Fractional Leibniz rules in quasi-Banach function spaces and weighted bi-parameter settings

by

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B.S., Truman State University, 2018

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Abstract

Fractional Leibniz rules are reminiscent of the product rule learned in calculus classes, describing properties of fractional differential operators applied to a product of functions. In particular, these inequalities traditionally give estimates in the Lebesgue norm for fractional derivatives of a product of functions in terms of the Lebesgue norms of each function and its fractional derivatives. Fractional Leibniz rules have applications in the study of the existence of solutions to PDEs such as Euler and Navier-Stokes equations. Moreover, partial fractional differential operators satisfy estimates whose structure resembles the Leibniz rule for classical partial derivatives and that also have applications in understanding the well-posedness of PDEs.

Results of this type have been studied in a variety of settings. For instance, Lebesgue norms can be replaced with norms associated to other function spaces. For single parameter differential operators, settings studied have included Hardy, Triebel-Lizorkin, Besov, and mixed Lebesgue spaces, as well as their weighted counterparts and weighted Lebesgue spaces. Bi-parameter results have been demonstrated in weighted and mixed Lebesgue spaces. Fractional Leibniz rules in each of these settings can be further generalized by replacing the product of functions with a bilinear pseudodifferential operator, with varying assumptions on the smoothness of the associated multiplier.

In this manuscript, we present fractional Leibniz rules associated to Coifman-Meyer bilinear pseudodifferential operators in a broad context that unifies many existing results in addition to obtaining new ones. In particular, the use of Nikol'skiĭ representations allows for a flexible approach by first obtaining estimates at the level of Triebel-Lizorkin spaces based on the function spaces of interest. We prove such estimates for Coifman-Meyer multiplier operators in the setting of Triebel-Lizorkin and Besov spaces based on quasi-Banach function spaces, which imply fractional Leibniz rules in the setting of quasi-Banach function spaces.

As corollaries, we obtain results in weighted mixed Lebesgue, weighted Morrey, and variable Lebesgue spaces. Another example is the class of rearrangement invariant quasi-Banach function spaces, which includes weighted Lebesgue, Lorentz, and Orlicz spaces. We further demonstrate the flexibility of this method by using it to prove bi-parameter fractional Leibniz rules in the setting of weighted Lebesgue spaces.

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Dedication

To my first teachers and biggest supporters – my parents.

Chapter 1

Introduction

The product rule for derivatives introduced in a typical calculus class says that the derivative of the product of two functions is given by the first function multiplied by the derivative of the second function plus the derivative of the first function multiplied by the second function. More precisely,

$$(fg)' = fg' + f'g.$$

Higher dimension analogs also hold for partial derivatives. For functions f and g on $\mathbb{R}^n$ and multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, denoting $\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$, we have

$$\partial^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)\,(\partial^\beta g) = (\partial^\alpha f)g + f(\partial^\alpha g) + \sum_{0<\beta<\alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)\,(\partial^\beta g),$$

where $\beta \leq \alpha$ if, and only if, $\beta_j \leq \alpha_j$, for all $j \in \{1, \ldots, n\}$ and $\beta < \alpha$ if, and only if, $\beta \leq \alpha$ and there exists $j \in \{1, \ldots, n\}$ such that $\beta_j < \alpha_j$. We note that in this more general setting, we preserve the structure of two terms with derivatives falling entirely on one function and not the other and introduce terms where the partial derivatives are distributed across the two functions. Known more generally as Leibniz rules, identities such as these describe the behavior of differential operators applied to the product of two functions in a variety of contexts.

Fractional Leibniz rules, also known as Kato-Ponce inequalities, give normed estimates for fractional differential operators applied to the product of two functions. In the setting of Lebesgue spaces, for $1 \leq p_1, p_2 \leq \infty$, $1/2 \leq p \leq \infty$ such that $1/p = 1/p_1 + 1/p_2$, and $s > \max(0, n(1/p - 1))$ or $s \in 2\mathbb{N}$, the inequality

$$\|E^s(fg)\|_{L^p} \lesssim \|E^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|E^s g\|_{L^{p_2}} \tag{1.1}$$

holds true with $E^s = D^s$ or $E^s = J^s$, where D^s and J^s are the homogeneous and inhomogeneous fractional differentiation operators of order s, respectively, defined through the Fourier transform by $\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$ and $\widehat{J^s(f)}(\xi) = (1 + |\xi|^2)^{s/2}\, \widehat{f}(\xi)$, $\xi \in \mathbb{R}^n$.

The inequality (1.1) and related commutator estimates have emerged as essential tools to study a number of nonlinear PDEs, including Euler and Navier-Stokes equations (see Kato–Ponce [44]) and Korteweg–de Vries equations (see Christ–Weinstein [20] and Kenig et al. [46]), as well as the study of smoothing properties of Schrödinger semigroups (see Gulisashvili–Kon [37]). The range $1 < p < \infty$ is addressed in the mentioned works and the case $\frac{1}{2} < p \leq 1$ is treated in Grafakos–Oh [36] and Muscalu–Schlag [57] (see also Koezuka–Tomita [47] and Naibo–Thomson [60]); for the endpoints $p = \infty$ and $p = \frac{1}{2}$, the reader is referred to Bourgain–Li [12] (see also Grafakos et al. [35]) and Oh–Wu [62], respectively.

The estimate (1.1) is a particular instance of inequalities in a variety of function spaces where the product fg is replaced by $T_\sigma(f, g)$; here, T_σ is a bilinear pseudodifferential operator associated to $\sigma = \sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, (called a symbol, or a multiplier if independent of x) and defined by

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta. \tag{1.2}$$

Note that for $\sigma \equiv 1$, we recover the product fg. Estimates using T_σ have the form

$$\|E^s(T_\sigma(f, g))\|_Z \lesssim \|E^s f\|_{Z_1} \|g\|_{Z_2} + \|f\|_{Z_1} \|E^s g\|_{Z_2} \tag{1.3}$$

for function spaces Z, Z_1, and Z_2. For example, variants of these estimates in weighted

Lebesgue spaces associated to Muckenhoupt weights are given in Naibo–Thomson [60] for Coifman-Meyer multiplier operators and in Cruz-Uribe–Naibo [23, 24] for $\sigma \equiv 1$; Hart et al. [40] proved estimates for multiplier operators in the context of Lebesgue and mixed Lebesgue spaces using minimal assumptions on the smoothness of the multipliers; Oh–Wu [63] obtained results with $\sigma \equiv 1$ in the setting of Lebesgue and mixed Lebesgue spaces associated to power weights; the works Koezuka–Tomita [47] and Naibo–Thomson [60] include estimates in the context of local Hardy spaces and weighted Hardy spaces, respectively. The estimates (1.3) have also been studied in the scale of Besov and Triebel-Lizorkin spaces for operators with symbols belonging to bilinear Hörmander classes; see, for instance, the works Bényi [8] and Naibo–Thomson [59] in the scale of Besov spaces, Bényi et al. [9] in the context of Sobolev spaces, and Naibo [58] and Koezuka–Tomita [47] for Besov and Triebel-Lizorkin spaces. For bilinear pseudodifferential operators with symbols closely related to the Hörmander classes, Brummer–Naibo [13] proved estimates in the setting of function spaces that admit a molecular decomposition and a φ-transform in the sense of Frazier–Jawerth [32, 33], and for Coifman-Meyer multiplier operators, Naibo–Thomson [60] worked in the context of weighted Besov and Triebel-Lizorkin spaces with weights in the Muckenhoupt classes. We refer the reader to the survey in Torres [67] for other considerations.

Other variations on (1.1) consider bi-parameter differential operators and have a structure comparable to the product rule for classical partial derivatives. That is, letting $n_1, n_2 \in \mathbb{N}$ be such that $n = n_1 + n_2$, for $1 \le p_1, p_2 \le \infty$, $1/2 \le p \le \infty$ such that $1/p = 1/p_1 + 1/p_2$, and $s_\ell > \max(0, n_\ell(1/p - 1))$, $\ell = 1, 2$, we have

$$\begin{aligned} \|D_1^{s_1} D_2^{s_2}(fg)\|_{L^p} \lesssim \|D_1^{s_1} D_2^{s_2} f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|D_1^{s_1} f\|_{L^{p_1}} \|D_2^{p_2} g\|_{L^{p_2}} \\ + \|D_2^{s_2} f\|_{L^{p_1}} \|D_1^{s_1} g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|D_1^{s_1} D_2^{s_2} g\|_{L^{p_2}}, \end{aligned} \tag{1.4}$$

where the partial fractional differential operators $D_1^{s_1}$ and $D_2^{s_2}$ are defined through

$$\widehat{D_\ell^{s_\ell} f}(\xi) = |\xi_\ell|^{s_\ell} \widehat{f}(\xi), \quad \xi_\ell \in \mathbb{R}^{n_\ell}, \ \ell = 1, 2.$$

The estimate in (1.4) was obtained for $n_1 = n_2 = 2$ by Muscalu et al. [56] and generalized by Grafakos–Oh [36] to allow for any $n_1, n_2 \in \mathbb{N}$, as well as partial differential operators for more than two parameters. Applications of (1.4) were studied by Kenig [45] for the existence of solutions to the KP-I equation.

Bi-parameter fractional Leibniz rules have also been proven in other settings and with a bilinear pseudodifferential operator $T_\sigma(f, g)$ replacing the product fg. Brummer–Naibo [14, 15] consider results with a Coifman-Meyer multiplier operator in weighted Lebesgue spaces associated to Muckenhoupt weights, while Yang et al. study estimates for pseudodifferential operators, whose symbols have limited Sobolev regularity, in the setting of Lebesgue spaces [70] and weighted Lebesgue spaces with flag weights [71]. Results with $\sigma \equiv 1$ have been proven in mixed Lebesgue spaces by Benea–Muscalu [4, 5], Di Plinio–Ou [28], and Oh–Wu [62], and in weighted mixed Lebesgue spaces with power weights by Oh–Wu [63].

In this manuscript, we prove fractional Leibniz rules of the type (1.3) for Coifman-Meyer multiplier operators in the setting of Triebel-Lizorkin and Besov spaces based on quasi-Banach function spaces, as in Hale–Naibo [38]. We also present bi-parameter fractional Leibniz rules in the spirit of (1.4) for bi-parameter Coifman-Meyer multiplier operators in the setting of weighted Lebesgue spaces, reflecting the work contained in Hale–Naibo [39].

A Coifman-Meyer multiplier operator of order $m \in \mathbb{R}$ is an operator of the type (1.2) with a smooth, complex-valued multiplier $\sigma(\xi, \eta), \xi, \eta \in \mathbb{R}^n$, that verifies

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)\right| \leq C_{\alpha,\beta}(|\xi| + |\eta|)^{m-(|\alpha|+|\beta|)}, \quad \forall (\xi, \eta) \in \mathbb{R}^{2n} \setminus \{(0,0)\}\,, \tag{1.5}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and some constant $0 < C_{\alpha,\beta} < \infty$. We also consider an inhomogeneous analog where σ is such that (1.5) holds with $1 + |\xi| + |\eta|$ instead of $|\xi| + |\eta|$.

One of the main results of this work is the following fractional Leibniz rule in the setting of Triebel-Lizorkin spaces based on quasi-Banach function spaces (we refer the reader to Chapters 2 and 3 for definitions of such spaces). Letting $\dot{F}^s_{Y,r}$ and H^Y denote the Triebel-

Lizorkin and Hardy spaces based on a quasi-Banach function space Y, we have

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{X^p,r}} \lesssim \|f\|_{\dot{F}^{s+m}_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}} + \|f\|_{H^{X_1^{p_1}}} \|g\|_{\dot{F}^{s+m}_{X_2^{p_2},r}}\,, \tag{1.6}$$

where $0 < p, p_1, p_2 < \infty$, $0 < r \leq \infty$, s satisfies a certain lower bound, σ is a Coifman-Meyer multiplier of order $m \in \mathbb{R}$, and X, X_1, and X_2 are quasi-Banach function spaces satisfying appropriate properties; corresponding estimates also hold in the Besov and inhomogeneous settings. See details in Chapter 4.

This result yields fractional Leibniz rules at the level of several function spaces. In particular, quasi-Banach function spaces include a diverse family of function spaces such as weighted mixed Lebesgue spaces, Morrey spaces, variable Lebesgue spaces, as well as the large class of rearrangement invariant quasi-Banach function spaces, of which weighted Lebesgue spaces, generalized versions of Lorentz spaces, and Orlicz spaces are specific examples. By proving the identification of quasi-Banach function spaces with spaces in the scale of the associated Triebel-Lizorkin spaces, (1.6) implies a plethora of fractional Leibniz rules in quasi-Banach function spaces, recovering in a unified way many results in the literature and providing new ones. For instance, we recover the following estimates proved in Naibo–Thomson [60]:

$$\|D^s(T_\sigma(f,g))\|_{H^p} \lesssim \|D^s f\|_{H^{p_1}} \|g\|_{H^{p_2}} + \|f\|_{H^{p_1}} \|D^s g\|_{H^{p_2}}\,, \tag{1.7}$$

for a Coifman-Meyer multiplier σ of order zero, $0 < p_1, p_2 < \infty$, $0 < p < \infty$ such that $1/p = 1/p_1 + 1/p_2$, $s > \max(0, n(1/p-1))$, and where $H^q = H^q(\mathbb{R}^n)$ denotes a Hardy space for $0 < q < \infty$ (recall that $H^q(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ for $1 < q < \infty$) and the Hardy norms may be replaced by $L^\infty(\mathbb{R}^n)$ in the terms without differential operators. Notice that, when $\sigma \equiv 1$, the estimate (1.7) improves (1.1) by allowing all indices to be in the wider range $(0, \infty]$ and by admitting the larger H^p-norm on the left hand side. A weighted version of (1.7) also holds with Hardy spaces associated to weights in the Muckenhoupt class $A_\infty(\mathbb{R}^n)$.

More generally, our estimates in Triebel-Lizorkin spaces lead to the following novel version of (1.3) in the setting of Hardy spaces based on weighted rearrangement invariant quasi-Banach function spaces,

$$\|D^s(T_\sigma(f,g))\|_{H^{X^p}(w)} \lesssim \|D^s f\|_{H^{X_1^{p_1}}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \|D^s g\|_{H^{X_2^{p_2}}(w_2)}, \tag{1.8}$$

where for a weight v and $0<q<\infty$, $H^{X^q}(v)$ denotes the Hardy space based on the weighted rearrangement invariant quasi-Banach function space $X^q(v)$, w, w_1, and w_2 are weights in the Muckenhoupt class $A_\infty(\mathbb{R}^n)$, the parameters s, p, p_1, and p_2 satisfy appropriate conditions, and σ is a Coifman-Meyer multiplier of order zero. In turn, (1.8) implies

$$\|D^s(T_\sigma(f,g))\|_{X^p(w)} \lesssim \|D^s f\|_{X_1^{p_1}(w_1)} \|g\|_{X_2^{p_2}(w_2)} + \|f\|_{X_1^{p_1}(w_1)} \|D^s g\|_{X_2^{p_2}(w_2)}, \tag{1.9}$$

for appropriate indices and weights in the Muckenhoupt classes. We refer the reader to Section 5.1 for more details.

The estimate in (1.6) also provides new results in the setting of weighted mixed Lebesgue spaces; for instance, if σ is a Coifman-Meyer multiplier of order zero, we obtain

$$\|D^s(T_\sigma(f,g))\|_{L^p(L^q(w))} \lesssim \|D^s f\|_{L^{p_1}(L^{q_1}(w_1))} \|g\|_{L^{p_2}(L^{q_2}(w_2))} + \|f\|_{L^{p_1}(L^{q_1}(w_1))} \|D^s g\|_{L^{p_2}(L^{q_2}(w_2))},$$

for $1<p,p_1,p_2,q,q_1,q_2<\infty$ such that $1/p=1/p_1+1/p_2$ and $1/q=1/q_1+1/q_2$, $s>0$, and appropriate weights w, w_1, and w_2 in 'mixed' versions of Muckenhoupt classes. Details can be found in Section 5.2. Other concrete examples implied by (1.6) include Leibniz rules in settings associated to weighted Lorentz and Orlicz spaces, as well as weighted Morrey and variable Lebesgue spaces. Details on these results are located in Sections 5.1.1, 5.3, and 5.4, respectively.

Some particular cases of (1.3) can be recast as

$$\|T_\sigma(f,g)\|_Y \lesssim \|f\|_Y \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_Y, \tag{1.10}$$

where Y is some function space associated to a smoothness parameter (for instance, a Sobolev space, or more generally a Besov or Triebel-Lizorkin space). Such estimates, in particular when $\sigma \equiv 1$, have played a fundamental role in the study of partial differential equations (see, for instance, Bahouri et al. [3], Brummer–Naibo [13], Ly–Naibo [51], and Naibo–Thomson [60], and the references therein), and they imply that $Y \cap L^\infty(\mathbb{R}^n)$ is an algebra under pointwise multiplication. Our result in (1.6) and its Besov counterpart give that (1.10) holds for Triebel-Lizorkin or Besov spaces based on a quasi-Banach function space; as a byproduct, the intersection of such spaces with $L^\infty(\mathbb{R}^n)$ is an algebra under pointwise multiplication.

Multiple approaches (which are based on Coifman-Meyer multiplier operators and the bilinear Calderón-Zygmund theory, square-function estimates, vector-valued multiplier theorems, among others) have been put forward to prove fractional Leibniz rules in the spirit of (1.3). We employ an alternative unifying approach used in Naibo–Thomson [60], presenting the work contained in Hale–Naibo [38]. This method is based on Nikol'skiĭ representations for function spaces and was pioneered for classical spaces in Bourdaud [11], Meyer [53], Nikol'skiĭ [61], Triebel [68], and Yamazaki [69]. We prove such representations for the general setting of Besov and Triebel-Lizorkin spaces based on quasi-Banach function spaces as given in Theorem 3.4.1.

This method based on Nikol'skiĭ representations is flexible and can be adapted to other settings. In particular, we adapt this method for application in the weighted bi-parameter setting. Thus, we obtain the following estimate for T_σ in Theorem 4.3.1, as proven in Hale–Naibo [39]:

$$\begin{aligned} \|D_1^{s_1} D_2^{s_2}(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|D_1^{s_1} D_2^{s_2} f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} + \|D_1^{s_1} f\|_{L^{p_1}(w_1)} \|D_2^{s_2} g\|_{L^{p_2}(w_2)} \\ + \|D_2^{s_2} f\|_{L^{p_1}(w_1)} \|D_1^{s_1} g\|_{L^{p_2}(w_2)} + \|f\|_{L^{p_1}(w_1)} \|D_1^{s_1} D_2^{s_2} g\|_{L^{p_2}(w_2)} . \end{aligned}$$

where s_1, s_2, p_1, p_2, and p satisfy appropriate conditions, w_1, w_2, and w are product Muckenhoupt weights, and σ is a bi-parameter Coifman-Meyer multiplier operator. That is, σ is such that, denoting $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, for all multi-indices $\alpha = (\alpha_1, \alpha_2), \beta =$

$(\beta_1, \beta_2) \in \mathbb{N}_0^{n_1} \times \mathbb{N}_0^{n_2}$, there exists a constant $0 < C_{\alpha,\beta} < \infty$ such that

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)\right| \leq \frac{C_{\alpha,\beta}}{(|\xi_1| + |\eta_1|)^{|\alpha_1|+|\beta_1|}(|\xi_2| + |\eta_2|)^{|\alpha_2|+|\beta_2|}}, \quad \forall (\xi_\ell, \eta_\ell) \neq (0, 0), \ell = 1, 2. \quad (1.11)$$

This result extends the work of Brummer–Naibo [14, 15] by expanding the range of values for the orders of differentiation s_1 and s_2 and contains as particular cases results by Muscalu et al. [56] and Grafakos–Oh [36] when $\sigma \equiv 1$ and $w = w_1 = w_2 \equiv 1$.

This manuscript is organized as follows. We introduce notation, definitions, and properties of weights, Littlewood-Paley operators, quasi-Banach function spaces, and rearrangement invariant quasi-Banach function spaces in Chapter 2. Chapter 3 contains notation and properties pertaining to weighted Triebel-Lizorkin and Besov spaces, Triebel-Lizorkin and Besov spaces based on quasi-Banach function spaces, and weighted bi-parameter Triebel-Lizorkin spaces, as well as the statements and proofs of Nikol'skiĭ representations for these spaces. The statements of the main results on fractional Leibniz rules in Triebel-Lizorkin and Besov spaces based on quasi-Banach function spaces as well as bi-parameter fractional Leibniz rules in weighted Lebesgue spaces are presented in Chapter 4, along with their proofs. In Chapter 5, we strengthen the single-parameter results in the quasi-Banach function space settings for the particular case of rearrangement invariant quasi-Banach function spaces, and present specific examples in weighted Lebesgue spaces, weighted Lorentz spaces and Orlicz spaces. Other particular applications of the general theory in the setting of quasi-Banach function spaces that are not rearrangement invariant are also given in Chapter 5 for weighted mixed Lebesgue spaces, weighted Morrey spaces, and variable Lebesgue spaces in Sections 5.2, 5.3, and 5.4, respectively.

Chapter 2

Preliminaries

In this chapter, we provide definitions, notations, and properties related to weighted Lebesgue spaces and Littlewood-Paley operators that will be of use in the single and bi-parameter settings, as well as the function spaces that will serve as the setting for many of our results. We begin here with a brief summary of notation and definitions pertaining to the Schwartz class and related families of functions and distributions, as well as the Hardy-Littlewood maximal operator. Section 2.1 contains a discussion on weights. Section 2.2 pertains to Littlewood-Paley theory. An overview of quasi-Banach function spaces is given in Section 2.3, and rearrangement invariant spaces are discussed in Section 2.4.

The Schwartz class of smooth, rapidly decreasing functions on $\mathbb{R}^n$ will be denoted by $\mathcal{S}(\mathbb{R}^n)$ and its dual, the space of tempered distributions, will be denoted by $\mathcal{S}'(\mathbb{R}^n)$. We use $\mathcal{S}_0(\mathbb{R}^n)$ to indicate the subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ with vanishing moments of all orders. That is, $\mathcal{S}_0(\mathbb{R}^n) = \left\{ f \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\alpha f(x)dx = 0, \ \ \forall \alpha \in \mathbb{N}_0^n \right\}$. Its dual space is the class of tempered distributions modulo polynomials, $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, which we represent by $\mathcal{S}_0'(\mathbb{R}^n)$. Also of interest are the bi-parameter counterparts of these spaces. For $n = n_1 + n_2$, $n_1, n_2 \in \mathbb{N}$, let $\mathcal{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ represent the Schwartz class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. We also define $\mathcal{S}_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as the subspace of functions $f \in \mathcal{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ such that, denoting $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $\int_{\mathbb{R}^{n_\ell}} x_\ell^{\alpha_\ell} f(x_1, x_2) dx_\ell = 0$ for any multi-index $\alpha \in \mathbb{N}_0^{n_\ell}$, $\ell = 1, 2$. Its dual will be denoted by $\mathcal{S}_\infty'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

The Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is denoted by $\widehat{f}$ or $\mathscr{F} f$. In particular, if f is integrable in $\mathbb{R}^n$, we have

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \forall \xi \in \mathbb{R}^n.$$

Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a compact subset A of $\mathbb{R}^n$, we represent the average value of f over A by

$$⨍_A f(x) dx = \frac{1}{|A|} \int_A f(x) dx,$$

where $|A|$ is the Lebesgue measure of A. We then denote the *Hardy-Littlewood maximal operator* and the *strong maximal operator* by $\mathcal{M}_n f$ and $\mathcal{M}_n^S f$, respectively. That is,

$$\mathcal{M}_n f(x) = \sup_{x \in Q} ⨍_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes containing x, and

$$\mathcal{M}_n^S f(x) = \sup_{x \in R} ⨍_R |f(y)| \, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all rectangles $R = Q_1 \times Q_2$ containing x with Q_1 and Q_2 cubes with sides parallel to the coordinate axes in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. For $0 < h < \infty$, we define $\mathcal{M}_{n,h} f(x) := (\mathcal{M}_n |f|^h)^{1/h}$, with $\mathcal{M}_{n,h}^S f(x)$ defined analogously.

Finally, for brevity, $n_1, n_2 \in \mathbb{N}$ will always be such that $n = n_1 + n_2$. Moreover, in the bi-parameter settings we often suppress the indication $\ell = 1, 2$ for expressions that are associated to both parameters. That is, any expression with the subscript ℓ is assumed to represent the expression for both parameters. For example, $\psi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ will be used to represent that both $\psi_1 \in \mathcal{S}(\mathbb{R}^{n_1})$ and $\psi_2 \in \mathcal{S}(\mathbb{R}^{n_2})$. Also, for two quantities A and B, we say that $A \lesssim B$ when $A \le cB$ for some constant $c > 0$; if $A \lesssim B$ and $B \lesssim A$, we use the notation $A \sim B$.

2.1 Weights

A *weight* on $\mathbb{R}^n$ is a locally integrable, nonnegative function defined on $\mathbb{R}^n$. Given a weight w on $\mathbb{R}^n$ and $0 < p \le \infty$, the *weighted Lebesgue space* $L^p(w)$ is the collection of measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty,$$

with the usual change when $p = \infty$.

For $1 < p < \infty$, $A_p(\mathbb{R}^n)$ will represent the *Muckenhoupt class* of weights; this is the collection of weights w on $\mathbb{R}^n$ such that the following constant is finite:

$$[w]_{A_p(\mathbb{R}^n)} = \sup_Q \left(\fint_Q w(x) dx \right) \left(\fint_Q w(x)^{1-p'} dx \right)^{p-1},$$

where the supremum is over all cubes $Q \subset \mathbb{R}^n$. We recall that $w \in A_p(\mathbb{R}^n)$ if, and only if, the Hardy-Littlewood maximal operator $\mathcal{M}_n$ is bounded on $L^p(w)$ (see Muckenhoupt [55]). We designate the class of weights $A_\infty(\mathbb{R}^n) = \cup_{p>1}^{\infty} A_p(\mathbb{R}^n)$, and given $w \in A_\infty(\mathbb{R}^n)$, denote $\tau_w = \inf \{\tau > 1 : w \in A_\tau(\mathbb{R}^n)\}$.

The weighted Fefferman-Stein inequality will also be of use: for $1 < p \le \infty$, $0 < r < \infty$, $0 < h < \min(p, r)$, and $w \in A_{p/h}(\mathbb{R}^n)$ (equivalently, $0 < h < \min(p/\tau_w, r)$), we have

$$\left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{M}_{n,h} f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)}, \tag{2.1}$$

where the implicit constant depends only on p, r, and $[w]_{A_p(\mathbb{R}^n)}$.

For $1 < p < \infty$, the *product Muckenhoupt class* $A_{p,\mathcal{R}}(\mathbb{R}^n)$ is the collection of weights on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that the following constant is finite:

$$[w]_{A_{p,\mathcal{R}}} = \sup_R \left(\fint_R w(x) dx \right) \left(\fint_R w(x)^{1-p'} dx \right)^{p-1}, \tag{2.2}$$

where the supremum is taken over all rectangles $R = Q_1 \times Q_2$ with Q_1 and Q_2 cubes in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. We then define $A_{\infty,\mathcal{R}}(\mathbb{R}^n)$ and τ_w analogously to the traditional Muckenhoupt weights.

The product Muckenhoupt class can be thought of as a bi-parameter analog of the classical Muckenhoupt class, and Bagby and Kurtz [2] characterize the relationship between $A_{p,\mathcal{R}}(\mathbb{R}^n)$ and $A_p(\mathbb{R}^{n_\ell})$ through the following lemma:

Lemma 2.1.1 (Lemma 1.2 from [2])**.** *Let $1 < p < \infty$. Then $w \in A_{p,\mathcal{R}}(\mathbb{R}^n)$ if, and only if, $w(\cdot, x_2) \in A_p(\mathbb{R}^{n_1})$ and $w(x_1, \cdot) \in A_p(\mathbb{R}^{n_2})$ for almost every $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with constants uniform in x_1 and x_2.*

This implies that for $1 < p < \infty$ and $w \in A_{p,\mathcal{R}}(\mathbb{R}^n)$, $\mathcal{M}_{n_\ell}$, $\mathcal{M}_{n_1}\mathcal{M}_{n_2}$, and $\mathcal{M}_{n_2}\mathcal{M}_{n_1}$ are bounded on $L^p(w)$. We also note that the strong maximal operator $\mathcal{M}_n^S$ is bounded on $L^p(w)$ and a corresponding Fefferman-Stein inequality holds (see Capri-Gutierrez [17]).

