CONSENSUS IN GROUP DECISION MAKING UNDER LINGUISTIC ASSESSMENTS

by

ZHIFENG CHEN

B.S., TONGJI UNIVERSITY, CHINA, 1997
M.S., TONGJI UNIVERSITY, CHINA, 2000

AN
ABSTRACT OF A DISSERTATION

Submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Industrial and Manufacturing Systems Engineering
College of Engineering

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Manhattan, Kansas

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ABSTRACT

Group decision-making is an essential activity is many domains such as financial, engineering, and medical fields. Group decision-making basically solicits opinions from experts and combines these judgments into a coherent group decision. Experts typically express their opinion in many different formats belonging to two categories: quantitative evaluations and qualitative ones. Many times experts cannot express judgment in accurate numerical terms and use linguistic labels or fuzzy preferences. The use of linguistic labels makes expert judgment more reliable and informative for decision-making.

In this research, a new linguistic label fusion operator has been developed. The operator helps mapping one set of linguistic labels into another. This gives decision makers more freedom to choose their own linguistic preference labels with different granularities and/or associated membership functions.

Three new consensus measure methods have been developed for group decision making problem in this research. One is a Markov chain based consensus measure method, the other is order based, and the last one is a similarity based consensus measure approach.

Also, in this research, the author extended the concept of Ordered Weighted Average (OWA) into a fuzzy linguistic OWA (FLOWA). This aggregation operator is more detailed and includes more information about the aggregate than existing direct methods.

After measuring the current consensus, we provide a method for experts to modify their evaluations to improve the consensus level. A cost based analysis gives the least cost suggestion for this modification, and generates a least cost of group consensus.

In addition, in this research I developed an optimization method to maximize two types of consensus under a budget constraint.

Finally considering utilization of the consensus provides a practical recommendation to the desired level of consensus, considering its cost benefits.

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CHAPTER 1

INTRODUCTION

Fuzzy sets theory, even though still very young, has already been applied to quite a number of problems in the area of operations research. Compared to other fields of application of fuzzy sets theory, mathematical programming can be considered as already advanced. Particularly, multiple experts decision making (Group decision making) in a fuzzy environment upon the framework of mathematical programming, is an area of high research potential.

In this study, we concentrate on group decision making in the linguistic assessment. An introduction to the problem, objective of the study, and the organization of the dissertation will be first outlined in this chapter.

1.1 Problem Description

Decision making, as a specialized field of Operations Research (OR), is the process of specifying a problem or opportunity, identifying alternatives and criteria, evaluating alternatives, and selecting a preferred alternative from among the possible ones. A definition of Decision theory from SJDM (Society for Judgment and Decision Making) is given as:

The process of specifying a problem or opportunity, identifying alternatives & criteria, evaluating alternatives, and selecting from among the alternatives

There are three decision objectives:

1. Optimize, find the best possible decision

2. Maximize, find decision that meets maximum number of criteria

3. Satisfize, find the first satisfactory solution

Group decision making is necessary to solve problems, especially for complicated problems, some benefits we can take advantage of come from group decision making are:

1. A group has more information (Knowledge) than any one member.

2. It provides learning. Groups are better than individuals at understanding problems.

3. Groups are better than individuals at catching errors.

4. It may produce synergy during problem solving.

5. Working in a group may stimulate the creativeness of the participants and the process.

6. Group members have their egos embedded in the decision, and so they will be committed to the solution.

7. Risk propensity is balanced. Groups moderate high-risk takers and encourage conservatives.

Group decision making is very time consuming. Common wisdom claims that experts in a given field should agree with each other. However, in practice, a consensus among experts implies that the expert community has largely solved the problems of the domain which does not happen. (Weiss and Shanteua, in press). A fable from Ivan Krylov (Russia) explains the importance of consensus.

Once a Swan, a Pike, and a Crab
Tried to pull a loaded cab, ...
They pulled hard, did not flinch,
But they gained not an inch ...
The Swan pulled hard toward the sky
The Crab to crawl backward did try,
The Pike made for the river nearby

The reason that the cab keeps unmoved is that the three animals could not get the consensus on which direction they should go. Consensus is very important in group decision making. This research is on how to measure and improve the consensus in group decision making under linguistic assessment.

1.2 Research Objectives and Contributions

1.2.1 Research Objectives

As discussed in the previous sections, group decision making is a common and hard problem. In this research, we study the group decision making under uncertainty. We use linguistic variables to handle the uncertainties, i.e. a group of decision makers or experts use linguistic labels to represent their preference over alternatives. The purpose of this study is:

1. Develop a simple but precise group decision making procedure that aims at a desired consensus.

2. Develop new approaches to aggregate individual evaluation represented by linguistic labels into a group opinion.

3. Use consensus to measure how good the group solution is, that is consensus to test if a solution represents the groups' opinions.

4. Develop new approaches to measure consensus for group decision making problem in linguistic environment.

5. Develop a measure of utility for the consensus that gives the optimal consensus with least cost.

6. Develop methods that can be used to choose the consensus level to seek for the best money value.

1.2.2 Research Contributions

In this research, three new consensus measure methods have been developed for group decision making problem in this research. One is a Markov chain based consensus measure method, the other is order based, and the last one is a similarity based consensus measure approach. The Markov chain based method takes the advantage of the steady-state property of the Markov chain to generate the ideal consensus matrix. By measuring the similarity between the current peer evaluation matrix and the ideal one, we have the consensus value. The order based method is to measure the group preference order rank of alternatives and the individual preference order. The difference indicates the consensus level. After measuring the current consensus, experts could modify their evaluations to improve the consensus level. The cost analysis gives the least cost suggestion for this modification, as well as the least cost of the final consensus and the group decision making.

Also, in this research, the author extended the concept of Ordered Weighted Average (OWA) into a fuzzy linguistic OWA (FLOWA). This aggregation operator is more detailed and includes more information about the aggregate than existing direct methods.

In addition, a new linguistic label fusion operator has been developed in this research. The operator helps mapping one set of linguistic labels into another. This gives decision makers more freedom to choose their own linguistic preference labels with different granularities and/or associated membership functions.

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Other contributions include collecting and comparing current existing decision making tools, collecting and comparing current existing fuzzy similarity measure methods, collecting and comparing current existing fuzzy outranking approaches.

1.3 Dissertation Overview

First, an introduction to fuzzy set theory will be given in chapter 2, which includes Fuzzy Set Operations, Fuzzy Similarity Measure and outranking, and Linguistic Variables and their Aggregation. These concepts and techniques are related to the fuzzy group decision making and will be needed in later chapters. Then, the fuzzy group decision making and the four-step procedure are explored. That are expressing fuzzy preference of alternatives, aggregating individual preferences into a group decision, comparison and selection (fuzzy outranking), and consensus and contribution measure. We performed the state-of-the-art literature review on current approaches that applied to group decision making.

A new linguistic label sets fusion operator is introduced in chapter 3. In chapter 4, a new linguistic labels aggregation method is proposed. An exhaustive and through literature survey has been carried out on consensus in Chapter 5. All consensus measure approaches are classified into two major categories: hard consensus measure and soft consensus measure. Three new consensus measure methods are developed. A cost based consensus improvement approach is introduced in chapter 6, where utility theory is applied to choose the best consensus.

Finally, a conclusion and future research are presented in chapter 7.

CHAPTER 2

Background Preliminaries And Literature Review

2.1 Decision Theory

Decision theory, as a specialized field of Operations Research (OR), is the process of specifying a problem or opportunity, identifying alternatives and criteria, evaluating alternatives, and selecting a preferred alternative from among the possible ones. A definition of Decision theory from SJDM (Society for Judgment and Decision Making) is given as:

"*Decision theory is a body of knowledge and related analytical techniques of different degrees of formality designed to help a decision maker choose among a set of alternatives in light of their possible consequences*."

Decision theory offers a rich collection of techniques and procedures to reveal preferences and to introduce them into models of decision. Decision theory is not concerned with defining objectives, designing the alternatives or assessing the consequences; it usually assumes them known. Given a set of alternatives, a set of consequences, and a correspondence between those sets, decision theory offers conceptually simple procedures for choice.

2.1.1 Categories of Decision Making

There are several ways to categorize decision making problems.

2.1.1.1 By the information available

By the information available, we have decision under certainty, decision under risk, and decision under uncertainty.

Decision under certainty means that each alternative leads to one and only one consequence, and a choice among alternatives is equivalent to a choice among consequences. In a decision situation under certainty the decision maker's preferences are simulated by a single-attribute or multiattribute value function that introduces ordering on the set of consequences and thus also ranks the alternatives.

For *Decision under risk,* each alternative will have one of several possible consequences, and the probability of occurrence for each consequence is known. Therefore, each alternative is associated with a probability distribution, and a choice among probability distributions. Decision theory for risk conditions is based on the concept of utility theory.

When the probability distributions are unknown, one speaks about *decision under uncertainty*.

2.1.1.2 By the properties of the problem

By the properties of the problem, there are Multiple Criteria Decision Making (MADM), Multiple Objective Decision Making (MODM), and Multiple Experts Decision Making (MEDM). Figure 2.1 shows the relationships between these three categories.

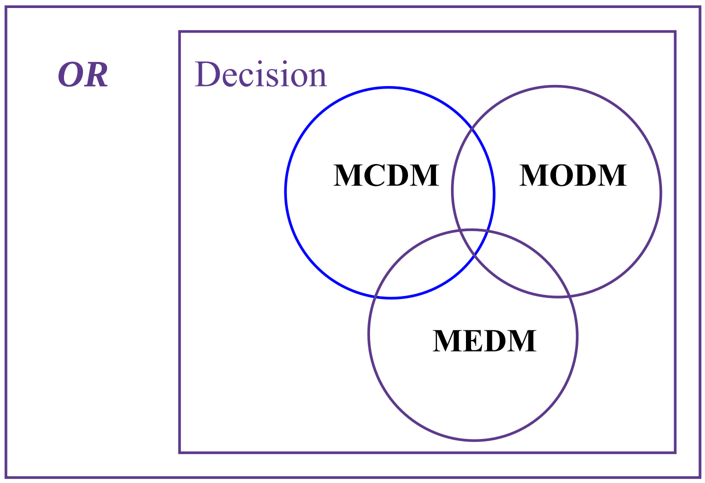


Figure 2.1 Category of decision making problems

2.1.1.3 Other category methods

By the importance roles of decision makers, there are heterogeneous and homogenous group decision making. Heterogeneous group decision making environment allows the opinions of individuals to have different weights, while homogenous not. Dubios and Prade (1979) pointed out that each individual is viewed as a subgroup, where the weight of an individual reflects the relative size of the subgroup, and reflect the relevance of the individual in the group.

By the number of decision makers, these are usually classified into single person, two-person and n-person (n>2) systems.

By the number of stages, we may usually examine one-stage and multi-stage decision making. One-stage systems are stationary in nature and only one-step decisions are considered. Multi-stage systems are dynamic systems, and decisions are performed in several steps.

By the preference information, we have hard decision systems and soft decision systems. Hard decision systems use precise and crisp numeric data and conventional mathematical models, more generally, quantitative methods. Soft decision systems apply qualitative methods which in practice mean the use of imprecise and linguistic entities and linguistic reasoning. Soft computing (SC) comprises fuzzy systems, neural networks, probabilistic reasoning and evolutionary computing, inter alias. In the computer milieu, these constituents can cope with imprecision, learning, uncertainty and optimization. The main goal of SC is to mimic human linguistic reasoning with a computer. By virtue of SC, we can apply quantitative and qualitative methods in combination within decision making. Hence, soft decision making can also mean decision making applying SC.

2.1.2 Multiple Criteria Decision Making

Multiple Criteria Decision Making (MCDM) is one of the most widely used methods in the decision-making area (Hwang and Yoon, 1981). The objective of MCDM is to select the best alternative from several mutually exclusive alternatives based on their general performance regarding various criteria (or attributes) decided by the decision maker. Depending on the type and the characteristic of the problem a number of MCDM methods have been developed such as *simple additive weighting method*, *Analytic Hierarchical Process (AHP) method*, *outranking methods*, *maximin methods*, and *lexicographic methods*. Introduced by Thomas Saaty in early 1970's, AHP has gained wide popularity and acceptance in decision-making. AHP is a procedure that supports a hierarchical structure of the problem, and uses pair-wise comparison of all objects and alternative solutions. Lexicographic method is appropriate for solving the problems in which the weight relationship among criteria is dominant and non-compensatory (Liu and Chi, 1995).

2.1.3 Multiple Objective Decision Making

In Multiple Objective Decision-Making the decision maker wants to attain more than one objective or goal in electing the course of action while satisfying the constraints dictated by environment, processes, and resources. This problem is often referred to as a vector maximum problem (VMP). There are two approaches for solving the VMP (Hwang and Masud, 1979). The first approach is to optimize one of the objectives while appending the other objectives to a constraint set so that the optimal solution would satisfy these objectives at least up to a predetermined level. This method requires the decision-maker to rank the objectives in order of importance. The preferred solution obtained by this method is one that maximizes the objectives starting with the most important and proceeding according to the order of importance of the objectives.

The second approach is to optimize a super-objective function created by multiplying each objective function with a suitable weight and then by adding them together. One well-known approach in this category is Goal Programming which requires the decision maker to set goals for each desired objective. A preferred solution is than defined as the one which minimizes the deviations from the set goals.

2.1.4 Group Decision Making (MEDM)

Group Decision Making has gained prominence due to the complexity of modern day decisions, which involve complex social, economical, technological, political and many other critical domains. Many times a group of experts needs to make a decision that represents the individual opinions and yet is mutually agreeable.

Such group decisions usually involve multiple criteria accompanied by multiple attributes. Clearly, the complexity of Multi Criteria Decision Making (MCDM) encourages group decision as a way to combine interdisciplinary skills and improve management of the decision process. The theory and practice of multiple objectives and multiple attribute decision making for a single decision maker has been studied extensively in the past 30 years. However, extending this methodology to group decision-making is not simple. This is due to the complexity introduced by the conflicting views of the decision makers, and their varying significance or weight in the decision process.

Moreover, the problem of group decision-making is complicated due to several additional factors. Usually, one expects such a decision model to follow a precise mathematical model. Such a model can enforce consistency and precision to the decision generated. Human decision makers, however, are quite reluctant to follow a decision generated by a formal model, unless they are confident in the model assumptions and methods. Many times, the input to such a decision model cannot be quantified precisely, conflicting with the perceived accuracy of the model. Intuitively, the act of optimization of the group decision – as a mathematical model would perform, is contradictory to the concept of consensus and a group agreement.

The benefits from Group decision-making however, are quire numerous justifying the additional effort required. Some of the benefits are:

1. Better learning. Groups are better than individuals at understanding problems.

2. Accountability. People are held accountable for decision in which they participate.

3. Fact screening. Groups are better than individuals at catching errors.

4. More knowledge. A group has more information (knowledge) than any one member. Groups can combine this knowledge to create new knowledge. More and more creative alternatives for problem solving can be generated, and better solutions can be derived (by group stimulation for example).

5. Synergy. The problem solving process may generate better synergy and communication among the parties involved.

6. Creativity. Working in a group may stimulate the creativity of the participants.

7. Commitment. Many times, group members have their egos embedded in the decision, and so they will be more committed to the solution.

8. Risk propensity is balanced. Groups moderate high-risk takers and encourage conservatives.

Generally there are three basic approaches towards group decision-making (Hwang and Lin, 1987):

1. *Game theory*. This approach implies a conflict or competition between the decision makers.

2. *Social Choice theory*. This approach represents voting mechanisms that allows the majority to express a choice.

3. *Group decision using expert judgment*. This approach deals with integrating the preferences of several experts into a coherent and just group position.

2.1.4.1 Game Theory

Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent and rational decision makers (Myerson, 1991). Modern game theory gained prominence after the work of Von Neumann in 1928 and later (Von Neumann and Morgenstern, 1944). Game theory became an important field during World War II and the Cold War that follows, culminating with the famous Nash Equilibrium. The objective of the games as a decision tool is to maximize some utility function for all decision makers under uncertainty. Since this technique does not explicitly accommodate multiple criteria for selection of alternatives, we will not consider it in this review.

2.1.4.2 Social Choice theory

Social Choice theory deals with multiple criteria decision making since this methodology considers votes of many individuals as the instrument for choosing a preferred candidate or alternative. The candidates can exhibit many characteristics such as honesty, wisdom, and experience as the criteria evaluated. The complexity of this seemingly simple problem of voting can be illustrated by the following example: a committee of 9 people need to select an office holder from three candidates, ***a, b***, and ***c***. The votes that rank the candidates are as follows:

3 votes have the order ***a, b, c***.

3 votes agree on the order ***b, c, a.***

2 votes have the preference of ***c, b, a***

1 votes prefers the order ***c, a, b***.

Quickly observing the results, one can realize that each candidate received three votes as the preferred option, resulting in an inconclusive choice.

The Theory of Social Choice was studies extensively with notable theories such as Arrow's Impossibility Theorem. (Arrow, 1963, Arrow and Raynaud, 1986). This type of decision-making is based on the ranking of choices by the individual voters, while the scores that each decision maker gives to each criterion of each alternative are not considered explicitly. Therefore, this methodology is less suitable for multi-criteria decision-making in which each criterion in each alternative is carefully weighed by the decision makers.

2.1.4.3 Expert Judgment Approach

Within the Expert Judgment approach, there are two minor styles denoted as Team Decision and Group Decision (terminology based on Zimmermann, 1987). Both styles differ in the degree of disagreement that the experts are allowed to have while constructing the common decision.

Generally, Expert Judgment methods can be divided into the following categories:

- Methods of generating ideas. These methods include brainstorming in verbal or written forms.
- Methods of polling ideas. These methods produce quick estimates of the preferences of the experts. Surveys, the Delphi method and conferencing are implementations of pooling ideas.
- Simulation models. These models include Cognitive Maps, and the SPAN method (Successive Proportional Additive Network also known as Social Participatory Allocative Network).

There is a vast amount of literature available on this topic, and this paper provides the most basic review in order to provide the background for the more detailed discussion on Fuzzy Group Decision Making. A good review of the general MCDM field can be found in Triantaphyllou (2000).

The essence of the group decision making can be summarized as follows: there is a set of options and a set of individuals (termed experts) who provide their preferences over the set of options. The problem is to find an option (or a set of options) that is best acceptable to the group of experts. Such a solution entertains the concept of majority that is further explored below.

A group decision-making problem is composed by the following elements: First, there exists a finite set of alternatives. $A = \{A_1, A_2 \ldots A_n\}$ as well as a finite set of experts $E = \{E_1, E_2 \ldots E_q\}$ and each expert $e_k \in E$ presents his/her preference relation on A_i as $x_{ik} \in S$. Where S is a finite but totally ordered term set of linguistic labels $S = \{s_0, s_1 \ldots s_T\}$, with $s_i > s_j$, if $i > j$. The experts may evaluate the alternatives on m criteria $C = \{C_1, C_2 \ldots C_m\}$.

2.1.5 Decision Making Tools

Table 2.1 summarizes some decision making tools categorized into three groups: information control, choosing preferred alternatives, and decision making process.

Table 2.1 Summary of decision making tools

Category	Method	References
Information Control	Brainstorming	mindtools.com
	Six Thinking Hats	mindtools.com
Choosing Preferred Alternatives	Analytic Hierarchy Process (AHP)	Chang, 1996
	Compromise programming	Prodanovic and Simonovic, 2003
	Dialectical inquiry	Modelling and Decision Support Tools
	Small Group	Modelling and Decision Support Tools
	Caucusing	Modelling and Decision Support Tools
	Grid Analysis	Modelling and Decision Support Tools
	Force field analysis	Modelling and Decision Support Tools
	Paired Comparison Analysis	Modelling and Decision Support Tools
	Criteria Rating Technique	Modelling and Decision Support Tools
Decision Making Process	Delphi method	Ishikawa, 1993
	Nominal Group Technique	Team Tools Objectives Decision Making
	Nemawashi	Group Decision Support and Groupware Technologies
	Devil's advocate technique	Team Tools Objectives Decision Making
	Arbitration	Group Decision Support and Groupware Technologies

2.1.5.1 Information Control

In this section, the following tools could be used to obtain information, especially to generate new alternatives.

- **Brainstorming (mindtools.com)**

This is a very useful technique when ideas need to be solicited from the whole group. The normal rule of waiting to speak until the facilitator recognizes you is suspended and everyone is encouraged to call out ideas to be written by the scribe for all to see. It is helpful if the atmosphere created is one in which all ideas, no matter how unusual or incomplete, are appropriate and welcomed. This is a situation in which suggestions can be used as catalysts, with ideas building one upon the next, generating very creative possibilities. Avoid evaluating each other's ideas during this time.

