ANALYSIS OF VARIATIONS OF INCOMPLETE OPEN CUBES BY SOL LEWITT

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Abstract

"Incomplete open cubes" is one of the major projects of the artist Sol Lewitt. It consists of a collection of frame structures and a presentation of their diagrams. Each structure in the project is a cube with some edges removed so that the structure remains three-dimensional and connected Structures are considered to be identical if one can be transformed into another by a space rotation (but not reflection).

The list of incomplete cubes consists of 122 structures. In this project, the concept of incomplete cubes was formulated in the language of graph theory. This allowed us to compare the problem posed by the artist with the similar questions of graph theory considered during the last decades. Classification of Incomplete cubes was then refined using the language of combinatorics. The list produced by the artist was then checked to be complete. And lastly, properties of Incomplete cubes in the list were studied.

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Chapter 1

Introduction

1.1 Overview

Many pieces of the art created by Sol Lewitt have heavy ties in mathematical principals. The focus of this paper is on one of his creations, the collection of sculptures of incomplete cubes. Combinatorial description of these sculptures is so complicated that a natural question is posed quite often by the observers: Is the list complete? Looking at this piece of work, we have come to ask the following questions. How do we formulate the concept of incomplete cubes in the language of graph theory? How can we compare the problem posed by the artist with the problems considered during the last decades in graph theory? Can we refine the classification of incomplete cubes using the language of combinatorics? Is the list produced by the artist complete? Lastly, we would like to study the properties of incomplete cubes in the list.

1.2 Sol Lewitt Biography

Solomon Lewitt is a conceptual artist of the 20th century. He was born September 9, 1928 in Hartford Connecticut. He was the child of Russian immigrants. “His mother took him to art classes at the Wadsworth Atheneum in Hartford” [4] when he was young. Syracuse University was where he studied before being drafted into the Special Services in 1951 to serve in Korea. The group he served in was in charge of peaceable operations and in his case, designing posters.

After his service in Korea, he worked for Seventeen magazine back in New York as a graphic designer and then in the office of the architect IM Pei. While working in the bookshop of the Museum of Modern Art with coworkers that where aspiring artists, he found his new profession in art.

His wall drawings and *three-dimensional* structures made a great impact on the art of twentieth century. It is clear from the first sight that many of Sol Lewitt's ideas are presented using combinatorics, and quite often are based on the answer to the question “list all possible cases”. He decided to reduce art to its essentials, “to recreate art, to start from square one,” he said, beginning literally with squares and cubes.” [4]

1.3 Other Work of Sol Lewitt

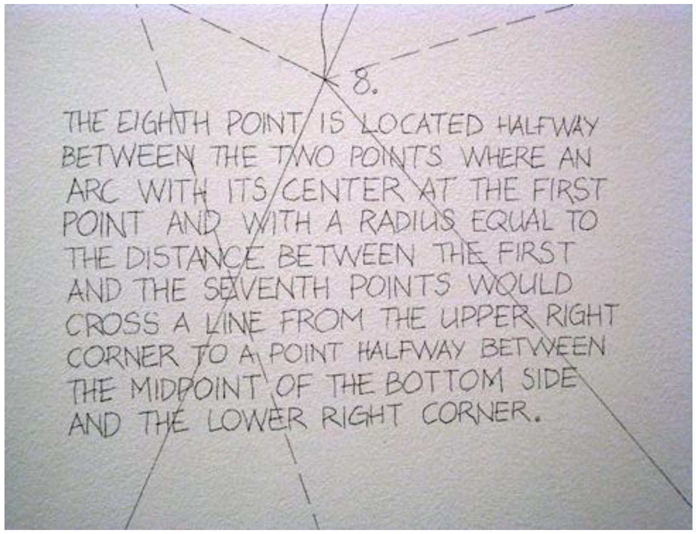


Figure 1.1 [7] wall drawing 305

Fig 1.1 is one of many wall drawings that he did. These where more about the creation of the piece then the work itself. Each one had a set of rules on how create the piece. This one was made by creating 100 points where each new point has a geometric description on where it's located. This description is made up at random during the creation of this point. For each point created, the description is written out by it.

Figure 1.2 [1] *49 Three-Part Variations on Three Different Kinds of Cubes* (1967–71)

Fig 1.2 shows solid cubes, cubes with opposite sides removed, and cubes with just one side removed. This display shows all possible permutations of these three types of cubes being stacked in towers of 3.

Chapter 2

Incomplete Cubes

2.1 Incomplete cubes

Incomplete cubes are one of the major projects of the artists that stand apart from the rest of his works. It consists of a collection of frame structures and a presentation of their diagrams. Each structure in the project is a cube with some edges removed so that the structure remains three-dimensional and connected. Two structures are considered to be identical if one can be transformed into another by a space rotation (but not a reflection).

The choice of restrictions aesthetically is very natural: a flat structure would break the unified look of the collection of the three dimensional objects, therefore two and one dimensional incomplete cubes are not included. Production of disconnected structures or the ones that would have to be placed in unstable positions would require complicated technical solutions.

The final list consists of 122 structures, and even from the first glance it is clear that combinatorial description of this project is far from being trivial.