It is useful to note that if $w_1 \in A_{p_1}(\mathbb{R}^n)$ and $w_2 \in A_{p_2}(\mathbb{R}^n)$, then $w_1^{\theta_1} w_2^{\theta_2} \in A_{\max(p_1,p_2)}(\mathbb{R}^n)$ for $\sigma_1, \sigma_2 \in (0,1)$ such that $\sigma_1 + \sigma_2 = 1$. A corresponding statement holds true for the product Muckenhoupt class of weights.

Finally, certain results, in particular extrapolation, are given in terms of pairs of functions (f, g). We use $\mathcal{F}$ to denote a family of pairs of measurable functions that are not identically zero. If for some $0 < p < \infty$ and $w \in A_q(\mathbb{R}^n)$ (or $w \in A_{q,\mathcal{R}}(\mathbb{R}^n)$), $1 \leq q \leq \infty$, we say that

$$\int_{\mathbb{R}^n} |f(x)|^p \, w(x)dx \lesssim \int_{\mathbb{R}^n} |g(x)|^p \, w(x)dx, \qquad \forall (f, g) \in \mathcal{F}, \tag{2.3}$$

we mean that (2.3) holds for all pairs of functions $(f, g) \in \mathcal{F}$ such that the left-hand side is finite, and the implicit constant depends only on p and $[w]_{A_q(\mathbb{R}^n)}$ (or $[w]_{A_{q,\mathcal{R}}(\mathbb{R}^n)}$). In the case that the $L^p(w)$ norm in (2.3) is replaced with another function space norm, the inequality should be interpreted the same way.

2.2 Littlewood-Paley Operators

Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ and $\Psi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{Z}$, we define the Littlewood-Paley operators

$$\widehat{P_j^{\Psi} f}(\xi) = \widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$
$$\widehat{P_j^{\Psi_\ell} f}(\xi) = \widehat{\Psi_\ell}(2^{-j}\xi_\ell)\widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

If $\widehat{\Psi}$ and $\widehat{\Psi_\ell}$ are supported in an annulus centered at the origin, we use the notation Δ_j^{Ψ} and $\Delta_j^{\Psi_\ell}$ rather than P_j^{Ψ} and $P_j^{\Psi_\ell}$, respectively. Similarly, if $\widehat{\Psi}$ and $\widehat{\Psi_\ell}$ are supported in a ball centered at the origin and are nonzero at the origin, we use S_j^{Ψ} and $S_j^{\Psi_\ell}$ instead of P_j^{Ψ} and $P_j^{\Psi_\ell}$. For $c \in \mathbb{R}^n$, we also denote with $\mathbb{T}_c$ the translation operator given by $\mathbb{T}_c f(x) = f(x+c)$ for $x \in \mathbb{R}^n$.

Remark 2.2.1. We note that for $\Psi_1 \in \mathcal{S}(\mathbb{R}^{n_1})$ and $y_2 \in \mathbb{R}^{n_2}$, we have

$$\widehat{P_j^{\Psi_1}}^{n_1}(\xi_1, y_2) = \widehat{\Psi_1}(2^{-j}\xi_1)\widehat{f}^{\,n_1}(\xi_1, y_2), \quad \forall \xi_1 \in \mathbb{R}^{n_1}, j \in \mathbb{Z},$$

where $\widehat{f}^{\,n_1}(\cdot_1, y_2)$ represents the Fourier transform of $f(\cdot_1, y_2)$ in $\mathbb{R}^{n_1}$. An analogous statement holds for $\Psi_2 \in \mathcal{S}(\mathbb{R}^{n_2})$ and $y_1 \in \mathbb{R}^{n_1}$.

The following lemmas describe some relationships between Littlewood-Paley operators and the Hardy-Littlewood maximal operator.

Lemma 2.2.2 (Lemma 3.1 in [60]). *Let $\Psi, \psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\Psi}$ and $\widehat{\psi}$ have compact supports and $\widehat{\Psi}\,\widehat{\psi} = \widehat{\Psi}$. If $0 < h \leq 1$ and $\varepsilon > 0$,it holds that*

$$\left|P_j^{\mathbb{T}_c\Psi} f(x)\right| \lesssim (1+|c|)^{\varepsilon+\frac{n}{h}} \mathcal{M}_{n,h}(P_j^{\psi} f)(x), \quad \forall x, c \in \mathbb{R}^n, j \in \mathbb{Z}, f \in \mathcal{S}'(\mathbb{R}^n).$$

Remark 2.2.3. It follows from Lemma 2.2.2 and Remark 2.2.1 that since for a fixed $y_2 \in \mathbb{R}^{n_2}$, $f \in \mathcal{S}(\mathbb{R}^n)$ implies that $f(\cdot_1, y_2) \in \mathcal{S}(\mathbb{R}^{n_1})$, if $\Psi_1, \psi_1 \in \mathcal{S}(\mathbb{R}^{n_1})$ have compact support and are such that $\widehat{\Psi_1}\widehat{\psi_1} = \widehat{\Psi_1}$, we have

$$\left| P_j^{\mathbb{T}_{c_1}\Psi_1} f(x_1, y_2) \right| \lesssim (1 + |c_1|)^{\varepsilon + \frac{n_1}{h}} \mathcal{M}_{n_1,h} \left(P_j^{\psi_1} f \right)(x_1, y_2), \tag{2.4}$$

for all $x_1, c_1 \in \mathbb{R}^{n_1}, y_2 \in \mathbb{R}^{n_2}, j \in \mathbb{Z}$, and $f \in \mathcal{S}(\mathbb{R}^n)$. An analogous statement for $\Psi_2, \psi_2 \in \mathcal{S}(\mathbb{R}^{n_2})$ with compact support and $\widehat{\Psi_2}\widehat{\psi_2} = \widehat{\Psi_2}$ also holds.

Also of use is a bi-parameter version of Lemma 2.2.2 giving estimates relating iterated Littlewood-Paley operators with the strong maximal operator.

Lemma 2.2.4. *Let $\Psi_\ell, \psi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ be such that $\widehat{\Psi_\ell}$ and $\widehat{\psi_\ell}$ have compact supports and $\widehat{\Psi_\ell}\,\widehat{\psi_\ell} = \widehat{\Psi_\ell}$. If $0 < h \le 1$ and $\varepsilon > 0$, it holds that*

$$\left| P_{j_1}^{\mathbb{T}_{c_1}\Psi_1} P_{j_2}^{\mathbb{T}_{c_2}\Psi_2} f(x) \right| \lesssim (1 + |c_1|)^{\varepsilon + \frac{n_1}{h}} (1 + |c_2|)^{\varepsilon + \frac{n_2}{h}} \mathcal{M}_{n,h}^S (P_{j_1}^{\psi_1} P_{j_2}^{\psi_2} f)(x),$$

for all $x \in \mathbb{R}^n$, $c_\ell \in \mathbb{R}^{n_\ell}$, $j_\ell \in \mathbb{Z}$, and $f \in \mathcal{S}'(\mathbb{R}^n)$, and where the implicit constant depends only on n_ℓ, ε, h, Ψ_ℓ, and ψ_ℓ.

The proof of Lemma 2.2.4 uses the following lemma.

Lemma 2.2.5. *Let $B_\ell > 0$, $R_\ell \ge 1$, $0 < h \le 1$, and $d_\ell > n_\ell/h$. Consider $\varphi(x) = \varphi_1(x_1)\varphi_2(x_2)$, for $\varphi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ and $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and $f \in \mathcal{S}'(\mathbb{R}^n)$ such that supp$(\widehat{f}) \subset \{\xi_1 \in \mathbb{R}^{n_1} : |\xi_1| \le B_1 R_1\} \times \{\xi_2 \in \mathbb{R}^{n_2} : |\xi_2| \le B_2 R_2\}$. Then, denoting $y = (y_1, y_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, it holds that*

$$|\varphi * f(x)| \lesssim B_1^{-n_1} B_2^{-n_2} R_1^{n_1(\frac{1}{h}-1)} R_2^{n_2(\frac{1}{h}-1)} \left\| (1 + B_1|\cdot_1|)^{d_1} (1 + B_2|\cdot_2|)^{d_2} \varphi \right\|_{L^\infty} \mathcal{M}_{n,h}^S(f)(x),$$

where the implicit constant depends only on n_ℓ, d_ℓ, and h.

Proof of Lemma 2.2.4. For any $j_\ell \in \mathbb{Z}$, we have $P_{j_\ell}^{\mathbb{T}_{c_\ell}\Psi_\ell} = P_{j_\ell}^{\mathbb{T}_{c_\ell}\Psi_\ell} P_{j_\ell}^{\psi_\ell}$. Applying Lemma 2.2.5 to the function $P_{j_1}^{\psi_1} P_{j_2}^{\psi_2} f$ with $s = \infty$, $\varphi_\ell(x_\ell) = 2^{j_\ell n_\ell} \mathbb{T}_{c_\ell} \Psi_\ell (2^{j_\ell} x_\ell)$, $d_\ell = \varepsilon + n_\ell/h$, $B_\ell = 2^{j_\ell}$, and $R_\ell > 1$ such that $\operatorname{supp}(\widehat{\psi_\ell}) \subset \{\xi_\ell \in \mathbb{R}^{n_\ell} : |\xi_\ell| \le R_\ell\}$, we have

$$\begin{aligned}
&\left|P_{j_1}^{\mathbb{T}_{c_1}\Psi_1} P_{j_2}^{\mathbb{T}_{c_2}\Psi_2} f(x)\right| \\
&\lesssim 2^{-j_1 n_1} 2^{-j_2 n_2} R_1^{n_1\left(\frac{1}{h}-1\right)} R_2^{n_2\left(\frac{1}{h}-1\right)} \\
&\quad \times \left\| \left(1+2^{j_1}|\cdot_1|\right)^{\varepsilon+\frac{n_1}{h}} \left(1+2^{j_2}|\cdot_2|\right)^{\varepsilon+\frac{n_2}{h}} 2^{j_1 n_1} \mathbb{T}_{c_1}\Psi_1\left(2^{j_1}\cdot_1\right) 2^{j_2 n_2} \mathbb{T}_{c_2}\Psi_2\left(2^{j_2}\cdot_2\right) \right\|_{L^\infty} \\
&\quad \times \mathcal{M}_{n,h}^S \left(P_{j_1}^{\psi_1} P_{j_2}^{\psi_2} f\right)(x) \\
&\sim \left\| \left(1+2^{j_1}|\cdot_1|\right)^{\varepsilon+\frac{n_1}{h}} \left(1+2^{j_2}|\cdot_2|\right)^{\varepsilon+\frac{n_2}{h}} \mathbb{T}_{c_1}\Psi_1\left(2^{j_1}\cdot_1\right) \mathbb{T}_{c_2}\Psi_2\left(2^{j_2}\cdot_2\right) \right\|_{L^\infty} \mathcal{M}_{n,h}^S \left(P_{j_1}^{\psi_1} P_{j_2}^{\psi_2} f\right)(x) \\
&\lesssim \left(1+|c_1|\right)^{\varepsilon+\frac{n_1}{h}} \left(1+|c_2|\right)^{\varepsilon+\frac{n_2}{h}} \mathcal{M}_{n,h}^S \left(P_{j_1}^{\psi_1} P_{j_2}^{\psi_2} f\right)(x),
\end{aligned}$$

where in the last inequality we have used that $|\mathbb{T}_{c_\ell}\Psi_\ell(x_\ell)| \lesssim (1+|c_\ell|)^{\varepsilon+n_\ell/h} / (1+|x_\ell|)^{\varepsilon+n_\ell/h}$ for any $x_\ell, c_\ell \in \mathbb{R}^{n_\ell}$. □

The next lemma gives estimates for the supremum over bi-parameter Littlewood-Paley operators associated to a given pair of Schwartz functions.

Lemma 2.2.6. *For $\varphi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ and an integrable function f on $\mathbb{R}^n$, we have*

$$\sup_{j_1, j_2 \in \mathbb{Z}} \left|P_{j_1}^{\varphi_1} P_{j_2}^{\varphi_2} f(x)\right| \lesssim \mathcal{M}_n^S f(x). \tag{2.5}$$

In particular, this implies that for $1 < p < \infty$ and $w \in A_{p,\mathcal{R}}(\mathbb{R}^n)$,

$$\left\| \sup_{j_1, j_2 \in \mathbb{Z}} \left|P_{j_1}^{\varphi_1} P_{j_2}^{\varphi_2} f\right| \right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}. \tag{2.6}$$

The proof of Lemma 2.2.6 requires the following result that gives an estimate associated to the strong maximal operator. A similar lemma is proven by Chen and Lu [19, Lemma 1.2]; here we allow for $n_1 \ne n_2$.

Lemma 2.2.7. *Let $\varepsilon_\ell > 0$; for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,*

$$\sup_{r_1, r_2 > 0} \left(r_1^{n_1} r_2^{n_2} \int_{\mathbb{R}^n} \frac{|f(y_1, y_2)|}{\left(1+r_1|x_1-y_1|\right)^{n_1+\varepsilon_1} \left(1+r_2|x_2-y_2|\right)^{n_2+\varepsilon_2}} dy_1 dy_2 \right) \lesssim \mathcal{M}_n^S f(x).$$

Proof. We will split the integral

$$r_1^{n_1} r_2^{n_2} \int_{\mathbb{R}^n} \frac{|f(y_1,y_2)|}{(1+r_1|x_1-y_1|)^{n_1+\varepsilon_1}(1+r_2|x_2-y_2|)^{n_2+\varepsilon_2}} dy_1 dy_2$$

into four regions.

First, Region 1 is $|x_1-y_1| < r_1^{-1}$ and $|x_2-y_2| < r_2^{-1}$. In this setting,

$$r_1^{n_1} r_2^{n_2} \int_{\substack{|x_1-y_1|<r_1^{-1} \\ |x_2-y_2|<r_2^{-1}}} \frac{|f(y_1,y_2)|}{(1+r_1|x_1-y_1|)^{n_1+\varepsilon_1}(1+r_2|x_2-y_2|)^{n_2+\varepsilon_2}} dy_1 dy_2 \lesssim \mathcal{M}_n^S f(x).$$

Next, Region 2 is $|x_1-y_1| \geq r_1^{-1}$ and $|x_2-y_2| < r_2^{-1}$. We have

$$\begin{aligned}
&\int_{\substack{|x_1-y_1|\geq r_1^{-1} \\ |x_2-y_2|<r_2^{-1}}} \frac{|f(y_1,y_2)|}{(1+r_1|x_1-y_1|)^{n_1+\varepsilon_1}(1+r_2|x_2-y_2|)^{n_2+\varepsilon_2}} dy_1 dy_2 \\
&\leq \sum_{k=0}^{\infty} \int_{\substack{2^k r_1^{-1} \leq |x_1-y_1| \leq 2^{k+1} r_1^{-1} \\ |x_2-y_2|<r_2^{-1}}} \frac{|f(y_1,y_2)|}{(1+r_1|x_1-y_1|)^{n_1+\varepsilon_1}(1+r_2|x_2-y_2|)^{n_2+\varepsilon_2}} dy_1 dy_2 \\
&\lesssim \sum_{k=0}^{\infty} \frac{1}{(1+2^k)^{n_1+\varepsilon_1}} \int_{\substack{|x_1-y_1| \leq 2^{k+1} r_1^{-1} \\ |x_2-y_2|<r_2^{-1}}} \frac{|f(y_1,y_2)|}{(1+r_2|x_2-y_2|)^{n_2+\varepsilon_2}} dy_1 dy_2 \\
&\sim \sum_{k=0}^{\infty} \frac{1}{(1+2^k)^{n_1+\varepsilon_1}} \frac{(2^{k+1})^{n_1} r_1^{-n_1} r_2^{-n_2}}{(2^{k+1})^{n_1} r_1^{-n_1} r_2^{-n_2}} \int_{\substack{|x_1-y_1| \leq 2^{k+1} r_1^{-1} \\ |x_2-y_2|<r_2^{-1}}} |f(y_1,y_2)| \, dy_1 dy_2 \\
&\sim r_1^{-n_1} r_2^{-n_2} \mathcal{M}_n^S f(x).
\end{aligned}$$

Region 3 is given by $|x_1-y_1| < r_1^{-1}$ and $|x_2-y_2| \geq r_2^{-1}$. The desired estimate in this setting follows from logic similar to that for Region 2. Finally, Region 4 is given by $|x_1-y_1| \geq r_1^{-1}$ and $|x_2-y_2| \geq r_2^{-1}$. To obtain the estimate in this region, we write the integral as a sum over the annuli $2^k r_\ell^{-1} \leq |x_\ell - y_\ell| \leq 2^k r_\ell^{-1}$ in both parameters and treat them both in the same manner as $2^k r_1^{-1} \leq |x_1-y_1| \leq 2^k r_1^{-1}$ in the computations for Region 2. □

We now prove Lemma 2.2.6.

Proof of Lemma 2.2.6. Beginning with (2.5), we have

$$\begin{aligned}
&\sup_{j_1,j_2\in\mathbb{Z}} \left|P_{j_1}^{\varphi_1}P_{j_2}^{\varphi_2}f(x)\right| \\
&= \sup_{j_1,j_2\in\mathbb{Z}} \left|\left(2^{j_1n_1}2^{j_2n_2}\left(\varphi_1(2^{j_1}\cdot_1)\varphi_2(2^{j_2}\cdot_2)\right)*f\right)(x)\right| \\
&= \sup_{j_1,j_2\in\mathbb{Z}} \left|2^{j_1n_1}2^{j_2n_2}\int_{\mathbb{R}^n}\varphi_1(2^{j_1}(x_1-y_1))\varphi_2(2^{j_2}(x_2-y_2))f(y_1,y_2)dy_1dy_2\right| \\
&\lesssim \sup_{j_1,j_2\in\mathbb{Z}} 2^{j_1n_1}2^{j_2n_2}\int_{\mathbb{R}^n}\left(1+2^{j_1}|x_1-y_1|\right)^{-n_1-\varepsilon_1}\left(1+2^{j_2}|x_2-y_2|\right)^{-n_2-\varepsilon_2}|f(y_1,y_2)|\,dy_1dy_2 \\
&\leq \sup_{r_1,r_2>0} r_1^{n_1}r_2^{n_2}\int_{\mathbb{R}^n}(1+r_1|x_1-y_1|)^{-n_1-\varepsilon_1}(1+r_2|x_2-y_2|)^{-n_2-\varepsilon_2}|f(y_1,y_2)|\,dy_1dy_2 \\
&\lesssim \mathcal{M}_n^S f(x),
\end{aligned}$$

where the final inequality follows from Lemma 2.2.7.

The estimate in (2.6) then follows from (2.5) and the boundedness of the strong maximal operator on $L^p(w)$ for $1<p<\infty$ and $w\in A_{p,\mathcal{R}}(\mathbb{R}^n)$. □

Finally, we note the following inequality: for an integrable function f defined on $\mathbb{R}^n$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$, we have

$$\sup_{t>0}\left|\frac{1}{t^n}\mathbb{T}_c\varphi\left(\frac{\cdot}{t}\right)*f(x)\right| \lesssim (1+|c|)^{n+1}\mathcal{M}_nf(x), \quad \text{a.e. } x\in\mathbb{R}^n. \tag{2.7}$$

2.3 Quasi-Banach Function Spaces

Many of our results hold in the setting of quasi-Banach function spaces; in this section, we introduce relevant definitions and notation, as well as describe additional conditions on the spaces of interest. We refer the reader to Bennett–Sharpley [7] and Cruz-Uribe et al. [26] for further details.

Let $(\mathbb{R}^n,\mu)$ be a totally σ-finite, nonatomic measure space and M represent the class of measurable functions on $(\mathbb{R}^n,\mu)$. A mapping $\rho: M\to[0,\infty]$ is called a *Banach function norm* if it satisfies the following properties for all $f,g\in M$:

P1. $\rho(f) = \rho(|f|)$ and $\rho(f) = 0$ if, and only if, $f = 0$ μ-a.e.;

P2. $\rho(f + g) \leq \rho(f) + \rho(g)$;

P3. $\rho(af) = |a|\, \rho(f)$, for all $a \in \mathbb{R}$;

P4. $|f| \leq |g|$ μ-a.e. implies $\rho(f) \leq \rho(g)$;

P5. If $\{f_j\}_{j \in \mathbb{Z}} \subset M$ is such that $|f_j|$ increases to $|f|$ μ-a.e., then $\rho(f_j)$ increases to $\rho(f)$;

P6. If $E \subset \mathbb{R}^n$ is measurable and $\mu(E) < \infty$, then

i. $\rho(\chi_E) < \infty$,

ii. $\int_E |f|\, d\mu \leq C_E \rho(f)$ for $0 < C_E < \infty$ depending only on E.

Given ρ, we define the function space

$$X = \{f \in M : \|f\|_X < \infty\},$$

where $\|f\|_X = \rho(f)$, and call X a *Banach function space* (BFS). Note that Properties P4. and P5. can be used to show that $(X, \|\cdot\|_X)$ is complete and hence a Banach space (see Bennett–Sharpley [7, Chap. 1, Theorem 1.6]).

The *associate space of* X, which we represent by X', is defined through the Banach function norm

$$\rho'(f) = \sup\left\{\int_{\mathbb{R}^n} |f(x)g(x)|\, d\mu : g \in X, \|g\|_X \leq 1\right\}.$$

Given $0 < p < \infty$, define the scale of spaces

$$X^p = \{f \in M : |f|^p \in X\},$$

with norm $\|f\|_{X^p} = \||f|^p\|_X^{1/p}$.

Now, if we replace Property P2. by

$$\rho(f + g) \lesssim \rho(f) + \rho(g),$$

with implicit constant depending only on ρ, and omit Property P6ii., X is called a *quasi-Banach function space* (QBFS). A QBFS is also complete (see Caetano et al. [16, Lemma 3.6]), making it a quasi-Banach space, and the definitions of X^p and X' extend to this setting. We note that when X is a BFS, then X^p for $1 \leq p < \infty$ and X' are BFSs, while X^p for $0 < p < 1$ can only be guaranteed to be a QBFS.

For many of our results, we require that a QBFS X is such that X^{p_0} is a Banach function space for some $1 \leq p_0 < \infty$. That is, denoting

$$p(X) = \inf \{p_0 \geq 1 : X^{p_0} \text{ is BFS}\},$$

we require that $p(X) < \infty$. We note that for X such that $p(X) < \infty$, if $0 < p, p_1, p_2 \leq \infty$ are such that $1/p = 1/p_1 + 1/p_2$, then the following Hölder-type inequality holds:

$$\|fg\|_{X^p} \leq \|f\|_{X^{p_1}} \|g\|_{X^{p_2}}. \tag{2.8}$$

We next discuss the validity of a Fefferman-Stein-type inequality in a QBFS X as well as the boundedness of the Hardy-Littlewood maximal operator on X.

Let X be a QBFS over $(\mathbb{R}^n, \mu)$ and suppose that for a given $0 < r \leq \infty$, there exists $0 < h < \infty$ such that the Fefferman-Stein inequality holds. That is, for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of measurable functions,

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_{n,h}(f_j)|^r \right)^{\frac{1}{r}} \right\|_X \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_X, \tag{2.9}$$

with the usual changes when $r = \infty$. If this inequality holds for some $0 < h < \infty$, then it is then valid for any $0 < h' < h$. Indeed, Hölder's inequality gives that $\mathcal{M}_{n,h'}f(x) \leq \mathcal{M}_{n,h}f(x)$ for $0 < h' < h$, and the monotonicity of the quasi-norms in ℓ^r and X (Property P4.) give (2.9) for $\mathcal{M}_{n,h'}$. We also note that if (2.9) holds, then $\mathcal{M}_{n,h}$ is bounded on X.

With this in mind, for a QBFS X and $0 < r \leq \infty$ such that (2.9) holds for some

$0 < h < \infty$, we define

$$h_X = \sup\{h > 0 : \mathcal{M}_{n,h} \text{ is bounded on } X\} \quad \text{and} \quad h_{X,r} = \sup\{h > 0 : \text{ (2.9) holds }\}.$$

We also define

$$\tau(X) = n\left(\frac{1}{\min(h_X, 1)} - 1\right) \quad \text{and} \quad \tau_r(X) = n\left(\frac{1}{\min(h_{X,r}, 1)} - 1\right).$$

Note that $h_X \geq h_{X,r}$ and $\tau_r(X) \geq \tau(X)$.

Extrapolation results use inequalities in weighted Lebesgue spaces to imply analogous estimates in the setting of other function spaces. Such results are central tools in obtaining Fefferman-Stein-type inequalities as well as equivalences between spaces (see also Section 2.4 and Section 3.2.4). The following result was obtained by Cruz-Uribe et al. [26, Corollary 4.8] for BFSs satisfying certain properties.

Theorem 2.3.1. *Let $\mathcal{F}$ be a family of pairs of measurable functions on $\mathbb{R}^n$ such that for some $1 \leq p_0 < \infty$ and every $w_0 \in A_{p_0}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)|^{p_0} w_0(x)dx \lesssim \int_{\mathbb{R}^n} |g(x)|^{p_0} w_0(x)dx, \quad \forall (f,g) \in \mathcal{F}.$$

If X is a BFS over $(\mathbb{R}^n, dx)$ such that the Hardy-Littlewood maximal operator is bounded on X', then for all $1 < p < \infty$,

$$\|f\|_{X^p} \lesssim \|g\|_{X^p}, \quad \forall (f,g) \in \mathcal{F}.$$

Extrapolation results are further strengthened in the setting of specific function spaces that allow for X to be a QBFS and relax the requirement for $\mathcal{M}_n$ to be bounded on X'. We direct the reader to Sections 2.4, 5.2, 5.3, and 5.4 for additional details on these instances.

2.4 Rearrangement Invariant QBFSs

A useful property of certain QBFSs is called rearrangement invariance, which allows the use of extrapolation theorems that imply the validity of the Fefferman-Stein inequality and boundedness properties for the Hardy-Littlewood maximal operator. In this section, we give definitions, notation, and properties of interest for rearrangement invariant quasi-Banach function spaces; for further details, we refer the reader to Bennett–Sharpley [7] and Cruz-Uribe et al. [26].

Let $(\mathbb{R}^n, \mu)$ be a measure space as in Section 2.3 such that $\mu(\mathbb{R}^n) = \infty$. The *distribution function* μ_f of a measurable function f on $\mathbb{R}^n$ is defined by

$$\mu_f(\lambda) = \mu\left(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right).$$

Measurable functions f on $(\mathbb{R}^n, \mu)$ and g on $(\mathbb{R}^d, \nu)$ are said to be *equimeasurable* when $\mu_f = \nu_g$. We call a BFS X over $(\mathbb{R}^n, \mu)$ *rearrangement invariant* (r.i.) if $\|f\|_X = \|g\|_X$ whenever f and g in X are equimeasurable.

The *decreasing rearrangement of* f is the function f_μ^* on $[0, \infty)$ given by

$$f_\mu^*(t) = \inf\left\{\lambda \geq 0 : \mu_f(\lambda) \leq t\right\}.$$

We note that f_μ^* is equimeasurable with f. For an r.i.BFS X, this gives a representation of X over the measure space $(\mathbb{R}^+, dt)$. Indeed, by the Luxemburg representation theorem (see Bennett–Sharpley [7]), there exists an r.i.BFS $\overline{X}$ over $(\mathbb{R}^+, dt)$ such that $f \in X$ if, and only if, $f_\mu^* \in \overline{X}$, and $\|f\|_X = \left\|f_\mu^*\right\|_{\overline{X}}$.