- **Six Thinking Hats (mindtools.com)**

'Six Thinking Hats' is used to look at decisions from a number of important perspectives. Each 'Thinking Hat' is a different style of thinking. White Hat focus on the data available, Red Hat looks at problems using intuition, gut reaction, and emotion. Using black hat thinking, decision maker will look at all the bad points of the decision. Black Hat thinking helps to make the decision 'tougher' and more resilient. The yellow hat helps to think positively. The Green Hat stands for creativity. And, the Blue Hat means for process control. If you look at a decision making problem with the 'Six Thinking Hats' technique, then you will solve it using all approaches. Your decisions and plans will mix ambition, skill in execution, public sensitivity, creativity and good contingency planning.

Choosing Preferred Alternatives

- **Analytic Hierarchy Process (AHP) (Chang, 1996)**

It is often difficult to conceptualize all the different elements of a problem, or there is not enough cognitive energy to prioritize those elements. The AHP was formulated to counter those situations, and is a mathematically-based theory. It employs two key aspects: data from the various variables that make up the decision and judgments about those variables.

- **Compromise programming (Prodanovic and Simonovic, 2003)**

Compromise programming attempts to preserve some level of transparency to problems. However, compromise programming only makes use of a limited amount of information. Extensive sensitivity analysis is necessary to recommend any kind of recommendation with confidence. The marriage of a transparent technique such as compromise programming with fuzzy sets is an example of a hybrid decision making tool available to future planners.

- **Dialectical inquiry (from Modeling and Decision Support Tools)**

In the dialectical inquiry approach, the team uses the same set of data to make two separate and opposing recommendations and then formally debates these recommendations based on the assumptions that were used to derive them. The philosophy behind this method is that a clearer understanding of the situation and an effective solution result when the assumptions underlying each recommendation are subjected to intense scrutiny and evaluation.

- **Small Group (from Modeling and Decision Support Tools)**

Breaking into smaller groups can be very useful. These small groups can be diads or triads or even larger. They can be selected randomly or self-selected. If used well, in a relatively short amount of time all participants have the opportunity to share their own point of view. Be sure to set clear time limits and select a note taker for each group. When the larger group reconvenes, the note takers relate the major points and concerns of their group. Sometimes, note takers can be requested to add only new ideas or concerns and not repeat something already covered in another report. It is also helpful for the scribe to write these reports so all can see the cumulative result and be sure every idea and concern gets on the list.

- **Caucusing (from Modeling and Decision Support Tools)**

A caucus might be useful to help a multifaceted conflict become clearer by unifying similar perspectives or defining specific points of departure without the focus of the whole group. It might be that only some people attend a caucus, or it might be that all are expected to participate in a caucus. The difference between caucuses and small groups is that caucuses are composed of people with similar viewpoints, whereas small group discussions are more useful if they are made up of people with diverse viewpoints or even a random selection of people.

- **Grid Analysis (from Modeling and Decision Support Tools)**

Grid Analysis is a useful technique to use for making a decision. It is most effective where the problem has a number of good alternatives and many factors to be taken into account. The first step is to list all alternatives and then the factors that are important for making the decision. Lay these out in a table, with options as the row labels, and factors as the column headings. Next, work out the relative importance of the factors in the decision. Show these as numbers. Use these to weight the preferences by the importance of the factor. These values may be obvious. If they are not, then use a technique such as Paired Comparison Analysis to estimate them. The next step is to work a way across the table, scoring each option for each of the important factors. Score each option from 0 (poor) to 3 (very good). Now multiply each of the scores by the values for the relative importance. This will give them the correct overall weight in final decision. Finally add up these weighted scores for the alternatives. The option that scores the highest wins!

- **Force field analysis (from Modeling and Decision Support Tools)**

Force Field Analysis is a useful technique for looking at all the forces for and against a decision. In effect, it is a specialized method of weighing pros and cons. By carrying out the analysis decision makers can plan to strengthen the forces supporting a decision, and reduce the impact of opposition to it. To carry out a force field analysis, follow these steps: first, list all forces for change in one column, and all forces against change in another column. Second, assign a score to each force, from 1 (weak) to 5 (strong). Then, draw a diagram showing the forces for and against change. Show the size of each force as a number next to it.

- **Paired Comparison Analysis (from Modeling and Decision Support Tools)**

Paired Comparison Analysis is also known as Paired Choice Analysis. Similar items are compared one against the next and the results are tallied to find an overall winner. This makes it easy to choose the most important problem to solve, or select the solution that will give the greatest advantage.

- **Criteria Rating Technique (from Modeling and Decision Support Tools)**

  The procedure of using the criteria rating technique is as the followings:

1) List the alternatives available
2) Brainstorm decision criteria. Decision makers will be judging all alternatives against what they feel are the most important qualities each one should have. These qualities are called decision criteria. Brainstorming may be a useful way for a group to agree appropriate criteria.
3) Determine the relative importance of each criterion. Rank the criteria and assign a relative importance (weight) to each. The total of the assigned weights should equal 100.
4) Establish a rating scale; rate the alternatives. A suitable rating scale might be, for instance: 1= low, 10=high. Each alternative should be weighed against each criterion, using the same scale for each.
5) Calculate the final score. Multiply the weight for each alternative by the score and write this in brackets. Add up the numbers in brackets for each alternative and write the sums in the appropriate total boxes. Add any summary comments in the appropriate summary box.
6) Select the best alternative. Select the alternative with the highest score. this alternative may not be the one ultimately chosen - if the group disagrees with the choice, they should review the weighting of the criteria and make the necessary changes. If necessary, repeat the process.

2.1.5.3 Decision Making Processes

- **Delphi method (Ishikawa, 1993)**

The Delphi Technique gathers and evaluates information from a group without physically assembling its members. Ensure anonymity of each member's input in the decision. Minimizing face-to-face interaction, such as when the issue is sensitive or required confidentiality. Communicate to each member the collective input of the rest of the team, so they can factor the team's position into their decision.

One of the weaknesses of Delphi method is that it requires repetitive surveys of the experts to allow the forecast value to converge. The more we repeat surveys, the more costly they become, especially for large (or complicated) problem.

- **Nominal Group Technique (from Team Tools Objectives Decision Making)**

Generate a large number of creative potential solutions to a problem or opportunity, evaluate these solutions, and rank them from most to least promising. Best for small group meetings, larger groups should be divided in subgroups.

- **Nemawashi (from Group Decision Support and Groupware Technologies)**

Nemawashi is a critical aspect of consensus-building in Japanese organizations, and the definition of it can be loosely translated several ways. Tomlinson (1996) offers this definition: a tactic implemented by the Japanese to bring about consensus through various pre-meeting consultations, where a strong foundation is being built so that the result will create a general agreement amongst those involved in the decision.

- **Devil's advocate technique (from Team Tools Objectives Decision Making)**

This is a technique that is also very useful during brainstorming and in consensus building processes as well. One member plays the devils advocate to the potential decision by stating all the opposite possibilities. This technique is useful in majority support because it prevents the board from falling into "groupthink". Groupthink occurs when members suppress their dissenting view because they believe no one will agree with them. By allowing someone to "play" the devils advocate it encourages members to discuss the merits of an action or potential decision without worrying about blocking the group's momentum.

- **Arbitration (from Group Decision Support and Groupware Technologies)**

Arbitration is a dispute resolution process in which the disputing parties present their case to a third party intermediary (or a panel of arbitrators) who examine all the evidence and then make a decision for the parties. This decision is usually binding. Like court-based adjudication, arbitration is adversarial. The presentations are made to prove one side right, the other wrong. Thus the parties assume they are working against each other, not cooperatively. Arbitration is generally not as formal as court adjudication, however, and the rules can be altered to some extent to meet the parties' needs.

2.2 Fuzzy Sets Theory

The term "fuzzy" was proposed by Zadeh in 1962. In 1965, Zadeh formally published the famous paper "Fuzzy Sets". The fuzzy set theory was intended to improve the oversimplified model, thereby developing a more robust and flexible model in order to solve real-world complex systems involving human aspects. Fuzzy set theory has been applied in almost every field, including control systems, optimization theory, artificial intelligence, human behavior, etc. According to *Berkeley Initiative in Soft Computing (BISC),* by 2004, the number of fuzzy-related patents issued in Japan is 4,801, and the number of fuzzy- related patents issued in the US is around 1,700. The Table 2.2 summaries the research publications of papers containing the word “fuzzy” in title cited in INSPEC and MATH.SCI.NET databases.

Table 2.2 Number of papers in INSPEC and MathSciNet which have "fuzzy" in their *titles, (Data for 2002 are not complete)*

	INSPEC	***MathSciNet***
1970-1979	*569*	*443*
1980-1989	*2,404*	*2,465*
1990-1999	*23,207*	*5,479*
2000-present	*9,945*	*2,865*
Total	*36,125*	*11,252*

**Compiled by Camille Wanat, Nov 20, 2003*

In this study we will concentrate fuzzy group decision making. To do so, we will first introduce the required knowledge such as fuzzy sets, fuzzy set theory, linguistic variable, membership functions, fuzzy relations, fuzzy operations, and fuzzy mathematics etc.

2.2.1 Definition

Traditional mathematics and logic assigns a membership of 1 to items which are members of a set, and 0 to those which are not. This is the dichotomy principle. Such a strong principle inevitably ran into philosophical problems. Fuzzy set theory offers a logic which more closely imitates the human thought process by allowing for possibilistic reasoning and vagueness. It allows a proposition to be neither fully true, nor fully false, but partly true and partly false to a given degree. It is common to restrict these degrees of membership to the real inclusive interval [0, 1].

A fuzzy subset *A* of a (crisp) set *X* is characterized by assigning to each element *x* of *X* the *degree of membership* of *x* in *A* (e.g. *X* is a group of people, *A* the fuzzy set of *old* people in *X*).

$$A = \{(x, \mu_F(x) \mid x \in X\} \tag{2.1}$$

where, $\mu_F(x)$ is the degree of membership or membership function (MF) of x in F. The closer the membership $\mu_F(x)$ is to 1, the more x belongs to F. The MF maps each elements of X to a continuous membership value between 0 and 1. Such representation is very flexible and allows arbitrary MF shape in real applications. That means the MF $\mu_F(x) = f(x)$ can be any function.

Now if *X* is a set of propositions then its elements may be assigned their *degree of truth*, which may be "absolutely true," "absolutely false" or some *intermediate* truth degree: a proposition may be more true than another proposition. This is obvious in the case of vague (imprecise) propositions like "this person is old" (beautiful, rich, etc.). In the analogy to various definitions of operations on fuzzy sets (intersection, union, complement, …) one may ask how propositions can be combined by *connectives* (conjunction, disjunction, negation, …) and if the truth degree of a composed proposition is determined by the truth degrees of its components, i.e. if the connectives have their corresponding *truth functions* (like truth tables of classical logic). Saying "yes" (which is the mainstream of fuzzy logic) one accepts the truth-functional approach; this makes fuzzy logic to something distinctly *different from probability theory* since the latter is not truth-functional (the probability of conjunction of two propositions is *not determined* by the probabilities of those propositions).

It is clear that the membership function is the key to designing meaningful fuzzy sets for image features.

2.2.1.1 Membership functions

The membership functions for fuzzy sets can have many different shapes, depending on definition. The popularly used fuzzy membership functions in the applications are triangular membership functions, trapezoidal membership function, Gaussian (Bell-shaped) membership function and Sigmoidal membership function. The general membership functions are described as:

1) Triangular membership function:

$$\mu_F(x) = \begin{cases} 1-(x^m - x)/e^L, & if\ x^L \le x \le x^m \\ 1-(x-x^m)/e^R, & if\ x^m \le x \le x^R \\ 0, & otherwise \end{cases} \tag{2.2}$$

where, x^m, $e^L > 0$, $e^R > 0$, represent the mode, the left-hand spread, the right-hand spread of the membership function, respectively; $x^L = x^m - e^L$, $x^R = x^m + e^R$.

In particular, if $e^L = e^R = e > 0$, then F becomes a symmetric triangular fuzzy membership function, which can be denoted as:

$$F = (x^m, e) \tag{2.3}$$

2) Trapezoidal membership function

A 4-tuple trapezoidal membership function is formed like (Figure 2.2):

$$\tilde{A} = (a_1, a_2, a_3, a_4, w_{\tilde{A}}) \tag{2.4}$$

Where a_2 and a_3 indicate the interval in which the membership function value is 1, while a_1 and a_4 are the left and right limits of the definition domain of trapezoidal membership function. w_A denotes the maximum membership value of the fuzzy number.

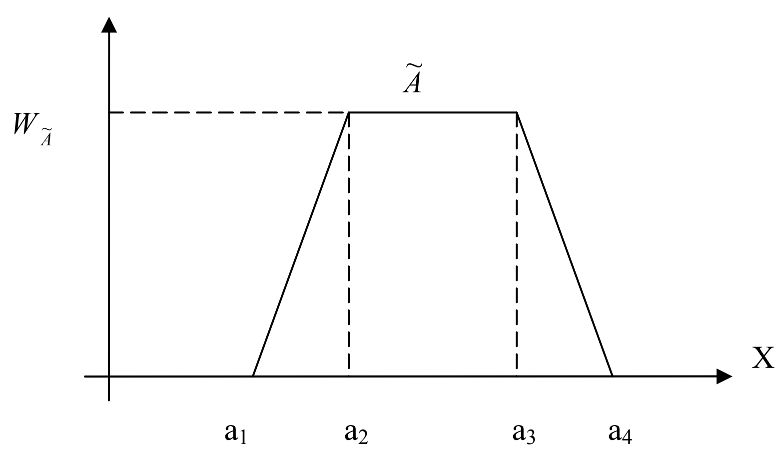


Figure 2.2 Trapezoidal membership function

3) Gaussian (Bell-shaped) membership function:

$$\mu_F(x) = \exp\left(-(\frac{x-v}{\sigma^2})^2\right) \tag{2.5}$$

where, ν and σ specify the center and spread of a Gaussian membership function, respectively.

4) Generalized Bell membership function.

A generalized bell MF is defined by three parameters (a, b, c):

$$\mu_F(x,a,b,c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}} \tag{2.6}$$

where the parameter b is usually positive. This membership function is a direct generalization of the Cauchy distribution used in probability theory.

5) Sigmoidal membership function

A sigmoidal MF is defined by:

$$\mu_F(x,a,c) = \frac{1}{1+\exp(-a(x-c))} \tag{2.7}$$

where the parameter a controls the slope at the crossover point x = c. This membership function is usually applied in the activation function in neural network.

In addition, there is one special fuzzy membership function – singleton MF, which is also most commonly used fuzzifier. It maps $x \in X$ into fuzzy set F in X with $\mu_F(b) = 1$ and $\mu_F(b') = 0$ for all $b' \in X$ with $b' \neq b$. A singleton MF can be represented as:

$$\mu_F(x) = \begin{cases} 1 & x = b \\ 0 & x \neq b \quad \text{and} \quad \forall x \in X \end{cases} \tag{2.8}$$

These five types of most commonly used membership functions are shown in Figure 2.3 using examples. In practice, compared with Gaussian MFs, triangular or trapezoidal functions needs low computatational cost. However, Gaussian MFs have some important advantages, for example, they can produce smooth mappings; universal approximation property can be easily proven; and normal distribution can be approximated well by Gaussian-type basis function. It is worthwhile noting that these five popularly used MFs are by no means exhaustive; one can create any specialized membership function for any special application if necessary. In particular, any types of continuous probability distribution functions can be used as a membership function, provided that a set of parameters are given to specify the appropriate meanings of the MFs.

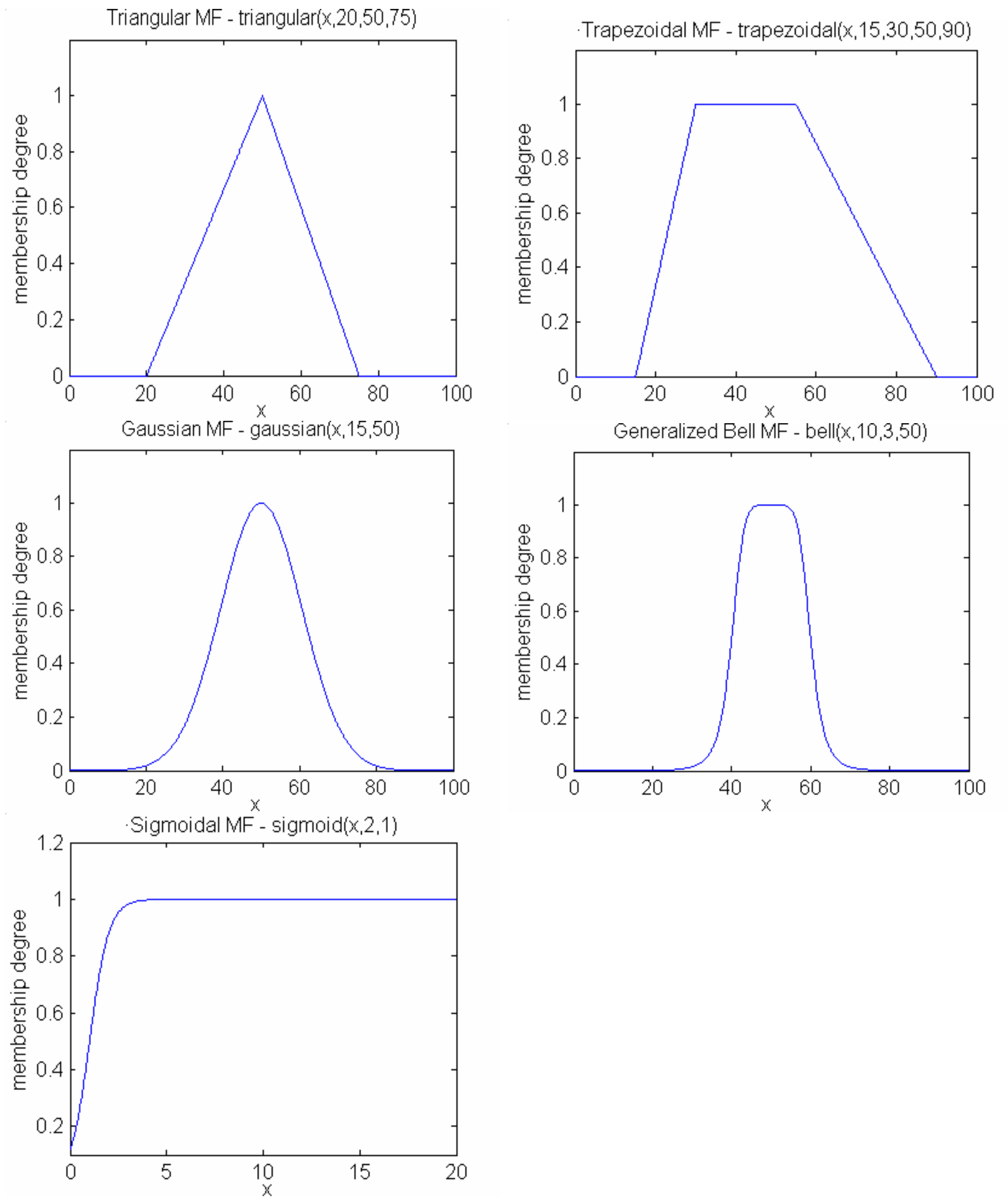


Figure 2.3 Five types of most commonly used membership functions

2.2.1.2 α -cut

It is the crisp domain in which we perform all computations with today's computers. The conversion from fuzzy to crisp sets can be done by two means, one of which is α -cut. Given a fuzzy set $\underset{\sim}{A}$, the α -cut set of $\underset{\sim}{A}$ is defined by:

$$A_\alpha = \left\{ x \middle| \mu_{\underset{\sim}{A}}(x) \geq \alpha \right\} \quad (2.9)$$

where $\alpha \in [0,1]$. Note that by virtue of the condition on $\mu_{\underset{\sim}{A}}(x)$ in above equation, i.e., a common property, the set A_α is now a crisp set. In fact, any fuzzy set can be converted to an infinite number of cut sets.

2.2.1.3 Convex fuzzy set

A set *A* is a convex fuzzy set**,** if and only if for all x_1, x_2 in X,

$$\mu_A(\lambda x_1 + (1 - \lambda)\, x_2\,) \geq Min(\mu_A(x_1), \mu_A(x_2\,)) \quad (2.10)$$

where $\lambda \in [0,1]$

2.2.1.4 Normal Fuzzy Set

A set *A* is a normal fuzzy set**,** if $\exists x_i \in X, \mu_A(x_i) = 1$

2.2.1.5 Fuzzy Number

Fuzzy Number is a fuzzy subset in the universe of discourse X that is both convex and normal.

