GEOMETRIC APPROACH TO HALL ALGEBRAS AND CHARACTER SHEAVES

by

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B.S., Harbin Engineering University, China, 2000
M.S., Harbin Engineering University, China, 2006

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

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Department of Mathematics
College of Arts and Sciences

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Abstract

A representation of a quiver Γ over a commutative ring R assigns an R-module to each vertex and an R-linear map to each arrow. In this dissertation, we consider $R = k[t]/(t^n)$ and all R-free representations of Γ which assign a free R-module to each vertex. The category, denoted by $Rep_R^f(\Gamma)$, containing all such representations is not an abelian category, but rather an exact category.

In this dissertation, we firstly study the Hall algebra of the category $Rep_R^f(\Gamma)$, denote by $\mathcal{H}(R\Gamma)$, for a loop-free quiver Γ. A geometric realization of the composition subalgebra of $\mathcal{H}(R\Gamma)$ is given under the framework of Lusztig's geometric setting. Moreover, the canonical basis and a monomial basis of this subalgebra are constructed by using perverse sheaves. This generalizes Lusztig's result about the geometric realization of quantum enveloping algebra. As a byproduct, the relation between this subalgebra and quantum generalized Kac-Moody algebras is obtained.

If Γ is a Jordan quiver, which is a quiver with one vertex and one loop, each representation in $Rep_R^f(\Gamma)$ gives a matrix over R when we fix a basis of the free R-module. An interesting case arises when considering invertible matrices. It then turns out that one is dealing with representations of the group $GL_m(k[t]/(t^n))$. Character sheaf theory is a geometric character theory of algebraic groups. In this dissertation, we secondly construct character sheaves on $GL_m(k[t]/(t^2))$. Then we define an induction functor and restriction functor on these perverse sheaves.

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Dedication

This dissertation is dedicated to
my parents, my wife and my daughter
for their love and endless support.

Chapter 1

Introduction

The first remarkable work in representation theory of quivers is due to Gabriel, see [15]. He proves that a quiver has finite many isomorphism classes of indecomposable representations if and only if its underlying graph is a Dynkin diagram. Such quivers are called of finite type. In this case the dimension vector of an indecomposable representation is a positive root of the semisimple Lie algebra sharing the same Dynkin diagram. Furthermore, the isomorphism classes of indecomposable representations are in one to one correspondence to positive roots of corresponding semisimple Lie algebra.

Gabriel's theorem is generalized to the extended Dynkin diagrams by Donovan, Frieslich, Nazarova, and finally to more general quivers by Kac, see [11, 22, 46]. The Lie algebra corresponding to an extended Dynkin diagram is an affine Kac-Moody Lie algebra. In this case, it has infinite many roots and its roots belong to two classes: real roots, obtained from the simple roots by the Weyl group action, and the rest–imaginary roots. There is a unique indecomposable representation corresponding to each positive real root. Moreover, for each positive imaginary root, the isomorphism classes of indecomposable representations are parametrized by the projective line together with finitely many points. An quiver is called of tame type if the indecomposable representations of a given dimension may be parametrized by a finite number of rational curves.

All other quivers are called wild type. Classifying all indecomposable representations of wild type quivers is a so-called "hopeless" problem. The representation theory of wild type quivers is as complicated as the representation theory of $k\langle x, y\rangle$, the non commutative free associative algebra in two variables.

The representations of quivers over local rings $R = k[t]/(t^n)$ have a close relation with the representations over field k. For example, the category of R-free representations of Jordan quiver over R is equivalent to the category of representations of algebra $k[x, y]/(x^n)$. On the other hand, a $k[t]/(t^n)$-free representation of quivers is a deformation of a representation of the same quiver over k.

Another remarkable work in representation theory of quivers is Ringel's work about Hall algebras. The Hall algebra $\mathcal{H}(\mathcal{A})$ of an finitary abelian category $\mathcal{A}$ is defined to be the $\mathbb{C}$-vector space with a basis consisting of all isomorphism classes $\{[M]\}_{M\in Ob(\mathcal{A})}$. The multiplication between two basis elements $[M]$ and $[N]$ is a linear combination of elements $[P]$ which runs through the set of extensions of M by N with coefficient counting the number of certain submodules. This multiplication is called a Hall multiplication. The first example of Hall algebra can be traced back to 1901. In [54], Steinitz described an idea of producing an algebra from isomorphism classes of finite abelian p-groups. In [19], Phillip Hall rediscovered the idea. The algebra is now called Hall-Steinitz algebra. Now many operations in algebra can be thought of as Hall multiplications. For instance, the parabolic induction for modular representations; the parabolic induction of Eisenstein series for function fields, see [26]; and the multiplication for symmetric functions etc., see [43].

Ringel shows that Hall algebra, $\mathcal{H}(k\Gamma)$, associated with a Dynkin quiver Γ is generated by simple representations. Moreover, after twisting by using the Euler character on the Grothendick group, $K_0(k\Gamma)$, one may obtain the twisted Hall algebra $\mathcal{H}_\star(k\Gamma)$ which is isomorphic to U_q^+ as an algebra, see [50, 51]. Here U_q^+ is the positive part of the quantum enveloping algebra of the Lie algebra corresponding to the given quiver. Later on, J.A. Green, in [17], defined a coalgebraic structure on $\mathcal{H}_\star(k\Gamma)$. Additionally Xiao, in [56], defined the antipode of $\mathcal{H}_\star(k\Gamma)$. This makes $\mathcal{H}_\star(k\Gamma)$ a Hopf algebra.

After Ringel's work relating Hall algebras with quantum groups, the interest for Hall algebras reaches to more general mathematics, including topics describing physics invariants.

Motivated by Ringel's work, in [35], Lusztig gives a geometric realization of U_q^+ by using perverse sheaves on moduli spaces of representations of quivers. Later on, Lusztig's work is generalized to affine case and more general quivers, see [29, 30, 37]. In [23], Kang and Schiffmann give a geometric realization of the positive part of the quantized enveloping algebra of a generalized Kac-Moody Lie algebra by using quivers with multiple loops. Via relations between representations of quivers over a field k and representations of quivers over the local rings $k[t]/(t^n)$, one may expect to approach Kang and Schiffmann's result through the representations of quiver over the local ring $R = k[t]/(t^n)$. This question is partially answered in Chapter 4.

More recently, Hall algebras are defined for different categories. Here are some examples. Hubery defines the Hall algebra over exact category, see [20]; Peng-Xiao, Töen and Kapranov construct Hall algebras for some derived categories, see [4, 24, 47, 55, 57]; Caldero, Chapoton and Keller build some relations between Hall algebra and cluster categories, see [5, 6]; Joyce defines the Hall algebra as the algebra of constructible functions on moduli stack of objects in an abelian category, see [21]. And more and more relations between Hall algebra and other mathematical objects are discovered. Kontsevich and Soibelman defines cohomological Hall algebra which is related to Donaldson-Thomas invariants, see [28, 49]; Kapranov, Schiffmann and Vasserot study the Hall algebras over the category of vector bundles on smooth irreducible projective curves and give a purely Galois theoretic interpretation of these Hall algebras by applying the Langlands correspondence for the groups GL_m over functional fields, see [25, 52].

What is an geometric approach to Hall algebras good for? Firstly, Hall numbers, which are structure constants of the Hall algebra, are numbers of rational points of certain varieties. So one doesn't need to prove the existence of Hall polynomials. Another generalization is the motivic version of Hall algebra. If we consider representations of quivers over infinite field, such as $\mathbb{C}$. The Hall numbers, which counts numbers of filtration, are infinite numbers.

Instead of numbers of rational points of varieties, one uses motivic measures of varieties as structure constants. The obtained Hall algebra is called motivic Hall algebra, see [27].

Secondly, it is easy to construct the canonical basis of Hall algebras through geometric approach. Lusztig constructs the canonical basis of U_q^+ by using simple perverse sheaves on moduli spaces of representations of quivers. Kashiwara constructs independently such basis, which is called the global crystal basis, by combinatorial methods. These bases provide a uniform description of irreducible finite-dimensional modules. Moreover these bases have many remarkable properties such as integrality and positivity of structure constants etc. Lusztig's construction can be thought of as a categorification of the Hall algebra (or U_q^+). The category $\mathcal{Q} := \oplus_V \mathcal{Q}_V$ (see definition in section 2.2) is now called Hall category.

The isomorphic classes of representations of quivers give a natural basis of Hall algebra associated to Dynkin quivers, which is a PBW type basis. Some entries of the transition matrix between the canonical basis and a PBW type basis is Poincare polynomials at a certain point in the representations varieties, see [34]. On the other hand, Lusztig's sheaves (see section 2.2) give a monomial basis of the Hall algebra.

In Chapter 4, we study the Hall algebra, $\mathcal{H}(R\Gamma)$, of the category of R-free representations of a loops free quiver Γ over $R = k[t]/(t^n)$. There is a certain relation between the composition subalgebra of $\mathcal{H}(R\Gamma)$ and quantum generalized Kac-Moody algebras. Under the framework of Lusztig's geometric approach, we construct the canonical basis and a monomial basis of the composition subalgebra of $\mathcal{H}(R\Gamma)$.

The geometry of conjugacy classes of $m \times m$-matrices for various m is related the full subcategory of category of representations of the Jordan quiver over k with a fixed dimension. Green in [16] gives all irreducible complex characters of finite groups $GL_m(\mathbb{F}_q)$, which are complex valued class functions on $GL_m(\mathbb{F}_q)$. Later on, different ways are used to approach to the irreducible characters of $GL_m(\mathbb{F}_q)$. In [58], Zelevensky uses Hopf algebra (subalgebra of Hall algebra associated the Jordan quiver) approach to the irreducible characters of $GL_m(\mathbb{F}_q)$ for all m. It is important to note that $GL_m(\mathbb{F}_q)$ is the fixed point set, usually denoted by

$GL_m(\overline{\mathbb{F}}_q)^F$, of Frobenius morphism F in $GL_m(\overline{\mathbb{F}}_q)$.

Motivated by Macdonald's conjecture, which claims that there should be a map from general position complex characters of F-stable maximal tori to irreducible complex representations of G^F, Deligne and Lusztig in [10] construct Deligne-Lusztig characters for any reductive algebraic group by using l-adic étale cohomology with compact support. Deligne-Lusztig characters are certain virtual characters. Furthermore, every irreducible character of G^F is a constituent of some Deligne-Lusztig characters.

Using l-adic cohomology to define the Deligne-Lusztig characters, Lusztig in [34] uses intersection l-adic cohomology complexes, and introduces certain simple perverse sheaves on connected reductive algebraic groups, which are called character sheaves. Character sheaf theory provides a remarkable geometric interpretation for the complex characters of finite groups of Lie type. The F-invariant character sheaves are closely related to the irreducible characters of the group G^F. Character sheaf theory allows us to study uniformly representations of G^F for various F. Later on, Lusztig generalizes his construction to disconnected reductive algebraic groups, parabolic and more general algebraic groups. In 2008, Boyarchenko and Drinfeld construct character sheaves for unipotent algebraic group in [3].

In [39], Lusztig considers the representations of reductive groups over the finite ring $R = \mathbb{F}_q[t]/(t^n)$ and gives some virtual characters of $GL_m(R)$. Lusztig further constructs in [40] some generalized character sheaves of $GL_m(\overline{\mathbb{F}}_q[t]/(t^n))$. It is called generalized character sheaves because they behave like character sheaves but it is not clear that they are intersection complexes. And he conjectures that all these generalized character sheaves are intersection cohomology complexes. Moreover, Lusztig mentions that the construction and conjecture also make sense when GL_m is replaced by any reductive group. In the same paper, he proves his conjecture for $n = m = 2$.

In this dissertation, we secondly consider $GL_m(\overline{\mathbb{F}}[t]/(t^n)) = GL_m(\overline{\mathbb{F}}) \ltimes H$, where H is a unipotent algebraic group. This is a mixed group. i.e. neither reductive nor solvable group. The complete description of representations of an arbitrary mixed group is a hopeless problem. In [12], Drozd treats such matrix problems in terms of representations of bocses. In general, one may ask if we can construct character sheaves of $G \ltimes H$ with knowing information of character sheaves of G and H, such as structure of stabilizers of character sheaves of H in G. From algebraic point of view, little group method gives a way to list irreducible characters of semidirect product of two finite groups. Little group method is a special case of Clifford theory. The question turns out to be a geometric version of Clifford theory. This question is partially answered in Chapter 5. In Chapter 5, we construct character sheaves on $GL_m(\overline{\mathbb{F}}_q[t]/(t^n))$. Then we define an induction functor and restriction functor on them.

More generally, for any reductive algebraic group G or even more general algebraic groups, there is a natural group homomorphism $G(k[t]/(t^n)) \xrightarrow{\pi} G(k)$. Then $H := \mathrm{Ker}(\pi)$ is a unipotent algebraic group. If $n = 2$, H is a tangent space of $G(k)$ and can be thought as the Lie algebra of $G(k)$. The approach in Chapter 5 should also apply to this case. This is one of the questions we will work later. This dissertation opens a lot of interesting questions, for example, character sheaves of algebraic group $G(k[t]/(t^n))$ for $n > 2$; character sheaves of algebraic group $G(k[t])$ etc. The answer for the second question will be related to the local Langlands program over function field.

Chapter 2

Perverse sheaves

In this chapter, we will quickly review the theory of perverse sheaves. For reference, we refer to Chapter 8 in [41]. The reader can also find these in [1, 9, 13].

Let k be the algebraic closure of $\mathbb{F}_q$, and let all algebraic varieties be over k and of finite type separable.

2.1 Perverse sheaves

Let X be an algebraic variety. Denote by $\mathcal{D}(X) = \mathcal{D}^b_c(X)$ the bounded derived category of $\overline{\mathbb{Q}}_l$-constructible sheaves. Here l is a fixed prime number which is invertible in k, and $\overline{\mathbb{Q}}_l$ is the algebraic closure of the field $\mathbb{Q}_l$ of l-adic numbers. Objects of $\mathcal{D}(X)$ are referred to as complexes. For a complex $K \in \mathcal{D}(X)$, denote by $\mathcal{H}^n(K)$ the n-th cohomology sheaf of K. For any integer j, let $[j] : \mathcal{D}(X) \to \mathcal{D}(X)$ be the shift functor which satisfies $\mathcal{H}^n(K[j]) = \mathcal{H}^{n+j}(K)$.

Let $f : X \to Y$ be a morphism of algebraic varieties. There are functors $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$, $f_* : \mathcal{D}(X) \to \mathcal{D}(Y)$, $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ (direct image with compact support), and $f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$.

Let $p_X : X \to \{\text{pt}\}$ be the morphism from an algebraic variety X to a point. Denote by $\mathbf{1} = \mathbf{1}_X$ the $\overline{\mathbb{Q}}_l$-constant sheaf on X. The complex $\dot{\omega}_X = (p_X)^!(\mathbf{1}_{\text{pt}})$ is called the dualizing complex on X. And $\mathbb{D}K = R\mathcal{H}om(K, \dot{\omega}_X) \in \mathcal{D}(X)$ is called the Verdier dual of $K \in \mathcal{D}(X)$.

In this dissertation, the perversity refers to the middle perversity. To define perverse sheaves, we first introduce two full subcategories which define a t-structure on $\mathcal{D}(X)$. An object $K \in \mathcal{D}(X)$ is said to satisfy

$$\begin{cases} (1) & \textit{support condition} \text{ if } \dim \mathrm{Supp}\mathcal{H}^n(K) \le -n, \forall n; \\ (2) & \textit{cosupport condition} \text{ if } \dim \mathrm{Supp}\mathcal{H}^n(\mathbb{D}K) \le -n, \forall n. \end{cases}$$

Let $\mathcal{D}(X)^{\le 0}$ be the full subcategory of $\mathcal{D}(X)$ whose objects satisfy support condition. In particular, $\mathcal{H}^n(K) = 0$ for $n > 0$. Let $\mathcal{D}(X)^{\ge 0}$ be the full subcategory of $\mathcal{D}(X)$ whose objects satisfy cosupport condition. Then $(\mathcal{D}(X)^{\le 0}, \mathcal{D}(X)^{\ge 0})$ defines a t-structure on $\mathcal{D}(X)$.

Let $\mathcal{M}(X)$ be the full subcategory of $\mathcal{D}(X)$ whose objects are in $\mathcal{D}(X)^{\le 0} \bigcap \mathcal{D}(X)^{\ge 0}$. The objects of $\mathcal{M}(X)$ are called perverse sheaves on X. $\mathcal{M}(X)$ is the heart of the t-structure and is actually an abelian category in which all objects have finite length. The simple objects of $\mathcal{M}(X)$ are given by the Deligne-Goresky-Macpherson intersection cohomology complexes corresponding to various smooth irreducible subvarieties of X and to irreducible local systems on them.

Let $\tau_{\le 0}$ (resp. $\tau_{\ge 0}$) : $\mathcal{D}(X) \to \mathcal{D}(X)$ be the truncation functor. Then we have a functor

$$\begin{aligned} {}^pH^0 : \mathcal{D}(X) &\to \mathcal{M}(X) \\ K &\mapsto \tau_{\ge 0}\tau_{\le 0}K. \end{aligned}$$

Define the perverse cohomology functor ${}^pH^n : \mathcal{D}(X) \to \mathcal{M}(X)$ as ${}^pH^n(K) = {}^pH^0(K[n])$.

A complex $K \in \mathcal{D}(X)$ is called semisimple if ${}^pH^n(K)$ is semisimple in $\mathcal{M}(X)$ for all n and K is isomorphic to $\oplus_n {}^pH^n(K)[-n]$ in $\mathcal{D}(X)$.

For any integer n, denote by $\mathcal{M}(X)[n]$ the full subcategory of $\mathcal{D}(X)$ whose objects are of the form $K[n]$ for some $K \in \mathcal{M}(X)$.

2.2 Properties of functors

Let $f : X \to Y$ be a morphism of algebraic varieties. The functors $f^*, f_!, f^!, f_*, [j]$ and the Verdier dual $\mathbb{D}$ satisfy the following properties.

2.2.1 Adjunction

If $f : X \to Y$, then (f^*, f_*) and $(f_!, f^!)$ are adjoint pairs. i.e. for any $A \in \mathcal{D}(X),\ B \in \mathcal{D}(Y)$,

(1) $\mathrm{Hom}_{\mathcal{D}(X)}(f^*B, A) = \mathrm{Hom}_{\mathcal{D}(Y)}(B, f_*A)$;

(2) $\mathrm{Hom}_{\mathcal{D}(Y)}(f_!A, B) = \mathrm{Hom}_{\mathcal{D}(X)}(A, f^!B)$.

2.2.2 Pull back

If $f : X \to Y$ is smooth with connected fibers of dimension d, let $\widetilde{f} = f^* \circ [d]$, then we have the following properties,

(1) $f^! = f^*[2d]$ and $\mathbb{D}f^*(B) = f^!(\mathbb{D}B)$. (We will ignore the Tate twist.)

(2) $K \in \mathcal{D}(Y)^{\leq 0} \Leftrightarrow \widetilde{f}K \in \mathcal{D}(X)^{\leq 0}$.

(3) $K \in \mathcal{D}(Y)^{\geq 0} \Leftrightarrow \widetilde{f}K \in \mathcal{D}(X)^{\geq 0}$.

(4) $K \in \mathcal{M}(Y) \Leftrightarrow \widetilde{f}K \in \mathcal{M}(X)$.

(5) ${}^pH^i(\widetilde{f}K) = \widetilde{f}({}^pH^i(K))$.

(6) If $K \in \mathcal{D}Y^{\leq 0}$ and $K' \in \mathcal{D}Y^{\geq 0}$, then

$\mathrm{Hom}_{\mathcal{D}(Y)}(K, K') = \mathrm{Hom}_{\mathcal{D}(X)}(\widetilde{f}K, \widetilde{f}K')$.

(7) $\widetilde{f} : \mathcal{M}(Y) \to \mathcal{M}(X)$ is a fully faithful functor.

(8) If $K \in \mathcal{M}(Y)$ and $K' \in \mathcal{M}(X)$ is a subquotient of $\widetilde{f}K \in \mathcal{M}(X)$, then K' is isomorphic to $\widetilde{f}K_1$ for some $K_1 \in \mathcal{M}(Y)$.

Lemma 1. *If $f : X \to Y$ is smooth with connected fibers of dimension d, then $\widetilde{f}$ sends irreducible perverse sheaves to irreducible perverse sheaves.*

Proof. Let $K \in \mathcal{M}(Y)$ be an irreducible perverse sheaf. Assume $\widetilde{f}K$ is not an irreducible perverse sheaf. Let K' be a proper subobject of $\widetilde{f}K$. By (8), $\exists K_1 \in \mathcal{M}(Y)$ such that

$\widetilde{f}K_1 \simeq K'$. There is a nonzero map $\phi \in \mathrm{Hom}(\widetilde{f}K_1, \widetilde{f}K)$. By (6), there is a nonzero map $\phi \in \mathrm{Hom}(K_1, K)$. But K is irreducible, so we have an exact sequence $K_1 \xrightarrow{\phi} K \to 0$. Since $\widetilde{f}$ is an exact functor, we have $\widetilde{f}K_1 \xrightarrow{\phi} \widetilde{f}K \to 0$. Since $\widetilde{f}K_1$ is a subobject of $\widetilde{f}K$, $\widetilde{f}K_1 \simeq \widetilde{f}K$. This is an contradiction. □

2.2.3 Pushforward and decomposition

(1) If $f : X \to Y$ is a proper morphism, then $f_* = f_!$ and $f_!(\mathbb{D}A) = \mathbb{D}f_!(A)$.

(2) If $f : X \to Y$ is a proper morphism with X smooth, then $f_!(\mathbf{1}) \in \mathcal{D}(Y)$ is a semisimple complex.

(3) Let $f : X \to Y$ be a morphism of varieties. If there is a partition $X = X_0 \cup X_1 \cup \cdots \cup X_m$ of locally closed subvarieties, such that $X_{\leq j} = X_0 \cup \cdots \cup X_j$ is closed for $j = 0, \cdots, m$, and for each j there are morphisms $X_j \xrightarrow{f_j} Z_j \xrightarrow{f'_j} Y_j$, such that Z_j is smooth, f_j is a vector bundle, f'_j is proper and $f'_j f_j = f|_{X_j}$, then $f_!(\mathbf{1}) \in \mathcal{D}(Y)$ is a semisimple complex. Additionally, for any n and j, there is a canonical exact sequence:

$$0 \longrightarrow {}^pH^n(f_j)_!\mathbf{1} \longrightarrow {}^pH^n(f_{\leq j})_!\mathbf{1} \longrightarrow {}^pH^n(f_{\leq j-1})_!\mathbf{1} \longrightarrow 0,$$

where $f_{\leq j}$ and f_j are the restrictions of f.

(4) Let X be an algebraic variety, U be an open subset of X, and Z be the complement of U in X. Let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be the inclusions. For any $K \in \mathcal{D}(X)$, there is a canonical distinguished triangle in $\mathcal{D}(X)$,

$$j_!j^*K \longrightarrow K \longrightarrow i_!i^*K \xrightarrow{[1]} .$$

If $f : X \to Y$, then we have a canonical distinguished triangle in $\mathcal{D}(Y)$,

$$f_!j_!j^*K \longrightarrow f_!K \longrightarrow f_!i_!i^*K \xrightarrow{[1]} .$$

2.2.4 Base change

If

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ Z & \xrightarrow{g} & W \end{array}$$

is a cartesian square and s is proper (resp. g is smooth), then

$$r_! f^* = g^* s_! : \mathcal{D}(Y) \to \mathcal{D}(Z).$$

Usually it is called a proper (resp. smooth) base change.

2.2.5 Projection formula

Let $f : X \to Y$ be a morphism of varieties. $C \in \mathcal{D}(X)$ and $K \in \mathcal{D}(Y)$ are constructible complexes, then

$$K \otimes f_! C \simeq f_!(f^* K \otimes C).$$

2.2.6 Künneth formula

If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are morphisms over a variety S, let $f := f_1 \times_S f_2 : X_1 \times_S X_2 \to Y_1 \times_S Y_2$, then we have the following properties (see [42]).

(1) If $A \in D^b(X_1)$ and $B \in D^b(X_2)$, then

$$f_!(A \boxtimes_S B) \simeq f_{1!} A \boxtimes_S f_{2!} B.$$

(2) If S is a point, and $F \in D^b(Y_1)$ and $G \in D^b(Y_2)$, then

$$f^!(F \boxtimes G) \simeq f_1^! F \boxtimes f_2^! G.$$

(3) If we further assume f_1 (resp. f_2) is smooth of relative dimension d_1 (resp. d_2), then

$$f^*(F \boxtimes G) \simeq f_1^* F \boxtimes f_2^* G.$$

(4) Under the same assumption as (3), we have

$$f_*(F \boxtimes G) \simeq f_{1*} F \boxtimes f_{2*} G.$$

2.3 G-equivariant complexes

Let $m : G\times X \to X$ be an action of a connected algebraic group G on X and $\pi : G\times X \to X$ be the second projection. Both maps are smooth with connected fiber of $\dim G$. A perverse sheaf K on X is said to be G-equivariant if the perverse sheaves $\pi^*K[\mathrm{dimG}]$ and $m^*K[\mathrm{dimG}]$ are isomorphic. More generally, a complex $K \in \mathcal{M}(X)[n]$ is said to be G-equivariant if the perverse sheaf $K[-n]$ is G-equivariant. Denote by $\mathcal{M}_G(X)$ the full subcategory of $\mathcal{M}(X)$ whose objects are the G-equivariant perverse sheaves on X. More generally, denote by $\mathcal{M}_G(X)[n]$ the full subcategory of $\mathcal{M}(X)[n]$ whose objects are of the form $K[n]$ with $K \in \mathcal{M}_G(X)$.

(1) If $A \in \mathcal{M}_G(X)$, and $B \in \mathcal{M}(X)$ is a subquotient of A, then $B \in \mathcal{M}_G(X)$.

(2) Assume $f : X \to Y$ is a G-equivariant morphism. If $K \in \mathcal{M}_G(Y)$, then ${}^pH^n(f^*K) \in \mathcal{M}_G(X)$ for all n. If $K' \in \mathcal{M}_G(X)$, then ${}^pH^n(f_!K') \in \mathcal{M}_G(Y)$ for all n.

(3) Assume that $f : X \to Y$ is a locally trivial principal G-bundle. Let $d = \dim(G)$. If $K \in \mathcal{M}(Y)[n+d]$, then $f^*K \in \mathcal{M}_G(X)[n]$. Furthermore, the functor $f^* : \mathcal{M}(Y)[n+d] \to \mathcal{M}_G(X)[n]$ defines an equivalence of categories. The inverse $f_\flat : \mathcal{M}_G(X)[n] \to \mathcal{M}(Y)[n+d]$ is given by $f_\flat(K) = H^{-n-d}(f_*K)[n+d]$.

2.4 Fourier Deligne transformations

The Artin-Schreier covering $k \to k$ sending x to $x^p - x$ has $\mathbb{F}_p$ as a group of covering transformations. Hence any non-trivial character $\phi : \mathbb{F}_p \to \overline{\mathbb{Q}}_l^*$ gives rise to a $\overline{\mathbb{Q}}_l$ local system, $\mathcal{E}$, of rank 1 on k. Let $T : X \to k$ be any morphism of algebraic varieties. Then $\mathcal{L}_T := T^*\mathcal{E}$ is a local system of rank 1 on X.

Now let $E \to X$ and $E' \to X$ be two vector bundles of constant fiber dimension d over X. Let $T : E \times_X E' \to k$ be a bilinear map which defines a duality between the two vector bundles. Consider the following diagram,

$$E \xleftarrow{s} E \times_X E' \xrightarrow{t} E' \ ,$$

where s, t are projection maps. Define

$$\Phi : \mathcal{D}(E) \to \mathcal{D}(E')$$
$$K \mapsto t_!(s^*K \otimes \mathcal{L}_T)[d].$$

This functor is called a Fourier-Deligne transform.

If we interchange the roles of E, E', then we have another Fourier-Deligne transform, by abuse of notation, which we still denote by $\Phi : \mathcal{D}(E') \to \mathcal{D}(E)$. Moreover, we have $\Phi(\Phi(K)) = j^*K$ for any $K \in \mathcal{D}(E)$, where $j : E \to E$ is multiplication by -1 on each fiber of E.

If we restricts Φ to perverse sheaves, then $\Phi|_{\mathcal{M}(E)} : \mathcal{M}(E) \simeq \mathcal{M}(E')$. Moreover ${}^pH^n(\Phi(K)) = \Phi({}^pH^nK)$ for $K \in \mathcal{D}(E)$.

We will use the following two properties in Chapter 4 (see [41]).

(1) Let A (resp. A') be an object of $\mathcal{D}(E)$ (resp. $\mathcal{D}(E')$). Let u (resp. $u', \dot{u}$) be the map of E (resp. $E', E \times_X E'$) to the point. Then we have

$$u_!(A \otimes \Phi(A')) = \dot{u}_!(s^*A \otimes t^*A' \otimes \mathcal{L}_T[d]) = u'_!(\Phi(A) \otimes A').$$

(2) Let $T : k^n \to k$ be a non-constant affine linear function. Let $u : k^n \to \{\mathrm{pt}\}$. Then $u_!(\mathcal{L}_T) = 0$.

2.5 Characteristic functions of complexes

For the definition of characteristic function and its properties, we refer to [13, 31].

Let X be an algebraic variety over k. Let F be a Frobenius morphism of X and X^F be the set of fixed points by F. For any complex $\mathcal{F} \in D^b(X^F, \overline{\mathbb{Q}}_l)$, such that $F^*\mathcal{F} \simeq \mathcal{F}$, we choose for each such $\mathcal{F}$ an isomorphism $\phi_{\mathcal{F}}$. The characteristic function of $\mathcal{F}$ with respect to $\phi_{\mathcal{F}}$, denote by $\chi_{\mathcal{F},\phi_{\mathcal{F}}}$ can be defined as follows

$$\chi_{\mathcal{F},\phi_{\mathcal{F}}}(x) = Tr(\phi_{\mathcal{F},x} : \mathcal{F}_x \to \mathcal{F}_x),\ \forall\ x \in X^F,$$

where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x.