The Boyd indices of an r.i.BFS X are defined using its Luxemburg representation $\overline{X}$. For $f \in \overline{X}$, the dilation operator D_t, $0 < t < \infty$, is given by $D_t f(x) = f(x/t)$, and we define

$$a_X(t) = \|D_t\|_{B(\overline{X})},$$

where $\|D_t\|_{B(\overline{X})}$ represents the norm of D_t. The *lower* and *upper Boyd indices* are then respectively given by

$$p_X = \lim_{t\to\infty} \frac{\log t}{\log a_X(t)} = \sup_{1<t<\infty} \frac{\log t}{\log a_X(t)} \quad \text{and} \quad q_X = \lim_{t\to 0^+} \frac{\log t}{\log a_X(t)} = \inf_{0<t<1} \frac{\log t}{\log a_X(t)}.$$

It holds that $1 \le p_X \le q_X \le \infty$, $p_{X'} = (q_X)'$, and $q_{X'} = (p_X)'$.

The Luxemburg representation is also used to define weighted versions of an r.i.BFS X over $(\mathbb{R}^n, dx)$. For a weight $w \in A_\infty(\mathbb{R}^n)$, define

$$X(w) = \left\{ f \in M : \|f_w^*\|_{\overline{X}} < \infty \right\},$$

with norm $\|f\|_X = \|f_w^*\|_{\overline{X}}$. This weighted space $X(w)$ is also an r.i.BFS over the measure space $(\mathbb{R}^n, w(x)dx)$, with associate space given by $(X(w))' = X'(w)$.

The Luxemburg representation, Boyd indices, and weighted spaces can be extended to the r.i.QBFS setting when the space X is such that $p(X) < \infty$. We also note that for $0 < r < \infty$ and $w \in A_\infty(\mathbb{R}^n)$, we have $0 < p_X \le q_X \le \infty$, $p_{X^r} = rp_X$, $q_{X^r} = rq_X$, and $(X(w))^r = X^r(w)$.

Rearrangement invariant quasi-Banach function spaces over $(\mathbb{R}^n, dx)$ enjoy additional properties related to the Hardy-Littlewood maximal operator and Fefferman-Stein inequality that are not guaranteed in the more general QBFS setting. These are in part due to the extrapolation results that hold in the rearrangement invariant setting. In particular, we use the following theorems from Curbera et al. [27, Theorem 2.1] and Cruz-Uribe et al. [26, Theorem 4.10], respectively.

Theorem 2.4.1. *Let $\mathcal{F}$ be a family of pairs of measurable functions on $\mathbb{R}^n$ such that for some $0 < p_0 < \infty$ and all $w_0 \in A_\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)|^{p_0} w_0(x)dx \lesssim \int_{\mathbb{R}^n} |g(x)|^{p_0} w_0(x)dx, \quad \forall (f,g) \in \mathcal{F}.$$

If X is a r.i.QBFS over $(\mathbb{R}^n, dx)$ with Boyd indices $0 < p_X \le q_X < \infty$ and $p(X) < \infty$, then *for all $w \in A_\infty(\mathbb{R}^n)$, we have*

$$\|f\|_{X(w)} \lesssim \|g\|_{X(w)}, \quad \forall (f,g) \in \mathcal{F}.$$

Theorem 2.4.2. *Let $\mathcal{F}$ be a family of pairs of measurable functions such that for some $1 \le p_0 < \infty$ and every $w_0 \in A_{p_0}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)|^{p_0} w_0(x)dx \lesssim \int_{\mathbb{R}^n} |g(x)|^{p_0} w_0(x)dx, \quad \forall (f,g) \in \mathcal{F}.$$

If X is a r.i.BFS over $(\mathbb{R}^n, dx)$ such that $1 < p_X \le q_X < \infty$, then for all $w \in A_{p_X}(\mathbb{R}^n)$, we have

$$\|f\|_{X(w)} \lesssim \|g\|_{X(w)}, \quad \forall (f,g) \in \mathcal{F}.$$

Moreover, the boundedness of $\mathcal{M}_n$ on an r.i.QBFS X is given in Montgomery-Smith [54].

Theorem 2.4.3 (Theorem 5 in [54])**.** *Let X be a r.i.QBFS over $(\mathbb{R}^n, dx)$. Then $\mathcal{M}_n$ is bounded on X if, and only if, $p_X > 1$.*

This leads to the following result for the boundedness of the operator $\mathcal{M}_{n,h}$ on X.

Corollary 2.4.4. *Let X be a r.i.QBFS over $(\mathbb{R}^n, dx)$. The operator $\mathcal{M}_{n,h}$ is bounded on X if, and only if, $0 < h < p_X$.*

Proof. First, assume that $0 < h < p_X$. Then $1 < p_X/h < \infty$, which by Theorem 2.4.3 gives

$$\|\mathcal{M}_{n,h} f\|_X = \left\| \left(\mathcal{M}_n |f|^h\right)^{\frac{1}{h}} \right\|_X = \left\|\mathcal{M}_n |f|^h\right\|_{X^{\frac{1}{h}}}^{\frac{1}{h}} \le \left\||f|^h\right\|_{X^{\frac{1}{h}}}^{\frac{1}{h}} = \|f\|_X.$$

Conversely, suppose that $\mathcal{M}_{n,h}$ is bounded on X. Then

$$\left\| \left(\mathcal{M}_n |f|^h\right)^{\frac{1}{h}} \right\|_X \lesssim \|f\|_X = \left\||f|^h\right\|_{X^{\frac{1}{h}}}^{\frac{1}{h}}.$$

Let $g \in X^{1/h}$; setting $f = |g|^{1/h}$, we have $f \in X$. Then

$$\|\mathcal{M}_n g\|_{X^{\frac{1}{h}}} = \left\|\mathcal{M}_n |f|^h\right\|_{X^{\frac{1}{h}}} = \left\|\left(\mathcal{M}_n |f|^h\right)^{\frac{1}{h}}\right\|_X^h \lesssim \|f\|_X^h = \left\||f|^h\right\|_{X^{\frac{1}{h}}} = \|g\|_{X^{\frac{1}{h}}}.$$

That is, $\mathcal{M}_n$ is bounded on $X^{1/h}$. It follows from Theorem 2.4.3 that $0 < h < p_X$. □

Thanks to Theorem 2.4.1, the Fefferman-Stein inequality also holds in weighted r.i.QBFSs under appropriate conditions.

Theorem 2.4.5. *If X is an r.i.BFS over $(\mathbb{R}^n, dx)$, $w \in A_{p_X}(\mathbb{R}^n)$, and $1 < r \leq \infty$, then for all sequences of measurable functions $\{f_j\}_{j\in\mathbb{Z}}$, we have*

$$\left\|\left(\sum_{j\in\mathbb{Z}} (\mathcal{M}_n f_j)^r\right)^{\frac{1}{r}}\right\|_{X(w)} \lesssim \left\|\left(\sum_{j\in\mathbb{Z}} |f_j|^r\right)^{\frac{1}{r}}\right\|_{X(w)}.$$

Furthermore, suppose X is a r.i.QBFS over $(\mathbb{R}^n, dx)$ such that $0 < p_X \leq q_X < \infty$ and $p(X) < \infty$. If $w \in A_\infty(\mathbb{R}^n)$, $0 < r \leq \infty$, and $0 < h < \min(p_X/\tau_w, 1/p(X), r)$, then for all sequences of measurable functions $\{f_j\}_{j\in\mathbb{Z}}$, we have

$$\left\|\left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_{n,h} f_j|^r\right)^{\frac{1}{r}}\right\|_{X(w)} \lesssim \left\|\left(\sum_{j\in\mathbb{Z}} |f_j|^r\right)^{\frac{1}{r}}\right\|_{X(w)},$$

with the sum in j replaced by the supremum in j when $r = \infty$.

Proof. We begin with the case where X is an r.i.BFS, $1 < r < \infty$, and $w \in A_{p_X}(\mathbb{R}^n)$. By the Fefferman-Stein inequality in weighted Lebesgue spaces in (2.1) with $h = 1$ and Theorem 2.4.2, we have

$$\left\|\left(\sum_{j\in\mathbb{Z}} (\mathcal{M}_n f_j)^r\right)^{\frac{1}{r}}\right\|_{X(w)} \lesssim \left\|\left(\sum_{j\in\mathbb{Z}} |f_j|^r\right)^{\frac{1}{r}}\right\|_{X(w)}.$$

For $w \in A_\infty(\mathbb{R}^n)$ and an r.i.QBFS X such that $0 < p_X \leq q_X < \infty$, $0 < r < \infty$, and

$0 < h < \min(p_X/\tau_w, 1/p(X), r)$, we have that

$$\begin{aligned}\left\|\left(\sum_{j\in\mathbb{Z}}(\mathcal{M}_{n,h}(f_j))^r\right)^{\frac{1}{r}}\right\|_{X(w)} &= \left\|\left(\sum_{j\in\mathbb{Z}}\left(\mathcal{M}_n|f_j|^h\right)^{\frac{r}{h}}\right)^{\frac{h}{r}}\right\|_{X^{\frac{1}{h}}(w)}^{\frac{1}{h}} \\ &\lesssim \left\|\left(\sum_{j\in\mathbb{Z}}\left(|f_j|^h\right)^{\frac{r}{h}}\right)^{\frac{h}{r}}\right\|_{X^{\frac{1}{h}}(w)}^{\frac{1}{h}} \\ &= \left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^r\right)^{\frac{1}{r}}\right\|_{X(w)},\end{aligned}$$

where the second line follows from the first case since $1 < r/h < \infty, 1 < p_X/h \le q_X/h < \infty$, $w \in A_{p_X/h}(\mathbb{R}^n)$, and $X^{1/h}(w)$ is an r.i.BFS since $1/h > p(X)$.

In the case that $r = \infty$, the summation in j is replaced by the supremum in j and the proof remains the same. □

Corollary 2.4.6. *Suppose X is such that $0 < p_X \le q_X < \infty$ and $p(X) < \infty$ and $w \in A_\infty(\mathbb{R}^n)$. Then, for $0 < h < \min(p_X/\tau_w, 1/p(X))$, $\mathcal{M}_{n,h}$ is bounded on $X(w)$.*

Note that the results above imply that

$$h_{X(w)} \ge \min(p_X/\tau_w, 1/p(X)) \tag{2.10}$$

and

$$h_{X(w),r} \ge \min(p_X/\tau_w, 1/p(X), r, 1), \tag{2.11}$$

which also gives

$$\tau(X(w)) \le n\left(\frac{1}{\min(p_X/\tau_w, 1/p(X), 1)} - 1\right) \tag{2.12}$$

and

$$\tau_r(X(w)) \le n\left(\frac{1}{\min(p_X/\tau_w, 1/p(X), r, 1)} - 1\right). \tag{2.13}$$

Chapter 3

Triebel-Lizorkin and Besov Spaces

In this chapter, we discuss Triebel-Lizorkin and Besov spaces based on weighted Lebesgue spaces and QBFSs, as well as weighted bi-parameter Triebel-Lizorkin spaces. Many tools in these spaces play a central role in the proofs of fractional Leibniz rules in the QBFS and weighted bi-parameter settings. Section 3.1 contains definitions, notations, and properties for the classical weighted Triebel-Lizorkin and Besov spaces. The QBFS setting is discussed in Section 3.2. Section 3.3 defines weighted bi-parameter homogeneous Triebel-Lizorkin spaces. Finally, Nikol'skiĭ representations in the QBFS and weighted bi-parameter settings are both discussed in Section 3.4.

3.1 Weighted Triebel-Lizorkin and Besov Spaces

Let $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}(\widehat{\varphi}) \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\}\,, \tag{3.1}$$

$$\operatorname{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : \tfrac{1}{2} < |\xi| < 2\}. \tag{3.2}$$

We define $\dot{\mathcal{A}}(\mathbb{R}^n)$ as the class of $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that ψ satisfies (3.2) and

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

and denote by $\mathcal{A}(\mathbb{R}^n)$ the class of pairs (φ, ψ) such that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, φ satisfies (3.1), ψ satisfies (3.2), and

$$\widehat{\varphi}(\xi) + \sum_{j\in\mathbb{N}} \widehat{\psi}(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

Let $\psi \in \dot{\mathcal{A}}(\mathbb{R}^n)$ and w be a weight on $\mathbb{R}^n$. For $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < r \le \infty$, the *weighted homogeneous Triebel-Lizorkin space* $\dot{F}^s_{p,r}(w)$ is the collection of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,r}(w)} = \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{sj} \Delta_j^{\psi} f \right|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} < \infty.$$

For $s \in \mathbb{R}$ and $0 < p, r \le \infty$, the *weighted homogeneous Besov space* $\dot{B}^s_{p,r}(w)$ is the collection of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{p,r}(w)} = \left(\sum_{j\in\mathbb{Z}} \left\| 2^{sj} \Delta_j^{\psi} f \right\|^r_{L^p(w)} \right)^{\frac{1}{r}} < \infty.$$

For $(\varphi, \psi) \in \mathcal{A}(\mathbb{R}^n)$ and a weight w on $\mathbb{R}^n$, the *weighted inhomogeneous Triebel-Lizorkin space* $F^s_{p,r}(w)$ is the class of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^s_{p,r}(w)} = \|S_0^{\varphi} f\|_{L^p(w)} + \left\| \left(\sum_{j\in\mathbb{N}} \left| 2^{sj} \Delta_j^{\psi} f \right|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} < \infty,$$

and the *weighted inhomogeneous Besov space* $B^s_{p,r}(w)$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,r}(w)} = \|S_0^{\varphi} f\|_{L^p(w)} + \left(\sum_{j\in\mathbb{N}} \left\| 2^{sj} \Delta_j^{\psi} f \right\|^r_{L^p(w)} \right)^{\frac{1}{r}} < \infty.$$

For each of these definitions, when $r = \infty$ the summation in j is replaced with the supremum in j. In the case that $w \equiv 1$, these definitions yield classical unweighted Triebel-Lizorkin and Besov spaces.

We note that the definitions of the Triebel-Lizorkin and Besov spaces are independent from the choice of the functions ψ and φ. Moreover, they are complete, making them quasi-Banach spaces. The following embeddings also hold:

$$\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{p,r}(w), \dot{B}^s_{p,r}(w) \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n);$$

in particular, $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\dot{F}^s_{p,r}(w)$ and $\dot{B}^s_{p,r}(w)$ for $0 < p, r < \infty$. Analogous statements hold for $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ with the inhomogeneous Triebel-Lizorkin and Besov spaces. Finally, lifting properties hold in these spaces. That is,

$$\|f\|_{\dot{F}^s_{p,r}(w)} \sim \|D^s f\|_{\dot{F}^0_{p,r}(w)} \quad \text{and} \quad \|f\|_{F^s_{p,r}(w)} \sim \|J^s f\|_{F^0_{p,r}(w)}, \tag{3.3}$$

where p, r, s, and w are as in the definitions, with corresponding equivalences in the Besov spaces.

We also define the Hardy spaces as follows. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x)dx \neq 0$. For $0 < p < \infty$ and $w \in A_\infty(\mathbb{R}^n)$, the *weighted Hardy space* $H^p(w)$ is defined as the class of tempered distributions such that

$$\|f\|_{H^p(w)} = \left\| \sup_{0<t<\infty} \left| t^{-n}\phi(t^{-1}\cdot) * f \right| \right\|_{L^p(w)} < \infty,$$

and the *weighted local Hardy space* $h^p(w)$ is given by the collection of tempered distributions such that

$$\|f\|_{h^p(w)} = \left\| \sup_{0<t<1} \left| t^{-n}\phi(t^{-1}\cdot) * f \right| \right\|_{L^p(w)} < \infty.$$

Theorem 1.4(vi) in Qui [64] gives that for $0 < p < \infty$ and $w \in A_\infty(\mathbb{R}^n)$,

$$\|f\|_{\dot{F}^0_{p,2}(w)} \sim \|f\|_{H^p(w)}, \tag{3.4}$$

while Remark 4.5 in Qui [64] gives that for $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$,

$$\|f\|_{\dot{F}^0_{p,2}(w)} \sim \|f\|_{H^p(w)} \sim \|f\|_{L^p(w)} . \tag{3.5}$$

Analogous equivalences hold for $F^0_{p,2}(w)$, $h^p(w)$, and $L^p(w)$.

We refer the reader to Qui [64] and Triebel [68] for further details on the theory of weighted and classical Triebel-Lizorkin and Besov spaces.

3.2 QBFS-Based Triebel-Lizorkin and Besov Spaces

Let X be a QBFS, $0 < r \leq \infty$, $s \in \mathbb{R}$, and $\psi \in \dot{\mathcal{A}}(\mathbb{R}^n)$. The *homogeneous Triebel-Lizorkin space based on a QBFS* X will be denoted by $\dot{F}^s_{X,r}$ and is the collection of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{X,r}} = \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{sj} \Delta_j^\psi f \right|^r \right)^{\frac{1}{r}} \right\|_X < \infty.$$

For $(\varphi, \psi) \in \mathcal{A}(\mathbb{R}^n)$, the *inhomogeneous Triebel-Lizorkin space based on a QBFS* X is represented by $F^s_{X,r}$ and is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^s_{X,r}} = \|S_0^\varphi f\|_X + \left\| \left(\sum_{j\in\mathbb{N}} \left| 2^{sj} \Delta_j^\psi f \right|^r \right)^{\frac{1}{r}} \right\|_X < \infty.$$

In the case that $r = \infty$, the summation in j is replaced with the supremum in j. For $s \in \mathbb{R}$ and $0 < p, r \leq \infty$, the *homogeneous and inhomogeneous Besov spaces based on a QBFS* X will be denoted by $\dot{B}^s_{X,r}$ and $B^s_{X,r}$, respectively, and are defined analogously to the Triebel-Lizorkin spaces with the quasi-norms for X and ℓ^r exchanged. In the case that these spaces are based on a weighted QBFS $X(w)$, we use the notation $\dot{F}^s_{X,r}(w)$, $F^s_{X,r}(w)$, $\dot{B}^s_{X,r}(w)$, and $B^s_{X,r}(w)$.

Many of the properties seen in the classical and weighted Triebel-Lizorkin and Besov spaces also hold in those based on QBFSs. We next give more details.

3.2.1 Independence of the Norm

The definitions of the Triebel-Lizorkin and Besov spaces do not depend on the choice of φ and ψ when $h_{X,r} > 0$ and $h_X > 0$, respectively.

Theorem 3.2.1. *The definition of the homogeneous and inhomogeneous Triebel-Lizorkin and Besov spaces based on a QBFS X are independent of the choice of φ and ψ used to define them. That is:*

i) For $\psi_1, \psi_2 \in \dot{\mathcal{A}}(\mathbb{R}^n)$ and a QBFS X such that $h_{X,r} > 0$, we have

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{sj} \Delta_j^{\psi_1} f \right|^r \right)^{\frac{1}{r}} \right\|_X \sim \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{sj} \Delta_j^{\psi_2} f \right|^r \right)^{\frac{1}{r}} \right\|_X ,$$

with an analogous expression holding for Besov spaces when X is such that $h_X > 0$.

ii) For $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \mathcal{A}(\mathbb{R}^n)$ and a QBFS X such that $h_{X,r} > 0$, we have

$$\|S_0^{\varphi_1} f\|_X + \left\| \left(\sum_{j\in\mathbb{N}} \left| 2^{sj} \Delta_j^{\psi_1} f \right|^r \right)^{\frac{1}{r}} \right\|_X \sim \|S_0^{\varphi_2} f\|_X + \left\| \left(\sum_{j\in\mathbb{N}} \left| 2^{sj} \Delta_j^{\psi_2} f \right|^r \right)^{\frac{1}{r}} \right\|_X ,$$

with an analogous expression holding for Besov spaces when X is such that $h_X > 0$.

Proof. We show here the proof for the homogeneous Triebel-Lizorkin setting; the homogeneous Besov and inhomogeneous settings are similar.

By symmetry, it is sufficient to show that

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{sj} \Delta_j^{\psi_1} f \right|^r \right)^{\frac{1}{r}} \right\|_X \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{sj} \Delta_j^{\psi_2} f \right|^r \right)^{\frac{1}{r}} \right\|_X .$$

Note that for $\psi \in \dot{\mathcal{A}}(\mathbb{R}^n)$ and $k \in \mathbb{Z}$, if $2^{k-1} \leq |\xi| \leq 2^{k+1}$, then

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = \widehat{\psi}(2^{-k-1}\xi) + \widehat{\psi}(2^{-k}\xi) + \widehat{\psi}(2^{-k+1}\xi) = 1.$$

Thus, defining ϕ_2 such that $\widehat{\phi_2}(\xi) = \widehat{\psi_2}(\frac{1}{2}\xi) + \widehat{\psi_2}(\xi) + \widehat{\psi_2}(2\xi)$, we have $\widehat{\psi_1}(\xi) = \widehat{\phi_2}(\xi)\widehat{\psi_1}(\xi)$. By Lemma 2.2.2, for $0 < h \leq 1$, we have

$$\left|\Delta_j^{\psi_1} f(x)\right| \lesssim \mathcal{M}_{n,h}\left(\Delta_j^{\phi_2} f\right)(x).$$

Therefore, taking $0 < h < \min(1, h_{X,r})$, we have

$$\begin{aligned}
\left\|\left(\sum_{j\in\mathbb{Z}}\left|2^{sj}\Delta_j^{\psi_1} f\right|^r\right)^{\frac{1}{r}}\right\|_X &\lesssim \left\|\left(\sum_{j\in\mathbb{Z}}\left[2^{sj}\mathcal{M}_{n,h}\left(\Delta_j^{\phi_2} f\right)\right]^r\right)^{\frac{1}{r}}\right\|_X \\
&\lesssim \left\|\left(\sum_{j\in\mathbb{Z}}\left|2^{sj}\Delta_j^{\phi_2} f\right|^r\right)^{\frac{1}{r}}\right\|_X \\
&= \left\|\left(\sum_{j\in\mathbb{Z}}\left|2^{sj}\left(\Delta_{j-1}^{\psi_2} f + \Delta_j^{\psi_2} f + \Delta_{j+1}^{\psi_2} f\right)\right|^r\right)^{\frac{1}{r}}\right\|_X \\
&\lesssim \left\|\left(\sum_{j\in\mathbb{Z}}\left|2^{sj}\Delta_j^{\psi_2} f\right|^r\right)^{\frac{1}{r}}\right\|_X .
\end{aligned}$$

□

3.2.2 Embeddings & Completeness

The Triebel-Lizorkin and Besov spaces based on a QBFS X satisfying certain conditions enjoy the same embeddings with $\mathcal{S}(\mathbb{R}^n), \mathcal{S}_0(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)$, and $\mathcal{S}_0'(\mathbb{R}^n)$ as those in the classical setting. Thus, we also consider the following properties for a QBFS X, with $s \in \mathbb{R}$ and $0 < r \leq \infty$:

P7. $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{X,r} \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$ and $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{X,r} \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$;

P8. $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_{X,r} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{X,r} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$;

The inclusions $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{X,r}$ and $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{X,r}$, their inhomogeneous counterparts, and the inclusion $\mathcal{S}(\mathbb{R}^n) \hookrightarrow X$ hold if $(1+|x|)^{-N} \in X$ for some $N > 0$. We also have $X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, $\dot{F}^s_{X,r} \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$ and $\dot{B}^s_{X,r} \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$ if $(1+|x|)^{-N} \in X'$ for some $N > 0$. These claims can be proved using arguments similar to those used for corresponding results in the classical setting (see Triebel [68]; see also Liang et al. [50]).

An additional important observation is that the Triebel-Lizorkin and Besov spaces based on X have the Fatou property. That is, letting Y be a quasi-Banach space such that $Y \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ (or $Y \hookrightarrow \mathcal{S}'_0(\mathbb{R}^n)$), the space Y is said to have the Fatou property if for every sequence $\{f_k\}_{k\in\mathbb{N}} \subset Y$ that converges in $\mathcal{S}'(\mathbb{R}^n)$ (or $\mathcal{S}'_0(\mathbb{R}^n)$) as $k \to \infty$, and that satisfies $\liminf_{k\to\infty} \|f_k\|_Y < \infty$, it follows that $\lim_{k\to\infty} f_k \in Y$ and $\|\lim_{k\to\infty} f_k\|_Y \lesssim \liminf_{k\to\infty} \|f_k\|_Y$, where the implicit constant is independent of $\{f_k\}_{k\in\mathbb{N}}$.

Theorem 3.2.2. *Let $s \in \mathbb{R}$, $0 < r \le \infty$, and X be a QBFS over $(\mathbb{R}^n, \mu)$.*

i) If $\dot{F}^s_{X,r} \hookrightarrow \mathcal{S}'_0(\mathbb{R}^n)$, then $\dot{F}^s_{X,r}$ has the Fatou property.

ii) If $\dot{B}^s_{X,r} \hookrightarrow \mathcal{S}'_0(\mathbb{R}^n)$, then $\dot{B}^s_{X,r}$ has the Fatou property.

Corresponding statements hold for $F^s_{X,r}$ and $B^s_{X,r}$.

Proof. We begin with the Triebel-Lizorkin setting. Let $\{f_k\}_{k\in\mathbb{Z}} \subset \dot{F}^s_{X,r}$ be such that $f_k \to f$ in $\mathcal{S}'_0(\mathbb{R}^n)$ and $\liminf_{k\to\infty} \|f_k\|_{\dot{F}^s_{X,r}} < \infty$. We first note that $f_k \to f$ in $\mathcal{S}'_0(\mathbb{R}^n)$ implies that $\Delta^\psi_j f_k \to \Delta^\psi_j f$ in $\mathcal{S}'_0(\mathbb{R}^n)$ and pointwise. Fixing $J \in \mathbb{N}$, we have

$$\left(\sum_{j=-J}^{J} \left|2^{sj}\Delta^\psi_j f(x)\right|^r\right)^{\frac{1}{r}} \lesssim \left(\sum_{j=-J}^{J} \left|2^{sj}\Delta^\psi_j (f-f_k)(x)\right|^r\right)^{\frac{1}{r}} + \left(\sum_{j=-J}^{J} \left|2^{sj}\Delta^\psi_j f_k(x)\right|^r\right)^{\frac{1}{r}},$$

where the implicit constant depends on r. Then

$$\left(\sum_{j=-J}^{J} \left|2^{sj}\Delta^\psi_j f(x)\right|^r\right)^{\frac{1}{r}} \lesssim \liminf_{k\to\infty} \left(\sum_{j=-J}^{J} \left|2^{sj}\Delta^\psi_j f_k(x)\right|^r\right)^{\frac{1}{r}}.$$

Therefore,

$$\begin{aligned}\left\|\left(\sum_{j=-J}^{J}\left|2^{sj}\Delta_j^{\psi}f\right|^{r}\right)^{\frac{1}{r}}\right\|_X &\lesssim \left\|\liminf_{k\to\infty}\left(\sum_{j=-J}^{J}\left|2^{sj}\Delta_j^{\psi}f_k\right|^{r}\right)^{\frac{1}{r}}\right\|_X \\ &\lesssim \liminf_{k\to\infty}\left\|\left(\sum_{j=-J}^{J}\left|2^{sj}\Delta_j^{\psi}f_k\right|^{r}\right)^{\frac{1}{r}}\right\|_X \\ &\leq \liminf_{k\to\infty}\|f\|_{\dot{F}^s_{X,r}},\end{aligned}$$

where the first and third inequalities follow from Property P4. and the second inequality follows from Properties P4. and P5. of X.

Property P5. also gives that $f \in \dot{F}^s_{X,r}$, since as $J \to \infty$,

$$\|f\|_{\dot{F}^s_{X,r}} \lesssim \liminf_{k\to\infty}\|f_k\|_{\dot{F}^s_{X,r}} < \infty.$$

Now, suppose $\{f_k\}_{k\in\mathbb{N}} \subset \dot{B}^s_{X,r}$ is such that $f_k \to f$ in $\mathcal{S}_0'(\mathbb{R}^n)$ and $\liminf_{k\to\infty}\|f_k\|_{\dot{B}^s_{X,r}} < \infty$. As before, note that for each $j \in \mathbb{Z}$, we have $\Delta_j^{\psi}f_k \to \Delta_j^{\psi}f$ pointwise as $k \to \infty$. We have

$$\left\|\Delta_j^{\psi}f\right\|_X = \left\|\lim_{k\to\infty}\Delta_j^{\psi}f_k\right\|_X \leq \liminf_{k\to\infty}\left\|\Delta_j^{\psi}f_k\right\|_X.$$

Then, using the Fatou property for ℓ_r,

$$\|f\|_{\dot{B}^s_{X,r}} \leq \liminf_{k\to\infty}\|f_k\|_{\dot{B}^s_{X,r}},$$

as desired. □

We next state and prove the completeness of the Triebel-Lizorkin and Besov spaces.