In addition, if $\widetilde{n}$ is a fuzzy number and $n_l^\alpha > 0$ for $\alpha \in [0,1]$, then $\widetilde{n}$ is called a positive fuzzy number.

Fuzzy number is a very important definition in fuzzy set theory. Since it is normal, convex, it simplifies the calculation. Here are some characteristic values of fuzzy numbers:

a). The value of fuzzy number: A central value that represents the value of the magnitude that the fuzzy number represents.

$$G(s_i) = \int_0^1 s(r)(L_{y_{s_i}}(r) + R_{y_{s_i}}(r))dr \tag{2.11}$$

b). Maximum value

$$height(s_i) = Sup\{\mu_{y_{s_i}}(v), \forall v\} = \max\{v \mid \mu_{y_{s_i}}(v)\} \tag{2.12}$$

c). Minimum Value

$$height(s_i) = \min\{v \mid \mu_{y_{s_i}}(v)\} \tag{2.13}$$

d). Center of gravity

$$G(s_i) = \frac{\int_v v\mu_{y_{s_i}}(v)dv}{\int_v \mu_{y_{s_i}}(v)dv} \tag{2.14}$$

2.2.2 Fuzzy Set Operations

As in the traditional crisp sets, logical operations, e.g., union, intersection, and complement, can be applied to fuzzy sets.

1) Union

The union operation (and the intersection operation as well) can be defined in many different ways. The union of two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ with the membership functions $\mu_{\underset{\sim}{A}}(x)$ and $\mu_{\underset{\sim}{B}}(x)$ is a fuzzy set $\underset{\sim}{C}$, written as $\underset{\sim}{C} = \underset{\sim}{A} \cup \underset{\sim}{B}$, whose membership function is related to those of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ as follows:

$$\forall x \in U : \mu_{\underset{\sim}{C}} = \max\left\lfloor \mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x) \right\rfloor \quad (2.15)$$

2) Intersection

According to the min-operator, the intersection of two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ with the membership functions $\mu_{\underset{\sim}{A}}(x)$ and $\mu_{\underset{\sim}{B}}(x)$, respectively, is a fuzzy set $\underset{\sim}{C}$, written as $\underset{\sim}{C} = \underset{\sim}{A} \cap \underset{\sim}{B}$, whose membership function is related to those of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ as follows:

$$\forall x \in U : \mu_{\underset{\sim}{C}} = \min\left[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x) \right] \quad (2.16)$$

3) Complement

The complement of a set $\underset{\sim}{A}$, denoted $\overline{\underset{\sim}{A}}$, is defined as the collection of all elements in the universe which do not reside in the set $\underset{\sim}{A}$.

$$\forall x \in U : \mu_{\overline{\underset{\sim}{A}}} = 1 - \mu_{\underset{\sim}{A}}(x) \quad (2.17)$$

Note that even though the equations of the union, intersection, and complement appear to be the same for classical and fuzzy sets, they differ in the fact that $\mu_{\underset{\sim}{A}}(x)$ and $\mu_{\underset{\sim}{B}}(x)$ can take only a value of zero or one in the case of classical set, while in fuzzy sets they include the whole interval from zero to one.

Although (and only) min and max operations satisfy the law of distributivity, min and max do incur some difficulty in analyzing fuzzy systems. But they are not the only ways that can be chosen to model the intersection and union of a fuzzy set. Two popular alternatives to them are algrbraic product and algrbraic sum, which were first defined by Zadeh in 1965 (Zadeh, 1965) as:

$$\mu_{A \wedge B}(x) = \mu_A(x) \cdot \mu_B(x) \tag{2.18}$$

$$\mu_{A \vee B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \tag{2.19}$$

Typically, the first algrbraic product logical operator gives much smoother approximations than MIN operator for fuzzy intersection operation. And it is interesting that a sum of fuzzy sets (obviously not a s-norm) is far more common choice than algrbraic sum, Max or other fuzzy union operators (s-norm) for fuzzy union operation, which may in a supernormal fuzzy set with height greater than one.

2.2.2.1 T-norms

In addition to the above two kinds of operators available, the more general forms of the operators for fuzzy intersections and fuzzy unions are called T-norms (triangular norm) and T-conorms or S-norms in fuzzy logic, which can be defined as

$$\mu_{A \wedge B}(x) = T[\mu_A(x), \mu_B(x)] \tag{2.20}$$

$$\mu_{A \vee B}(x) = S[\mu_A(x), \mu_B(x)] \tag{2.21}$$

The T-norm operators should satisfy the following basic requirements (Smolíková and Wachowiak, 2002):

- Boundary: $T(0,0) = 0$, $T(a,1) = T(1,a) = a$
- Monotonicity: $T(a,b) \leq T(c,d)$, if $a \leq c$ and $b \leq d$
- Commutativity: $T(a,b) = T(b,a)$
- Associativity: $T(a,T(b,c)) = T(T(a,b),c)$.

The first requirement ensures the correct generalization of crisp sets. The second requirement implies that a decrease in the membership values in A and B can not produce an increase in the membership value of the intersection of sets A and B. The third requirement specifies that the operation is insensitive to the order in which fuzzy sets are combined, and the fourth requirement enables us to take the intersection of any number of fuzzy sets and any order of pairwise grouping (Kulkarni, 2001).

Four of the most frequently used T-norm operators are (Jang, Sun and Mizutani, 1997):

- Minimum T-norm: $T_{\min}(a,b) = \min(a,b) = a \wedge b$
- Algebraic product T-norm: $T_{ap}(a,b) = ab$
- Bounded product (or Lukasiewicz) T-norm: $T_{bp}(a,b) = 0 \vee (a+b-1)$
- Drastic product (or Degenerate): $T_{dp}(a,b) = \begin{cases} a, & if \quad b = 1 \\ b, & if \quad a = 1 \\ 0, & if \; a,b < 1 \end{cases}$

And it can be proved that $T_{dp}(a,b) \le T_{bp}(a,b) \le T_{ap}(a,b) \le T_{\min}(a,b)$ is satisfied.

2.2.2.2 S-norms

Similar to T-norm, S-norm operators should meet the following requirements (Kulkarni, 2001):

- Boundary: $S(1,1) = 0$, $S(a,0) = S(0,a) = a$
- Monotonicity: $S(a,b) \le S(c,d)$, if $a \le c$ and $b \le d$
- Commutativity: $S(a,b) = S(b,a)$
- Associativity: $S(a, S(b,c)) = S(S(a,b),c)$.

Four of the most frequently used S-norm operators are (Jang, Sun and Mizutani, 1997):

- Maximum T-conorm: $S_{\max}(a,b) = \max(a,b) = a \vee b$
- Algebraic sum T-conorm: $S(a,b) = a + b - ab$

- Bounded sum (or Lukasiewicz) T-conorm: $S(a,b) = 1 \wedge (a+b)$

- Drastic sum (or Degenerate) T-conorm: $S(a,b) = \begin{cases} a, & if \quad b = 0 \\ b, & if \quad a = 0 \\ 1, & if\ a,b > 0 \end{cases}$.

And it can be proved that $S_{\max}(a,b) \le S_{as}(a,b) \le S_{bs}(a,b) \le S_{ds}(a,b)$ is satisfied.

The above described T-norms and S-norms are only part of the operators appeared in literatures. It is pretty hard to illustrate all of them. Some classes of T-norms and S-norms are summarized in Table 2.3.

Table 2.3 Definitions of T-norms and S-norms

	T-Norm T(a, b)	S-Norm S(a, b)
Zadeh	Min (a, b)	Max (a, b)
Lukasiewicz	$\max\{a+b-1, 0\}$	$\min\{a+b, 1\}$
Product/Probabilistic	ab	$a+b-ab$
	$\dfrac{ab}{1+\overline{a}\cdot\overline{b}}$	$\dfrac{a+b}{1+a\cdot b}$
	$0\vee(a+b-1)$	$1\wedge(a+b)$
Dubois and Prade	$\dfrac{ab}{\max\{a,b,\alpha\}}, \alpha\in(0,1)$	$1-\dfrac{(1-a)(1-b)}{\max\{(1-a),(1-b),\alpha\}}, \alpha\in(0,1)$
Yager	$1-\min\{1,[(1-a)^p+(1-b)^p]^{1/p}\}, p>0$	$\min\{1,[a^p+b^p]^{1/p}\}, p>0$
	$\dfrac{ab}{\gamma+(1-\gamma)(a+b-ab)}, \gamma\in[0,\infty)$	$\dfrac{(a+b-ab)-(1-\gamma)ab}{\gamma+(1-\gamma)(1-ab)}, \gamma\in[0,\infty)$
	$1-[1\wedge(a^{-\lambda}+b^{-\lambda})^{\frac{1}{\gamma}}], \gamma\in[1,\infty)$	$1\wedge(a^{\lambda}+b^{\gamma})^{\frac{1}{\gamma}}, \gamma\in[1,\infty)$
	$(a^{-\rho}+b^{-\rho}-1)^{-\frac{1}{\rho}}, \rho>0$	$1-(\overline{a}^{-\rho}+\overline{b}^{-\rho}-1)^{-\frac{1}{\rho}}, \rho<0, \overline{a}^{-\rho}+\overline{b}^{-\rho}\geq 1$ $1, \rho<0, \overline{a}^{-\rho}+\overline{b}^{-\rho}\leq 1$
Drastic	$\min(a,b)$ if $\max(a,b)$=1, 0 otherwise	$\max(a,b)$ if $\min(a,b)$=0, 1 otherwise
Einstein	$a\cdot b/(2-(a+b-a\cdot b))$	$(a+b)/(1+ab)$
Hamacher	$a\cdot b/(a+b)-a\cdot b)$	$(a+b-2ab)/(1-ab)$

Relation of T-norms and S-norms

- $T(a, b) \le S(a, b)$
- If $T(a, 1) = a$, then $T(a, b) \le a \wedge b$
- If $S(a, 0) = a$, then $S(a, b) \ge a \vee b$
- If $T(a, 1) = a \Rightarrow T(a, 0) = 0$
- If $S(a, 0) = a \Rightarrow S(a, 1) = 1$
- If $T(a, a) \le a \Rightarrow T(a, b) \le a \vee b$
- If $S(a, a) \ge a \Rightarrow S(a, b) \ge a \wedge b$

Generation of T-norms and S-norms

Given function g(s), where g(0)=0 and g(1)=1, then,

$S(a, b) = G(1 \wedge (g(a) + g(b)))$ is a S-norm and

$T(a, b) = G(0 \vee (g(a) + g(b) - 1))$ is a T-norm

Example:

$g(s) = s^2$ satisfy g(0)=0 and g(1)=1, then, $s = \sqrt{g}$ or $G(s) = \sqrt{s}$. So, $S(a, b) = G(1 \wedge g(a) + g(b)) = G(1 \wedge (a^2 + b^2)) = (1 \wedge (a^2 + b^2))^{\frac{1}{2}}$

If S(a, b) is a S-norm, then, $F(a, b) = G(S(g(a), g(b)))$ is also a triangular norm (S-norm).

If T(a, b) is a T-norm, then, $F(a,b) = G(T(g(a), g(b)))$ is also T-norm.

Example:

$S(a,b) = a + b - ab$, and taking $g(s) = s^2$, then $G(s) = \sqrt{s}$

$F(a,b) = G(S(g(a) + g(b))) = G(S(a^2, b^2)) = G(a^2 + b^2 - a^2b^2) = (a^2 + b^2 - a^2b^2)^{\frac{1}{2}}$ is a S-norm.

Given function h(s), where $h(0) \to \infty, h(1) = 1$ and $h(+\infty) = 0$, and $H(s) = h^{-1}(s)$, then, $T(a,b) = H(h(a), h(b))$ is a T-norm, $T(a,b) = H(h(a) + h(b) - 1)$ is a T-norm

Example:

$$h(s) = \frac{1}{s}, H(s) = \frac{1}{s},\ T(a,b) = H(h(a), h(b)) = H(\frac{1}{a} \cdot \frac{1}{b}) = \frac{1}{\frac{1}{a} \cdot \frac{1}{b}} = ab$$

$$T(a,b) = H(h(a) + h(b) - 1) = H(\frac{1}{a} + \frac{1}{b} - 1) = \frac{1}{\frac{1}{a} + \frac{1}{b} - 1} = \frac{ab}{a + b - ab}$$

2.3 Fuzzy Similarity Measure

A similarity measure is a function that associates a numeric value with a pair of fuzzy sets, with the intention that a higher value indicates greater similarity. Measuring similarity is an important topic, common to many areas of applications. The importance of this topic emerges from the fact that in real life many of the attributes describing objects are not quantitative, and using fuzzy logic provides an essential tool to utilize qualitative judgment in practice. Fuzzy similarity methods are useful in applications of fuzzy expert systems, fuzzy pattern recognition, fuzzy database and information retrieval and many types of comparative studies using fuzzy set theory.

A similarity measure function must conform to the following five axioms:

1) Non-negativity, $s \geq 0$
2) Symmetry, $s(A,B) = s(B,A)$
3) Identity, $s(A,A) = 1$
4) Opposite, $s(A,\overline{A}) = 0$
5) Definiteness, $s(A,B) = 0 \Leftrightarrow$ A∩B=Ø, $s(A,B) = 1 \Leftrightarrow$ A=B

These constraints are not counter-intuitive. The first and second constraints define a metric as a scalar and not a directional vector quantity. The third property reflects the axiomatic belief that the similarity between two objects must be zero when they are the identical. The fourth property strengthens the previous one by enforcing the constraint that the similarity metric must only be zero when two objects have no commons. The final constraint strengths the third and the fourth constraint.

During the evolution of fuzzy set theory numerous approaches for comparison or ranking of fuzzy sets have been proposed. However, the methods are not consistent, difficult to compare, and usually result in different similarity values between the same sets.

The methods presented in this research are classified into six categories (an extension of the classification provided by Chang and Lee, 1994):

1) Methods using α-cut.
2) Set-theoretic approaches.
3) Fuzzy Max-Min integration (operations of union and intersection).
4) Methods using distance functions.
5) Vector based methods.
6) Multi-dimensional methods.

The Table 2.4 summarizes 26 fuzzy similarity measure methods and compared some of important properties.

Table 2.4 Important properties of different fuzzy similarity measures

No	Method	Symmetry	Simplicity	S=0 ⇔ A∩B=Ø	S=1 ⇔ A=B	Non-convex	Non-normal
1	2.22	N	M	Y	N	N	N
2	2.23	Y	M	N	N	Y	Y
3	2.24	Y	H	Y	Y	Y	Y
4	2.25	Y	H	N	Y	Y	Y
5	2.26	Y	H	Y	Y	Y	Y
6	2.27	Y	H	N	Y	Y	Y
7	2.28	Y	H	Y	Y	Y	Y
8	2.29	Y	M	N	Y	Y	Y
9	2.30	Y	H	N	Y	Y	Y
10	2.31	Y	H	N	Y	Y	Y
11	2.32	Y	H	N	N	Y	Y

12	2.33	Y	L	N	Y	Y	Y
13	2.36	N	M	N	Y	Y	Y
14	2.37	Y	M	N	Y	Y	Y
15	2.38	Y	M	N	Y	Y	Y
16	2.39	N	M	N	N	Y	Y
17	2.40	Y	L	N	N	Y	Y
18	2.42	Y	M	Y	Y	Y	Y
19	2.43	Y	L	N	N	Y	Y
20	2.46	Y	L	Y	Y	Y	Y
21	2.47	Y	M	Y	Y	Y	Y
22	2.49	Y	M	Y	Y	Y	Y
23	2.51	Y	M	N	N	Y	Y
24	2.54	Y	L	N	N	Y	Y
25	2.57	Y	L	N	N	Y	Y
26	2.58	Y	L	N	N	Y	Y

Note: Y-Yes N-No H-High M-Medium L-Low

Note: The method's number is represented by the equation number.

Note: The method's number is represented by the equation number.

2.3.1 Methods using α-cut

This approach is demonstrated using the following similarity measure (Lee and Hyung 1994):

$$S(A,B)=\int_0^1 \frac{|A^\alpha \cap B^\alpha|}{|B^\alpha|}d\alpha \qquad (2.\ 22)$$

where $A^\alpha=\{x \mid \mu_A(x)\geq \alpha\}$ and $|A^\alpha|$ is the interval length of A^α

2.3.2 Set-theoretic measures

Inspired by the equivalence $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$ for classical sets, Wang et al. (1995) defined the similarity measure as:

$$S(A,B) = \min(\inf_{x \in X} \wp(A(x), B(x)), \inf_{x \in X}(\wp(B(x), A(x)))) \tag{2. 23}$$

Where $\wp$ is an implication operator such that $\wp[0,1]^2 \rightarrow [0,1]$ is a mapping satisfying $\wp(0,0) = \wp(0,1) = \wp(1,1) = 1$ and $\wp(1,0) = 0$.

Such a mapping can be for example the Lukasiewicz implication operator: $\wp(x, y) = \min(1, 1 - x + y)$.

2.3.3 Logic-Based measures

This family of methods is very popular among researchers and several similarity measures using this approach are proposed.

Pappis and Karacapilidis (1993) propose three measures of similarity between fuzzy values as follows:

$$S(A,B) = 1 - \max_i(|\mu_A(x_i) - \mu_B(x_i)|) \tag{2. 24}$$

$$S(A,B) = 1 - \frac{\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|}{\sum_{i=1}^{n} (\mu_A(x_i) + \mu_B(x_i))} \tag{2. 25}$$

$$S(A,B) = \frac{|A \cap B|}{|A \cup B|} = \frac{\sum_{i=1}^{n} |\mu_A(x_i) \wedge \mu_B(x_i)|}{\sum_{i=1}^{n} (\mu_A(x_i) \vee \mu_B(x_i))} \tag{2. 26}$$

Other measures from this family are (Chen et al. 1995):

$$S(A,B) = 1 - \frac{\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|}{n} \tag{2. 27}$$

Two additional similarity measures from the same source are:

$$S(A,B) = \sup_{x \in U} \mu_{A \cap B}(x) \tag{2. 28}$$

$$S(A,B) = \frac{\sum (\mu_A(x_i) \cdot \mu_B(x_i))}{\max(\sum (\mu_A(x_i) \cdot \mu_A(x_i)), \sum (\mu_B(x_i) \cdot \mu_B(x_i)))} \tag{2. 29}$$

Two additional similarity measures were proposed by Wang (1997):

$$S(A,B) = \frac{\sum_{i=1}^{n} \frac{\min(\mu_A(x_i), \mu_B(x_i))}{\max(\mu_A(x_i), \mu_B(x_i))}}{n} \tag{2. 30}$$

$$S(A,B) = \frac{\sum_{i=1}^{n} [1 - |\mu_A(x_i) - \mu_B(x_i)|]}{n} \tag{2. 31}$$

Other measures are as follows:

Hyung, Song and Lee (1994):

$$S(A,B) = \max_{x \in X}(\min(\mu_A(x), \mu_B(x))) \quad (2.\ 32)$$

Gerstenkorn and Man'ko's correlation method (1991):

$$S(A,B) = \frac{C(A,B)}{\sqrt{[T(A).T(B)]}} \quad (2.\ 33)$$

Where $T(A) = \sum_{i=1}^{n} [\mu_A{}^2(x_i) + \nu_A{}^2(x_i)]$ (2. 34)

$$C(A,B) = \sum_{i=1}^{n} [\mu_A(x_i) \cdot \mu_B(x_i) + \nu_A(x_i) \cdot \nu_B(x_i)] \quad (2.\ 35)$$

Where $\nu_A(x_i) = 1 - \mu_A(x_i)$

Yeung and Tsang's Equality and Cardinality (EC) Method (1997):

$$S(A,B) = 1 - \frac{\sum_{x \in X} | \mu_{a_i}(x) - \mu_{b_i}(x) |}{\sum_{x \in X} \mu_{b_i}(x)} \quad (2.\ 36)$$

This is another asymmetric approach, ($S(A,B) \neq S(B,A)$).

We propose to change the denominator into the bigger one of the two fuzzy sets:

$$S(A,B) = S(B,A) = 1 - \frac{\sum_{x \in X} | \mu_{a_i}(x) - \mu_{b_i}(x) |}{\max(\sum_{x \in X} \mu_{b_i}(x), \sum_{x \in X} \mu_B(x))} \quad (2.\ 37)$$

Chen's Function T (FT) Method (1994):

$$S(A,B,W) = \sum_{i=1}^{n} \left[T(\mu_A(x_i), \mu_B(x_i)) * \frac{w_i}{\sum_{k=1}^{n} w_k} \right] \tag{2. 38}$$

where $T(\mu_A(x_i), \mu_B(x_i)) = 1 - |\, \mu_A(x_i) - \mu_B(x_i) \,|$

Yeung and Tsang's Inclusion and Cardinality (IC) Method (1997):

$$S(A,B) = \frac{\sum_{x \in X} \min(1, \mu_{\bar{a}_i}(x) + \mu_{b_i}(x))}{\sum_{x \in X} \mu_X(x)} \tag{2. 39}$$

where $\mu_{\bar{a}_i}(x) = 1 - \mu_{a_i}(x)$.