If $X = G$ is an algebraic group, we will choose a unique isomorphism $\phi : F^*\mathcal{F} \to \mathcal{F}$ which induces identity on the stalk at $1 \in G$. We will simply denote the characteristic function by $\chi_{\mathcal{F},F}$. We list some properties of characteristic functions in the following.

(1) If $f : X \to Y$ is a morphism defined over $\mathbb{F}_q$, and $\mathcal{F} \in D^b(X^F, \overline{\mathbb{Q}}_l)$, then for any $y \in Y^F$,

$$\chi_{f_!\mathcal{F},F}(y) = \sum_{x\in f^{-1}(y)^F} \chi_{\mathcal{F},F}(x).$$

(2) If $f : X \to Y$ is a morphism defined over $\mathbb{F}_q$, and $\mathcal{G} \in D^b(Y^F, \overline{\mathbb{Q}}_l)$, then for any $x \in X^F$,

$$\chi_{f^*\mathcal{G},F}(x) = \chi_{\mathcal{G},F}(f(x)).$$

(3) For any $\mathcal{F}, \mathcal{G} \in D^b(X^F, \overline{\mathbb{Q}}_l)$ and $x \in X^F$, we have $\chi_{\mathcal{F}\otimes\mathcal{G},F}(x) = \chi_{\mathcal{F},F}(x)\chi_{\mathcal{G},F}(x)$.

(4) For any $\mathcal{F} \in D^b(X^F, \overline{\mathbb{Q}}_l)$ and $x \in X^F$, we have $\chi_{\mathcal{F}[1],F}(x) = (-1)\chi_{\mathcal{F},F}(x)$.

(5) If $\mathcal{F}, \mathcal{G}$ are semisimple complexes, then $\chi_{\mathcal{F},F}(x) = \chi_{\mathcal{G},F}(x)$ for all $x \in X^F$ and all Frobenius morphisms if and only if $\mathcal{F} \simeq \mathcal{G}$.

(6) If $(\mathcal{F}, \mathcal{E}, \mathcal{G})$ is a distinguished triangle in $\mathcal{D}(X)$, then for all $x \in X^F$,

$$\chi_{\mathcal{E},F}(x) = \chi_{\mathcal{F},F}(x) + \chi_{\mathcal{G},F}(x).$$

Chapter 3

Preliminary

3.1 Ringel Hall algebra

A *quiver* $\Gamma = (I, H, s, t)$ consists of a set of vertices I, a set of arrows H and two maps $s, t : H \to I$, such that $s(h)$ is the source and $t(h)$ is the target of $h \in H$. In order to simplify the notation, denote $s(h) = h', t(h) = h''$. If $h \in H$, $s(h) = i$, and $t(h) = j$, then the arrow h is commonly presented as $i \xrightarrow{h} j$.

A *representation* (V, x) of $\Gamma = (I, H, s, t)$ over field k is an I-graded k vector space V together with a set $\{x_h\}_{h\in H}$ of linear transformations $x_h : V_{h'} \to V_{h''}$. A *homomorphism* from one representation (V, x) to another representation (W, y) is a collection $\{g_i\}_{i\in I}$ of linear maps $g_i : V_i \to W_i$, such that $g_{h''} x_h = y_h g_{h'}$ for all $h \in H$. If all g_i are isomorphisms, (V, x) and (W, y) are said to be *isomorphic*.

Let $\mathfrak{A}$ be a finitary abelian category, where finitary means $\forall M, N \in \mathrm{ob}(\mathfrak{A})$, $|\operatorname{Hom}(M, N)| < \infty$, and $|\operatorname{Ext}^1(M, N)| < \infty$. Let χ be the set of isomorphism classes $[M]$ of objects M in $\mathfrak{A}$. Set

$$\mathcal{H}_{\mathfrak{A}} = \bigoplus_{[M]\in\chi} \mathbb{Z}[M]$$

and define a multiplication on $\mathcal{H}_{\mathfrak{A}}$ as

$$[M] \cdot [N] = \Sigma_{[E]\in\chi} F^E_{M,N} [E],$$

where $F^E_{M,N} = \sharp\{L \text{ is a subobject of } E \mid L \cong N, E/L \cong M\}$. $\mathcal{H}_{\mathfrak{A}}$ is called the Hall algebra of $\mathfrak{A}$ and $F^E_{M,N}$ is called a Hall number.

If $\mathfrak{A}$ is the category of finite dimensional representations of an algebra Λ over a finite field, then the finiteness condition holds. In this case, we denote $\mathcal{H}_{\mathfrak{A}}$ by $\mathcal{H}(\Lambda)$.

Fix a Dynkin quiver Γ. For given representations M, N, E, the Hall number $F^E_{M,N}$ depends on the cardinality, q, of k. Precisely, one may find polynomials of q as structure constants of the Hall algebra $\mathcal{H}(k\Gamma)$. Such polynomials are called Hall polynomials, see [50]. Thus the free $\mathbb{Z}[q]$-module $\mathcal{H}_q(k\Gamma)$, regarding q as indeterminant, is well defined and is called the generic Hall algebra.

3.2 Lusztig's geometric approach to Hall algebras

Fix a loop-free quiver $\Gamma = (I, H, s, t)$ and an I-graded k vector space $V = \bigoplus_{i\in I} V_i$, where k is the algebraic closure field of $\mathbb{F}_q$. Let

$$E_V = \bigoplus_{h\in H} \mathrm{Hom}_k(V_{h'}, V_{h''})$$

and

$$G_V = \bigoplus_{i\in I} GL_k(V_i).$$

A *flag* of type $(\underline{i}, \underline{k}) = ((i_1, k_1), \cdots, (i_m, k_m)) \in (I \times \mathbb{N})^m$ in V is a sequence

$$\mathfrak{f} = (V = V^0 \supset V^1 \supset \cdots \supset V^m = 0)$$

of I-graded vector spaces such that $V^{l-1}/V^l \simeq k^{\oplus k_l}$ concentrated at vertex i_l for all $l = 1, 2, \cdots, m$.

Let $\mathcal{F}_{V,\underline{i},\underline{k}}$ be the variety of all flags of type $(\underline{i}, \underline{k})$ in V. Then $\mathcal{F}_{V,\underline{i},\underline{k}}$ is a product of certain Grassmannians.

Let $\widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}} = \{(x, \mathfrak{f}) \in E_V \times \mathcal{F}_{V,\underline{i},\underline{k}} \mid \mathfrak{f} \text{ is } x\text{-stable}\}$, where x-stable means $x_h(V^l_{h'}) \subset V^l_{h''}$, for all $h \in H,\ l = 1, \cdots, m$.

The G_V-actions are defined as follows. G_V acts on E_V by conjugation. i.e. $g \cdot (x_h)_{h\in H} = (g_{h''} x_h g^{-1}_{h'})_{h\in H}$ for any $g \in G_V$ and $(x_h)_{h\in H} \in E_V$; G_V acts on $\mathcal{F}_{V,\underline{i},\underline{k}}$ by

$$g \cdot (V^0 \supset V^1 \supset \cdots \supset V^m = 0) \mapsto (gV^0 \supset gV^1 \supset \cdots \supset gV^m = 0),$$

for any $g \in G_V$ and $(V^0 \supset V^1 \supset \cdots \supset V^m = 0) \in \mathcal{F}_{V,\underline{i},\underline{k}}$; and G_V acts diagonally on $\widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}}$, i.e., $g \cdot (x, \mathfrak{f}) \mapsto (gx, g\mathfrak{f})$.

Let $L_{V,\underline{i},\underline{k}} = (\pi_{V,\underline{i},\underline{k}})_! \mathbf{1}[\dim_k(\widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}})]$, where $\pi_{V,\underline{i},\underline{k}} : \widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}} \to E_V$ is the first projection map. Since $\pi_{V,\underline{i},\underline{k}}$ is a G_V-equivariant map, $L_{V,\underline{i},\underline{k}}$ is a G_V-equivariant perverse sheaf. By decomposition theorem in [1], $L_{V,\underline{i},\underline{k}}$ is semisimple since $\pi_{V,\underline{i},\underline{k}}$ is a proper map. $L_{V,\underline{i},\underline{k}}$ is called a Lusztig sheaf.

Let $\mathcal{Q}_V$ be the full subcategory of $\mathcal{D}^b_{G_V}(E_V)$ whose objects are isomorphic to finite direct sums of $L[d]$ for various $d \in \mathbb{Z}$ and various simple perverse sheaves L which are direct summands of $L_{V,\underline{i},\underline{k}}$ for some $(\underline{i}, \underline{k}) \in (I \times \mathbb{N})^m$. Let $\mathcal{K}_V$ be the graded Grothendieck group of the category $\mathcal{Q}_V$. Let $\mathcal{M}_V$ be the graded Grothendieck group of the category which consists of all direct sums of $L_{V,\underline{i},\underline{k}}$ up to shift for various $(\underline{i}, \underline{k}) \in (I \times \mathbb{N})^m$. Let v be an indeterminate and $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Define an $\mathcal{A}$-action on $\mathcal{D}^b(E_V)$ as $v^n \cdot L = L[n]$. Then $\mathcal{M}_V$ (resp. $\mathcal{K}_V$) is an $\mathcal{A}$-module generated by $L_{V,\underline{i},\underline{k}}$ (resp. simple direct summands of $L_{V,\underline{i},\underline{k}}$) for various $(\underline{i}, \underline{k}) \in (I \times \mathbb{N})^m$.

Fix an I-graded subspace $W \subset V$ and let $T = V/W$. Let P be the stabilizer of W in G_V and U be the unipotent radical of P. Consider the following diagrams:

$$E_T \times E_W \xleftarrow{p_1} G_V \times^U F \xrightarrow{p_2} G_V \times^P F \xrightarrow{p_3} E_V.$$

$$E_T \times E_W \xleftarrow{\pi} F \xrightarrow{\iota} E_V.$$

Here $F = \{x \in E_V \mid x_h(W_{h'}) \subset W_{h''}, \forall h \in H\}$, and the maps are defined as follows.

$p_1(g, x) = (x_T,\ x_W)$, where $x_W = x|_W$ and x_T is the induced map $\overline{x} : V/W \to V/W$; $p_2(g, x) = (g, x)$; $p_3(g, x) = g(\iota(x))$, where ι is the embedding $F \hookrightarrow E_V$; and $\pi(x) = (x_T,\ x_W)$.

For any $A \in \mathcal{D}^b_{G_T \times G_W}(E_T \times E_W)$, define $\mathrm{Ind}^V_{T,W} A := p_{3!} p_{2\flat} p_1^* A[d_1 - d_2]$, where d_1 (resp. d_2) is the dimension of fibers of p_1 (resp. p_2). Here $p_{2\flat}$ is well defined by proposition 2.3(3) since p_2 is a principle $G_T \times G_W$-bundle. The functor $\mathrm{Ind}^V_{T,W} : \mathcal{Q}_T \times \mathcal{Q}_W \to \mathcal{Q}_V$ is called an induction functor.

For any $A \in \mathcal{D}^b_{G_V}(E_V)$, define $\mathrm{Res}^V_{T,W} := \pi_! \iota^* A[d_1 - d_2 - 2\dim G/P]$, where d_1, d_2 are the same as ones in the inductive functor. The functor $\mathrm{Res}^V_{T,W} : \mathcal{Q}_V \to \mathcal{Q}_T \times \mathcal{Q}_W$ is called a restriction functor.

Now let $\mathcal{K} = \oplus_V \mathcal{K}_V$. Define a multiplication as follows.

$$\mathrm{Ind} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$$
$$(A, B) \mapsto \mathrm{Ind}^V_{T,W}(A \boxtimes B),$$

where A, B are homogenous elements with $A \in \mathcal{K}_T$ and $B \in \mathcal{K}_W$, V is a I-graded vector space such that $W \subset V$ and $V/W = T$.

Define a comultiplication as follows.

$$\mathrm{Res} : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$$
$$A \mapsto \oplus_{T,W} \mathrm{Res}^V_{T,W}(A),$$

where A are homogenous elements in $\mathcal{K}_V$.

Now define an algebraic structure on $\mathcal{K} \otimes \mathcal{K}$ by

$$(x \otimes y)(x' \otimes y') = q^{-\frac{(|x'|,|y|)}{2}} xx' \otimes yy'$$

for x, x', y, y' homogeneous, where the symmetric bilinear form $(-,-) : \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ is defined by

$$(a, b) = 2\sum_{i \in I} a_i b_i - \sum_{h \in H} (a_{h'} b_{h''} + a_{h''} b_{h'}).$$

Theorem 1 ([34, 36, 38, 41])**.** (1) *If* Γ *is a quiver without loops, then* $\mathcal{K}$ *equipped multiplication* Ind *is isomorphic to* $\mathcal{H}_\star(\Gamma)$ *(see Chapter 1) as an* I*-graded* $\mathcal{A}$ *algebra.*

(2) Res : $\mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ *is an* $\mathcal{A}$*-algebra homomorphism. i.e.* Res *defines a coalgebra structure on* $\mathcal{K}$*.*

(3) $\{L_{V,\underline{i},\underline{k}} \mid$ for all V and $(\underline{i}, \underline{k})\}$ *are additive generators of* $\mathcal{K}$*. Hence* $\{L_{V,\underline{i},\underline{k}}\}$ *for various* V *and* $(\underline{i}, \underline{k})$ *contains an* $\mathcal{A}$*-basis of* $\mathcal{K}$*. This basis is called a monomial basis.*

(4) *All simple perverse sheaves which are direct summands of* $L_{V,\underline{i},\underline{k}}$ *for various* V *and* $(\underline{i}, \underline{k})$*, form an* $\mathcal{A}$*-basis of* $\mathcal{K}$*. This basis is called the canonical basis.*

3.3 Character sheaves

Roughly speaking, character sheaf theory is a geometric character theory of algebraic groups. Constructions of character sheaves for different algebraic groups are different. In this section, we will introduce characteristic functions of character sheaves which connect geometric objects, character sheaves, to algebraic objects, class functions.

Let k be the algebraic closure of $\mathbb{F}_q$, and G be an algebraic group over k. An $\mathbb{F}_q$-structure on G is given by a Frobenius map $F : G \to G$. Let G^F be the fixed point set of G by F which is a finite subgroup of G, see [7].

Character sheaves are some G-equivariant perverse sheaves $\mathcal{F}$ whose characteristic functions are certain generalized characters of the group G^F. Here G acts on itself by conjugation.

Let $CS(G)$ be the set of character sheaves on G. Let $CS^F(G) = \{\mathcal{F} \in CS(G) \mid F^*\mathcal{F} \simeq \mathcal{F}\}$. For any $\mathcal{F} \in CS^F(G)$, choose an isomorphism $\phi_{\mathcal{F}} : F^*\mathcal{F} \to \mathcal{F}$. Here $\phi_{\mathcal{F}}$ is unique up to nonzero scalar. Characteristic functions, $\chi_{\mathcal{F},F}$, of $\mathcal{F}$ with respect to F is defined by

$$\chi_{\mathcal{F},F}(g) = \sum_i (-1)^i Tr(\phi_{\mathcal{F}}, \mathcal{H}^i_g(\mathcal{F})),$$

where $\mathcal{H}^i_g(\mathcal{F})$ is the stalk of the cohomology sheaf, $\mathcal{H}^i(\mathcal{F})$ of $\mathcal{F}$ at $g \in G^F$. See more details in [32]. Since $\mathcal{F}$ is a G-equivariant complex, the function $\chi_{\mathcal{F},F}(g) : G^F \to \overline{\mathbb{Q}}_l$ satisfies

$$\chi_{\mathcal{F},F}(hgh^{-1}) = \chi_{\mathcal{F},F}(g),\ \forall h \in G^F.$$

i.e. $\chi_{\mathcal{F},F}$ is a $\overline{\mathbb{Q}}_l$-valued class function of G^F.

For some algebraic groups, such as abelian groups and $GL_m(k)$, the characteristic functions of character sheaves are irreducible characters. For general reductive groups, one cannot always get irreducible characters with value in $\overline{\mathbb{Q}}_l$ from character sheaves. However, characteristic functions of character sheaves are a linear combination of a "small" number of irreducible characters of G^F, where "small" means the number is independent of $q := |k^F|$. Moreover, the characteristic functions form a basis of vector space of class functions $G^F \to \overline{\mathbb{Q}}_l$.

Chapter 4

Geometric approach to Hall algebras

In this chapter, we fix $R = k[t]/(t^n)$ and consider R-free representations of loop-free quivers. Denoted $Rep_R^f(\Gamma)$ the category consisting of all R-free representations of Γ. $Rep_R^f(\Gamma)$ is not an abelian category but rather an exact category. Hubery defines the Hall algebra over an exact category in [20]. Let $\mathcal{H}_R(\Gamma)$ be the Hall algebra on $Rep_R^f(\Gamma)$. One can ask if there exists a coalgebra structure on the Hall algebra over exact category. In general, this is not true (even the homological dimension of the exact category is 1). The category $Rep_R^f(Q)$ serves as a counterexample to that the Hall algebra on it has no coalgebra structure. We will give a geometric realization of the composition subalgebra of $\mathcal{H}_R(\Gamma)$.

4.1 Hall algebra over an exact category

4.1.1 Exact category

Let $\mathcal{A}$ be an additive category which is a full subcategory of an abelian category $\mathcal{B}$ and closed under extension in $\mathcal{B}$. Let $\mathcal{E}$ be a class of sequences

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{j} M'' \longrightarrow 0$$

in $\mathcal{A}$ which are exact in the abelian category $\mathcal{B}$. A map f is called an inflation (resp. a deflation) if it occurs as the map i (resp. j) of some members in $\mathcal{E}$. Inflations and deflations will be denoted by $M' \rightarrowtail M$ and $M \twoheadrightarrow M''$ respectively. The pair $M' \rightarrowtail M \twoheadrightarrow M''$ is called a conflation. The following is Quillen's definition of an exact category. See [4] for more properties of an exact category.

Definition 1. [48] An exact category is the additive category $\mathcal{A}$ equipped with a family $\mathcal{E}$ of the short exact sequences of $\mathcal{A}$, such that the following properties hold:

(i) Any sequence in $\mathcal{A}$ which is isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$, and the split sequences in $\mathcal{A}$ are in $\mathcal{E}$.

(ii) The class of deflations is closed under composition and under base change by an arbitrary map in $\mathcal{A}$. Dually, the class of inflations is closed under composition and base change by an arbitrary map in $\mathcal{A}$.

(iii) Let $M \to M''$ be a map possessing a kernel in $\mathcal{A}$. If there exists a map $N \to M$ in $\mathcal{A}$ such that $N \to M \to M''$ is a deflation, then $M \to M''$ is a deflation. Dually, let $M' \to M$ be a map possessing a cokernel in $\mathcal{A}$. If there exists a map $M \to L$ in $\mathcal{A}$ such that $M' \to M \to L$ is an inflation, then $M' \to M$ is an inflation.

4.1.2 Representation of quivers over commutative rings

A *representation* (V, x) of $\Gamma = (I, H, s, t)$ over a commutative ring R is an I-graded R-module V together with a set $\{x_h\}_{h\in H}$ of R-linear transformations $x_h : V_{h'} \to V_{h''}$.

A *homomorphism* from one representation (V, x) to another representation (W, y) is a collection $\{g_i\}_{i\in I}$ of R-linear maps $g_i : V_i \to W_i$, such that $g_{h''}x_h = y_h g_{h'}$ for all $h \in H$. If all g_i are R-isomorphisms, (V, x) and (W, y) are said to be *isomorphic*.

Let $Rep_R(\Gamma)$ be the category of representations of Γ over R. $Rep_R(\Gamma)$ is an abelian category. If V is an I-graded free R-module, then the representation (V, x) is called an *R-free representation*. All such representations form a full subcategory $Rep^f_R(\Gamma)$ of $Rep_R(\Gamma)$. Unfortunately, this subcategory is not an abelian category anymore, but rather an exact category. In the following, all representations are assumed to be free over R. In this case, we can define the dimension vector $|V| := (\mathrm{Rank}_R(V_i))_{i\in I}$.

Lemma 2. *$Rep^f_R(\Gamma)$ is an exact category with homological dimension 1.*

Proof. It is easy to see $Rep^f_R(\Gamma)$ is an additive category. Let $\mathcal{E}$ be the set of all possible short exact sequences in $Rep^f_R(\Gamma)$. Then $Rep^f_R(\Gamma)$ with the class $\mathcal{E}$ is an exact category. In this case, an inflation is an injective map, such that the cokernel is an I-graded free R-module and a deflation map is a surjective map, such that the kernel is an I-graded free R-module.

Let $A = R\Gamma$. To show the homological dimension of $Rep^f_R(\Gamma)$ is 1, it is enough to show that sequence,

$$0 \longrightarrow \oplus_{\rho\in H} Ae_{\rho''} \otimes_R e_{\rho'}X \xrightarrow{f} \oplus_{i\in I} Ae_i \otimes_R e_iX \xrightarrow{g} X \longrightarrow 0 \tag{4.1}$$

is exact for any R-free left A-module X, where e_i is the trivial path for the vertex i, and $g(a\otimes x) = ax$, $f(a\otimes x) = a\rho\otimes x - a\otimes \rho x$.

In fact, the proof is the same as Crawley-Boevey's proof for the stand resolution in [2]. Firstly, g is clearly onto, since for any $x\in X$, $g(\oplus_i e_i\otimes e_ix) = 1\cdot x = x$. Secondly, $g\circ f = 0$. i.e. $\mathrm{Im}(f)\subset \mathrm{Ker}(g)$. We will show the converse inclusion.

Any element $\xi\in \oplus_{i\in I} Ae_i\otimes_R e_iX$ can be uniquely written into

$$\xi = \sum_i \sum_{\text{paths } a \text{ with } s(a)=i} a\otimes x_a,$$

where $a\in Ae_i$ and $x_a\in e_{s(a)}X$.

Define degree(ξ) to be the length of the longest path a with $x_a\neq 0$. If a is a non-trivial path with $s(a) = i$, then $a = a'\rho$ for some path a' and arrow ρ such that $s(\rho) = i$ and $s(a') = t(\rho)$. Since $a'\otimes x_a \in \oplus_{\rho\in H} Ae_{\rho''}\otimes_R e_{\rho'}X$, we have $f(a'\otimes x_a) = a\otimes x_a - a'\otimes \rho x_a$.

Since degree$(\xi - f(\sum_i\sum_{\text{paths } a \text{ with } s(a)=i} a'\otimes x_a)) <$ degree(ξ), by repeating this process, $\xi + \mathrm{Im}(f)$ always contains an element of degree 0.

Now for any $\xi\in\mathrm{Ker}(g)$, let $\xi'\in\xi+\mathrm{Im}(f)$ has degree zero. Then

$$0 = g(\xi) = g(\xi') = g(\sum_i e_i\otimes x'_{e_i}) = \sum x'_{e_i}.$$

So $x'_{e_i} = 0$ for all i. Therefore $\xi' = 0$. This shows $\mathrm{Ker}(g)\subset\mathrm{Im}(f)$.

We will next show $\mathrm{Ker}(f) = 0$. Any element $\xi \in \mathrm{Ker}(f)$ can be written into

$$\xi = \sum_{\rho \in H} \sum_{\text{paths } a \text{ with } s(a)=t(\rho)} a \otimes x_{\rho,a},$$

where $x_{\rho,a} \in e_{s(\rho)}$. Let a be a path with maximal length such that $x_{\rho,a} \neq 0$ for some ρ. Since

$$f(\xi) = \sum_{\rho} \sum_{a} a\rho \otimes x_{\rho,a} - \sum_{\rho} \sum_{a} a \otimes \rho x_{\rho,a},$$

the coefficient of $a\rho$ in $f(\xi)$ is $x_{\rho,a}$, but the length of $a\rho$ is one more than the length of a. This is a contradiction. $\square$

Here the homological dimension 1 refers to $\mathrm{Ext}^n(X, Y)$ vanishing for all $n \geq 2$ and $X, Y \in \mathcal{A}$. See Chapter 6 in [14] for the definition of $\mathrm{Ext}^n(X, Y)$ in an exact category.

4.1.3 Hall algebra over an exact category

In this section, we first deform Hubery's definition of the Hall algebra over a finitary exact category, then give a counterexample to show that a coalgebra structure of the Hall algebra can not be obtained by twisting. We will always assume $\mathcal{A}$ is an exact category which is a full subcategory of an abelian category $\mathcal{B}$.

The algebra structure

Let $\mathcal{A}$ be a finitary and small exact category. Denote by W_{XY}^L the set of all conflations $Y \rightarrowtail L \twoheadrightarrow X$. The group $Aut(X) \times Aut(Y)$ acts on W_{XY}^L via:

$$\begin{CD} Y @>f>> L @>g>> X \\ @V{\eta}VV @| @VV{\varepsilon}V \\ Y @>\overline{f}>> L @>\overline{g}>> X. \end{CD}$$

Denote by V_{XY}^L the quotient set of W_{XY}^L by the group $Aut(X) \times Aut(Y)$. Since f is an inflation and g is a deflation, this action is free. So

$$F_{XY}^L := |V_{XY}^L| = \frac{|W_{XY}^L|}{a_X a_Y},$$

where $a_X = |Aut(X)|$. The Ringle-Hall algebra $\mathcal{H}(\mathcal{A})$ is defined as the free $\mathbb{Z}$-module on the set of isomorphism classes of objects. By abuse of notation, we will write X for the isomorphism classes $[X]$, and use the numbers F^L_{XY} as the structure constants of multiplication. Define

$$X \circ Y := \sum_L F^L_{XY} L,$$

Theorem 2 ([20]). *The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ of a finitary and small exact category $\mathcal{A}$ is an associate, unital algebra.*

If $\mathcal{A} = Rep^f_R(\Gamma)$, we want to deform the Ringel-Hall algebra $\mathcal{H}(Rep^f_R(\Gamma))$. Firstly, for $\alpha = (a_i)_{i\in I}, \beta = (b_i)_{i\in I}$, define

$$\langle \alpha, \beta \rangle := \sum_{i\in I} a_i b_i - \sum_{h\in H} a_{h'} b_{h''}. \tag{4.2}$$

It is easy to check this is a bilinear form on $\mathbb{N}^I$.

For any $X, Y \in Rep^f_R(\Gamma)$, define a deformed multiplication as

$$XY := q^{n\langle |X|,|Y|\rangle} X \circ Y.$$

Here $|X|$ is the dimension vector of X which is defined in last section.

Theorem 3. *$\mathcal{H}(Rep^f_R(\Gamma))$ equipped with the deformed multiplication is an associate, unital algebra.*

Proof. By Theorem 2, it is enough to prove that for any $\alpha, \beta, \gamma \in \mathbb{N}^I$,

$$\langle \alpha, \beta \rangle + \langle \alpha + \beta, \gamma \rangle = \langle \beta, \gamma \rangle + \langle \alpha, \beta + \gamma \rangle.$$

From the bilinearity of $\langle -, - \rangle$, both sides are equal to $\langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$. □

The coalgebra structure

Let $\Delta : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$ be the map as following,

$$\Delta(E) := \sum_{M,N} q^{\frac{n}{2}\langle M,N \rangle} F^E_{MN} \frac{a_M a_N}{a_E} M \otimes N, \tag{4.3}$$

where M, N run through all conflations $M \rightarrowtail E \twoheadrightarrow N$. If one defines the twisted multiplication on $\mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$ to be

$$(A \otimes B) \cdot (C \otimes D) := q^{\frac{n}{2}(\langle B,C\rangle + \langle C,B\rangle)} AC \otimes BD, \tag{4.4}$$

then as Green shows, in [17], the map Δ defined in (4.3) is an algebra homomorphism with respect to this twisted multiplication on $\mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$ when $\mathcal{A}$ is a hereditary abelian category. i.e. Δ gives a coalgebra structure on $\mathcal{H}(\mathcal{A})$. Unfortunately, Δ is not a homomorphism of algebras if $\mathcal{A}$ is an exact category. In the rest of this section, let's focus on the case of the exact category $\mathcal{A} = Rep_R^f(\Gamma)$.

The following counterexample shows that $\Delta : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$, defined in (4.3), cannot be an algebra homomorphism under any twist in the case of $\mathcal{A} = Rep_R^f(\Gamma)$.

Example 1. Let $\Gamma = A_2 : 1 \to 2$, $R = k[t]/(t^n)$ $(n > 2)$, and $M = N = (R \xrightarrow{t} R)$.

If Δ is an algebra homomorphism, we must have

$$\Delta(MN) = \Delta(M)\Delta(N). \tag{4.5}$$

On the right hand side of (4.5), we have

$$\begin{array}{ccccc} D & \rightarrowtail & X & \twoheadrightarrow & B \\ \downarrow & & & & \downarrow \\ M & & & & N \\ \downarrow & & & & \downarrow \\ C & \rightarrowtail & Y & \twoheadrightarrow & A, \end{array}$$

where the only possible choices for B and D are $0 \xrightarrow{0} R$, $R \xrightarrow{t} R$, and $0 \xrightarrow{0} 0$. Thus, all possible choices for X are $0 \xrightarrow{0} R^2$, $0 \xrightarrow{0} R$, $0 \xrightarrow{0} 0$, $R \xrightarrow{t} R$, $R \xrightarrow{\begin{bmatrix} t \\ 0 \end{bmatrix}} R^2$, and $R^2 \xrightarrow{\begin{bmatrix} t & a \\ 0 & t \end{bmatrix}} R^2$, where $a \in R$.