Theorem 3.2.3. *Let X be a QBFS over $(\mathbb{R}^n,\mu)$ such that $p(X) < \infty$, $0 < r \leq \infty$, and $s \in \mathbb{R}$.*

i) If $\dot{F}^s_{X,r} \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$, then $\dot{F}^s_{X,r}$ is complete.

ii) If $\dot{B}^s_{X,r} \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$, then $\dot{B}^s_{X,r}$ is complete.

Corresponding statements hold for $F^s_{X,r}$ and $B^s_{X,r}$.

Proof. We show here the details proving the completeness of $\dot{F}^s_{X,r}$; the Besov and inhomogeneous settings are similar. Let $\{f_k\}_{k\in\mathbb{Z}}$ be a Cauchy sequence in $\dot{F}^s_{X,r}$. Since $\dot{F}^s_{X,r} \hookrightarrow \mathcal{S}'_0(\mathbb{R}^n)$, there exists $f \in \mathcal{S}'_0(\mathbb{R}^n)$ such that $f_k \to f$ in $\mathcal{S}'_0(\mathbb{R}^n)$. Noting that $\liminf_{k\to\infty} \|f_k\|_{\dot{F}^s_{X,r}} < \infty$, Theorem 3.2.2 gives that $f \in \dot{F}^s_{X,r}$ and

$$\|f\|_{\dot{F}^s_{X,r}} \lesssim \liminf_{k\to\infty} \|f_k\|_{\dot{F}^s_{X,r}}. \tag{3.6}$$

It remains to show that $f_k \to f$ in $\dot{F}^s_{X,r}$. Applying (3.6) to the sequence $\{f - f_{k_0}\}_{k\in\mathbb{N}}$ for fixed $k_0 \in \mathbb{N}$, we have

$$\|f - f_{k_o}\|_{F^s_{X,r}} \lesssim \liminf_{k\to\infty} \|f_k - f_{k_0}\|_{\dot{F}^s_{X,r}}.$$

Let $\varepsilon > 0$. Then there exists $K > 0$ such that for $k, k_0 > K$,

$$\inf_{m\geq k} \|f_m - f_{k_0}\|_{\dot{F}^s_{X,r}} < \varepsilon.$$

This implies that for all $k_0 > K$,

$$\liminf_{k\to\infty} \|f_k - f_{k_0}\|_{\dot{F}^s_{X,r}} \leq \varepsilon.$$

Therefore, for all $k_0 > K$,

$$\|f - f_{k_0}\|_{\dot{F}^s_{X,r}} \lesssim \varepsilon,$$

and we conclude that $f_k \to f$ in $\dot{F}^s_{X,r}$. □

Remark 3.2.4. Let $w \in A_\infty(\mathbb{R}^n)$. Theorem 6.6 in Bennett-Sharpley [7] gives that for an r.i.BFS X

$$L^1(w) \cap L^\infty \hookrightarrow X(w) \hookrightarrow L^1(w) + L^\infty. \tag{3.7}$$

Now, for any $N \in \mathbb{N}$, $(1 + |x|)^{-N} \in L^\infty$ since $(1 + |x|)^{-N}$ is bounded. Moreover, using the boundedness properties of the Hardy-Littlewood maximal operator, it can be shown that for

N sufficiently large,

$$\left\|(1+|x|)^{-N}\right\|_{L^1(w)} \lesssim \int_{|x|<1} w(x)dx < \infty,$$

giving $(1+|x|)^{-N} \in L^1(w) \cap L^\infty$ for sufficiently large N. Therefore, for an r.i.QBFS X such that $p(X) < \infty$ and $p > p(X)$, (3.7) implies that $(1+|x|)^{-N} \in X^p(w)$ for some $N > 0$; the same is true of $(X^p(w))'$ as $(X^p(w))' = (X^p)'(w)$.

3.2.3 Lifting Property

The following lifting properties hold in Triebel-Lizorkin and Besov spaces based on QBFSs:

Theorem 3.2.5. *Let $0 < r \leq \infty$ and suppose X is a QBFS over $(\mathbb{R}^n, \mu)$ such that $h_{X,r} > 0$. For $\sigma, s \in \mathbb{R}$, the following hold:*

i) D^σ maps $\dot{F}^s_{X,r}$ isomorphically onto $\dot{F}^{s-\sigma}_{X,r}$ and $\|D^\sigma f\|_{\dot{F}^{s-\sigma}_{X,r}} \sim \|f\|_{\dot{F}^s_{X,r}}$;

ii) J^σ maps $F^s_{X,r}$ isomorphically onto $F^{s-\sigma}_{X,r}$ and $\|J^\sigma f\|_{F^{s-\sigma}_{X,r}} \sim \|f\|_{F^s_{X,r}}$.

Analogous statements hold in the Besov settings.

The proof follows similar computations to those for such equivalences in the classical Triebel-Lizorkin and Besov setting (see Triebel [68, Sections 2.3.8 and 5.2.3]), so we do not include the details here.

3.2.4 Equivalences between Spaces

We define Hardy spaces in the setting of QBFSs and establish equivalences between these spaces, the Triebel-Lizorkin spaces, and BFSs.

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x)dx \neq 0$. Given a QBFS X, the *Hardy space based on X* is denoted H^X and defined as the collection of tempered distributions such that

$$\|f\|_{H^X} = \left\| \sup_{0<t<\infty} \left| t^{-n}\phi(t^{-1}\cdot) * f \right| \right\|_X < \infty,$$

while the *local Hardy space based on* X, denoted h^X, is the collection of tempered distributions such that

$$\|f\|_{h^X} = \left\| \sup_{0<t<1} \left| t^{-n}\phi(t^{-1}\cdot) * f \right| \right\|_X < \infty.$$

Note that we have

$$\|f\|_X \leq \|f\|_{h^X} \leq \|f\|_{H^X}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n), \tag{3.8}$$

due to Property P4. of X and the fact that

$$|f(x)| \leq \sup_{0<t<1} \left| t^{-n}\phi(t^{-1}x) * f \right| \leq \sup_{0<t<\infty} \left| t^{-n}\phi(t^{-1}x) * f \right|.$$

When X satisfies appropriate conditions, it follows immediately from (3.5) and extrapolation in Theorem 2.3.1 that we also have equivalences between the Triebel-Lizorkin spaces, Hardy spaces, and X^p for $1 < p < \infty$.

Theorem 3.2.6. *Let* X *be a BFS over* $(\mathbb{R}^n, dx)$ *such that the Hardy-Littlewood maximal operator is bounded on* X'*. Then for* $1 < p < \infty$*,*

$$\dot{F}^0_{X^p,2} = F^0_{X^p,2} = H^{X^p} = h^{X^p} = X^p,$$

with equivalent norms.

We note that in the rearrangement invariant setting, stronger extrapolation results are available and we can state equivalences between the Triebel-Lizorkin and Hardy spaces based on r.i.QBFSs. In particular, Theorem 2.4.1 and Theorem 2.4.2 imply the following result:

Theorem 3.2.7. *Let* X *be a r.i.QBFS over* $(\mathbb{R}^n, dx)$ *such that* $p(X) < \infty$*.*

i) If X *has Boyd indices* $0 < p_X \leq q_X < \infty$ *and* $w \in A_\infty(\mathbb{R}^n)$*, then*

$$\dot{F}^0_{X,2}(w) = H^X(w) \quad \text{and} \quad F^0_{X,2}(w) = h^X(w),$$

with equivalent quasi-norms.

ii) If X is a r.i.BFS with Boyd indices $1 < p_X \leq q_X < \infty$ and $w \in A_{p_X}(\mathbb{R}^n)$, then

$$\dot{F}^0_{X,2}(w) = F^0_{X,2}(w) = X(w),$$

with equivalent norms.

Similarly, in other specific settings, such as Morrey and variable Lebesgue spaces, extrapolation results that exist in these spaces permit the use of quasi-norms in the equivalences with the corresponding Hardy spaces; see Section 5.3 and Section 5.4 for further details.

3.3 Weighted Bi-Parameter Triebel-Lizorkin Spaces

Our results on fractional Leibniz rules in the weighted bi-parameter setting rely only on tools in the context of Triebel-Lizorkin spaces. Let $\psi_\ell \in \dot{\mathcal{A}}(\mathbb{R}^{n_\ell})$, $s_\ell \in \mathbb{R}$, $0 < p < \infty$, $0 < r \leq \infty$, and w be a weight on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The *weighted homogeneous bi-parameter Triebel-Lizorkin space* $\dot{F}^{s_1,s_2}_{p,r}(w)$ is the collection of all $f \in \mathcal{S}'_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ such that

$$\|f\|_{\dot{F}^{s_1,s_2}_{p,r}(w)} = \left\| \left(\sum_{j_1,j_2 \in \mathbb{Z}} \left| 2^{s_1 j_1 + s_2 j_2} \Delta^{\psi_1}_{j_1} \Delta^{\psi_2}_{j_2} f \right|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} < \infty.$$

Properties analogous to the classical setting hold in such spaces as well. In particular, the definition of the weighted bi-parameter Triebel-Lizorkin spaces is independent of the functions ψ_1 and ψ_2 used in their definition. We also note that the following inclusions hold for $1 < p < \infty$, $0 < q < \infty$, and $s_1, s_2 \in \mathbb{R}$

$$\mathcal{S}_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \hookrightarrow \dot{F}^{s_1,s_2}_{p,r}(w) \hookrightarrow \mathcal{S}'_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}).$$

Moreover, a lifting property holds in the bi-parameter setting analogous to that in the traditional setting.

Theorem 3.3.1. *If $\sigma_\ell, s_\ell \in \mathbb{R}$, $0 < p, r < \infty$, and $w \in A_{\infty,\mathcal{R}}(\mathbb{R}^n)$, then $D_1^{\sigma_1} D_2^{\sigma_2}$ maps*

$\dot{F}^{s_1,s_2}_{p,r}(w)$ *isomorphically onto* $\dot{F}^{s_1-\sigma_1,s_2-\sigma_2}_{p,r}(w)$ *and*

$$\|D_1^{\sigma_1} D_2^{\sigma_2} f\|_{\dot{F}^{s_1-\sigma_1,s_2-\sigma_2}_{p,r}(w)} \sim \|f\|_{\dot{F}^{s_1,s_2}_{p,r}(w)}.$$

Proof of the properties above follow the same arguments as in the classical and QBFS case and we do not include them here.

Finally, due to Lemma 2.2b from Brummer–Naibo [14], the following relationships between the weighted bi-parameter Triebel-Lizorkin and weighted Lebesgue spaces hold. First, for $1 < p < \infty$ and $w \in A_{p,\mathcal{R}}(\mathbb{R}^n)$, we have

$$\|f\|_{L^p(w)} \sim \|f\|_{\dot{F}^{0,0}_{p,2}(w)}. \tag{3.9}$$

Moreover, for $0 < p < \infty$ and $w \in A_{\infty,\mathcal{R}}(\mathbb{R}^n)$,

$$\|f\|_{L^p(w)} \lesssim \|f\|_{\dot{F}^{0,0}_{p,2}(w)}. \tag{3.10}$$

3.4 Nikol'skiĭ Representations

Nikol'skiĭ representations give estimates for Triebel-Lizorkin and Besov space norms. These estimates play a central role in the proofs to obtain fractional Leibniz rules in QBFSs and the bi-parameter setting. In this section, we state and prove these results.

First, we have Nikol'skiĭ representations for the QBFS-based Triebel-Lizorkin and Besov spaces.

Theorem 3.4.1. *For* $D > 0$*, let* $\{u_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$ *be such that*

$$supp(\widehat{u}_j) \subset B(0, D2^j), \quad j \in \mathbb{Z},$$

and let X *be a QBFS over* $(\mathbb{R}^n, \mu)$ *satisfying Properties P7. and P8. for* r *and* s *as given below.*

i) Let $0 < r \leq \infty$. *If* $h_{X,r} > 0$, $s > \tau_r(X)$, *and* $\left\| \left(\sum_{j\in\mathbb{Z}} |2^{sj} u_j|^r\right)^{\frac{1}{r}} \right\|_X < \infty$, *then the series* $\sum_{j\in\mathbb{Z}} u_j$ *converges in* $\mathcal{S}_0'(\mathbb{R}^n)$ *to an element in* $\dot{F}^s_{X,r}$ *and*

$$\left\| \sum_{j\in\mathbb{Z}} u_j \right\|_{\dot{F}^s_{X,r}} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |2^{sj} u_j|^r \right)^{\frac{1}{r}} \right\|_X, \tag{3.11}$$

where the implicit constant depends only on n, D, s, r, X, *and the function* ψ *used in the definition of* $\dot{F}^s_{X,r}$. *An analogous statement with* $j \in \mathbb{N}_0$ *holds true for* $F^s_{X,r}$ *(where convergence is in* $\mathcal{S}'(\mathbb{R}^n)$*).*

ii) Let $0 < r \leq \infty$. *If* $h_X > 0$, $s > \tau(X)$, *and* $\left(\sum_{j\in\mathbb{Z}} \|2^{sj} u_j\|_X^r\right)^{\frac{1}{r}} < \infty$, *then the series* $\sum_{j\in\mathbb{Z}} u_j$ *converges in* $\dot{B}^s_{X,r}$ *(in* $\mathcal{S}_0'(\mathbb{R}^n)$ *if* $r = \infty$*) and*

$$\left\| \sum_{j\in\mathbb{Z}} u_j \right\|_{\dot{B}^s_{X,r}} \lesssim \left(\sum_{j\in\mathbb{Z}} \|2^{sj} u_j\|_X^r \right)^{\frac{1}{r}}, \tag{3.12}$$

where the implicit constant depends only on n, D, s, r, X, *and the function* ψ *used in the definition of* $\dot{B}^s_{X,r}$. *An analogous statement with* $j \in \mathbb{N}_0$ *holds true for* $B^s_{X,r}$ *(when* $r = \infty$, *the convergence is in* $\mathcal{S}'(\mathbb{R}^n)$*).*

Corresponding Nikol'skiĭ representations are also valid for weighted bi-parameter Triebel-Lizorkin spaces.

Theorem 3.4.2. *Given* $D_\ell > 0$, *suppose* $\{u_{j_1,j_2}\}_{j_1,j_2\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *satisfies*

$$supp(\widehat{u_{j_1,j_2}}) \subset \{\xi_1 \in \mathbb{R}^{n_1} : |\xi_1| \leq D_1 2^{j_1}\} \times \{\xi_2 \in \mathbb{R}^{n_2}\ |\xi_2| \leq D_2 2^{j_2}\}, \quad j_\ell \in \mathbb{Z}.$$

Let $w \in A_{\infty,\mathcal{R}}(\mathbb{R}^n)$, $0 < p, r < \infty$, *and* $s_\ell > n_\ell\,(1/\min(p/\tau_w, r, 1) - 1)$. *If*

$$\left\| \left(\sum_{j_1,j_2\in\mathbb{Z}} \left|2^{s_1 j_1 + s_2 j_2} u_{j_1,j_2}\right|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} < \infty,$$

then $\sum_{j_1,j_2\in\mathbb{Z}} u_{j_1,j_2}$ *converges in* $\dot{F}^{s_1,s_2}_{p,r}(w)$ *and*

$$\left\| \sum_{j_1,j_2\in\mathbb{Z}} u_{j_1,j_2} \right\|_{\dot{F}^{s_1,s_2}_{p,r}(w)} \lesssim \left\| \left(\sum_{j_1,j_2\in\mathbb{Z}} \left| 2^{s_1 j_1 + s_2 j_2} u_{j_1,j_2} \right|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)},$$

where the implicit constant depends only on n_ℓ, s_ℓ, D_ℓ, p, r, *and* ψ_ℓ *used in the definition of* $\dot{F}^{s_1,s_2}_{p,r}(w)$.

The proof of Theorem 3.4.1 follows the same ideas as those for the weighted Lebesgue spaces (see Naibo–Thomson [60, Theorem 3.2]) with some modifications due to the fact that a dominated convergence theorem may not hold in X. The proof of the weighted bi-parameter Nikol'skiĭ representations follow the same argument as that for Theorem 3.4.1, with appropriate alterations for the bi-parameter setting. In this section, we give the proof of the estimates in the QBFS setting and omit the proof for the bi-parameter estimates.

Before proving Theorem 3.4.1, we introduce some notation. For a QBFS X, $0 < r \leq \infty$, and a sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$, denote

$$\left\| \{f_j\}_{j\in\mathbb{Z}} \right\|_{X(\ell^r)} = \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_X \quad \text{and} \quad \left\| \{f_j\}_{j\in\mathbb{Z}} \right\|_{\ell^r(X)} = \left(\sum_{j\in\mathbb{Z}} \|f_j\|_X^r \right)^{\frac{1}{r}}.$$

We also use the following lemmas. The first is Lemma A.1 and the second is similar to Lemma A.2, modified for the QBFS setting, both from Naibo–Thomson [60].

Lemma 3.4.3. *Suppose* $0 < h < 1$, $A > 0$, $R \geq 1$, *and* $d > n/h$. *If* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *and* f *is such that supp*$(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq AR\}$, *it holds that*

$$|\phi * f(x)| \lesssim R^{n(\frac{1}{h}-1)} A^{-n} \left\| (1 + |A\cdot|)^d \phi \right\|_{L^\infty} \mathcal{M}_{n,h} f(x),$$

where the implicit constant is independent of A, R, ϕ, *and* f.

Lemma 3.4.4. *Suppose* X *is a QBFS such that* $h_X > 0$. *Let* $A > 0, R \geq 1$, *and* $d > b >$

$n/\min(h_X,1)$. *If* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *and* f *is such that supp*$(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \le AR\}$*, it holds that*

$$\|\phi * f\|_X \lesssim R^{b-n} A^{-n} \left\| (1 + |A\,\cdot\,|)^d \phi \right\|_{L^\infty} \|f\|_X \,,$$

where the implicit constant is independent of A, R, ϕ*, and* f*.*

We will also use Lemma A.3 from Naibo–Thomson [60].

Lemma 3.4.5. *Let* $\gamma < 0$*,* $\lambda \in \mathbb{R}$*,* $0 < r \le \infty$*, and* $k_0 \in \mathbb{Z}$*. Then for any sequence* $\{d_j\}_{j\in\mathbb{Z}} \subset [0,\infty)$*, it holds that*

$$\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\gamma k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^r} \lesssim \left\| \left\{ 2^{\lambda j} d_j \right\}_{j\in\mathbb{Z}} \right\|_{\ell^r} ,$$

where the implicit constant only depends on k_0, γ, λ*, and* r*.*

We now prove Theorem 3.4.1.

Proof of Theorem 3.4.1. We begin by proving the result for finite families of functions. Here, we show the homogeneous case, but the logic for the inhomogeneous case is similar.

Let $\{u_j\}_{j\in\mathbb{Z}}$ be such that $u_j \equiv 0$, except for finitely many $j \in \mathbb{Z}$. Suppose D, X, r, and s are as in the hypotheses of the theorem and that $\psi \in \dot{\mathcal{A}}(\mathbb{R}^n)$. Choosing $k_0 \in \mathbb{Z}$ such that $2^{k_0-1} < D \le 2^{k_0}$, for any $m \in \mathbb{Z}$, we have

$$\operatorname{supp}(\widehat{u_m}) \subset B(0, 2^m D) \subset B(0, 2^{m+k_0}).$$

Defining $u = \sum_{m\in\mathbb{Z}} u_m$, we note that

$$\operatorname{supp}\left(\widehat{\psi}(2^{-j}\cdot)\widehat{u_m} \right) \subset B(0, 2^{m+k_0}) \cap \left\{ 2^{j-1} < |\xi| < 2^{j+1} \right\}.$$

This intersection is empty for any $m < j - k_0$; therefore, we have the following identity:

$$\Delta_j^\psi u = \sum_{m\in\mathbb{Z}} \Delta_j^\psi u_m = \sum_{m=j-k_0}^{\infty} \Delta_j^\psi u_m = \sum_{k=-k_0}^{\infty} \Delta_j^\psi u_{j+k}. \tag{3.13}$$

We first prove (3.11). Fix $0 < h < \min(h_{X,r}, 1)$ such that $s > n(1/h - 1)$. We apply Lemma 3.4.3 with $\phi(x) = 2^{jn}\psi(2^j x)$, $f = u_{j+k}$, $A = 2^j$, $R = 2^{k+k_0}$, $k, j \in \mathbb{Z}$, $k \geq -k_0$, and $d > n/h$ to obtain

$$\begin{aligned}\left|\Delta_j^{\psi} u_{j+k}(x)\right| &\lesssim \left(2^{k+k_0}\right)^{n(\frac{1}{h}-1)} \left(2^j\right)^{-n} \left\|\left(1 + |2^j \cdot|\right)^d 2^{jn}\psi(2^j \cdot)\right\|_{L^\infty} \mathcal{M}_{n,h} u_{j+k}(x) \\ &\lesssim 2^{kn(\frac{1}{h}-1)} \mathcal{M}_{n,h} u_{j+k}(x),\end{aligned}$$

where the implicit constant depends only on the parameters stated and ψ. This gives

$$\left|2^{sj}\Delta_j^{\psi} u_{j+k}(x)\right| \lesssim 2^{kn(\frac{1}{h}-1)} 2^{sj} \mathcal{M}_{n,h} u_{j+k}(x) = 2^{kn(\frac{1}{h}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_{n,h} u_{j+k}(x).$$

Therefore, by (3.13), we have

$$\left|2^{sj}\Delta_j^{\psi} u(x)\right| \lesssim \sum_{k=-k_0}^{\infty} 2^{kn(\frac{1}{h}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_{n,h} u_{j+k}(x).$$

We now apply Lemma 3.4.5 with $\gamma = n(1/h - 1 - s/n)$, $\lambda = s$, and $d_{j+k} = \mathcal{M}_{n,h} u_{j+k}(x)$. Note that by definition $\gamma < 0$. This yields

$$\begin{aligned}\left\|\left\{\left|2^{sj}\Delta_j^{\psi} u(x)\right|\right\}_{j\in\mathbb{Z}}\right\|_{\ell^r} &\lesssim \left\|\left\{\sum_{k=-k_0}^{\infty} 2^{kn(\frac{1}{h}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_{n,h} u_{j+k}(x)\right\}_{j\in\mathbb{Z}}\right\|_{\ell^r} \\ &\lesssim \left\|\left\{2^{sj}\mathcal{M}_{n,h} u_j(x)\right\}_{j\in\mathbb{Z}}\right\|_{\ell^r}.\end{aligned}$$

The desired inequality follows from the monotonicity of the quasi-norm associated to X and the Fefferman-Stein inequality.

We now prove (3.12) for finite families. We apply similar logic, but working instead with the norm inequality from Lemma 3.4.4. Using $\phi(x) = 2^{jn}\psi(2^j x)$, $f = u_{j+k}$, $A = 2^j$, $R = 2^{k+k_0}$, $k, j \in \mathbb{Z}$, $k \geq -k_0$, $d > b$, and $n/\min(h_X, 1) < b < n + s$, we have

$$\left\|\Delta_j^{\psi} u_{j+k}\right\|_X \lesssim 2^{(k+k_0)(b-n)} 2^{-jn} \left\|\left(1 + \left|2^j \cdot\right|\right)^d 2^{jn}\psi\left(2^j \cdot\right)\right\|_{L^\infty} \|u_{j+k}\|_X \sim 2^{k(b-n)} \|u_{j+k}\|_X\,,$$

where the implicit constants depend only on the parameters stated and ψ.

Setting p^* such that $|||\cdot||| \sim \|\cdot\|_X$ and $|||f+g|||^{p^*} \leq |||f|||^{p^*} + |||g|||^{p^*}$ (Aoki-Rolewicz Theorem), we obtain

$$\begin{aligned}
\left\|2^{sj}\Delta_j^{\psi} u\right\|_X^{p^*} &\lesssim \sum_{k=-k_0}^{\infty} \left\|2^{sj}\Delta_j^{\psi} u_{j+k}\right\|_X^{p^*} \\
&\lesssim \sum_{k=-k_0}^{\infty} 2^{sjp^*} 2^{k(b-n)p^*} \|u_{j+k}\|_X^{p^*} \\
&= \sum_{k=-k_0}^{\infty} 2^{sp^*(j+k)} 2^{k(b-n-s)p^*} \|u_{j+k}\|_X^{p^*}.
\end{aligned}$$

Taking ℓ^{r/p^*} norms (quasi-norms when $r/p^* < 1$) and applying Lemma 3.4.5 with $\gamma = (b-n-s)p^*$, $\lambda = sp^*$, and $d_{j+k} = \|u_{j+k}\|_X^{p^*}$, we have the desired result.

We now show that the result holds for infinite families of functions. We first show the homogeneous Besov space case for $0 < r < \infty$. Let $\{u_j\}_{j\in\mathbb{Z}}, X, r$, and s be as in the hypotheses.

Let $U_N = \sum_{j=-N}^{N} u_j$. For $M < N$, $\{u_j\}_{M+1\leq|j|\leq N}$ satisfies the conditions of the theorem, and since the theorem has been shown to hold for finite families of functions, we have

$$\|U_N - U_M\|_{\dot{B}^s_{X,r}} \lesssim \left\|\{2^{sj}u_j\}_{M+1\leq|j|\leq N}\right\|_{\ell^r(X)},$$

where the implicit constant is independent of M, N, and the family $\{u_j\}_{j\in\mathbb{Z}}$.

By the assumption that $\left\|\{2^{sj}u_j\}_{j\in\mathbb{Z}}\right\|_{\ell^r(X)} < \infty$, the value of $\left\|\{2^{sj}u_j\}_{M+1\leq|j|\leq N}\right\|_{\ell^r(X)}$ must tend to zero as M approaches ∞. Therefore, $\{U_N\}_{N\in\mathbb{Z}}$ is a Cauchy sequence in $\dot{B}^s_{X,r}$, and by the completeness of $\dot{B}^s_{X,r}$, the sum $\sum_{j\in\mathbb{Z}} u_j$ converges in $\dot{B}^s_{X,r}$.

Similarly, we see that

$$\|U_N\|_{\dot{B}^s_{X,r}} \lesssim \left\|\{2^{sj}u_j\}_{-N\leq j\leq N}\right\|_{\ell^r(X)},$$

where the implicit constant is independent of N and the family $\{u_j\}_{j\in\mathbb{Z}}$, implying that

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot{B}^s_{X,r}} \lesssim \left\|\left\{2^{sj}u_j\right\}_{j\in\mathbb{Z}}\right\|_{\ell^r(X)},$$

with implicit constant independent of $\{u_j\}_{j\in\mathbb{Z}}$.

We now consider the case of infinite families for $\dot{F}^s_{X,r}$ with $0 < r \le \infty$ as well as $\dot{B}^s_{X,\infty}$. Note that $\left\{2^{(s-\varepsilon)j}u_j\right\}_{j\ge 0}$ and $\left\{2^{(s+\varepsilon)j}u_j\right\}_{j<0}$ belong to $\ell^1(X)$ for any $\varepsilon > 0$. Indeed, we have

$$\left\|\left\{2^{(s-\varepsilon)j}u_j\right\}_{j\ge 0}\right\|_{\ell^1(X)} = \sum_{j=0}^{\infty} 2^{sj}2^{-\varepsilon j}\left\|u_j\right\|_X \lesssim \left\|\{2^{sj}u_j\}_{j\ge 0}\right\|_{\ell^\infty(X)} \lesssim \left\|\{2^{sj}u_j\}_{j\ge 0}\right\|_{\ell^r(X)},$$

where the final expression is finite by assumption. Similar logic gives that $\left\|\left\{2^{(s+\varepsilon)j}u_j\right\}_{j<0}\right\|_{\ell^1(X)}$ is also finite.