2.3.4 Methods using distance functions

Zwick et al. (1987) propose the following similarity definition:

$$S(A,B) = (1 + D_r(A,B))^{-1} \tag{2. 40}$$

Using this method, one needs to calculate the distance first:

$$D_r(A,B) = \sqrt[r]{\sum_{i=1}^{n} |\, \mu_A(x_i) - \mu_B(x_i) \,|^r}, \qquad r \geq 1 \tag{2. 41}$$

Turksen and Zhong's (1990) Approximate Analogical Reasoning Schema (AARS) is another distance/similarity measure defined as:

$$S(A,B) = 1 - \sup_{x \in X} \mu_{A \cap B}(x) \qquad (2.42)$$

Chang and Lee (1994) summarize several distance measure approaches. For instance, after comparing the existence concepts from Yager (1978), Kaufmann (1986), Campos and Gonzalez (1980), Gonzalez (1990) and definitions of difference from Mabuchi (1988), Nakamura (1986) and Yuan (1991), they proposed their own definition as following,

$$D(A,B) = OM(A) - OM(B) = \int_0^{w^*_{hgt}} g_A(\{\mu^{-1}{}_A(w)\})dw - \int_0^{w^*_{hgt}} g_B(\{\mu^{-1}{}_B(w)\})dw \qquad (2.43)$$

Where, $g_i(\{\mu^{-1}{}_{Ai}(w)\}) = \omega(w)[\chi_1(w)x_i'(w) + \chi_2(w)x_i''(w)]$

$$x'(w) = \mu^{-1}{}_{A_L}(w),\ x''(w) = \mu^{-1}{}_{A_R}(w),\ w^*_{hgt} = \min v[hgt(A), hgt(B)] \qquad (2.44)$$

The weighting measures $\omega(w), \chi_1(w)\ and\ \chi_2(w)$ must be determined subjectively by the decision maker. Here OM represents the individual measurement for each fuzzy set.

This methods is different from the other methods mentioned, especially when non-convex fuzzy sets are involved. Chang and Lee used the following example in their paper:

$$\mu_A(x) = \begin{cases} x-5 & \text{for } 5 \le x \le 6 \\ 7-x & \text{for } 6 \le x \le 7 \\ 0 & \text{elsewhere} \end{cases} \qquad \mu_B(x) = \begin{cases} \frac{x}{7} & \text{for } 0 \le x \le 7 \\ 8-x & \text{for } 7 \le x \le 8 \\ 0 & \text{elsewhere} \end{cases} \qquad (2.45)$$

Table 2.5 summarized the results for different kinds of weighting measures

Table 2.5 Results using various weights from Chang and Lee's similarity method

		Fixed type weighting			**Linear type weighting**			**Nonlinear type weighting**		
$\chi_1(w)$	$\chi_2(w)$	**OM(A)**	**OM(B)**	**D(A,B)**	**OM(A)**	**OM(B)**	**D(A,B)**	**OM(A)**	**OM(B)**	**D(A,B)**
0.9	0.9	5.73	4.93	0.8	5.87	5.47	0.4	5.81	5.25	0.56
0.7	0.3	5.87	5.47	0.4	5.93	5.73	0.2	5.91	5.63	0.28
0.5	0.5	6.0	6.0	0.0						
0.3	0.7	6.13	6.53	-0.4	6.07	6.27	-0.2	6.09	6.37	-0.28
0.1	0.9	6.27	7.07	-0.8	6.13	6.53	-0.4	6.19	6.75	-0.56

The example shows that the similarity between the two fuzzy sets varies greatly (from -0.8 to 0.8) when using different weighting measures. This allows the decision maker to introduce subjective preferences into the similarity analyzes.

2.3.5 Representing the Fuzzy Sets as Two Vectors

This approach is demonstrated using Chen's Matching Function (MF) (1988):

$$S(A,B) = \frac{|A||B|\cos\theta}{\max(|A||A|,|B||B|)} \qquad (2.46)$$

where $\cos\theta$ is the cosine of the angle between the two vectors of A and B.

2.3.6 Multi-dimensional methods

Di Nola et al. (1994) propose an equality index ($a \equiv b$) matching method. This method defines the similarity between corresponding slots of a case and a prototype frame.

The equality index $(a \equiv b)$ determines the similarity of slots a and b:

$$a \equiv b = (a\, \varphi\, b) \wedge (b\, \varphi\, a) \tag{2. 47}$$

The $\equiv$ relation is defined as the t-norm (φ-operator).

One example of the φ-operator is Lukasiewicz implication (Pedrycz, 1990a):

$$a\, \varphi\, b = \begin{cases} 1 & if\ a \le b, \\ b - a + 1 & if\ a > b. \end{cases} \text{ So, } a \equiv b = \begin{cases} b - \mathrm{a} + 1 & \text{if b < a,} \\ 1 & \text{if a = b,} \\ a - \mathrm{b} + 1 & \text{if a < b.} \end{cases} \tag{2. 48}$$

Thus,

$$a \equiv b = \frac{((0.4 - 0.6 + 1) + (0.3 - 0.8 + 1) + (0.0 - 0.9 + 1))}{3} = 0.467$$

By adding the s-norm, the similarity is defined as:

$$a \equiv b = \frac{1}{2}(a \equiv_1 b\ +\ a \equiv_2 b) = \frac{1}{2}[(a\, \varphi\, b) \wedge (b\, \varphi\, a)] + \frac{1}{2}[1 - (a\, \beta\, b)\, s\, (b\, \beta\, a)] \tag{2. 49}$$

which can be also implemented using:

$$a \equiv b = \frac{1}{2}(a \equiv_1 b + a \equiv_2 b) = \frac{1}{2}[a \equiv_1 b + (1-a) \equiv_1 (1-b)] \qquad (2.50)$$

Using Lukasiewicz implication, the example is calculates as:

$$a \equiv b = \frac{1}{2}(0.467 + 0.467) = 0.467$$

Another similarity measure in this category was proposed by Cho and Lehto (1992):

$$S(A,B)_i = \begin{cases} CERT_+ & \text{if } (\text{CERT}_+ \geq 0.5) \wedge (\text{CERT}_- \leq 0.5) \\ (CERT_+ - CERT_-) + 0.5 & \text{if } (\text{CERT}_+ \geq 0.5) \wedge (\text{CERT}_- \geq 0.5) \\ 1 - CERT_+ & \text{if } (\text{CERT}_+ \leq 0.5) \wedge (\text{CERT}_- \geq 0.5) \end{cases} \qquad (2.51)$$

$$\text{CERT}_+ = \max\{\min(\mu_A, \mu_B)\} \qquad (2.52)$$

$$\text{CERT}_- = \max\{\min(1-\mu_A, \mu_B)\} \qquad (2.53)$$

$S = f(S_1, S_2, \ldots S_n)$ is the overall similarity of all n slots.

S_i is the degree of match between the ith slot and its reference slot, and the default function for f is minimum.

Nompto et al.'s Auto-fuzzy thesauri method (1999):

This method defines a new similarity measure between a query and a case, in which the relations among the values are taken into account. This derivative of the nearest neighbor algorithm measures the similarity by counting the number of the indices in which the values of the case are the same to those of the query and normalizing it by the number of the total indices K.

$$s_p^{\ A} = \frac{1}{K}\sum_{k=1}^{K} a_{p,k}^{\ \ T} \circ T_{k\times k} \circ b_k \tag{2. 54}$$

Where $a_{p,k} = [a_{p,k,1},...,a_{p,k,i_k},...,a_{p,k,m_k}]^T$, $b_k = [b_{k,1},...,b_{k,i_k},...,b_{k,m_k}]^T$ and a_{p,k,i_k} is the ith element of the pth case appearance vector.

$$a_{p,k,i_k} = \begin{cases} 1 & \text{the i}^{\text{th}} \text{ value of the k}^{\text{th}} \text{ index is given to the p}^{\text{th}} \text{ case} \\ 0 & \text{otherwise} \end{cases} \tag{2. 55}$$

Like a_{p,k,i_k} , $b_{k,i_k} \in \{0,1\}$ is 1 when the ith value of the kth index is given to the query and that is 0 otherwise.

$$[A \circ B]_{i,k} = \bigvee_{j=1}^{q} (a_{i,j} \wedge b_{j,k}) \tag{2. 56}$$

An extension of this method considers the relations between all the values from slot A and slot B. In this case:

$$s_p^{\ A/C} = \frac{1}{K}\sum_{k=1}^{K}\bigvee_{l=1}^{K} a_{p,l}^{\ \ T} \circ T_{k\times l} \circ b_k \tag{2. 57}$$

As an extension, instead of using Max as the combination of relation between slot values, we suggest using the sum of all the slot relations and then divided by the number of the slots.

$$s_p^{A/C} = \frac{1}{K^2} \sum_{K=1}^{K} \sum_{K=1}^{K} a_{p,l}^{T} \circ T_{k \times l} \circ b_k \tag{2. 58}$$

Different similarity measures usually have different properties. Thus, the selection of the right similarity measure is not obvious, at time depending on the application.

2.3.7 Sensitivity Analysis

The sensitivity of a similarity measure is an indication of the rate at which the similarity increases as the two fuzzy sets move closer to each other. This property is measured by moving one of the sets relative to a stationary set. Thus, the sensitivity is defined as the relation between the similarity S(A,B) and the distance D between the sets.. This sensitivity analysis can be used to examine the robustness of the similarity measure.

2.3.7.1 Measuring Distance between Sets

The approach developed in this research is to use the distance between the two centers of gravity as the distance measure.

Generally, for a polygonal shape with n discrete nodes the location of the center of gravity $Mi(x_i, y_i)$ =is defined as:

$$X(G) = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad (2.\ 59)$$

Where m_i is the mass of the i[th] node.

For an arc s as shown in Figure 2.4, this location is defined by equation (2. 60).

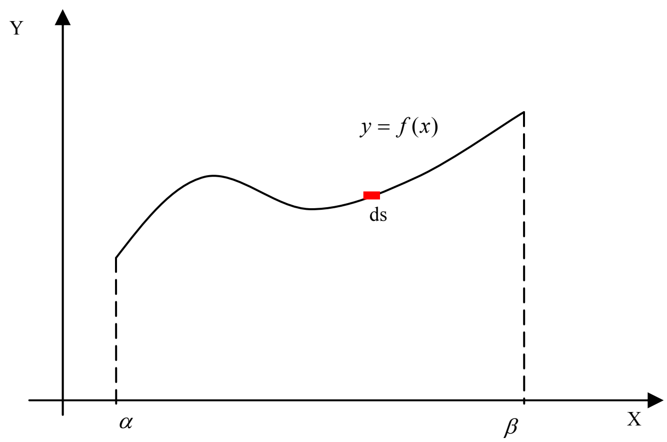


Figure 2.4 Center of Gravity of an arc

$$X(G) = \frac{\int_{\alpha}^{\beta} x\,\rho\,ds}{\sum_{i=1}^{n} m_i} = \frac{\int_{\alpha}^{\beta} x\,\rho\,ds}{\rho\,s} = \frac{\int_{\alpha}^{\beta} x\sqrt{1+f'^2(x)}dx}{\int_{\alpha}^{\beta} \sqrt{1+f'^2(x)}dx} \qquad (2.\ 60)$$

Where s is the length of the arc with $s = \int_{\alpha}^{\beta} \sqrt{x'^2(t) + y'^2(t)}\,dt$, given $x = x(t)$, $y = y(t)$ and $\alpha \le t \le \beta$, and ρ is the density of the arc.

Thus, the location of the center of gravity of the arc is:

$$X(G_A) = \frac{\int_{\underline{X}_A}^{\overline{X}_A} x\sqrt{1+\mu_A'^2(x)}dx}{\int_{\underline{X}_A}^{\overline{X}_A} \sqrt{1+\mu_A'^2(x)}dx}, \; X(G_B) = \frac{\int_{\underline{X}_B}^{\overline{X}_B} x\sqrt{1+\mu_B'^2(x)}dx}{\int_{\underline{X}_B}^{\overline{X}_B} \sqrt{1+\mu_B'^2(x)}dx} \tag{2. 61}$$

where $\mu_A(x)$ and $\mu_B(x)$ are membership functions of two fuzzy numbers.

2.3.7.2 Calculating the Sensitivity of the Distance Measures

The procedure for calculating the sensitivity is as follows:

1) Calculate similarity of two fuzzy sets A and B using a specific measure.

2) Measure distance between two fuzzy sets A and B

$$D = x(G_A) - x(G_B) \tag{2. 62}$$

3) Develop the relationship between similarity and distance:

$$S(A,B) = f(D) \tag{2. 63}$$

2.3.7.3 Example using Triangular Fuzzy Sets

The following example demonstrates the process of measuring the sensitivity between two triangular fuzzy sets using seven similarity measures including the proposed area-based similarity measure. The two fuzzy sets are presents in Figure 2.5, and have the following membership function.

$$\mu_A(x) = \begin{cases} x-1 & \text{for } 1 \le x \le 2 \\ -x+3 & \text{for } 2 \le x \le 3 \\ 0 & elsewhere \end{cases} \qquad \mu_B(x) = \begin{cases} x-2 & \text{for } 2 \le x \le 3 \\ -x+4 & \text{for } 3 \le x \le 4 \\ 0 & elsewhere \end{cases}$$

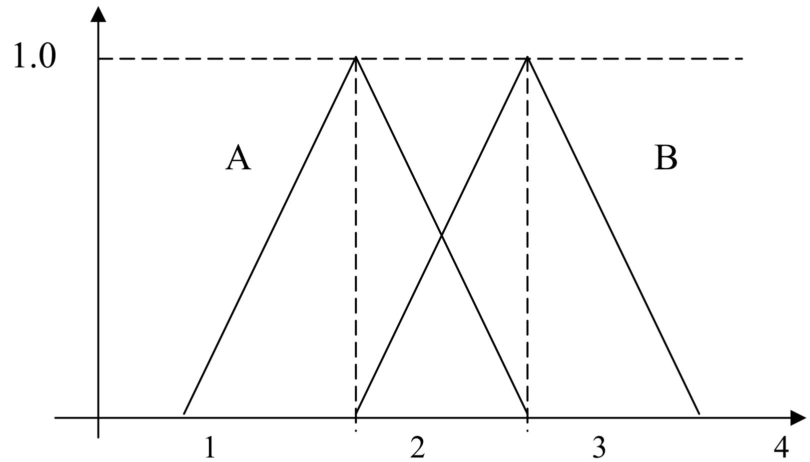


Figure 2.5 Membership Functions of two Fuzzy Numbers A and B

For comparison purposes we choose 6 similarity measures from the six categories, (one from each category), to analyze the sensitivity. Table 2.6 shows the computational results.

Table 2.6 Similarity values in several distance points

Distance	-2	-1.5	-1	-0.5	0	0.5	1.0	1.5	2
Lee and Hyung. (1994)	0	0.063	0.25	0.563	1	0.563	0.25	0.063	0
Wang et al. (1995)	0	0	0	0.50	1	0.50	0	0	0
Wang (1997)	0	0.055	0.129	0.269	1	0.269	0.129	0.055	0
Zwick et al. (1987)	0.08	0.081	0.091	0.140	1	0.140	0.091	0.081	0.08
Chen's MF (1988)	0	0.031	0.250	0.719	1	0.719	0.250	0.031	0
Di Nola et al. (1994)	0.497	0.463	0.498	0.649	1	0.649	0.498	0.463	0.497
New Area-based method	0	0.004	0.0625	0.32	1	0.32	0.0625	0.004	0

A continuous analysis of the sensitivity is depicted in Figure 2.6.

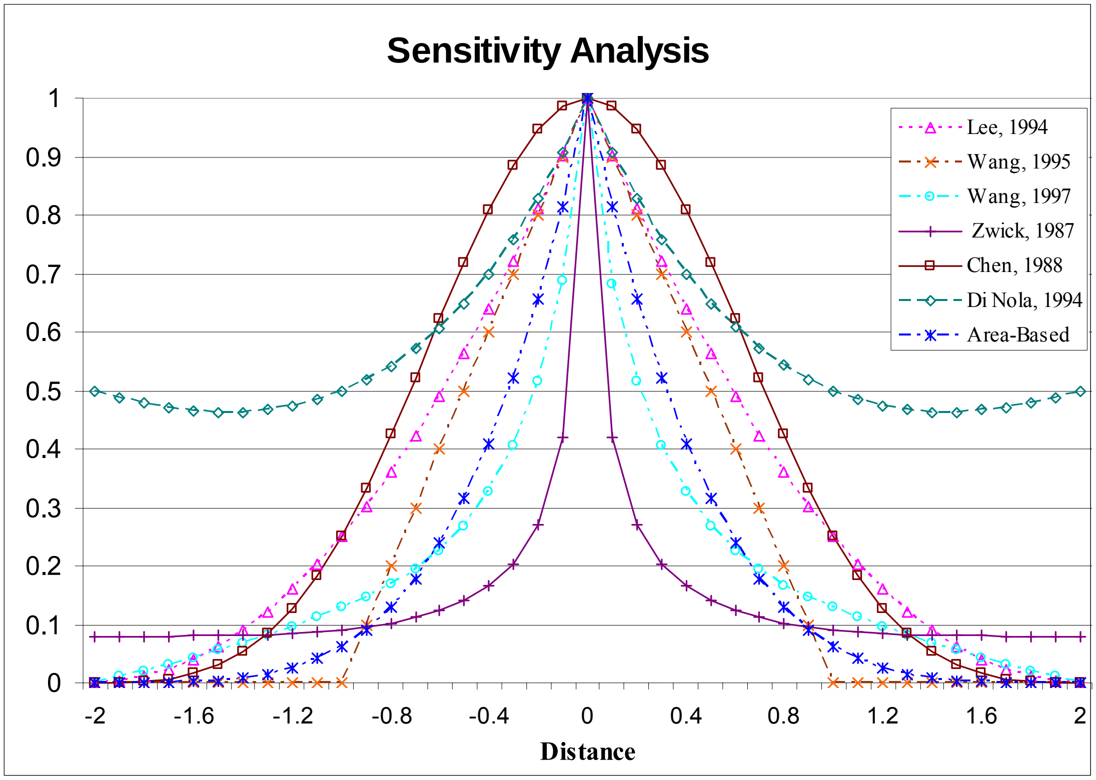


Figure 2.6 Sensitivity analyses of seven fuzzy similarity measure methods

2.3.7.4 Analysis of sensitivity Results

The example demonstrates the case in which the two fuzzy sets have identical shapes. Therefore, the similarity curves should be symmetric (if the distance measure is such). When the two fuzzy numbers do not have identical shape the curve is not symmetric. In this case, the maximum similarity value could be smaller than 1.0.

Also, the steeper the curve is, the more sensitive the similarity measure is. That means the measure has more acute discrimination ability.

Figure 2.6 shows that some approaches, (for example Zwick et al.'s (1987)) never converge to 0, even for far apart sets. The figure also reveals an undesired property of convex sensitivity function (e.g. Di Nola et al.'s, 1994). In such a case, there is a range in which the similarity increases as the distance increases.

Usually it is hard to develop a close form representation of the sensitivity function (similarity vs. distance). However, the new area-based similarity measure has a closed form representation of this relationship. The value of the sensitivity function (similarity as a function of distance) for the example is as follows:

$$S = \begin{cases} \frac{1}{16}(2-D)^4 & D \geq 0 \\ \frac{1}{16}(2+D)^4 & D \leq 0 \end{cases} \tag{2. 64}$$

2.4 Linguistic variables

2.4.1 Introduction

There are decision situations in which the information cannot be assessed precisely in a quantitative form but may be in a qualitative one, and thus, the use of a linguistic approach is necessary. For example, when attempting to qualify phenomena related to human perception, we are often led to use words in natural language instead of numerical values, e.g., when evaluating the "comfort" or "design" of a car, terms like "good", "medium", "bad" can be used (Herrera and Herrera-Viedma, 2000).

Precise quantitative information may not be stated because either it is unavailable or the cost of its computation is too high, so an "approximate value" may be tolerated (e.g., when evaluating the speed of a car, linguistic terms like “fast”, “very fast”, “slow” may be used instead of numerical values).

2.4.2 Linguistic variable

Since the concept was introduced by Zadeh in 1975, linguistic variables have been widely used. Briefly speaking, linguistic variables mean that variables whose values are not numbers but words or sentences in a natural or artificial language; and these values of linguistic variables are called linguistic labels.