On the left hand side of (4.5), we have

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow & & \\ M & \rightarrowtail & E & \twoheadrightarrow & N \\ & & \downarrow & & \\ & & Y & & \end{array}$$

Here $E \simeq (R^2 \xrightarrow{\begin{bmatrix} t & a \\ 0 & t \end{bmatrix}} R^2)$. If $a \in tR$, then

$$\begin{bmatrix} t & a \\ 0 & t \end{bmatrix} \simeq \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}.$$

If a is invertible in R, then

$$\begin{bmatrix} t & a \\ 0 & t \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ 0 & t^2 \end{bmatrix}.$$

Let $E_1 \simeq (R^2 \xrightarrow{\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}} R^2)$ and $E_2 \simeq (R^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & t^2 \end{bmatrix}} R^2)$. Then, $MN = \alpha E_1 + \beta E_2$ for some nonzero number α and β. It is clear that $\Delta(E_2)$ has a summand $(R \xrightarrow{1} R) \otimes (R \xrightarrow{t^2} R)$. This term, however, never appears on the right hand side of (4.5). This shows Δ cannot be an algebra homomorphism.

This counterexample shows that a coalgebra structure of $\mathcal{H}(RA_2)$ cannot be defined by (4.3) no matter how the multiplication of $\mathcal{H}(RA_2) \otimes \mathcal{H}(RA_2)$ is twisted.

Let $C\mathcal{H}_R(\Gamma)$ be the subalgebra of $\mathcal{H}(R\Gamma)$ generated by all S_i, for $i \in I$. We will give a geometric approach of $C\mathcal{H}_R(\Gamma)$ in the rest of this chapter.

4.2 Lusztig's geometric setting

In this section, we will fix $R = k[t]/(t^n)$, a loop-free quiver $\Gamma = (I, H, s, t)$, and an I-graded free R-module $V = \bigoplus_{i \in I} V_i$ which can be thought of as an I-graded k-vector space. We define

$$E_V^k = \bigoplus_{h \in H} \mathrm{Hom}_k(V_{h'}, V_{h''}), \tag{4.6}$$

$$E_V^R = \bigoplus_{h \in H} \mathrm{Hom}_R(V_{h'}, V_{h''}), \tag{4.7}$$

$$G_V^k = \bigoplus_{i \in I} GL_k(V_i), \tag{4.8}$$

and

$$G_V^R = \bigoplus_{i \in I} GL_R(V_i). \tag{4.9}$$

G_V^R (resp. G_V^k) acts on E_V^R (resp. E_V^k) by conjugation, i.e., $gx = x'$ and $x'_h = g_{h''} x_h g_{h'}^{-1}$ for all $h \in H$.

Given R-modules V_1 and V_2, $\mathrm{Hom}_k(V_1, V_2)$ has an R-module structure as follows,

$$(rf)(v) = f(rv) - rf(v),$$

for all $r \in R$, $v \in V_1$ and $f \in \mathrm{Hom}_k(V_1, V_2)$. Then

$$\mathrm{Hom}_R(V_1, V_2) = \{f \in \mathrm{Hom}_k(V_1, V_2) \mid rf - fr = 0,\ \forall r \in R\}.$$

Since E_V^k is an affine k-variety and $rf - fr = 0$ for different $r \in R$ are algebraic equations, E_V^R is a closed k-subvariety of E_V^k. Similarly, G_V^R is a closed algebraic k-subgroup of G_V^k.

4.2.1 Flags

A generalized *k-flag* of type $(\underline{i}, \underline{k}) = ((i_1, k_1), \cdots, (i_m, k_m)) \in (I \times \mathbb{N})^m$ in an I-graded k-vector space V is a sequence

$$\mathfrak{f} = (V = V^0 \supset V^1 \supset \cdots \supset V^m = 0)$$

of I-graded vector spaces such that $V^{l-1}/V^l \simeq k^{\oplus k_l}$ concentrated at vertex i_l for all $l = 1, 2, \cdots, m$.

Let $\mathcal{F}^k_{V,\underline{i},\underline{k}}$ be the k-variety of all generalized k-flags of type $(\underline{i}, \underline{k})$ in V.

Let $\widetilde{\mathcal{F}}^k_{V,\underline{i},\underline{k}} = \{(x, \mathfrak{f}) \in E_V^k \times \mathcal{F}^k_{V,\underline{i},\underline{k}} \mid \mathfrak{f} \text{ is } x\text{-stable}\}$, where $\mathfrak{f}$ is x-stable if $x_h(V^l_{h'}) \subset V^l_{h''}$, for all $h \in H, l = 1, \cdots, m$.

G_V^k acts on $\mathcal{F}^k_{V,\underline{i},\underline{k}}$ by $g \cdot \mathfrak{f} \mapsto g\mathfrak{f}$, where

$$g\mathfrak{f} = (gV^0 \supset gV^1 \supset \cdots \supset gV^m = 0)$$

if $\mathfrak{f} = (V = V^0 \supset V^1 \supset \cdots \supset V^m = 0)$. And G_V^k acts diagonally on $\widetilde{\mathcal{F}}^k_{V,\underline{i},\underline{k}}$, i.e., $g \cdot (x, \mathfrak{f}) \mapsto (gx, g\mathfrak{f})$.

An *R-flag* of type $(\underline{i}, \underline{k}) = ((i_1, k_1), \cdots, (i_m, k_m)) \in (I \times \mathbb{N})^m$ in an I-graded R-module V is a sequence

$$\mathfrak{f} = (V = V^0 \supset V^1 \supset \cdots \supset V^m = 0)$$

of I-graded R-modules such that $V^{l-1}/V^l \simeq k^{\oplus k_l}$ as k-vector spaces concentrated at vertex i_l for all $l = 1, 2, \cdots, m$.

Similarly, let $\mathcal{F}^R_{V,\underline{i},\underline{k}}$ be the k-variety of all R-flags of type $(\underline{i}, \underline{k})$ in V.

For any free R-module V, V/tV is a k vector space; we will denote it by V_0. Moreover, we can define the evaluation map as follows,

$$e : \mathcal{F}^R_{V,\underline{i},\underline{k}} \to \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$$
$$\mathfrak{f} = (V \supset V^1 \supset \cdots \supset V^m = 0) \mapsto e(\mathfrak{f}) = (V_0 \supset V_0^1 \supset \cdots \supset V_0^m = 0).$$

Denote

$$\mathfrak{f}_0 = e(\mathfrak{f}) \otimes_k R := (V_0 \otimes_k R \supset V_0^1 \otimes_k R \supset \cdots \supset V_0^m \otimes_k R). \tag{4.10}$$

Let $\widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}} = \{(x, \mathfrak{f}) \in E^k_V \times \mathcal{F}^R_{V,\underline{i},\underline{k}} \mid \mathfrak{f}_0 \text{ is } x\text{-stable}\}$. Moreover, we can define G^R_V actions on $\mathcal{F}^R_{V,\underline{i},\underline{k}}$ and $\widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}}$ in a similar way.

Note that if V is an R-module, then a k-subspace $W \subset V$ is an R-submodule if and only if $(1 + t)W = W$. This gives an algebraic equation. So $\mathcal{F}^R_{V,\underline{i},\underline{k}}$ (resp. $\widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}}$) is a closed subvariety of $\mathcal{F}^k_{V,\underline{i},\underline{k}}$ (resp. $\widetilde{\mathcal{F}}^k_{V,\underline{i},\underline{k}}$).

A *R-free flag* of type $(\underline{i}, \underline{k}) = ((i_1, k_1), \cdots, (i_m, k_m)) \in (I \times \mathbb{N})^m$ in an I-graded free R-module V is a sequence

$$\mathfrak{f} = (V = V^0 \supset V^1 \supset \cdots \supset V^m = 0)$$

of I-graded free R-modules such that $V^{l-1}/V^l \simeq R^{\oplus k_l}$ concentrated at vertex i_l as R-modules for all $l = 1, 2, \cdots, m$.

Let $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} \subset \mathcal{F}^{R}_{V,\underline{i},n\underline{k}}$ be the subvariety of all R-free flags of type $(\underline{i}, \underline{k})$, where $(\underline{i}, n\underline{k}) = ((i_1, nk_1), \cdots, (i_m, nk_m))$ if $(\underline{i}, \underline{k}) = ((i_1, k_1), \cdots, (i_m, k_m))$.

Let $\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} = \widetilde{\mathcal{F}}^{R}_{V,\underline{i},\underline{k}} \cap (E^R_V \times \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}})$ and $\mathcal{F}^{RNf}_{V,\underline{i},\underline{k}}$ (resp. $\widetilde{\mathcal{F}}^{RNf}_{V,\underline{i},\underline{k}}$) be the complement of $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ (resp. $\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}}$) in $\mathcal{F}^{R}_{V,\underline{i},n\underline{k}}$ (resp. $\widetilde{\mathcal{F}}^{R}_{V,\underline{i},n\underline{k}}$). We can define G^R_V actions on these k-varieties in a similar way.

Remark 1. Notice that $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ is an $(n-1)$th-Jet scheme over $\mathcal{F}^{k}_{V_0,\underline{i},\underline{k}}$ (see [45]). So $\dim_k \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} = n \dim_k \mathcal{F}^{k}_{V_0,\underline{i},\underline{k}}$. Moreover, the dimension (resp. shift functors for perverse sheaves) argument in Lusztig's papers can be adapted here by multiplying by n. In the rest of this chapter, we will skip the proof of the statements about dimension and shift degree.

To simplify the notations, for any $(\underline{i},\underline{k}) \in (I \times \mathbb{N})^m$ and each $i \in I$, let $N_i(\underline{i},\underline{k}) = \sum_{r<r'} k_r k_{r'} \delta_{ii_r} \delta_{ii_{r'}}$; for each $h \in H$, let $N_h(\underline{i},\ \underline{k}) = \sum_{r'<r} k_{r'} k_r \delta_{h' i_{r'}} \delta_{h'' i_r}$, where δ is the Kronecker delta. In the following, dimension always refers to k-dimension, so we will denote it by dim instead of $\dim_k$. Rank always refers to the rank of free R-modules, and we will therefore denote it by Rank.

Proposition 1. (1) *$\mathcal{F}^{R}_{V,\underline{i},\underline{k}}$ is a projective variety.*

(2) *$\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ is an open smooth subvariety of $\mathcal{F}^{R}_{V,\underline{i},n\underline{k}}$ and $\mathcal{F}^{RNf}_{V,\underline{i},\underline{k}}$ is a closed subvariety of $\mathcal{F}^{R}_{V,\underline{i},n\underline{k}}$.*

(3) *The evaluation map $e : \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} \to \mathcal{F}^{k}_{V_0,\underline{i},\underline{k}}$ is a vector bundle with rank $(n-1)\sum_i N_i(\underline{i},\underline{k})$. Hence the dimension of $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ is $n\sum_i N_i(\underline{i},\underline{k})$.*

Proof. (1) Choose $\mathfrak{f} \in \mathcal{F}^{k}_{V,\underline{i},\underline{k}}$. Define $P := \mathrm{Stab}_{G^k_V}(\mathfrak{f})$, the stabilizer of $\mathfrak{f}$ in G^k_V. P is a parabolic subgroup, so $\mathcal{F}^{k}_{V,\underline{i},\underline{k}} = G^k_V/P$ is a projective variety. $\mathcal{F}^{R}_{V,\underline{i},\underline{k}}$ is a closed subvariety, so it is also a projective variety.

(2) To show $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} \subset \mathcal{F}^{R}_{V,\underline{i},n\underline{k}}$ is an open subset, let's first consider the Grassmannian.

Let $G_k(sn, ln)$ be the set of all sn-dimensional k-subspaces in V with $\dim V = ln$ and $G_R(sn, ln) = \{\mathfrak{f} \in G_k(sn, ln) \mid (1+t)\mathfrak{f} = \mathfrak{f}\}$. Let $G_{Rf}(s, l)$ be the set of all free R-submodules with Rank s in V, where $\mathrm{Rank}(V) = l$. Clearly, $G_{Rf}(s, l) \subset G_R(sn, ln) \subset G_k(sn, ln)$.

Let $\widetilde{G}_R(sn, ln) = \{(W, b_W) \mid W \in G_R(sn, ln)$ and b_W is a k-basis of $W\}$.

The first projection $\pi : \widetilde{G}_R(sn, ln) \to G_R(sn, ln)$ is a frame bundle.

Define

$$\phi : \widetilde{G}_R(sn, ln) \to \mathrm{Mat}(sn),$$
$$(W, b_W) \mapsto M(b_W, t),$$

where $\mathrm{Mat}(sn)$ is the set of all $sn \times sn$ matrices and $M(b_W, t)$ is the matrix of t under the basis b_W. Clearly, ϕ is a morphism of algebraic varieties.

In general, for any free R-module $V \simeq R^{\oplus r}$, the R-module structure induces a nilpotent k-linear map $t : V \to V$ where $\dim(\mathrm{Ker}(t)) = r$.

For any R-submodule $W \subset V$ with k-dimension ns, W is a free R-module if and only if $\dim(\mathrm{Ker}(t|_W)) = s$, i.e., $t|_W$ has maximal rank $(n-1)s$. Therefore, $G_{Rf}(s, l) = \pi(\phi^{-1}(\mathrm{Mat}(sn)_{rk=s(n-1)}))$, where $\mathrm{Mat}(sn)_{rk=s(n-1)}$ is the set of all matrices with rank $s(n-1)$. $\phi^{-1}(\mathrm{Mat}(sn)_{rk=s(n-1)})$ is open in $\phi^{-1}(\mathrm{Mat}(sn)_{rk\leq s(n-1)}) = \widetilde{G}_R(sn, ln)$ since $\mathrm{Mat}(sn)_{rk=s(n-1)}$ is an open subset in $\mathrm{Mat}(sn)_{rk\leq s(n-1)}$. Moreover, π is a principle $GL_{sn}(k)$-bundle and $\phi^{-1}(\mathrm{Mat}(sn)_{rk=s(n-1)})$ is $GL_{sn}(k)$-stable, so $G_{Rf}(s, l)$ is open in $G_R(sn, ln)$.

Now for any flag $\mathfrak{f} = (V^0 \supset V^1 \supset \cdots \supset V^m) \in \mathcal{F}^R_{V,\underline{i},n\underline{k}}$, each entry V^l gives an open condition when it is a free R-module. So $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ is the intersection of m many such open subsets. The smoothness follows from Remark 1 and the notes after Lemma 1.2 in [45]. This proves the first statement. The second statement follows from the first one.

(3) Recall for any R-free module V, we denote $V_0 = V/tV$, then $V \simeq V_0 \otimes_k R$. Without loss of generality, we will simply assume $V = V_0 \otimes_k R$. For any element $\mathfrak{f} = (V_0 \supset V_0^1 \supset \cdots \supset V_0^m) \in \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$, let $P^k(\mathfrak{f})$ be its stabilizer in $G^k_{V_0}$. Let $P^R(\mathfrak{f})$ be the stabilizer of $\mathfrak{f} \otimes_k R$ (see (4.10)) in G^R_V, where we identify V with $V_0 \otimes_k R$. So $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ (resp. $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$) can be identified with $G^R_V/P^R(\mathfrak{f})$ (resp. $G^k_{V_0}/P^k(\mathfrak{f})$) for a fixed $\mathfrak{f}$.

Now consider the map $\iota : \mathcal{F}^k_{V_0,\underline{i},\underline{k}} \to \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$, $\mathfrak{f} \mapsto \mathfrak{f} \otimes_k R$. Since ι is an injective map, in the rest of this chapter, we will identify $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ with $\iota(\mathcal{F}^k_{V_0,\underline{i},\underline{k}})$ and consider $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ as a subset of $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$. Similarly, we will consider $G^k_{V_0}/P^k(\mathfrak{f})$ as a subset of $G^R_V/P^R(\mathfrak{f})$ via the map ι.

Now any element $A \in G^R_V$ can be written as $A = h \cdot A_0$, where $A_0 \in G^k_{V_0}$ and $h \in H := \{Id + tB | B \in \mathrm{End}_R(V)\}$. Then the evaluation map e sends $h \cdot A_0$ to A_0. Therefore, $\forall x \in G^k_{V_0}$, we have $e^{-1}(x) = H/(H \cap P^R) \cdot x$. As a set, $H/(H \cap P^R) \cdot x$ is in 1-1 correspondence to $H/(H \cap P^R)$, and $H/(H \cap P^R)$ is a direct sum of quasi-lower triangular matrices with entries in tR for all $i \in I$, which is clearly a k-vector space of dimension $(n-1)\sum_i N_i(\underline{i}, \underline{k})$.

Now for any open subset $U \subset G^k_{V_0}/P^k(\mathfrak{f})$, define

$$\phi_U : U \times H/(H \cap P^R) \to e^{-1}(U)$$
$$(x, a) \mapsto a \cdot x.$$

It is easy to check this gives a vector bundle structure. □

Remark 2. From the above proof, we have $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} = \{gh \cdot \mathfrak{f} \mid g \in G^k_{V_0}/P^k, h \in H/(H \bigcap P^R)\}$ for a fixed $\mathfrak{f} \in \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ and $e : \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} \to \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ sending $gh \cdot \mathfrak{f}$ to $g \cdot \mathfrak{f}$. Since $G^k_{V_0}$ acts on $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ transitively, we have $\mathcal{F}^k_{V_0,\underline{i},\underline{k}} = \{g \cdot \mathfrak{f} \mid g \in G^k_{V_0}/P^k(\mathfrak{f})\}$ for a fixed $\mathfrak{f} \in \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$.

Proposition 2. (1) *$\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}}$ is a smooth irreducible variety, and the second projection p_2 : $\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} \to \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$ is a vector bundle of dimension $n\sum_h N_h(\underline{i}, \underline{k})$. So the k-dimension of $\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}}$ is $d(V, \underline{i}, \underline{k}) := n\sum_i N_i(\underline{i}, \underline{k}) + n\sum_h N_h(\underline{i}, \underline{k})$.*

(2) *Let $\pi^f_{V,\underline{i},\underline{k}} : \widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} \to E^R_V$ be the first projection map. Then $(\pi^f_{V,\underline{i},\underline{k}})_! \mathbf{1}$ is semisimple.*

Proof. Let $\widetilde{\mathcal{F}}^k_R = \{(x, \mathfrak{f}) \in E^R_V \times \mathcal{F}^k_{V_0,\underline{i},\underline{k}} \mid \mathfrak{f} \text{ is } x-\text{stable}\}$. Here we consider $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ as a subset of $\mathcal{F}^{Rf}_{V,\underline{i},\underline{k}}$. By using Lusztig's argument for Lemma 1.6 in [36], we want to show the second projection $p_2 : \widetilde{\mathcal{F}}^k_R \to \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ is a vector bundle. In fact, for any $\mathfrak{f} = (V \supset V^1 \supset \cdots \supset V^m = 0) \in \mathcal{F}^k_{V_0,\underline{i},\underline{k}}$, let Z be the fiber of p_2. The first projection identifies Z with the set of all $x \in E^R_V$ such that $x_h(V^l_{h'}) \subset V^l_{h''}$ for all $h \in H$ and all $l = 0, 1, \cdots, m$. This is a linear subspace of E^R_V because we can choose a basis for each V_i such that x_h are upper triangle matrices for each $h \in H$. Hence its dimension is equal to

$$n \sum_{l' \leq l, h \in H} (\mathrm{Rank}(V_{h'}^{l'-1}) - \mathrm{Rank}(V_{h'}^{l'}))(\mathrm{Rank}(V_{h''}^{l'-1}) - \mathrm{Rank}(V_{h''}^{l'}))$$

which is equal to $n \sum_h N_h(\underline{i}, \underline{k})$. Since G_V^R acts on $\mathcal{F}^k_{V_0,\underline{i},\underline{k}}$ transitively, this is independent of $\mathfrak{f}$ but only dependent on $(\underline{i}, \underline{k})$. This shows that p_2 is a vector bundle.

Now consider the following cartesian square,

$$\begin{array}{ccc} \widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} & \xrightarrow{p_1} & \mathcal{F}^{Rf}_{V,\underline{i},\underline{k}} \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle e} \\ \widetilde{\mathcal{F}}^k_R & \xrightarrow{p_2} & \mathcal{F}^k_{V_0,\underline{i},\underline{k}}. \end{array} \tag{4.11}$$

Since p_2 is a vector bundle with rank $n \sum_h N_h(\underline{i}, \underline{k})$, p_1 is a vector bundle with rank $n \sum_h N_h(\underline{i}, \underline{k})$. The smoothness and irreducibility follow Proposition 1. This proves the first statement. The second statement follows from the first one.

(2) Consider cartesian square (4.11). By Proposition 1, e is a vector bundle, then b is also a vector bundle. Now consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} & \xrightarrow{b} & \widetilde{\mathcal{F}}^k_R \\ & {\scriptstyle \pi^f_{V,\underline{i},\underline{k}}} \searrow & \downarrow{\scriptstyle p} \\ & & E^R_V \end{array} \tag{4.12}$$

Here the first projection map, p, is a proper map. By Proposition 2.2.3(3), $(\pi^f_{V,\underline{i},\underline{k}})_! \mathbf{1} = p_! b_! \mathbf{1}$ is semisimple. Proposition follows. □

Denote $\widetilde{L}^f_{V,\underline{i},\underline{k}} := (\pi^f_{V,\underline{i},\underline{k}})_! \mathbf{1} \in \mathcal{D}(E_V)$. By Proposition 2, $\widetilde{L}^f_{V,\underline{i},\underline{k}}$ is semisimple.

Proposition 3. *Let* $L^f_{V,\underline{i},\underline{k}} = \widetilde{L}^f_{V,\underline{i},\underline{k}}[d(V, \underline{i}, \underline{k}) + (n-1)\sum_i N_i(\underline{i}, \underline{k})]$. *Then* $L^f_{V,\underline{i},\underline{k}}$ *is a perverse sheaf. In particular,* $\mathbb{D}(L^f_{V,\underline{i},\underline{k}}) = L^f_{V,\underline{i},\underline{k}}$.

Proof. From the proof of Proposition 2, we have $\pi^f_{V,\underline{i},\underline{k}} = pb$, where p is a proper map and b

is a vector bundle with rank $(n-1)\sum_i N_i(\underline{i},\underline{k})$. Therefore,

$$\begin{aligned} L^f_{V,\underline{i},\underline{k}} &= (\pi^f_{V,\underline{i},\underline{k}})_!\mathbf{1}[d(V,\underline{i},\underline{k})+(n-1)\sum_i N_i(\underline{i},\underline{k})] \\ &= p_!b_!\mathbf{1}[d(V,\underline{i},\underline{k})+(n-1)\sum_i N_i(\underline{i},\underline{k})] \\ &= p_!b_!b^*\mathbf{1}_{\widetilde{\mathcal{F}}^k_R}[d(V,\underline{i},\underline{k})+(n-1)\sum_i N_i(\underline{i},\underline{k})] \\ &= p_!\mathbf{1}_{\widetilde{\mathcal{F}}^k_R}[d(V,\underline{i},\underline{k})-(n-1)\sum_i N_i(\underline{i},\underline{k})] \\ &= p_!\mathbf{1}_{\widetilde{\mathcal{F}}^k_R}[\dim_k(\widetilde{\mathcal{F}}^k_R)]. \end{aligned}$$

Since p is a proper map and $\mathbf{1}_{\widetilde{\mathcal{F}}^k_R}[\dim_k(\widetilde{\mathcal{F}}^k_R)]$ is a perverse sheaf, $L^f_{V,\underline{i},\underline{k}}$ is a perverse sheaf. □

Similarly, let $\pi^R_{V,\underline{i},\underline{k}} : \widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}} \to E^R_V$ be the first projection. We define $\widetilde{L}^R_{V,\underline{i},\underline{k}} = (\pi^R_{V,\underline{i},\underline{k}})_!\mathbf{1}_{\widetilde{\mathcal{F}}^R_{V,\underline{i},\underline{k}}}$.

Let $\mathcal{P}^f_V$ (resp. $\mathcal{P}^R_V$) be the full subcategory of $\mathcal{M}(E^R_V)$ consisting of perverse sheaves which are direct sums of the simple perverse sheaves L that are the direct summands of $\widetilde{L}^f_{V,\underline{i},\underline{k}}$ (resp. $\widetilde{L}^R_{V,\underline{i},\underline{k}}$) up to shift for some $(\underline{i},\underline{k}) \in (I\times\mathbb{N})^m$. Let $\mathcal{Q}^f_V$ (resp. $\mathcal{Q}^R_V$) be the full subcategory of $\mathcal{D}(E^R_V)$ whose objects are isomorphic to finite direct sums of $L[d]$ for various simple perverse sheaves $L \in \mathcal{P}^f_V$ (resp. $\mathcal{P}^R_V$) and various $d \in \mathbb{Z}$.

4.2.2 Restriction functor

To define the restriction functor, Lusztig considers the following diagram

$$E_T \times E_W \xleftarrow{\kappa} F \xrightarrow{\iota} E_V,$$

where ι is an embedding and $\kappa(x) = (x_W, x_T)$. Recall $x_W = x|_W$ and x_T is the induced map $\overline{x} : V/W \to V/W$. For any $B \in \mathcal{D}(E^R_V)$, define $\overline{\mathrm{Res}}^V_{T,W}B := \kappa_!\iota^*B$. However, it is no longer true that $\overline{\mathrm{Res}}^V_{T,W}B \in \mathcal{Q}^f_{T,W}$, even for $B \in \mathcal{Q}^f_V$. In fact, given a free flag $\mathfrak{f} = (V^0 \supset V^1 \supset \cdots \supset V^m = 0)$, and $W \subset V$, $T = V/W$ being free R-modules, the induced flags

$$\mathfrak{f}_T := ((V^0+W)/W \supset (V^1+W)/W \supset \cdots \supset (V^m+W)/W = 0) \tag{4.13}$$

$$\mathfrak{f}_W := (V^0 \bigcap W \supset V^1 \bigcap W \cdots \supset V^m \bigcap W = 0) \tag{4.14}$$

are no longer free flags, since $V^l \bigcap W$ and $(V^l+W)/W$ are no longer free modules in general.

Lemma 3. *$\overline{\mathrm{Res}}^V_{T,W}(B)$ is semisimple in $\mathcal{D}^b_{G^R_T \times G^R_W}(E^R_T \times E^R_W)$ for $B \in \mathcal{Q}^f_V$.*

Proof. It is sufficient to prove that $\kappa_! \iota^*(\widetilde{L}^f_{V,\underline{i},\underline{k}})$ is semisimple, that is, $k_! \iota^*(\pi^f_{V,\underline{i},\underline{k}})_!(\mathbf{1}_{\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}}})$ is semisimple. Consider the following diagram,

$$\begin{array}{ccccc} & & \widetilde{F}^R & \xrightarrow{\iota'} & \widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}} \\ & \swarrow^{\kappa\pi'} & \downarrow^{\pi'} & & \downarrow^{\pi^f_{V,\underline{i},\underline{k}}} \\ E_T \times E_W & \xleftarrow{\kappa} & F & \xrightarrow{\iota} & E_V, \end{array} \tag{4.15}$$

where $\widetilde{F}^R = (\pi^f_{V,\underline{i},\underline{k}})^{-1}(F)$.

Using base change, we have

$$\kappa_! \iota^*(\pi^f_{V,\underline{i},\underline{k}})_! \mathbf{1} = \kappa_! \pi'_! \iota'^* \mathbf{1} = (\kappa\pi')_! \mathbf{1}_{\widetilde{F}^R}.$$

We now prove that $(\kappa\pi')_! \mathbf{1}_{\widetilde{F}^R}$ is semisimple.

Recall from the proof of Proposition 2(2), we have a vector bundle $\widetilde{F}^R \xrightarrow{b} \widetilde{\mathcal{F}}^k_R$ sending $(x, h\mathfrak{f}_0)$ to $(x, \mathfrak{f}_0)$, where $\widetilde{\mathcal{F}}^k_R = \big\{(x,\mathfrak{f}) \in F \times \mathcal{F}^k_{V_0,\underline{i},\underline{k}} \mid \mathfrak{f} \text{ is } x-\text{stable}\big\}$.

For any $\underline{k}_1$ and $\underline{k}_2$ satisfying $\underline{k} = \underline{k}_1 + \underline{k}_2$, we set

$$\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2} = \Big\{(x,\mathfrak{f}) \in \widetilde{\mathcal{F}}^k_R \mid (x_T, \mathfrak{f}_{T_0}) \in \widetilde{\mathcal{F}}^k_{T_0,\underline{i},\underline{k}_1},\ (x_W, \mathfrak{f}_{W_0}) \in \widetilde{\mathcal{F}}^k_{W_0,\underline{i},\underline{k}_2}\Big\},$$

where $\mathfrak{f}_{T_0}$ and $\mathfrak{f}_{W_0}$ are defined in (4.13), (4.14) by using T_0 (resp. W_0) instead of T (resp. W). $\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2}$ is a locally closed subvariety of $\widetilde{\mathcal{F}}^k_R$. For various $(\underline{i},\underline{k}_1)$ and $(\underline{i},\underline{k}_2)$, $\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2}$ form a partition of $\widetilde{\mathcal{F}}^k_R$. Then $b^{-1}(\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2})$ for various $(\underline{i},\underline{k}_1)$ and $(\underline{i},\underline{k}_2)$ form a partition of $\widetilde{\mathcal{F}}^R$.

Now define $\alpha_{\underline{i},\underline{k}_1,\underline{k}_2} : \widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2} \to \widetilde{\mathcal{F}}^k_{T_0,\underline{i},\underline{k}_1} \times \widetilde{\mathcal{F}}^k_{W_0,\underline{i},\underline{k}_2}$, which sends $(x,\mathfrak{f})$ to $((x_T,\mathfrak{f}_{T_0}),(x_W,\mathfrak{f}_{W_0}))$. It is easy to check that $\alpha_{\underline{i},\underline{k}_1,\underline{k}_2}$ is a vector bundle.

