Define $P_X \coloneqq \sum_{v \in X} s_v$ and note that $P_X \in M(C^*(\Lambda))$. Then $P_X C^*(\Lambda) P_X \cong C^*(\Lambda, X)$.*

Proof. Define a map $\phi\colon C^*(\Lambda, X) \to P_X C^*(\Lambda) P_X$ via $\phi(T_{\alpha,\beta}) = s_\alpha s_\beta^*$. Then ϕ is a surjective homomorphism which commutes with the gauge action. Furthermore, $\phi(T_{\lambda,\lambda}) = s_\lambda s_\lambda^* \neq 0$ for all $\lambda \in X\Lambda v$. So by Theorem 5.1.2, ϕ is also injective. □

Definition 5.1.8. Let $v, w \in \Lambda^0$. We say $w \leq v$ if there exists a finite path $\lambda \in \Lambda$ such that $s(\lambda) = v$ and $r(\lambda) = w$.

We generalize the definition of the saturation of a vertex set (see Definition **??**) from k-graphs to $\mathbb{N}$-graphs.

Definition 5.1.9. Let Λ be an $\mathbb{N}$-graph and let $X \subseteq \Lambda^0$. We define the saturation of X (denoted $\Sigma(X)$) as the smallest set $S \subseteq \Lambda^0$ such that $X \subseteq S$ and which satisfies the following properties:

(Heredity) If $v \in S$ and $\lambda \in v\Lambda$, then $s(\lambda) \in S$.

(Saturation) If $\{s(\lambda) \colon \lambda \in v\Lambda^n\} \subseteq S$ for some $n \in \mathbb{N}^{\mathbb{N}}$, then $v \in S$.

In order to invoke the relevant gauge-invariant uniqueness theorem in the delay and reduction cases, the following lemma is useful.

Lemma 5.1.2.1. *Let Λ be a row-finite source-free $\mathbb{N}$-graph. Let $R \colon \Lambda \to \mathbb{Z}^{\mathbb{N}}$ and consider the function $\beta \colon \mathbb{T}^{\mathbb{N}} \to Aut(C^*(\Lambda))$ via $\beta_z(t_\mu t_\nu^*) = z^{R(\mu)-R(\nu)} t_\mu t_\nu^*$ for all $\mu, \nu \in \Lambda$ and for all $z \in \mathbb{T}^{\mathbb{N}}$. Then β is an action of $\mathbb{T}^{\mathbb{N}}$ on $C^*(\Lambda)$.*

The proofs which show Morita equivalence between an original $\mathbb{N}$-graph Λ and the transformation rely on a few results derived from [Allen, 2005] and [Eckhardt et al., 2020].

Proposition 5.1.1. *Let Λ be a row-finite $\mathbb{N}$-graph and let $X \subseteq \Lambda^0$. Then $C^*(\Lambda, X)$ is Morita equivalent to $C^*(\Lambda, \Sigma(X))$.*

Proof. The ideal generated by P_X is the same as the ideal generated by $P_{\Sigma(X)}$. □

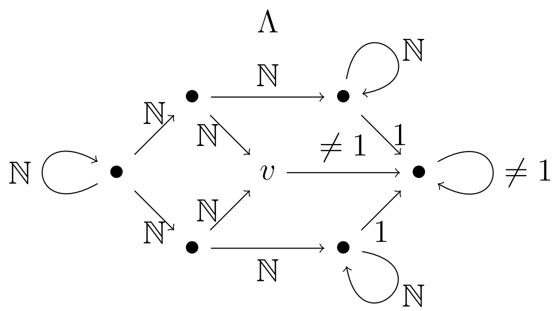
Theorem 5.1.3. *Let Λ be an row-finite $\mathbb{N}$-graph and let $X \subseteq \Lambda^0$. Define $P_X \coloneqq \sum_{v \in X} p_v$ (note that $P_X \in M(C^*(\Lambda))$). Then $P_X C^*(\Lambda) P_X$ is Morita equivalent to $C^*(\Lambda)$ if $\Sigma(X) = \Lambda^0$.*

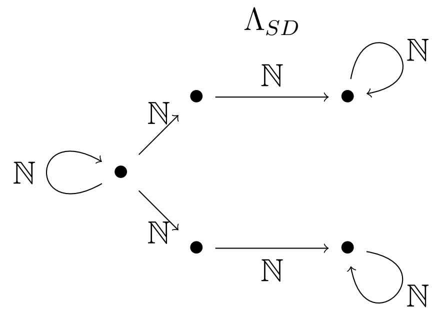
Proof. By Corollary 5.1.1, we have $P_X C^*(\Lambda) P_X \cong C^*(\Lambda, X)$. Then by Proposition 5.1.1, we have $C^*(\Lambda, X)$ is Morita equivalent to $C^*(\Lambda, \Sigma(X))$. Finally observe from Corollary 5.1.1 that $C^*(\Lambda, \Sigma(X)) \cong P_{\Sigma(X)} C^*(\Lambda) P_{\Sigma(X)}$. But note that $P_{\Lambda^0} C^*(\Lambda) P_{\Lambda^0} = C^*(\Lambda)$. □

5.2 Sink Deletion

Definition 5.2.1. Suppose Λ is a row-finite $\mathbb{N}$-graph with a sink $v \in \Lambda^0$. Define $\Lambda^0_{SD} \coloneqq \Lambda^0 \backslash \{w \colon w \leq v\}$ and $\Lambda^{e_i}_{SD} \coloneqq \Lambda^{e_i} \backslash \bigcup_{w \leq v} \{e \colon s(e) = w\} \cup \{e \colon r(e) = w\}$. Inherit edge relations from Λ to Λ_{SD}. This is defined as the sink deletion of Λ at v.

In the figure below, observe that v is a 1-sink. This is the vertex chosen in forming Λ_{SD}.





It will be useful to have the following lemma later.

Lemma 5.2.0.1. *Let* Λ *be an* $\mathbb{N}$*-graph and let* $v \in \Lambda^0$ *be an* e^i*-sink. Then any* $w \leq v$ *is also an* e^i*-sink.*

Proof. First suppose $w \in \Lambda^{e_j} v$ for some $j \in \mathbb{N}$ (by assumption $j \neq i$). Call the edge h. Assume for contradition that there exists $e \in \Lambda^{e_i} \cap s^{-1}(w)$. By the factorization property of Λ, there edges f, g such that $fg = eh$ with $f \in \Lambda^{e_j}, g \in \Lambda^{e_i}$. In particular, $s(g) = v$, thereby making v not an e^i-sink. This is a contradiction. Thus any vertex one edge away from v must be an e_i-sink. Induct on the rest of the vertices $w \leq v$ to finish the proof. □

Claim 5.2.1. *If* Λ *is a source-free, row-finite* $\mathbb{N}$*-graph and* $v \in \Lambda^0$ *is a sink, then* Λ_{SD} *is a source-free, row-finite* $\mathbb{N}$*-graph.*

Proof. Let $i \in \mathbb{N}$ be the color for which v is a sink (so $\Lambda^{e_i} v = \emptyset$). Suppose $w \leq v$ and note that $\Lambda^{e_i} w = \emptyset$ also since the paths in Λ satisfy the factorization property. Let $\iota \colon \Lambda_{SD} \to \Lambda$ be the standard inclusion map.

We wish to show that the paths in Λ_{SD} also satisfy the factorization property. Suppose μ is a finite path in Λ_{SD}. Since $\iota(\mu)$ avoids any vertex $w \leq v$ in Λ, any factorization of $\iota(\mu)$ in Λ will also be present in Λ_{SD} as appropriate edges. Thus Λ_{SD} satisfies the factorization property, and so Λ_{SD} is a $\mathbb{N}$-graph. Also, since no edges were added, Λ_{SD} must be row-finite.