Choosing $\varepsilon > 0$ such that $s - \varepsilon > \tau(X)$, by the case when $0 < r < \infty$, it follows that $\sum_{j=0}^{N} u_j$ and $\sum_{j=-N}^{-1} u_j$ converge in $\dot{B}^{s-\varepsilon}_{X,1}$ and $\dot{B}^{s+\varepsilon}_{X,1}$, respectively. Therefore, $\{U_N\}_{N\in\mathbb{Z}}$ converges in $\mathcal{S}_0'(\mathbb{R}^n)$. Applying the case for finite sequences for the space $\dot{F}^s_{X,r}$, we have $U_N \in \dot{F}^s_{X,r}$ and

$$\|U_N\|_{\dot{F}^s_{X,r}} \lesssim \left\|\left\{2^{sj}u_j\right\}_{-N\le j\le N}\right\|_{X(\ell^r)} \le \left\|\left\{2^{sj}u_j\right\}_{j\in\mathbb{Z}}\right\|_{X(\ell^r)}.$$

By the Fatou Property for $\dot{F}^s_{X,r}$, we have

$$\lim_{N\to\infty} U_N = \sum_{j\in\mathbb{Z}} u_j \in \dot{F}^s_{X,r} \quad \text{and} \quad \left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot{F}^s_{X,r}} \lesssim \left\|\left\{2^{sj}u_j\right\}_{j\in\mathbb{Z}}\right\|_{X(\ell^r)}.$$

For $\dot{B}^s_{X,\infty}$, similar reasoning can be applied. $\square$

Chapter 4

Fractional Leibniz Rule Results

In this chapter, we state and prove fractional Leibniz rules in QBFS and weighted bi-parameter settings. Section 4.1 contains the statement of the main result in QBFS-based Triebel-Lizorkin and Besov spaces, as well as inferring particular cases in BFSs; its proof is presented in Section 4.2. The generality of this result implies estimates in the setting of a variety of QBFSs; we direct the reader to Chapter 5 for applications in the specific instances of r.i.QBFSs and weighted mixed Lebesgue spaces, weighted Morrey spaces, and variable Lebesgue spaces. Bi-parameter fractional Leibniz rules in the setting of weighted Lebesgue spaces are proven in Section 4.3, where we employ analogous arguments adapted to the bi-parameter setting.

4.1 Fractional Leibniz Rules in QBFSs

The main result of this section is the following theorem giving estimates associated to T_σ in the setting of Triebel-Lizorkin and Besov spaces based on powers of QBFSs; from this, we obtain fractional Leibniz rules in the setting of BFSs.

Theorem 4.1.1. *Let $0 < r \leq \infty$, $0 < p, p_1, p_2 < \infty$, and $\sigma(\xi, \eta), \xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m, with $m \in \mathbb{R}$. Suppose X, X_1, and X_2 are QBFSs over $(\mathbb{R}^n, \mu)$, $(\mathbb{R}^n, \mu_1)$, and $(\mathbb{R}^n, \mu_2)$, respectively, such that $p(X), p(X_1), p(X_2) < \infty$, Properties P7. and*

P8. are satisfied by X^p with r as given and s as below, and the following Hölder-type inequality is valid:

$$\|fg\|_{X^p} \lesssim \|f\|_{X_1^{p_1}} \|g\|_{X_2^{p_2}}, \quad \forall f \in X_1^{p_1},\ g \in X_2^{p_2}. \tag{4.1}$$

i) If $h_{X^p,r}, h_{X_1^{p_1},r}, h_{X_2^{p_2},r} > 0$ and $s > \tau_r(X^p)$, then

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{X^p,r}} \lesssim \|f\|_{\dot{F}^{s+m}_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}} + \|f\|_{H^{X_1^{p_1}}} \|g\|_{\dot{F}^{s+m}_{X_2^{p_2},r}}. \tag{4.2}$$

ii) If $h_{X^p}, h_{X_1^{p_1}}, h_{X_2^{p_2}} > 0$ and $s > \tau(X^p)$, then

$$\|T_\sigma(f,g)\|_{\dot{B}^s_{X^p,r}} \lesssim \|f\|_{\dot{B}^{s+m}_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}} + \|f\|_{H^{X_1^{p_1}}} \|g\|_{\dot{B}^{s+m}_{X_2^{p_2},r}}. \tag{4.3}$$

Moreover, if $h_{X^p,r} > 0$ and $s > \tau_r(X^p)$,

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{X^p,r}} \lesssim \|f\|_{\dot{F}^{s+m}_{X^p,r}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{X^p,r}}, \tag{4.4}$$

with a corresponding statement for the Besov spaces if $h_X > 0$ and $s > \tau(X^p)$.

We can apply the lifting property in Theorem 3.2.5 to recast (4.2) and (4.3) in the form of fractional Leibniz rules as

$$\|D^s(T_\sigma(f,g))\|_{\dot{F}^0_{X^p,r}} \lesssim \|D^s f\|_{\dot{F}^m_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}} + \|f\|_{H^{X_1^{p_1}}} \|D^s g\|_{\dot{F}^m_{X_2^{p_2},r}}, \tag{4.5}$$

$$\|D^s(T_\sigma(f,g))\|_{\dot{B}^0_{X^p,r}} \lesssim \|D^s f\|_{\dot{B}^m_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}} + \|f\|_{H^{X_1^{p_1}}} \|D^s g\|_{\dot{B}^m_{X_2^{p_2},r}}. \tag{4.6}$$

Analogous estimates hold for (4.4) and its Besov counterpart.

Moreover, due to Theorem 3.2.6, if X, X_1, and X_2 are BFSs over $(\mathbb{R}^n, dx)$ such that $\mathcal{M}_n$ is bounded on X', X_1', and X_2', and if $r = 2$ and σ is a Coifman-Meyer multiplier of order 0, then for $1 < p, p_1, p_2 < \infty$, we write (4.5) and (4.4) in the setting of BFSs as

$$\|D^s(T_\sigma(f,g))\|_{X^p} \lesssim \|D^s f\|_{X_1^{p_1}} \|g\|_{X_2^{p_2}} + \|f\|_{X_1^{p_1}} \|D^s g\|_{X_2^{p_2}}, \tag{4.7}$$

$$\|D^s(T_\sigma(f,g))\|_{X^p} \lesssim \|D^s f\|_{X^p} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{X^p}. \tag{4.8}$$

In the particular case when $\sigma \equiv 1$, (4.7) and (4.8) give the following fractional Leibniz rules:

$$\|D^s(fg)\|_{X^p} \lesssim \|D^s f\|_{X_1^{p_1}} \|g\|_{X_2^{p_2}} + \|f\|_{X_1^{p_1}} \|D^s g\|_{X_2^{p_2}}, \tag{4.9}$$

$$\|D^s(fg)\|_{X^p} \lesssim \|D^s f\|_{X^p} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{X^p}. \tag{4.10}$$

Moreover, a version of Theorem 4.1.1 along with the corresponding estimates (4.5) - (4.10) hold in the inhomogeneous setting with an inhomogeneous Coifman-Meyer multiplier and J^s instead of D^s.

Remark 4.1.2. Due to (2.8), if X is a QBFS over $(\mathbb{R}^n, \mu)$ such that $p(X) < \infty$, the assumptions in Items i) and ii) are satisfied, and Properties P7. and P8. are satisfied for X^p, then Theorem 4.1.1 holds true with $X_1 = X_2 = X$ and $1/p = 1/p_1 + 1/p_2$.

Remark 4.1.3. The proof of Theorem 4.1.1 shows that different pairs of p_1 and p_2 and X_1 and X_2 can be used in each term on the right hand side of (4.2) and (4.3) as long as the corresponding Hölder-type inequality in (4.1) holds for both pairs.

4.2 Proof of Theorem 4.1.1

We now prove Theorem 4.1.1, following ideas presented in Naibo–Thomson [60] and extending the logic to the more general QBFS setting. We show here the details for the homogeneous Triebel-Lizorkin and Besov space settings. The proof of the corresponding result for the inhomogeneous spaces is similar.

Proof of Theorem 4.1.1. As in Naibo–Thomson [60], we begin with a decomposition of T_σ

due to the work of Coifman–Meyer [21]. Fix $\Psi \in \dot{\mathcal{A}}(\mathbb{R}^n)$ and choose $\Phi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\widehat{\Phi}(0) = 1, \quad \widehat{\Phi}(\xi) = \sum_{j \le 0} \widehat{\Psi}(2^{-j}\xi), \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given sufficiently large N, we write $T_\sigma = T_\sigma^1 + T_\sigma^2$ where, for $f, g \in \mathcal{S}_0(\mathbb{R}^n)$,

$$T_\sigma^1(f,g)(x) = \sum_{a,b \in \mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j \in \mathbb{Z}} C_j(a,b)(\Delta_j^{\mathbb{T}_a \Psi} f)(x)(S_j^{\mathbb{T}_b \Phi} g)(x),$$

with coefficients $C_j(a,b)$ that are bounded uniformly in $a, b \in \mathbb{Z}^n$. That is,

$$|C_j(a,b)| \lesssim 2^{jm}, \quad \forall a, b \in \mathbb{Z}^n, j \in \mathbb{Z},$$

with implicit constant depending on σ. The term T_σ^2 is defined analogously, with the roles of f and g interchanged and constants $C_j(a,b)$ that are slightly different but still bounded uniformly in $a, b \in \mathbb{Z}^n$.

It suffices to work with T_σ^1 and show that

$$\left\| T_\sigma^1(f,g) \right\|_{\dot{F}^s_{X^p,r}} \lesssim \|f\|_{\dot{F}^{s+m}_{X_1^{p_1},r}} \|g\|_{HX_2^{p_2}},$$

$$\left\| T_\sigma^1(f,g) \right\|_{\dot{B}^s_{X^p,r}} \lesssim \|f\|_{\dot{B}^{s+m}_{X_1^{p_1},r}} \|g\|_{HX_2^{p_2}},$$

with corresponding estimates for (4.4) and its Besov counterpart. Moreover, $\left\| \sum_{j \in \mathbb{Z}} f_j \right\|^{p/p_0}_{\dot{F}^s_{X^p,r}} \le \sum_{j \in \mathbb{Z}} \|f_j\|^{p/p_0}_{\dot{F}^s_{X^p,r}}$ for $p_0 > \max(p(X), p, p/r)$, with a similar inequality holding in $\dot{B}^s_{X^p,r}$. Thus, it suffices to prove that given $\varepsilon > 0$, there exist $0 < h_1, h_2 \le 1$ such that for any $f, g \in \mathcal{S}_0(\mathbb{R}^n)$,

$$\left\| T^{a,b}(f,g) \right\|_{\dot{F}^s_{X^p,r}} \lesssim (1+|a|)^{\varepsilon + \frac{n}{h_1}} (1+|b|)^{\varepsilon + \frac{n}{h_2}} \|f\|_{\dot{F}^{s+m}_{X_1^{p_1},r}} \|g\|_{HX_2^{p_2}}, \tag{4.11}$$

$$\left\| T^{a,b}(f,g) \right\|_{\dot{B}^s_{X^p,r}} \lesssim (1+|a|)^{\varepsilon + \frac{n}{h_1}} (1+|b|)^{\varepsilon + \frac{n}{h_2}} \|f\|_{\dot{B}^{s+m}_{X_1^{p_1},r}} \|g\|_{HX_2^{p_2}}, \tag{4.12}$$

where

$$T^{a,b}(f,g) = \sum_{j\in\mathbb{Z}} C_j(a,b)\left(\Delta_j^{\mathbb{T}_a\Psi} f\right)\left(S_j^{\mathbb{T}_b\Phi} g\right)$$

with implicit constants independent of $a, b \in \mathbb{Z}^n$. Estimates corresponding to (4.11) and (4.12) suffice for the fractional Leibniz rule in (4.4) and its Besov counterpart as well.

We show here the case when r is finite; the usual changes apply when $r = \infty$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\varphi}$ has compact support and $\widehat{\varphi} \equiv 1$ on $\mathrm{supp}(\widehat{\Phi})$. Define ψ such that $\widehat{\psi}(\xi) = \widehat{\Psi}(\frac{1}{2}\xi) + \widehat{\Psi}(\xi) + \widehat{\Psi}(2\xi)$, then $\widehat{\psi} \equiv 1$ on $\mathrm{supp}(\widehat{\Psi})$. Note that due to the supports of Ψ and Φ, we have that for all $j \in \mathbb{Z}$ and $a, b \in \mathbb{Z}^n$,

$$\mathrm{supp}\left(\mathscr{F}\left[C_j(a,b)\left(\Delta_j^{\mathbb{T}_a\Psi} f\right)\left(S_j^{\mathbb{T}_b\Phi} g\right)\right]\right) \subset \left\{\xi \in \mathbb{R}^n : |\xi| \lesssim 2^j\right\}.$$

We start with the proof of (4.11). Theorem 3.4.1, the bound on the coefficients $|C_j(a,b)|$, and (4.1) give us that

$$\begin{aligned}
\left\|T^{a,b}(f,g)\right\|_{\dot{F}^s_{X^p,r}} &\lesssim \left\|\left(\sum_{j\in\mathbb{Z}}\left[2^{sj}C_j(a,b)\left(\Delta_j^{\mathbb{T}_a\Psi} f\right)\left(S_j^{\mathbb{T}_b\Phi} g\right)\right]^r\right)^{\frac{1}{r}}\right\|_{X^p} \\
&\lesssim \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr}\left|\left(\Delta_j^{\mathbb{T}_a\Psi} f\right)\left(S_j^{\mathbb{T}_b\Phi} g\right)\right|^r\right)^{\frac{1}{r}}\right\|_{X^p} \qquad (4.13)\\
&\leq \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr}\left|\Delta_j^{\mathbb{T}_a\Psi} f\right|^r\right)^{\frac{1}{r}}\right\|_{X_1^{p_1}} \left\|\sup_{j\in\mathbb{Z}}\left|S_j^{\mathbb{T}_b\Phi} g\right|\right\|_{X_2^{p_2}}.
\end{aligned}$$

Let $0 < h_1 < \min(h_{X_1^{p_1},r}, 1)$; by Lemma 2.2.2 and the Fefferman-Stein inequality, we have

$$\begin{aligned}
\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr}\left|\Delta_j^{\mathbb{T}_a\Psi} f\right|^r\right)^{\frac{1}{r}}\right\|_{X_1^{p_1}} &\lesssim (1+|a|)^{\varepsilon+n/h_1}\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr}\left|\mathcal{M}_{n,h_1}\left(\Delta_j^{\psi} f\right)\right|^r\right)^{\frac{1}{r}}\right\|_{X_1^{p_1}} \\
&\lesssim (1+|a|)^{\varepsilon+n/h_1}\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr}\left|\Delta_j^{\psi} f\right|^r\right)^{\frac{1}{r}}\right\|_{X_1^{p_1}} \\
&\sim (1+|a|)^{\varepsilon+n/h_1}\|f\|_{\dot{F}^{s+m}_{X_1^{p_1},r}},
\end{aligned}$$

where the implicit constants are independent of a and f.

Now let $0 < h_2 < \min(h_{X_2^{p_2}}, 1)$. Applying Lemma 2.2.2 and the boundedness of $\mathcal{M}_{n,h_2}$ on $X_2^{p_2}$, we have

$$\begin{aligned} \left\| \sup_{j\in\mathbb{Z}} \left| S_j^{\mathbb{T}_b\Phi} g \right| \right\|_{X_2^{p_2}} &\lesssim (1+|b|)^{\varepsilon+n/h_2} \left\| \mathcal{M}_{n,h_2} \left(\sup_{j\in\mathbb{Z}} \left| S_j^{\varphi} g \right| \right) \right\|_{X_2^{p_2}} \\ &\lesssim (1+|b|)^{\varepsilon+n/h_2} \left\| \sup_{j\in\mathbb{Z}} \left| S_j^{\varphi} g \right| \right\|_{X_2^{p_2}} \\ &\sim (1+|b|)^{\varepsilon+n/h_2} \|g\|_{H^{X_2^{p_2}}}, \end{aligned} \tag{4.14}$$

where the constants are independent of b and g.

All together, this gives (4.11).

For (4.12), we again apply Theorem 3.4.1, the bound on $|C_j(a,b)|$, and the Hölder-type inequality in (4.1) to obtain

$$\begin{aligned} \left\| T^{a,b}(f,g) \right\|_{\dot{B}^s_{X^p,r}} &\lesssim \left(\sum_{j\in\mathbb{Z}} \left\| 2^{sj} C_j(a,b) \left(\Delta_j^{\mathbb{T}_a\Psi} f \right) \left(S_j^{\mathbb{T}_b\Phi} g \right) \right\|_{X^p}^r \right)^{\frac{1}{r}} \\ &\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr} \left\| \left(\Delta_j^{\mathbb{T}_a\Psi} f \right) \left(S_j^{\mathbb{T}_b\Phi} g \right) \right\|_{X^p}^r \right)^{\frac{1}{r}} \\ &\le \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr} \left\| \left(\Delta_j^{\mathbb{T}_a\Psi} f \right) \right\|_{X_1^{p_1}}^r \right)^{\frac{1}{r}} \left\| \sup_{j\in\mathbb{Z}} \left| S_j^{\mathbb{T}_b\Phi} g \right| \right\|_{X_2^{p_2}}. \end{aligned} \tag{4.15}$$

Set $0 < h_1 < \min(h_{X_1^{p_1}}, 1)$; apply Lemma 2.2.2 and the boundedness of $\mathcal{M}_{n,h_1}$ on $X_1^{p_1}$ to get

$$\left\| \Delta_j^{\mathbb{T}_a\Psi} f \right\|_{X_1^{p_1}} \lesssim (1+|a|)^{\varepsilon+n/h_1} \left\| \mathcal{M}_{n,h_1} \left(\Delta_j^{\psi} f \right) \right\|_{X_1^{p_1}} \lesssim (1+|a|)^{\varepsilon+n/h_1} \left\| \Delta_j^{\psi} f \right\|_{X_1^{p_1}},$$

where the implicit constant is independent of a and f. Therefore,

$$\begin{aligned} \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr} \left\| \Delta_j^{\mathbb{T}_a\Psi} f \right\|_{X_1^{p_1}}^r \right)^{\frac{1}{r}} &\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr} (1+|a|)^{(\varepsilon+n/h_1)r} \left\| \Delta_j^{\psi} f \right\|_{X_1^{p_1}}^r \right)^{\frac{1}{r}} \\ &\sim (1+|a|)^{\varepsilon+n/h_1} \|f\|_{\dot{B}^{s+m}_{X_1^{p_1},r}}. \end{aligned}$$

The factor $\left\|\sup_{j\in\mathbb{Z}}\left|S_j^{\mathbb{T}_b\Phi}g\right|\right\|_{X_2^{p_2}}$ is treated as in (4.14). This gives the desired inequality (4.12).

For (4.4) and its Besov counterpart, we proceed as in (4.13) and (4.15) with X^p instead of $X_1^{p_1}$ and $\sup_{j\in\mathbb{Z}}\left\|\left|S_j^{\mathbb{T}_b\Phi}g\right|\right\|_{L^\infty}$ instead of $\left\|\sup_{j\in\mathbb{Z}}\left|S_j^{\mathbb{T}_b\Phi}g\right|\right\|_{X_2^{p_2}}$. □

4.3 Weighted Bi-Parameter Fractional Leibniz Rules

In this section, we state and prove fractional Leibniz rules associated to partial fractional differential operators in the setting of weighted Lebesgue spaces. Analogous to the QBFS case, the proof of these estimates employs tools in homogeneous weighted bi-parameter Triebel-Lizorkin spaces. We first present the statement of the main result, followed by its proof broken into three subsections.

Theorem 4.3.1. *Denoting* $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, *let* $\sigma = \sigma(\xi, \eta)$ *be a bi-parameter Coifman-Meyer multiplier, and suppose* $1 < p_1, p_2 < \infty$ *and* $\frac{1}{2} < p < \infty$ *are such that* $1/p = 1/p_1 + 1/p_2$. *If* $w_1 \in A_{p_1,\mathcal{R}}(\mathbb{R}^n)$, $w_2 \in A_{p_2,\mathcal{R}}(\mathbb{R}^n)$, $w = w_1^{p/p_1} w_2^{p/p_2}$, *and* $s_\ell > n_\ell\,(1/\min(p/\tau_w, 1) - 1)$, *then for all* $f, g \in \mathcal{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *it holds that*

$$\begin{aligned}\|D_1^{s_1}D_2^{s_2}(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|D_1^{s_1}D_2^{s_2}f\|_{L^{p_1}(w_1)}\|g\|_{L^{p_2}(w_2)} + \|D_1^{s_1}f\|_{L^{p_1}(w_1)}\|D_2^{s_2}g\|_{L^{p_2}(w_2)} \\ + \|D_2^{s_2}f\|_{L^{p_1}(w_1)}\|D_1^{s_1}g\|_{L^{p_2}(w_2)} + \|f\|_{L^{p_1}(w_1)}\|D_1^{s_1}D_2^{s_2}g\|_{L^{p_2}(w_2)}.\end{aligned}$$

The proof of Theorem 4.3.1 follows ideas contained in Hale–Naibo [39] and Naibo–Thomson [60], modified using techniques from Brummer–Naibo [14] to accommodate bi-parameter function spaces and operators. The argument is divided into three parts. In Section 4.3.1, we obtain a paraproduct decomposition of T_σ and in Sections 4.3.2 and 4.3.3, we analyze representative components of the decomposition to obtain the desired estimates.

4.3.1 Decomposition of T_σ

Let $\Psi_\ell \in \dot{\mathcal{A}}(\mathbb{R}^{n_\ell})$ and define $\Phi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ via

$$\widehat{\Phi_\ell}(0) := 1, \quad \widehat{\Phi_\ell}(\xi_\ell) := \sum_{j<-2} \widehat{\Psi_\ell}(2^{-j}\xi_\ell), \quad \forall \xi_\ell \in \mathbb{R}^{n_\ell} \setminus \{0\}.$$

Then $\widehat{\Phi_\ell} \equiv 1$ on $\{\xi_\ell \in \mathbb{R}^{n_\ell} : |\xi_\ell| < \frac{1}{16}\}$, $\operatorname{supp}(\widehat{\Phi_\ell}) \subseteq \{\xi_\ell \in \mathbb{R}^{n_\ell} : |\xi_\ell| < \frac{1}{4}\}$, and for any $k \in \mathbb{Z}$,

$$\widehat{\Phi_\ell}(2^{-k}\xi_\ell) = \sum_{j<k-2} \widehat{\Phi_\ell}(2^{-j}\xi_\ell), \quad \forall \xi_\ell \in \mathbb{R}^{n_\ell} \setminus \{0\}.$$

Following the work in Brummer–Naibo [14], we write T_σ as a finite sum of bilinear multiplier operators, which will be grouped into cases for study in Sections 4.3.2 and 4.3.3. We have

$$\begin{aligned}
T_\sigma(f,g)(x) &= \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta \\
&= \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\left(\sum_{j_1,k_1\in\mathbb{Z}} \widehat{\Psi_1}(2^{-j_1}\xi_1)\widehat{\Psi_1}(2^{-k_1}\eta_1)\right)\left(\sum_{j_2,k_2\in\mathbb{Z}} \widehat{\Psi_2}(2^{-j_2}\xi_2)\widehat{\Psi_2}(2^{-k_2}\eta_2)\right) \\
&\quad\times \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta \\
&= \sum_{t_1,t_2=1}^{3} \Pi_{t_1,t_2}(f,g)(x),
\end{aligned}$$

where Π_{t_1,t_2} is the bilinear multiplier operator with symbol

$$\sigma_{t_1,t_2}(\xi,\eta) = \sigma(\xi,\eta)M_1^{t_1}(\xi_1,\eta_1)M_2^{t_2}(\xi_2,\eta_2),$$

and

$$M_\ell^1(\xi_\ell,\eta_\ell) = \sum_{\substack{j_\ell,k_\ell\in\mathbb{Z}\\ k_\ell<j_\ell-2}} \widehat{\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\Psi_\ell}(2^{-k_\ell}\eta_\ell) = \sum_{j_\ell\in\mathbb{Z}} \widehat{\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\Phi_\ell}(2^{-j_\ell}\eta_\ell),$$

$$M_\ell^2(\xi_\ell,\eta_\ell) = \sum_{\substack{j_\ell,k_\ell\in\mathbb{Z}\\ j_\ell<k_\ell-2}} \widehat{\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\Psi_\ell}(2^{-k_\ell}\eta_\ell) = \sum_{k_\ell\in\mathbb{Z}} \widehat{\Phi_\ell}(2^{-k_\ell}\xi_\ell)\widehat{\Psi_\ell}(2^{-k_\ell}\eta_\ell),$$

$$M_\ell^3(\xi_\ell, \eta_\ell) = \sum_{\delta=-2}^{2} \left[\sum_{j_\ell \in \mathbb{Z}} \widehat{\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\Psi_\ell}(2^{-(\delta+j_\ell)}\eta_\ell) \right].$$

Without loss of generality, we focus on the case $\delta = 0$ for M_ℓ^3. That is, we work with

$$M_\ell^3(\xi_\ell, \eta_\ell) = \sum_{j_\ell \in \mathbb{Z}} \widehat{\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\Psi_\ell}(2^{-j_\ell}\eta_\ell).$$

We prove appropriate estimates for each bilinear multiplier operator Π_{t_1,t_2}, focusing our attention on analyzing Π_{t_1,t_2} that are representative of the computations needed for analogous paraproducts. In particular, we look at $\Pi_{1,1}$ to represent $\Pi_{2,2}$, $\Pi_{1,2}$ to represent $\Pi_{2,1}$, $\Pi_{1,3}$ to represent $\Pi_{2,3}, \Pi_{3,1}$ and $\Pi_{3,2}$, and finally $\Pi_{3,3}$ stands on its own.