Let $D_j = \Big\{(\underline{i},\underline{k}_1,\underline{k}_2) \mid \dim_k(\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2}) = j\Big\}$. Let $\widetilde{\mathcal{F}}^k_{Rj}$ be the disjoint union of $\widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2}$ for various $(\underline{i},\underline{k}_1,\underline{k}_2) \in D_j$. i.e. $\widetilde{\mathcal{F}}^k_{Rj} = \coprod_{(\underline{i},\underline{k}_1,\underline{k}_2)\in D_j} \widetilde{\mathcal{F}}_{R,\underline{i},\underline{k}_1,\underline{k}_2}$. Let $Z_j = \coprod_{(\underline{i},\underline{k}_1,\underline{k}_2)\in D_j} (\widetilde{\mathcal{F}}^k_{T_0,\underline{i},\underline{k}_1} \times \widetilde{\mathcal{F}}^k_{W_0,\underline{i},\underline{k}_2})$. There is a well-defined map $\alpha_j := \coprod_{(\underline{i},\underline{k}_1,\underline{k}_2)\in D_j} \alpha_{\underline{i},\underline{k}_1,\underline{k}_2} : \widetilde{\mathcal{F}}^k_{Rj} \to Z_j$. Since these are disjoint union, α_j is a vector bundle. Moreover, the composition map $\alpha_j \circ b : b^{-1}(\widetilde{\mathcal{F}}^k_{Rj}) \to Z_j$ is a vector bundle. Therefore, we have the following diagram,

$$b^{-1}(\widetilde{\mathcal{F}}^k_{Rj}) \xrightarrow{\alpha_j \circ b} Z_j \xrightarrow{\pi_j} E_T \times E_W,$$

where $\pi_j := \coprod_{(\underline{i},\underline{k}_1,\underline{k}_2)\in D_j}(\pi_{T,\underline{i}_1,\underline{k}_1} \times \pi_{W,\underline{i}_2,\underline{k}_2})$ is a proper map.

By Proposition 2.2.3(3), $(\kappa\pi')_!\mathbf{1}_{\widetilde{F}^R}$ is semisimple. □

Since the objects in $\mathcal{Q}^R_V$ are semisimple complexes, every object $A \in \mathcal{Q}^R_V$ can be uniquely written into $A = A^f \oplus A^{Nf}$ such that $A^f \in \mathcal{Q}^f_V$, $A^{Nf} \in \mathcal{Q}^R_V \setminus \mathcal{Q}^f_V$ and A^f is the maximal subobject of A which is in $\mathcal{Q}^f_V$. Therefore we can define a projection map $P_f : \mathcal{Q}^R_V \to \mathcal{Q}^f_V$ sending A to A^f.

Definition 2. $\widetilde{\mathrm{Res}}^V_{T,W}(B) := P_f(\overline{\mathrm{Res}}^V_{T,W}(B))$.

Proposition 4. *$\widetilde{\mathrm{Res}}^V_{T,W}(B) \in \mathcal{Q}^f_{T,W}$ if $B \in \mathcal{Q}_V$.*

Proof. This follows directly from the definition of $\widetilde{\mathrm{Res}}^V_{T,W}$. □

Proposition 5. *If $E_T = 0$, i.e., $E_W \simeq F$, then $\overline{\mathrm{Res}}^V_{T,W}(B) = \widetilde{\mathrm{Res}}^V_{T,W}(B)$ for all $B \in \mathcal{Q}^f_V$.*

Proof. Since any simple object $B \in \mathcal{Q}^f_V$ is a direct summand of $\widetilde{L}^f_{V,\underline{i},\underline{k}}$ for some $(\underline{i},\underline{k})$ up to shift, it is enough to prove the proposition for $B = \widetilde{L}^f_{V,\underline{i},\underline{k}}$.

From the proof of Lemma 3 and Diagram (4.15), if $E_W \simeq F$, κ is an isomorphism, then

$$\overline{\mathrm{Res}}^V_{T,W}(\widetilde{L}^f_{V,\underline{i},\underline{k}}) = \kappa_!\iota^*(\pi^f_{V,\underline{i},\underline{k}})_!\mathbf{1} = (\kappa\pi')_!\iota'^*\mathbf{1} = (\kappa\pi')_!\mathbf{1}_{\widetilde{F}^R} \in \mathcal{Q}^f_{T,W}.$$

Here $\mathcal{Q}^f_{T,W}$ is defined similarly as $\mathcal{Q}^f_V$ for $E_T \times E_W$.

So $\overline{\mathrm{Res}}^V_{T,W}(\widetilde{L}^f_{V,\underline{i},\underline{k}}) = \widetilde{\mathrm{Res}}^V_{T,W}(\widetilde{L}^f_{V,\underline{i},\underline{k}})$ by the definition of $\widetilde{\mathrm{Res}}^V_{T,W}$. □

4.2.3 Induction functor

By abuse of notation, in the rest of this chapter, we will write E_V (resp. G_V) instead of E^R_V (resp. G^R_V) unless we specify. And $\widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}}$ always means $\widetilde{\mathcal{F}}^{Rf}_{V,\underline{i},\underline{k}}$ unless we specify.

Let W be an I-graded free R-submodule of V such that $T = V/W$ is also a free R-module. Let P be the stabilizer of W in G_V and U be the unipotent radical of P. Consider the following diagram:

$$E_T \times E_W \xleftarrow{p_1} G_V \times^U F \xrightarrow{p_2} G_V \times^P F \xrightarrow{p_3} E_V. \tag{4.16}$$

Here $p_1(g,x) = \kappa(x)$, $p_2(g,x) = (g,x)$, and $p_3(g,x) = g(\iota(x))$, where κ and ι are the maps introduced in Section 4.2.2. For any $A \in \mathcal{D}_{G_T \times G_W}(E_T \times E_W)$, define $\widetilde{\operatorname{Ind}}_{T,W}^V A := p_{3!} p_{2\flat} p_1^* A$. Here $p_{2\flat}$ is well defined since p_2 is a principle $G_T \times G_W$-bundle.

Proposition 6. $\widetilde{\operatorname{Ind}}_{T,W}^V A \in \mathcal{Q}_V^f$ *if* $A \in \mathcal{Q}_{T,W}^f$.

Proof. Since $\widetilde{\operatorname{Ind}}_{T,W}^V$ is additive, it is enough to prove the proposition for $A = \widetilde{L}_{T,\underline{i}',\underline{k}'} \boxtimes \widetilde{L}_{W,\underline{i}'',\underline{k}''}$, where $(\underline{i}',\underline{k}') = ((i_1,k_1),\cdots,(i_m,k_m))$ and $(\underline{i}'',\underline{k}'') = ((i_{m+1},k_{m+1}),\cdots,(i_{m+s},k_{m+s}))$. Let $(\underline{i},\underline{k}) = ((\underline{i}',\underline{k}'),(\underline{i}'',\underline{k}'')) := ((i_1,k_1),\cdots,(i_m,k_m),(i_{m+1},k_{m+1}),\cdots,(i_{m+s},k_{m+s}))$.

Let $\mathcal{F}^0_{V,\underline{i},\underline{k}} = \{(V^0 \supset V^1 \supset \cdots \supset V^m \cdots \supset V^{m+s} = 0) \in \mathcal{F}_{V,\underline{i},\underline{k}} \mid V^m = W\}$ and $\widetilde{\mathcal{F}}^0_{V,\underline{i},\underline{k}} = \widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}} \bigcap (F \times \mathcal{F}^0_{V,\underline{i},\underline{k}})$.

Now consider the following diagram,

$$\begin{array}{ccccccc}
\widetilde{\mathcal{F}}_{T,\underline{i}',\underline{k}'} \times \widetilde{\mathcal{F}}_{W,\underline{i}'',\underline{k}''} & \xleftarrow{\widetilde{p_1}} & G_V \times^U \widetilde{\mathcal{F}}^0_{V,\underline{i},\underline{k}} & \xrightarrow{\widetilde{p_2}} & G_V \times^P \widetilde{\mathcal{F}}^0_{V,\underline{i},\underline{k}} & \xrightarrow{i} & \widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}} \\
\downarrow{\scriptstyle \pi_{T,W}} & \boxed{1} & \downarrow{\scriptstyle u'} & \boxed{2} & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \pi_{V,\underline{i},\underline{k}}} \\
E_T \times E_W & \xleftarrow{p_1} & G_V \times^U F & \xrightarrow{p_2} & G_V \times^P F & \xrightarrow{p_3} & E_V.
\end{array} \tag{4.17}$$

Here the vertical maps are all projection maps and i is an identity map. The squares $\boxed{1}$ and $\boxed{2}$ are both cartesian squares and $\widetilde{p_2}$ is a principle $G_T \times G_W$-bundle. It follows that

$$p_1^*(\pi_{T,W})_! \mathbf{1} = u'_! \widetilde{p_1}^* \mathbf{1} = u'_! \widetilde{p_2}^* \mathbf{1} = p_2^* u_! \mathbf{1}.$$

So

$$p_{3!} p_{2\flat} p_1^* A = p_{3!} p_{2\flat} p_1^* (\pi_{T,W})_! \mathbf{1} = p_{3!} u_! \mathbf{1} = (\pi_{V,\underline{i},\underline{k}})_! \mathbf{1} \in \mathcal{Q}_V^f.$$

$\square$

Remark 3. The above proof also shows $\widetilde{\mathrm{Ind}}_{T,W}^{V}(\widetilde{L}_{T,\underline{i}',\underline{k}'} \boxtimes \widetilde{L}_{W,\underline{i}'',\underline{k}''}) = \widetilde{L}_{V,\underline{i},\underline{k}}$, where $(\underline{i},\underline{k}) = ((\underline{i}',\underline{k}'),(\underline{i}'',\underline{k}''))$.

Lemma 4. *Let $b : Y \to X$ be a fiber bundle with d dimensional connected smooth irreducible fiber. If $B = b^*A$ for some $A \in \mathcal{D}^b(X)$, then $\mathbb{D}b_!B = (b_!\mathbb{D}B)[2d]$.*

Proof. Since $b_!B = b_!b^*A = A[-2d]$, we have

$$\mathbb{D}b_!B = \mathbb{D}(A[-2d]) = (\mathbb{D}A)[2d],$$

and

$$b_!\mathbb{D}B = b_!\mathbb{D}b^*A = b_!b^!(\mathbb{D}A) = b_!b^*(\mathbb{D}A)[2d] = \mathbb{D}A.$$

□

Denote d_1 (resp. d_2) the dimension of the fibers of p_1 (resp. p_2), where p_1 and p_2 are the maps defined in Diagram (4.16). After simple calculations, $d_2 = \dim P/U$ and $d_1 = \dim G_V/U + n\sum_{h\in H} \mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''})$.

Proposition 7. *Let A be a direct summand of $\widetilde{L}_{T,\underline{i},\underline{k}} \boxtimes \widetilde{L}_{W,\underline{j},\underline{l}}$, then*

$$\mathbb{D}(\widetilde{\mathrm{Ind}}_{T,W}^{V}(A)) = \widetilde{\mathrm{Ind}}_{T,W}^{V}(\mathbb{D}(A))[2d_1 - 2d_2 + 2(n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i)].$$

Proof. Since $\mathbb{D}$ is additive, it is enough to consider $A = \widetilde{L}_{T,\underline{i},\underline{k}} \boxtimes \widetilde{L}_{W,\underline{j},\underline{l}}$. From the proof of Proposition 6, we have

$$\mathbb{D}(\widetilde{\mathrm{Ind}}_{T,W}^{V}(A)) = \mathbb{D}(p_{3!}p_{2\flat}p_1^*(\pi_{T,W})_!\mathbf{1}) = \mathbb{D}((\pi_{V,(\underline{i},\underline{k}),(\underline{j},\underline{l})})_!i_!\widetilde{p}_{2\flat}\widetilde{p}_1^*\mathbf{1}).$$

From the proof of Proposition 2, $\pi_{V,(\underline{i},\underline{k}),(\underline{j},\underline{l})} = p \circ b$ such that p is a proper map and b is a vector bundle with rank $(n-1)\sum_i N_i((\underline{i},\underline{k}),(\underline{j},\underline{l}))$. By Lemma 4,

$$\begin{aligned}\mathbb{D}((\pi_{V,(\underline{i},\underline{k}),(\underline{j},\underline{l})})_!i_!\widetilde{p}_{2\flat}\widetilde{p}_1^*\mathbf{1}) &= p_!b_!i_!\mathbb{D}\widetilde{p}_{2\flat}\widetilde{p}_1^*\mathbf{1}[2(n-1)\sum_i N_i((\underline{i},\underline{k}),(\underline{j},\underline{l}))] \\ &= (\pi_{V,(\underline{i},\underline{k}),(\underline{j},\underline{l})})_!i_!\widetilde{p}_{2\flat}\widetilde{p}_1^*(\mathbb{D}\mathbf{1})[2d_1 - 2d_2 + 2(n-1)\sum_i N_i((\underline{i},\underline{k}),(\underline{j},\underline{l}))].\end{aligned}$$

On the other hand, by the similar reason,

$$\begin{aligned}
\widetilde{\mathrm{Ind}}^V_{T,W}(\mathbb{D}A) &= p_{3!}p_{2\flat}p_1^*(\mathbb{D}(\pi_{T,W})_!\mathbf{1}) \\
&= p_{3!}p_{2\flat}p_1^*p'_!b'_!(\mathbb{D}\mathbf{1})[2(n-1)\sum_i(N_i(\underline{i},\underline{k})+N_i(\underline{j},\underline{l}))] \\
&= p_{3!}p_{2\flat}p_1^*(\pi_{T,W})_!(\mathbb{D}\mathbf{1})[2(n-1)\sum_i(N_i(\underline{i},\underline{k})+N_i(\underline{j},\underline{l}))] \\
&= (\pi_{V,(\underline{i},\underline{k}),(\underline{j},\underline{l})})_!i_!\widetilde{p}_{2\flat}\widetilde{p}_1^*(\mathbb{D}\mathbf{1})[2(n-1)\sum_i(N_i(\underline{i},\underline{k})+N_i(\underline{j},\underline{l}))].
\end{aligned}$$

Here we use a similar decomposition $\pi_{T,W}=p'\circ b'$ such that p' is a proper map and b' is a vector bundle with rank $(n-1)\sum_i(N_i(\underline{i},\underline{k})+N_i(\underline{j},\underline{l}))$.

By the definition of $N_i(\underline{i},\underline{k})$, it is easy to check that

$$N_i((\underline{i},\underline{k}),(\underline{j},\underline{l}))-N_i(\underline{i},\underline{k})-N_i(\underline{j},\underline{l})=\sum_{r,r'}k_rl_{r'}\delta_{ii_r}\delta_{ij_{r'}}=\mathrm{Rank}(T_i)\mathrm{Rank}(W_i). \tag{4.18}$$

The proposition follows. $\square$

Let

$$\mathrm{Ind}^V_{T,W}A=\widetilde{\mathrm{Ind}}^V_{T,W}A[d_1-d_2+(n-1)\sum_i\mathrm{Rank}(T_i)\mathrm{Rank}(W_i)], \tag{4.19}$$

and

$$\mathrm{Res}^V_{T,W}A=\overline{\mathrm{Res}}^V_{T,W}A[d_1-d_2-2\mathrm{dim}G_V/P+(n-1)\sum_i\mathrm{Rank}(T_i)\mathrm{Rank}(W_i)]. \tag{4.20}$$

Then we have the following corollary.

Corollary 1. $\mathbb{D}(\mathrm{Ind}^V_{T,W}(A))=\mathrm{Ind}^V_{T,W}(\mathbb{D}(A))$.

Proof.

$$\begin{aligned}
\mathbb{D}(\mathrm{Ind}^V_{T,W}(A)) &= \mathbb{D}(\widetilde{\mathrm{Ind}}^V_{T,W}A[d_1-d_2+(n-1)\sum_i\mathrm{Rank}(T_i)\mathrm{Rank}(W_i)]) \\
&= \mathbb{D}(\widetilde{\mathrm{Ind}}^V_{T,W}A)[-(d_1-d_2+(n-1)\sum_i\mathrm{Rank}(T_i)\mathrm{Rank}(W_i))] \\
&= \widetilde{\mathrm{Ind}}^V_{T,W}(\mathbb{D}(A))[d_1-d_2+(n-1)\sum_i\mathrm{Rank}(T_i)\mathrm{Rank}(W_i)] \\
&= \mathrm{Ind}^V_{T,W}(\mathbb{D}(A)).
\end{aligned}$$

$\square$

Corollary 2. $\mathrm{Ind}_{T,W}^{V}(L_{T,\underline{i},\underline{k}} \boxtimes L_{W,\underline{j},\underline{l}}) = L_{V,(\underline{i}\ \underline{j}),(\underline{k}\ \underline{l})}$.

Proof. Denote $M := d_1 - d_2 + (n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i)$, then we have

$$\begin{aligned}
&\mathrm{Ind}_{T,W}^{V}(L_{T,\underline{i},\underline{k}} \boxtimes L_{W,\underline{j},\underline{l}}) \\
&= \widetilde{\mathrm{Ind}}_{T,W}^{V}(L_{T,\underline{i},\underline{k}} \boxtimes L_{W,\underline{j},\underline{l}})[M] \\
&= \widetilde{\mathrm{Ind}}_{T,W}^{V}(\widetilde{L}_{T,\underline{i},\underline{k}} \boxtimes \widetilde{L}_{W,\underline{j},\underline{l}})[d(T,\underline{i},\underline{k}) + d(W,\underline{j},\underline{l}) + (n-1)\sum_i (N_i(\underline{i},\underline{k}) + N_i(\underline{j},\underline{l})) + M] \\
&= \widetilde{L}_{V,(\underline{i}\ \underline{j}),(\underline{k}\ \underline{l})}[d(T,\underline{i},\underline{k}) + d(W,\underline{j},\underline{l}) + M + (n-1)\sum_i (N_i(\underline{i},\underline{k}) + N_i(\underline{j},\underline{l}))] \\
&= L_{V,(\underline{i}\ \underline{j}),(\underline{k}\ \underline{l})}[(n-1)\sum_i (N_i(\underline{i},\underline{k}) + N_i(\underline{j},\underline{l}) - N_i((\underline{i}\ \underline{j}),(\underline{k}\ \underline{l}))) + M - d_1 + d_2].
\end{aligned}$$

The last equality holds because

$$d(T,\underline{i},\underline{k}) + d(W,\underline{j},\underline{l}) + d_1 - d_2 - d(V,(\underline{i}\ \underline{j}),(\underline{k}\ \underline{l})) = 0$$

which follows from Remark 1 and Lusztig's argument in 9.2.7 in [41]. Therefore the proposition follows from (4.18). □

4.2.4 Bilinear form

Let A and B be two G-equivariant semisimple complexes on algebraic variety X. Let's choose an integer m and a smooth irreducible algebraic variety Γ with a free action of G such that $H^i(\Gamma, \overline{\mathbb{Q}}_l) = 0$ for $i = 1, \cdots, m$. G acts diagonally on $\Gamma \times X$. Consider the diagram

$$X \xleftarrow{s} \Gamma \times X \xrightarrow{t} G \setminus (\Gamma \times X)$$

with the obvious projection maps s and t. We have $s^*A = t^*({}_\Gamma A)$ and $s^*B = t^*({}_\Gamma B)$ for well defined semisimple complexes ${}_\Gamma A$ and ${}_\Gamma B$ on ${}_\Gamma X := G \setminus (\Gamma \times X)$.

By the argument in [18, 33], if m is large enough, then

$$\mathrm{dim}H_c^{j+2\mathrm{dim}G-2\mathrm{dim}\Gamma}({}_\Gamma X, {}_\Gamma A \otimes {}_\Gamma B) = \mathrm{dim}H_c^{j}({}_\Gamma X, {}_\Gamma A[\mathrm{dim}(G\setminus\Gamma)] \otimes {}_\Gamma B[\mathrm{dim}(G\setminus\Gamma)])$$

is independent of m and Γ. Denote this by $d_j(X, G; A, B)$.

Suppose A, A' and B are semisimple G-equivariant complexes on X. Then we have the following properties for $d_j(X, G; A, B)$ (see [18, 32, 33, 41]):

(1) $d_j(X, G; A, B) = d_j(X, G; B, A)$.

(2) $d_j(X, G; A[n], B[m]) = d_{j+n+m}(X, G; A, B)$ for any $m, n \in \mathbb{Z}$.

(3) $d_j(X, G; A \oplus A', B) = d_j(X, G; A, B) + d_j(X, G; A', B)$.

(4) If A and B are perverse sheaves, then so are ${}_\Gamma A[\dim(G \setminus \Gamma)]$ and ${}_\Gamma B[\dim(G \setminus \Gamma)]$. Moreover, we have $d_j(X, G; A, B) = 0$ for all $j > 0$. If, in addition, A and B are simple and $B \simeq \mathbb{D}A$, then $d_0(X, G; A, B)$ is 1 and is zero otherwise.

(5) If A' and B' are in $\mathcal{Q}_T^f$ and A'' and B'' are in $\mathcal{Q}_W^f$, then

$d_j(E_T \times E_W, G_T \times G_W; A' \otimes A'', B' \otimes B'')$

$= \Sigma_{j'+j''=j} d_{j'}(E_T, G_T; A', B') d_{j''}(E_W, G_W; A'', B'')$.

(6) Let $K, K' \in \mathcal{Q}_V^f$. The following two conditions are equivalent:

  (i) $K \simeq K'$;

  (ii) $d_j(E_V, G_V; K, B) = d_j(E_V, G_V; K', B)$ for all simple objects $B \in \mathcal{P}_V^f$ and $j \in \mathbb{Z}$.

Lemma 5 ([18]). *Let $A \in \mathcal{Q}_{T,W}^f$ and $B \in \mathcal{Q}_V^f$. Then for any $j \in \mathbb{Z}$,*

$$d_j(E_T \times E_W, G_T \times G_W; A, \overline{\mathrm{Res}}_{T,W}^V B) = d_{j'}(E_V, G_V; \widetilde{\mathrm{Ind}}_{T,W}^V A, B),$$

where $j' = j + 2\dim G_V/P$.

Proposition 8. *Let $A \in \mathcal{Q}_{T,W}^f$ and $B \in \mathcal{Q}_V^f$. Then for any $j \in \mathbb{Z}$,*

$$d_j(E_T \times E_W, G_T \times G_W; A, \mathrm{Res}_{T,W}^V B) = d_j(E_V, G_V; \mathrm{Ind}_{T,W}^V A, B).$$

Proof. This follows directly from definitions (4.19), (4.20) and Lemma 5. □

Remark 4. The algebra structure of the composition subalgebra of the Hall algebra associated quivers is independent of the orientation of the given quiver. To give a geometric realization of this subalgebra, one must show that the algebra constructed by using perverse sheaves is also independent of the orientation. The Fourier Deligne transform, later denoted by Φ, is the tool used to prove this. Let $\mathcal{Q}^f = \oplus_V \mathcal{Q}^f_V$, the functors $\mathrm{Ind}^V_{T,W}$ and $\mathrm{Res}^V_{T,W}$ give an algebra and an coalgebra structure of $\mathcal{Q}^f$. With this point of view, one needs to show Φ commutes with $\mathrm{Ind}^V_{T,W}$ and $\mathrm{Res}^V_{T,W}$.

4.2.5 Fourier Deligne transform

In this section, let's consider a new orientation of the given quiver. Denote the source of the arrow h by $s(h) = {}'h$ and its target by $t(h) = {}''h$ for the new orientation. Recall that we denote the source of the arrow h by $s(h) = h'$ and its target by $t(h) = h''$ for the old orientation. Let $H_1 = \{h \in H \mid {}'h = h', {}''h = h''\}$ and $H_2 = \{h \in H \mid {}'h = h'', {}''h = h'\}$. For a given I-graded free R-module V, denote

$$E_V = \oplus_{h\in H_1} \mathrm{Hom}_R(V_{h'}, V_{h''}) \oplus \oplus_{h\in H_2} \mathrm{Hom}_R(V_{h'}, V_{h''}),$$

$${}'E_V = \oplus_{h\in H_1} \mathrm{Hom}_R(V_{h'}, V_{h''}) \oplus \oplus_{h\in H_2} \mathrm{Hom}_R(V_{h''}, V_{h'}),$$

and

$$\dot{E}_V = \oplus_{h\in H_1} \mathrm{Hom}_R(V_{h'}, V_{h''}) \oplus \oplus_{h\in H_2} \mathrm{Hom}_R(V_{h'}, V_{h''}) \oplus \oplus_{h\in H_2} \mathrm{Hom}_R(V_{h''}, V_{h'}).$$

Then we have the natural projection maps

$$E_V \xleftarrow{s} \dot{E}_V \xrightarrow{t} {}'E_V.$$

Consider E^R_V as a subset of E^k_V, then we can define a map $\mathcal{T}_V : \dot{E}_V \to k$ by

$$\mathcal{T}_V(a, b, c) = \sum_{h\in H_2} tr(V_{h'} \xrightarrow{b} V_{h''} \xrightarrow{c} V_{h'}), \tag{4.21}$$

where tr is the trace function of the endomorphism of k-vector space. Clearly, $\mathcal{T}_V$ is a bilinear map.

Define

$$\begin{aligned}\Phi : \mathcal{D}(E_V) &\to \mathcal{D}({}'E_V)\\ A &\mapsto t_!(s^*(A) \otimes L_{\mathcal{T}_V})[d_V],\end{aligned}$$

where $d_V = \dim(\oplus_{h \in H_2} \operatorname{Hom}_R(V_{h'}, V_{h''})) = n \sum_{h \in H_2} \operatorname{Rank}(V_{h'})\operatorname{Rank}(V_{h''})$.

Similarly, we have the projection maps

$$E_T \times E_W \xleftarrow{\overline{s}} \dot{E}_T \times \dot{E}_W \xrightarrow{\overline{t}} {}'E_T \times {}'E_W.$$

Define $\overline{\mathcal{T}} : \dot{E}_T \times \dot{E}_W \to k$ by $\overline{\mathcal{T}} := \mathcal{T}_T + \mathcal{T}_W$, where $\mathcal{T}_T : \dot{E}_T \to k$ (resp. $\mathcal{T}_W : \dot{E}_W \to k$) is defined in (4.21) replacing V by T (resp. W). In a similar fashion, one can define

$$\Phi : \mathcal{D}(E_T \times E_W) \to \mathcal{D}({}'E_T \times {}'E_W)$$
$$A \mapsto \overline{t}_!(\overline{s}^*(A) \otimes L_{\overline{\mathcal{T}}})[d_T + d_W].$$

Proposition 9. *For any $B \in \mathcal{Q}_V^f$, we have*

$$\Phi \overline{\operatorname{Res}}_{T,W}^V(B) = \overline{\operatorname{Res}}_{T,W}^V \Phi(B)[\pi],$$

where $\pi = n\Sigma_{h \in H_2}(\operatorname{Rank}(T_{h''})\operatorname{Rank}(W_{h'}) - \operatorname{Rank}(T_{h'})\operatorname{Rank}(W_{h''}))$.

Proof. The following proof is based on Lusztig's proof for Proposition 10.1.2 in [41]. Consider the following diagram,

$$\begin{array}{ccccccccc}
E_T \times E_W & \xleftarrow{p} & F & & & \xrightarrow{\iota} & & & E_V \\
\uparrow{\scriptstyle \overline{s}} & & \uparrow{\scriptstyle \dot{s}} & & & & & & \uparrow{\scriptstyle s} \\
\dot{E}_T \times \dot{E}_W & \xleftarrow{\dot{p}} & \psi & \xleftarrow{\dot{q}} & \dot{F} & \xrightarrow{\dot{\xi}} & \Xi & \xrightarrow{i} & \dot{E}_V \\
\downarrow{\scriptstyle \overline{t}} & & & & & & \downarrow{\scriptstyle \dot{i}} & & \downarrow{\scriptstyle t} \\
{}'E_T \times {}'E_W & & & \xleftarrow{'p} & & & {}'F & \xrightarrow{'\iota} & {}'E_V.
\end{array}$$

Here $F = \{x \in E_V \mid x_h(W_{h'}) \subset W_{h''},\ \forall h \in H\}$;

${}'F = \{x \in {}'E_V \mid x_h(W_{'h}) \subset W_{''h},\ \forall h \in H\}$;

$\dot{F} = \left\{(x_1, x_2, x_3) \in \dot{E}_V \mid (x_1, x_2) \in F,\ (x_1, x_3) \in\ {}'F\right\}$, where $x_1 \in \oplus_{h \in H_1} \operatorname{Hom}_R(V_{h'}, V_{h''})$, $x_2 \in \oplus_{h \in H_2} \operatorname{Hom}_R(V_{h'}, V_{h''})$, $x_3 \in \oplus_{h \in H_2} \operatorname{Hom}_R(V_{h''}, V_{h'})$. Similarly, in the rest of proof, the subscript $1, 2, 3$ always mean the first, second and third component in $\dot{E}_V$ or its subset respectively and super-script T (resp. W) means the object is in $\dot{E}_T$ (resp. $\dot{E}_W$).

$\Xi = \left\{(y_1, y_2, y_3) \in \dot{E}_V \mid (y_1, y_3) \in {}'F\right\}$;

$\psi = \Big\{(x, y^T, y^W) \in F \times \dot{E}_T \times \dot{E}_W \mid x'' = (y_1^T, y_2^T),\ x' = (y_1^W, y_2^W)\Big\}$, where $x' = x_W, x'' = x_T$ and in the rest of proof, single prime always means restriction to W and double prime means the induce map $V/W \to V/W$. Maps are defined as follows.