Lastly, we show Λ_{SD} is source-free by contradiction. Let $z \in \Lambda^0_{SD}$ be a source for color $j \in \mathbb{N}$. Since Λ is source-free, then $\exists \lambda \in r^{-1}(\iota(z)) \cap \Lambda^{e_j}$ which was deleted in forming Λ_{SD}. Thus $\iota(z) \leq v$, meaning $z \notin \Lambda^0_{SD}$. This is a contradiction. □

Theorem 5.2.1. $C^*(\Lambda) \cong_{ME} C^*(\Lambda_{SD})$.

Proof. Let $\{p_v, s_\lambda\}$ be the canonical projections and isometries for $C^*(\Lambda)$. For $v \in \Lambda^0_{SD}, \lambda \in \Lambda^1_{SD}$, let $Q_v := p_{\iota(v)}$ and $T_\lambda := s_{\iota(\lambda)}$. We claim that $\{Q_v, T_\lambda\}$ is a Cuntz-Krieger Λ_{SD} family in $C^*(\Lambda)$.

To prove this claim, we first show that $\{Q_v, T_\lambda\}$ satisfies the four Cuntz-Krieger properties outlined in 5.1.4.

- Since the projections $\{p_v\}$ are mutually orthogonal, so are $\{Q_v\}$ by definition.
- Let $e \in \Lambda^1_{SD}$. Then $T^*_\lambda T_\lambda = s^*_{\iota(\lambda)} s_{\iota(\lambda)} = p_{s(\iota(\lambda))} = Q_{s(\lambda)}$.
- Let $v \in \Lambda^0_{SD}, i \in \mathbb{N}$. Then $Q_v = p_{\iota(v)} = \sum_{r^{-1}(\iota(v)) \cap \Lambda^{e_i}} s_{\iota(e)} s^*_{\iota(e)} = \sum_{r^{-1}(v) \cap \Lambda^{e_i}_{SD}} T_e T^*_e$.
- Suppose $af = bg$ in Λ^1_{SD}. Then $T_a T_f = s_{\iota(a)} s_{\iota(f)} = s_{\iota(b)} s_{\iota(g)} = T_b T_g$.

Let $\{q_v, t_e\}$ be the canonical projections and isometries for $C^*(\Lambda_{SD})$. By the universal property in $C^*(\Lambda_{SD})$, there exists a homomorphism $\pi\colon C^*(\Lambda_{SD}) \to C^*(\Lambda)$ satisfying $\pi(q_v) = Q_v$ and $\pi(t_e) = T_e$ for $v \in \Lambda^0_{SD}$ and $e \in \Lambda^1_{SD}$. Furthermore, note that π intertwines the standard gauge actions on $C^*(\Lambda)$ and $C^*(\Lambda_{SD})$. Therefore, the gauge invariant uniqueness theorem (Theorem 3.1 from [Schenkel, 2022]) applies. So π must be injective.

Now consider $p := \sum_{v \in \Lambda^0_{SD}} p_{\iota(v)} \in \operatorname{span}(\{p_v\}) \subseteq M(C^*(\Lambda))$. We wish to show that $pC^*(\Lambda)p \cong C^*(\Lambda_{SD})$ to complete the proof of Morita equivalence. Note the following:

$$\begin{aligned} pC^*(\Lambda)p &= \overline{\operatorname{span}}\{s_\lambda s^*_\mu \colon r(\lambda), r(\mu) \in X\} \\ &= \overline{\operatorname{span}}\{s_\lambda s^*_\mu \colon r(\lambda), r(\mu) \in \Lambda^0 \backslash \{w \colon w \leq v\}\} \\ &= \overline{\operatorname{span}}\{s_\lambda s^*_\mu \colon r(\lambda), r(\mu) \not\leq v\} \end{aligned}$$

Furthermore, if $r(\lambda) \not\leq v$ then $s(\lambda) \not\leq v$. This shows that $pC^*(\Lambda)p \subseteq \operatorname{Im}(\pi) \cong C^*(\Lambda_{SD})$. For the other direction, let s_λ be a generator of $\operatorname{Im}(\pi)$. Then $s_\lambda \in C^*(\Lambda)$ and must be formed from generators which avoid v. Therefore $s_\lambda \in pC^*(\Lambda)p$. Lastly, we claim $\Sigma(X) = \Lambda^0$ and invoke Corollary 5.1.3 to finish the proof. To prove the claim, observe that $w \notin X$ if and only if $w \leq v$. By Lemma 5.2.0.1, any such w is an e_i-sink. Also, since Λ is source-free, we have that $w\lambda^{e_i} \neq \emptyset$. But every $z \in s^{-1}(w\Lambda^{e_i})$ is by defintion not an e_i-sink, therefore every such z is in X. So by the saturation property of $\Sigma(X)$, any $w \leq v$ is in $\Sigma(X)$. This completes the claim. □

5.3 Delay

For directed graphs, an edge can be delayed by splitting the edge into two, and creating a new vertex in the middle. The work done in [Eckhardt et al., 2020] shows that the delay transformation is much more involved in k-graphs, but can still be done.

Definition 5.3.1. Let Λ be a row-finite, source-free $\mathbb{N}$-graph with fixed edge $f \in \Lambda^{e_i}$. (Without loss of generality, let $i = 1$ for the rest of this definition.) The e_1-delay collection of f is the set recursively defined as follows:

- $A_1 = \{f\} \cup \{g \in \Lambda^{e_1} \colon ag = fb \text{ or } ga = bf \text{ where } a, b \in \Lambda^{e_i} \text{ for } i \geq 2\}$;
- $A_m = \{e \in \Lambda^{e_1} \colon ag = eb \text{ or } ga = be \text{ where } a, b \in \Lambda^{e_i} \text{ for } i \geq 2, g \in A_{m-1}\}$;
- $\mathcal{E}^{e_1} = \bigcup_{j \in \mathbb{N}} A_j$.

Observe that $\mathcal{E}^{e_1} \subseteq \Lambda^{e_1}$. From the above definition, sets of affected edges of other colors can also be defined via $\mathcal{E}^{e_j} := \{g \in \Lambda^{e_j} \colon g \in \Lambda^1 r(a) \cup s(a)\Lambda^1, a \in \mathcal{E}^{e_1}\}$.

Lastly, we define $\mathcal{E}_D^{e_1} := \{g^1, g^2 \colon g \in \mathcal{E}^{e_1}\}$ and $(i \neq 1)$ $\mathcal{E}_D^{e_i} := \{e_\alpha \colon \alpha = ga \in \Lambda^{e_1+e_i}, g \in \mathcal{E}^{e_1}, a \in \Lambda^{e_i}\}$ to be the delayed edge sets (to be used in the construction of the new graph).

Definition 5.3.2. Let Λ be a row-finite, source-free $\mathbb{N}$-graph with fixed edge $f \in \Lambda^{e_i}$. (Without loss of generality, let $i = 1$ for the rest of this definition.) Let $\mathcal{E}^{e_i}$ denote the e_i-delay collections as defined above. Define a new $\mathbb{N}$-graph (labeled Λ_D) according to the following:

- $\Lambda_D^0 = \Lambda^0 \cup \{v_g \colon g \in \mathcal{E}^{e_1}\}$
- $\Lambda_D^{e_1} = (\Lambda^{e_1} \backslash \mathcal{E}^{e_1}) \cup \mathcal{E}_D^{e_1}$

- $(\forall e \in \Lambda_D^{e_1})\ s_D(e) = \begin{cases} s(e) \in \Lambda^0 & \text{if } e \in \Lambda^{e_1} \backslash \mathcal{E}^{e_1} \\ s(e) & \text{if } e = g^1 \in \mathcal{E}_D^{e_1}, g \in \mathcal{E}^{e_1} \\ v_g & \text{if } e = g^2 \in \mathcal{E}_D^{e_1}, g \in \mathcal{E}^{e_1} \end{cases}$