Zadeh (1975) define the linguistic variable as:

A linguistic variable is characterized by a quintuple (H, T (H), U, G, M) in which H is the name of the variable; T (H) denotes the term set of H, i.e., the set of names of linguistic values of H, which each value being a fuzzy variable denoted generically by X and ranging across a universe of discourse U which is associated with the base variable u; G is a syntactic rule (which usually takes the form of a grammar) for generating the names of values of H; and M is a semantic rule for associating its meaning with each H, M(X), which is a fuzzy subset of U.

For example, the linguistic variable "temperature" might have the values "*hot*," "*cold*," "*freezing*," and so on. A membership function describes these linguistic values in terms of numerals. For example, to describe the word "*cold*", we can say, "*Any temperature above 60 degrees is not cold at all; as the temperature lowers, the coldness gradually increases; below 40 degrees, we can say it's definitely cold."* To describe the above as a function, we make the x-axis as temperature in degrees, and the y-axis as the "membership" of a given temperature in the definition of "cold." At temperatures above 60, the membership (or "degree of truth") is 0, meaning those temperatures above 60 don't belong to the definition of "cold" at all; at temperatures below 40, membership is 1, meaning that they definitely belong to the definition of cold; we will probably have a line between 60 and 40 degrees, to show the gradual increase in membership of the temperature in the definition of "coldness." The linguistic variables and their membership functions are what allow fuzzy logic to perform the imprecise, non-numerical reasoning performed by humans.

2.4.3 Linguistic label set

Linguistic label set *S* is a finite but totally ordered term set of linguistic labels S= $\{s_0, s_1 \ldots s_T\}$, with $s_i > s_j$, for i>j (Delgado et al., 1998, Herrera and Martinez, 2000). A linguistic label set has the following properties:

1) Ordered, $s_i > s_j$, if i>j
2) Negation operator, $Neg(s_i) = s_j$, such that j=T-i
3) T is an even number, the cardinality of the linguistic label set is odd (T+1),
4) Maximization Operator, $MAX(s_i, s_j) = s_i$, if $s_i > s_j$
5) Minimization operator, $MIN(s_i, s_j) = s_j$, if $s_i > s_j$
6) Symmetric, s_i and s_{T-i+2} have symmetrical meaning with regard to the middle label $s_{T/2+1}$.
7) Order reversal, for any $S_i > S_j$, $Neg(S_i) \le Neg(S_j)$

8) Involution, for any $\text{Neg}(\text{Neg}(S_i)) = S_i$, *for all i*

For example, Herrera et al. (1996a) defines the linguistic label set $S = \{s_0 = I, s_1 = SW, s_2 = WO, s_3 = SI, s_4 = EQ, s_5 = SB, s_6 = SU, s_7 = SS, s_8 = CS\}$. The associated trapezoidal membership functions are showed in Figure 2.7 with the numeric membership functions listed in Table 2.7.

s_0=I	Incomparable
s_1=SW	Significantly Worse
s_2=WO	Worse
s_3=SI	Somewhat Inferior
s_4=EQ	Equivalent
s_5=SB	Somewhat Better
s_6=SU	Superior
s_7=SS	Significantly Superior
s_8=CS	Certainly Superior

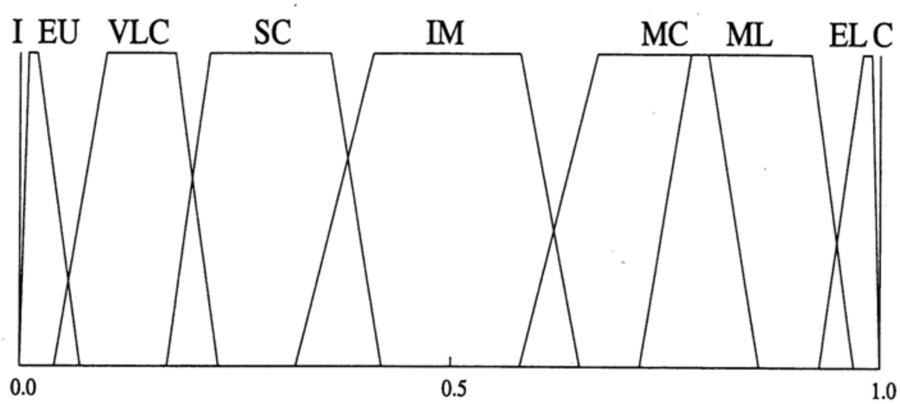


Figure 2.7 Membership functions of a linguistic label set

The mapping of linguistic label into the corresponding TFN

Table 2.7 Membership functions of a linguistic label set

	Linguistic Label	TFN
s_0	I	(0, 0, 0, 0)
s_1	SW	(0.01, 0.02, 0.01, 0.05)
s_2	WO	(0.1, 0.18, 0.06, 0.05)
s_3	SI	(0.22, 0.36, 0.05, 0.06)
s_4	EQ	(0.41, 0.58, 0.09, 0.07)
s_5	SB	(0.63, 0.80, 0.05, 0.06)
s_6	SU	(0.78, 0.92, 0.06, 0.05)
s_7	SS	(0.98, 0.99, 0.05, 0.01)
s_8	CS	(1, 1, 0, 0)

Here, a 4-tuple (x_0, x_1, x_2, x_3) trapezoidal membership functions are used, where x_1 and x_2 indicate the interval in which the membership function value is 1, and x_0 and x_3 are the left and right limits of the definition domain of trapezoidal membership function.

A linear trapezoidal membership functions are good enough to capture the vagueness of those linguistic assessments, since it may be impossible or unnecessary to obtain more accurate assessments.

2.5 Linguistic Quantifiers

2.5.1 Introduction

According to Zadeh (1983b), linguistic quantifiers, Q(r), can be viewed as linguistic probability, which determines the degree that the concept Q has been satisfied by r. In exploring this concept, Zadeh also proposed the concepts of absolute and relative or proportional quantifiers. The absolute quantifier represents the linguistic terms which related to an absolute count such as “At least 5” and “More than 10”. The relative or proportional quantifier represents the term containing the proportion r where r belongs to the unit interval. Examples of relative quantifiers are “at least 0.5” and “more than 0.3”, as well as “many” and “few”. Yager (1991) categorized the relative quantifiers into 3 categories;

2.5.2 Categories of Linguistic Quantifiers

Yager (1991) categorized the relative quantifiers into 3 categories;

- Regular monotonically non-decreasing

As mentioned, the quantifier $Q(r)$ can be perceived as the degree that the concept Q has been satisfied by r. In this type of quantifiers, as more criteria are satisfied, the higher the value of the quantifier. Examples for this type of quantifier are “Most”, “All”, “More than α”, “There exists”, and “At least α”. This type of quantifier has the following properties

  - $Q(0) = 0$;
  - $Q(1) = 1$;
  - If $r_1 > r_2$ then $Q(r_1) \geq Q(r_2)$.

- Regular monotonically non-increasing

These quantifiers are used to express linguistic terms such as “Few”, “Less than α” “Not all” and “None” in which the quantifier prefers fewer criteria to be satisfied. Such a quantifier has these properties:

  - $Q(0) = 1$;

- $Q(1) = 0$;
- If $r_1 < r_2$ then $Q(r_1) \geq Q(r_2)$

- Regular unimodal

These quantifiers are used to express linguistic terms such as “About α” or “Close to α” which implies that the maximum satisfaction is achieved when exactly α is satisfied.

2.5.2.1 Regular monotonically non-decreasing

As mentioned, the quantifier Q(r) can be perceived as the degree that the concept Q has been satisfied by r. In this type of quantifiers, as more criteria are satisfied, the higher the value of the quantifier. Examples for this type of quantifier are “Most”, “All”, “More than α”, “There exists”, and “At least α”. This type of quantifier has the following properties

1) $Q(0) = 0$;
2) $Q(1) = 1$;
3) If $r_1 > r_2$ then $Q(r_1) \geq Q(r_2)$.

2.5.2.2 Regular monotonically non-increasing

These quantifiers are used to express linguistic terms such as “Few”, “Less than α” “Not all” and “None” in which the quantifier prefers fewer criteria to be satisfied. Such a quantifier has these properties:

1) $Q(0) = 1$;

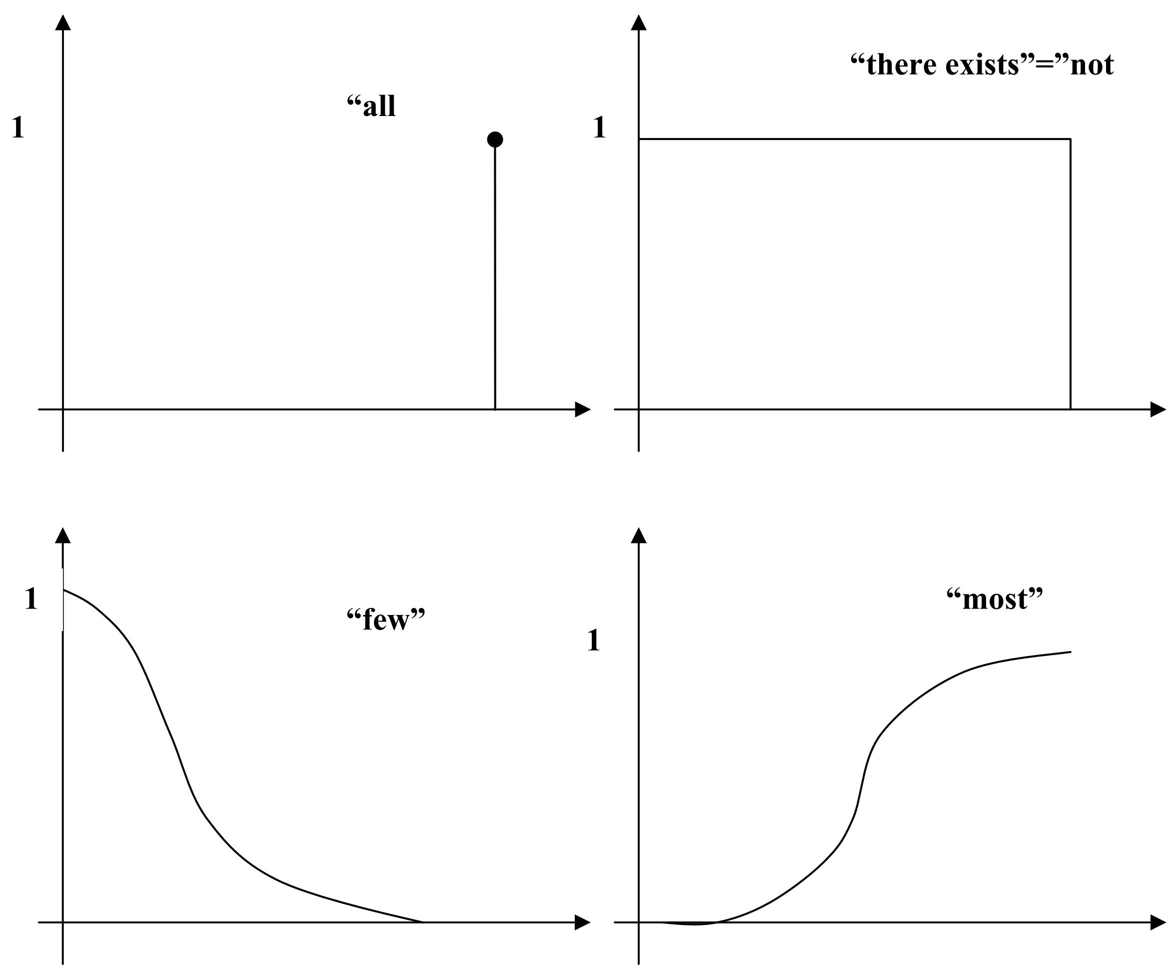
2) $Q(1) = 0$;

3) If $r_1 < r_2$ then $Q(r_1) \geq Q(r_2)$

2.5.2.3 Regular unimodal

These quantifiers are used to express linguistic terms such as “About α” or “Close to α” which implies that the maximum satisfaction is achieved when exactly α is satisfied.

Figure 2.8 below gives some ecamples of the common used linguistic quantifiers.



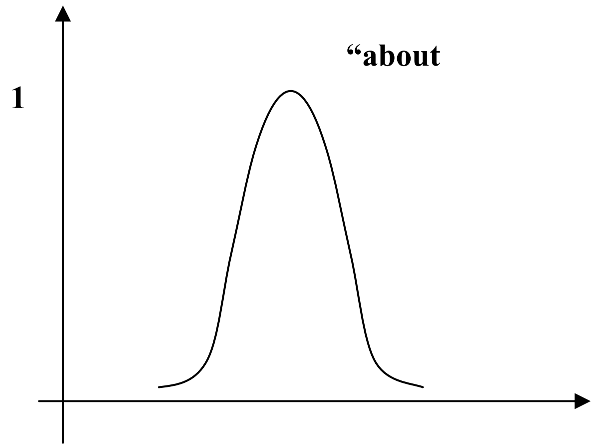


Figure 2.8 Some examples of lingustic quantifiers

Linguistically quantified statements belong to the category of soft statements defining the degree of agreement among the experts. These statements, essential in everyday life, can be represented in general as

Qy's are F

Where Q is a linguistic quantifier (such as "most"), y belongs to a set of objects (such as experts), and F is a verb property (such as convinced) (Kacprzyk et al., 1992b). In addition to this definition, it is possible to add more information regarding the importance of the experts in the quantified statement. The importance B can be added, resulting in the statement:

QBy's are F.

Such a statement can represent the understanding that "Most of the important experts are convinced". This statement can support the group decision "that alternative A is superior" or a similar decision outcome.

The application and some operators of linguisitc quantifiers will be further discussed in chapter 5.

2.5.3 Examples of Some Basic Quantifiers

In this subsection, we will introduce three of them as samples.

- "All"

This quantifier is also defined as the logical "AND" quantifier and can be represented as (Kacprzyk and Yager, 1984; Yager, 1988, 1993a, 1996):

$$Q_*\left(\frac{i}{n}\right)=\begin{cases}0 & \text{for } \frac{i}{n}<1 \\ 1 & \text{for } \frac{i}{n}=1\end{cases} \tag{2.65}$$

This representation shows that the satisfaction is a step function achieved only when all the criteria are included (as expected).

- "There exists"

This quantifier is equivalent to the term "At least one" and can be represented as:

$$w_i=\begin{cases}1 & \text{for } i=1 \\ 0 & \text{for } i\neq 1\end{cases} \tag{2.66}$$

The level of satisfaction Q(r) is presented in Figure 2.8. Thus this quantifier exhibits complete satisfaction when one criterion is included.

- “More than α”

The term “More than α” can be represented using the weights shown in Figure 2.9 (for example, Wang and Lin, 2003).

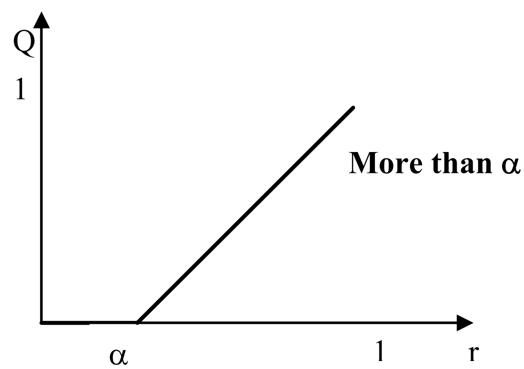


Figure 2.9 Satisfaction level of the “More than α” linguistic quantifier

All these weights are used in the OWA process to generate the overall score of each alternative. This is done using the ordered scores the criteria of each alternative.

2.5.4 Calculate weights using linguistic Quantifiers

Since the OWA aggregation method requires a set of weights w_i, these weight have a profound effect on the solution (the ranking of the alternatives in order of preference). One approach for generating the weights has been proposed in (Yager, 1993a, 1996) for the regular monotonically non-decreasing quantifiers. Using this approach the weights are calculated using

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1,\ldots,n \tag{2.67}$$

Calculating the weights for the regular monotonically non-increasing quantifiers is base on the fact that these quantifiers are the antonyms to the regular monotonically non-decreasing quantifiers.

The generated weights have the following properties

- $\sum w_i = 1;$
- $w_i = [0, 1]$

There are some useful measures to evaluate the weighting vectors, for instance Liu (1992) uses entropy defined as:

$$Entr(W) = -\sum_i w_i \ln w_i \tag{2.68}$$

In this subsection, we assume the weighting vector W and the scores X are all crisp numbers, then the weighting vector W should have the following properties:

$$w_i \in [0,1] \tag{2.69}$$

$$\sum_i w_i = 1 \tag{2.70}$$

2.6 Fuzzy Group Decision Making

Fuzzy set theory was first applied in decision making by Bellman and Zadeh in 1970. They noted that "Much of the decision making in the real world takes place in an environment in which the goals, the constraints and the consequences of possible actions are not known precisely" and fuzzy set theory can be used to deal with imprecision in decision making. One of the current topics is to apply fuzzy set theory into group decision making methods for dealing with the imprecision, uncertainty and fuzziness in human decision making.

Classic paper on fuzzy decision making is from Baas and Kwakernaak (1977). In this paper, they proposed a rating and ranking algorithms for ranking multiple aspect alternatives using fuzzy sets. In 1983, Zadeh published "The Role of Fuzzy Logic in the Management of Uncertainty in Expert Systems" where he states "...the conventional approaches to the management of uncertainty in expert systems are intrinsically inadequate because they fail to come to grips with the fact that much of the uncertainty in such systems is possibilistic rather than probabilistic in nature. As an alternative, it is suggested that a fuzzy-logic-based computational framework be employed to deal with both possibilistic and probabilistic uncertainty within a single conceptual System".

During this period, we begin to see substantial works on "decision making and expert systems", e.g., Zimmermann (1987), Negoita (1981, 1983), good summaries in fuzzy group decision making such as Kickert (1978), Zimmermann (1987), Kacprzyk et al. (1993), Bezdek et al. (1978), Blin (1973), Montero and Tejada (1986), Nurmi (1981), Tanino (1984, 1988),

Most of these researches were focused on how to apply fuzzy sets in group decision making and how to handle these fuzzy information. Then, researches continue on information aggregation and decision procedure development in fuzzy group decision making. For instance, Delgado et al. (1994) proposed a model for linguistic partial information in decision-making problems, the general procedure includes three steps: representation, aggregation, and comparison.

Herrera and Herrera-Viedma (2000) use only two phases: the aggregation phase and the exploitation phase. The aggregation phase acquires of the performance values with respect to all the criteria for obtaining a collective performance value for the alternatives. Exploitation phase of the collective performance value for obtaining a rank ordering, sorting or choice among the alternatives. Aggregation phase of linguistic information consists of obtaining a collective linguistic performance value on the alternatives by aggregating the linguistic performance values provided according to all the criteria by means of the chosen aggregation operator - of linguistic information. The exploitation phase consists of establishing a rank ordering among the alternatives according to the collective linguistic performance value for choosing the best alternatives.

Wang and Chuu (2004) proposes two algorithms, direct and indirect, for determining the degree of manufacturing flexibility in a fuzzy environment using a fuzzy linguistic approach based on LOWA aggregation method. All experts give the rating of performance and grade of importance for each criterion; both are linguistic labels from linguistic label set S. Direct approach: The final solution is based on the individual preference relations. Indirect approach: The final solution is based on an aggregated preference relation of the group.

Geldermann et al. (2000) research the Fuzzy outranking in group decision making with the application to iron and steel making industry.

Bozdag et al. (2003) compares four group decision making methods using an example in selection among computer integrated manufacturing systems: Blin's fuzzy relations, Fuzzy synthetic, Yager's weighted goals, and Fuzzy analytic hierarchy process.

Prodanovic and Simonovic (2003) combine fuzzy compromise programming with group decision making under fuzziness. Each decision maker is to specify his/her fuzzy weights, wz, deviation parameter, p, as well as positive and negative ideals concerning the criteria of the problem. Also, experts overall degree of risk is to be specified here as well (parameter χ_1). It should be noted that these parameters are entirely subjective and are based on the preferences of the expert. For each expert, a set of fuzzy alternatives is generated via fuzzy compromise programming equation. This means that the fuzzy compromise programming equation takes in t (fuzzy) criteria (for each alternative, for each expert), and produces one (fuzzy) distance metric—one distance metric for every alternative of the problem, for each expert. (It should be mentioned that alternatives are the same for each expert.) After this, for each individual, a fuzzy preference relation matrix is generated. Finally, after everyone's fuzzy preference relation matrix is obtained, Q-core, /Q-core and s/Q-core algorithms are performed, and a group decision is made.

Shamsuzzaman et al. (2003) presents a computational framework that combines both fuzzy sets and analytical hierarchy process (AHP) for selecting the best-ranked flexible manufacturing system from a number of feasible alternatives. Here, a value of a criterion is measured not by the value itself, but by the linguistic measures.