$\dot{s} : \psi \to F,\ (x, y^T, y^W) \mapsto x;$

$\dot{p} : \psi \to \dot{E}_T \times \dot{E}_W,\ (x, y^T, y^W) \mapsto (x', y_3^T, x'', y_3^W);$

$\dot{q} : \dot{F} \to \psi,\ (x_1, x_2, x_3) \mapsto ((x_1, x_2), (x_1', x_2', x_3'), (x_1'', x_2'', x_3''));$

$i : \Xi \to \dot{E}_V$ is an embedding;

$\dot{t} : \Xi \to {}'F, (y_1, y_2, y_3) \mapsto (y_1, y_3);$

$\dot{\xi} : \dot{F} \to \Xi,\ (y_1, y_2, y_3) \mapsto (y_1, y_2, y_3).$

Let $Z = \{(y_1, y_2, y_3) \in \Xi \mid y_3(W_{h''}) = 0, y_3(T_{h''}) = 0, y_1 = y_2 = 0\}$ and let $c : \Xi \to \Xi/Z$ be the canonical projection map. Define $\widetilde{\mathcal{T}} : \Xi \to k$ sending x to $\mathcal{T}_V i(x)$. It is clear that $\widetilde{\mathcal{T}}|_Z = 0$. Let $\widetilde{\mathcal{T}}_1 : \Xi/Z \to k$ be the induce map of $\widetilde{\mathcal{T}}$.

We are going to show $\widetilde{\mathcal{T}}_1$ is constant if and only if $tr(y_3 x_2) = 0$ for all y_3.

In fact, let $\overline{x} = x + y, x \in \Xi, y \in Z$, then

$$\begin{aligned}
\widetilde{\mathcal{T}}(\overline{x}) &= \mathcal{T}_V(i(\overline{x})) \\
&= \mathcal{T}_V((x+y)_1, (x+y)_2, (x+y)_3) \\
&= \mathcal{T}_V(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
&= \Sigma_{h \in H_2} tr((x_3 + y_3)(x_2 + y_2)) \\
&= \Sigma_{h \in H_2} tr(x_3 x_2 + x_3 y_2 + y_3 x_2 + y_3 y_2) \\
&= \Sigma_{h \in H_2} tr(x_3 x_2) + tr(y_3 x_2).
\end{aligned}$$

Since $\widetilde{\mathcal{T}}_1$ is affine linear function, $\widetilde{\mathcal{T}}_1$ is constant if and only if $tr(y_3 x_2) = 0$ for all y_3.

Next we want to show that $tr(y_3 x_2) = 0$ for all y_3 satisfying $y_3(W_{h''}) = 0, y_3(T_{h''}) = 0$ if and only if $\{x + y \mid \forall y \in Z\} \subseteq \dot{\xi}(\dot{F})$, i.e. $(x_1, x_2) \in F$.

Since $V_{h''}, W_{h''}$ and $T_{h''}$ are all free modules, we can fix a basis of $W_{h''}$ and extend it to a basis of $V_{h''}$. Under this basis, we have a block matrix decomposition $y_3 = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$. Here

$*$ is any block. For any $x_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $tr(y_3 x_2) = 0$ if and only if $tr(*c) = 0$. Since $*$ is an arbitrary matrix, this is true if and only if $c = 0$. Therefore $(x_1, x_2) \in F$.

Now let $\Xi' = \Xi - \dot{\xi}(\dot{F})$. Denote $c' = c|_{\Xi'}$ and $\mathcal{T}' = \widetilde{\mathcal{T}}|_{\Xi'}$. Then the restriction of $\mathcal{T}'$ to any fibre of c' is a non-constant affine linear function. Hence by Section 2.4(2), the local system $\mathcal{L}_{\mathcal{T}'}$ on Ξ' satisfies $c'_!(\mathcal{L}_{\mathcal{T}'}) = 0$.

Since $\dot{\xi} : \dot{F} \to \Xi$ is a closed embedding, applying 2.2.3(4) to the partition $\Xi = \Xi' \cup \dot{\xi}(\dot{F})$, we have a distinguished triangle

$$c_! j_! j^* \mathcal{L}_{\widetilde{\mathcal{T}}} \longrightarrow c_! \mathcal{L}_{\widetilde{\mathcal{T}}} \longrightarrow c_! \dot{\xi}_! (\dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}}) \xrightarrow{[1]},$$

where $j : \Xi' \to \Xi$ is the open embedding. By the above argument, $c_! j_! j^* \mathcal{L}_{\widetilde{\mathcal{T}}} = 0$. Therefore $c_! \dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} = c_! \mathcal{L}_{\widetilde{\mathcal{T}}}$

Clearly the composition $si : \Xi \to E_V, (x_1, x_2, x_3) \mapsto (x_1, x_2)$ factors through Ξ/Z since $c : \Xi \to \Xi/Z$ sends (x_1, x_2, x_3) to $(x_1, x_2, \overline{x}_3)$. Let $si = gc$, where $g : \Xi/Z \to E_V$. By projection formula, we have

$$c_!(\dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes c^* g^* B) = c_! \dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes g^* B = c_! \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes g^* B = c_!(\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes c^* g^* B).$$

Therefore,

$$c_!(\dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B) = c_!(\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B). \tag{4.22}$$

The composition $'p\dot{t} : \Xi \to {}'E_T \times {}'E_W, (x_1, x_2, x_3) \mapsto (x'_1, x'_3, x''_1, x''_3)$ also factor through Ξ/Z. Because if $\overline{z}_3 = \overline{x}_3 \in \Xi/Z$, then there exists $y_3 \in Z$ such that $x_3 - z_3 = y_3$. Hence $(x_3 - z_3)(W_{h''}) = 0$ and $(x_3 - z_3)(T_{h''}) = 0$, i.e. $x_3(a) = z_3(a), \forall a \in W_{h''}$ and $a \in T_{h''}$. Therefore $x''_3 = z''_3, x'_3 = z'_3$. Let $'p\dot{t} = hc$ for some $h : \Xi/Z \to {}'E_T \times {}'E_W$, then, by (4.22), we have

$$h_! c_!(\dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B) = h_! c_!(\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B).$$

i.e.

$$'p_! \dot{t}_!(\dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B) = {}'p_! \dot{t}_!(\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B).$$

Since $\mathcal{T}_V i \dot{\xi} = \overline{\mathcal{T}} \dot{p} \dot{q}$, we have $\dot{q}^* \dot{p}^* \mathcal{L}_{\overline{\mathcal{T}}} = \dot{\xi}^* i^* \mathcal{L}_{\mathcal{T}_V} = \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}}$.

Since $\dot{q}$ is a vector bundle with rank $m := n\sum_{h\in H_2}\mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''})$, we have $\dot{q}_!\dot{q}^*L = L[-2m]$ for all $L \in \mathcal{D}(\psi)$. Therefore,

$$\begin{aligned}
\Phi(\overline{\mathrm{Res}}^V_{T,W}B) &= \bar{t}_!(\mathcal{L}_{\overline{\mathcal{T}}} \otimes \bar{s}^* p_! \iota^* B)[d_T + d_W] \\
&= \bar{t}_!(\mathcal{L}_{\overline{\mathcal{T}}} \otimes \dot{p}_! \dot{s}^* \iota^* B)[d_T + d_W] \\
&= \bar{t}_!(\mathcal{L}_{\overline{\mathcal{T}}} \otimes \dot{p}_! \dot{q}_! \dot{q}^* \dot{s}^* \iota^* B[2m])[d_T + d_W] \\
&= \bar{t}_! \dot{p}_! \dot{q}_!(\dot{q}^* \dot{p}^*(\mathcal{L}_{\overline{\mathcal{T}}}) \otimes \dot{q}^* \dot{s}^* \iota^* B)[2m + d_T + d_W] \\
&= {}'p_! \dot{t}_! \dot{\xi}_!(\dot{q}^* \dot{p}^*(\mathcal{L}_{\overline{\mathcal{T}}}) \otimes \dot{\xi}^* i^* s^* B)[2m + d_T + d_W] \\
&= {}'p_! \dot{t}_!(\dot{\xi}_!(\dot{q}^* \dot{p}^* \mathcal{L}_{\overline{\mathcal{T}}}) \otimes i^* s^* B)[2m + d_T + d_W] \\
&= {}'p_! \dot{t}_!(\dot{\xi}_! \dot{\xi}^* \mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B)[2m + d_T + d_W] \\
&= {}'p_! \dot{t}_!(\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B)[2m + d_T + d_W]
\end{aligned}$$

and

$$\begin{aligned}
\overline{\mathrm{Res}}^V_{T,W}(\Phi(B))[\pi] &= {}'p_!\, {}'\iota^* t_!(\mathcal{L}_{\mathcal{T}} \otimes s^* B)[\pi + d_V] \\
&= {}'p_! \dot{t}_! i^*(\mathcal{L}_{\mathcal{T}} \otimes s^* B)[\pi + d_V] \\
&= {}'p_! \dot{t}_!(\mathcal{L}_{\widetilde{\mathcal{T}}} \otimes i^* s^* B)[\pi + d_V].
\end{aligned}$$

Using $\mathrm{Rank}(V_i) = \mathrm{Rank}(T_i) + \mathrm{Rank}(W_i)$, we have $\pi + d_V = 2m + d_T + d_W$. This finishes the proof. $\square$

Lemma 6. *$\Phi(\widetilde{L}^f_{V,\underline{i},\underline{k}}) = {}'\widetilde{L}^f_{V,\underline{i},\underline{k}}[M]$ for some M.*

Proof. This proof is the same as Lusztig's proof for 10.2.2 in [41]. Consider the following diagram,

$$\begin{array}{ccccc}
\widetilde{\mathcal{F}}^f_{V,\underline{i},\underline{k}} & \xleftarrow{b} & \Xi & \xrightarrow{c} & \Xi' \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle p} \\
E_V & \xleftarrow{s} & \dot{E}_V & \xrightarrow{t} & {}'E_V.
\end{array}$$

Here

$$\Xi = \left\{(x, y, \mathfrak{f}) \in E_V \times {}'E_V \times \mathcal{F}^f_{V,\underline{i},\underline{k}} \mid \mathfrak{f}_0 \text{ is an } x-\text{stable and } y_h = x_h, \forall h \in H_2\right\},$$

and

$$\Xi' = \left\{ (y, \mathfrak{f}) \in {}'E_V \times \mathcal{F}^f_{V,\underline{i},\underline{k}} \mid y_h(V^l_{h'0}) \subset V^l_{h''0}, \forall l \text{ and } h \in H_1 \right\}.$$

Maps are defined as follows. $b(x, y, \mathfrak{f}) = (x, \mathfrak{f}), c(x, y, \mathfrak{f}) = (y, \mathfrak{f})$ and π, ρ, p are obvious projection maps. Then the left square is a cartesian square and the right square is commutative. By the definition of $\widetilde{L}^f_{V,\underline{i},\underline{k}}$, we have

$$\Phi(\widetilde{L}^f_{V,\underline{i},\underline{k}}) = t_!(\mathcal{L}_T \otimes s^* \pi_! \mathbf{1})[d_V] = t_!(\mathcal{L}_T \otimes \rho_! \mathbf{1}_\Xi)[d_V].$$

By projection formula in Section 2.2.5,

$$t_!(\mathcal{L}_T \otimes \rho_! \mathbf{1})[d_V] = t_! \rho_! (\rho^* \mathcal{L}_T \otimes \mathbf{1})[d_V] = p_! c_! (\mathcal{L}_{T'})[d_V].$$

The last equality follows from $T' = T\rho$ and $pc = t\rho$.

Let $\Xi_0 = \{(x, y, \mathfrak{f}) \in \Xi \mid \mathfrak{f}_0 \text{ is } y-\text{stable}\}$ and $\Xi_1 = \Xi - \Xi_0$. Clearly, $T'|_{\Xi_1}$ is not a constant function. By Property 2.4(2), $c_!(\mathcal{L}_{T'}|_{\Xi_1}) = 0$. Since $j : \Xi_1 \to \Xi$ is an open embedding, $c_! j_! j^* \mathcal{L}_{T'} = 0$. Applying 2.2.3(4) to the partition $\Xi = \Xi_1 \coprod \Xi_0$, we have a distinguish triangle,

$$c_! j_! j^* \mathcal{L}_{T'} \longrightarrow c_! \mathcal{L}_{T'} \longrightarrow c_! i_! i^* \mathcal{L}_{T'} \xrightarrow{[1]},$$

where $i : \Xi_0 \to \Xi$ is the closed embedding. Then $c_! \mathcal{L}_{T'} = c_! i_! i^* \mathcal{L}_{T'}$.

For any $(x, y, \mathfrak{f}) \in \Xi_0$,

$$T'(x, y, \mathfrak{f}) = T(x, y) = \sum_{h \in H_2} tr(y_h x_h : V_{h'} \to V_{h'}).$$

Let $\mathfrak{f} = g_\mathfrak{f} \mathfrak{f}_0$ for some $g_\mathfrak{f} \in H/(H \bigcap P^R)$ (see Remark 2). Since $\mathfrak{f}_0$ is stable under both x and y, $\mathfrak{f}$ is stable under both $g_\mathfrak{f}^{-1} \cdot x$ and $g_\mathfrak{f}^{-1} \cdot y$. Since $(g_\mathfrak{f}^{-1} \cdot y)_h (g_\mathfrak{f}^{-1} \cdot x)_h = (g_\mathfrak{f}^{-1})_{h'} y_h x_h (g_\mathfrak{f})_{h'}$, $tr(y_h x_h) = tr((g_\mathfrak{f}^{-1})_{h'} y_h x_h (g_\mathfrak{f})_{h'})$. Moreover, we have

$$tr((g_\mathfrak{f}^{-1})_{h'} y_h x_h (g_\mathfrak{f})_{h'} : V_{h'} \to V_{h'}) = \sum_l tr((g_\mathfrak{f}^{-1})_{h'} y_h x_h (g_\mathfrak{f})_{h'} : V^{l-1}_{h'}/V^l_{h'} \to V^{l-1}_{h'}/V^l_{h'}).$$

Since V^{l-1}/V^l concentrate on one vertex, for any l, at least one of $V^{l-1}_{h'}/V^l_{h'}$ and $V^{l-1}_{h''}/V^l_{h''}$ is zero. Therefore, $tr(y_h x_h : V_{h'} \to V_{h'}) = 0$ for each $h \in H_2$. i.e. $T'(x, y, \mathfrak{f}) = 0$. Hence

$\mathcal{L}_{T'}|_{\Xi_0} = \mathbf{1}$. i.e. $i_! i^* \mathcal{L}_{T'} = \mathbf{1}$. Therefore,

$$p_! c_! (\mathcal{L}_{T'})[d_V] = p_! c_! i_! i^* \mathcal{L}_{T'}[d_V] = p_! (c|_{\Xi_0})_! \mathbf{1}[d_V].$$

Since $c(\Xi_0) = \widetilde{\mathcal{F}}^f_{V,\underline{i},\underline{k}}$ and $c|_{\Xi_0}$ is a vector bundle. Denote the rank of $c|_{\Xi_0}$ by M'. Then

$$p_! (c|_{\Xi_0})_! \mathbf{1}[d_V] = p_! \mathbf{1}[d_V - 2M'].$$

Since $p|_{'\widetilde{\mathcal{F}}^f_{V,\underline{i},\underline{k}}} = {'\pi}$, where ${'\pi} : {'\widetilde{\mathcal{F}}^f_{V,\underline{i},\underline{k}}} \to {'E_V}$ is the first projection map,

$$p_! \mathbf{1}[d_V - 2M'] = {'\widetilde{L}^f_{V,\underline{i},\underline{k}}}[d_V - 2M'].$$

Let $M = d_V - 2M'$. The proposition follows. ∎

Corollary 3. $\Phi(\widetilde{\mathrm{Res}}^V_{T,W}(B)) = \widetilde{\mathrm{Res}}^V_{T,W}(\Phi(B))[\pi]$.

Proof. From Lemma 6, $\Phi(\mathcal{Q}^f_V) \subset {'\mathcal{Q}^f_V}$, where ${'\mathcal{Q}^f_V}$ is defined similarly as $\mathcal{Q}^f_V$ for ${'E_V}$. By the same argument $\Phi({'\mathcal{Q}^f_V}) \subset \mathcal{Q}^f_V$. Since $\Phi(\Phi(K)) = K$ (see 10.2.3 in [41]), for any $K \in \mathcal{Q}^{Nf}_V \setminus \mathcal{Q}^f_V$, if $\Phi(K) \in {'\mathcal{Q}^f_V}$, then $K = \Phi(\Phi(K)) \in \mathcal{Q}^f_V$. This is a contradiction. Therefore, $\Phi(K) \notin {'\mathcal{Q}^f_V}$ for any $K \in \mathcal{Q}^{Nf}_V \setminus \mathcal{Q}^f_V$.

By definition of $\widetilde{\mathrm{Res}}^V_{T,W}$, the corollary follows from Proposition 9. ∎

Corollary 4. $\Phi(\mathrm{Res}^V_{T,W}(B)) = \mathrm{Res}^V_{T,W}(\Phi(B))$.

Proof. From (4.20) and Corollary 3, it is enough to show $\pi + d_1 - {'d_1} = 0$, where ${'d_1}$ is defined similarly as d_1 for the new orientation. Recall $d_1 = \mathrm{dim} G_V/U + n\sum_{h\in H}\mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''})$. Since $\mathrm{dim} G_V/U$ has nothing to do with orientations and $H = H_1 \bigcup H_2$, it is enough to show

$$\pi = n\sum_{h\in H_2}\mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''}) - n\sum_{h\in {'H_2}}\mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''}).$$

Here ${'H_2}$ is the set of all arrows with opposite orientation of arrows in H_2. The corollary follows that $\sum_{h\in {'H_2}}\mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''}) = \sum_{h\in H_2}\mathrm{Rank}(T_{h''})\mathrm{Rank}(W_{h'})$. ∎

Lemma 7. *Let $A \in \mathcal{Q}_V$, $A' \in {'\mathcal{Q}_V}$, then for any $j \in \mathbb{Z}$, we have*

$$d_j(E_V, G_V; A, \Phi(A')) = d_j({'E_V}, G_V; \Phi(A), A').$$

Proof. Let u (resp. $u', \dot{u}$) be the map of ${}_\Gamma E_V$ (resp. ${}_\Gamma{}'E_V$, ${}_\Gamma(E_V \times {}'E_V)$) to the point (see Section 4.2.4 for notations). By definition of $d_j(E_V, G_V; A, A')$, we have

$$d_j(E_V, G_V; A, \Phi(A')) = \dim H^{j+2\dim G - 2\dim \Gamma}(\mathrm{pt}, u_!({}_\Gamma A \otimes {}_\Gamma \Phi(A'))).$$

The lemma follows from Section 2.4(1). □

Corollary 5. $\Phi(\mathrm{Ind}^V_{T,W}(B)) = \mathrm{Ind}^V_{T,W}(\Phi(B))$

Proof. By Proposition 4.2.4(6), it is enough to prove

$$d_j({}'E_V, G_V; \Phi(\mathrm{Ind}^V_{T,W}(B)), \Phi(K)) = d_j({}'E_V, G_V; \mathrm{Ind}^V_{T,W}(\Phi(B)), \Phi(K))$$

for all simple objects $K \in \mathcal{P}^f_V$ and $j \in \mathbb{Z}$.

By Lemma 7 and Proposition 8,

$$\begin{aligned} &d_j({}'E_V, G_V; \Phi(\mathrm{Ind}^V_{T,W}(B)), \Phi(K)) \\ &\quad = d_j(E_V, G_V; (\mathrm{Ind}^V_{T,W}(B)), K) \\ &\quad = d_j(E_T \times E_W, G_T \times G_W; B, \mathrm{Res}^V_{T,W}(K)). \end{aligned}$$

By Proposition 8,

$$\begin{aligned} &d_j({}'E_V, G_V; \mathrm{Ind}^V_{T,W}(\Phi(B)), \Phi(K)) \\ &\quad = d_j({}'E_T \times {}'E_W, G_T \times G_W; \Phi(B), \mathrm{Res}^V_{T,W}(\Phi(K))) \\ &\quad = d_j(E_T \times E_W, G_T \times G_W; B, \Phi(\mathrm{Res}^V_{T,W}(\Phi(K)))) \text{ (by Lemma 7)} \\ &\quad = d_j(E_T \times E_W, G_T \times G_W; B, \mathrm{Res}^V_{T,W}(K)) \text{ (by Corollary 4)}. \end{aligned}$$

□

4.2.6 Additive generators

In this section, we fix a vertex $i \in I$ and assume W is an I-grade free R-submodule of V such that $T = V/W$ is also a free R-module.

Remark 5. By the Fourier-Deligne transform, we can assume W satisfy that $W_{h'} = V_{h'}$, $\forall h \in H$ and by induction we can further assume $\mathrm{Supp}(T) = \{i\}$. Hence, $E_T = 0$, and $E_W \simeq F$.

Given any matrix X with entries in R, any k-th minor D_k of X can be written into $D_k(X) = f_{i0}(X) + f_{i1}(X)t + \cdots + f_{ir}(X)t^r$. We will use super-script to distingue the different k-th minors and their coefficients. For example, $D_k^s(X)$ and $f_{kl}^s(X)$. Note that we take all minors with value in $k[t]$ but not those in R since we are studying the coordinate ring of the k-variety E_V.

If we fix an R-basis for each V_i, then all x_h can be written as a matrix, denoted by X_h, with entries in R. Moreover, $\sum_{h\in H, h''=i} x_h$ corresponds to the matrix $X_i := (X_{h_1}, X_{h_2}, \cdots, X_{h_s})$, where each subscript h_j is an arrow with target vertex i. Given $i \in I$, let

$$B_{V,i,k} = \{x \in E_V \mid D_k^r(X_i) = 0 \text{ for all } r\}.$$

Notice that X_i depends on the choice of basis of V_i, but $B_{V,i,k}$ doesn't depend on the choice of basis of V_i. Because equivalent transformations of matrixes change a k-th minor into another k-th minor which is obtained by multiplying by an invertible element in R. Moreover, $B_{V,i,k}$ is a closed subset of E_V. Given $(k,l) \in \mathbb{N} \times \mathbb{N}$, let

$$C_{V,i,(k,l)} = \{x \in E_V \mid f_{ks}^r(X_i) = 0 \text{ for all } r, \text{ and all } s \leq l\}.$$

By the same reason, this set doesn't depend on the choice of basis of V_i and it is a closed subset of E_V.

Now define a total order on $\mathbb{N} \times \mathbb{N}$ by

$$(k,l) < (r,s) \text{ if and only if } k < r \text{ or } k = r, s < l.$$

Let $E_{V,i,\leq(k,l)} = C_{V,i,(k,l)} \bigcap B_{V,i,k+1}$. This is a closed subset. It is clear that

$$B_{V,i,k} \subset \cdots \subset E_{V,i,\leq(k,l)} \subset E_{V,i,\leq(k,l-1)} \subset \cdots \subset B_{V,i,k+1} \subset \cdots \subset E_V. \tag{4.23}$$

Furthermore, for $x \in E_V$, there exists (k,l) such that $x \in E_{V,i,\leq(k,l)}$.

Let $E_{V,i,(k,l)} = E_{V,i,\leq(k,l)} \setminus E_{V,i,\leq(k,l+1)}$. This is a locally closed subset of E_V and its closure $\overline{E_{V,i,(k,l)}} = E_{V,i,\leq(k,l)}$. From linear algebra, $E_{V,i,(k,l)}$ is stable under G_V-action.

Recall $p : G_V \times^P E_W \to E_V$ is a G_V-equivariant map sending (g,x) to $g\iota(x)$, where $\iota : E_W \to E_V$ is an embedding. Let $p_0 := p|_{G_V \times^P E_{W,i,(k,l)}} : G_V \times^P E_{W,i,(k,l)} \to E_{V,i,(k,l)}$.

Lemma 8. *p_0 is a vector bundle with rank $d_0 = (v_i - w_i)(l + n(w_i - k))$, where $v_i = \mathrm{Rank(V_i)}$ and $w_i = \mathrm{Rank(W_i)}$.*

Proof. For any $y \in E_{V,i,(k,l)}$, $p_0^{-1}(y) = \{(g,x) \mid g\iota(x) = y\}$.

If $g_1\iota(x_1) = g_2\iota(x_2) = y$, then $g_1^{-1}g_2\iota(x_2) = \iota(x_1)$. This implies that x_1 is equivalent to x_2. i.e. there exists $g \in G_W$ such that $gx_1 = x_2$. Hence there exists $h \in P$ such that $h\iota(x_1) = \iota(x_2)$. Then $g_1\iota(x_1) = g_2h\iota(x_1)$. This means that $s := g_1^{-1}g_2h$ is in the stabilizer, $\mathrm{Stab}_{G_V}(\iota(x_1))$, in G_V of $\iota(x_1)$. i.e. $g_2 = g_1sh^{-1}$ for some $s \in \mathrm{Stab}_{G_V}(\iota(x_1))$ and $h \in P$. Hence $(g_1, x_1) \in p_0^{-1}(y)$ if and only if $(g_1sh^{-1}, hx_1) \in p_0^{-1}(y)$ for some $s \in \mathrm{Stab}_{G_V}(\iota(x_1))$. Therefore, $\dim(p_0^{-1}(y)) = \dim(\mathrm{Stab}_{G_V}(\iota(x_1))) - \dim(P \bigcap \mathrm{Stab}_{G_V}(\iota(x_1)))$.

If we fix a basis of W and extend it to a basis of V, then

$$\dim(\mathrm{Stab}_{G_V}(\iota(x_1)) = \dim \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right\}),$$

and

$$\dim(P \bigcap \mathrm{Stab}_{G_V}(\iota(x_1)) = \dim \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right\}).$$

Therefore,

$$\dim(p_0^{-1}(y)) = \dim(\{c \in \mathrm{Hom}_R(W_i, T_i) \mid cx_1 = 0\}) \tag{4.24}$$

Given $\underline{j} = (j_0, j_1, \cdots, j_{n-1})$, let

$$M_{i,\underline{j}} = \left\{ x \in E_V \mid X_i \simeq \mathrm{Diag}(1^{\oplus j_0}, t^{\oplus j_1}, \cdots, (t^{n-1})^{\oplus j_{n-1}}) \right\}.$$

If $y \in M_{i,\underline{j}}$ for some $\underline{j}$ with $|\underline{j}| := \sum_r j_r = k$. Then $y \in E_{V,i,(k,\lambda(\underline{j}))}$, where $\lambda(\underline{j}) = \sum_r rj_r$. From the above argument,

$$\dim(p_0^{-1}(y)) = (v_i - w_i)(\lambda(\underline{j}) + n(w_i - |\underline{j}|))$$

which only depends on $\lambda(\underline{j})$ and $|\underline{j}|$.

On the other hand, from linear algebra, if $y \in E_{V,i,(k,l)}$, then $y \in M_{i,\underline{j}}$ such that $|\underline{j}| = k$ and $\lambda(\underline{j}) = l$. i.e. the dimension of fibers of any element in $E_{V,i,(k,l)}$ only depends on k and l. By (4.24), p_0 has a vector bundle structure. This finishes proof. □

Let $\iota_1 : E_{V,i,\leq(k,l)} \to E_V$ be the closed embedding. Applying base change formula to the following cartesian square,

$$\begin{array}{ccc} \widetilde{F}' & \longrightarrow & \widetilde{\mathcal{F}}_{V,\underline{i},\underline{k}} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ E_{V,i,\leq(k,l)} & \xrightarrow{\iota_1} & E_V, \end{array} \tag{4.25}$$

we have

$$\iota_1^* \widetilde{L}_{V,\underline{i},\underline{k}} = \iota_1^* \pi_! \mathbf{1} = \pi'_! \mathbf{1}.$$

Remark 6. For any simple perverse sheaf $A \in \mathcal{P}_V^f$ (see Section 4.2.1), i.e. A is a direct summand of $\widetilde{L}_{V,\underline{i},\underline{k}}$ for some $(\underline{i}, \underline{k})$ up to shift, $\iota_1^* A$ is a direct summand of $\iota_1^* \widetilde{L}_{V,\underline{i},\underline{k}} = \pi'_! \mathbf{1}$ up to shift. By the same argument as we show Proposition 3, $\iota_1^* A[d_A]$ is a perverse sheaf on $E_{V,i,\leq(k,l)}$ for some d_A.

Let $\mathcal{P}^f_{V,i,(k,l)}$ be the full subcategory of $\mathcal{M}(E_{V,i,\leq(k,l)})$ consisting of direct sums of perverse sheaves $\iota_1^* A$ up to shifts for some $A \in \mathcal{P}_V^f$. Let

$$\mathcal{P}^{f0}_{V,i,(k,l)} = \{B \in \mathcal{P}^f_{V,i,(k,l)} \mid \mathrm{Supp}(B) \bigcap E_{V,i,(k,l)} \neq \emptyset\}$$

and

$$\mathcal{P}^{f1}_{V,i,(k,l)} = \{B \in \mathcal{P}^f_{V,i,(k,l)} \mid \mathrm{Supp}(B) \bigcap E_{V,i,(k,l)} = \emptyset\}.$$

Then any object $A \in \mathcal{P}^f_{V,i,(k,l)}$ can be decomposed into $A = A^0 \oplus A^1$, where $A^0 \in \mathcal{P}^{f0}_{V,i,(k,l)}$ and $A^1 \in \mathcal{P}^{f1}_{V,i,(k,l)}$. Furthermore, if we require A^1 is the maximal subobject of A in $\mathcal{P}^{f1}_{V,i,(k,l)}$, then such decomposition is unique since A is a semisimple perverse sheaf.

One can similarly define $\mathcal{P}^f_{W,i,(k,l)}$ (resp. $\mathcal{P}^{f0}_{W,i,(k,l)}$ and $\mathcal{P}^{f1}_{W,i,(k,l)}$).