- $(\forall e \in \Lambda_D^{e_1})\ r_D(e) = \begin{cases} r(e) \in \Lambda^0 & \text{if } e \in \Lambda^{e_1} \backslash \mathcal{E}^{e_1} \\ v_g & \text{if } e = g^1 \in \mathcal{E}_D^{e_1}, g \in \mathcal{E}^{e_1} \\ r(e) & \text{if } e = g^2 \in \mathcal{E}_D^{e_1}, g \in \mathcal{E}^{e_1} \end{cases}$

- $(i \neq 1)\ \Lambda_D^{e_i} = \Lambda^{e_i} \cup \mathcal{E}_D^{e_i}$

- $(\forall e \in \Lambda_D^{e_i}, i \neq 1)\ s_D(e) = \begin{cases} s(e) & \text{if } e \in \Lambda^{e_i} \\ v_g & \text{if } e = e_\alpha \text{ and } \alpha = bg \text{ for some } g \in \Lambda^{e_1} \end{cases}$

- $(\forall e \in \Lambda_D^{e_i}, i \neq 1)\ r_D(e) = \begin{cases} r(e) & \text{if } e \in \Lambda^{e_i} \\ v_h & \text{if } e = e_\alpha \text{ and } \alpha = ha \text{ for some } h \in \Lambda^{e_1} \end{cases}$

For such an $\mathbb{N}$-graph and its delay, it will be helpful to use a partially defined inclusion map $\iota\colon \Lambda_D \to \Lambda$ for $v \in \Lambda_D^0 \backslash \{v_e \colon e \in \mathcal{E}^{e_1}\}$ and $e \in \Lambda_D^1 \backslash \{\bigcup_{i \in \mathbb{N}} \mathcal{E}_D^{e_i}\}$. Lastly, edge relations for Λ_D are defined for the bicolor path fe in the following way (assume $j, n \neq 1$):

- If $(e \in \Lambda^{e_1}, f \in \Lambda^{e_j})$, then by the factorization property of Λ, there exists $b, g \in \Lambda^1$ such that $\iota(f)\iota(e) = gb$. These paths exist in Λ_D as well, so the edge relation $fe = \iota^{-1}(g)\iota^{-1}(b)$ is inherited.

- If $(e = e^2 \in \mathcal{E}^{e_1}, f \in \Lambda^{e_j})$, then by the factorization property of Λ, there exists $b, g \in \Lambda^1$ such that $\alpha \coloneqq \iota(f)e = gb$. (Note $g \in \mathcal{E}^{e_1}$.) The edge relation is inherited via $fe^2 = g^2 e_\alpha$.

- If $(e = e^1 \in \mathcal{E}_D^{e_1}, f \in \mathcal{E}_D^{e_j})$, then $f = e_\alpha$ for some α, which means there exists $h, b, g \in \Lambda^1$ such that $\alpha = he = gb$. (Note $g \in \mathcal{E}^{e_1}$ and $b \in \mathcal{E}^{e_j}$.) The edge relation is inherited via $e_\alpha e^1 = g^1 \iota^{-1}(b)$.

- If $(e \in \Lambda^{e_n}, f \in \Lambda^{e_j})$, then follow the first bullet point.

- If $(e_\alpha \in \mathcal{E}_D^{e_n}, e_\beta \in \mathcal{E}_D^{e_j})$, then there exist paths $a, b, c, d \in \mathcal{E}^{e_1}$, $w, x \in \mathcal{E}^{e_n}$, $y, z \in \mathcal{E}^{e_j}$ such that $\alpha = wa = bx$ and $\beta = yc = dz$. Furthermore, since $v_b = r_D(e_\alpha) = s_D(e_\beta) = v_c$, it must be that $b = c$. Consider the tricolor path $ycx \in \Lambda^{e_1+e_j+e_n}$. By the factorization property of Λ, there exist paths $f \in \Lambda^{e_1}$, $g, g', \in \Lambda^{e_j}$, and $h, h' \in \Lambda^{e_n}$ such that $ycx = dzx = ywa = dhg = h'g'a = h'fg$ and such that all included squares work (specifically, $dh = h'f =: \gamma$ and $fg = g'a =: \delta$ and $h'g' = yw$). In particular, $f \in \mathcal{E}^{e_1}$ so there exist $e_\gamma \in \mathcal{E}_D^{e_n}$, $e_\delta \in \mathcal{E}_D^{e_j}$ with $s_D(e_\gamma) = v_f = r_D(e_\delta)$ and $s_D(e_\delta) = v_a = s_D(e_\alpha)$ and $r_D(e_\gamma) = v_d = r_D(e_\beta)$. So define the factorization via $e_\beta e_\alpha = e_\gamma e_\delta$.

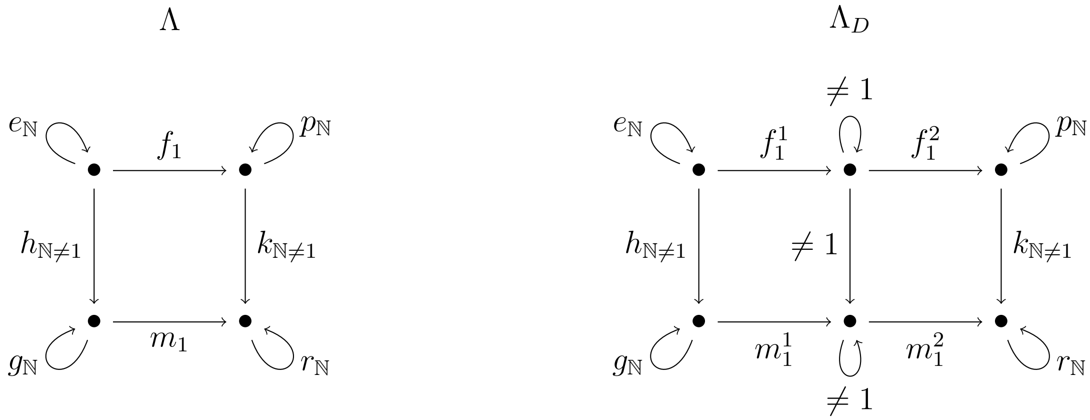
(All other options are impossible since the sources/ranges of the two edges do not align.) The new $\mathbb{N}$-graph is called the delayed graph of Λ at f.

Below is an example of an $\mathbb{N}$-graph Λ with a path $f \in \Lambda^{e_1}$ which is delayed to form Λ_D. (In this figure, the subscript denotes the color of the edge, i.e. x_i denotes an edge with color i.) The edge relations are unique, except two possibilities with $\{h_j e_i = h_i e_j\}$ and $\{h_j e_i = g_i h_j\}$ if $i \neq 1$. Similar choices are available on the other side of the graph: $\{k_j p_i = k_i p_j\}$ and $\{k_j p_i = r_i k_j\}$. In order for the factorization property to hold in Λ, the choices on either side must be the same. Consider the path $k_j f_1 e_i \in \Lambda^3$. Note that $f_1 e_n = p_n f_1$, $k_n f_1 = m_1 h_n$, and $m_1 g_n = r_n m_1$ are required for all $n \neq 1$. If $h_j e_i = h_i e_j$ is chosen, then we get