Let $\tilde{\psi}_\ell, \tilde{\phi}_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ be such that $\widehat{\tilde{\psi}_\ell}$ is supported on an annulus with $\widehat{\tilde{\psi}_\ell} \equiv 1$ on the support of $\widehat{\Psi_\ell}$ and $\widehat{\tilde{\phi}_\ell}$ is supported on a ball centered at the origin with $\widehat{\tilde{\phi}_\ell} \equiv 1$ on the support of $\widehat{\Phi_\ell}$. We then write the symbol of $\Pi_{1,1}$ as

$$\begin{aligned}
&\sigma(\xi, \eta) \left(\sum_{j_1 \in \mathbb{Z}} \widehat{\Psi_1}(2^{-j_1}\xi_1)\widehat{\Phi_1}(2^{-j_1}\eta_1) \right) \left(\sum_{j_2 \in \mathbb{Z}} \widehat{\Psi_2}(2^{-j_2}\xi_2)\widehat{\Phi_2}(2^{-j_2}\eta_2) \right) \\
&= \sum_{j_1, j_2 \in \mathbb{Z}} \sigma_{j_1,j_2}(2^{-j_1}\xi_1, 2^{-j_2}\xi_2, 2^{-j_1}\eta_1, 2^{-j_2}\eta_2)\widehat{\Psi_1}(2^{-j_1}\xi_1)\widehat{\Phi_1}(2^{-j_1}\eta_1) \\
&\quad \times \widehat{\Psi_2}(2^{-j_2}\xi_2)\widehat{\Phi_2}(2^{-j_2}\eta_2),
\end{aligned}$$

where

$$\sigma_{j_1,j_2}(\xi, \eta) = \sigma(2^{j_1}\xi_1, 2^{j_2}\xi_2, 2^{j_1}\eta_1, 2^{j_2}\eta_2)\widehat{\tilde{\psi}_1}(\xi_1)\widehat{\tilde{\phi}_1}(\eta_1)\widehat{\tilde{\psi}_2}(\xi_2)\widehat{\tilde{\phi}_2}(\eta_2).$$

Now let $D = [-\frac{d}{2}, \frac{d}{2}]^{2n}$, where d is such that $\operatorname{supp}(\sigma_{j_1,j_2}) \subset D$, and let $c_{j_1,j_2}(a,b), a, b \in \mathbb{Z}^n$, be the Fourier coefficients of a periodic extension of $\sigma_{j_1,j_2}\chi_D$. Then we have

$$\begin{aligned}
&\sigma_{j_1,j_2}(2^{-j_1}\xi_1, 2^{-j_2}\xi_2, 2^{-j_1}\eta_1, 2^{-j_2}\eta_2) \\
&= \left(\sum_{a,b \in \mathbb{Z}^n} c_{j_1,j_2}(a,b) e^{\frac{2\pi i}{d}(a,b)\cdot(2^{-j_1}\xi_1, 2^{-j_2}\xi_2, 2^{-j_1}\eta_1, 2^{-j_2}\eta_2)} \right) \\
&\quad \times \chi_D(2^{-j_1}\xi_1, 2^{-j_2}\xi_2, 2^{-j_1}\eta_1, 2^{-j_2}\eta_2).
\end{aligned}$$

Defining $C_{j_1,j_2}(a,b) = (1+|a|^2+|b|^2)^N c_{j_1,j_2}(a,b)$ for sufficiently large N, it can be shown that $|C_{j_1,j_2}(a,b)|$ are uniformly bounded in $j_\ell \in \mathbb{Z}$ and $a,b \in \mathbb{Z}^n$. Thus, we represent $\Pi_{1,1}$ as

$$\Pi_{1,1}(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j_1,j_2\in\mathbb{Z}} C_{j_1,j_2}(a,b)[\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1}\Delta_{j_2}^{\mathbb{T}_{a_2}\Psi_2} f](x)[S_{j_1}^{\mathbb{T}_{b_1}\Phi_1}S_{j_2}^{\mathbb{T}_{b_2}\Phi_2} g](x),$$

where we have set $\mathbb{T}_u F(\cdot) = F(\cdot + \frac{u}{d})$ and denote $a=(a_1,a_2), b=(b_1,b_2) \in \mathbb{Z}^{n_1}\times\mathbb{Z}^{n_2}$ (see also Brummer–Naibo [14]). Analogous computations give

$$\Pi_{1,2}(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j_1,j_2\in\mathbb{Z}} C_{j_1,j_2}(a,b)[\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1}S_{j_2}^{\mathbb{T}_{a_2}\Phi_2} f](x)[S_{j_1}^{\mathbb{T}_{b_1}\Phi_1}\Delta_{j_2}^{\mathbb{T}_{b_2}\Psi_2} g](x),$$

$$\Pi_{1,3}(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j_1,j_2\in\mathbb{Z}} C_{j_1,j_2}(a,b)[\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1}\Delta_{j_2}^{\mathbb{T}_{a_2}\Psi_2} f](x)[S_{j_1}^{\mathbb{T}_{b_1}\Phi_1}\Delta_{j_2}^{\mathbb{T}_{b_2}\Psi_2} g](x),$$

$$\Pi_{3,3}(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j_1,j_2\in\mathbb{Z}} C_{j_1,j_2}(a,b)[\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1}\Delta_{j_2}^{\mathbb{T}_{a_2}\Psi_2} f](x)[\Delta_{j_1}^{\mathbb{T}_{b_1}\Psi_1}\Delta_{j_2}^{\mathbb{T}_{b_2}\Psi_2} g](x),$$

where the coefficients $C_{j_1,j_2}(a,b)$ are slightly different in each line.

For $a,b\in\mathbb{Z}^n$, we denote

$$\Pi_{1,1}^{a,b}(f,g)(x) = \sum_{j_1,j_2\in\mathbb{Z}} C_{j_1,j_2}(a,b)[\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1}S_{j_2}^{\mathbb{T}_{a_2}\Phi_2} f](x)[S_{j_1}^{\mathbb{T}_{b_1}\Phi_1}\Delta_{j_2}^{\mathbb{T}_{b_2}\Psi_2} g](x),$$

with $\Pi_{1,2}^{a,b}, \Pi_{1,3}^{a,b}$, and $\Pi_{3,3}^{a,b}$ defined analogously.

In view of the supports of Ψ and Φ, it follows that $\operatorname{supp}(\widehat{u_{j_1,j_2}}) \subset \{\xi_1 \in \mathbb{R}^{n_1} : |\xi_1| \lesssim 2^{j_1}\} \times \{\xi_2 \in \mathbb{R}^{n_2} : |\xi_2| \lesssim 2^{j_2}\}$ for all $j_\ell \in \mathbb{Z}$, where u_{j_1,j_2} is the (j_1,j_2) term in $\Pi_{1,1}^{a,b}$, $\Pi_{1,2}^{a,b}$, $\Pi_{1,3}^{a,b}$ or $\Pi_{3,3}^{a,b}$.

The bi-parameter lifting property in Theorem 3.3.1 along with (3.10) imply that

$$\|D_1^{s_1}D_2^{s_2}(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|T_\sigma(f,g)\|_{\dot{F}^{s_1,s_2}_{p,2}(w)}.$$

Thus, it is enough to prove that given $\varepsilon > 0$, there exists $0 < h \le 1$ such that for all

$f, g \in \mathcal{S}_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, we have

$$\left\|\Pi_{1,1}^{a,b}(f,g)\right\|_{\dot{F}_{p,2}^{s_1,s_2}(w)} \lesssim (1+|a_1|)^{\varepsilon+\frac{n_1}{h}}(1+|a_2|)^{\varepsilon+\frac{n_2}{h}}(1+|b_1|)^{\varepsilon+\frac{n_1}{h}}(1+|b_2|)^{\varepsilon+\frac{n_2}{h}}$$

$$\times \|D_1^{s_1}D_2^{s_2}f\|_{L^{p_1}(w_1)}\|g\|_{L^{p_2}(w_2)},$$

$$\left\|\Pi_{1,2}^{a,b}(f,g)\right\|_{\dot{F}_{p,2}^{s_1,s_2}(w)} \lesssim (1+|a_1|)^{1+n_1}(1+|a_2|)^{1+n_2}(1+|b_1|)^{1+n_1}(1+|b_2|)^{1+n_2}$$

$$\times \|D_1^{s_1}f\|_{L^{p_1}(w_1)}\|D_2^{s_2}g\|_{L^{p_2}(w_2)}$$

$$\left\|\Pi_{1,3}^{a,b}(f,g)\right\|_{\dot{F}_{p,2}^{s_1,s_2}(w)} \lesssim (1+|a_1|)^{\varepsilon+\frac{n_1}{h}}(1+|a_2|)^{\varepsilon+\frac{n_2}{h}}(1+|b_1|)^{\varepsilon+\frac{n_1}{h}}(1+|b_2|)^{\varepsilon+\frac{n_2}{h}}$$

$$\times \|D_1^{s_1}D_2^{s_2}f\|_{L^{p_1}(w_1)}\|g\|_{L^{p_2}(w_2)},$$

$$\left\|\Pi_{3,3}^{a,b}(f,g)\right\|_{\dot{F}_{p,2}^{s_1,s_2}(w)} \lesssim (1+|a_1|)^{\varepsilon+\frac{n_1}{h}}(1+|a_2|)^{\varepsilon+\frac{n_2}{h}}(1+|b_1|)^{\varepsilon+\frac{n_1}{h}}(1+|b_2|)^{\varepsilon+\frac{n_2}{h}}$$

$$\times \|D_1^{s_1}D_2^{s_2}f\|_{L^{p_1}(w_1)}\|g\|_{L^{p_2}(w_2)},$$

where $s_\ell > n_\ell(1/\min(p/\tau_w,1)-1)$.

The estimates associated to $\Pi_{1,1}^{a,b}, \Pi_{1,3}^{a,b}$, and $\Pi_{3,3}^{a,b}$ can be treated similarly, and are addressed in Section 4.3.2, with the estimates corresponding to $\Pi_{1,2}^{a,b}$ discussed in Section 4.3.3.

4.3.2 Estimates for $\Pi_{1,1}^{a,b}, \Pi_{1,3}^{a,b}$, and $\Pi_{3,3}^{a,b}$

Fixing $0 < h < \min(p_1/\tau_{w_1}, p_2/\tau_{w_2}, 1)$ and $\varepsilon > 0$, we begin with $\Pi_{1,1}^{a,b}(f,g)$. Theorem 3.4.2, the uniform bound for $|C_{j_1,j_2}(a,b)|$, and Hölder's inequality give us that

$$\begin{aligned}
&\left\|\Pi_{1,1}^{a,b}(f,g)\right\|_{\dot{F}^{s_1,s_2}_{p,2}(w)}\\
&\lesssim \left\|\left(\sum_{j_1,j_2\in\mathbb{Z}}\left|2^{s_1j_1+s_2j_2}[\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1}\Delta_{j_2}^{\mathbb{T}_{a_2}\Psi_2}f][S_{j_1}^{\mathbb{T}_{b_1}\Phi_1}S_{j_2}^{\mathbb{T}_{b_2}\Phi_2}g]\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(w)}\\
&\le \left\|\sup_{j_1,j_2\in\mathbb{Z}}\left|S_{j_1}^{\mathbb{T}_{b_1}\Phi_1}S_{j_2}^{\mathbb{T}_{b_2}\Phi_2}g\right|\left(\sum_{j_1,j_2\in\mathbb{Z}}\left|2^{s_1j_1+s_2j_2}\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1}\Delta_{j_2}^{\mathbb{T}_{a_2}\Psi_2}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(w)}\\
&\le \left\|\left(\sum_{j_1,j_2\in\mathbb{Z}}\left|2^{s_1j_1+s_2j_2}\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1}\Delta_{j_2}^{\mathbb{T}_{a_2}\Psi_2}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p_1}(w_1)}\left\|\sup_{j_1,j_2\in\mathbb{Z}}\left|S_{j_1}^{\mathbb{T}_{b_1}\Phi_1}S_{j_2}^{\mathbb{T}_{b_2}\Phi_2}g\right|\right\|_{L^{p_2}(w_2)}.
\end{aligned}$$

Let $\psi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ be defined through $\widehat{\psi_\ell}(\xi_\ell) = \widehat{\Psi_\ell}(\frac{1}{2}\xi_\ell) + \widehat{\Psi_\ell}(\xi_\ell) + \widehat{\Psi_\ell}(2\xi_\ell)$; then $\widehat{\Psi_\ell}\widehat{\psi_\ell} = \widehat{\Psi_\ell}$. Applying Lemma 2.2.4 and the Fefferman–Stein inequality associated to the strong maximal operator, we obtain

$$\begin{aligned}
&\left\|\left(\sum_{j_1,j_2\in\mathbb{Z}}\left|2^{s_1j_1+s_2j_2}\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1}\Delta_{j_2}^{\mathbb{T}_{a_2}\Psi_2}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p_1}(w_1)}\\
&\quad\lesssim (1+|a_1|)^{\varepsilon+\frac{n_1}{h}}(1+|a_2|)^{\varepsilon+\frac{n_2}{h}}\left\|\left(\sum_{j_1,j_2\in\mathbb{Z}}\left|2^{s_1j_1+s_2j_2}\mathcal{M}^S_{n,h}(\Delta_{j_1}^{\psi_1}\Delta_{j_2}^{\psi_2}f)\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p_1}(w_1)}\\
&\quad\lesssim (1+|a_1|)^{\varepsilon+\frac{n_1}{h}}(1+|a_2|)^{\varepsilon+\frac{n_2}{h}}\left\|\left(\sum_{j_1,j_2\in\mathbb{Z}}\left|2^{s_1j_1+s_2j_2}\Delta_{j_1}^{\psi_1}\Delta_{j_2}^{\psi_2}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p_1}(w_1)}\\
&\quad\sim (1+|a_1|)^{\varepsilon+\frac{n_1}{h}}(1+|a_2|)^{\varepsilon+\frac{n_2}{h}}\|f\|_{\dot{F}^{s_1,s_2}_{p_1,2}(w_1)}\\
&\quad\sim (1+|a_1|)^{\varepsilon+\frac{n_1}{h}}(1+|a_2|)^{\varepsilon+\frac{n_2}{h}}\|D_1^{s_1}D_2^{s_2}f\|_{L^{p_1}(w_1)},
\end{aligned}$$

where the final inequality follows from the bi-parameter lifting property in Theorem 3.3.1 and (3.9).

Choosing $\phi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ supported in a ball centered at the origin satisfying $\widehat{\Phi_\ell}\widehat{\phi_\ell} = \widehat{\Phi_\ell}$, we also get

$$\begin{aligned}
&\left\| \sup_{j_1,j_2\in\mathbb{Z}} \left| S_{j_1}^{\mathbb{T}_{b_1}\Phi_1} S_{j_2}^{\mathbb{T}_{b_2}\Phi_2} g \right| \right\|_{L^{p_2}(w_2)} \\
&\lesssim (1+|b_1|)^{\varepsilon+\frac{n_1}{h}}(1+|b_2|)^{\varepsilon+\frac{n_2}{h}} \left\| \mathcal{M}_{n,h}^S \left(\sup_{j_1,j_2\in\mathbb{Z}} \left| S_{j_1}^{\phi_1} S_{j_2}^{\phi_2} g \right| \right) \right\|_{L^{p_2}(w_2)} \\
&\lesssim (1+|b_1|)^{\varepsilon+\frac{n_1}{h}}(1+|b_2|)^{\varepsilon+\frac{n_2}{h}} \left\| \sup_{j_1,j_2\in\mathbb{Z}} \left| S_{j_1}^{\phi_1} S_{j_2}^{\phi_2} g \right| \right\|_{L^{p_2}(w_2)} \\
&\lesssim (1+|b_1|)^{\varepsilon+\frac{n_1}{h}}(1+|b_2|)^{\varepsilon+\frac{n_2}{h}} \|g\|_{L^{p_2}(w_2)},
\end{aligned}$$

with the last line following from Lemma 2.2.6.

Similar logic can also be applied to obtain the desired estimates corresponding to $\Pi_{1,3}^{a,b}(f,g)$ and $\Pi_{3,3}^{a,b}(f,g)$.

4.3.3 Estimates for $\Pi_{1,2}^{a,b}$

Beginning as with the estimates corresponding to $\Pi_{1,1}^{a,b}, \Pi_{1,3}^{a,b}$, and $\Pi_{3,3}^{a,b}$, by Theorem 3.4.2, the uniform bound for $|C_{j_1,j_2}(a,b)|$, the estimate (2.7), and Hölder's inequality, we obtain

$$\begin{aligned}
&\left\| \Pi_{1,2}^{a,b}(f,g) \right\|_{\dot{F}_{p,2}^{s_1,s_2}(w)} \\
&\lesssim \left\| \left(\sum_{j_1,j_2\in\mathbb{Z}} \left| 2^{s_1j_1+s_2j_2} C_{j_1,j_2}(a,b) \left[\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1} S_{j_2}^{\mathbb{T}_{a_2}\Phi_2} f \right] \left[S_{j_1}^{\mathbb{T}_{b_1}\Phi_1} \Delta_{j_2}^{\mathbb{T}_{b_2}\Psi_2} g \right] \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \\
&\lesssim (1+|a_2|)^{1+n_2}(1+|b_1|)^{1+n_1} \\
&\quad \times \left\| \left(\sum_{j_1\in\mathbb{Z}} \left[2^{s_1j_1} \mathcal{M}_{n_2} \Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1} f \right]^2 \right)^{\frac{1}{2}} \left(\sum_{j_2\in\mathbb{Z}} \left[2^{s_2j_2} \mathcal{M}_{n_1} \Delta_{j_2}^{\mathbb{T}_{b_2}\Psi_2} g \right]^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \\
&\lesssim (1+|a_2|)^{1+n_2}(1+|b_1|)^{1+n_1} \\
&\quad \times \left\| \left(\sum_{j_1\in\mathbb{Z}} \left[2^{s_1j_1} \mathcal{M}_{n_2} \Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1} f \right]^2 \right)^{\frac{1}{2}} \right\|_{L^{p_1}(w_1)} \left\| \left(\sum_{j_2\in\mathbb{Z}} \left[2^{s_2j_2} \mathcal{M}_{n_1} \Delta_{j_2}^{\mathbb{T}_{b_2}\Psi_2} g \right]^2 \right)^{\frac{1}{2}} \right\|_{L^{p_2}(w_2)}.
\end{aligned}$$

For $w_1 \in A_{p_1,\mathcal{R}}(\mathbb{R}^n)$ and $w_2 \in A_{p_2,\mathcal{R}}(\mathbb{R}^n)$, Lemma 2.1.1 along with the Fefferman-Stein inequality for $\mathcal{M}_{n_\ell}$ imply that

$$\begin{aligned}\left\|\Pi_{1,2}^{a,b}(f,g)\right\|_{\dot{F}^{s_1,s_2}_{p,2}(w)} &\lesssim (1+|a_2|)^{1+n_2}(1+|b_1|)^{1+n_1}\\ &\times \left\|\left(\sum_{j_1\in\mathbb{Z}}\left|2^{s_1j_1}\Delta_{j_1}^{\mathbb{T}_{a_1}\Psi_1}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p_1}(w_1)}\left\|\left(\sum_{j_2\in\mathbb{Z}}\left|2^{s_2j_2}\Delta_{j_2}^{\mathbb{T}_{b_2}\Psi_2}g\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p_2}(w_2)}.\end{aligned}$$

As before, let $\psi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ be defined through $\widehat{\psi_\ell}(\xi_\ell) = \widehat{\Psi_\ell}(\frac{1}{2}\xi_\ell) + \widehat{\Psi_\ell}(\xi_\ell) + \widehat{\Psi_\ell}(2\xi_\ell)$; then $\widehat{\Psi_\ell}\widehat{\psi_\ell} = \widehat{\Psi_\ell}$. Using Remark 2.2.3, Lemma 2.1.1, and the Fefferman-Stein inequality on each of the factors above, we obtain

$$\begin{aligned}\left\|\Pi_{1,2}^{a,b}(f,g)\right\|_{\dot{F}^{s_1,s_2}_{p,2}(w)} &\lesssim (1+|a_1|)^{1+n_1}(1+|a_2|)^{1+n_2}(1+|b_1|)^{1+n_1}(1+|b_2|)^{1+n_2}\\ &\times \left\|\left(\sum_{j_1\in\mathbb{Z}}\left[2^{s_1j_1}\mathcal{M}_{n_1}(\Delta_{j_1}^{\psi_1}f)\right]^2\right)^{\frac{1}{2}}\right\|_{L^{p_1}(w_1)}\left\|\left(\sum_{j_2\in\mathbb{Z}}\left[2^{s_2j_2}\mathcal{M}_{n_2}(\Delta_{j_2}^{\psi_2}g)\right]^2\right)^{\frac{1}{2}}\right\|_{L^{p_2}(w_2)}\\ &\lesssim (1+|a_1|)^{1+n_1}(1+|a_2|)^{1+n_2}(1+|b_1|)^{1+n_1}(1+|b_2|)^{1+n_2}\\ &\times \left\|\left(\sum_{j_1\in\mathbb{Z}}\left|2^{s_1j_1}\Delta_{j_1}^{\psi_1}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p_1}(w_1)}\left\|\left(\sum_{j_2\in\mathbb{Z}}\left|2^{s_2j_2}\Delta_{j_2}^{\psi_2}g\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p_2}(w_2)}\end{aligned}$$

Now, note that

$$\left\|\left(\sum_{j_1\in\mathbb{Z}}\left|2^{s_1j_1}\Delta_{j_1}^{\psi_1}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p_1}(w_1)} = \left\|\left\|\left(\sum_{j_1\in\mathbb{Z}}\left|2^{s_1j_1}\Delta_{j_1}^{\psi_1}f(\cdot_1,y_2)\right|^2\right)^{\frac{1}{2}}\right\|^{p_1}_{L^{p_1}(w_1(\cdot_1,y_2))}\right\|_{L^{p_1}_{y_2}}$$

and that $w_1 \in A_{p_1,\mathcal{R}}(\mathbb{R}^n)$ implies $w_1(\cdot_1, y_2) \in A_{p_1}(\mathbb{R}^{n_1})$ for almost every $y_2 \in \mathbb{R}^{n_2}$ with uniform constants by Lemma 2.1.1. Therefore, by Remark 2.2.1, the lifting property in weighted Lebesgue spaces (3.3), and the equivalences in (3.5),

$$\left\| \left(\sum_{j_1 \in \mathbb{Z}} \left| 2^{s_1 j_1} \Delta_{j_1}^{\psi_1} f(\cdot_1, y_2) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p_1}(w_1(\cdot_1, y_2))} \sim \| D_1^{s_1} f(\cdot_1, y_2) \|_{L^{p_1}(w_1(\cdot_1, y_2))} .$$

Hence,

$$\left\| \left(\sum_{j_1 \in \mathbb{Z}} \left| 2^{s_1 j_1} \Delta_{j_1}^{\psi_1} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p_1}(w_1)} \sim \| D_1^{s_1} f \|_{L^{p_1}(w_1)} .$$

The same logic gives that

$$\left\| \left(\sum_{j_2 \in \mathbb{Z}} \left| 2^{s_2 j_2} \Delta_{j_2}^{\psi_2} g \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p_2}(w_2)} \sim \| D_2^{s_2} g \|_{L^{p_2}(w_2)} .$$

Therefore,

$$\begin{aligned} \left\| \Pi_{1,2}^{a,b}(f,g) \right\|_{\dot{F}_{p,2}^{s_1,s_2}(w)} &\lesssim (1+|a_1|)^{1+n_1}(1+|a_2|)^{1+n_2}(1+|b_1|)^{1+n_1}(1+|b_2|)^{1+n_2} \\ &\quad \times \| D_1^{s_1} f \|_{L^{p_1}(w_1)} \| D_2^{s_2} g \|_{L^{p_2}(w_2)} . \end{aligned}$$

Chapter 5

Fractional Leibniz Rules in Specific Function Spaces

Theorem 4.1.1 gave fractional Leibniz rules in Triebel-Lizorkin and Besov spaces based on QBFSs. In this chapter, we apply these estimates in different function spaces. Section 5.1 gives fractional Leibniz rules in the setting of r.i.QBFSs. Examples of rearrangement invariant settings include Lebesgue, Lorentz, and Orlicz spaces, as well as weighted versions of each of these. There are also applications of Theorem 4.1.1 in non-rearrangement invariant settings, particular instances of which are discussed in this chapter as well. Specifically, Section 5.2 contains results in the setting of weighted mixed Lebesgue spaces and Section 5.3 shows results in the context of weighted Morrey spaces, both including applications with power weights. Finally, Section 5.4 gives estimates in the variable Lebesgue spaces.

5.1 Results in Rearrangement Invariant Spaces

We now present our results for fractional Leibniz rules in the setting of r.i.QBFSs. While we show only the results in the homogeneous case, corresponding results also hold with an inhomogeneous Coifman-Meyer multiplier and the inhomogeneous differential operator J^s.

We first have a corollary to Theorem 4.1.1 giving estimates associated to T_σ in the setting of Triebel-Lizorkin and Besov spaces based on r.i.QBFSs.

Corollary 5.1.1. *Let $m \in \mathbb{R}$, $0 < r \leq \infty$, $0 < p, p_1, p_2 < \infty$, $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m, and $w, w_1, w_2 \in A_\infty(\mathbb{R}^n)$. Suppose X, X_1, and X_2 are r.i.QBFSs over $(\mathbb{R}^n, dx)$ with finite Boyd indices such that $p(X), p(X_1), p(X_2) < \infty$, Properties P7. and P8. are satisfied by $X^p(w)$ with r as given and s as below, and the following Hölder inequality holds:*

$$\|fg\|_{X^p(w)} \lesssim \|f\|_{X_1^{p_1}(w_1)} \|g\|_{X_2^{p_2}(w_2)}, \quad \forall f \in X_1^{p_1}(w_1),\ g \in X_2^{p_2}(w_2).$$

i) If $s > n\left(\frac{1}{\min(pp_X/\tau_w, 1/p(X^p), r)} - 1\right)$, then

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{X^p,r}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{X_1^{p_1},r}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \|g\|_{\dot{F}^{s+m}_{X_2^{p_2},r}(w_2)}. \tag{5.1}$$

ii) If $s > n\left(\frac{1}{\min(pp_X/\tau_w, 1/p(X^p))} - 1\right)$, then

$$\|T_\sigma(f,g)\|_{\dot{B}^s_{X^p,r}(w)} \lesssim \|f\|_{\dot{B}^{s+m}_{X_1^{p_1},r}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \|g\|_{\dot{B}^{s+m}_{X_2^{p_2},r}(w_2)}. \tag{5.2}$$

Moreover, if $s > n\left(\frac{1}{\min(pp_X/\tau_w, 1/p(X^p), r)} - 1\right)$,

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{X^p,r}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{X^p,r}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{X^p,r}(w)}, \tag{5.3}$$

with a corresponding estimate for the Besov spaces if $s > n\left(\frac{1}{\min(pp_X/\tau_w, 1/p(X^p))} - 1\right)$.

Proof. This follows by applying Theorem 4.1.1 with the r.i.QBFSs $X(w)$, $X_1(w_1)$, and $X_2(w_2)$. Indeed, since $(X(w))^{p_0} = X^{p_0}(w)$, whenever X^{p_0} is a BFS, $(X(w))^{p_0}$ is as well, giving that $p(X(w)) \leq p(X) < \infty$; similarly, $p(X_1(w_1)), p(X_2(w_2)) < \infty$. Moreover, (2.11) applied to X^p, $X_1^{p_1}$, and $X_2^{p_2}$ implies that $h_{X^p(w),r}, h_{X_1^{p_1}(w_1),r}, h_{X_2^{p_2}(w_2),r} > 0$, while (2.13) applied to X^p implies $s > \tau_r(X^p(w))$. The argument for (5.2) is similar. $\square$

Applying the lifting property, we obtain the following versions of (5.1) and (5.2):

$$\|D^s(T_\sigma(f,g))\|_{\dot F^0_{X^p,r}(w)} \lesssim \|D^s f\|_{\dot F^m_{X_1^{p_1},r}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \|D^s g\|_{\dot F^m_{X_2^{p_2},r}(w_2)}, \quad (5.4)$$

$$\|D^s(T_\sigma(f,g))\|_{\dot B^0_{X^p,r}(w)} \lesssim \|D^s f\|_{\dot B^m_{X_1^{p_1},r}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \|D^s g\|_{\dot B^m_{X_2^{p_2},r}(w_2)}. \quad (5.5)$$

Moreover, using Theorem 3.2.7 and (5.4), we obtain the following estimates when the symbol σ is of order zero:

$$\|D^s(T_\sigma(f,g))\|_{H^{X^p}(w)} \lesssim \|D^s f\|_{H^{X_1^{p_1}}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \|D^s g\|_{H^{X_2^{p_2}}(w_2)}. \quad (5.6)$$

In particular, if $\sigma \equiv 1$, this implies

$$\|D^s(fg)\|_{H^{X^p}(w)} \lesssim \|D^s f\|_{H^{X_1^{p_1}}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \|D^s g\|_{H^{X_2^{p_2}}(w_2)}. \quad (5.7)$$

Estimates analogous to (5.4) - (5.7) also hold for (5.3) and its Besov counterpart.

Finally, due to Theorem 3.2.7 and (3.8), the estimate in (5.6) implies the following fractional Leibniz rule in weighted r.i.QBFSs.