According to [5], Sol Lewitt did not realize the full complexity of his mathematical problem at the beginning of the project, but once he started, he wanted to solve it completely: “In the first place, I thought it’d be so easy that it wouldn’t be necessary. Secondly, I did not know any mathematician to ask. Thirdly, it was a kind of challenge to be able to do it and to work it all out. It got to be a game or a puzzle that I wanted to solve.” (p.25)

He organized the list of incomplete cubes by the number of edges of the structures, and in order to check that there are no repetitions, the artist made small three-dimensional models of each structure. The final list of structures was checked twice by professional mathematicians: Dr. Erna Herrey and later by Professor Arthur Babakhanian ([5], p.25). The pages with calculations by Professor Babakhanian later became a part of another conceptual exhibition “Working Drawings and Other Visible Things on Paper Not Necessarily to Be Viewed as Art” which was organized by Mel Bochner. Fig 2.1 shows the original list created by Sol Lewitt.

Figure 2.1 Variations of Incomplete Open Cubes

2.2 Examples

In fig 2.2 there is a picture of the actual sculpture created by Sol Lewitt paired with a graph (see Ch.3.1) that represents the sculpture.

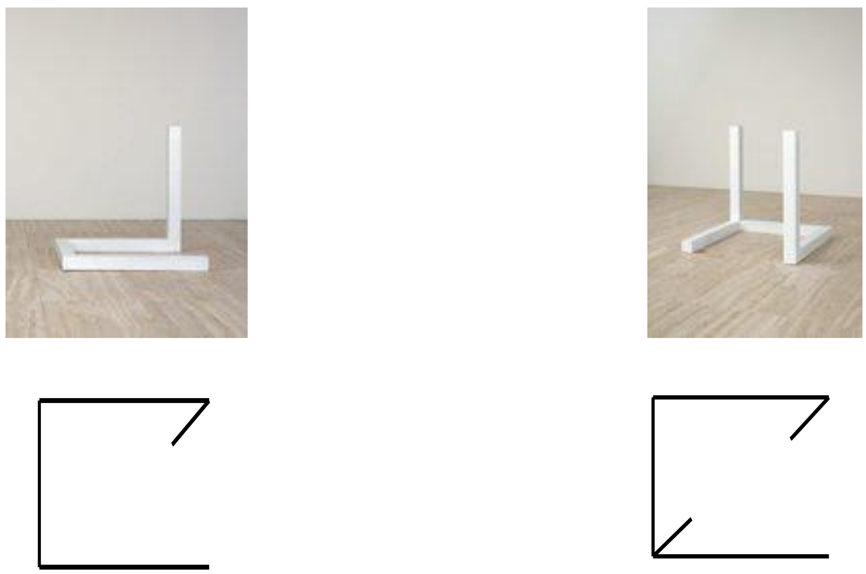


Figure 2.2 [3][2] Incomplete cube Sculpture and Graph Comparison

Chapter 3

Statement of the Problem of Incomplete Cubes

3.1 Definitions

Definition 3.1 The *N-dimensional cube* I_n is the Cartesian product of $n \geq 2$ copies of I: $I_n = I \times \ldots \times I$ (n factors), where I is a non-degenerate line segment.

Definition 3.2 The *unit cube* is the set $[0,1]^n = \{(x_1, \ldots, x_n) \in R^n : 0 \leq x_i \leq 1, 1 \leq i \leq n\}$ which is in the n-dimensional Euclidean space R^n.

Definition 3.3 *Vertices* are points of $[0,1]^n$ with all coordinates 0 or 1.

Definition 3.4 A *graph* G is an ordered pair (V,E) consisting of a nonempty set of vertices V and a set of edges E. Each edge in E is an unordered pair uv of distinct vertices of G. To distinguish between different sets of vertices and edges between different graphs, V(G) and E(G) will represent the sets of vertices and edges respectively for the graph G.

Definition 3.5 A *distance* $d_G(u,v)$ between u and v is the length of a shortest uv-path. If no such path exists, $d_G(u,v)$=inf.

A common question that arises in graph theory is the embedding of certain classes of graphs into graphs of another type. Embedding in the most general sense means a mapping between two graphs that preserves some topological properties. Here are the most often considered types of embeddings.

Definition 3.6 A *homomorphism* of graph G into G' is a function h from V(G) into V(G') such that if u and v are adjacent in G, then h(u) and h(v) are adjacent in G'.

Definition 3.7 *Isomorphic embedding* of G into G' means that G is isomorphic to a subgraph of G'.

Definition 3.8 *Cubical graphs* are graphs that can be isomorphically embedded in an n-cube I_n.

Figure 3.1 Examples of cubical and non-cubical graphs

Definition 3.9 An *isometric embedding* of a graph G into a graph G' means that the distance between verticies is preserved.

Definition 3.10 A *partial cube* is a graph that can be isometrically embedded into an n-dimensional cube I_n.

Definition 3.11 The *dimension* of a partial cube G is the minimal n such that G is embeddable in I_n.

Since the structures of Sol Lewitt are classified up to rotations of a cube, it is worth mentioning the following theorem.

Theorem 3.1 Aut(I_3) isomorphic to S_4

Proof. For the cube, the position of one corner determines the position of its opposite corner, so the diagonals going between these opposite corners (4 in total) determine the orientation of the cube so there is a homomorphism f: Aut(I_3) --> S_4. Any vertex can be moved to another and there are 3 rotations that fix any one vertex so all that needs to be shown is that all 24 permutations of the diagonals come from rotations.