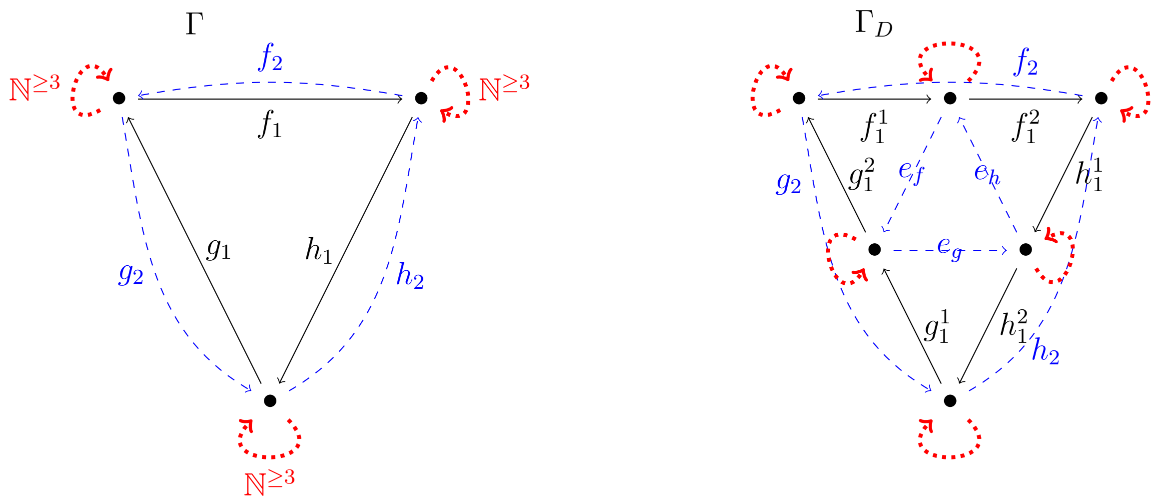
$$k_j p_i f_1 = k_j f_1 e_i = m_1 h_j e_i = m_1 h_i e_j = k_i f_1 e_j = k_i p_j f_1$$

and therefore $k_j p_i = k_i p_j$. A similar argument shows that if $h_j e_i = g_i h_j$ is chosen instead, then $k_j p_i = r_i k_j$ is forced.

In forming Λ_D, observe that since $\mathcal{E}^{e_1} = \{f, m\}$, then m is delayed as well. Furthermore, for $i \neq 1$ we have $\mathcal{E}_D^{e_i}$ consisting of the affected length-2 paths, so the new vertices have new edges added in order to preserve the factorization property. Note that $0 < |\mathcal{E}_D^{e_i}| < \infty$ for all $i \in \mathbb{N}$.



Here is another example. Again the delay occurs at $f \in \Gamma^{e_1}$, but notice that $\mathcal{E}^{e_1} = \{f, g, h\}$ so the other two paths are delayed also. For readability, the loops which contain edges of colors $i \geq 3$ have the labels omitted in Γ_D. The edge relations do not need to be specified since they are unique.



Theorem 5.3.1. *Let Λ be a row-finite, source-free $\mathbb{N}$-graph with fixed edge $f \in \Lambda^{e_i}$. Then the delayed $\mathbb{N}$-graph Λ_D at f is also a row-finite, source-free $\mathbb{N}$-graph.*

Proof. Without loss of generality, assume $f \in \Lambda^{e_1}$. We wish to show that the paths in Λ_D satisfy the factorization property. By Lemma 5.1.0.1, it is sufficient to show that any $\lambda \in \Lambda_D^3$ can be factorized. There are four cases:

- Case 1: If $\lambda \cap \left(\bigcup_{j\in\mathbb{N}} \mathcal{E}_D^{e_j}\right) = \emptyset$, then λ avoids the affected edges altogether and so the path $\iota^{-1}(\lambda) \in \Lambda$ exists. Follow the factorization from Λ.

- Case 2a: Assume $\lambda \cap \mathcal{E}_D^{e_1} \neq \emptyset$. (Note that $d_1(\lambda) = 1$ in this case.) If $s(\lambda) \in \Lambda^0$, then $\lambda = e_\beta e_\alpha f^1 = e_\delta e_\gamma f^1$ for some $f \in \mathcal{E}^{e_1}$ and $\alpha = af = gb$, $\beta = cg = hd$, $\gamma = pf = xq$, $\delta = rx = ys$ (where $g, h, x, y \in \mathcal{E}^{e_1}$, $a, b, r, s \in \mathcal{E}^{e_j}$, $c, d, p, q \in \mathcal{E}^{e_k}$). Complete the "cubes" formed via the paths $caf, rpf \in \Lambda$ using the factorization property from Λ. Namely, there exist $\tilde{h}, \hat{y} \in \mathcal{E}^{e_1}$, $\tilde{c}, \tilde{d}, \hat{p}, \hat{q} \in \mathcal{E}^{e_k}$, $\tilde{a}, \tilde{b}, \hat{r}, \hat{s} \in \mathcal{E}^{e_j}$ such that

$$caf = cgb = \tilde{a}\tilde{c}f = \tilde{a}\tilde{h}\tilde{d} = hdb = h\tilde{b}\tilde{d} \qquad \text{and} \qquad rpf = rxq = ysq = y\hat{q}\hat{s} = \hat{p}\hat{y}\hat{s} = \hat{p}\hat{r}f$$

as well as satisfying all inner edge relations. Label $\alpha' = \tilde{a}\tilde{h} = h\tilde{b}$ so that $e_{\alpha'} \in \mathcal{E}_D^{e_j}$ and note that the following factorization works:

$$e_\beta e_\alpha f^1 = e_\delta e_\gamma f^1 = e_{\alpha'} \iota^{-1}(\tilde{h})\iota^{-1}(\tilde{d}) = h^1 \iota^{-1}(\tilde{b})\iota^{-1}(\tilde{d}) = e_\beta g^1 \iota^{-1}(b) = h^1 \iota^{-1}(d)\iota^{-1}(b).$$

(A similar factorization comes from the cube with γ, δ which will show that many of the edges between the two cubes are in the same.)

- Case 2b: Assume $\lambda \cap \mathcal{E}_D^{e_1} \neq \emptyset$. (Again note $d_1(\lambda) = 1$.) If $s(\lambda) = v_g$, then $\lambda = yxg^2 = x'y'g^2$ for some $g \in \mathcal{E}^{e_1}$, $x, x' \in \Lambda^{e_j}$, $y, y' \in \Lambda^{e_k}$. Thus there are edge relations $\alpha, \beta, \gamma, \delta$ so that $xg^2 = g^2 e_\alpha$ and $y'g^2 = g^2 e_\beta$ and $x'g^2 = g^2 e_\gamma$ and $y'g^2 = g^2 e_\delta$. Thus we get

$$yxg^2 = x'y'g^2 = yg^2 e_\alpha = x'g^2 e_\beta = g^2 e_\gamma e_\beta = g^2 e_\delta e_\alpha.$$

It remains to be shown that $e_\gamma e_\beta = e_\delta e_\alpha$ which can be done by applying a similar method from Case 2a, describing the cubes formed from the paths in Λ.

- Case 3: Assume $\lambda \cap \mathcal{E}_D^{e_1} = \emptyset$ but $\lambda \cap \left(\bigcup_{j \neq 1} \mathcal{E}_D^{e_j} \right) \neq \emptyset$. Then $\lambda = e_\gamma e_\beta e_\alpha$. By edge relations, there exist $\alpha', \beta', \gamma', \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \tilde{\alpha}\tilde{\beta}, \tilde{\gamma}, \alpha^\diamond$ such that

$$e_\gamma e_\beta e_\alpha = e_\gamma e_{\alpha'} e_{\beta'} = e_{\hat{\alpha}} e_{\gamma'} e_{\beta'} = e_{\hat{\alpha}} e_{\hat{\beta}} e_{\hat{\gamma}} = e_{\tilde{\beta}} e_{\tilde{\alpha}} e_{\hat{\gamma}} = e_{\tilde{\beta}} e_{\tilde{\gamma}} e_{\alpha^\diamond}$$

For this factorization to work, we need $\alpha = \alpha^\diamond$ (equivalently, $e_\gamma e_\beta = e_{\tilde{\beta}} e_{\tilde{\gamma}}$). Again describe the cubes as in Step 2a to check that everything aligns.

In all four cases, the factorization property is satisfied.