Corollary 5.1.2. *Let $\sigma(\xi,\eta), \xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order zero and let $w \in A_\infty(\mathbb{R}^n)$. Suppose X, X_1, and X_2 are r.i.QBFSs over $(\mathbb{R}^n, dx)$ with finite Boyd indices such that $p(X), p(X_1), p(X_2) < \infty$ and Properties P7. and P8. are satisfied by $X^p(w)$ with $r = 2$ and s as given below. Suppose that $0 < p < \infty$, $p(X_1) < p_1 < \infty$, $p(X_2) < p_2 < \infty$, $w_1 \in A_{p_1 p_{X_1}}(\mathbb{R}^n)$, $w_2 \in A_{p_2 p_{X_2}}(\mathbb{R}^n)$, and*

$$\|fg\|_{X^p(w)} \lesssim \|f\|_{X_1^{p_1}(w_1)} \|g\|_{X_2^{p_2}(w_2)}, \qquad \forall f \in X_1^{p_1}(w_1),\ g \in X_2^{p_2}(w_2).$$

Then if $s > n\left(\frac{1}{\min(pp_X/\tau_w, 1/p(X^p))} - 1\right)$,

$$\|D^s(T_\sigma(f,g))\|_{X^p(w)} \lesssim \|D^s f\|_{X_1^{p_1}(w_1)} \|g\|_{X_2^{p_2}(w_2)} + \|f\|_{X_1^{p_1}(w_1)} \|D^s g\|_{X_2^{p_2}(w_2)}.$$

In particular,

$$\|D^s(fg)\|_{X^p(w)} \lesssim \|D^s f\|_{X_1^{p_1}(w_1)} \|g\|_{X_2^{p_2}(w_2)} + \|f\|_{X_1^{p_1}(w_1)} \|D^s g\|_{X_2^{p_2}(w_2)} .$$

Moreover, if $p(X) < p < \infty$, $w \in A_{pp_X}(\mathbb{R}^n)$, *and* $s > n\left(\frac{1}{\min(pp_X/\tau_w, 1/p(X^p))} - 1\right)$,

$$\|D^s(T_\sigma(f,g))\|_{X^p(w)} \lesssim \|D^s f\|_{X^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{X^p(w)} , \tag{5.8}$$

and in particular,

$$\|D^s(fg)\|_{X^p(w)} \lesssim \|D^s f\|_{X^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{X^p(w)} . \tag{5.9}$$

Remark 5.1.3. As a consequence of Remark 4.1.2, Corollary 5.1.1 holds in the particular case that X is r.i.QBFS over $(\mathbb{R}^n, dx)$ with finite Boyd indices, $p(X) < \infty$, $X_1 = X_2 = X$, $w = w_1 = w_2$, $w \in A_\infty(\mathbb{R}^n)$, $0 < p, p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$, and $X^p(w)$ satisfies Properties P7. and P8. for $0 < r \leq \infty$ and s as given in the statement of Corollary 5.1.1.

Remark 5.1.4. Due to Remark 4.1.3, different pairs of X_1, X_2 and p_1, p_2 can be used on the right hand side of (5.1) and (5.2).

5.1.1 Examples

We now give explicit examples of r.i.QBFSs where Corollary 5.1.1 may be applied, specifically considering weighted Lebesgue spaces, classical weighted Lorentz spaces, Lorentz Λ-spaces, and weighted Orlicz spaces.

Weighted Lebesgue Spaces

Corollary 5.1.1 gives as a particular case the previously known fractional Leibniz rules in Triebel-Lizorkin and Besov spaces based on weighted Lebesgue spaces proved in Naibo–Thomson [60], as well as (1.7) in the Hardy space setting and its weighted version. In this case, we have $X = X_1 = X_2 = L^1(\mathbb{R}^n)$, $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$, $w_1, w_2 \in A_\infty(\mathbb{R}^n)$, and $w = w_1^{p/p_1} w_2^{p/p_2}$. Therefore, $p(X) = p(X_1) = p(X_2) = 1$, $p_X = q_X = 1$, $X^p(w) = L^p(w), X^{p_1}(w_1) = L^{p_1}(w_1)$, and $X^{p_2}(w_2) = L^{p_2}(w_2)$; the lower bounds for s are $n\left(\frac{1}{\min(p/\tau_w, r, 1)} - 1\right)$ in the Triebel-Lizorkin case and $n\left(\frac{1}{\min(p/\tau_w, 1)} - 1\right)$ in the Besov setting.

Corollary 5.1.2 then gives the fractional Leibniz rules in the weighted Lebesgue spaces for $1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}(\mathbb{R}^n)$, $w_2 \in A_{p_2}(\mathbb{R}^n)$, and $s > n\left(\frac{1}{\min(p/\tau_w, 1)} - 1\right)$ and versions with L^∞ for $p > 1$, $w \in A_p(\mathbb{R}^n)$, and $s > n\left(\frac{1}{\min(p/\tau_w, 1)} - 1\right)$; in particular, we recover the unweighted version (1.1) presented in the introduction when $\sigma \equiv 1$.

Classical Lorentz Spaces

Given $0 < p, q < \infty$, the *classical Lorentz spaces* $L^{p,q}(\mathbb{R}^n)$ are r.i.QBFSs defined through the quasi-norm given by

$$\|f\|_{L^{p,q}} = \left(\int_0^\infty (f^*(s) s^{\frac{1}{p}})^q \frac{ds}{s}\right)^{\frac{1}{q}}, \tag{5.10}$$

where $f^* = f_w^*$ with $w \equiv 1$. These spaces extend the scale of Lebesgue spaces since $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

The Boyd indices for $L^{p,q}(\mathbb{R}^n)$ are given by $p_X = q_X = p$. We note that if $1 \leq p, q < \infty$, $L^{p,q}(\mathbb{R}^n)$ is a r.i.BFS, and since $(L^{p,q}(\mathbb{R}^n))^{p_0} = L^{pp_0, qp_0}(\mathbb{R}^n)$, we have $p(L^{p,q}(\mathbb{R}^n)) = 1/\min(p, q, 1)$. If $X = L^{p,q}(\mathbb{R}^n)$, then $X(w)$ is given by (5.10) by replacing f^* with f_w^*. Therefore, Corollary 5.1.1 gives fractional Leibniz rules for Triebel-Lizorkin and Besov spaces based on weighted Lorentz spaces (see also Naibo–Thomson [60]). In this case, we have $0 < p, p_1, p_2, q, q_1, q_2 < \infty$ satisfying $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$, $X = L^{1,q/p}, X_1 = L^{1,q_1/p_1}$, $X_2 = L^{1,q_2/p_2}$, and $w = w_1 = w_2$ with $w \in A_\infty(\mathbb{R}^n)$. Therefore, $p_X = q_X = 1$, $X^p(w) = L^{p,q}(w), X_1^{p_1}(w) = L^{p_1,q_1}(w)$, and $X_2^{p_2}(w) = L^{p_2,q_2}(w)$ (see Hunt [41, Theorem 4.5] for Hölder's inequality between these spaces). The lower bound for s is $n\left(\frac{1}{\min(p/\tau_w, q, r, 1)} - 1\right)$ for the Triebel-Lizorkin case and $n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right)$ for the Besov setting.

Corollary 5.1.2 then gives the following fractional Leibniz rule for weighted Lorentz spaces:

$$\|D^s(T_\sigma(f,g))\|_{L^{p,q}(w)} \lesssim \|D^s f\|_{L^{p_1,q_1}(w)} \|g\|_{L^{p_2,q_2}(w)} + \|f\|_{L^{p_1,q_1}(w)} \|D^s g\|_{L^{p_2,q_2}(w)},$$

with $w \in A_{\min(p_1,p_2)}(\mathbb{R}^n)$, $1 < p_1, p_2, q_1, q_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$, and $s > n\left(\frac{1}{\min(p/\tau_w,q,1)} - 1\right)$, with corresponding counterparts for (5.8) and (5.9) if $1 < p, q < \infty$ and $w \in A_p(\mathbb{R}^n)$. See also Cruz-Uribe–Naibo [23, 24] for the case $\sigma \equiv 1$.

Lorentz Λ-Spaces

The *Lorentz Λ-spaces* Λ_v^q are defined to be the collection of measurable functions f defined on $\mathbb{R}^n$ such that

$$\|f\|_{\Lambda_v^q} = \left(\int_0^\infty f^*(s)^q v(s) ds\right)^{\frac{1}{q}} < \infty,$$

where $0 < q < \infty$ and v is a weight on $(0,\infty)$ (see Carro et al. [18]).

The classical Lorentz spaces presented above are a specific case of the Lorentz-Λ spaces, since $\Lambda_v^q = L^{p,q}(\mathbb{R}^n)$ for $v(s) = s^{q/p-1}$. Choosing $v(s) = s^{q/p-1}(1+\log^+(1/s))^\alpha$, we obtain the Lorentz-Zygmund spaces $\Lambda_v^q = L^{p,q}(\log L)^\alpha$ (see Bennett–Rudnick [6]). Alternatively, letting $v(s) = s^{q/p-1}(1+\log^+(1/s))^\alpha(1+\log^+\log^+(1/s))^\beta$, then $\Lambda_v^q = L^{p,q}(\log L)^\alpha(\log\log L)^\beta$ are the generalized Lorentz-Zygmund spaces (see Evans et al. [31]).

As shown in Curbera et al. [27], $X = \Lambda_v^q$ has finite upper Boyd index q_X whenever

$$\frac{1}{t}\int_0^t v(x)dx \lesssim v(t), \qquad t > 0.$$

Moreover, if v satisfies

$$\int_t^\infty v(x)x^{-p_0}dx \lesssim \frac{1}{t^{p_0}}\int_0^t v(x)dx, \qquad t > 0,$$

for large enough p_0, $(\Lambda_v^q)^{p_0}$ is a Banach space, so $p(\Lambda_v^q) < \infty$ (see Sawyer [66] and Carro et al. [18]).

Orlicz Spaces

Let ϕ be a Young function; that is, $\phi : [0,\infty) \to [0,\infty)$ is continuous, convex, strictly increasing, and

$$\lim_{t\to 0^+} \frac{\phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty.$$

The *Orlicz space* L^ϕ is the collection of measurable functions f defined on $\mathbb{R}^n$ such that

$$\|f\|_{L^\phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\} < \infty. \tag{5.11}$$

It can be shown that Orlicz spaces are r.i.BFSs. For $X = L^\phi$ and w a weight in $\mathbb{R}^n$, the weighted Orlicz space $X(w)$ is given by replacing dx with $w(x)dx$ in (5.11) (see Bennett–Sharpley [7], Cruz-Uribe et al. [26], and Curbera et al. [27]).

We see that the Orlicz spaces also extend Lebesgue spaces, since in the case that $\phi(x) = x^p$, $1 < p < \infty$, we obtain the Lebesgue space $L^p(\mathbb{R}^n)$. Moreover, the Zygmund spaces $L^p(\log L)^\alpha$ for $1 < p < \infty$ and $\alpha \in \mathbb{R}$, a particular case of the Lorentz-Zygmund spaces of Section 5.1.1, result when $\phi(t) = t^p(1+\log^+ t)^\alpha$. These spaces have Boyd indices $p_X = q_X = p$, and $(L^p(\log L)^\alpha)^{p_0} = L^{pp_0}(\log L)^\alpha$. Other examples of Orlicz spaces include $L^p + L^q$ and $L^p \cap L^q$, which are associated with $\phi(t) \sim \max(t^p, t^q)$ and $\phi(t) = \min(t^p, t^q)$, respectively, and have Boyd indices $p_X = \min(p,q)$ and $q_X = \max(p,q)$.

5.2 Results in Weighted Mixed Lebesgue Spaces

Theorem 4.1.1 also has applications to QBFSs that are not rearrangement invariant. In this section, we obtain fractional Leibniz rules in Triebel-Lizorkin and Besov spaces based on weighted mixed Lebesgue spaces as corollaries of Theorem 4.1.1 and show that particular cases of these estimates include fractional Leibniz rules in weighted mixed Lebesgue spaces. We then analyze these results for spaces with power weights.

5.2.1 Preliminaries

For $0 < p, q < \infty$ and a weight w on $\mathbb{R}^n$, we define the *weighted mixed Lebesgue space* $L^p(L^q(w))$ as the collection of all measurable functions f defined on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(L^q(w))} = \left(\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} |f(x_1, x_2)|^q w(x_1, x_2) dx_2 \right)^{\frac{p}{q}} dx_1 \right)^{\frac{1}{p}} < \infty.$$

We note that taking $p = q$ recovers the weighted Lebesgue space $L^p(w)$.

In this setting, we consider a 'mixed' version of the Muckenhoupt classes, which we denote $A_p(A_q)$. Following the work of Kurtz [49], we define $A_p(A_q)$ to be the collection of weights on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that the following constant is finite:

$$[w]_{A_p(A_q)} = \sup_{Q_1, Q_2} \left(\fint_{Q_1} \left(\fint_{Q_2} w(x_1, x_2) dx_2 \right)^{\frac{p}{q}} dx_1 \right) \left(\fint_{Q_1} \left(\fint_{Q_2} w(x_1, x_2)^{1-q'} dx_2 \right)^{\frac{p'}{q'}} dx_1 \right)^{p-1},$$

where the supremum is taken over all cubes $Q_1 \subset \mathbb{R}^{n_1}$ and $Q_2 \subset \mathbb{R}^{n_2}$. In the case that $p = q$, we have $A_p(A_p) = A_{p,\mathcal{R}}(\mathbb{R}^n)$.

We restrict our attention to weights of the following form: if $0 < p, q < \infty$, we consider $w(x_1, x_2) = u(x_1)v(x_2)$, where $u^{p/q} \in A_\infty(\mathbb{R}^{n_1})$ and $v \in A_\infty(\mathbb{R}^{n_2})$. The following relationship between these types of weights and the traditional Muckenhoupt weights was proven by Kurtz [49].

Lemma 5.2.1 (Lemma 3 from [49])**.** *The weight $w(x_1, x_2) = u(x_1)v(x_2)$ is in $A_p(A_q)$ if, and only if, $u^{p/q} \in A_p(\mathbb{R}^{n_1})$ and $v \in A_q(\mathbb{R}^{n_2})$. Moreover, $[u^{p/q}]_{A_p(\mathbb{R}^{n_1})} \le [w]_{A_p(A_q)}$, $[v]_{A_q(\mathbb{R}^{n_2})} \le [w]_{A_p(A_q)}^{q/p}$, and $[w]_{A_p(A_q)} \le [u^{p/q}]_{A_p(\mathbb{R}^{n_1})} [v]_{A_q(\mathbb{R}^{n_2})}^{p/q}$.*

In general, the mixed Lebesgue spaces $L^p(L^q(w))$ are not necessarily rearrangement invariant (see Blozinski [10]); however, it easily follows that $L^p(L^q(w))$ is a QBFS over $(\mathbb{R}^n, u^{p/q} \times v)$. In this setting, Property P6i. is only required for measurable sets $E \subset \mathbb{R}^n$ such that $E \subset I_1 \times I_2$, where I_1 and I_2 are measurable sets in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ with finite measures with respect to $u^{p/q}(x_1)dx_1$ and $v(x_2)dx_2$, respectively (see Blozinski [10]). In the case that $1 \le p, q < \infty$, $L^p(L^q(w))$ also fulfills Properties P2. and P6ii., where the same change made for P6i. is implemented for P6ii.. We next note that $(L^p(L^q(w)))^{p_0} = L^{pp_0}(L^{qp_0}(w))$, and therefore $p(L^p(L^q(w))) = 1/\min(p, q, 1)$.

The following extrapolation theorem in weighted mixed Lebesgue spaces allows us to also obtain the Fefferman-Stein inequality and necessary equivalences between spaces. The result shown here is similar to that by Kurtz [49, Theorem 2], but for pairs of functions; its proof follows the same argument.

Theorem 5.1. *Let $\mathcal{F}$ be a family of pairs of measurable functions such that for some $1 < p_0 < \infty$ and for all $w_0 \in A_{p_0,\mathcal{R}}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} |f(x)|^{p_0} w_0(x) dx \lesssim \int_{\mathbb{R}^n} |g(x)|^{p_0} w_0(x) dx, \quad \forall (f, g) \in \mathcal{F}.$$

If $1 < p, q < \infty$, then for every $w \in A_p(A_q)$ such that $w(x_1, x_2) = u(x_1)v(x_2)$, we have

$$\|f\|_{L^p(L^q(w))} \lesssim \|g\|_{L^p(L^q(w))}, \quad \forall (f, g) \in \mathcal{F}.$$

Using this, the following Fefferman-Stein inequality holds. Its proof is similar to that of Theorem 2.4.5.

Theorem 5.2.2. *Suppose $0 < p, q < \infty$, $0 < r \le \infty$, $w(x_1, x_2) = u(x_1)v(x_2)$ with $u^{p/q} \in A_\infty(\mathbb{R}^{n_1})$ and $v \in A_\infty(\mathbb{R}^{n_2})$, and $0 < h < \min(p/\tau_{u^{p/q}}, q/\tau_v, r)$. Then for all sequences of measurable functions $\{f_j\}_{j\in\mathbb{Z}}$, we have*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_{n,h}(f_j)|^r \right)^{\frac{1}{r}} \right\|_{L^p(L^q(w))} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(L^q(w))}, \tag{5.12}$$

with the sum in j replaced by the supremum in j when $r = \infty$.

Remark 5.2.3. We note that for $p = q$, $0 < h < \min(p, r)$, and $w \in A_{p/h,\mathcal{R}}(\mathbb{R}^n)$, (2.1) implies (5.12), since $A_{p/h,\mathcal{R}}(\mathbb{R}^n) \subset A_{p/h}(\mathbb{R}^n)$.

This Fefferman-Stein inequality in the weighted mixed Lebesgue spaces immediately implies the boundedness of $\mathcal{M}_{n,h}$ on $L^p(L^q(w))$.

Theorem 5.2.4. *If $0 < p, q < \infty$, $w(x_1, x_2) = u(x_1)v(x_2)$ with $u^{p/q} \in A_\infty(\mathbb{R}^{n_1})$ and $v \in A_\infty(\mathbb{R}^{n_2})$, and $0 < h < \min(p/\tau_{u^{p/q}}, q/\tau_v)$, then*

$$\|\mathcal{M}_{n,h}(f)\|_{L^p(L^q(w))} \lesssim \|f\|_{L^p(L^q(w))}.$$

We then define

$$\tau_{p,q,r}(w) = n\left(\frac{1}{\min(p/\tau_{u^{p/q}}, q/\tau_v, r, 1)} - 1\right) \text{ and } \tau_{p,q}(w) = n\left(\frac{1}{\min(p/\tau_{u^{p/q}}, q/\tau_v, 1)} - 1\right).$$

This implies that for $X = L^p(L^q(w))$,

$$h_{X,r} \geq \min\left(\frac{p}{\tau_{u^{p/q}}}, \frac{q}{\tau_v}, r\right) \quad \text{and} \quad h_X \geq \min\left(\frac{p}{\tau_{u^{p/q}}}, \frac{q}{\tau_v}\right),$$

as well as

$$\tau_r(X) \leq \tau_{p,q,r}(w) \quad \text{and} \quad \tau(X) \leq \tau_{p,q}(w).$$

Let $0 < r \leq \infty$, $s \in \mathbb{R}$, w be a weight on $\mathbb{R}^n$, and $0 < p, q < \infty$. When $X = L^p(L^q(w))$, we denote the weighted homogeneous Triebel-Lizorkin space $\dot{F}^s_{X,r}$ as $\dot{F}^s_{p,q,r}(w)$ and the weighted inhomogeneous Triebel-Lizorkin space $F^s_{X,r}$ as $F^s_{p,q,r}(w)$. Analogous notation applies to the scale of Besov spaces. The corresponding Hardy space is denoted by $H^{p,q}(w)$ and the local Hardy space is denoted by $h^{p,q}(w)$. We observe that since $L^p(L^p(w)) = L^p(w)$, $\dot{F}^s_{p,p,r}(w)$ yields the classical weighted homogeneous Triebel-Lizorkin space defined in Section 3.1, and analogous associations apply for $F^s_{p,p,r}(w)$, $\dot{B}^s_{p,p,r}(w)$, $B^s_{p,p,r}(w)$, $H^{p,p}(w)$, and $h^{p,p}(w)$.

We can establish equivalences between Triebel-Lizorkin spaces, Hardy spaces, and the weighted mixed Lebesgue spaces. These equivalences follow from an application of extrapolation in Theorem 5.1, as well as the relationship between Triebel-Lizorkin and Hardy spaces based on weighted Lebesgue spaces with those based on weighted mixed Lebesgue spaces.

That is, for $1 < p, q < \infty$ and $w(x_1, x_2) = u(x_1)v(x_2) \in A_p(A_q)$,

$$\dot{F}^0_{p,q,2}(w) = F^0_{p,q,2}(w) = H^{p,q}(w) = h^{p,q}(w) = L^p(L^q(w)), \tag{5.13}$$

with equivalent norms.

5.2.2 Fractional Leibniz Rules in $L^p(L^q(w))$

We first state a corollary of Theorem 4.1.1 in the Triebel-Lizorkin and Besov spaces based on weighted mixed Lebesgue spaces. We then present Leibniz rules in weighted mixed Lebesgue spaces.

Corollary 5.2.5. *Let $m \in \mathbb{R}$, $\sigma(\xi, \eta), \xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m, $0 < r \le \infty$, and $0 < p, p_1, p_2, q, q_1, q_2 < \infty$ be such that $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$. Suppose $w_1(x_1, x_2) = u_1(x_1)v_1(x_2)$ and $w_2(x_1, x_2) = u_2(x_1)v_2(x_2)$ with $u_1^{p_1/q_1}, u_2^{p_2/q_2} \in A_\infty(\mathbb{R}^{n_1})$ and $v_1, v_2 \in A_\infty(\mathbb{R}^{n_2})$; set $w(x_1, x_2) = (w_1(x_1, x_2))^{q/q_1}(w_2(x_1, x_2))^{q/q_2}$ and assume $L^p(L^q(w))$ satisfies Properties P7. and P8. with r as given and s as below.*

i) If $s > \tau_{p,q,r}(w)$, then

$$\|T_\sigma(f, g)\|_{\dot{F}^s_{p,q,r}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1,q_1,r}(w_1)} \|g\|_{H^{p_2,q_2}(w_2)} + \|f\|_{H^{p_1,q_1}(w_1)} \|g\|_{\dot{F}^{s+m}_{p_2,q_2,r}(w_2)}. \tag{5.14}$$

ii) If $s > \tau_{p,q}(w)$, then

$$\|T_\sigma(f, g)\|_{\dot{B}^s_{p,q,r}(w)} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1,q_1,r}(w_1)} \|g\|_{H^{p_2,q_2}(w_2)} + \|f\|_{H^{p_1,q_1}(w_1)} \|g\|_{\dot{B}^{s+m}_{p_2,q_2,r}(w_2)}. \tag{5.15}$$

In particular, (5.14) and (5.15) hold for $u = u_1 = u_2$, $v = v_1 = v_2$ with $u^{p_1/q_1}, u^{p_2/q_2} \in A_\infty(\mathbb{R}^{n_1})$ and $v \in A_\infty(\mathbb{R}^{n_2})$, in which case $w = w_1 = w_2$. Moreover, if $s > \tau_{p,q,r}(w)$, then

$$\|T_\sigma(f, g)\|_{\dot{F}^s_{p,q,r}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p,q,r}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{p,q,r}(w)}, \tag{5.16}$$

with corresponding analogous estimates for the Besov spaces if $s > \tau_{p,q}(w)$.

Proof. We first note that $w(x_1,x_2) = u(x_1)v(x_2)$ where $u(x_1) = (u_1(x_1))^{q/q_1}(u_2(x_1))^{q/q_2}$ and $v(x_2) = (v_1(x_2))^{q/q_1}(v_2(x_2))^{q/q_2}$. We then have that $u^{p/q} = (u_1^{p_1/q_1})^{p/p_1}(u_2^{p_2/q_2})^{p/p_2}$ belongs to $A_\infty(\mathbb{R}^{n_1})$ since $u_1^{p_1/q_1}, u_2^{p_2/q_2} \in A_\infty(\mathbb{R}^{n_1})$ and $p/p_1 + p/p_2 = 1$; similarly, $v \in A_\infty(\mathbb{R}^{n_2})$ since $v_1, v_2 \in A_\infty(\mathbb{R}^{n_2})$ and $q/q_1 + q/q_2 = 1$. Moreover, a simple computation shows that

$$\|fg\|_{L^p(L^q(w))} \leq \|f\|_{L^{p_1}(L^{q_1}(w_1))} \|g\|_{L^{p_2}(L^{q_2}(w_2))}.$$

We then apply Theorem 4.1.1 with $X = L^1(L^{q/p}(w)), X_1 = L^1(L^{q_1/p_1}(w_1))$, and $X_2 = L^1(L^{q_2/p_2}(w_2))$, which verify all assumptions required. Therefore, (5.14), (5.15), and (5.16) with its Besov counterpart follow. □

Remark 5.2.6. Corollary 5.2.5 requires $L^p(L^q(w))$ to satisfy P7. and P8. for r and s as stated. We first note that if $0 < p, q < \infty$, $w(x_1,x_2) = u(x_1)v(x_2)$ with $u^{p/q} \in A_\infty(\mathbb{R}^{n_1})$ and $v \in A_\infty(\mathbb{R}^{n_2})$, $0 < r \leq \infty$, and $s \in \mathbb{R}$, then the inclusions $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{p,q,r}(w)$, $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{p,q,r}(w)$, and their inhomogeneous counterparts hold since it can be proved that there exists $N > 0$ such that $(1+|x_1|+|x_2|)^{-N} \in L^p(L^q(w))$. Moreover, under the same assumptions on the weights and indices, the inclusions $\dot{F}^s_{p,q,r}(w) \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$, $\dot{B}^s_{p,q,r}(w) \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$, and their inhomogeneous counterparts, as well as the completeness of the spaces hold in the following cases:

1. In the case that $1 \leq p, q < \infty$, it can be proved that $(1+|x_1|+|x_2|)^{-N} \in (L^p(L^q(w)))'$. The desired inclusions and completeness follow.

2. When $0 < p, q < \infty$ and u and v satisfy

$$\int_{|x_1-y_1|\leq t} u^{p/q}(y_1)dy_1 \geq t^{d_1} \quad \text{and} \quad \int_{|x_2-y_2|\leq t} v(y_2)dy_2 \geq t^{d_2}, \tag{5.17}$$

for all $t > 0$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, and some $d_1, d_2 > 0$, it can be proved that if $f \in L^p(L^q(w)) \cap \mathcal{S}'(\mathbb{R}^n)$ is such that $\operatorname{supp}(\widehat{f}) \subset [-a,a]^n$ for some $a > 0$, then

$$\|f\|_{L^\infty} \lesssim a^{\frac{d_1}{p}+\frac{d_2}{q}} \|f\|_{L^p(L^q(w))}, \tag{5.18}$$

where the implicit constant is independent of f and a. With the estimate (5.18), the proofs of the desired inclusions and completeness follow similar ideas as in those for the classical settings (see Triebel [68, Section 2.3.3]). A proof of (5.18) can be obtained using analogous steps to those in Qui [64, Lemma 2.5]; the unweighted case of (5.18) was treated in Johnsen–Sickel [43]. For later use, we note that if $u(x_1) = |x_1|^a$ and $v(x_2) = |x_2|^b$ with $a \geq 0$ and $b \geq 0$, then u and v satisfy (5.17) with $d_1 = n_1 + ap/q$ and $d_2 = n_2 + b$ (see Grafakos [34, p. 505 - 506]).

Applying the lifting property, the estimates (5.14) and (5.15) can be recast as

$$\begin{aligned} \|D^s(T_\sigma(f,g))\|_{\dot{F}^0_{p,q,r}(w)} \lesssim & \|D^s f\|_{\dot{F}^m_{p_1,q_1,r}(w_1)} \|g\|_{H^{p_2,q_2}(w_2)} \\ & + \|f\|_{H^{p_1,q_1}(w_1)} \|D^s g\|_{\dot{F}^m_{p_2,q_2,r}(w_2)}, \end{aligned} \tag{5.19}$$

$$\begin{aligned} \|D^s(T_\sigma(f,g))\|_{\dot{B}^0_{p,q,r}(w)} \lesssim & \|D^s f\|_{\dot{B}^m_{p_1,q_1,r}(w_1)} \|g\|_{H^{p_2,q_2}(w_2)} \\ & + \|f\|_{H^{p_1,q_1}(w_1)} \|D^s g\|_{\dot{B}^m_{p_2,q_2,r}(w_2)}. \end{aligned} \tag{5.20}$$

Using (5.13), Remark 5.2.6, and that $\tau_{p,q,2}(w) = 0$ for $1 < p, q < \infty$ and $w(x_1, x_2) = u(x_1)v(x_2) \in A_p(A_q)$, we obtain the following Leibniz rules in weighted mixed Lebesgue spaces.