For the 4 cycle the following is an example for the cycle (1324), a rotation of $\pi/2$ radians on the a axis corresponds to (1324) and a rotation in the opposite direction corresponds to (1423). Similar rotations labeled a and b in fig 3.1 and there inverses will get us the other 4-cycles.

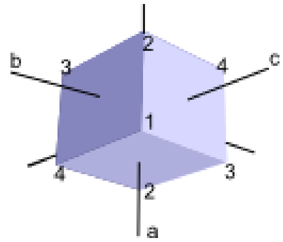


Figure 3.2 Rotation Reference

(adcb)(acbd)=(abc) and since all the 4-cycles are in the symmetry group of the cube, so are all of the 3-cycles.

(acbd)(acbd)=(ab)(cd) so for the same reasonings, all cycles of the form (ab)(cd) are also included. There are six 4-cycles, eight 3-cycles, three cycles of the form (ab)(cd), and one identity permutation so the order of the symmetry group of the cube is at least 18. The only subgroup of S_4 of order $\geq$ 18 is S_4, so Aut(I_3) is isomorphic to S_4.

3.2 Applying Graph theory to the Incomplete cubes

Now the problem of incomplete cube can be reformulated on the language of graph theory as follows:

<u>Problem</u>: *Classify all three-dimensional isomorphic embeddings of cubical graphs in I_3 up to rotations of I_3.*

We approach the classification problem in two steps.

1) we list all isomorphism classes of cubical graphs of dimension at most 3.
2) for each graph in the list we study its isomorphic embeddings in I_3.

The diagram in fig 3.3 gives the answer to the 1st question. It lists isomorphism classes of cubical graphs of dimension at most 3. Two graphs on the diagram are connected if and only if they differ from each other by exactly one edge.