To show that Λ_D is source-free, note that $v \in \Lambda^0 \subseteq \Lambda_D^0$ are not sources. Consider $v \in \{v_g : g \in \mathcal{E}^{e_1}$ and observe via construction that $r^{-1}(v_g) \cap \Lambda_D^{e_i} \neq \emptyset$. To show that Λ_D is row-finite, observe that $|s^{-1}(v) \cap \Lambda_D^{e_i}| = |s^{-1}(v) \cap \Lambda^{e_i}| < \infty$ if $v \in \Lambda^0 \subseteq \Lambda_D^0$. Consider $v \in \{v_g : g \in \mathcal{E}^{e_1}$ and again observe via construction that $|s^{-1}(v_g) \cap \Lambda_D^{e_i}| < \infty$. □

Theorem 5.3.2. *Let Λ be a row-finite,source-free $\mathbb{N}$-graph and let Λ_D be a delay of Λ at the edge $f \in \Lambda^{e_i}$. Then $C^*(\Lambda_D) \cong_{ME} C^*(\Lambda)$.*

Proof. Without loss of generality, assume $f \in \Lambda^{e_1}$. Let $\{p_v, s_e : v \in \Lambda_D^0, e \in \Lambda_D^1\}$ be the canonical Cuntz-Krieger Λ_D-framily generating $C^*(\Lambda_D)$. Let $\{Q_v, T_e\}$ be defined for $v \in \Lambda^0, e \in \Lambda^1$ via

$$Q_v = p_v \qquad\qquad T_e = \begin{cases} s_{\iota^{-1}(e)} & \text{if } e \notin \mathcal{E}^{e_1} \\ s_{e^2} s_{e^1} & \text{if } e \in \mathcal{E}^{e_1} \end{cases}$$

We claim that $\{Q_v, T_e\}$ is a Cuntz-Krieger Λ-family in $C^*(\Lambda_D)$.

- Since the projections $\{p_v\}$ are orthogonal, so are $\{Q_v\}$ by definition.
- Let $e \in \Lambda^1$. Then $T_e^* T_e = \begin{cases} s^*_{\iota^{-1}(e)} s_{\iota^{-1}(e)} & \text{if } e \notin \mathcal{E}^{e_1} \\ s^*_{e^1} s^*_{e^2} s_{e^2} s_{e^1} & \text{if } e \in \mathcal{E}^{e_1} \end{cases} = \begin{cases} p_{s(\iota^{-1}(e))} & \text{if } e \notin \mathcal{E}^{e_1} \\ p_{s(\iota^{-1}(e^1))} & \text{if } e \in \mathcal{E}^{e_1} \end{cases} = Q_{s(e)}$.
- Let $v \in \Lambda^0, i \in \mathbb{N}$. Note that for any $h \in \mathcal{E}^{e_1}$ we have $r_D^{-1}(v_h) \cap \Lambda_D^{e_1} = \{h_1\}$. Thus

$p_{v_h} = s_{h^1}s_{h^1}^*$. Thus, if $i = 1$ we get

$$\begin{aligned} Q_v = p_{\iota^{-1}(v)} = \sum_{\iota^{-1}(v)\Lambda_D^{e_1}} s_h s_h^* &= \sum_{v\Lambda^{e_1}\setminus\mathcal{E}^{e_1}} s_{\iota^{-1}(h)}s_{\iota^{-1}(h)}^* + \sum_{v\mathcal{E}^{e_1}} s_{h^2}s_{h^2}^* \\ &= \sum_{v\Lambda^{e_1}\setminus\mathcal{E}^{e_1}} T_hT_h^* + \sum_{v\mathcal{E}^{e_1}} s_{h^2}s_{h^1}s_{h^1}^*s_{h^2}^* \\ &= \sum_{v\Lambda^{e_1}\setminus\mathcal{E}^{e_1}} T_hT_h^* + \sum_{v\mathcal{E}^{e_1}} T_hT_h^* \\ &= \sum_{\iota^{-1}(v)\Lambda^{e_1}} T_hT_h^*. \end{aligned}$$

Furthermore, if $i \neq 1$ then there are no concerns, since $\iota^{-1}(v)\mathcal{E}^{e_i} = \emptyset$. So $Q_v = p_{\iota^{-1}(v)} = \sum t_{\iota^{-1}(h)}t_{\iota^{-1}(h)}^* = \sum T_hT_h^*$.

- Suppose $ah = bg$ in Λ^1. Then

$$\begin{aligned} T_aT_h &= \begin{cases} s_{\iota^{-1}(a)}s_{\iota^{-1}(h)} & \text{if } a,h \notin \mathcal{E}^{e_1} \\ s_{\iota^{-1}(a)}s_{h^2}s_{h^1} & \text{if } a \notin \mathcal{E}^{e_1}, h \in \mathcal{E}^{e_1} \\ s_{a^2}s_{a^1}s_{\iota^{-1}(h)} & \text{if } a \in \mathcal{E}^{e_1}, h \notin \mathcal{E}^{e_1} \end{cases} \\ &= \begin{cases} s_{\iota^{-1}(b)}s_{\iota^{-1}(g)} & \text{if } a,h \notin \mathcal{E}^{e_1} \\ s_{b^2}s_{b^1}s_{\iota^{-1}(g)} & \text{if } a \notin \mathcal{E}^{e_1}, h \in \mathcal{E}^{e_1} \\ s_{\iota^{-1}(b)}s_{g^1}s_{g^2} & \text{if } a \in \mathcal{E}^{e_1}, h \notin \mathcal{E}^{e_1} \end{cases} \\ &= T_bT_g. \end{aligned}$$

This shows that $\{Q_v, T_e\}$ is a Cuntz-Krieger Λ-family in $C^*(\Lambda_D)$. Let $\{q_v, t_e\}$ be the generators of the universal algebra $C^*(\Lambda)$ and observe that because of the universal property in $C^*(\Lambda_D)$ there exists a homomorphism $\pi\colon C^*(\Lambda) \to C^*(\Lambda_D)$ such that $\pi(q_v) = Q_v$ and $\pi(t_e) = T_e$. We wish to show that π is injective which will require the use of the gauge-invariant uniqueness theorem. The canonical actions between the two algebras may not intertwine, so Lemma 5.1.2.1 will be helpful. Let α be the canonical gauge action on $C^*(\Lambda)$ and construct β as an action on $C^*(\Lambda_D)$ via the following:

$$\beta_z(p_v) = p_v \qquad\qquad \beta_z(s_e) = \begin{cases} z^{d(e)} s_e & \text{if } e \notin \{g^1 \colon g \in \mathcal{E}^{e_1}\} \\ s_e & \text{if } e \in \{g^1 \colon g \in \mathcal{E}^{e_1}\} \end{cases}$$

Then we get

$$\begin{aligned} \pi(\alpha_z(t_e)) = \pi(z^{d(e)} t_e) = z^{d(e)} T_e &= \begin{cases} z^{d(e)} s_{\iota^{-1}(e)} & \text{if } e \notin \mathcal{E}^{e_1} \\ z^{d(e)} s_{e^2} s_{e^1} & \text{if } e \in \mathcal{E}^{e_1} \end{cases} \\ &= \begin{cases} \beta_z(s_{\iota^{-1}(e)}) & \text{if } e \notin \mathcal{E}^{e_1} \\ \beta_z(s_{e^2} s_{e^1}) & \text{if } e \in \mathcal{E}^{e_1} \end{cases} \\ &= \beta_z(T_e) = \beta_z(\pi(t_e)) \end{aligned}$$

The reader can check that the gauge actions intertwine similarly on the vertex projections. So we get that π is injective (in particular, $C^*(\Lambda) \cong \operatorname{Im}(\pi)$).

Now let $X = \iota^{-1}(\Lambda^0) = \Lambda_D^0 \backslash \{v_g \colon g \in \mathcal{E}^{e_1}\}$. We wish to show that $\Sigma(X) = \Lambda_D^0$. Note that for any v_g, we get $v_g \Lambda^{e_1} = \{g^1\}$ and $s(g^1) = s(g) \in X$. Thus since $\Sigma(X)$ is saturated, we get $\{v_g\} \subseteq \Sigma(X)$ and therefore $\Sigma(X) = \Lambda_D^0$. Let $P_X = \sum_{v \in X} p_v$ and observe via Theorem 5.1.3 that $P_X C^*(\Lambda_D) P_X \approx_{ME} C^*(\Lambda_D)$.