Corollary 5.2.7. *Let $\sigma(\xi,\eta), \xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order zero and $1 < p, p_1, p_2, q, q_1, q_2 < \infty$ be such that $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$. Suppose $w_1(x_1, x_2) = u_1(x_1)v_1(x_2) \in A_{p_1}(A_{q_1}), w_2(x_1, x_2) = u_2(x_1)v_2(x_2) \in A_{p_2}(A_{q_2})$, and $w(x_1, x_2) = (w_1(x_1, x_2))^{q/q_1}(w_2(x_1, x_2))^{q/q_2} \in A_p(A_q)$. If $s > 0$, then*

$$\begin{aligned} \|D^s(T_\sigma(f,g))\|_{L^p(L^q(w))} \lesssim & \|D^s f\|_{L^{p_1}(L^{q_1}(w_1))} \|g\|_{L^{p_2}(L^{q_2}(w_2))} \\ & + \|f\|_{L^{p_1}(L^{q_1}(w_1))} \|D^s g\|_{L^{p_2}(L^{q_2}(w_2))}. \end{aligned}$$

Versions of Corollaries 5.2.5 and 5.2.7 and the corresponding estimates for (5.19) and (5.20) also hold in the inhomogeneous setting with an inhomogeneous Coifman-Meyer multiplier and the operator J^s.

5.2.3 Example: Power Weights

Of particular interest are power weights, or weights of the form $|x_1|^a|x_2|^b$ in the homogeneous setting and $\langle x_1\rangle^a\langle x_2\rangle^b$ in the inhomogeneous setting, where $\langle y\rangle^a = (1+|y|^2)^{a/2}$. In this section, we present examples of fractional Leibniz rules for weighted mixed Lebesgue spaces associated to power weights.

We first recall that for $1<\tau<\infty$, a power weight $|x|^a$, $x\in\mathbb{R}^n$, is in $A_\tau(\mathbb{R}^n)$ if, and only if, $-n<a<n(\tau-1)$. Therefore, in order for $u_j(x_1)=|x_1|^{a_j}$ and $v_j(x_2)=|x_2|^{b_j}$, $j=1,2$, to meet the conditions in Corollary 5.2.5 that $u_j^{p_j/q_j}\in A_\infty(\mathbb{R}^{n_1})$ and $v_j\in A_\infty(\mathbb{R}^{n_2})$, we require that

$$-n_1\frac{q_j}{p_j}<a_j<\infty \quad \text{and} \quad -n_2<b_j<\infty.$$

With these conditions on a_j and b_j, $j=1,2$, Corollary 5.2.5 holds with $w_1(x_1,x_2) = |x_1|^{a_1}|x_2|^{b_1}$, $w_2(x_1,x_2)=|x_1|^{a_2}|x_2|^{b_2}$, and $w(x_1,x_2)=|x_1|^a|x_2|^b$ where

$$\frac{a}{q}=\frac{a_1}{q_1}+\frac{a_2}{q_2} \quad \text{and} \quad \frac{b}{q}=\frac{b_1}{q_1}+\frac{b_2}{q_2}, \tag{5.21}$$

if $L^p(L^q(w))$ satisfies P7. and P8. for r and s as needed, in particular, if $1\le p,q<\infty$ or, if $0<p<1$ or $0<q<1$ and $a,b\ge 0$ (see Remark 5.2.6).

To obtain Leibniz rules in mixed Lebesgue spaces with power weights we use Corollary 5.2.7, which requires $w_1\in A_{p_1}(A_{q_1})$, $w_2\in A_{p_2}(A_{q_2})$, and $w\in A_p(A_q)$. Therefore, we impose further conditions on the exponents a_1,a_2,b_1, and b_2. Using Lemma 5.2.1, we require, for $j=1,2$,

$$\begin{aligned} -n_1\frac{q_j}{p_j}<a_j<\frac{q_jn_1}{p_j'} \quad &\text{and} \quad -n_2<b_j<n_2(q_j-1),\\ -\frac{n_1}{p}<\frac{a_1}{q_1}+\frac{a_2}{q_2}<\frac{n_1}{p'} \quad &\text{and} \quad -\frac{n_2}{q}<\frac{b_1}{q_1}+\frac{b_2}{q_2}<\frac{n_2}{q'}. \end{aligned} \tag{5.22}$$

In particular, in the case $\sigma\equiv 1$ and for those values of a,a_1,a_2,b,b_1, and b_2 as in (5.21) and

(5.22), Corollary 5.2.7 gives

$$\begin{aligned} \|D^s(fg)\|_{L^p(L^q(|x_1|^a|x_2|^b))} &\lesssim \|D^s f\|_{L^{p_1}(L^{q_1}(|x_1|^{a_1}|x_2|^{b_1}))} \|g\|_{L^{p_2}(L^{q_2}(|x_1|^{a_2}|x_2|^{b_2}))} \\ &\quad + \|f\|_{L^{p_1}(L^{q_1}(|x_1|^{a_1}|x_2|^{b_1}))} \|D^s g\|_{L^{p_2}(L^{q_2}(|x_1|^{a_2}|x_2|^{b_2}))} . \end{aligned} \tag{5.23}$$

An analogous result also holds in the inhomogeneous settings.

We note that when $a = a_1 = a_2$ and $b = b_1 = b_2$, the conditions in (5.22) translate to

$$-n_1 \min\left(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q}{p}\right) < a < n_1 \min\left(\frac{q_1}{p_1'}, \frac{q_2}{p_2'}, \frac{q}{p'}\right) \quad \text{and} \quad -n_2 \ < b < n_2(q-1).$$

Using different methods of proof, fractional Leibniz rules in weighted mixed Lebesgue spaces with power weights were also proved in Oh–Wu [63, Theorem 1.6]. In this work, they let $1/2 \le p, q \le \infty$, $1 \le p_1, p_2, q_1, q_2 \le \infty$, and $0 \le a, a_1, a_2, b, b_1, b_2 < \infty$ be such that $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$, and satisfy (5.21). For $s > \max\left(n\left(\frac{1}{\min(p,q,1)} - 1\right), 0\right)$ or s a positive even integer, they obtain

$$\begin{aligned} \|J^s(fg)\|_{L^p(L^q(\langle x_1\rangle^a\langle x_2\rangle^b))} &\lesssim \|J^s f\|_{L^{p_1}(L^{q_1}(\langle x_1\rangle^{a_1}\langle x_2\rangle^{b_1}))} \|g\|_{L^{p_2}(L^{q_2}(\langle x_1\rangle^{a_2}\langle x_2\rangle^{b_2}))} \\ &\quad + \|f\|_{L^{p_1}(L^{q_1}(\langle x_1\rangle^{a_1}\langle x_2\rangle^{b_1}))} \|J^s g\|_{L^{p_2}(L^{q_2}(\langle x_1\rangle^{a_2}\langle x_2\rangle^{b_2}))} . \end{aligned}$$

5.3 Fractional Leibniz Rules in Weighted Morrey Spaces

In this section, we present Leibniz rules in weighted Morrey spaces. For $0 < p \le t < \infty$ and $w \in A_\infty(\mathbb{R}^n)$, the *weighted Morrey space* $M_p^t(w)$ consists of measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{M_p^t(w)} = \sup_{B \subset \mathbb{R}^n} w(B)^{\frac{1}{t}-\frac{1}{p}} \left(\int_B |f(x)|^p\, w(x)dx\right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all balls B contained in $\mathbb{R}^n$. In the case that $t = p$, it is easy to see that we recover the traditional weighted Lebesgue space, $L^p(w)$. In this setting, we denote the homogeneous Triebel-Lizorkin and Besov spaces as $\dot{F}^s_{[p,t],r}(w)$ and $\dot{B}^s_{[p,t],r}(w)$, respectively, and the Hardy space as $H^{[p,t]}(w)$. The inhomogeneous spaces are represented analogously. We refer the reader to Rosenthal-Schmeisser [65] for more details about weighted Morrey spaces and to the works of Kozono-Yamazaki [48], Mazzucato [52], and Izuki et al. [42] regarding Morrey-based Triebel-Lizorkin and Besov spaces.

Applying the same argument as that in Theorem 4.1.1, we obtain the following estimates which recover the result in Naibo–Thomson [60, Theorem 6.2].

Theorem 5.3.1. *Let $m \in \mathbb{R}$, and suppose $\sigma(\xi, \eta), \xi, \eta \in \mathbb{R}^n$, is a Coifman-Meyer multiplier of order m.*

i) If $w \in A_\infty(\mathbb{R}^n)$, $0 < p \le t < \infty$, $0 < p_1 \le t_1 < \infty$, $0 < p_2 \le t_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $1/t = 1/t_1 + 1/t_2$, then for $0 < r \le \infty$ and $s > n\left(\frac{1}{\min(p/\tau_w, r, 1)} - 1\right)$,

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{[p,t],r}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{[p_1,t_1],r}(w)} \|g\|_{H^{[p_2,t_2]}(w)} + \|f\|_{H^{[p_1,t_1]}(w)} \|g\|_{\dot{F}^{s+m}_{[p_2,t_2],r}(w)}, \tag{5.24}$$

where different pairs of p_1, p_2 and t_1, t_2 can be used on the right hand side of the inequality above. Moreover,

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{[p,t],r}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{[p,t],r}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{[p,t],r}(w)}. \tag{5.25}$$

ii) If $w_1, w_2 \in A_\infty(\mathbb{R}^n)$, $w = w_1^{p/p_1} w_2^{p/p_2}$, $0 < p \le t < \infty$, $0 < p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $s > n\left(\frac{1}{\min(p/\tau_w, r, 1)} - 1\right)$, then

$$\begin{aligned} \|T_\sigma(f,g)\|_{\dot{F}^s_{[p,t],r}(w)} \lesssim{}& \|f\|_{\dot{F}^{s+m}_{[p_1,p_1 t/p],r}(w_1)} \|g\|_{H^{[p_2,p_2 t/p]}(w_2)} \\ &+ \|f\|_{H^{[p_1,p_1 t/p]}(w_1)} \|g\|_{\dot{F}^{s+m}_{[p_2,p_2 t/p],r}(w_2)}. \end{aligned} \tag{5.26}$$

Estimates analogous to (5.24) - (5.26) hold in the Besov setting when $s > n\left(\frac{1}{\min(p/\tau_w, 1)} - 1\right)$.

From Theorem 5.3.1, we deduce Leibniz rules in weighted Morrey spaces and Hardy spaces based on weighted Morrey spaces. Through an extrapolation theorem in Morrey spaces given in Duoandikoetxea–Rosenthal [30, Corollary 4.3], for $0 < p \leq t < \infty$ and $w \in A_\infty(\mathbb{R}^n)$, we obtain

$$H^{[p,t]}(w) = \dot{F}^0_{[p,t],2}(w) \quad \text{and} \quad h^{[p,t]}(w) = F^0_{[p,t],2}(w), \tag{5.27}$$

with equivalent quasi-norms. We also note that the lifting property holds in Triebel-Lizorkin and Besov spaces based on weighted Morrey spaces.

These equivalences and (5.24) combined with the lifting property give that under the hypotheses of Theorem 5.3.1 for a Coifman-Meyer multiplier σ of order zero,

$$\|D^s(T_\sigma(f,g))\|_{H^{[p,t]}(w)} \lesssim \|D^s f\|_{H^{[p_1,t_1]}(w)} \|g\|_{H^{[p_2,t_2]}(w)} + \|f\|_{H^{[p_1,t_1]}(w)} \|D^s g\|_{H^{[p_2,t_2]}(w)}. \tag{5.28}$$

From (5.25) and (5.26) we also have

$$\|D^s(T_\sigma(f,g))\|_{H^{[p,t]}(w)} \lesssim \|D^s f\|_{H^{[p,t]}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{H^{[p,t]}(w)} \tag{5.29}$$

and

$$\begin{aligned} \|D^s(T_\sigma(f,g))\|_{H^{[p,t]}(w)} \lesssim & \|D^s f\|_{H^{[p_1,p_1 t/p]}(w_1)} \|g\|_{H^{[p_2,p_2 t/p]}(w_2)} \\ & + \|f\|_{H^{[p_1,p_1 t/p]}(w_1)} \|D^s g\|_{H^{[p_2,p_2 t/p]}(w_2)}. \end{aligned} \tag{5.30}$$

Similarly, for $1 < p \leq t < \infty$ and $w \in A_p(\mathbb{R}^n)$, we have that, through extrapolation in Duoandikoetxea–Rosenthal [30, Theorem 4.1],

$$\dot{F}^0_{[p,t],2}(w) = M^t_p(w) \quad \text{and} \quad F^0_{[p,t],2}(w) = M^t_p(w).$$

Using this, (5.27), (5.28), and the fact that $\|\cdot\|_{M^t_p(w)} \leq \|\cdot\|_{H^{[p,t]}(w)}$ for $0 < p \leq t < \infty$, under the hypotheses of Theorem 5.3.1 with $1 < p_1, p_2 < \infty$, $w \in A_{\min(p_1,p_2)}(\mathbb{R}^n)$, and $m = 0$, we have

$$\|D^s(T_\sigma(f,g))\|_{M^t_p(w)} \lesssim \|D^s f\|_{M^{t_1}_{p_1}(w)} \|g\|_{M^{t_2}_{p_2}(w)} + \|f\|_{M^{t_1}_{p_1}(w)} \|D^s g\|_{M^{t_2}_{p_2}(w)}.$$

as well as an analog to (5.29):

$$\|D^s(T_\sigma(f,g))\|_{M^t_p(w)} \lesssim \|D^s f\|_{M^t_p(w)}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|D^s g\|_{M^t_p(w)}.$$

Moreover, if $0 < p \le t < \infty$ and $1 < p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}(\mathbb{R}^n)$, $w_2 \in A_{p_2}(\mathbb{R}^n)$, and $w = w_1^{p/p_1} w_2^{p/p_2}$ then

$$\|D^s(T_\sigma(f,g))\|_{M^t_p(w)} \lesssim \|D^s f\|_{M^{p_1 t/p}_{p_1}(w_1)}\|g\|_{M^{p_2 t/p}_{p_2}(w_2)} + \|f\|_{M^{p_1 t/p}_{p_1}(w_1)}\|D^s g\|_{M^{p_2 t/p}_{p_2}(w_2)}.$$

We can apply these results with power weights as a specific example. For $1 < p_1, p_2 < \infty$ and $w(x) = |x|^a$, we require that $w \in A_{\min(p_1,p_2)}(\mathbb{R}^n)$; equivalently,

$$-n < a < n\left(\min(p_1,p_2) - 1\right).$$

Then for $\frac{1}{2} < p \le t < \infty$, $1 < p_1 \le t_1 < \infty$, and $1 < p_2 \le t_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $1/t = 1/t_1 + 1/t_2$, and $w(x) = |x|^a$, with a as above,

$$\|D^s(T_\sigma(f,g))\|_{M^t_p(|x|^a)} \lesssim \|D^s f\|_{M^{t_1}_{p_1}(|x|^a)}\|g\|_{M^{t_2}_{p_2}(|x|^a)} + \|f\|_{M^{t_1}_{p_1}(|x|^a)}\|D^s g\|_{M^{t_2}_{p_2}(|x|^a)},$$

where $s > n\left(\frac{1}{\min(p,1)} - 1\right)$ if $a \le 0$ and $s > n\left(\frac{1}{\min\left(\frac{p}{a/n+1},1\right)} - 1\right)$ if $a > 0$. Similarly, we have

$$\|D^s(T_\sigma(f,g))\|_{M^t_p(|x|^a)} \lesssim \|D^s f\|_{M^t_p(|x|^a)}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|D^s g\|_{M^t_p(|x|^a)}.$$

Further, suppose $0 < p \le t < \infty$ and $1 < p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$; also let $w_1(w) = |x|^{a_1}$ and $w_2(x) = |x|^{a_2}$ with

$$-n < a_j < n(p_j - 1), \quad j = 1, 2,$$

and $w = w_1^{p/p_1} w_2^{p/p_2} = |x|^b$, where $b = p(a_1/p_1 + a_2/p_2)$. Then we have

$$\begin{aligned} \|D^s(T_\sigma(f,g))\|_{M_p^t(|x|^b)} \lesssim & \|D^s f\|_{M_{p_1}^{p_1 t/p}(|x|^{a_1})} \|g\|_{M_{p_2}^{p_2 t/p}(|x|^{a_2})} \\ & + \|f\|_{M_{p_1}^{p_1 t/p}(|x|^{a_1})} \|D^s g\|_{M_{p_2}^{p_2 t/p}(|x|^{a_2})}, \end{aligned} \tag{5.31}$$

with $s > n\left(\frac{1}{\min(p,1)} - 1\right)$ if $b \leq 0$ and $s > n\left(\frac{1}{\min(\frac{p}{b/n+1},1)} - 1\right)$ if $b > 0$.

Moreover, corresponding versions of Corollary 5.3.1 and (5.28) - (5.31) also hold with an inhomogeneous Coifman-Meyer multiplier and the inhomogeneous differential operator J^s.

5.4 Fractional Leibniz Rules in Variable Lebesgue Spaces

We now discuss applications of Theorem 4.1.1 in the setting of variable Lebesgue spaces. We begin with some definitions and notation followed by results in this context.

Let $\mathcal{P}_0$ be the collection of measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 0 \quad \text{and} \quad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty.$$

For $p(\cdot) \in \mathcal{P}_0$, the *variable Lebesgue space* $L^{p(\cdot)}$ is the class of all measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\} < \infty.$$

With this quasi-norm, $L^{p(\cdot)}$ is a QBFS (BFS when $p_- \geq 1$). Note that if $p(x) = p_0$, $0 < p_0 < \infty$, then $L^{p(\cdot)}$ coincides with $L^{p_0}(\mathbb{R}^n)$ with equality of quasi-norms. Similar to the traditional Lebesgue spaces,

$$\||f|^r\|_{L^{p(\cdot)}} = \|f\|_{L^{rp(\cdot)}}^r, \tag{5.32}$$

and, if $p_- \geq 1$, $(L^{p(\cdot)})' = L^{p'(\cdot)}$, where $p'(\cdot)$ is defined to be the pointwise conjugate exponent of $p(\cdot)$; that is, $p'(\cdot)$ is such that

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad \forall x \in \mathbb{R}^n.$$

Let $\mathcal{D}$ be the collection of $p(\cdot) \in \mathcal{P}_0$ such that the Hardy-Littlewood maximal operator $\mathcal{M}_n$ is bounded on $L^{p(\cdot)}$. A necessary condition for $p(\cdot) \in \mathcal{D}$ is $p_- > 1$, while log-Hölder continuity conditions are sufficient. Moreover, it can be proved that the following conditions are equivalent for $p(\cdot) \in \mathcal{P}_0$ such that $p_- > 1$:

a) $p(\cdot) \in \mathcal{D}$;

b) $p'(\cdot) \in \mathcal{D}$;

c) $p(\cdot)/q \in \mathcal{D}$ for some $1 < q < p_-$;

d) $(p(\cdot)/q)' \in \mathcal{D}$ for some $1 < q < p_-$.

See Cruz-Uribe et al. [22, Theorem 1.2] and references therein.

A version of Hölder's inequality holds for variable Lebesgue spaces: if $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}_0$ are such that $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, then

$$\|fg\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}}, \quad \forall f \in L^{p_1(\cdot)},\ g \in L^{p_2(\cdot)}.$$

The case for exponents in $\mathcal{P}_0$ such that $p_- \geq 1$ is given in Cruz-Uribe–Fiorenza [25]; the general case follows from the latter case and (5.32).

Jensen's inequality combined with (5.32) give that if $p(\cdot) \in \mathcal{P}_0$ and $0 < \tau_0 < \infty$ is such that $p(\cdot)/\tau_0 \in \mathcal{D}$, then $p(\cdot)/\tau \in \mathcal{D}$ for $0 < \tau < \tau_0$. Therefore, we define $\mathcal{P}_0^*$ to be the collection of $p(\cdot) \in \mathcal{P}_0$ such that $p(\cdot)/\tau_0 \in \mathcal{D}$ for some $\tau_0 > 0$ and, for $p(\cdot) \in \mathcal{P}_0^*$, we set

$$\tau_{p(\cdot)} = \sup \{\tau > 0 : p(\cdot)/\tau \in \mathcal{D}\}.$$

We observe that $\tau_{p(\cdot)} \leq p_-$. The following version of the Fefferman-Stein inequality is obtained by using Cruz-Uribe–Fiorenza [25, Section 5.6.8] and (5.32). For $p(\cdot) \in \mathcal{P}_0^*$, $0 < r \leq \infty$, and $0 < h < \min(\tau_{p(\cdot)}, r)$,

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_{n,h}(f_j)|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}}.$$

In particular, for $0 < h < \tau_{p(\cdot)}$, $\mathcal{M}_{n,h}$ is bounded on $L^{p(\cdot)}$; i.e.,

$$\|\mathcal{M}_{n,h}(f)\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}} .$$

As a consequence, if $X = L^{p(\cdot)}$ and $p(\cdot) \in \mathcal{P}_0^*$, we have $h_{X,r} \geq \min(\tau_{p(\cdot)}, r)$ and $h_X \geq \tau_{p(\cdot)}$, as well as

$$\tau_r(X) \leq n\left(\frac{1}{\min(\tau_{p(\cdot)}, r, 1)} - 1\right) \quad \text{and} \quad \tau(X) \leq n\left(\frac{1}{\min(\tau_{p(\cdot)}, 1)} - 1\right).$$

For $s \in \mathbb{R}$, $0 < r \leq \infty$, and $p(\cdot) \in \mathcal{P}_0$, we denote the homogeneous Triebel-Lizorkin and Besov spaces in this setting as $\dot{F}^s_{p(\cdot),r}$ and $\dot{B}^s_{p(\cdot),r}$, respectively. More general variable exponent Triebel-Lizorkin and Besov spaces, where r and s are replaced with functions are considered in Diening et al. [29] and Almeida–Hästö [1]. The Hardy space with variable exponent $p(\cdot) \in \mathcal{P}_0$ will be denoted $H^{p(\cdot)}$. The corresponding inhomogeneous spaces are denoted analogously.

We then obtain the following fractional Leibniz rules in variable exponent Triebel-Lizorkin and Besov spaces as a corollary to Theorem 4.1.1. This result was also proven directly in Naibo–Thomson [60, Theorem 6.4] using methods similar to those for Theorem 4.1.1.

Corollary 5.4.1. *Let $m \in \mathbb{R}, \sigma(\xi, \eta), \xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m, $0 < r \leq \infty$, $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}_0^*$ be such that $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, and assume $L^{p(\cdot)}$ satisfies Properties P7. and P8..*

i) If $s > n\left(\frac{1}{\min(\tau_{p(\cdot)}, r, 1)} - 1\right)$, it holds that

$$\|T_\sigma(f, g)\|_{\dot{F}^s_{p(\cdot),r}} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1(\cdot),r}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \|g\|_{\dot{F}^{s+m}_{p_2(\cdot),r}} ; \tag{5.33}$$

ii) if $s > n\left(\frac{1}{\min(\tau_{p(\cdot)}, 1)} - 1\right)$, it holds that

$$\|T_\sigma(f, g)\|_{\dot{B}^s_{p(\cdot),r}} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1(\cdot),r}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \|g\|_{\dot{B}^{s+m}_{p_2(\cdot),r}} , \tag{5.34}$$

where different pairs of $p_1(\cdot)$ *and* $p_2(\cdot)$ *can be used on the right hand sides of* (5.33) *and* (5.34)*. Moreover, if* $s > n\left(\frac{1}{\min(\tau_{p(\cdot)},r,1)} - 1\right)$*, then*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p(\cdot),r}} \lesssim \|f\|_{\dot{F}^s_{p(\cdot),r}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^s_{p(\cdot),r}}, \tag{5.35}$$

with a corresponding estimate holding in the Besov setting for $s > n\left(\frac{1}{\min(\tau_{p(\cdot)},1)} - 1\right)$*.*

Proof. We apply Theorem 4.1.1 with $X = L^{p(\cdot)}$, $X_1 = L^{p_1(\cdot)}$, $X_2 = L^{p_2(\cdot)}$, and $p = p_1 = p_2 = 1$. Then $X^p = L^{p(\cdot)}$, $X_1^{p_1} = L^{p_1(\cdot)}$, and $X_2^{p_2} = L^{p_2(\cdot)}$ fulfill all conditions of Theorem 4.1.1, implying that (5.33), (5.34), and (5.35) with its Besov space counterpart follow. Finally, Remark 4.1.3 implies that different pairs of $p_1(\cdot)$ and $p_2(\cdot)$ can be used on the right hand sides of the estimates in (5.33) and (5.34), provided that both pairs satisfy the Hölder relationship with $p(\cdot)$. □

As in the settings of weighted mixed Lebesgue and weighted Morrey spaces, we can apply the lifting property (see also Diening et al. [29, Lemma 4.4]) and write the estimates (5.33) and (5.34) as

$$\|D^s(T_\sigma(f,g))\|_{\dot{F}^0_{p(\cdot),r}} \lesssim \|D^s f\|_{\dot{F}^m_{p_1(\cdot),r}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \|D^s g\|_{\dot{F}^m_{p_2(\cdot),r}}, \tag{5.36}$$

$$\|D^s(T_\sigma(f,g))\|_{\dot{B}^0_{p(\cdot),r}} \lesssim \|D^s f\|_{\dot{B}^m_{p_1(\cdot),r}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \|D^s g\|_{\dot{B}^m_{p_2(\cdot),r}}; \tag{5.37}$$

(5.35) and its Besov counterpart can be also be rewritten in a similar manner.

Now, by using Theorem 1.3 in Cruz-Uribe et al. [22], an extrapolation theorem that allows to deduce inequalities in variable Lebesgue spaces from weighted inequalities in classical Lebesgue spaces, it follows that if $p(\cdot) \in \mathcal{P}_0^*$, then

$$\dot{F}^0_{p(\cdot),2} = H^{p(\cdot)} \quad \text{and} \quad F^0_{p(\cdot),2} = h^{p(\cdot)}. \tag{5.38}$$

With this in mind, using (5.36) and (5.38), when σ is a Coifman-Meyer multiplier of order zero, Corollary 5.4.1 gives

$$\|D^s(T_\sigma(f,g))\|_{H^{p(\cdot)}} \lesssim \|D^s f\|_{H^{p_1(\cdot)}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \|D^s g\|_{H^{p_2(\cdot)}}\,; \tag{5.39}$$

in particular, when $\sigma \equiv 1$,

$$\|D^s(fg)\|_{H^{p(\cdot)}} \lesssim \|D^s f\|_{H^{p_1(\cdot)}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \|D^s g\|_{H^{p_2(\cdot)}}\,. \tag{5.40}$$

Moreover, for $p(\cdot) \in \mathcal{D}$, by Theorem 3.2.6 applied with power q to $X = L^{p(\cdot)/q}$, where q is as in Item c) on Page 79, we have

$$\dot{F}^0_{p(\cdot),2} = F^0_{p(\cdot),2} = H^{p(\cdot)} = h^{p(\cdot)} = L^{p(\cdot)}, \tag{5.41}$$

with equivalence in norm (see also Diening et al. [29, Theorem 4.2]).

Thus, when $p_1(\cdot), p_2(\cdot) \in \mathcal{D}$, using (5.39), (5.41), and (3.8), we obtain

$$\|D^s(T_\sigma(f,g))\|_{L^{p(\cdot)}} \lesssim \|D^s f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}} + \|f\|_{L^{p_1(\cdot)}} \|D^s g\|_{L^{p_2(\cdot)}}\,, \tag{5.42}$$

and, in the specific case that $\sigma \equiv 1$,

$$\|D^s(fg)\|_{L^{p(\cdot)}} \lesssim \|D^s f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}} + \|f\|_{L^{p_1(\cdot)}} \|D^s g\|_{L^{p_2(\cdot)}}\,.$$

Corresponding estimates for (5.35) also hold.

We note that (5.42) was proved in Cruz-Uribe–Naibo [24, Theorem 3.1] using bilinear extrapolation techniques.

Versions of Corollary 5.4.1, (5.36), (5.37), (5.39), (5.40), and (5.42) hold in the inhomogeneous setting with an inhomogeneous Coifman-Meyer multiplier and the operator J^s.

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