Ishikawa (1993) develops a process of the fuzzy Delphi method to reduce the number of iterations. The author set up a particular item at the outset which is called "the extent of expertise" which is associated with a membership function.

Cheng (1999) utilizes fuzzy Delphi method to adjust the fuzzy rating of every expert to achieve the consensus condition. The experts' opinions are described by linguistic terms, which can be expressed in trapezoidal fuzzy numbers.

Iggland (1991) also applies fuzzy Delphi method in coupling of customer preferences and production cost information.

More work on how to aggregate fuzzy or linguistic information in group decision making includes Calvo and Mesiar (2003)'s weighted triangular norms-based aggregation operators, ordered weighted average (OWA) from Yager (1993a), Linguistic OWA Combinations (LOWA) from Delgado et al. (1993), Neat OWA Operator (Marimin et al., 2002), Yager's IOWA (1998a, 2003), quasi-arithmetic means and quasi-linear means aggregation (Bullen et al., 1988; Marichal et al., 1999), Yager' s weighted median (1994), Sugeno integral, (Sugeno, 1974) and the Leximin ordering from Dubois et al. (1996) etc. All these aggregation operators will be introduced in Chapter 5.

In next subsections, we will summarize several fuzzy group decision making procedures including Fuzzy Delphi, Fuzzy Analytical Hierarchy Process (Fuzzy AHP), Fuzzy compromise programming etc.

2.6.1 Introduction

Decision-making is an important subject in business, manufacturing, and service. Group decision-making (i.e. multi-expert) is a typical one where the inherent complexity and uncertainty necessitates the participation of many experts in the decision-making process. The essence of the group decision making can be summarized as follows: there is a set of options and a set of individuals (termed experts) who provide their preferences over the set of options. The problem is to find an option (or a set of options) that is best acceptable to the group of experts. Such a solution entertains the concept of majority.

In the real world, the uncertainty, constraints and even unclear knowledge of the experts imply that decision makers cannot give exact numbers to express their opinions. The use of linguistic labels makes expert judgment more reliable and consistent. The motivation behind using fuzzy sets in group decision-making comes from several sources:

- The available information about the true state of nature lacks evidence and thus the representation of such piece of knowledge by a probability function is not possible.
- The user preferences are assessed by means of linguistic terms instead of numerical values. These terms in many cases are subjective and inconsistent.
- Decision maker's objectives, criteria, or preferences are vaguely established and cannot be induced a crisp relation.

2.6.1.1 Sources of Uncertainty

Fuzzy decision making techniques have addressed some uncertainties, such as the vagueness and conflict of preferences common in group decision making (Blin, 1974; Siskos, 1982; Seo and Sakawa, 1985; Felix, 1994; and others), and at least one effort has been made to combine decision problems with both stochastic and fuzzy components (Munda, 1995). Application, however, demands some level of intuitiveness for the decision makers, and encourages interaction or experimentation such as that found in Nishizaki and Seo (1994). Authors such as Leung (1982) and many others have explored fuzzy decision making environments. This is not always so intuitive to many people involved in practical decisions because the decision space may be some abstract measure of fuzziness, instead of a tangible measure of alternative performance. The alternatives to be evaluated are rarely fuzzy. Their performance is fuzzy. In other words, a fuzzy decision making environment may not be as generically-relevant as a fuzzy evaluation of a decision making problem.

Uncertainty is a source of complexity in decision making which can be found in many forms. Typical types of uncertainty include uncertainty in model assumptions, and uncertainty in data or parameter values. There may also be uncertainty in the interpretation of results. While some uncertainties can be modeled as stochastic variables in a simulation, other forms of uncertainty may simply be vague or imprecise.

Traditional techniques for evaluating discrete alternatives such as ELECTRE (Benayoun et. at., 1966), AHP (Saaty, 1980), Compromise Programming (Zeleny, 1973, 1982), and others do not normally consider uncertainties involved in procuring criteria values.

Sensitivity analysis can be used to express decision maker uncertainty (such as uncertain preferences and ignorance), but this form of sensitivity analysis can be inadequate at expressing decision complexity. There have been efforts to extend traditional techniques, such as PROTRADE (Goicocchea et al., 1982), which could be described as a stochastic compromise programming technique. A remaining problem is that not all uncertainties easily fit the probabilistic classification.

2.6.1.2 Fuzzy method to handle Uncertainty

A fuzzy membership function acknowledges that we may not be completely sure what values we are talking about. Statistical precision can be independent of our classification of an event. For example, we may predict 90% probability of the occurrence of a good value. What qualifies as a good value? Qualification of good can be subjective. Also, in many practical applications, there is not enough data to make probabilistic predictions with confidence. The dependence of stochastic applications on distribution functions can be restricting and misleading because of the intensity of data requirements. A fuzzy membership function may be used here in place of numeric data. In general, fuzzy sets provide an intuitive, and flexible framework for interactively exploring a problem that is either ill-defined or has limited available data.

There are typically 3 main forms of imprecision identified in fuzzy decision making (Ribeiro et. al., 1995):

- Incompleteness, such as insufficient data
- Fuzziness, where precise concepts are difficult to define
- Illusion of validity, such as detection of erroneous outputs (Tversky and Kahneman, 1992)

The use of fuzzy representation of systems must be considered for planning decisions involving uncertainties.

- The available information about the true state of nature is some kind of evidence and thus the representation of such piece of knowledge by a probability id not possible.
- The gains are assessed by means of linguistic terms instead of numerical values.
- Decision maker's objectives, criteria, or preferences are vaguely established and can not be induced a crisp relation.

2.6.2 Fuzzy Delphi

The Delphi approach uses expert opinion surveys with three special features: anonymous response, iteration and controlled feedback, and statistical group response. The number of iterations of Delphi questionnaires may vary from three to five, depending on the degree of agreement and the amount of additional information being sought or obtained. Generally, the first questionnaire asks individuals to response to a broad question. Each subsequent questionnaire is built upon responses to the preceding questionnaire. The process stops when consensus has been approached among participants, or when sufficient information exchange has been obtained. Thus one of the most attractive properties of this approach is the ability to gather and evaluate information from a group of experts without requiring a face-to-face meeting.

The Delphi approach typically involves three different groups: decision makers, staff, and experts (Hwang and Lin, 1987). Decision makers are responsible for the outcome of the Delphi study. A work group of five to nine members, composed of both staff and decision makers, develops and analyzes all questionnaires, evaluates collected data, and revises the questionnaires if necessary. The staff group is directed by a coordinator who should have experience in designing and conducting the Delphi method and is familiar with the problem area. The staff coordinator's duties also involve supervising a support staff in typing, mailing questionnaires, receiving and processing of results, and scheduling meetings. Experts who are also called respondents are recognized as experts on the problem and agree to answer the questionnaires.

The Delphi method is suitable for decision domains with the following properties:

- Subjective expertise and judgmental inputs.
- Complex, large, multidisciplinary problems with considerable uncertainties.
- Possibility of unexpected breakthroughs.
- Causal models cannot be built or validated.
- Particularly long time frames.
- Opinions required from a large group. Anonymity is deemed beneficial.

One of the weaknesses of the Delphi method is that it requires repetitive surveys of the experts to allow the evaluations to converge. The cost of this method rapidly increases with repetitive surveys especially in large and complicated problems (Ishikawa, 1993).

Fuzzy Delphi method is applied to alleviate this problem. Using fuzzy numbers or linguistic labels for evaluating the experts' opinions allows a faster convergence to an agreeable group decision. An example presented in Cheng (1999) uses linguistic terms to express the experts' response.

2.6.3 Fuzzy Analytical Hierarchy Process (Fuzzy AHP)

In some instances decision problems are hard to conceptualize or even clearly define. The Analytical Hierarchy Process (AHP) was formulated to support the decision-maker in these situations.

The AHP is based on following two steps: structuring the decision as a hierarchical model; and, then, using pair-wise comparison of all criteria and alternative, finding the calculated weight of the criteria and the score of each alternative. This approach allows decision makers to examine complex problem in a detailed rational manner. The hierarchical representation helps in dealing with large systems, which are usually complex in nature. The decisions are made one level at a time, from the bottom up, to more aggregate strategic levels.

The advantages of AHP include highly structured and more easily understood models, and consistent decision-making (or at least a measure of the level of consistency – the decision maker is always free to remain inconsistent in preferences and scores).

The disadvantages of AHP focus mainly on the decision maker who has to make many pair-wise comparisons to reach a decision while possibly using subjective preferences.

Fuzzy AHP methods are systematic approaches to the alternative selection and justification problem by using the concepts of fuzzy set theory and hierarchical structure analysis. The Fuzzy AHP approach uses the concepts of fuzzy set theory for evaluation of alternatives, and defining the weights of criteria. Shamsuzzaman et al. (2003) integrated fuzzy sets and the Analytical Hierarchy Process for selecting the best-ranked flexible manufacturing system from a number of feasible alternatives. Fuzzy sets are employed to recognize the selection criteria as linguistic variables rather than numerical ones. The AHP is used to determine the weights of the selection criteria, in accordance with their relative importance.

The earliest work in fuzzy AHP appeared in van Laarhoven and Pedrycz (1983), which compared fuzzy ratios described by triangular membership functions. Buckley (1985) determines fuzzy priorities of comparison ratios whose membership functions trapezoidal. Stam et al. (1996) explore how recently developed artificial intelligence techniques can be used to determine or approximate the preference ratings in AHP. Chang (1996) introduces a new approach for handling fuzzy AHP, with the use of the extent analysis method for the synthetic extent values of the pairwise comparisons.

Cheng (1997) proposes a new fuzzy analytical hierarchy process based on grade value of membership function. Weck et al. (1997) present a method by adding the mathematics of fuzzy logic to the classical AHP. Deng (1999) presents a fuzzy approach for tackling qualitative multi-criteria analysis problems in a simple and straightforward manner. Lee et al. (1999) introduce the concept of comparison interval and propose a methodology based on stochastic optimization to achieve global consistency. Cheng et al. (1999) propose a new method by analytical hierarchy process based on linguistic variable weight. Chan et al. (2000) present a technology selection algorithm to quantify both tangible and intangible benefits in fuzzy environment.

Hierarchies are very important tools for dealing with large systems which are usually complex in nature. They involve identifying the elements of a problem, grouping the elements into homogeneous sets, and arranging these sets in different levels. Each set of elements occupies a level of the hierarchy.

- The top level, i.e., the focus consists of only one element: the broad, overall objective.
- The second level is a set of criteria. This level may be divided into sub-criteria levels depending on the problem size.
- The last level is a set of alternatives, which are considered in the analysis.

Fuzzy AHP methods are systematic approaches to the alternative selection and justification problem by using the concepts of fuzzy set theory and hierarchical structure analysis.

$X = \{x_1, x_2, ..., x_n\}$ be an object set, and

$U = \{u_1, u_2, ..., u_m\}$ be a goal set.

- Obtain m extent analysis values for each object

$$M^1_{gi}, M^2_{gi}, ..., M^m_{gi} \quad \text{i} = 1, 2, ..., \text{n} \tag{2.71}$$

All are triangular fuzzy numbers.

- The value of fuzzy synthetic extent with respect to the ith object

$$S_i = \sum_{j=1}^{m} M^j_{gi} \otimes \left[\sum_{i=1}^{n} \sum_{j=1}^{m} M^j_{gi} \right]^{-1} \tag{2.72}$$

Chai and Wei (2001)'s Fuzzy Analytical Hierarchy Process or Fuzzy AHP is a multi criteria method which uses hierarchic structures to represent a decision problem and then develops priorities for the factors based on DM's judgment.

The arithmetic operations on these fuzzy numbers are based on interval, arithmetic.

By using a α-cut on the performance matrix, an interval performance matrix can be derived as Z, where $0 \le \alpha \le 1$. The value of α represents the DM's degree of confidence in his/her fuzzy assessments regarding alternative ratings and criteria weights. A larger α value indicates a more confident DM. Incorporated with he DM's attitude towards risk using an optimism index λ, an overall crap performance matrix is calculated by $z^{\lambda'}_{i\alpha} = \lambda z^{\alpha}_{ir} + (1-\lambda)\lambda z^{\alpha}_{il}$.

$$Z = \begin{bmatrix} w_1 x_{11} + w_2 x_{12} + ... + w_n x_{1m} \\ w_1 x_{21} + w_2 x_{22} + ... + w_n x_{2m} \\ ... \\ w_1 x_{n1} + w_2 x_{n2} + ... + w_n x_{nm} \end{bmatrix}, \ Z = \begin{bmatrix} [z^{\alpha}_{1l}, z^{\alpha}_{1r}] \\ [z^{\alpha}_{2l}, z^{\alpha}_{2r}] \\ ... \\ [z^{\alpha}_{nl}, z^{\alpha}_{nr}] \end{bmatrix}, \ Z^{\lambda'}_{\alpha} = \begin{bmatrix} z^{\lambda'}_{1\alpha} \\ z^{\lambda'}_{2\alpha} \\ ... \\ z^{\lambda'}_{n\alpha} \end{bmatrix} \tag{2.73}$$

$\lambda = 1, \lambda = 0.5, \lambda = 0$ are used to indicate that the DM involved has an optimistic, moderate, or pessimistic view respectively. An optimistic DM is apt to prefer higher values of his/her fuzzy assessments, while a pessimistic DM tends to favor lower values. After the facilitation of the vector matching process, a normalization process in regard to each criterion is applied to $z_{i\alpha}^{\lambda}$, resulting in a normalized performance matrix expressed as N_{α}^{λ}.

$$z_{i\alpha}^{\lambda} = \frac{z_{i\alpha}^{\lambda'}}{\sqrt{\sum_{i=1}^{n}(z_{i\alpha}^{\lambda'})^2}}, \quad N_{\alpha}^{\lambda} = \begin{bmatrix} z_{1\alpha}^{\lambda} \\ z_{2\alpha}^{\lambda} \\ \cdots \\ z_{n\alpha}^{\lambda} \end{bmatrix} \tag{2.74}$$

The values of N_{α}^{λ} indicate the degree of preference with respect to the alternatives for fixed α and λ respectively where $\alpha \in [0,1], \lambda \in [0,1]$. Indeed, this value is the Enhanced Compactness Index (ECI), which considers both of the compactness measurements or evaluation criteria earlier. Therefore, the larger the value, the more the preference of the alternative.

2.6.4 Fuzzy compromise programming

Classical compromise programming is a multi-criteria decision analysis technique used to identify the best compromise solution from a set of solutions by some measure of distance. As the Figure 2.10 indicates, solution should be the closeness of a particular solution to a generally infeasible (ideal) solution.

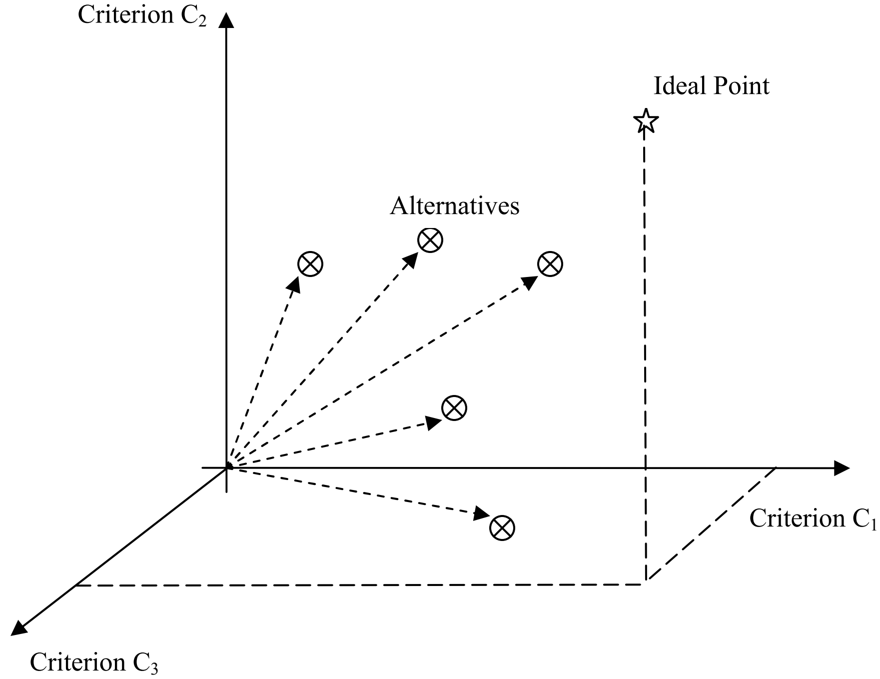


Figure 2.10 Compromise programming

$$L_j = \left[\sum_{z}^{t} \left\{ w_z^p \left(\frac{f_z^* - f_z}{f_z^* - f_z^-} \right)^p \right\} \right]^{\frac{1}{p}}$$

Where

z	1, 2, 3, …, t represents t criteria or objectives;
j	1, 2, 3, …, n represents n alternatives;
L_j	distance metric of alternative j;
w_z	corresponds to a weight of a particular criteria or objective;
p	parameter (p=1, 2, …, ∞);
f_z^* *and* f_z^-	best and the worst value for criteria z, respectively (also referred to as positive and negative ideals);
f_z	actual value of criterion .

The parameter p is used to represent the importance of the maximal deviation from the ideal point. If p=1, all deviations are weighted equally; if p=2, the deviations are weighted in proportion to their magnitude. As p increases, so does the weighting of the deviations. Varying the parameter p from 1 to infinity, allows one to move from minimizing the sum of individual regrets to minimizing the maximum regret in the decision making process. The choice of a particular value of this compensation parameter p depends on the type of problem and desired solution. In general, the greater the conflict between players is, the smaller the possible compensation becomes.

Fuzzy Compromise Programming merges of fuzzy with group decision making under fuzziness algorithms. Fuzzy Compromise Programming considers all input parameters as fuzzy sets, not just criteria values (as fuzzy composite programming does).

Fuzzy Compromise Programming uses of fuzzy sets in representation of these parameters insures that as much as possible of relevant information is used. The more certain the expert is (about a particular parameter value), the less fuzziness is assigned to the fuzzy number. But, the distance metrics are also fuzzy, it is a range of lengths.

2.6.5 Blin's fuzzy relations (1974)

A fuzzy relation $\widetilde{R}$ is a mapping from the Cartesian space $X \times Y$ to the interval [0, 1]. Blin's fuzzy relations (1974) use a composition operator

$$T = \widetilde{R} \cdot \widetilde{S} \tag{2.75}$$

with Max-min composition

$$\mu_{\widetilde{T}}(x,z) = \underset{y \in Y}{\vee} (\mu_{\widetilde{R}}(x,z) \wedge \mu_{\widetilde{S}}(x,z)) \tag{2.76}$$

or Max-product composition.

$$\mu_{\tilde{T}}(x,z) = \vee_{y \in Y}(\mu_{\tilde{R}}(x,z) \cdot \mu_{\tilde{S}}(x,z)) \tag{2.77}$$

Then a decision making procedure with Blin's fuzzy relations can be defined as:

- Each member of a group of n individual decision makers is assumed to have a reflexive, anti-symmetric, and transitive preference ordering $P_k, k \in N_n$
- Define the social preference S as a fuzzy binary relation with membership grade function:

$$\mu_S : X \times X \rightarrow [0,1] \tag{2.78}$$

  The membership grade $\mu_S(x_i, x_j)$ indicating the degree of group preference of alternative x_i over alternative x_j.

  Each value α essentially represents the level of agreement between the individuals concerning the particular crisp ordering S_α.

- Intersect the classes of crisp total orderings that are compatible with the pairs in the a-cuts S_α for increasingly smaller values of α until a single crisp total ordering is achieved. In this process. Any pairs (x_i, x_j) that lead to an intransitivity are removed.
- The largest value α for which the unique compatible ordering on $X \times X$ is found represents the maximized agreement level of the group (Consensus).

2.6.6 Fuzzy synthetic (Chen et al., 2002, Chang et al., 2001)

The term synthetic is used to connote the process of evaluation whereby several individual elements and components of an evaluation are synthesized into an aggregate form; the whole is a synthesis of the parts.

A decision making procedure using fuzzy synthetic is defined as:

- Each member of a group of n individual decision makers is assumed to have a reflexive, anti-symmetric, and transitive preference ordering $P_k, k \in N_n$
- Define the fuzzy social preference S as a fuzzy binary relation with membership grade function:

$$\mu_S : X \times X \rightarrow [0,1] \tag{2.79}$$

$\widetilde{R} = [r_{ij}], where\ i = 1, 2, ..., n$ (# of criteria) and $j = 1, 2, ..., m$ (# of evaluations, linguistic labels)

- The matrix of relative weights of subjective estimates

$$\widetilde{w} = \{w_1, w_2, ..., w_n\}\ \ where\ \sum_n w_i = 1 \tag{2.80}$$

Here, we can use Saaty's Analytic Hierarchy Process to calculate fuzzy vector of scoring factors. First, construct a questionnaire form, which might facilitate the answering of pairwise comparison questions. The, compute a vector of priorities or weighting of elements in the matrix, this consists of calculating the principal vector (eigenvector) of the matrix, and then normalizing it to sum to 1.0.