Lastly we wish to show that $\operatorname{Im}(\pi) = P_X C^*(\Lambda_D) P_X$. Clearly any $T_e \in \operatorname{Im}(\pi)$ is also in $P_X C^*(\Lambda_D) P_X$, so the first inclusion is automatically satisfied. Therefore, we focus on the other inclusion, $P_X C^*(\Lambda_D) P_X \subseteq \operatorname{Im}(\pi)$.

First, recall from earlier that $p_{v_g} = s_{g^1} s_{g^1}^*$ and observe that for any $g \in \mathcal{E}^{e_1}$, we have $s_{g^2} s_{g^2}^* = s_{g_2} s_{g^1} s_{g^1}^* s_{g^2}^* = T_g T_g^* \in \operatorname{Im}(\pi)$. Next, we claim that any path $\lambda = \lambda_{|\lambda|} \lambda_{|\lambda|-1} \cdots \lambda_1 \in \Lambda_D$ of length $|\lambda| = n \geq 2$ with $r(\lambda) \in X$ can be written in a way that contains no $h \in \bigcup_{i=2}^{\infty} \mathcal{E}_D^{e_i}$.

Since $r(\lambda) \in X$, then $\lambda_n \in \Lambda^{e_i}$ (in which case $s(\lambda_n) \in X$) or $\lambda_n = g_2$ for some $g \in \mathcal{E}^{e_1}$. If $s(\lambda_n) \in X$ then we continue the argument with the preceding edge, λ_{n-1}. If $\lambda_n = g^2$, then $r(\lambda_{n-1}) = v_g$ and so $\lambda_{n-1} = g^1$ (so $s(\lambda_{n-1} \in X$ and continue the argument with λ_{n-2}) or $\lambda_{n-1} = e_\alpha$ for some $\alpha = ha$ and $h \in \Lambda^{e_1}$. In particular, $d(e_\alpha) \neq e^1$ so the pair $\lambda_n\lambda_{n-1} = g^2 e_\alpha$ can be rewritten using the factorization property of Λ_D. So there exists $\lambda'_n\lambda'_{n-1} = g^2 e_\alpha$. Note that $r(\lambda'_n) = r(g^2) \in X, d(\lambda'_n) = d(e_\alpha)$, and $s(\lambda'_{n-1}) = s(e_\alpha), d(\lambda'_{n-1}) = d(g^2) = e_1$. Observe that $\lambda'_n \notin \mathcal{E}^{e_{d(e_\alpha)}}$ and λ'_{n-1} is h^2 for some $h \in \mathcal{E}^{e_1}$ (in which case, use the argument on λ_{n-2}). In this way, the entire path λ can be written without using edges from $\bigcup_{i=2}^{\infty} \mathcal{E}^{e_i}$. Observe that for any such λ, if g^2 is present in the path then it must immediately be preceded by g^1 unless $s(\lambda) \notin X$. So, if $s(\lambda) \in X$ then s_λ can be written using the above method to describe a path $\lambda' \in \Lambda$ with $s_\lambda = T_{\lambda'} \in \mathrm{Im}(\pi)$.

Consider $s_\mu s_\nu^* \in P_X C^*(\Lambda_D) P_X$. (Note that $s(\mu) = s(\nu)$ and $r(\mu), r(\nu) \in X$.) If $s(\mu) = s(\nu) \in X$, then by above, then there exist μ', ν' with $s_\mu s_\nu^* = T_{\mu'} T_{\nu'}^* \in \mathrm{Im}(\pi)$. On the other hand, if $s(\mu) = s(\nu) = v_g$, then $\tilde{\mu} = \mu g^1$ and $\tilde{\nu} = \nu g^1$ fits the above description, and $s_\mu s_\nu^* = s_\mu s_{g^1} s_{g^1}^* s_\nu^* = s_{\tilde{\mu}} s_{\tilde{\nu}}^* \in \mathrm{Im}(\pi)$. □

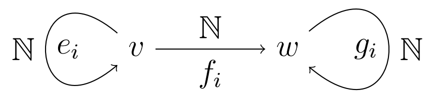
5.4 Reduction

For directed graphs, a reduction occurs by deleting a vertex between two edges and "gluing" the two related edges together. (Put another way, the reduction inverts a delay.) However, in order for the modification to preserve the $\mathbb{N}$-graph structure, only certain edges can be reduced.

Definition 5.4.1. Let Λ big an $\mathbb{N}$-graph. The set $E \subseteq \Lambda^1$ is called a complete edge if it has the following properties:

- E contains exactly one edge of each color;
- $s(e) = s(f)$ and $r(e) = r(f)$ for every $e, f \in E$;
- if $e \in E$ and $a, b \in \Lambda^1$ such that $ea = fb$, then $f \in E$.

Essentially put, the idea of a complete edge describes $\mathbb{N}$ arrows from one vertex to another. Notice that the definition depends on the edge relations which describe the factorization property.



Observe there are two possibilities for relations in the above picture which can define an $\mathbb{N}$-graph: $\{f_j e_i = f_i e_j, g_j f_i = g_i f_j\}$ or $\{f_j e_i = g_i f_j\}$. With the first option, there are three complete edges (namely, $\{e_i\}_{i\in\mathbb{N}}, \{f_i\}_{i\in\mathbb{N}}, \{g_i\}_{i\in\mathbb{N}}$) but with the second option there are none.

Definition 5.4.2. Let Λ be an $\mathbb{N}$-graph. Fix $v \in \Lambda^0$ such that both $\Lambda^1 v$ and $v\Lambda^1$ are complete edges and such that $w := r(\Lambda^1 v) \neq v$. Let $\{f_i\} = \Lambda^{e_i} \cap \Lambda^1 v$. We define a new $\mathbb{N}$-graph via the following:

$$\Lambda^0_R = \Lambda^0 \backslash \{v\} \qquad \Lambda^{e_i}_R = \Lambda^{e_i} \backslash \{f_i\} \qquad s_R(e) = s(e) \qquad r_R(e) = \begin{cases} r(e) & \text{if } r(e) \neq v \\ w & \text{if } r(e) = v \end{cases}$$

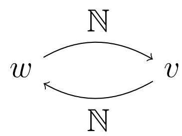
(we use the inclusion map $\iota\colon \Lambda^1_R \cup \Lambda^0_R \to \Lambda^1 \cup \Lambda^0$). To establish edge relations so that the factorization property is satisfied, we say:

$$ae =_R bg \text{ if } \begin{cases} \iota(a)\iota(e) = \iota(b)\iota(g) & r(\iota(e)), r(\iota(g)) \neq v \\ \exists f \in \Lambda^1 v\colon \iota(a) f \iota(e) = \iota(b) f \iota(g) & r(\iota(e)), r(\iota(g)) = v \end{cases}$$

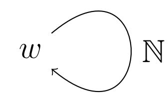
(Note that since $\Lambda^1 v$ is a complete edge, the chosen $f = f_i$ ends up being the unique e_i-choice for any related path $ae =_R bg$.) When this occurs, we call Λ_R the graph of Λ reduced at $v \in \Lambda^0$.

Below are two examples of reduction for $\mathbb{N}$-graphs. In both cases, v is the “redundant” vertex which is removed through the reduction process.