Now consider the following diagrams,

$$E_{W,i,\le(k,l)} \xleftarrow{p'_1} G_V \times^U E_{W,i,\le(k,l)} \xrightarrow{p'_2} G_V \times^P E_{W,i,\le(k,l)} \xrightarrow{p'_3} E_{V,i,\le(k,l)} \tag{4.26}$$

$$E_{W,i,\le(k,l)} \xrightarrow{\iota'} E_{V,i,\le(k,l)}. \tag{4.27}$$

Here G_V, U, P are the same as in Section 4.2.3 and the maps are defined similarly as in the Section 4.2.3 and 4.2.2. Define the functors $\widetilde{\mathrm{Ind}}^V_{T,W,i,(k,l)}(A) = p'_{3!}p'_{2\flat}p'^*_1(A)$ and $\widetilde{\mathrm{Res}}^V_{T,W.i,(k,l)}(B) = \iota'^*(B)$. This is the induction (resp. restriction) functor defined on $E_{W,i,\le(k,l)}$ (resp. $E_{V,i,\le(k,l)}$) instead of E_W (resp. E_V).

Let

$$\mathrm{Ind}^V_{T,W,i,(k,l)}(A) = \widetilde{\mathrm{Ind}}^V_{T,W,i,(k,l)}(A)[\dim(G_V/P) + d_0], \tag{4.28}$$

where d_0 is the dimension of fibers of p_0, and

$$\mathrm{Res}^V_{T,W,i,(k,l)}(A) = \widetilde{\mathrm{Res}}^V_{T,W,i,(k,l)}(A)[d_0 - \dim(G_V/P)]. \tag{4.29}$$

Lemma 9. *Let $\iota_2 : E_{W,i,\le(k,l)} \to E_W$ be the closed embedding, then for any $A \in \mathcal{P}^f_{W,i,(k,l)}$,*

$$\widetilde{\mathrm{Ind}}^V_{T,W}(\iota_{2!}A) = \iota_{1!}\widetilde{\mathrm{Ind}}^V_{T,W,i,(k,l)}(A).$$

Proof. Consider the following diagram:

$$\begin{array}{ccccccc}
E_{W,i,\le(k,l)} & \xleftarrow{p'_1} & G_V \times^U E_{W,i,\le(k,l)} & \xrightarrow{p'_2} & G_V \times^P E_{W,i,\le(k,l)} & \xrightarrow{p'_3} & E_{V,i,\le(k,l)} \\
\downarrow{\scriptstyle \iota_2} & \boxed{1} & \downarrow{\scriptstyle \iota_3} & \boxed{2} & \downarrow{\scriptstyle \iota_4} & \boxed{3} & \downarrow{\scriptstyle \iota_1} \\
E_W & \xleftarrow{p_1} & G_V \times^U E_W & \xrightarrow{p_2} & G_V \times^P E_W & \xrightarrow{p_3} & E_V.
\end{array} \tag{4.30}$$

Here vertical maps are all closed embedding. Since the squares $\boxed{1}$ and $\boxed{2}$ are cartesian squares and $\boxed{3}$ is commutative, we have

$$p_1^*\iota_{2!}A = \iota_{3!}p'^*_1A = \iota_{3!}p'^*_2(\widetilde{A}) = p_2^*\iota_{4!}(\widetilde{A}).$$

Here $\widetilde{A}$ is the unique complex such that $p'^*_1A = p'^*_2\widetilde{A}$ by Property 2.3(3).

Hence $\iota_{4!}\widetilde{A} = p_{2\flat}p_1^*\iota_{2!}A$. Therefore,

$$\widetilde{\mathrm{Ind}}^V_{T,W}(\iota_{2!}A) = p_{3!}\iota_{4!}\widetilde{A} = \iota_{1!}p'_{3!}\widetilde{A} = \iota_{1!}\widetilde{\mathrm{Ind}}^V_{T,W,i,(k,l)}(A).$$

□

Corollary 6. *For any* $A \in \mathcal{P}^f_{W,i,(k,l)}$,

$$\mathrm{Ind}^V_{T,W}(\iota_{2!}A) = \iota_{1!}\,\mathrm{Ind}^V_{T,W,i,(k,l)}(A)[N],$$

where $N = (n-1)\sum_i \mathrm{Rank}(T_i)\mathrm{Rank}(W_i) + n\sum_{h\in H}\mathrm{Rank}(T_{h'})\mathrm{Rank}(W_{h''}) - d_0$.

Proof. By Lemma 9, (4.19) and (4.28). □

Lemma 10. *Let* $b: Y \to X$ *be a fiber bundle with* d *dimensional connected smooth irreducible fiber. If* $B = b^*A$ *is a perverse sheaf on* Y, *then* $b_!B[d]$ *is a perverse sheaf on* X.

Proof. By the definition of perverse sheaves, $B \in \mathcal{D}^{\geq 0}(Y)\bigcap \mathcal{D}^{\leq 0}(Y)$. Then $b_!B \in \mathcal{D}^{\leq d}(X)$. i.e. $b_!B[d] \in \mathcal{D}^{\leq 0}(X)$.

On the other hand, by Lemma 4,

$$\mathbb{D}(b_!B[d]) = \mathbb{D}(b_!B)[-d] = (b_!\mathbb{D}B)[d] \in \mathcal{D}^{\leq d}(X)[d] = \mathcal{D}^{\leq 0}(X).$$

This proves the lemma. □

Proposition 10. (1) *Let* $A \in \mathcal{P}^{f0}_{W,i,(k,l)}$. *Then* $H^n\,\mathrm{Ind}^V_{T,W,i,(k,l)}(A) \in \mathcal{P}^{f1}_{V,i,(k,l)}$ *if* $n \neq 0$, *and* $H^0\,\mathrm{Ind}^V_{T,W,i,(k,l)}(A) \in \mathcal{P}^{f0}_{V,i,(k,l)}$ *So one can define a functor*

$$\xi : \mathcal{P}^{f0}_{W,i,(k,l)} \to \mathcal{P}^{f0}_{V,i,(k,l)}$$
$$A \mapsto (H^0\,\mathrm{Ind}^V_{T,W,i,(k,l)}(A))^0.$$

(2) *Let* $B \in \mathcal{P}^{f0}_{V,i,(k,l)}$. *Then* $H^n\,\mathrm{Res}^V_{T,W,i,(k,l)}(B) \in \mathcal{P}^{f1}_{W,i,(k,l)}$ *if* $n \neq 0$ *and* $H^0\,\mathrm{Res}^V_{T,W,i,(k,l)}(B) \in \mathcal{P}^{f0}_{W,i,(k,l)}$. *So one can define a functor*

$$\rho : \mathcal{P}^{f0}_{V,i,(k,l)} \to \mathcal{P}^{f0}_{W,i,(k,l)}$$
$$B \mapsto (H^0\,\mathrm{Res}^V_{T,W,i,(k,l)}(B))^0.$$

(3) *The functors* $\xi : \mathcal{P}^{f0}_{W,i,(k,l)} \to \mathcal{P}^{f0}_{V,i,(k,l)}$ *and* $\rho : \mathcal{P}^{f0}_{V,i,(k,l)} \to \mathcal{P}^{f0}_{W,i,(k,l)}$ *give an equivalence of categories* $\mathcal{P}^{f0}_{V,i,(k,l)}$ *and* $\mathcal{P}^{f0}_{W,i,(k,l)}$.

Proof. The proof is based on Lusztig's idea for proving Proposition 9.3.3 in [41]. Consider the following diagram,

$$
\begin{array}{ccccc}
G_V \times^P E_{W,i,(k,l)} & \xrightarrow{p_0} & E_{V,i,(k,l)} & \xleftarrow{\iota_0} & E_{W,i,(k,l)} \\
\downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle m} \\
G_V \times^P E_{W,i,\leq(k,l)} & \xrightarrow{p_3'} & E_{V,i,\leq(k,l)} & \xleftarrow{\iota'} & E_{W,i,\leq(k,l)}.
\end{array}
\tag{4.31}
$$

Here ι_0, ι', j, j_0 and m are all inclusions, both squares are cartesian squares. Additionally, both j and m are open embeddings.

(1) For any $A \in \mathcal{P}^{f0}_{W,i,(k,l)}$,

$$j^*\widetilde{\mathrm{Ind}}^V_{T,W,I',\eta}A = j^*p'_{3!}p'_{2\flat}p'^*_1(A) = p_{0!}j_0^*p'_{2\flat}p'^*_1(A).$$

By Property 2.3(3), $p'_{2\flat}p'^*_1(A)[\dim(G_V/P)]$ is a perverse sheaf. j_0 is an open embedding, so $j_0^*p'_{2\flat}p'^*_1(A)[\dim(G_V/P)]$ is a perverse sheaf. Moreover $j_0^*p'_{2\flat}p'^*_1(A)[\dim(G_V/P)]$ is a G_V-equivariant perverse sheaf.

We claim that $j_0^*p'_{2\flat}p'^*_1(A) = p_0^*\iota_{0*}m^*A$.

By the following commutative diagram,

$$
\begin{array}{ccccc}
E_{W,i,\leq(k,l)} & \xleftarrow{p_1'} & G_V \times^U E_{W,i,\leq(k,l)} & \xrightarrow{p_2'} & G_V \times^P E_{W,i,\leq(k,l)} \\
\uparrow{\scriptstyle m} & & \uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle j_0} \\
E_{W,i,(k,l)} & \xleftarrow{p_1''} & G_V \times^U E_{W,i,(k,l)} & \xrightarrow{p_2''} & G_V \times^P E_{W,i,(k,l)}
\end{array}
$$

we have

$$j_0^*p'_{2\flat}p'^*_1A = p''_{2\flat}j_1^*p'^*_1A = p''_{2\flat}p''^*_1m^*A. \tag{4.32}$$

We next consider the following commutative diagram,

$$
\begin{array}{ccccc}
 & & G_V \times^U E_{W,i,(k,l)} & & \\
 & \swarrow{\scriptstyle p} & \uparrow{\scriptstyle \pi} & \searrow{\scriptstyle u} & \\
E_V & \xleftarrow{q_1} & G_V \times E_{W,i,(k,l)} & \xrightarrow{u_1} & E_V \\
 & \nwarrow{\scriptstyle q_2} & \downarrow{\scriptstyle \iota} & \nearrow{\scriptstyle u_2} & \\
 & & G_V \times E_{V,i,(k,l)} & &
\end{array}
\tag{4.33}
$$

Here p (resp. q_1, q_2) is the projection map sending (g,x) to $\iota_0(x)$ (resp. $\iota_0(x), x$) and u (resp. u_1, u_2) is the G_V-action map sending (g,x) to $g\iota_0(x)$ (resp. $g\iota_0(x), gx$). π is the quotient map and ι is the embedding. p is well-defined since U acts on E_W trivially in this case.

For any G_V-equivariant complex K, $q_2^*K = u_2^*K$. Then

$$q_1^*K = \iota^* q_2^* K = \iota^* u_2^* K = u_1^* K.$$

Therefore $\pi^*p^*K = \pi^*u^*K$ which implies $p^*K = u^*K$ since π is a principle U-bundle.

Now consider the following diagram,

$$\begin{array}{ccc} E_{W,i,(k,l)} & \xleftarrow{p_1''} & G_V \times^U E_{W,i,(k,l)} \\ \downarrow{\scriptstyle \iota_0} & \overset{p}{\underset{u}{\swarrow}} & \downarrow{\scriptstyle p_2''} \\ E_{V,i,(k,l)} & \xleftarrow{p_0} & G_V \times^P E_{W,i,(k,l)} \end{array} \tag{4.34}$$

where $p_1''(g,x) = x$; $u(g,x) = g\iota_0(x)$ and $p(g,x) = \iota_0(x)$. By commutativity, for any G_V-equivariant complex K, we have

$$p_0^*K = p_{2\flat}''u^*K = p_{2\flat}''p^*K = p_{2\flat}''p_1''^*\iota_0^*K \tag{4.35}$$

Since $\iota_0 : E_{W,i,(k,l)} \to E_{V,i,(k,l)}$ is the inclusion of a locally closed subset, by [44], we have

$$\iota_0^*\iota_{0*}m^*A = m^*A \tag{4.36}$$

By (4.32), (4.35) and (4.36), we have

$$j_0^*p_{2\flat}'p_1'^*A = p_{2\flat}''p_1''^*\iota_0^*\iota_{0*}m^*A = p_0^*\iota_{0*}m^*A.$$

This proves the claim.

Therefore, by Lemma 10, $j^*\widetilde{\mathrm{Ind}}_{T,W,I',\eta}^V A[\dim(G_V/P)+d_0]$ is a perverse sheaf on $E_{V,i,(k,l)}$. Since j is an open embedding, j^* is exact. if $n \neq 0$,

$$j^*(H^n(\widetilde{\mathrm{Ind}}_{T,W,i,(k,l)}^V A[\dim(G_V/P)+d_0])) = H^n(j^*\widetilde{\mathrm{Ind}}_{T,W,i,(k,l)}^V A[\dim(G_V/P)+d_0]) = 0.$$

i.e.

$$j^*(H^n(\operatorname{Ind}^V_{T,W,i,(k,l)} A)) = 0.$$

Therefore, the support of $H^n \operatorname{Ind}^V_{T,W,i,(k,l)} A$ is disjoint from $E_{V,i,(k,l)}$.

(2) For any $B \in \mathcal{P}^{f0}_{V,i,(k,l)}$, j^*B is a perverse sheaf since j is an open embedding.

We claim that $\iota_0^* j^* B[d_0 - \dim(G_V/P)]$ is a perverse sheaf on $E_{W,i,(k,l)}$.

In Diagram (4.34), p_1'' is a fiber bundle of relative dimension $\dim(G_V/U)$ since $\operatorname{Supp}(T) = \{i\}$ and U acts on E_W trivially.

By the commutativity, we have

$$p_2''^* p_0^* j^* B = u^* j^* B = p^* j^* B = p_1''^* \iota_0^* j^* B.$$

Then

$$\iota_0^* j^* B[d_0 - \dim(G_V/P)] = p''_{1!} p_2''^* p_0^* j^* B[d_0 + \dim(G_V/U) + \dim(G_V/P)].$$

By Lemma 8, $p_2''^* p_0^* j^* B[d_0 + \dim(P/U)]$ is a perverse sheaf. From Lemma 10, $\iota_0^* j^* B[d_0 - \dim(G_V/P)]$ is a perverse sheaf on $E_{W,i,(k,l)}$. This proves the claim.

Since right hand square in Diagram (4.31) is commutative,

$$m^* \iota'^* B[d_0 - \dim(G_V/P)] = \iota_0^* j^* B[d_0 - \dim(G_V/P)]$$

which is a perverse sheaf. Since m is open embedding,

$$m^*(H^n \iota'^* B[d_0 - \dim(G_V/P)]) = H^n(m^* \iota'^* B[d_0 - \dim(G_V/P)]).$$

If $n \neq 0$, support of $H^n \iota'^* B[d_0 - \dim(G_V/P)]$ is disjoint form $E_{V,i,(k,l)}$.

(3) From the proof of (1), we have

$$j^* \xi(A) = j^* p'_{3!} p'_{2\flat} p_1'^* A[\dim(G_V/P) + d_0] = p_{0!} j_0^* p'_{2\flat} p_1'^* A[\dim(G_V/P) + d_0].$$

Hence

$$j^*(\xi(\rho(B))) = p_{0!} j_0^* p'_{2\flat} p_1'^* \rho(B)[\dim(G_V/P) + d_0] = p_{0!} j_0^* p'_{2\flat} p_1'^* \iota'^*(B)[2d_0].$$

Consider the following diagram,

$$\begin{array}{ccc} E_{W,i,\leq(k,l)} & \xleftarrow{p'_1} & G_V \times^U E_{W,i,\leq(k,l)} \\ \downarrow{\scriptstyle \iota'} & \swarrow{\scriptstyle p,\ u} & \downarrow{\scriptstyle p'_2} \\ E_{V,i,\leq(k,l)} & \xleftarrow{p'_3} & G_V \times^P E_{W,i,\leq(k,l)}. \end{array}$$

Here $u(g,x) = g\iota'(x)$ and $p(g,x) = \iota_0(x)$. By the same reason as above, p is well-defined. By commutativity, we have

$$p_3'^* B = p'_{2\flat} u^* B = p'_{2\flat} p^* B = p'_{2\flat} p_1'^* \iota'^* B \tag{4.37}$$

From Diagram (4.31), $p_0^* j^* B = j_0^* p_3'^* B$. Then

$$j_0^* p'_{2\flat} p_1'^* \iota'^*(B) = j_0^* p_3'^* B = p_0^* j^* B.$$

Therefore,

$$j^*(\xi(\rho(B))) = p_{0!} p_0^* j^* B[2d_0].$$

By Lemma 8, $j^*(\xi(\rho(B))) = j^* B$. Since $B \in \mathcal{P}^{f0}_{V,i,(k,l)}$ and $E_{V,i,(k,l)}$ is open in $E_{V,i,\leq(k,l)}$, we have $\xi(\rho(B)) = B$.

On the other hand, from the proof of (1), we have

$$m^*(\rho(B)) = \iota_0^* j^* B[d_0 - \dim(G_V/P)].$$

Hence,

$$m^*(\rho(\xi(A))) = \iota_0^* j^* \xi(A)[d_0 - \dim(G_V/P)] = \iota_0^* p_{0!} j_0^* p'_{2\flat} p_1'^* A[2d_0].$$

From the proof of (1), we have

$$m^*(\rho(\xi(A))) = \iota_0^* p_{0!} p_0^* \iota_{0*} m^* A[2d_0] = \iota_0^* \iota_{0*} m^* A = m^* A.$$

Since $A \in \mathcal{P}^{f0}_{W,i,(k,l)}$ and $E_{W,i,(k,l)}$ is open in $E_{W,i,\leq(k,l)}$, we have $\rho(\xi(A)) = A$. □

Remark 7. By Proposition 5, if $E_W \simeq F$, then $\mathrm{Res}^V_{T,W,i,(k,l)}$ send any element of $\mathcal{P}^f_{V,i,(k,l)}$ into $\mathcal{P}^f_{W,i,(k,l)}$, and the map ρ is well defined.

Let v be an indeterminate and $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Let $\mathcal{M}_V$ be the Grothendieck group of the category which consists of all direct sums of $L_{V,\underline{i},\underline{k}}$ for various $(\underline{i}, \underline{k})$ and their shifts. Define an $\mathcal{A}$-action on $\mathcal{M}_V$ by $v^n \cdot L = L[n]$. Then $\mathcal{M}_V$ is an $\mathcal{A}$-module generated by $L_{V,\underline{i},\underline{k}}$. Let $\mathcal{K}_V$ be the Grothendieck group of category $\mathcal{Q}_V^f$ (see Section 4.2.1). Then under the same $\mathcal{A}$-action, $\mathcal{K}_V$ is an $\mathcal{A}$-module generated by the simple perverse sheaves in $\mathcal{P}_V^f$.

Theorem 4. *$\mathcal{M}_V \simeq \mathcal{K}_V$ as an $\mathcal{A}$-module, i.e., $\{L_{V,\underline{i},\underline{k}} \mid \forall (\underline{i}, \underline{k})\}$ are the additive generators of $\mathcal{K}_V$.*

Proof. Clearly, $\mathcal{M}_V \subseteq \mathcal{K}_V$ since $L_{V,\underline{i},\underline{k}}$ is a direct sum of simple perverse sheaves in $\mathcal{P}_V^f$. By abuse of notation, we will denote by the same B the isomorphism class of B in $\mathcal{K}_V$ (resp. $\mathcal{M}_V$). One only needs to show $B \in \mathcal{M}_V$ for any simple perverse sheaf $B \in \mathcal{P}_V^f$.

We first use induction on $Td(V) := \sum_{i \in I} \mathrm{Rank}(V_i)$. If $V = 0$, then $E_V = \{\mathrm{pt}\}$. So there is only one simple perverse sheaf and thus the theorem is true. Now assume the theorem is true for any I-graded proper R-submodule W of V. We want to show the theorem is true for V.

Suppose B is a simple direct summand of $L_{V,\underline{j},\underline{l}}$, where $(\underline{j}, \underline{l}) = ((i,k), (\underline{j}', \underline{l}'))$. Recall $L_{V,\underline{j},\underline{l}} = \mathrm{Ind}_{T,W}^V(L_{T,i,k} \boxtimes L_{W,\underline{j}',\underline{l}'})$. Since $\mathrm{Supp}(T) = \{i\}$, $E_T = \{\mathrm{pt}\}$. We will simply write $L_{T,i,k} \boxtimes L_{W,\underline{j}',\underline{l}'}$ as $L_{W,\underline{j}',\underline{l}'}$. By Fourier-Deligne transform, we can further assume i is a sink. By the definition of $\mathrm{Ind}_{T,W}^V$, we have

$$\mathrm{Supp}(B) \subset \mathrm{Supp}(L_{V,\underline{j},\underline{l}}) \subset \mathrm{Supp}(\mathrm{Ind}_{T,W}^V(L_{W,\underline{j}',\underline{l}'})) \subset E_{V,i,\leq(m,l)}$$

for some $(m,l) \in \mathbb{N} \times \mathbb{N}$. By (4.23), we can choose maximal (m,l) such that $\mathrm{Supp}(B) \subset E_{V,i,\leq(m,l)}$ and $\mathrm{Supp}(B)$ meets $E_{V,i,(m,l)}$.

Recall $\iota_1 : E_{V,i,\leq(m,l)} \hookrightarrow E_V$ is the closed embedding. Since $\mathrm{Supp}(B) \subset E_{V,i,\leq(m,l)}$, by the definition of $\mathcal{P}_{V,i,(m,l)}^{f0}$, we have $\iota_1^* B \in \mathcal{P}_{V,i,(m,l)}^{f0}$. Let $\rho(\iota_1^* B) = A$. By Proposition 10, $\xi(A) = \iota_1^* B$. i.e.,

$$\mathrm{Ind}_{T,W,i,m}^V A = \iota_1^* B \oplus (\oplus_j C_j[j]) \tag{4.38}$$

for some C_j satisfying $\mathrm{Supp}(C_j) \subset E_{V,i,\leq(m,l)}$ and disjoint with $E_{V,i,(m,l)}$. Therefore $\mathrm{Supp}(C_j) \subset E_{V,i,\leq(r,s)}$ with $(r,s) < (m,l)$. By applying $\iota_{1!}$ to (4.38) and Corollary 6,

$$\mathrm{Ind}_{T,W}^{V} \iota_{2!}A = \iota_{1!}\, \mathrm{Ind}_{T,W,i,(m,l)}^{V} A[N] = \iota_{1!}\iota_1^* B[N] \oplus (\oplus_j \iota_{1!} C_j[N+j]),$$

where $\iota_2 : E_{W,i,\leq(m,l)} \hookrightarrow E_W$ is the closed embedding and N is defined in Corollary 6. Since $\mathrm{Supp}(B) \subset E_{V,i,\leq(m,l)}$,

$$\iota_{1!}\iota_1^* B = B|_{E_{V,i,\leq(m,l)}} = B.$$

Now we want to show $\iota_{2!}A \in \mathcal{M}_W$.

By the definition of ρ, in fact, $\iota_{2!}A$ is a direct summand of $\iota_{2!}\iota'^*\iota_1^* B[d_0 - \dim(G_V/P)]$. Applying base change formula to the following cartesian square,

$$\begin{array}{ccc} E_{W,i,\leq(m,l)} & \xrightarrow{\iota'} & E_{V,i,\leq(m,l)} \\ \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_1} \\ E_W & \xrightarrow{\iota} & E_V, \end{array}$$

we have

$$\iota_{2!}\iota'^*\iota_1^* B = \iota^* \iota_{1!}\iota_1^* B = \iota^* B = \widetilde{\mathrm{Res}}_{T,W}^{V} B.$$

By Proposition 4, $\iota_{2!}A \in \mathcal{K}_W$. Since $Td(W) < Td(V)$, by the assumption, $\iota_{2!}A \in \mathcal{M}_W$. Therefore, $\mathrm{Ind}_{T,W}^{V} \iota_{2!}A \in \mathcal{M}_V$ by Corollary 2.

To show $B \in \mathcal{M}_V$, it is enough to show $\iota_{1!}C_j \in \mathcal{M}_V$. Since $\iota_{1!}C_j$ is a direct summand of $\mathrm{Ind}_{T,W}^{V} \iota_{2!}A$ and $\iota_{2!}A \in \mathcal{Q}_W^f$, by Proposition 6, $\iota_{1!}C_j \in \mathcal{Q}_V^f$.

To apply induction on (m,l), it is enough to show $C_j[j] \in \mathcal{M}_V$ if $\mathrm{Supp}(C_j[j]) \subset E_{V,i,\leq(0,l)}$. By a similar argument as above, there exists $K \in \mathcal{P}_{W,i,\leq(0,l)}^{f0}$ such that

$$\mathrm{Ind}_{T,W}^{V}(\iota'_{2!}K) = \iota'_{1!}\iota_1'^* C_j[M+j] = C_j[M+j]$$

for some M, where ι'_1 and ι'_2 are the embedding maps. By induction on $Td(V)$, $C_j[j] \in \mathcal{M}_V$ since $\iota'_{2!}K \in \mathcal{M}_W$ as we have shown above.

The theorem follows from the induction on (m,l). □

4.3 Geometric approach to Hall algebras and quantum generalization Kac-Moody algebras

4.3.1 The algebra ($\mathcal{K}$, Ind)

Recall the dimension vector of an I-graded free R-module V is defined as $|V| := (\mathrm{Rank}(V_i))_{i\in I} \in \mathbb{N}I$. It is important to notice that, given two different I-graded free R-modules V and V' with the same dimension vector, $\mathcal{K}_V \simeq \mathcal{K}_{V'}$ since E_V and $E_{V'}$ are isomorphism spaces. So one may denote $\mathcal{K}_V$ by $\mathcal{K}_{|V|}$. Moreover, the functors $\mathrm{Ind}^V_{T,W}$ and $\mathrm{Res}^V_{T,W}$ can be rewritten as $\mathrm{Ind}^{|T|+|W|}_{|T|,|W|}$ and $\mathrm{Res}^{|T|+|W|}_{|T|,|W|}$ respectively. Now let $\mathcal{K} = \oplus_{|V|\in\mathbb{N}I}\mathcal{K}_{|V|}$. Define multiplication as follows.

$$\mathrm{Ind} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$$
$$(A, B) \mapsto \mathrm{Ind}^{|T|+|W|}_{|T|,|W|}(A \otimes B)$$

for homogenous elements A, B with $A \in \mathcal{K}_{|T|}$ and $B \in \mathcal{K}_{|W|}$.

Theorem 5. (1) $\mathcal{K}$ *equipped the multiplication* Ind *is an I-graded associated $\mathcal{A}$-algebra.*

(2) $\{L_{V,\underline{i},\underline{k}}$ | for all V and $(\underline{i}, \underline{k})\}$ *contains an $\mathcal{A}$-basis of $\mathcal{K}$. This basis is called a monomial basis.*

(3) *All simple perverse sheaves in $\mathcal{P}^f_V$ for various V form an $\mathcal{A}$-basis of $\mathcal{K}$. This basis is called the canonical basis.*

Proof. (1) follows from Theorem 4, Corollary 2 and additivity of Ind. (2) follows from Theorem 4. (3) follows from the definition of $\mathcal{K}$. □

In the rest of this section we will give the relation among the Hall algebra $C\mathcal{H}_R$ (see Section 4.1), the algebra $\mathcal{K}$, and the quantum generalized Kac-Moody algebra.

4.3.2 Relation between $\mathcal{K}$ and U_v^-

Let I be a countable index set. A simply laced *generalized root datum* (see [23]) is a matrix $A = (a_{ij})_{i,j\in I}$ satisfying the following conditions:

(i) $a_{ii} \in \{2, 0, -2, -4, \cdots\}$, and

(ii) $a_{ij} = a_{ji} \in \mathbb{Z}_{\leq 0}$.

Such a matrix is a special case of Borcherds-Cartan matrix. Let $I^{re} = \{i \in I \mid a_{ii} = 2\}$ and $I^{im} = I \setminus I^{re}$. A collection of positive integers $m = (m_i)_{i \in I}$ with $m_i = 1$ whenever $i \in I^{re}$ is called the charge of A.

The *quantum generalized Kac-Moody algebra* (see [23]) associated with (A, m) is the $\mathbb{Q}(v)$-algebra $U_v(\mathfrak{g}_{A,m})$ generated by the elements K_i, K_i^{-1}, $E_{i,k}$, and $F_{i,k}$ for $i \in I, k = 1, \cdots, m_i$ subject to the following relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \ K_i K_j = K_j K_i, \tag{4.39}$$

$$K_i E_{jk} K_i^{-1} = v^{a_{ij}} E_{jk}, \ K_i F_{jk} K_i^{-1} = v^{-a_{ij}} F_{jk}, \tag{4.40}$$

$$E_{ik} F_{jl} - F_{jl} E_{ik} = \delta_{lk} \delta ij \frac{K_i - K_i^{-1}}{v - v^{-1}}, \tag{4.41}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix} E_{ik}^{1-a_{ij}-n} E_{jl} E_{ik}^n = 0, \forall i \in I^{re}, j \in I, i \neq j, \tag{4.42}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix} F_{ik}^{1-a_{ij}-n} F_{jl} F_{ik}^n = 0, \forall i \in I^{re}, j \in I, i \neq j, and \tag{4.43}$$

$$E_{ik} E_{jl} - E_{jl} E_{ik} = F_{ik} F_{jl} - F_{jl} F_{ik} = 0, \text{ if } a_{ij} = 0. \tag{4.44}$$

Here $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}$, $[n]! = \prod_{i=1}^n [i]$, and $[n] = \frac{v^n - v^{-n}}{v - v^{-1}}$.

In this dissertation, we only consider the case in which all $m_i = 1$ and all indices are in I^{im}.