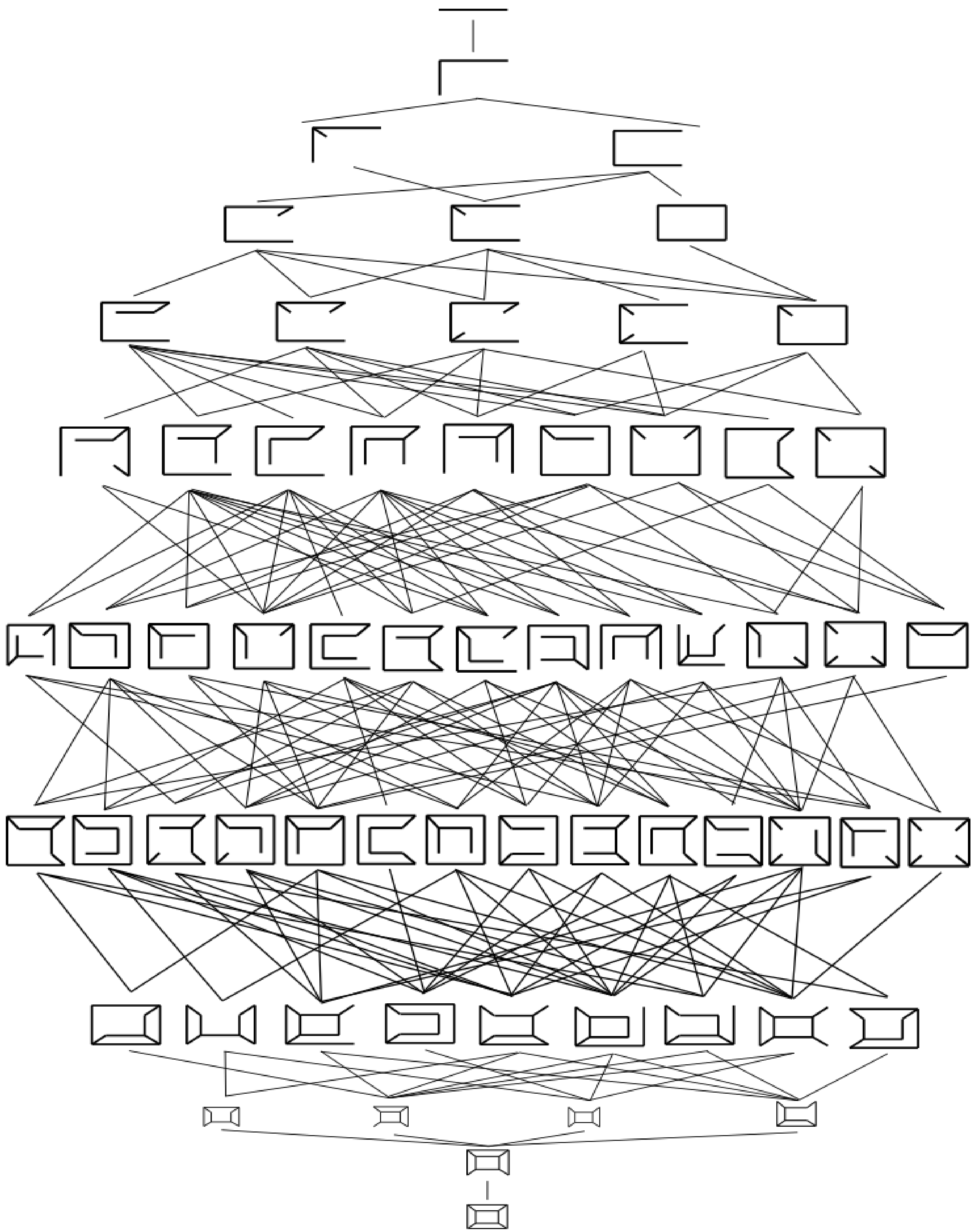


Figure 3.3 Graph of Incomplete Open Cubes

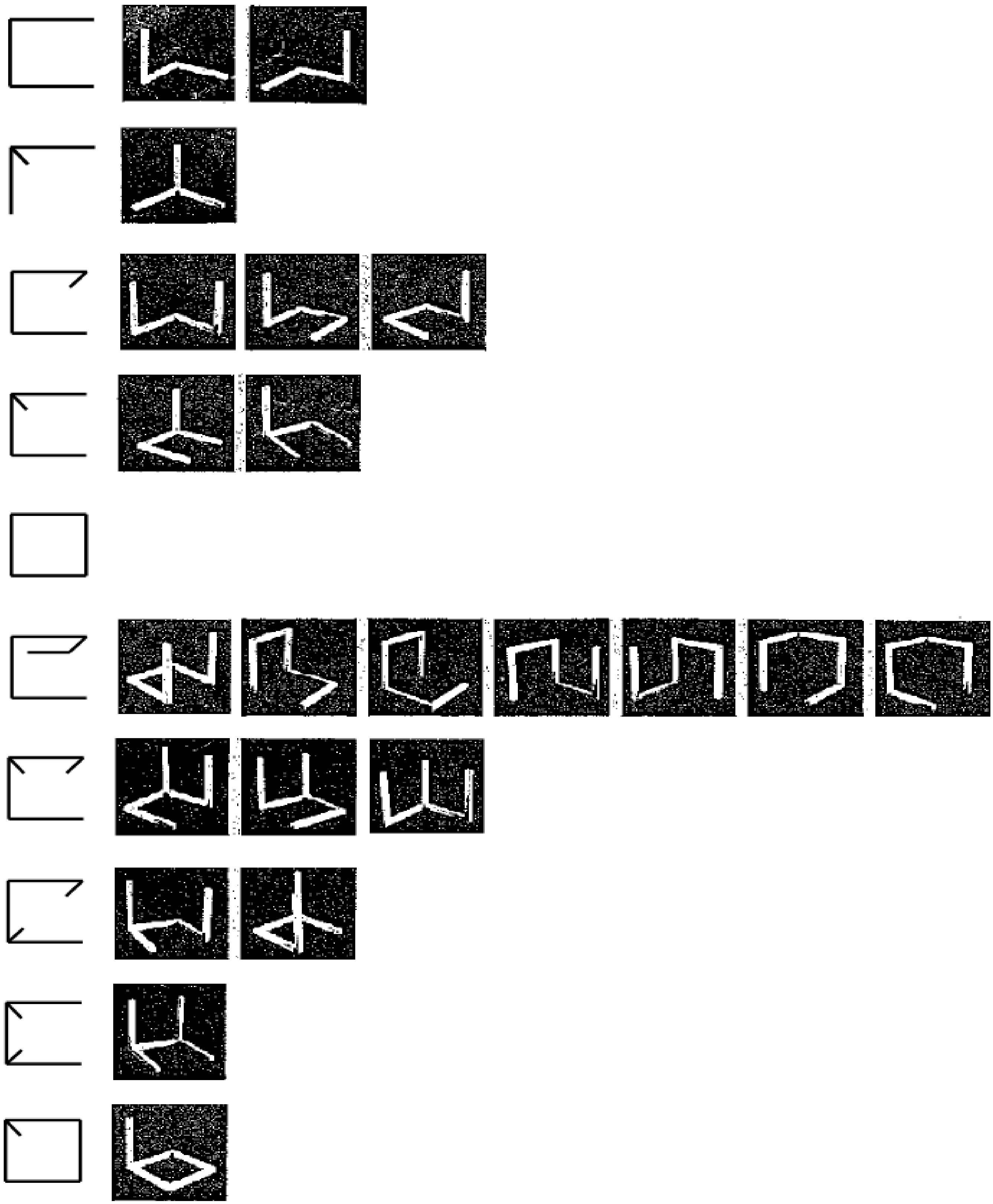
Note that the list contains 63 isomorphism classes. Also note that while the shapes in fig 3.4 are included in fig 1, they are not included in the list of Sol Lewitt due to his aesthetic restrictions (see Ch.2).