- Calculated the composition $\widetilde{e}$ for each alternative

$$\widetilde{e} = \widetilde{w} \cdot \widetilde{R} \tag{2.81}$$

- The alternative whose highest membership in higher category wins.

2.6.7 Yager's weighted goals (Yager, 1978)

In this method, the decision is defined as the intersection of all fuzzy goals. The optimal alternative is defined as that achieving the highest degree of membership in weighted goals (Yager, 1978).

The weights are used as exponents to express the importance of a goal. The higher the importance of a goal the larger should be the exponent of its representing fuzzy set, at least for normalized fuzzy sets and when using the min-operator for the intersection of the fuzzy goals. The solution procedure can be described as follows:

- Establish by pairwise comparison the relative importance, α_i, of the goals among themselves. Arrange the α_i in a matrix M.

$$M = \begin{bmatrix} \frac{\alpha_1}{\alpha_1} & \frac{\alpha_1}{\alpha_2} & \cdots & \frac{\alpha_1}{\alpha_n} \\ \frac{\alpha_2}{\alpha_1} & \frac{\alpha_2}{\alpha_2} & \cdots & \frac{\alpha_2}{\alpha_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\alpha_n}{\alpha_1} & \frac{\alpha_n}{\alpha_2} & \cdots & \frac{\alpha_n}{\alpha_n} \end{bmatrix} \tag{2.82}$$

- Determine consistent weights w_j for each goal by employing Saaty's eigenvector method.

- Weight the degrees of goal attainment, $\mu_{\widetilde{G}_j}(x_i)$ exponentially by the respective w_j. The resulting fuzzy sets are $(\widetilde{G}_j(x_i))^{w_j}$.

- Determine the intersection of all $(\widetilde{G}_j(x_i))^{w_j}$:

$$\widetilde{D} = \{(x_i, \min_j(\mu_{\widetilde{G}_j}(x_i))^{w_j} \mid i = 1,2,...,n; j = 1,2,...,m\} \tag{2.83}$$

- Select the x_i with largest degree of membership in $\widetilde{D}$ as the optimal alternative.

2.6.8 Fuzzy Relations Approach

A fuzzy relation $\widetilde{R}$ is a mapping from the Cartesian space $X \times Y$ to the interval (0, 1]. Blin (1974) proposed to represent a relative group preference as a fuzzy preference matrix from individual preferences. The decision-making procedure is defined as following:

- Each member of a group of n individual decision makers is assumed to have a reflexive, anti-symmetric, and transitive preference ordering $P_k, k \in N_n$
- Define the social preference S as a fuzzy binary relation with membership grade function:

$$\mu_S : X \times X \rightarrow [0,1] \tag{2.84}$$

- The membership grade $\mu_S(x_i, x_j)$ indicating the degree of group preference of alternative x_i over alternative x_j.
- Each value α essentially represents the level of agreement between the individuals concerning the particular crisp ordering S_α.
- Intersect the classes of crisp total orderings that are compatible with the pairs in the a-cuts S_α for increasingly smaller values of α until a single crisp total ordering is achieved. In this process.
- Any pairs (x_i, x_j) that lead to an intransitivity are removed.
- The largest value α for which the unique compatible ordering on $X \times X$ is found represents the maximized agreement level of the group (Consensus level).

2.6.9 Distance Aggregation Method

Fan et al.'s (2002) method is to minimize the difference between the decision maker's fuzzy preference information and the fuzzy preference information calculated from the decision matrix. By solving the optimization model, find the optimal weights to the criteria. Then we can rank the alternatives and make the decision.

The approaches to solve MADM problem can be classified into three categories according to different forms of preference information given by a decision maker:

the approaches without preference information

the approaches with information on attributes

the approaches with information on alternatives

The decision maker's preference relation matrix P_{nxn}. p_{ik} denotes the preference degree of alternative A_i over A_k. (Table 2.8)

Table 2.8 Preference relation matrix from one decision maker

	A_1	**…**	**A_k**	**…**	**A_n**
A_1	p_{11}	…	p_{1k}	…	p_{1n}
…	…	…	…	…	…
A_i	p_{i1}	…	p_{ik}	…	p_{in}
…	…	…	…	…	…
A_n	p_{n1}		p_{nk}		p_{nn}

The following procedure is developed for the weights calculation:

- Calculate the degree of membership, get the new matrix $B=[b_{ij}]_{nxm}$

For benefit criterion:

$$b_{ijk} = \frac{x_{ij}^{k} - x_{jk}^{\min}}{x_{jk}^{\max} - x_{jk}^{\min}}, \ i = 1,...n \tag{2.85}$$

For cost criterion:

$$b_{ijk} = \frac{x_{jk}^{\max} - x_{ij}^{k}}{x_{jk}^{\max} - x_{jk}^{\min}}, \ i = 1,...n \tag{2.86}$$

Where,

$$x_{jk}^{\max} = \max(x_{1j}^{k}, x_{2j}^{k}, ..., x_{nj}^{k}), \ \ j = 1,...m \tag{2.87}$$

$$x_{jk}^{\min} = \min(x_{1j}^{k}, x_{2j}^{k}, ..., x_{nj}^{k}), \ \ j = 1,...m \tag{2.88}$$

- Transform the overall values of alternatives into fuzzy preference relations.

$$d_i = \sum_{j=1}^{m} b_{ij}^{k} w_j, \ i = 1,...n \tag{2.89}$$

$$\overline{p}_{it} = \frac{d_i}{d_i + d_t} = \frac{\sum_{j=1}^{m} b_{ijk} w_j}{\sum_{j=1}^{m} (b_{ijk} + b_{tjk}) w_j}, \ i \neq t \tag{2.90}$$

$$g_{it}(w) = p_{it} - \overline{p}_{it} = p_{it} - \frac{\sum_{j=1}^{m} b_{ij}^{k} w_j}{\sum_{j=1}^{m} (b_{ij}^{k} + b_{tj}^{k}) w_j}, \ i \neq t \tag{2.91}$$

- Construct the optimization model

$$Minimize: H(w)=\sum_{i=1}^{n}\sum_{\substack{t=1\\ t\neq i}}^{n}\left[p_{it}\sum_{j=1}^{m}(b_{ij}^{k}+b_{tj}^{k})w_j-\sum_{j=1}^{m}b_{ij}^{k}w_j\right]^2 \tag{2.92}$$

Subject to

$$\sum_{j=1}^{n} w_j = 1$$

$$0 \le w_j \le 1$$

- Solve the optimization model and get the weights to the criteria

$$w^* = Q^{-1}e / e^T Q^{-1}e \tag{2.93}$$

$$\lambda^* = -1 / e^T Q^{-1} e \tag{2.94}$$

Where Q is positive definite and invertible

- Substituted by the weights w* to

$$d_i = \sum_{j=1}^{m} b_{ij}^{k} w_j,\ i = 1,...n \tag{2.95}$$

We can obtain the overall values of every alternative.

2.6.10 Similarity Aggregation Method (SAM)

Basic idea of Hsu's SAM method (1996) is that weight of an expert's opinion should be larger if his opinion is closer to other opinions.

Assume X_j = (a_j, b_j, c_j, d_j) is a positive trapezoidal fuzzy number representing j^{th} expert's subjective estimate of the rating to an alternative under a given criterion. And X = *F* (X_1, $X_{2...}$ X_q) is the consensus of opinions. The main problem of aggregation is to determine appropriate weights for each opinion. Hsu defines the average similarity between expert E_i and other experts as:

$$A(E_j) = \frac{1}{n-1} \sum_{i=1 \ and \ j \neq i}^{q} S(X_i, X_j) \tag{2.96}$$

Where $S(X_i, X_j)$ is a similarity measure function defined by *Zwick* et al. (1987) as:

$$S(X_i, X_j) = \frac{\int_x (\min\{\mu_{X_i}(x), \mu_{X_j}(x)\})dx}{\int_x (\max\{\mu_{X_i}(x), \mu_{X_j}(x)\})dx} \tag{2.97}$$

Based on the similarity values from (2.97), we can calculate the aggregation weight for expert E_i

$$W(E_j) = \frac{A(E_j)}{\sum_{j=1}^{q} A(E_j)} \tag{2.98}$$

Then the combined opinion of group opinions is computed by:

$$X = \sum_{j=1}^{q} X_j w_j \tag{2.99}$$

Lee (2002) improves Hsu's similarity aggregation method (SAM). The new optimal consensus model can deal with the situation where the supports do not intersect and tell whether the aggregation weights of individual opinions derived from SAM are optimal or not.

Lee chooses Tong's distance metric (1980):

$$d_p(X_i, X_j) = \left(\sum_{k=1}^{4} (|a_i - a_j|)^p \right)^{1/p} \tag{2.100}$$

And the similarity between X_i and X_j is defined as:

$$S_p(X_i, X_j) = 1 - \frac{1}{4u^p}(d_p(X_i, X_j))^p \tag{2.101}$$

Where U is the universe of discourse and u=max(U)-min(U). Also, the dissimilarity between A and B is defined as:

$$\mathrm{c} - \mathrm{S}_2(X_i, X_j) \tag{2.102}$$

Where *c* is a constant and c>1

An optimal aggregated opinion is to minimize the sum of weighted dissimilarity between aggregated opinion and each individual opinion. So, the optimization model is constructed like:

$$\underset{\mathrm{M}\times\mathrm{X}^4}{Minimize} : Z_{m,c}(W, X) = \sum_{j=1}^{q} (w_j)^m (c - S_2(X_j, X)) \tag{2.103}$$

Subject to:

$$\sum_{j=1}^{q} w_j = 1$$
$$0 \le w_j \le 1$$

Where m is an integer and m>1.

Notice that this analytical problem is quite similar to fuzzy c-means problem. It is easy to construct but not easy to solve the optimization model.

Lee proposes an algorithm OAM to solve the optimization model with the degree of importance of experts.

- Each expert E_j (j=1, 2… q) constructs a positive trapezoidal fuzzy number $X_j = (a_j, b_j, c_j, d_j)$ to represent the subjective estimate of the rating to the alternative under a given criterion.

- Set initial aggregation weights such that $0 \le w_j^{(0)} \le 1$ (j=1,2,…q) and $\sum_{j=1}^{q} w_j^{(0)} = 1$. Each iteration in the algorithm will be labeled l, where l=0, 1, 2, …

- Calculate

$$X^{(l+1)} = \frac{1}{\sum_{j=1}^{q} (w_j^{(l)})^p} \sum_{j=1}^{q} (w_j^{(l)})^m X_j \tag{2.104}$$

- Let $W^{(l)} = (w_1^{(l)}, w_2^{(l)}, \ldots, w_q^{(l)})$. Calculate $W^{(l+1)}$ as follows:

$$w_j^{(l+1)} = \frac{(1/(c - S_2(X_j, X^{(l+1)})))^{1/(m-1)}}{\sum_{i=1}^{q} (1/(c - S_2(X_i, X^{(l+1)})))^{1/(m-1)}} \tag{2.105}$$

- If $\left\| W^{(l+1)} - W^{(l)} \right\| > \varepsilon$, set l=l+1 and go to (2.104).

- Let $W^{(l+1)} = (w_1^{(l)}, w_2^{(l)}, \ldots, w_q^{(l)})$. Calculate aggregation coefficient AC_j (j=1, 2… q) by:

$$AC_j = \beta \frac{(w_j)^m}{\sum_{j=1}^{q} (w_j)^m} + (1 - \beta) e_i \tag{2.106}$$

- Aggregate opinions of experts by:

$$X = \sum_{j=1}^{q} (AC_j \cdot X_j) \tag{2.107}$$

2.6.11 Marimin et al.'s Semi-numeric Method

Marimin et al.'s semi-numeric Method (2002) is defined as:

Linguistic label representation with fuzzy sets computation for pairwise fuzzy group decision making.

In a single-level analysis of pairwise fuzzy group decision making, each decision maker expresses his or her evaluation on each pair of alternatives based on whole criteria or based on each criterion when the criteria are considered explicitly. In the explicit criteria consideration, solutions based on each criterion are then aggregated into the final solution. The criteria may have the same or different weights. The weight for each criterion is determined separately based on the decision makers' consensus or by adjusting a decision parameter for the aggregation operator used.

The difference of the single-level analysis from the numeric single-point computation models:

- The core concept is extended into a fuzzy core concept which allows fuzzy membership of alternatives in a solution set.
- The thresholds are eliminated from the computation models.
- Neat OWA aggregation operators are used both for aggregating the individual preferences into a group preference and for representing the aggregation guided by a linguistic quantifier such as most.

- A set of criteria which may have the same or different weights are considered explicitly. The weights may be assigned by the decision makers in label form or they will be automatically determined based on the preference data and the adjusted decision parameter α of the neat OWA operator. A higher α means that a value is more highly weighted.

2.6.12 Niskanen (2002)

Niskanen (2002) compares four different decision models with the ZZ data sets. ZZ data comprises 24 data vectors, these being the degrees of membership of 24 objects with respect to three fuzzy sets. The objects are tiles. The components of these vectors are the two inputs and the output. The first input variable is solidity, and it will be assessed on the basis of the tile's color. The second input variable is dovetailing, which means that the tiles should cling to each other as tightly as possible. The output variable is an ideal tile, and this feature is assessed on the basis of the input variables.

The four models he used are:

- The conventional linear approach (Linear Regression)
- Semi-conventional non-linear approach (non-linear regression analysis, neural network approach)
- Fuzzy associative memory approach (Fuzzy Control)
- Neuro-fuzzy associative memory approach (Fuzzy Associate Rule)

2.6.13 Comparing Fuzzy and Crisp group decision making

From a few sections' introduction, we can conclude that fuzzy and crisp group decision making have the following properties:

Crisp group decision making methods

- All decision data are assumed to be known and must be represented by crisp numbers
- The methods are to effectively aggregate performance scores.

Fuzzy group decision making methods

- Have difficulty in judging the preferred alternatives because all aggregated scores are fuzzy data
- Should do fuzzy outranking

Other approaches can be applied on group decision making under linguistic assessments include Fuzzy Analytical Hierarchy Process (Mikhailov, 2004; Shamsuzzaman et al., 2003), Fuzzy Delphi (Cheng, 1999; Iggland, 1991; Ishikawa, 1993), Fuzzy Compromising (Prodanovic and Simonovic, 2003), etc. Blin (1974) proposed to represent a relative group preference as a fuzzy preference matrix from individual preferences. Fan et al.'s (2002) weighting construction method tries to minimize the difference between the decision maker's fuzzy preference information and the fuzzy preference information calculated from the decision matrix. By solving the optimization model, the optimal weights to the criteria are calculated. Then the alternatives are ranked based on the aggregation results. Hsu and Chen (1996) proposed a similarity aggregation method. The basic idea of the method is that weight of an expert's opinion should be larger if his opinion is closer to other opinions.

2.7 Fuzzy Group Decision Making Process

In a fuzzy environment, the group decision-making problem can be solved in four steps (Herrera et al., 1996a). As shown in Figure 2.11, first, one should unify the evaluations from each expert. The second step is to aggregate the opinions of all group members to a final score for each alternative. This score is usually a fuzzy set or a linguistic label, which is used to order the alternatives. The third step is to rank the linguistic labels or fuzzy sets and select the preferred alternatives based on this order. Finally, the decision manager assesses the consensus level and the individual contribution to the group decision.

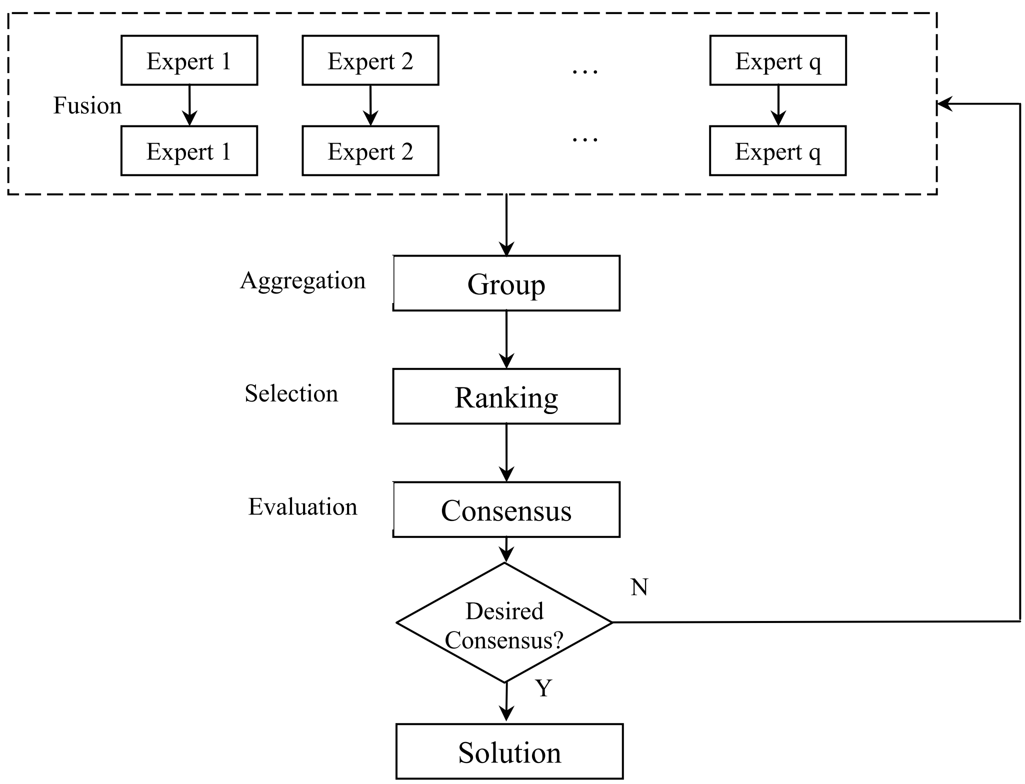


Figure 2.11 Fuzzy group decision making process

2.7.1 Expressing Fuzzy Preference of Alternatives

Several ways can be used to express group members' preference of alternatives. Herrera-Viedma et al. (Herrera-Viedma et al., 2002) present four ways for group members to express their opinions: preference ordering, utility values, fuzzy preference relations and multiplicative preference relations. These opinions can be converted into the various representations using different transformations (Chiclana et al., 1998, 2001). For instance in (Herrera-Viedma et al., 2002), the function of transforming the multiplicative preference relations a_{ij}^k into the fuzzy preference relations p_{ij}^k is given as $p_{ij}^k = \frac{1}{2}(1 + \log_9 a_{ij}^k)$. As mentioned before, group members can also produce linguistic opinions, especially when the problem could not be evaluated by exact numbers. In this way, expert E_k can choose s_j from the linguistic label set S as the score x_{ik} to alternative A_i. So, before we resume to the next step, we should uniform experts' evaluations.

2.7.2 Aggregating Individual Preferences into a Group Decision

In this stage, all experts' opinions are combined to get a final rating for each alternative. The selection of aggregation function plays an important role in the accuracy of the final solution. Many methods for aggregation can be found in Baas and Kwakernaak (1977), Chen and Hwang (1992), Cheng (1999), Delgado et al. (1998), Herrera and Herrera-Viedma (1997), Herrera et al. (1995, 1996a), Hwang and Lin (1987), Smolikova and Wachowiak (2002), Wang et al. (2000), and Yager (1993b). Regardless whether the weights and scores are linguistic or numeric, the general form of the aggregation function is:

$$f_i = \sum_{k=1}^{q} u_k \otimes x_{ik} \tag{2.108}$$

where f_i is the final score for alternative i. The weights of experts u_k could have quantitative or qualitative values. The qualitative weights can be rationalized using some algorithms introduced in Bordogna et al. (1997), Herrera and Herrera-Viedma (1997), Herrera et al. (1996b), and Yager (1993b, 1994, 1998b).

The use of linguistic variables makes decision makers' evaluations more flexible and reliable, but makes the aggregation of linguistic labels complicate, especially when applies the weighting associated with the evaluations.

Generally, there are two main approaches to aggregate linguistic labels in group decision making. Most methods use the associated membership functions. Among them, Baas and Kwakernaak's rating algorithm (1977) aggregates fuzzy scores and weights at differentα-cut levels with their associated membership functions. Chen and Hwang (1989) present a conversion scales approach to transform the linguistic expression into fuzzy numbers attribute by attribute. They give eight conversion scales and find a scale from the pool contains all linguistic terms with the principle that the scale should be as simple as possible. By assigning crisp scores to fuzzy numbers, they then apply classical MCDM method or TOPSIS method. After Yager introduced the OWA operator in (1988), a lot of aggregation researches have been done by applying OWA and linguistic quantifiers, such as Yager (1993a, 1994, 1995), and Kacprzyk et al. (1992b).