Define a bilinear form on $\mathcal{K}$ as follows,

$$(A, B)_{\mathcal{K}} = \sum_j d_j(E_V, G_V; A, B) v^{-j}.$$

Proposition 11. *The bilinear form $(-, -)_{\mathcal{K}}$ defined above is non-degenerate.*

Proof. Firstly, by the properties of $d_j(E, G; A, B)$ (see Section 4.2.4), this is a bilinear form. Secondly, by Theorem 5, all simple perverse sheaves in $\mathcal{P}_V^f$ for various V form a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{K}$. So, for any $A \in \mathcal{K}$, A can be written as $A = \sum_K c_K K$ for $c_K \in \mathbb{Z}[v, v^{-1}]$. Now for any $A \in \mathcal{K}$, let B is a direct summand of A. Then by Property (3) of $d_j(E, G; A, B)$ in Section 4.2.4, $(A, B)_{\mathcal{K}} = c_B \neq 0$. Hence the bilinear form is non-degenerate. □

Let U_v^- be the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $U_v(\mathfrak{g}_{A,m})$ generated by all $F_{i,k}$ with $k = 1, \cdots, m_i$. Because we only consider the case that the charge $m_i = 1$ for all $i \in I$, there is only one generator $F_{i,1}$ for each i, which we will simply denote by F_i. It is clear that U_v^- only subjects to one relation, namely $F_iF_j = F_jF_i$ if $a_{ij} = 0$.

Define a multiplication of $U_v^- \otimes U_v^-$ as

$$(A \otimes B)(C \otimes D) := v^{-n(|B|,|C|)}(AC) \otimes (BD),$$

where $|B|$ is the grading of B when B is a homogeneous element and

$$(|B|, |C|) = \langle |B|, |C| \rangle + \langle |C|, |B| \rangle,$$

where $\langle |B|, |C| \rangle$ is defined in (4.2).

Let $\mathfrak{F}$ be the free algebra generated by $\{F_i \mid i \in I\}$. Let $r' : \mathfrak{F} \to U_v^- \otimes U_v^-$ be the algebra homomorphism sending F_i to $F_i \otimes 1 + 1 \otimes F_i$. Since $r'(F_iF_j) = r'(F_jF_i)$ if $a_{ij} = 0$, the map r' induces an algebra homomorphism $r : U_v^- \to U_v^- \otimes U_v^-$. This gives a coalgebra structure on U_v^-.

Define a map

$$f : U_v^- \to \mathcal{K}$$
$$F_i \mapsto L_{i,1}.$$

It is easy to check that $f(F_iF_j) = f(F_jF_i)$. So this map can be extended to an algebra homomorphism. In addition, f preserves the grading, where the grading of $B \in \mathcal{K}$ is defined as dimension vector $|W|$ when B is a homogeneous element in $\mathcal{K}_W$. Now define a bilinear form $(-, -)_U$ on U_v^- as $(A, B)_U := (f(A), f(B))_{\mathcal{K}}$.

Theorem 6. Ker$(f) = Rad(-,-)_U =: \mathcal{I}_1$, *so* $U_v^-/\mathcal{I}_1 \simeq \mathcal{K}$.

Proof. Obviously, Ker$(f) \subset \mathcal{I}_1$.

Let us pick any $x \in \mathcal{I}_1$. Then for any $y \in \mathcal{K}$, there exists $z \in U_v^-$ such that $f(z) = y$ due to the fact that f is surjective. Therefore,

$$0 = (x, z)_U = (f(x), f(z))_{\mathcal{K}} = (f(x), y)_{\mathcal{K}}.$$

This means $f(x) \in Rad(-,-)_{\mathcal{K}}$. Since the bilinear form $(-,-)_{\mathcal{K}}$ is non-degenerate, $f(x) = 0$. i.e., $x \in$ Ker(f). Hence $U_v^-/\mathcal{I}_1 \simeq \mathcal{K}$. □

4.3.3 Relation between $\mathcal{K}$ and $C\mathcal{H}_R$

Let $\mathcal{H}(R\Gamma)^*$ be the dual Hall algebra of $\mathcal{H}(R\Gamma)$, i.e., $\mathcal{H}(R\Gamma)^* = \oplus_\nu \mathcal{H}(R\Gamma)^*_\nu$. Here $\mathcal{H}(R\Gamma)^*_\nu$ is the set of all $\mathbb{C}$-valued functions on the set of isomorphism classes of all representations M of Γ over R with dimension vector $|M| = \nu$. The multiplication on $\mathcal{H}(R\Gamma)^*$ is defined as follows:

$$(f_1 \cdot f_2)(E) = \sum_{N \subset E} f_1(E/N) f_2(N).$$

See [31] for more information. Let $C\mathcal{H}_R^*$ be the subalgebra of $\mathcal{H}(R\Gamma)^*$ generated by $\delta_{S_i}, \forall i \in I$, where δ_{S_i} is the characteristic function of S_i. i.e.

$$\delta_{S_i}(x) = \begin{cases} 1 & \text{if } x = S_i \\ 0 & \text{otherwise.} \end{cases}$$

By the formular,

$$(\delta_M \cdot \delta_N)(E) = \#\{L \subset E \mid L \simeq N, E/L \simeq M\} = F^E_{M,N}.$$

$C\mathcal{H}_R^*$ is isomorphic to the algebra $C\mathcal{H}_R$.

Now define

$$\chi : \mathcal{K} \to C\mathcal{H}_R^*$$
$$A \mapsto \chi_A,$$

where $\chi_A(x) = Tr(Fr : A^{\cdot}_x \to A^{\cdot}_x)$ (see Section 2.5).

Lemma 11 (Theorem 4.1(b) in [31]). *$\chi : \mathcal{K} \to C\mathcal{H}_R^*$ is an algebra homomorphism.*

Theorem 7. *χ is a surjective algebra homomorphism.*

Proof. By Lemma 11, it is enough to show $\chi_{L_i} = \delta_{S_i}$ for any δ_{S_i}. In fact,

$$\chi_{L_i}(x) = \chi_{\pi_{i!}\mathbf{1}}(x) = \sum_{y \in \pi_i^{-1}(x)} \chi_{\mathbf{1}}(y) = \chi_{\mathbf{1}}(\pi_i^{-1}(x)) = \delta_{S_i}.$$

Here π_i is the obvious projection map. The penultimate equality is true because both E_V and $\widetilde{\mathcal{F}}_V$ contain a single point. □

Let us denote $\mathcal{I}_2 = \mathrm{Ker}(\chi)$. Then $C\mathcal{H}_R \simeq \mathcal{K}/\mathcal{I}_2$.

Remark 8. One may ask what the kernel $\mathcal{I}_1$ of the above map f is. For the field case, Lusztig and Ringel show this is the ideal generated by the quantum Serre relations. However, for the case where we have the local ring $R = k[t]/(t^n)$, the kernel $\mathcal{I}_1$ is much more complicated. Let's finish this chapter with the following example which gives some idea of what $\mathcal{I}_1$ is.

Example 2. Fix $R = \mathbb{F}_q[t]/(t^n)$, consider quiver $A_2 : 1 \to 2$. Then $S_1 : R \to 0$ and $S_2 : 0 \to R$ are all simple objects. By computation, one has,

$S_1^2 = q^{n/2}(q^n + q^{n-1})(R^2 \to 0)$,

$S_1S_2 = q^{-n/2}[(R \xrightarrow{0} R) + (R \xrightarrow{1} R) + (R \xrightarrow{t} R) + \cdots + (R \xrightarrow{t^{n-1}} R)]$,

$S_2S_1 = R \xrightarrow{0} R$,

$S_1^2S_2 = q^{-n/2}(q^n + q^{n-1})[(R^2 \xrightarrow{0} R) + (R^2 \xrightarrow{(1,0)} R) + (R^2 \xrightarrow{(t,0)} R) + \cdots + (R^2 \xrightarrow{(t^{n-1},0)} R)]$,

$S_1S_2S_1 = (q^n + q^{n-1})(R^2 \xrightarrow{0} R) + (R^2 \xrightarrow{(1,0)} R) + q(R^2 \xrightarrow{(t,0)} R) + \cdots + q^{n-1}(R^2 \xrightarrow{(t^{n-1},0)} R)$,

and

$S_2S_1^2 = q^{n/2}(q^n + q^{n-1})(R^2 \xrightarrow{0} R)$.

There is no quantum Serre relation at this time.

Chapter 5

Character sheaves of $GL_m(k[t]/(t^2))$

In this chapter, we will construct character sheaves of $GL_m(k[t]/(t^2))$ and define induction and restriction functors on them. It is an important observation that $GL_m(k[t]/(t^n)) = GL_m(k) \ltimes H$, where H is a unipotent algebraic group (see Section 5.3). In the special case that $n = 2$, H is an abelian group. Little group method gives a way to list irreducible characters of semidirect product of two groups. We will give a weak version of geometric little group method.

Throughout this chapter, all algebraic group are over $k := \overline{\mathbb{F}}_q$. F is a Frobenius map on the given algebraic group G, and G^F is the finite subgroup of G consisting of all fixed points by F.

5.1 Character sheaves of abelian groups

In this section, we review the definition of character sheaves of abelian algebraic group. For the construction, we refer to [40].

Let H be an abelian algebraic group over k and defined over finite field $\mathbb{F}_q$, and $L : H \to H$ be the Lang map sending x to $F(x)x^{-1}$, where F is a Frobenius map on H. For any $y \in H$, the stalk of the local system $E := L_!\mathbf{1}_H$ at y is the vector space E_y consisting of all functions $f : L^{-1}(y) \to \overline{\mathbb{Q}}_l$. Then

$$E_y = \oplus_{\phi \in \mathrm{Hom}(H^F, \overline{\mathbb{Q}}_l^*)} E_y^{\phi},$$

where

$$E_y^\phi = \left\{ f \in E_y \mid f(zx) = \phi(z)f(x), \text{ for all } z \in H^F \text{ and } x \in L^{-1}(y) \right\}.$$

We have a decomposition $E = \oplus_\phi E^\phi$, where E^ϕ is a local system of rank 1 on H whose stalk at $y \in H$ is E_y^ϕ.

For any $y \in H$, and $f \in (F^* E^\phi)_y = E_{F(y)}^\phi$, let

$$\begin{aligned} \psi : E_{F(y)}^\phi &\to E_y^\phi \\ f(x) &\mapsto f'(x) := f(F(x)), \ \forall x \in L^{-1}(y). \end{aligned}$$

If $y = 1$ is the identity element in H, then $F(x)x^{-1} = 1$ implies $F(x) = x$. So $f' = f$. i.e. ψ is an identity map.

Now for any $y \in H^F$,

$$\chi_{E^\phi, F}(y) = Tr(\psi : E_{F(y)}^\phi \to E_y^\phi) = Tr(\psi : E_y^\phi \to E_y^\phi) = Tr(\psi : f \mapsto f(F(x))).$$

By the construction of E^ϕ,

$$f(F(x)) = f(yx) = \phi(y)f(x), \forall x \in L^{-1}(y).$$

So ψ is a multiplication by $\phi(y)$. Therefore, $\chi_{E^\phi, F}(y) = \phi(y)$ which gives a irreducible character of H.

For $r \in \mathbb{N}$, let $L^r : H \to H$ be the map $x \mapsto F^r(x)x^{-1}$ and $E^r = L^r_! \mathbf{1}$. Similarly, the stalk of E^r at $y \in H$ is the vector space E_y^r consisting of all functions $f' : (L^r)^{-1}(y) \to \overline{\mathbb{Q}}_l$. Now define a map $\xi_y : E_y \to E_y^r$ sending f to $f \circ N_{F^r, F}$, where $N_{F^r, F} : H \to H$ is the map $x \mapsto xF(x) \cdots F^{r-1}(x)$. It is clear that $N_{F^r, F}((L^r)^{-1}(y)) \subset L^{-1}(y)$. So ξ_y is well-defined. Moreover, it gives an isomorphism $E^\phi \simeq (E^r)^{\phi'}$, where $\phi' = \phi \circ N_{F^r, F} \in \mathrm{Hom}(H^{F^r}, \overline{\mathbb{Q}}_l^*)$.

The rank 1 local system E^ϕ on H for some (F, ϕ) is called a character sheaves on H. Let $CS(H)$ be the set of isomorphism classes of character sheaves of H.

5.2 Little group method

5.2.1 Induced character for finite group

Given a finite group G, denote by $\mathrm{Irr}(G)$ the set of all isomorphism classes of irreducible characters of G. Let H be a subgroup of G. For any $f \in \mathrm{Irr}(H)$, the induced character of f from H to G, denoted it by $\mathrm{Ind}_H^G f$, can be calculated by the following formula, more detail of induced character can be found in [8].

$$(\mathrm{Ind}_H^G f)(b) = \frac{1}{|H|} \sum_{g \in G} \widetilde{f}(gbg^{-1}), \forall\, b \in G,$$

where $\widetilde{f}(a) = \begin{cases} f(a) & \text{if } a \in H \\ 0 & \text{if } a \notin H \end{cases}$. By abuse of notation, we still denote $\widetilde{f}$ by f in the rest of this chapter.

5.2.2 Algebraic little group method

Let $G = A \ltimes H$ be a finite group such that H is an abelian group. The group G acts on $\mathrm{Irr}(H)$ by

$$g \cdot \chi(h) = \chi(g^{-1}hg), \forall g \in G, \chi \in \mathrm{Irr}(H),\ h \in H.$$

Let (χ_i) be the set of representative of orbits of A in $\mathrm{Irr}(H)$. For each χ_i, let A_i be the stabilizer of χ_i in A, and $G_i = A_i \ltimes H$. Let ρ be an irreducible character of A_i. We will extend χ_i (resp. ρ) to G_i as follows. By abuse of notation, we will use the same notation to denote the extended functions.

$$\chi_i(g) = \chi_i(p_2(g)),\ \rho(g) = \rho(p_1(g)), \forall g \in G_i. \tag{5.1}$$

Here $p_1 : G_i \to A_i$ (resp. $p_2 : G_i \to H$) is the obvious projection map. It is easy to check that $\rho \otimes \chi_i$ is an irreducible character of G_i. Let $\theta_{i,\rho} = \mathrm{Ind}_{G_i}^G(\rho \otimes \chi_i)$.

Proposition 12 (proposition 25 in [53]). (1) *$\theta_{i,\rho}$ is an irreducible character of G.*

(2) *If $\theta_{i,\rho}$ and $\theta_{j,\rho'}$ are isomorphic, then $i = j$ and ρ is isomorphic to ρ'.*

(3) *Every irreducible character of G is isomorphic to one of $\theta_{i,\rho}$.*

5.2.3 Geometric little group method

Let $G = A \ltimes H$ be an algebraic group such that H is an abelian algebraic group. We are going to construct character sheaves of G. Let $CS(H)$ be the set of isomorphism classes of character sheaves of H. There is a natural G-action on H. Hence, for any element $g \in G$, we can define a map $g : H \to H$ sending x to $g^{-1}xg$. The following lemma shows that pull back along $g \in G^F$ is compatible with g-action on $\mathrm{Irr}(H^F)$.

Lemma 12. *For any Frobenius map F on G, and $\mathcal{F} \in CS(H)$, if $g \in G^F$, then*

$$\chi_{g^*\mathcal{F},F} = g \cdot \chi_{\mathcal{F},F}.$$

Proof. For any $y \in H^F$,

$$\chi_{g^*\mathcal{F},F}(y) = \chi_{\mathcal{F},F}(g^{-1}yg) = (g \cdot \chi_{\mathcal{F},F})(y).$$

□

We want to define a G-action on $CS(H)$. Firstly, for any $g \in G$, there exists $r \in \mathbb{Z}$ such that $g \in G^{F^r}$, then $(g \circ F \circ g^{-1})^r = g \circ F^r \circ g^{-1} = F^r$, where we consider g as an automorphism of H as above. Now for any character sheaf $\mathcal{F} \in CS(H)$, by the construction in Section 5.1, $\mathcal{F} = E^{\phi}$ for some (F, ϕ), then $\chi_{\mathcal{F},F} = \phi$. Moreover $\mathcal{F} \simeq E^{\phi'}$, where $\phi' = \phi \circ N_{F^r,F}$, then $\chi_{\mathcal{F},F^r} = \phi'$. By Lemma 12, $\chi_{g^*\mathcal{F},F^r} = g \cdot \chi_{\mathcal{F},F^r} \in \mathrm{Hom}(H^{F^r}, \overline{\mathbb{Q}}_l^*)$. Therefore, we can define a G-action on $CS(H)$ as following,

$$g \cdot \mathcal{F} = g^*\mathcal{F}, \text{ for all } g \in G, \quad \mathcal{F} \in CS(H).$$

Let $CS(H)/G$ be the set of orbits of G in $CS(H)$. For any representative $\mathcal{F}$ of an orbit in $CS(H)/G$, let $A_{\mathcal{F}}$ be the stabilizer of $\mathcal{F}$ in A, and $G_{\mathcal{F}} = A_{\mathcal{F}} \ltimes H$. Clearly, $G_{\mathcal{F}}$ is a closed subgroup. Let $CS(G_{\mathcal{F}})$ be the set of isomorphism classes of character sheaves of $G_{\mathcal{F}}$. Consider the following diagram.

$$G_{\mathcal{F}} \xrightarrow{\iota} G \xleftarrow{u} G \times G \xrightarrow{p} G,$$

where ι is the closed embedding; p is the first projection and $u(g,h) = hgh^{-1}$.

The G-actions on the above diagram as follows. G acts on the second G as $g' \cdot g = g'gg'^{-1}$; G acts on $G_{\mathcal{F}}$ and the first G trivially and G acts on $G \times G$ as $g' \cdot (g,h) = (g'gg'^{-1}, hg'^{-1})$. Then all maps in the above diagram are G-equivariant maps.

For any $\mathcal{G} \in CS(A_{\mathcal{F}})$, define

$$\mathrm{Ind}_{G_{\mathcal{F}}}^{G}(\mathcal{G} \boxtimes \mathcal{F}) = p_! u^* \iota_! (\mathcal{G} \boxtimes \mathcal{F}).$$

Proposition 13. *For any $\mathcal{F} \in CS(H)$ and $\mathcal{G} \in CS(A_{\mathcal{F}})$, we have*

$$\chi_{\mathrm{Ind}_{G_{\mathcal{F}}}^{G}(\mathcal{G}\boxtimes\mathcal{F}),F} = \mathrm{Ind}_{G_{\mathcal{F}}^{F}}^{G^F}(\chi_{\mathcal{G},F} \otimes \chi_{\mathcal{F},F}),$$

where $\chi_{\mathcal{G},F}$ (resp. $\chi_{\mathcal{F},F}$) is consider as the class function on $G_{\mathcal{F}}^{F}$ by trivial extension as in (5.1).

Proof. By the definition of $\mathrm{Ind}_{G_{\mathcal{F}}}^{G}$ and properties of characteristic function, for any $g \in G^F$,

$$\begin{aligned}
\chi_{\mathrm{Ind}_{G_{\mathcal{F}}}^{G}(\mathcal{G}\boxtimes\mathcal{F}),F}(g) &= \chi_{p_! u^* \iota_!(\mathcal{G}\boxtimes\mathcal{F}),F}(g) \\
&= \sum_{h \in G^F} \chi_{u^*\iota_!(\mathcal{G}\boxtimes\mathcal{F}),F}(g,h) \\
&= \sum_{h \in G^F} \chi_{\iota_!(\mathcal{G}\boxtimes\mathcal{F}),F}(hgh^{-1}) \\
&= \sum_{h \in G^F} \chi_{(\mathcal{G}\boxtimes\mathcal{F}),F}(hgh^{-1}) \\
&= \mathrm{Ind}_{G_{\mathcal{F}}^{F}}^{G^F}(\chi_{\mathcal{G},F} \otimes \chi_{\mathcal{F},F})(g).
\end{aligned}$$

The penultimate equality holds because ι is a closed embedding and $\iota_!$ is just extension by 0. □

5.3 Character sheaves of $T \ltimes H$

Consider $G := GL_m(k[t]/(t^n))$ as an algebraic group over k. Let $G_0 = GL_m(k)$ which is a subgroup of G. Denote by H the quotient group G/G_0. It is clear that H is a unipotent group. In the case that $n = 2$, H is an abelian connected algebraic subgroup of G. Moreover,

$G = G_0 \ltimes H$. Let T be a maximal torus of G_0. In this section, we will construct character sheaves of $T \ltimes H$ for $n = 2$ as an example of semidirect product of groups.

5.3.1 Construction of character sheaves $T \ltimes H$

Let $CS(H)$ be the set of isomorphism classes of character sheaves of H. There is natural T-action on H since H is a normal subgroup of G. Hence, each element $g \in T$ defines a map $g : H \to H$ sending h to $g^{-1}hg$. For any $\mathcal{F} \in CS(H)$, let

$$T_{\mathcal{F}} = \{g \in T \mid g^*\mathcal{F} = \mathcal{F}\}.$$

On the other hand, there is an induced T^F-action on $\mathrm{Irr}(H^F)$ defined by

$$(g \cdot \chi)(h) = \chi(g^{-1}hg), \ \forall \chi \in \mathrm{Irr}(H^F), \ g \in T^F, \ h \in H^F.$$

For each $\chi \in \mathrm{Irr}(H^F)$, let $T_\chi^F = \left\{g \in T^F \mid g \cdot \chi = \chi\right\}$.

Consider the following diagram,

$$H \xleftarrow{u} T \times H \xrightarrow{p} H \xrightarrow{\iota} T \ltimes H. \tag{5.2}$$

Here $u(t,h) = t^{-1}ht$; p is the second projection and ι is the closed embedding. We define T-actions and H-actions as follows. T acts on the first H trivially; T acts on the second H as $t' \cdot h = t'ht'^{-1}$; T acts on $T \ltimes H$ as $t' \cdot th = tt'ht'^{-1}$ and T acts on $T \times H$ as,

$$t' \cdot (t,h) = (t't, t'ht'^{-1}), \forall t, t' \in T, \quad h \in H.$$

H acts on H and $T \times H$ trivially and acts on $T \ltimes H$ as

$$h' \cdot th = h'thh'^{-1} = t(t^{-1}h'thh'^{-1}).$$

T-actions and H-actions induce a $T \ltimes H$-actions on the above diagram. It is easy to check that u, p and ι are H-equivariant maps and $T \ltimes H$-equivariant maps.

For any $\mathcal{F} \in CS(H)$, define

$$\mathrm{Ind}_H^{T \ltimes H} \mathcal{F} = \iota_! p_! u^* \mathcal{F}.$$

Lemma 13. *For any $\mathcal{F} \in CS(H)$, we have*

$$\mathrm{Ind}_{H^F}^{(T \ltimes H)^F} \chi_{\mathcal{F},F} = \chi_{\mathrm{Ind}_H^{T \ltimes H} \mathcal{F},F}.$$

Proof. For any $b \in (T \ltimes H)^F$,

$$\chi_{\mathrm{Ind}_H^{T \ltimes H} \mathcal{F},F}(b) = \chi_{\iota_! p_! u^* \mathcal{F},F}(b) = \sum_{h \in \iota^{-1}(b)} \chi_{p_! u^* \mathcal{F},F}(h).$$

If $b \notin H^F$, this is just 0. If $b \in H^F$, then

$$\sum_{h \in \iota^{-1}(b)} \chi_{p_! u^* \mathcal{F},F}(h) = \chi_{p_! u^* \mathcal{F},F}(b) = \sum_{t \in T} \chi_{u^* \mathcal{F},F}(t,b) = \sum_{t \in T} \chi_{\mathcal{F},F}(tbt^{-1}).$$

On the other hand,

$$(\mathrm{Ind}_{H^F}^{(T \ltimes H)^F} \chi_{\mathcal{F},F})(b) = \frac{1}{|H^F|} \sum_{th \in (T \ltimes H)^F} \chi_{\mathcal{F},F}(thbh^{-1}t^{-1}).$$

Since $thbh^{-1}t^{-1} \in H^F$ if and only if $b \in H^F$. If $b \notin H^F$, then

$$(\mathrm{Ind}_{H^F}^{(T \ltimes H)^F} \chi_{\mathcal{F},F})(b) = 0.$$

If $b \in H^F$, then

$$(\mathrm{Ind}_{H^F}^{(T \ltimes H)^F} \chi_{\mathcal{F},F})(b) = \sum_{t \in T^F} \chi_{\mathcal{F},F}(tbt^{-1}).$$

□

Lemma 14. *Given $f \in \mathrm{Irr}(H^F)$, let $S = T_f$, then for all $g \in \mathrm{Irr}(S^F)$ and $ab \in T \ltimes H$,*

$$(\mathrm{Ind}_{(S \ltimes H)^F}^{(T \ltimes H)^F}(g \otimes f))(ab) = \frac{1}{|T^F|}(\mathrm{Ind}_{S^F}^{T^F} g)(a)(\mathrm{Ind}_{H^F}^{(T \ltimes H)^F} f)(b).$$

Proof.

$$\begin{aligned}
&(\mathrm{Ind}_{(S \ltimes H)^F}^{(T \ltimes H)^F}(g \otimes f))(ab) \\
&= \frac{1}{|S^F||H^F|} \sum_{th \in (T \ltimes H)^F} (g \otimes f)(thabh^{-1}t^{-1}) \\
&= \frac{1}{|S^F||H^F|} \sum_{th \in (T \ltimes H)^F} (g \otimes f)((tat^{-1})((ta^{-1})h(at^{-1})t(bh^{-1})t^{-1})) \\
&= \frac{1}{|S^F||H^F|} \sum_{th \in (T \ltimes H)^F} g(tat^{-1}) f((ta^{-1})h(at^{-1})t(bh^{-1})t^{-1}) \\
&= \frac{1}{|S^F||H^F|} \sum_{th \in (T \ltimes H)^F} g(a) f((ta^{-1})h(at^{-1})t(bh^{-1})t^{-1}).
\end{aligned}$$

If $a \notin S^F$, both sides are 0. The lemma is true.

Since H is an abelian group, f is a group homomorphism. If $a \in S^F$, then the left hand side is

$$\begin{aligned}
&\frac{1}{|S^F||H^F|}\sum_{th\in(T\ltimes H)^F} g(a)f((ta^{-1})h(at^{-1}))f(tbt^{-1})f(th^{-1}t^{-1})\\
&=\frac{1}{|S^F||H^F|}\sum_{th\in(T\ltimes H)^F} g(a)f(a^{-1}(tht^{-1})a)f(tbt^{-1})f(th^{-1}t^{-1})\\
&=\frac{1}{|S^F||H^F|}\sum_{th\in(T\ltimes H)^F} g(a)f(tht^{-1})f(tbt^{-1})f(th^{-1}t^{-1}) \text{ (since } a\in S^F)\\
&=\frac{1}{|S^F||H^F|}\sum_{th\in(T\ltimes H)^F} g(a)f(tbt^{-1})\\
&=\frac{1}{|S^F|}\sum_{t\in T^F} g(a)f(tbt^{-1})\\
&=\frac{1}{|S^F|}g(a)\sum_{t\in T^F} f(tbt^{-1})\\
&=\frac{1}{|T^F|}(\mathrm{Ind}_{S^F}^{T^F} g)(a)(\mathrm{Ind}_{H^F}^{(T\ltimes H)^F} f)(b).
\end{aligned}$$

□

Let $CS(T)$ be the set of isomorphism classes of character sheaves of T. For any $\mathcal{F} \in CS(H)$, $T_{\mathcal{F}}$ is a closed subgroup of T. Let $j_{\mathcal{F}} : T_{\mathcal{F}} \hookrightarrow T$ be the inclusion map.

Let

$$\mathcal{P}(T \ltimes H) = \{j_{\mathcal{F}*}j_{\mathcal{F}}^*\mathcal{G} \boxtimes p_!u^*\mathcal{F} \mid \mathcal{F} \in CS(H), \mathcal{G} \in CS(T)\},$$

where p, u are the maps defined in Diagram (5.2).

Let

$$(\mathcal{P}(T \ltimes H))^F = \{A \in \mathcal{P}(T \ltimes H) \mid F^*A \simeq A\}.$$

Definition 3. For any $A \in \mathcal{P}(T \ltimes H)$, a simple constituent of ${}^pH^i(A)$ for some i is called a *character sheaf* on $T \ltimes H$.

Denote by $CS(T \ltimes H)$ the set of representatives of isomorphism class of character sheaves on $T \ltimes H$.

Theorem 8. *If $A = j_{\mathcal{F}*}j_{\mathcal{F}}^*\mathcal{G} \boxtimes p_!u^*\mathcal{F} \in (\mathcal{P}(T \ltimes H))^F$ satisfies $F^*\mathcal{G} \simeq \mathcal{G}$ and $F^*\mathcal{F} \simeq \mathcal{F}$, then $\frac{1}{|T^F_{\chi_{\mathcal{F},F}}|}\chi_{A,F}$ is an irreducible character of $(T \ltimes H)^F$. Moreover, all irreducible characters arise in this way.*

Proof. Let $p_1 : T \ltimes H \to T$ and $p_2 : T \ltimes H \to H$ be the first and second projection maps respectively. Then, for any $ab \in T \ltimes H$,

$$\chi_{A,F}(ab) = \chi_{p_1^* j_{\mathcal{F}*}j_{\mathcal{F}}^*\mathcal{G},F}(ab)\chi_{p_2^* p_! u^*\mathcal{F},F}(ab) = \chi_{j_{\mathcal{F}*}j_{\mathcal{F}}^*\mathcal{G},F}(a)\chi_{p_! u^*\mathcal{F},F}(b).$$

If $a \notin T_{\mathcal{F}}$, the first term is 0 since $j_{\mathcal{F}*}j_{\mathcal{F}}^*\mathcal{G}$ is just the restriction of $\mathcal{G}$ to $T_{\mathcal{F}}$.