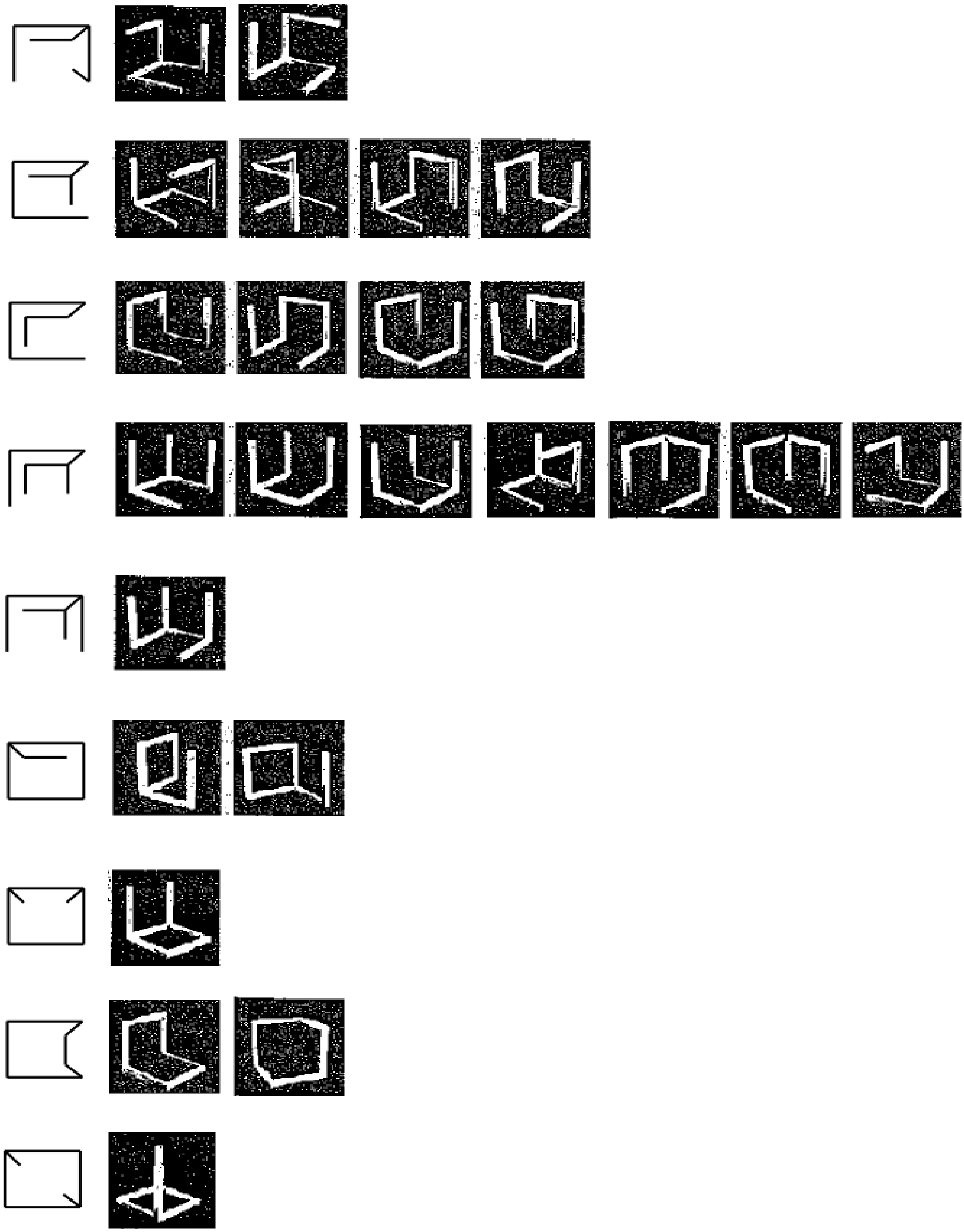
Figure 3.4 Graphs not included

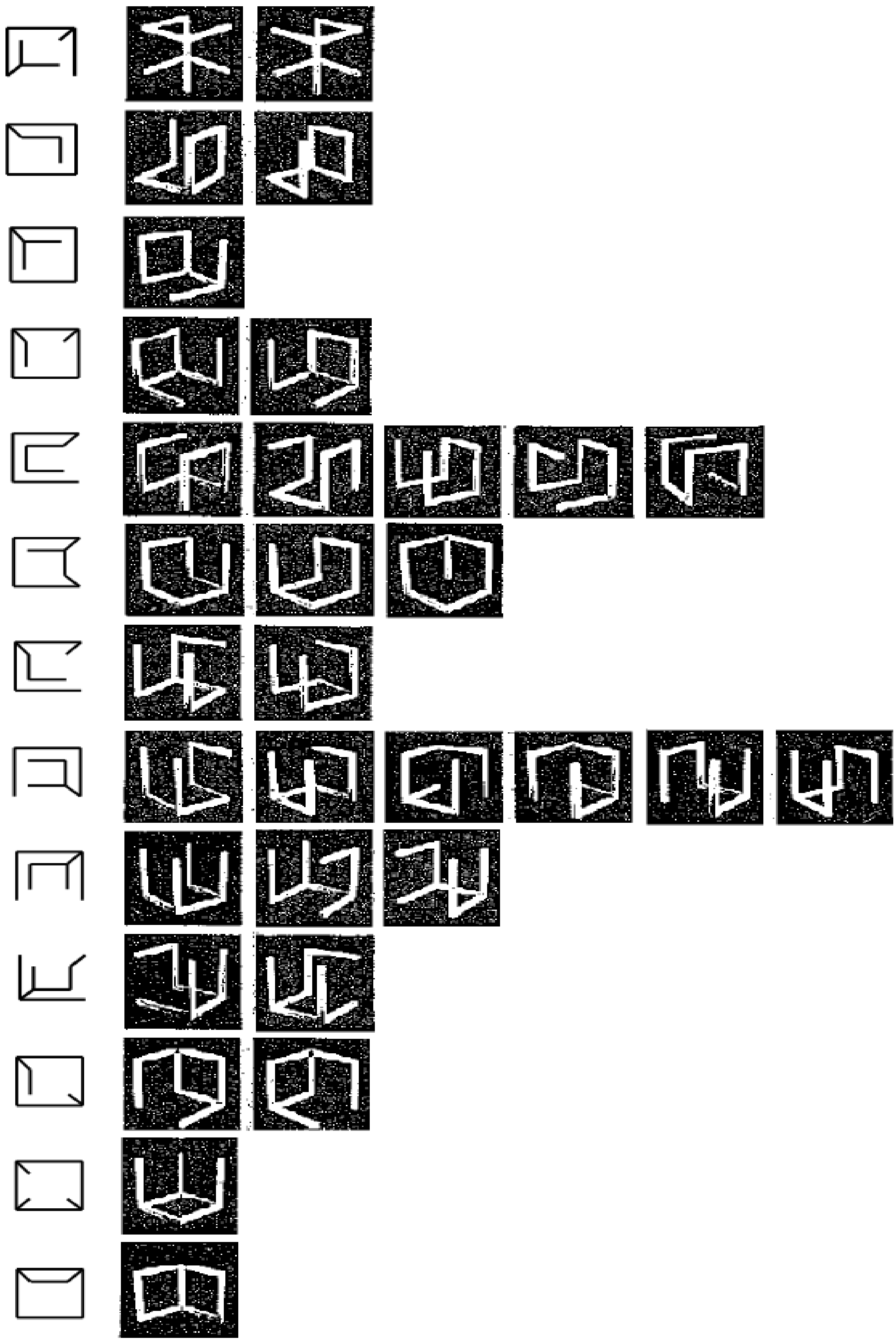
The diagram structure of the list allows checking its completeness. For example, in the 4 edged cubical graph on the left of the 4th line, three more options exist to add an edge and all of these options has been accounted for by the connections in the diagram so these cubical graphs connections are completely included and since any cubical graph can be made by adding edges to the one edged graph, all cubical graph are included.

3.3 Embeddings

In the original list created by the artist (see fig 2.1) the embeddings of cubical graphs are divided into groups by the number of edges in the graphs. It is almost impossible to track the completeness of the list in this form. The refinement of the list by breaking the graphs into classes of isomorphisms and then listing all embeddings for each particular isomorphism class makes the task of checking completeness manageable. Fig 3.5 below shows all classes of isomorphic embeddings in I_3 of cubical graphs listed in the diagram in fig 3.3.