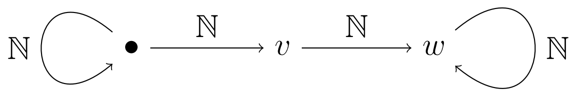
Λ



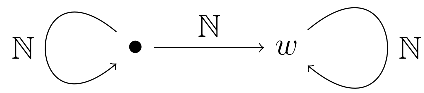
Λ_R



Γ



Γ_R



Theorem 5.4.1. *Let Λ be a row-finite, source-free $\mathbb{N}$-graph and let Λ_R be a reduction of Λ at $v \in \Lambda^0$. Then Λ_R is also a row-finite, source-free $\mathbb{N}$-graph.*

Proof. The factorization is established so that Λ_R satisfies the factorization property. Suppose $\lambda \in \Lambda_R$ and let $d_1+d_2 = d(\lambda)$. Then there exists a path $\lambda' \in \Lambda$ which coincides with $\iota(\lambda)$ when possible (note that λ' might include additional edges from $\Lambda^1 v$ in order to coincide with $\iota(\lambda)$). Then via the factorization property of Λ, there exist paths μ', ν' such that $\mu'\nu' = \lambda'$ and $d(\mu') = d_1', d(\nu') = d_2'$ for any $d_1' + d_2' = d(\lambda')$. Projecting these paths down to Λ_R will yield a unique factorization $\mu, \nu \in \Lambda_R$ such that $\mu\nu = \lambda$ and $d(\mu) = d_1, d(\nu) = d_2$. Furthermore, no edges were added during the reduction process so the result is still row-finite. Lastly any edges which were deleted are from $w\Lambda^1$ (thus w is the only possible candidate for being a source in Λ_R). However, $w\Lambda_R^1$ inherits all edges in $v\Lambda^1$. Since v was not a source in Λ, then w cannot be a source in Λ_R. □

Theorem 5.4.2. *Let Λ be a row-finite, source-free $\mathbb{N}$-graph and let Λ_R be a reduction of Λ at $v \in \Lambda^0$. Then $C^*(\Lambda_R) \cong_{ME} C^*(\Lambda)$.*

Proof. Let $\{p_v, s_e \colon v \in \Lambda^0, e \in \Lambda^1\}$ be the canonical Cuntz-Krieger Λ-family generating

$C^*(\Lambda)$. Fix $f \in \Lambda^1 v$. For $v \in \Lambda_R^0, e \in \Lambda_R^1$, define

$$Q_v = p_{\iota(v)} \qquad\qquad T_e = \begin{cases} s_{\iota(e)} & \text{if } r(\iota(e)) \neq v \\ s_f s_{\iota(e)} & \text{if } r(\iota(e)) = v \end{cases}$$

and observe that $\{Q_v, T_e\}$ is a Cuntz-Krieger Λ_R-family in $C^*(\Lambda)$:

- Since the projections p_v are mutually orthogonal, so are Q_v by definition.
- Let $e \in \Lambda_R^1$. Then,

$$T_e^* T_e = \begin{cases} s_{\iota(e)}^* s_{\iota(e)} & \text{if } r(\iota(e)) \neq v \\ s_{\iota(e)}^* s_f^* s_f s_{\iota(e)} & \text{if } r(\iota(e)) = v \end{cases} = \begin{cases} p_{s(\iota(e))} & \text{if } r(\iota(e)) \neq v \\ s_{\iota(e)}^* p_v s_{\iota(e)} & \text{if } r(\iota(e)) = v \end{cases} = Q_{s(e)}$$

- Let $x \in \Lambda_R^0, i \in \mathbb{N}$. Note that if $\iota(x) \neq w$ then $Q_x = p_{\iota(x)} = \sum_{r^{-1}(\iota(x)) \cap \Lambda^{e_i}} s_e s_e^* = \sum_{r^{-1}(x) \cap \Lambda_R^{e_i}} T_e T_e^*$ as desired since $T_e = s_{\iota(e)}$. So consider $x = w$. Then, note that since $v\Lambda^1 =: \{g_i\}$ is also a complete edge, we have $p_v = s_{g_i} s_{g_i}^*$ for any $i \in \mathbb{N}$. Furthermore, we have the relation $g_i f_j = g_j f_i$, so we get $s_{f_j} = s_{f_i} s_{g_j} s_{g_i}^*$ for any $i, j \in \mathbb{N}$. In particular, since $f = f_j$ for some $j \in \mathbb{N}$, we get $s_f s_f^* = (s_{f_i} s_{g_j} s_{g_i}^*)(s_{f_i} s_{g_j} s_{g_i}^*)^* = s_{f_i} s_{g_j} s_{g_i}^* s_{g_i} s_{g_j}^* s_{f_i}^* = s_{f_i} s_{f_i}^*$ for all $i \in \mathbb{N}$. Therefore,

$$\begin{aligned} Q_w = p_{\iota(w)} = \sum_{r^{-1}(\iota(w)) \cap \Lambda^{e_i}} s_{\iota(e)} s_{\iota(e)}^* &= \sum_{\Lambda^{e_i} v} s_{\iota(e)} s_{\iota(e)}^* + \sum_{w\Lambda^{e_i} \setminus \Lambda^{e_i} v} s_{\iota(e)} s_{\iota(e)}^* \\ &= s_{f_i} s_{f_i}^* + \sum_{\iota(w)\Lambda^{e_i} \setminus \Lambda^{e_i} v} s_{\iota(e)} s_{\iota(e)}^* \\ &= s_f s_f^* + \sum_{\iota(w)\Lambda^{e_i} \setminus \Lambda^{e_i} v} T_e T_e^* \\ &= s_f s_{\iota(g_i)} s_{\iota(g_i)}^* s_f^* + \sum_{\iota(w)\Lambda^{e_i} \setminus \Lambda^{e_i} v} T_e T_e^* \\ &= T_{g_i} T_{g_i}^* + \sum_{w\Lambda_R^{e_i} \setminus \Lambda^{e_i} v} T_e T_e^* = \sum_{r^{-1}(w) \cap \Lambda_R^{e_i}} T_e T_e^* \end{aligned}$$

- Suppose $ae =_R bg$ in Λ_R^2. Recall that $\iota(a)f\iota(e) = \iota(b)f\iota(g)$ in Λ^2 if $r(\iota(e)), r(\iota(g)) = v$. Then,

$$
\begin{aligned}
T_aT_e &= \begin{cases} s_{\iota(a)}s_{\iota(e)} & \text{if } r(\iota(a)), r(\iota(e)) \neq v \\ s_{\iota(a)}s_f s_{\iota(e)} & \text{if } r(\iota(a)) \neq v, r(\iota(e)) = v \\ s_f s_{\iota(a)}s_{\iota(e)} & \text{if } r(\iota(a)) = v, r(\iota(e)) \neq v \\ s_f s_{\iota(a)}s_f s_{\iota(e)} & \text{if } r(\iota(a)), r(\iota(e)) = v \end{cases} \\
&= \begin{cases} s_{\iota(b)}s_{\iota(g)} & \text{if } r(\iota(a)), r(\iota(e)) \neq v \\ s_{\iota(b)}s_f s_{\iota(g)} & \text{if } r(\iota(a)) \neq v, r(\iota(e)) = v \\ s_f s_{\iota(b)}s_{\iota(g)} & \text{if } r(\iota(a)) = v, r(\iota(e)) \neq v \\ s_f s_{\iota(b)}s_f s_{\iota(g)} & \text{if } r(\iota(a)), r(\iota(e)) = v \end{cases} \\
&= T_bT_g
\end{aligned}
$$

Let $C^*(\Lambda_R) = C^*(\{q_v, t_e\})$. By the universal property of $C^*(\Lambda_R)$, there exists a homomorphism $\pi\colon C^*(\Lambda_R) \to C^*(\Lambda)$ with $\pi(q_v) = Q_v$ and $\pi(t_e) = T_e$. There is a typical gauge action α on $C^*(\Lambda_R)$, i.e. $\alpha_z(\pi_R(t_e)) = zT_e = \pi_R(\alpha_z(t_e))$ and $\alpha_z(\pi_R(q_v)) = Q_v = \pi_R(\alpha_z(q_v))$. This gauge action may not intertwine with the canonical gauge action on $C^*(\Lambda)$ and the projection π. But we can use a different gauge action on $C^*(\Lambda)$ and invoke Lemma 5.1.2.1. Define $R\colon \Lambda \to \mathbb{Z}^{\mathbb{N}}$ via the following:

- $R(v) = 0$ for $v \in \Lambda^0$
- $R(e) = \begin{cases} d(e) & \text{if } e \in \Lambda^1, s(e) \neq v \\ d(e) - d(f) & \text{if } e \in \Lambda^1 v \end{cases}$
- $R(\lambda) = \sum_{i=1}^{|\lambda|} R(\lambda_i)$ for $\lambda = \lambda_{|\lambda|} \cdots \lambda_1 \in \Lambda$

Observe that $R(\lambda)$ simply counts the number of edges which move from v to w and subtracts the color of the chosen edge f that many times. So then R is a well-defined function and thus $\beta_z(s_\mu s_\nu^*) = z^{R(\mu)-R(\nu)} s_\mu s_\nu^*$ is a gauge action on $C^*(\Lambda)$. For $e \in \Lambda_R^1$, note that $s(\iota(e)) \neq v$ and so $R(\iota(e)) = d(\iota(e))$. Furthermore,

$$\pi(\alpha_z(t_e)) = \pi(z^{d_R(e)} t_e) = z^{d(\iota(e))} T_e = \begin{cases} z^{d(\iota(e))} s_{\iota(e)} & \text{if } r(\iota(e)) \neq v \\ z^{d(\iota(e))} s_f s_{\iota(e)} & \text{if } \iota(e) \in v\Lambda^1 \end{cases}$$

$$= \begin{cases} z^{R(\iota(e))} s_{\iota(e)} & \text{if } r(\iota(e)) \neq v \\ z^{R(f)} z^{R(\iota(e))} s_f s_{\iota(e)} & \text{if } \iota(e) \in v\Lambda^1 \end{cases}$$

$$= \begin{cases} \beta_z(s_\iota(e)) & \text{if } r(\iota(e)) \neq v \\ \beta_z(s_f s_{\iota(e)}) & \text{if } \iota(e) \in v\Lambda^1 \end{cases}$$

$$= \beta_z(T_e) = \beta_z(\pi(t_e))$$

since $R(f) = 0$. The reader can easily check that the actions intertwine on the vertex projections also. Thus, since these gauge actions are intertwined, the gauge-invariant uniqueness theorem tells us that π is injective. (In particular, $C^*(\Lambda_R) \cong \text{Im}(\pi)$.) Next, we use Theorem 5.1.3 to show Morita equivalence. Let $X = \iota(\Lambda_R^0) = \Lambda^0 \backslash \{v\}$ and observe that $w \in X$, therefore $s(w\Lambda^1) \subseteq \Sigma(X)$ by heredity. But $v \in s(w\Lambda^1)$. Lastly, we wish to show that $P_X C^*(\Lambda) P_X \cong \text{Im}(\pi)$ to finish the proof. For the first inclusion, consider $\lambda \in \Lambda_R$. Then as before there exists a corresponding path $\lambda' \in \Lambda$ with possibly added edges from $\Lambda^1 v$ to coincide with $\iota(\lambda)$. Observe that $s(\lambda'), r(\lambda') \neq v$. Thus $\pi(t_\lambda) = T_\lambda = s_{\lambda'} \in P_X C^*(\Lambda) P_X$. To prove the other inclusion, we first consider the following two claims.

Claim 5.4.1. *For any $f_i, f_j \in \Lambda^1 v$, we have $s_{f_i} s_{f_j}^* \in Im(\pi)$.*

Recall that $s_{f_i} = s_{f_j} s_{g_i} s_{g_j}^*$, thus

$$s_{f_i} s_{f_j}^* = s_{f_j} s_{g_i} s_{g_j}^* s_{f_j}^* = s_{f_j} s_{g_i} (s_{f_j} s_{g_j})^*.$$

If $f = f_j$, then $s_{f_j} s_{g_i} = s_f s_{g_i} = T_{\iota^{-1}(g_i)}$ and $s_{f_j} s_{g_j} = s_f s_{g_j} = T_{\iota^{-1}(g_j)}$, both in $\text{Im}(\pi)$. On the other hand, suppose $f \neq f_j$. As seen in Theorem 5.2.1, we can assume that Λ has no sinks without sacrificing Morita equivalence, and in particular, w is not a sink. Then $\exists e, g \in \Lambda^1 w$ such that $ef_j = gf$. In that case, we get

$$
\begin{aligned}
s_{f_j}s_{g_i} = p_w s_{f_j}s_{g_i} = s_e^* s_e s_{f_j} s_{g_i} = s_e^* s_g s_f s_{g_i} \\
&= \begin{cases} s_e^*(s_g s_f)s_{g_i} & \text{if } r(e) = r(g) \in X \\ s_e^*(s_f^* s_f)s_g s_f s_{g_i} & \text{if } r(e) = r(g) = v \end{cases} \\
&= \begin{cases} T^*_{\iota^{-1}(e)} T_{\iota^{-1}(g)} T_{\iota(g_i)} & \text{if } r(e) = r(g) \in X \\ (s_f s_e)^*(s_f s_g)(s_f s_{g_i}) & \text{if } r(e) = r(g) = v \end{cases} \\
&= T^*_{\iota^{-1}(e)} T_{\iota^{-1}(g)} T_{\iota^{-1}(g_i)} \\
&\in \operatorname{Im}(\pi)
\end{aligned}
$$

as desired. A similar calculation shows that $s_{f_j}s_{g_j} \in \operatorname{Im}(\pi)$ as well.

Claim 5.4.2. *For any $\lambda \in \Lambda$ with $s(\lambda) \neq v$, then $s_\lambda \in Im(\pi)$.*

There are two possibilities. If λ does not go through v, then $s_\lambda = T_{\iota^{-1}(\lambda)} \in \operatorname{Im}(\pi)$. Suppose λ goes through v. Then because $\Lambda^1 v$ and $v\Lambda^1$ are complete edges, every possible "rainbow" way to rewrite λ results in going through v. Write $s_\lambda = s_{|}\lambda| \cdots s_{\lambda_2}s_{\lambda_1}$ where each λ_i is an edge. If $\lambda_i \in \Lambda^1 v$, then $\lambda_i = f_j$ for some $j \in \mathbb{N}$. Replace s_{λ_i} via $s_{f_j} = s_e^* s_h s_f$ for some $e, h \in \Lambda^1 w$, as seen Claim 5.4.1. Now s_λ is written using only s_f and not s_{f_j} for $f_j \neq f$. By Claim 5.4.1, $s_\lambda \in \operatorname{Im}(\pi)$.

With the above two claims, we can prove the other inclusion. Let $s_\lambda s_\mu^* \in P_X C^*(\Lambda) P_X$ (so $r(\mu), r(\lambda) \in X$). If $s(\lambda) = s(\mu) \neq v$, then by Claim 5.4.2, $s_\lambda s_\mu^* \in \operatorname{Im}(\pi)$, as desired. On the other hand, write $s_\lambda s_\mu^* = s_{\lambda_{2+}} s_{\lambda_1} s_{\mu_1}^* s_{\mu_{2+}}^*$ where $s_{\lambda_{2+}}, s_{\mu_{2+}} \in \operatorname{im}(\pi)$ by Claim 5.4.2. Furthermore, since $\lambda_1, \mu_1 \in \Lambda^1 v$, we have that $\lambda_1, \mu_1 = f_i, f_j$ for some $i, j \in \mathbb{N}$. Thus by Claim 5.4.1, $s_{\lambda_1} s_{\mu_1}^* = s_{f_i} s_{f_j}^* \in \operatorname{Im}(\pi)$. Thus every generator of $p_X C^*(\Lambda) p_X$ is in $\operatorname{Im}(\pi)$, completing the inclusion. This completes the proof. □

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