If $a \in T_{\mathcal{F}}$, by Lemma 13,

$$\chi_{A,F}(ab) = \chi_{\mathcal{G},F}(a)\chi_{p_! u^*\mathcal{F},F}(b) = \chi_{\mathcal{G},F}(a)(\mathrm{Ind}_{H^F}^{(T \ltimes H)^F}\chi_{\mathcal{F},F})(b).$$

By the proof of Lemma 14 and Proposition 12, $\frac{1}{|T^F_{\chi_{\mathcal{F},F}}|}\chi_{A,F}$ is an irreducible character of $(T \ltimes H)^F$ and all irreducible characters of $(T \ltimes H)^F$ arise in this way. The theorem follows. □

5.3.2 Induction functor

Recall $G = G_0 \ltimes H$ with H a unipotent normal subgroup. Let $p_1 : G \to G_0$, and $p_2 : G \to H$ be the projection maps. For any Levi subgroup L of G, let P be a parabolic subgroup of G containing L. Denote $TL = T \ltimes p_2(L)$ and $TP = T \ltimes p_2(P)$. Using this notation, we also have $TG = T \ltimes H$. We will define induction functor and restriction functor on $CS(TG)$.

Consider the following diagram,

$$TL \xleftarrow{\pi} TV \xrightarrow{p} TG. \tag{5.3}$$

Here $TV = \{(g,h) \in TG \times TG \mid h^{-1}gh \in TP\}$ and p is the first projection map. $\pi : (g,h) \mapsto \pi_1(h^{-1}gh)$, where $\pi_1 : TP \to TL$ is the obvious map. We define TL-actions and TG-actions as follows.

TL acts on itself by conjugation, i.e. $t \cdot h = t^{-1}ht$; acts on TG trivially; and acts on TV as $t \cdot (g,h) = (g,ht)$. Then both π and p are TL-equivariant maps.

TG acts on TL trivially; acts on TG by conjugation, i.e. $t \cdot g = tgt^{-1}$; and acts on TV as $t \cdot (g, h) = (tgt^{-1}, th)$. Then both π and p are TG-equivariant maps.

Define

$$\operatorname{Ind}_{TL}^{TG} A = p_* \pi^* A, \forall A \in CS(TL).$$

Lemma 15. *Consider the following diagram,*

$$\begin{array}{ccccc} TL & \xleftarrow{\pi} & TV & \xrightarrow{p} & TG \\ \uparrow{\scriptstyle \pi_1} & \swarrow{\scriptstyle p_1} & & \searrow{\scriptstyle \iota} & \uparrow{\scriptstyle p_2} \\ TP & \xrightarrow{\iota_1} & TG & \xleftarrow{u} & TG \times TG. \end{array} \tag{5.4}$$

Then,

$$\operatorname{Ind}_{TL}^{TG} A = p_{2*} u^* \iota_{1!} \pi_1^* A.$$

Proof. The middle square is a cartesian square, so $\iota_! p_1^* B = u^* \iota_{1!} B$ for any $B \in \mathcal{D}(TP)$. Therefore,

$$\operatorname{Ind}_{TL}^{TG} A = p_{2*} \iota_* p_1^* \pi_1^* A = p_{2*} \iota_! p_1^* \pi_1^* A = p_{2*} u^* \iota_{1!} \pi_1^* A.$$

□

5.3.3 Restriction functor

Using the same notation as last section, let $\pi^0 : p_1(P) \to p_1(L)$ and $\pi^1 : p_2(P) \to p_2(L)$ be the natural projection maps, and let $\iota^0 : p_1(P) \to p_1(G)$ and $\iota^1 : p_2(P) \to p_2(G)$ be the natural embedding. Consider the following diagram,

$$TL \xleftarrow{\pi} TP \xrightarrow{\iota} TG. \tag{5.5}$$

Here $\pi = Id \times \pi^1$ and $\iota = Id \times \iota^1$.

Define

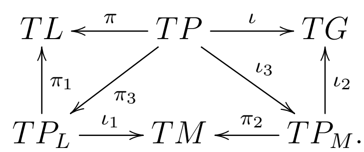
$$\operatorname{Res}_{TL}^{TG} A = \pi_! \iota^* A[2d_1],$$

where d_1 is the dimension of fibers of π.

Proposition 14. *Assume L is a Levi subgroup of M and M is a Levi subgroup of G. Then, for any $A \in \mathcal{D}(TG)$,*

$$\operatorname{Res}_{TL}^{TG} A = \operatorname{Res}_{TL}^{TM}(\operatorname{Res}_{TM}^{TG} A).$$

Proof. Denote TP_L (resp. TP_M, TP) the parabolic subgroup of TM (resp. TG, TG) containing TL (resp. TM, TL). Consider the following diagram,



The middle square is a cartesian square, so

$$\pi_{1!}\iota_1^*\pi_{2!}\iota_2^*A = \pi_{1!}\pi_{3!}\iota_3^*\iota_2^*A = \pi_!\iota^*A.$$

Since the dimension of fibers of π is equal to the sum of the dimension of fibers of π_1 and the dimension of fibers of π_2. The proposition follows the definition of restriction functor. □

5.3.4 Adjunction

Proposition 15. *For any $A \in D_{TG}^b(TG)$ and $B \in D_{TL}^b(TL)$, we have*

$$\operatorname{Hom}(A, \operatorname{Ind}_{TL}^{TG} B) = \operatorname{Hom}(\operatorname{Res}_{TL}^{TG} A, B).$$

Proof. Consider Diagram (5.4). Let d_1 (resp. d_2) be the dimesnsion of fibers of p_2 (resp.

π_1). By Lemma 15,

$$\begin{aligned}
&\mathrm{Hom}(A, \mathrm{Ind}_{TL}^{TG} B) \\
&= \mathrm{Hom}(A, p_{2*}u^*\iota_{1!}\pi_1^*B) \\
&= \mathrm{Hom}(p_2^*A, u^*\iota_{1!}\pi_1^*B) \\
&= \mathrm{Hom}(p_2^*A, p_2^*\iota_{1!}\pi_1^*B) \text{ (Since } \iota_{1!}\pi_1^*B \text{ is a } TG - \text{equivariant complex)} \\
&= \mathrm{Hom}(p_2^*A, p_2^!\iota_{1!}\pi_1^*B[-2d_1]) \\
&= \mathrm{Hom}(p_{2!}p_2^*A, \iota_{1!}\pi_1^*B[-2d_1]) \\
&= \mathrm{Hom}(A[-2d_1], \iota_{1!}\pi_1^*B[-2d_1]) \text{ (Since } p_2 \text{ is a vector bundle)} \\
&= \mathrm{Hom}(A, \iota_{1!}\pi_1^*B) \\
&= \mathrm{Hom}(A, \iota_{1*}\pi_1^*B) \text{ (Since } \iota_1 \text{ is a closed embedding)} \\
&= \mathrm{Hom}(\iota_1^*A, \pi_1^*B) \\
&= \mathrm{Hom}(\iota_1^*A, \pi_1^!B[-2d_2]) \\
&= \mathrm{Hom}(\pi_{1!}\iota_1^*A, B[-2d_2]) \\
&= \mathrm{Hom}(\mathrm{Res}_{TL}^{TG} A, B).
\end{aligned}$$

□

5.4 Character sheaves of $GL_m(k[t]/(t^2))$

5.4.1 Character sheaves of reductive algebraic groups

In this section, we will review Lusztig's construction for connected reductive algebraic groups. We refer to [32] for this section.

We fix a Borel subgroup $B \subset G_0 = GL_m(k)$, a unipotent radical U and a maximal torus $T \subset B$. Let $W = N_{G_0}(T)/T$ be the weyl group of G_0. We will fix a representative $\dot{w}$ of w. By abuse of notation, we still denote it by w.

Now any $w \in W$ can be regarded as an automorphism $w : T \to T$ sending t to wtw^{-1}.

For any $\mathcal{L} \in CS(T)$, let

$$W_{\mathcal{L}} = \{w \in W \mid w^*\mathcal{L} \simeq \mathcal{L}\}.$$

Now consider the following diagram,

$$T \xleftarrow{p_2} \dot{Y}_w \xrightarrow{p_1} Y_w \xrightarrow{\pi_w} G_0. \tag{5.6}$$

Here

$$\dot{Y}_w = \left\{(g, hU) \in G_0 \times (G_0/U) \mid h^{-1}gh \in BwB\right\}.$$

and

$$Y_w = \left\{(g, hB) \in G_0 \times G_0/B \mid h^{-1}gh \in BwB\right\}.$$

Maps are defined as follows. π_w is the first projection; $p_1(g, hU) = (g, hBh^{-1})$; and $p_2(g, hU) = pr(h^{-1}gh)$, where $pr : BwB \to T$ sending u_1wtu_2 to t.

T acts on T as $a \cdot t = w^{-1}awta^{-1}$; T acts on $\dot{Y}_w$ as $t \cdot (g, hU) = (g, ht^{-1}U)$ and T acts on Y_w, G trivially. Then p_1, p_2, π_w are all T-equivariant maps. Moreover, p_1 is a principle T-bundle.

G_0 acts on itself by conjugation, i.e. $a \cdot g = aga^{-1}$; G_0 acts on T trivially; G_0 acts on Y_w as $a \cdot (g, hB) = (aga^{-1}, ahB)$; G_0 acts on $\dot{Y}_w$ as $a \cdot (g, hU) = (aga^{-1}, ahU)$. Then all maps are G_0-equivariant maps.

For any $\mathcal{L} \in CS(T)$ and $w \in W_{\mathcal{L}}$, define $K_w^{\mathcal{L}} = (\pi_w)_! p_{1\flat} p_2^*(\mathcal{L})$.

A simple constituent of ${}^pH^i(K_w^{\mathcal{L}})$ for some $w \in W_{\mathcal{L}}$, $i \in \mathbb{Z}$ and $\mathcal{L} \in CS(T)$ is called a character sheaf of G_0. Denote by $CS(G_0)$ the set of all character sheaves of G_0.

Now we want to construct character sheaves of $G = GL_m(k[t]/(t^2))$.

Consider the following diagram

$$G \xleftarrow{u} G_0 \times G \xrightarrow{p} G.$$

Here $u(g,(a,b)) = (gag^{-1}, gbg^{-1})$ and p is the second projection.

Denote $\mathfrak{K}_w^{\mathcal{F},\mathcal{L}} = \mathcal{K}_w^{j_{\mathcal{F}*}j_{\mathcal{F}}^*\mathcal{L}} \boxtimes \mathcal{F}$.

A simple constituent of ${}^pH^i(p_!u^*\mathfrak{K}_w^{\mathcal{F},\mathcal{L}})$ for some $w \in W_{j_{\mathcal{F}*}j_{\mathcal{F}}^*\mathcal{L}}$, $i \in \mathbb{Z}$, $\mathcal{L} \in CS(T)$ and $\mathcal{F} \in CS(H)$ is called a character sheaf of G. Denote by $CS(G)$ the set of all character sheaves of G.

5.4.2 Restriction

For any Levi subgroup L of G, let P be the parabolic subgroup of G containing L. Recall $G = G_0 \ltimes H$, and $p_1 : G \to G_0$ and $p_2 : G \to H$ are the projection maps respectively. Denote $LP = p_1(L) \ltimes p_2(P)$. Similarly, we have PG, LG, TG, TP, TL. We also write $G = GG$, $P = PP$ and $L = LL$.

Recall that $\pi^0 : p_1(P) \to p_1(L)$ and $\pi^1 : p_2(P) \to p_2(L)$ are the natural projection maps, and $\iota^0 : p_1(P) \to p_1(G)$ and $\iota^1 : p_2(P) \to p_2(G)$ are the natural embedding.

Consider the following diagram,

$$LL \xleftarrow{Id\times\pi^1} LP \xrightarrow{Id\times\iota^1} LG. \tag{5.7}$$

Define

$$\mathrm{Res}_{LL}^{LG} A = (Id \times \pi^1)_!(Id \times \iota^1)^* A[d_1],$$

where d_1 is the dimension of fibers of $Id \times \pi^1$.

Consider the following diagram,

$$LG \xleftarrow{\pi^0\times Id} PG \xrightarrow{\iota^0\times Id} GG. \tag{5.8}$$

Define

$$\mathrm{Res}_{LG}^{GG} A = (\pi^0 \times Id)_!(\iota^0 \times Id)^* A[d_2],$$

where d_2 is the dimension of fibers of $\pi^0 \times Id$.

Consider the following diagram,

$$LL \xleftarrow{\pi^0\times\pi^1} PP \xrightarrow{\iota^0\times\iota^1} GG.$$

Define

$$\operatorname{Res}_{LL}^{GG} A = (\pi^0 \times \pi^1)_!(\iota^0 \times \iota^1)^* A[d],$$

where d is dimension of fibers of $\pi^0 \times \pi^1$.

Proposition 16. *For any $A \in \mathcal{D}(GG)$, we have*

$$\operatorname{Res}_{LL}^{GG} A = \operatorname{Res}_{LL}^{LG} \operatorname{Res}_{LG}^{GG} A.$$

Proof. Consider the following commutative diagram,

$$\begin{array}{ccccc} LL & \xleftarrow{\pi} & PP & \xrightarrow{\iota} & GG \\ {\scriptstyle \pi_2}\uparrow & \swarrow{\scriptstyle \pi_3} & & {\scriptstyle \iota_3}\searrow & \uparrow{\scriptstyle \iota_1} \\ LP & \xrightarrow{\iota_2} & LG & \xleftarrow{\pi_1} & LG. \end{array}$$

Here π_i are obvious projection maps; ι_i are obvious embedding maps and the middle square is a cartesian square. Therefore,

$$\pi_! \iota^* A = \pi_{2!}\pi_{3!}\iota_3^*\iota_1^* A = \pi_{2!}\iota_2^*\pi_{1!}\iota_1^* A.$$

Since the dimension of fibers of $\pi^0 \times \pi^1$ is the sum of dimension of fibers of $Id \times \pi^1$ and dimension of fibers of $\pi^0 \times Id$. Proposition follows. □

Proposition 17. *Let L (resp. M) be a Levi subgroup of M (resp. G), then, for any $A \in D(GG)$, we have*

$$\operatorname{Res}_{LL}^{MM} \operatorname{Res}_{MM}^{GG} A = \operatorname{Res}_{LL}^{GG} A.$$

Proof. Consider the following diagram,

$$\begin{array}{ccccc} LL & \xleftarrow{\pi} & PP & \xrightarrow{\iota} & GG \\ {\scriptstyle \pi_2}\uparrow & \swarrow{\scriptstyle \pi_3} & & {\scriptstyle \iota_3}\searrow & \uparrow{\scriptstyle \iota_1} \\ PM & \xrightarrow{\iota_2} & MM & \xleftarrow{\pi_1} & QQ. \end{array}$$

Here QQ (resp. PP, PM) is a parabolic subgroup of GG (resp. GG, MM) containing MM (resp. LL, LL); π_i are obvious projection maps; ι_i are obvious embedding maps and the middle square is a cartesian square. Therefore,

$$\pi_! \iota^* A = \pi_{2!}\pi_{3!}\iota_3^*\iota_1^* A = \pi_{2!}\iota_2^*\pi_{1!}\iota_1^* A.$$

Since the dimension of fibers of π is the sum of the dimension of fibers of π_1 and the dimension of fibers of π_2. Proposition follows. □

5.4.3 Induction

The induction functor is defined by two steps. Namely, one is an induction on "unipotent" part, another one is an induction on "reductive" part. Let P, L be the same as last section. Consider the following diagram

$$LL \xleftarrow{q_1} LV \xrightarrow{q_2} LG.$$

Here

$$LV = \{(g,h) \in LG \times LG \mid h^{-1}gh \in LP\}.$$

Maps are defined as follows. q_2 is the first projection and $q_1(g,h) = (Id \times \pi^1)(h^{-1}gh)$.

We define LL-actions as follows. LL acts on itself by conjugation, i.e. $g' \cdot g = g'^{-1}gg'$; acts on LG trivially and acts on LV as $g' \cdot (g,h) = (g, hg')$. Then all maps are LL-equivariant maps.

We define LG-actions as follows. LG acts on LL trivially; acts on itself by conjugation, i.e. $g' \cdot g = g'gg'^{-1}$ and acts on LV as $g' \cdot (g,h) = (g'gg'^{-1}, g'h)$. Then all maps are LG-equivariant maps.

Define

$$\mathrm{Ind}_{LL}^{LG} A = q_{2*}q_1^* A[d_1],$$

where d_1 is the dimension of fibers of $Id \times \pi^1 : LP \to LL$.

Consider the following diagram,

$$LG \xleftarrow{p_1} GV_1 \xrightarrow{p_2} GV_2 \xrightarrow{p_3} GG. \tag{5.9}$$

Here

$$GV_1 = \left\{(g,h) \in GG \times GG \mid h^{-1}gh \in PG\right\},$$

and

$$GV_2 = \left\{(g,\overline{h}) \in GG \times GG/PG \mid h^{-1}gh \in PG\right\}.$$

Maps are defined as follows. p_3 is the first projection; $p_2(g,h) = (g,\overline{h})$ and $p_1(g,h) = (\pi^0 \times Id)(h^{-1}gh)$.

We define the LG-actions as follows. LG acts on itself by conjugation, i.e. $a \cdot g = a^{-1}ga$; acts on GV_2 and GG trivially and acts on GV_1 as $a \cdot (g,h) = (g,ha)$. Then all maps are LG-equivariant maps.

We define the GG-actions as follows. GG acts on itself by conjugation, i.e. $b \cdot g = bgb^{-1}$; acts on LG trivially; acts on GV_1 as $b \cdot (g,h) = (bgb^{-1}, bh)$ and acts on GV_2 as $b \cdot (g,\overline{h}) = (bgb^{-1}, \overline{bh})$. Then all maps are GG-equivariant maps. Moreover, p_2 is a principle PG-bundle. p_3 is a proper map.

Define

$$\operatorname{Ind}_{LG}^{GG} A = p_{3!}p_{2\flat}p_1^* A[d_2],$$

where d_2 is the dimension of fibers of map $\pi^0 \times Id : PG \to LG$.

Define

$$\operatorname{Ind}_{LL}^{GG} A = \operatorname{Ind}_{LG}^{GG} \operatorname{Ind}_{LL}^{LG} A.$$

Lemma 16. *Consider the following diagram,*

$$\begin{array}{ccccc} LL & \xleftarrow{q_1} & LV & \xrightarrow{q_2} & LG \\ \uparrow{\scriptstyle \pi_1} & \swarrow{\scriptstyle q} & & \searrow{\scriptstyle \iota} & \uparrow{\scriptstyle p} \\ LP & \xrightarrow{\iota_1} & LG & \xleftarrow{u} & LG \times LG. \end{array} \tag{5.10}$$

Then

$$\operatorname{Ind}_{LL}^{LG} A = p_* u^* \iota_{1!} \pi_1^* A[d_1].$$

Proof. See proof of Lemma 15. □

5.4.4 Adjunction

Proposition 18. *For any* $A \in D^b_{LG}(LG), B \in D^b_{LL}(LL)$*, we have*

$$\operatorname{Hom}(A, \operatorname{Ind}_{LL}^{LG} B) = \operatorname{Hom}(\operatorname{Res}_{LL}^{LG} A, B).$$

Proof. Consider Diagram (5.10). Let d_1 (resp. d_2) be the dimension of fibers π_1 (resp. p), by Lemma 16,

$$
\begin{aligned}
&\operatorname{Hom}(A, \operatorname{Ind}_{LL}^{LG} B) \\
&\quad = \operatorname{Hom}(A, p_* u^* \iota_{1!} \pi_1^* B[d_1]) \\
&\quad = \operatorname{Hom}(p^* A, u^* \iota_{1!} \pi_1^* B[d_1]) \\
&\quad = \operatorname{Hom}(p^* A, p^* \iota_{1!} \pi_1^* B[d_1]) \text{ (Since } \iota_{1!}\pi_1^* B \text{ is a } LG \text{ equivariant complex)} \\
&\quad = \operatorname{Hom}(p^* A, p^! \iota_{1!} \pi_1^* B[d_1 - 2d_2]) \\
&\quad = \operatorname{Hom}(p_! p^* A, \iota_{1!} \pi_1^* B[d_1 - 2d_2]) \\
&\quad = \operatorname{Hom}(A[-2d_2], \iota_{1!} \pi_1^* B[d_1 - 2d_2]) \text{ (Since } p \text{ is a vector bundle)} \\
&\quad = \operatorname{Hom}(A, \iota_{1*} \pi_1^* B[d_1]) \text{ (Since } \iota_1 \text{ is a closed embedding)} \\
&\quad = \operatorname{Hom}(\iota_1^* A, \pi_1^! B[-d_1]) \\
&\quad = \operatorname{Hom}(\pi_{1!} \iota_1^* A[d_1], B).
\end{aligned}
$$

□

For any algebraic variety X, given a stratification $\mathcal{S}$ of X, let p be a $\mathcal{S}$-perversity function. Then one can define a t-structure on $\mathcal{D}^b(X)$. The category equipped with this t-structure is denoted by ${}^p\mathcal{D}^b(X)$.

Lemma 17 ([42]). *The functor* $\boxtimes : {}^p\mathcal{D}^b(X) \times {}^q\mathcal{D}^b(Y) \to {}^{p+q}\mathcal{D}^b(X \times Y)$ *is* t*-exact, where* $(p+q)(S \times T) = p(S) + q(T)$*. In particular, if* p, q *both are middle perversity functions, i.e.* $p(S) = -\dim(S)$*, this is true.*

Lemma 18. *For any* $A \in CS(GG)$*,* $\operatorname{Res}_{LG}^{GG} A \in \mathcal{D}(LG)^{\leq 0}$*.*

Proof. Let $A = C \boxtimes B$, by Künneth formula,

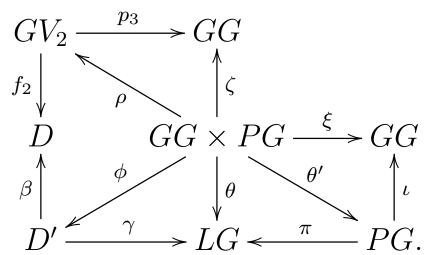
$$\operatorname{Res}_{LG}^{GG} A = (\pi_!^0 \iota^{0*} C) \boxtimes B[d_1] = (\operatorname{Res}_{L_0}^{G_0} C) \boxtimes B,$$

where $\mathrm{Res}_{L_0}^{G_0}$ is the restriction functor which Lusztig defines for the character sheaves on G_0 (see [32]) and d_1 is the dimension of fibers of $PG \to LG$. Since $\mathrm{Res}_{L_0}^{G_0} C \in D(L_0)^{\leq 0}$ and B is a perverse sheaf, by Lemma 17, $\mathrm{Res}_{LG}^{GG} A \in \mathcal{D}(LG)^{\leq 0}$. □

Proposition 19. *For any $A \in \mathcal{M}_{GG}(GG)$ and $B \in \mathcal{D}_{LG}^{\geq 0}(LG)$, then*

$$\mathrm{Hom}(A, \mathrm{Ind}_{LG}^{GG} B) = \mathrm{Hom}(\mathrm{Res}_{LG}^{GG} A, B).$$

Proof. This proof is based on Lusztig's argument of proving theorem 4.4 in [32]. Consider the following commutative diagram,



(5.11)

Here $D = GG \times LG$ modulo the PG-action as

$$h \cdot (x, l) = (xh^{-1}, (\pi^0 \times Id)(h)x(\pi^0 \times Id)(h^{-1}));$$

and $D' = GG \times LG$. The maps are defined as follows. $f_2(g, \overline{x}) = (x, (\pi^0 \times Id)(x^{-1}gx));$

$\rho(x, p) = (xpx^{-1}, \overline{x});$

$\phi(x, p) = (x, (\pi^0 \times Id)(p));$

$\theta(x, p) = (\pi^0 \times Id)(p);$

$\gamma(x, l) = l;$

$p_3(g, \overline{x}) = g;$

$\xi(x, p) = xpx^{-1};$

$\zeta(x, p) = p;$

$\theta'(x, p) = p.$

Firstly, we claim that $\beta^* f_{2!} p_3^* A = \gamma^* \pi_! \iota^* A$ for any $A \in \mathcal{M}_{GG}(GG)$. In fact, it is easy to check the following two squares are cartesian squares.

$$\begin{array}{ccccc}
GV_2 & \xleftarrow{\rho} & GG \times PG & \xrightarrow{\theta'} & PG \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \pi} \\
D & \xleftarrow{\beta} & D' & \xrightarrow{\gamma} & LG
\end{array}$$

By base change formula,

$$\gamma^* \pi_! \iota^* A = \phi_! \theta'^* \iota^* A = \phi_! \xi^* A$$

and

$$\beta^* f_{2!} p_3^* A = \phi_! \rho^* p_3^* A = \phi_! \zeta^* A.$$

Since A is a G-equivariant perverse sheaf, i.e. $\zeta^* A = \xi^* A$, the claim follows.

Secondly, we claim that $\mathrm{Ind}_{LG}^{GG} A = p_{3!} f_2^* \beta_\flat \gamma^* A[d_2]$, where d_2 is the dimension of fibers of f_2. This can be shown by the following commutative diagram,

$$\begin{array}{ccccccc}
LG & \xleftarrow{p_1} & GV_1 & \xrightarrow{p_2} & GV_2 & \xrightarrow{p_3} & GG \\
\Vert & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \\
LG & \xleftarrow{\gamma} & D' & \xrightarrow{\beta} & D. & &
\end{array}$$

In fact, by commutativity, we have

$$p_2^* f_2^* \beta_\flat \gamma^* A = f_3^* \beta^* \beta_\flat \gamma^* A = f_3^* \gamma^* A = p_1^* A.$$

The claim follows that the dimension of fibers of π is equal to the dimension of fibers of f_2. Therefore,

$$\begin{aligned}
&\mathrm{Hom}(A,\ \mathrm{Ind}_{LG}^{GG} B) \\
&\quad = \mathrm{Hom}(A,\ p_{3*} f_2^* \beta_\flat \gamma^* B[d_2]) \text{ (Since } p_3 \text{ is a proper map)} \\
&\quad = \mathrm{Hom}(p_3^* A,\ f_2^* \beta_\flat \gamma^* B[d_2]) \\
&\quad = \mathrm{Hom}(p_3^* A,\ f_2^! \beta_\flat \gamma^* B[-d_2]) \\
&\quad = \mathrm{Hom}(f_{2!} p_3^* A, \beta_\flat \gamma^* B[-d_2]) \\
&\quad = \mathrm{Hom}(\beta^* f_{2!} p_3^* A, \gamma^* B[-d_2]) \\
&\quad = \mathrm{Hom}(\gamma^* \pi_! \iota^* A, \gamma^* B[-d_2]) \\
&\quad = \mathrm{Hom}(\pi_! \iota^* A[d_2], B).
\end{aligned}$$

The last equality holds because $\pi_! \iota^* A[d_2] \in D^{\leq 0}$ and $B \in D^{\geq 0}$ by Lemma 18. $\square$

Proposition 20. *For any $A \in \mathcal{M}_{GG}(GG)$ and $B \in \mathcal{D}^{\geq 0}_{LL}(LL)$, we have*

$$\operatorname{Hom}(A, \operatorname{Ind}_{LL}^{GG} B) = \operatorname{Hom}(\operatorname{Res}_{LL}^{GG} A, B).$$

Proof. Consider Diagram (5.10). By Lemma 16, $\operatorname{Ind}_{LL}^{LG} B = p_* u^* \iota_{1!} \pi_1^* B[d_1]$. Since $B \in \mathcal{D}^{\geq 0}_{LL}(LL)$, $\iota_{1!} \pi_1^* B[d_1] \in \mathcal{D}^{\geq 0}$. Hence $u^* \iota_{1!} \pi_1^* B[d_1] \in \mathcal{D}^{\geq d}$, where $d = \dim(LG)$. Therefore $p_* u^* \iota_{1!} \pi_1^* B[d_1] \in \mathcal{D}^{\geq 0}_{LG}(LG)$. i.e. $\operatorname{Ind}_{LL}^{LG} B \in \mathcal{D}^{\geq 0}_{LG}(LG)$.

By Propositions 18, 19, we have

$$\operatorname{Hom}(A, \operatorname{Ind}_{LG}^{GG} \operatorname{Ind}_{LL}^{LG} B) = \operatorname{Hom}(\operatorname{Res}_{LL}^{LG} \operatorname{Res}_{LG}^{GG} A, B).$$

Proposition follows the definition of $\operatorname{Ind}_{LL}^{GG}$ and Proposition 16. $\square$

Chapter 6

Conclusion

In Chapter 4, a geometric realization of the composition subalgebra of $\mathcal{H}(R\Gamma)$ is given. And the canonical basis and a monomial basis of the composition subalgebra are constructed by using perverse sheaves. This gives an example to indicate that perverse sheaves theory can be used to study algebraic object. Character sheaf theory is another example.

In Chapter 5, we construct character sheaves, which are some perverse sheaves, on $GL_m(k[t]/(t^2))$. There are still many interesting problems to be investigated. Here we list some of them.

(1) Do the characteristic functions of character sheaves form a basis of the vector space of class functions: $GL_m(k[t]/(t^2)) \to \bar{\mathbb{Q}}_l$?

(2) The characteristic functions of character sheaves are only virtual characters. Which irreducible characters will be direct summands of given character sheaves?

(3) What are the character sheaves of $GL_m(k[t]/(t^n))$ for $n > 2$?

(4) More generally, the approach in Chapter 5 should also apply to any reductive algebraic group G or even more general algebraic groups through the group homomorphism $G(k[t]/(t^n)) \xrightarrow{\pi} G(k)$ induced from the k-algebraic homomorphism $k[t]/(t^n) \to k$ and $H := \mathrm{Ker}(\pi)$ is a unipotent algebraic group. Can we use Boyarchenko-Drinfeld method to characterize the character sheaves of H in term of geometric properties of G?

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