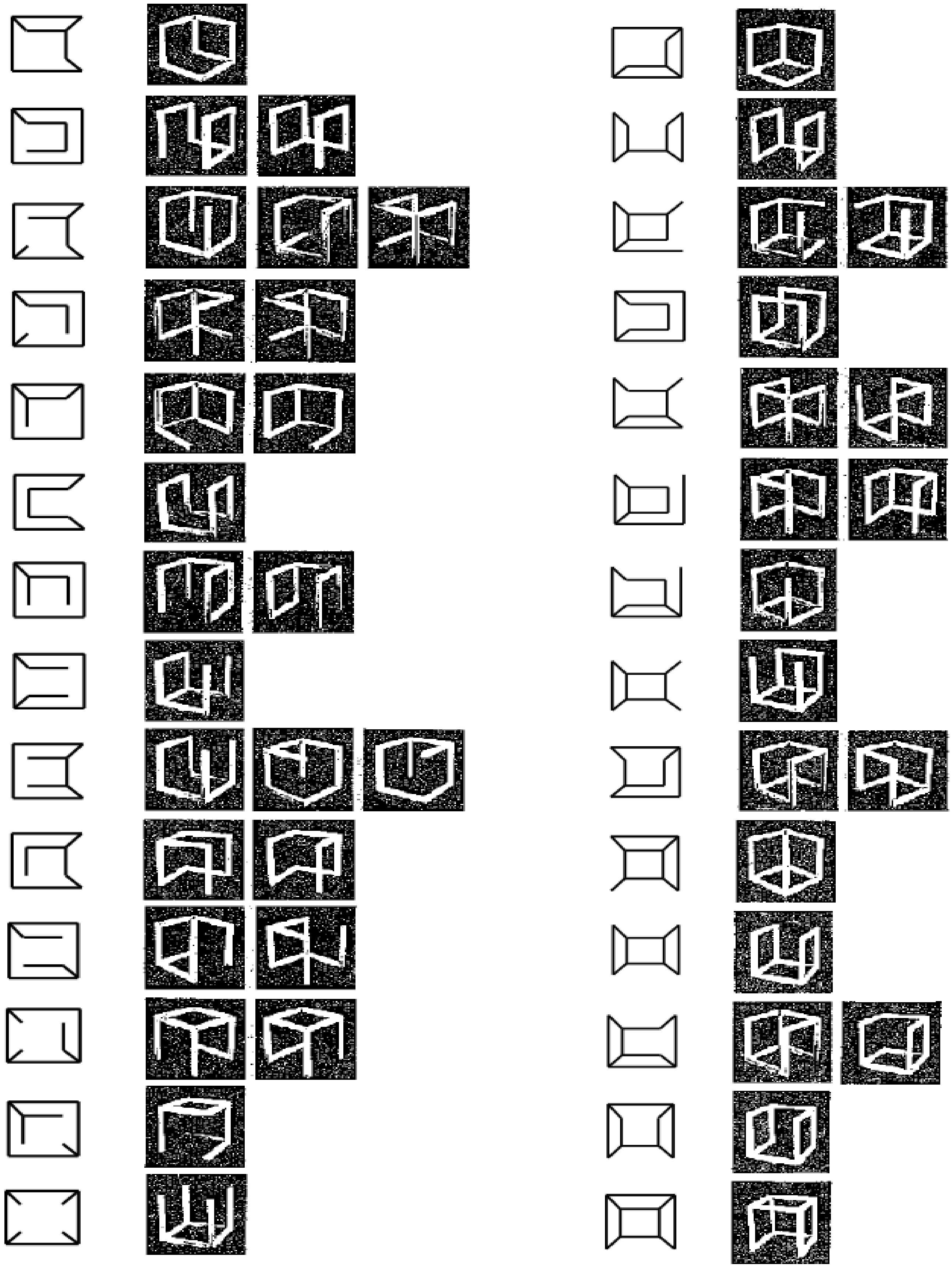


Figure 3.5 Isomorphic Embedding

Chapter 4

Partial Cubes

4.1 Djokovic-Winkler Criterion

Applications of graph theory in computer science justify the importance of embeddings of graphs that preserve distance (isometric embeddings). Note that the majority of the embeddings of the graphs of the fig 3.3 into three-dimensional cube are not isometric. Moreover, some of the graphs actually cannot be embedded isometrically in a hypercube of any dimension (in other words, such graphs are not partial cubes).

In this chapter we review the Djokovic - Winkler criterion and use it to identify partial cubes in the list of cubical graphs in fig 3.3 and their dimensions.

Definition 4.1. Let e=xy and f=uv be two edges of a connected graph G=(V,E). We say that the edge e is in *relation* Θ to the edge f if

$$d(x,u)+ d(y,v) \neq d(x,v)+ d(y,u).$$

The relation Θ reflexive and symmetric. However, in general it is not transitive.

Theorem 4.1. ([Ovchinnikov , p.30]) Let G be a connected bipartite graph and e=xy, f=uv be two edges of G with e Θf. There are two possible cases:

(i) d(x,v)=d(x,u)+1=d(y,v)+1=d(y,u)

(ii) d(x,u)=d(x,v)+1=d(y,u)+1=d(y,v)

The following theorem is known as Djokovic – Winkler criterion for partial cubes.

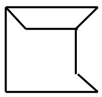
Theorem 4.2. ([Ovchinnikov , p.136]) Let G be a connected graph. The following statements are equivalent:

(i) G is a partial cube

(ii) G is bipartite and Θ is an equivalence relation on the set of edges of G.

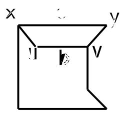
The dimension of the partial cube equals the number of equivalence classes of the relation Θ.

4.2 Example



This graph is one of the incomplete cubes and it is not a partial cube.

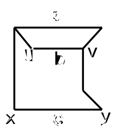
Proof: Mark the end points of edges a and b like it is shown in the image below.



Let d(x,u) denote the distance between the verticies x and u. We have:

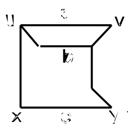
d(x,u)+d(y,v)=1+1=2 and d(x,v)+d(y,u)=2+2=4 so d(x,u)+d(y,v)≠d(x,v)+d(y,u). Thus a and b are Θ -related.

Now we look to see if edges b and c are Θ-related by labeling them as follows:



Then d(x,u)+d(y,v)=2+2=4 and d(x,v)+d(y,u)=3+3=6, so d(x,u)+d(y,v) ≠ d(x,v)+d(y,u). This implies that b and c are Θ -related.

Now for transitivity to hold, a must be Θ-related to c with the labeling of the graph below:



We have: d(x,u)+d(y,v)=1+3=4 and d(x,v)+d(y,u)=2+2=4 so d(x,u)+d(y,v)=d(x,v)+d(y,u), and so a is not Θ-related to c. Hence the relation is not transitive on this graph, and it is not a partial cube.

4.3 List of Partial Cubes

In order to list partial cubes and their dimensions among cubical graphs in fig 3.3, the following corollaries are useful:

Corollary 4.1 ([Ovchinnikov , p.177, Exercise 5.15]) A connected bipartite graph in which every edge is contained in at most one cycle is a partial cube.

Corollary 4.2 A connected bipartite graph that is a tree has dimension of the number of edges it has.

Proof: Let e=xy and f=uv be two edges of a connected tree G=(V,E). There are no cycles so there is only 1 path between edges e and f. Without loss of generality; let x and v be the verticies that are the farthest apart.

$$d(x,u)+ d(y,v)=(d(y,u)+1)+(d(y,u)+1)=2(d(y,u))+2$$

$$d(x,v)+ d(y,u)=(d(y,u)+2)+d(y,u)=2(d(y,u))+2$$

$$d(x,u)+ d(y,v)=d(x,v)+ d(y,u)$$

So e is not related to f and so every edge is its own equivalence class making the dimension of G equal to the number of edges.

Example 4.1. Paths, and cycles are cubical graphs.

The diagram in fig 4.1 lists all cubical graphs of dimension at most 3 that are partial cubes. The number below each graph is the ``partial cube dimension'' of that graph: the minimal dimension of the cube that admits an isometric embedding of that graph. The partial cube dimension equals the number of equivalence classes of the Θ-relation.

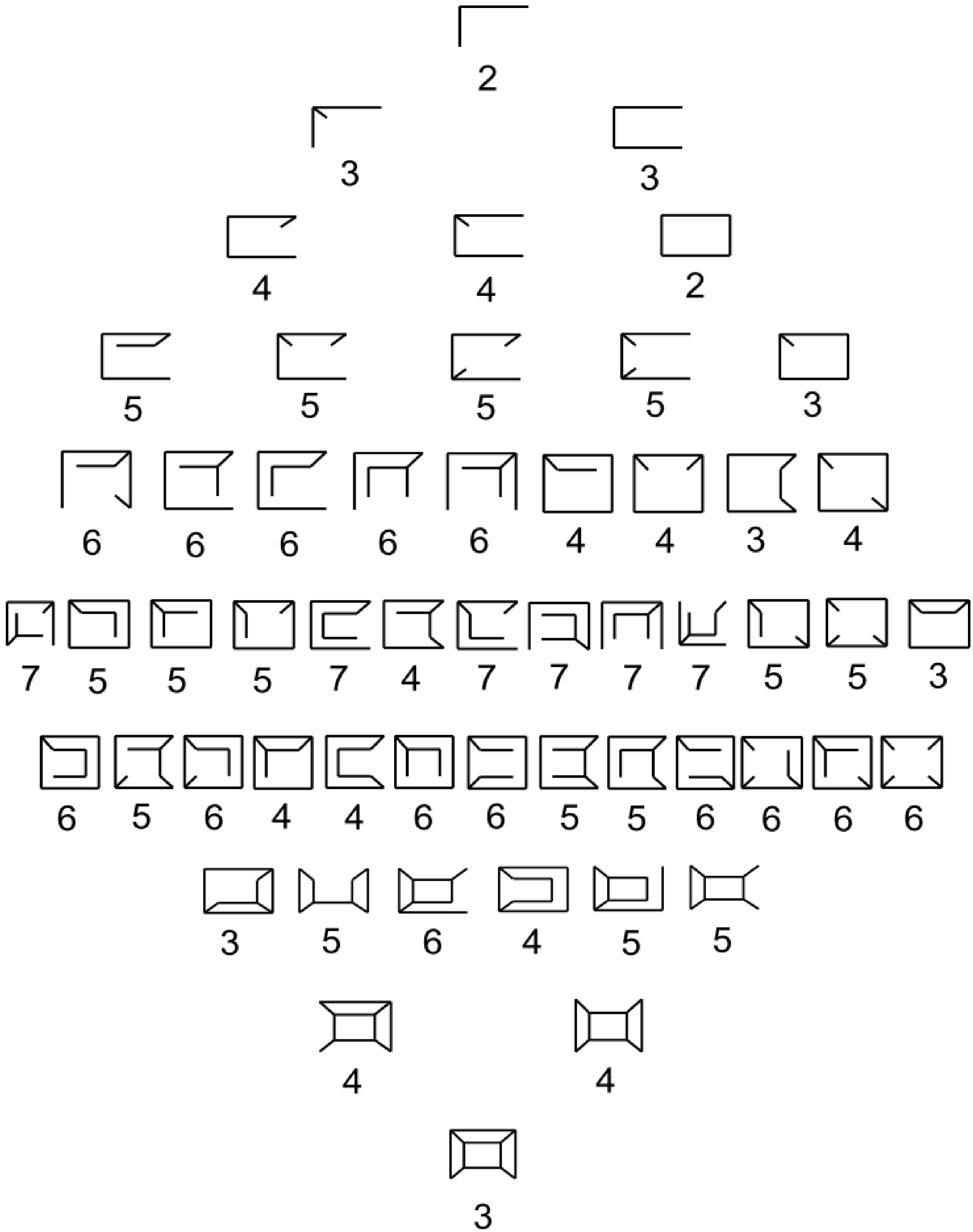


Figure 4.1 Partial Cubes Dimensions

Fig 4.2 lists all the incomplete cubes that are not partial cubes.

Figure 4.2 Non-Partial Cubes

Looking at fig. 4.1, we can see that no incomplete cube is of greater dimension than 7 and these only appear on the middle row. The average dimension is smaller the farther you are from the center line. Fig 3.3 also shows the complexity being greater towards the center. Also of note is fig 4.2 where you can see that there are very few incomplete cubes that are not partial cubes.

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[3]Incomplete Open Cube 5/6. 1974. Photograph. John Kaldor Family Collection at the Art Gallery of New South Wales, Sydney, Australia. *Incomplete Open Cube 5/6, (1974) by Sol LeWitt*. Web. 20 Apr. 2013. <http://www.artgallery.nsw.gov.au/collection/works/347.2011/>.

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