The other category is to calculate linguistic labels directly. Defined in Herrera and Herrera-Viedma (1997) and Herrera et al. (1996b), Linguistic OWA (LOWA) is based on the OWA (Yager, 1983) and the convex combination of linguistic labels (Delgado et al., 1998). The idea is that the combination resulting from two linguistic labels should be itself an element in the set S. So, given si, sj$\in$S and i, j $\in$[0, T], the LOWA method finds an index k in the set S representing a single resulting label. Another FLOWA method also based on OWA is from Ben-Arieh and Chen (2004). This model assigns membership functions to all linguistic labels in S by linearly spread the weights from the labels to be aggregated. The aggregating result is not a single label in S, but a fuzzy set with degrees to each label in S. A 2-tuple Fuzzy linguistic representation model based on the symbolic translation is introduced by Herrera and Martinez (2000). A linguistic 2-tuple (s, α) is used where s is a linguistic term and α is a numeric value representing the symbolic translation. A new approach to extend different classical aggregation operators with the 2-tuple linguistic model is developed. When both scores and weights are not crisp numbers, Yager (1998a) uses the fuzzy modeling technology to develop a model for the inclusion of importance in OWA aggregations. The so-called IOWA method suggested involves a transformation of the scores to be aggregated by their respective importance.

2.7.3 Comparison and Selection

The objective of the group decision making is to find one or several best alternatives, which is accomplished by ranking all the alternatives based on the aggregated result from the group members. A fuzzy outranking method may be necessary in ranking both fuzzy sets and groups of linguistic labels. Based on the ranking result, the alternatives can be selected.

Many methods for ranking fuzzy sets have been developed so far. Good summaries of fuzzy sets ranking methods can be found in books (Chen and Hwang, 1992; Hwang and Lin, 1987) and papers (Chang and Lee, 1994; Lee and Li, 1998, Lee-Kwang and Lee, 1999). After summarizing nearly 40 fuzzy sets ranking method, Chang and Lee proposed the following classification in (Chang and Lee, 1994):

1) α-cut methods. Usually a method developed by this approach is easy and fast to calculate.
2) Methods based on possibility concept.
3) Method by integration. Measure a fuzzy set by its mean value.
4) Multiple indices approach. Rank fuzzy sets using the results of multiple ranking or comparison functions.
5) Linguistic approach. This method is developed mainly due to the desire to maintain the fuzzy characteristics of the problem.

2.7.4 Consensus and Contribution Measure

After the group decision is created, we evaluate how good it is by checking whether it represents the majority of the group members' opinions. It is very rare when all individuals in a group share the same opinion about the alternatives, since a diversity of opinions commonly exists. It could happen that when the group members have conflicting opinions, the solution could be a medium one that no expert in the group likes.

Consensus is traditionally meant as a strict and unanimous agreement of all the experts regarding all possible alternatives. Consensus makes it possible for a group to reach a final decision that all group members can support among these differing opinions.

Any group decision-making process is basically aimed at reaching a "consensus". Consensus has become a major area of research in group decision making (Baas and Kwakernaak, 1977; Herrera et al., 1997; Herrera-Viedma et al., 2002; Kacprzyk et al., 1997). Generally, the approaches towards consensus in the literature can be divided into two groups. The first treats consensus as a "mathematical aggregated consensus" (Ng and Abramson, 1992). This type of consensus requires some kind of a binding arbitration so the contributing experts do not need to converge in their opinions. In most cases, the consensus is achieved by changing the weights of the experts (e.g. Lee, 2002). In the other type, the experts are encouraged to modify their opinion to reach a closer agreement in opinions (e.g. Hsu and Chen, 1996).

This is a general procedure for group decision-making in fuzzy environment. Most researches focus on the aggregation and selection steps. Good reviews on group decision making can be found in Zimmermann (1987), Hwang and Lin (1987), Hwang and Yoon (1981), Kacprzyk and Fedrizzi (1990) etc.

2.8 Summary

In this chapter, we summarized current research on fuzzy group decision making, introduced some fuzzy group decision making approaches such as fuzzy Delphi, fuzzy AHP, and fuzzy compromise programming. We introduced a general four-step procedure for group decision-making in fuzzy environment: unify the evaluations from each expert, aggregate the opinions of all group members to a final score for each alternative, rank the linguistic labels or fuzzy sets and select the preferred alternatives based on this order, measure consensus level and the individual contribution to the group decision. We gave a brief literature review on each step, detailed will be in the following four chapters.

In this chapter, a quick review of fuzzy sets, fuzzy sets operations, fuzzy arithmatics, and linguistic variables were given. Main similarities and differences between classical and fuzzy sets were introduced. In general, set operations are the same for classical and fuzzy sets. The exceptions were excluded middle laws. Alpha-cut sets and extension principle were presented followed by a brief summry of fuzzy similarities. This section presented issues that are important in understanding fuzzy sets and their advantages over classical sets. Most of the tools needed to form an idea about fuzzy logic and its operation have been introduced. These tools are essential in understanding fuzzy group decision making and linguistic group decision making approaches in the text. Fuzzy group decision making are very desirable in situations where precise mathematical models are not available and the human involvement is necessary. In that case linguistic variables could be used to mimic human behavior and actions.

CHAPTER 3

Unifying Fuzzy Preference of Alternatives

In a fuzzy environment, a group decision-making problem is composed by the following elements: a finite set of alternatives A= $\{A_1, A_2 \ldots A_n\}$, a finite set of experts E= $\{E_1, E_2 \ldots E_q\}$ with each expert $e_k \in E$ presents his/her preference relation on A_i as $x_{ik} \in S$. Where S is a finite but totally ordered term set of linguistic labels S= $\{s_0, s_1 \ldots s_T\}$, with $s_i > s_j$, if i>j. Usually each label has a membership function. The set S has odd number of elements. Also, we have the importance u_k, k= [1, 2 …q] assigned to each expert k. Yager introduces the weights b_i, i= [1, 2 …q] to experts. These weights represent the importance or trust that each expert carries (they can also be calculated from a required degree of orness) (Yager, 1988).

In this chapter, we give a literature review on current research in expressing experts' preference, introduce five ways for evaluation: preference ordering of the alternatives, fuzzy preference relation, multiplicative preference relation, utility function, and linguistic variable with functions to convert into each other. We then propose a new fusion approach of multi-granularity linguistic information for managing information assessed in different linguistic term sets.

3.1 Introduction

As each expert has their own ideas, attitudes, motivations, and personality, it is quite natural to consider that different experts will give their preferences in a different way.

Because the decision makers or experts often have limited knowledge of the domain and therefore limited background understanding, communication between the knowledge representation and the expert is impeded as is the process of transferring the expertise into a their evaluation preference. Therefore, the experts' abilities to understand what is being conveyed is a constraining factor.

The ability of an expert to convey knowledge is constrained by the experts communication abilities (i.e. their ability to express the knowledge that they posses). Experts often have trouble expressing or formalizing their knowledge, or communicating it in a form understandable to a novice. Experts also have a problem describing their knowledge in terms that are precise, complete, and consistent enough for use in a computer program (Sestito and Dillon, 1994). Experts sometimes give explanations for their decisions that sometimes do not correspond with the actual reasons for making their decisions. Often their explanations are approximate, incomplete, or inconsistent.

There is a difference in the way that a human expert's knowledge is structured as compared to the way that knowledge is represented in a different preference formats.

Current technology to adequately represent knowledge is limited. For example, the expressive capability of the representation used to encode knowledge limits the ability to capture knowledge.

There is a problem of verifying and validating knowledge after it has been input into the system. Isolating a reasoning fault may require a trace through dozens of inferences and hundreds of facts, and correcting such a fault may cause other reasoning faults through a domino / knock-on effect. Due to the complexity of interrelations between knowledge, it is often difficult to ascertain the implications of changes in the knowledge base, and changes in the knowledge base could therefore cause inconsistencies or performance degradation.

Knowledge discovery involves sifting through data in order to discover implicit (i.e. non-obvious), potentially useful, previously unknown, and non-trivial knowledge. (Usually there is a large volume of data to sift through.) Knowledge discovery is a multi-disciplinary field incorporating machine learning, statistics, database technology, expert systems, and data visualisation. The knowledge discovery process involves three main phases:

Pre-processing same data requires some domain knowledge and involves different expertise. Different expert may convert the data into a different form and represent the final results. This involves the presentation of the knowledge in a user friendly, understandable, and useful fashion. During this phase testing of the relevance, accuracy, comprehensibility, quality, validity, novelty, and generalisability of the results also occurs.

3.2 Methods

Currently, there are five common used ways that the experts can use to express their preferences over the set of alternatives: preference ordering of the alternatives, fuzzy preference relation, multiplicative preference relation, utility function, and linguistic variable.

3.2.1 Preference Ordering of the Alternatives

An expert here gives his/her preferences on alternatives as an individual preference ordering,

$$O^k = \{o^k(1), \ldots, o^k(n)\} \tag{3.1}$$

where $o^k(i)$ is a permutation function over the index set $\{1, \ldots, n\}$ showing the place of alternative i in the sequence (Chiclana et al., 1998; Seo and Sakawa, 1985).

Therefore, according to this point of view, an ordered vector of alternatives, from best to worst, is given. For example, an expert may give a four alternatives evaluation as O={1, 3, 4, 2} which means alternative A_1 is the best, A_4 is in the second place, A_2 is in third place, and alternative A_3 is the last, i.e. $A_1 > A_4 > A_2 > A_3$. So, the orders of the four alternatives are: $O(A_1)=1$, $O(A_2)=3$, $O(A_3)=4$, and $O(A_4)=2$.

3.2.2 Fuzzy Preference Relation

In this case, the expert's preferences on alternatives are described by a fuzzy preference relation P^k, with membership function:

$$\mu_{P^k}(A_i, A_j) = P_{ij}^k \tag{3.2}$$

This membership function denotes the preference degree or intensity of alternative A_i over A_j. Here $P_{ij}^k = 1/2$ indicates indifference between A_i and A_j , $P_{ij}^k = 1$ indicates that A_i is unanimously preferred to A_j, and $P_{ij}^k > 1/2$ indicates that A_i is preferred to A_j. It is usual to assume that $P_{ij}^k + P_{ji}^k = 1$ and $P_{ii}^k = 1/2$ (Orlovsky, 1978; Tanino, 1990).

For instance, the fuzzy preference relation to four alternatives can be:

$$P = \begin{bmatrix} 0.5 & 0.55 & 1.0 & 0.25 \\ 0.45 & 0.5 & 0.6 & 0.2 \\ 0 & 0.4 & 0.5 & 0.95 \\ 0.75 & 0.8 & 0.05 & 0.5 \end{bmatrix}$$

Where $P_{12} = 0.55 > 1/2$ means that A_1 is slightly preferred to A_2 while $P_{13} = 1.0$ indicates that A_1 is unanimously preferred to A_3.

3.2.3 Multiplicative Preference Relation

In this case, the preferences of alternatives of expert E_k are described by a positive preference relation

$$A^k = (a_{ij}^k) \tag{3.3}$$

where a_{ij}^k indicates a ratio of the preference intensity of alternative A_i to that of A_j, i.e., it is interpreted as A_i is a_{ij}^k times as good as A_j. Saaty suggests in (Saaty, 1980) to use a scale of 1 to 9 where $a_{ij}^k = 1$ indicates indifference between A_i and A_j and

$a_{ij}^{k} = 9$ indicates that A_i is unanimously preferred to A_j. An example of multiplicative preference relation can be expressed as:

$$A = \begin{bmatrix} 1 & 2 & 9 & 1/5 \\ 1/2 & 1 & 4 & 1/7 \\ 1/9 & 1/4 & 1 & 8 \\ 5 & 7 & 1/8 & 1 \end{bmatrix}$$

One can observe that the preference matrix has the property of multiplicative reciprocity relationship (i.e. $a_{ij}^{k} \cdot a_{ji}^{k} = 1$).

3.2.4 Utility Function

In this case the expert provides the preferences as a set of n utility values,

$$U^{k} = \{u_{i}^{k}, i = 1,..., n\} \tag{3.4}$$

where $u_{i}^{k} \in [0,1]$ represents the utility evaluation given by expert E_k to alternative A_i (Luce and Suppes, 1965; Tanino, 1990). For example, the utility function of four alternatives can be: U={0.7, 0.4, 0.2, 0.6}. So, the preference order of the four alternatives is: $A_1 > A_4 > A_2 > A_3$.

3.2.5 Linguistic Variable

As we introduced in chapter 2, the linguistic variables can be used as the evaluation scores by experts. Usually, a linguistic label set *S* is a finite but totally ordered term set of linguistic labels S= $\{s_0, s_1 \ldots s_T\}$, with $s_i > s_j$, for i>j (Delgado et al., 1998, Herrera and

Martinez, 2001). For example, Herrera et al. (1996a) defines the linguistic label set $S = \{s_0 = I,\ s_1 = SW,\ s_2 = WO,\ s_3 = SI,\ s_4 = EQ,\ s_5 = SB,\ s_6 = SU,\ s_7 = SS,\ s_8 = CS\}$. The associated trapezoidal membership functions are showed in Figure 2.7 with the numeric membership functions listed in Table 2.7.

3.3 Transformation functions

Experts have the freedom to use different preference formats for their evaluation, either preference order, fuzzy preference relation, multiplicative preference relation, utility function, or linguistic variable. But, for calculation, especially before aggregation, we should make all information uniform, thus all evaluations are comparable.

Since the experts may provide their preferences in different ways, there is a need to convert the various representations to a unified form. A common transformation between the various preferences is as follows (Chiclana et al., 1998):

$$p_{ij}^k = \frac{1}{2}\left(1 + \frac{o_j^k - o_i^k}{n-1}\right) \tag{3.5}$$

$$p_{ij}^k = \frac{(u_i^k)^2}{(u_i^k)^2 + (u_j^k)^2} \tag{3.6}$$

$$p_{ij}^k = \frac{1}{2}(1 + \log_9 a_{ij}^k) \tag{3.7}$$

These transformations allow all the experts' judgments to be converted into Fuzzy Preference Relations. Here is an example from Herrera-Viedma et al. (2002), we have three experts give their evaluations as O={2, 1, 3, 6, 4, 5}, U={0.3, 0.2, 0.8, 0.6, 0.4, 0.5},

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 & 4 & 3 & 5 \\ 2 & 1 & 1/3 & 1/4 & 4 & 6 \\ 3 & 3 & 1 & 7 & 6 & 9 \\ 1/4 & 4 & 1/7 & 1 & 1/2 & 3 \\ 1/3 & 1/4 & 1/6 & 2 & 1 & 4 \\ 1/6 & 1/6 & 1/9 & 1/3 & 1/4 & 1 \end{pmatrix}$$

Applying the formulas (3.5)~(3.7), we get the following fuzzy preference relation matrix:

$$P^O = \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.9 & 0.7 & 0.8 \\ 0.6 & 0.5 & 0.7 & 1 & 0.8 & 0.9 \\ 0.4 & 0.3 & 0.5 & 0.8 & 0.6 & 0.7 \\ 0.1 & 0 & 0.2 & 0.5 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.4 & 0.7 & 0.5 & 0.6 \\ 0.2 & 0.1 & 0.3 & 0.6 & 0.4 & 0.5 \end{pmatrix}$$

$$P^U = \begin{pmatrix} 0.5 & 0.69 & 0.12 & 0.2 & 0.36 & 0.9 \\ 0.31 & 0.5 & 0.06 & 0.1 & 0.2 & 0.8 \\ 0.88 & 0.94 & 0.5 & 0.64 & 0.8 & 0.98 \\ 0.8 & 0.9 & 0.36 & 0.5 & 0.69 & 0.97 \\ 0.64 & 0.8 & 0.2 & 0.31 & 0.5 & 0.94 \\ 0.1 & 0.2 & 0.02 & 0.03 & 0.06 & 0.5 \end{pmatrix}$$

$$P^A = \begin{pmatrix} 0.5 & 0.34 & 0.25 & 0.82 & 0.75 & 0.87 \\ 0.66 & 0.5 & 0.25 & 0.18 & 0.82 & 0.91 \\ 0.75 & 0.75 & 0.5 & 0.94 & 0.91 & 1 \\ 0.18 & 0.82 & 0.065 & 0.5 & 0.34 & 0.75 \\ 0.25 & 0.18 & 0.09 & 0.66 & 0.5 & 0.82 \\ 0.13 & 0.09 & 0 & 0.25 & 0.18 & 0.5 \end{pmatrix}$$

3.4 Linguistic fusion functions

As we mentioned before, generally, there are two main approaches to aggregate linguistic labels in group decision making. Most methods use the associated membership functions. The other category is to calculate linguistic labels directly.

Those approaches work well for homogeneous data type, but will have problems when compositive data, for example different linguistic definitions are used. An information fusion operator is necessary before aggregation. A fusion operation in group decision making is needed. As we discussed before, this is the first step of group decision-making and should be done before aggregating all experts' evaluations to make opinions uniform (Herrer et al., 2000).

3.4.1 Introduction

Data fusion is a technology of combining various forms of data such as sound, image, numerical, and linguistic, and acquires knowledge by combining these data. Usually, we cannot use only one form of data. For group decision-making problems, we just use numerical, and/or linguistic data. Currently, most researches on data fusion are in the fields of sensing and communicating (Chen and Luo, 1999, Goodridge et al., 1994, Hussien et al., 1994a, Hussien et al., 1994b, Singh and Bailey, 1997, Valet et al., 2003). Figure 3.1 demonstrates the transformation directions between and within the numeric and linguistic data in group decision making.

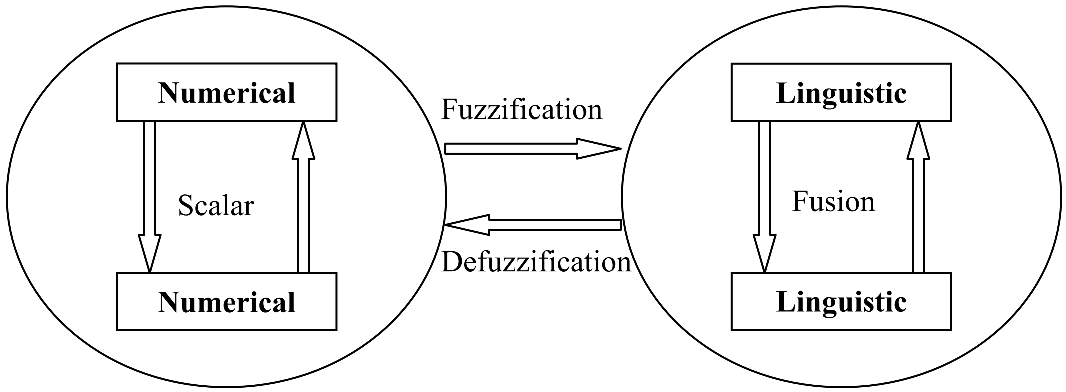


Figure 3.1 Transformation directions between numerical and linguistic values

When there are both linguistic and numeric data in group decision making problems, we need a fusion algorithm to transform either from linguistic to numeric or reverse before we could aggregate experts' opinions. There are a lot of fuzzificaion and defuzzificaion approaches such as Maximum Defuzzifier, Centroid Defuzzifier etc, in fuzzy logic domain (Mendel, 1995). In the following subsection, we review the current researches on fusion models applied in group decision making.

3.4.2 Fusion between Linguistic and Numeric data

Akiyama (2000) presents an object-oriented fusion and diffusion algorithms. The fusion algorithm glues a set of linguistically defined object types to create the new object type that transforms crisp information to the linguistic and original format. As the opposite operation of the fusion operation, the object diffusion is to un-glue a linguistic object type, it transforms a linguistic format of input and results into constitute crisp and primitive formats. Chen and Chen (2002) present a new information fusion algorithm based on a similarity measure. The algorithm can handle heterogeneous fuzzy group decision-making problems in a more flexible and intelligent manner. Delgado et al. (1998) introduces two transformation functions, from linguistic to numerical and from numerical to linguistic based on the fuzzy number characteristic values. They also propose a group decision-making process based on the fusion operator. Grabisch and Saveant (1998) propose a framework for handling uncertainty in data fusion based on possibility theory and also present a linguistic interface to translate possibility distributions into a natural language form. Torrez et al. (2002) combine "Boolean related event algebra" and "one point random set coverage representations of fuzzy sets" together to integrate fuzzy input into probability input. Hathaucey et al. (1996) presents a model to integrate numbers, intervals and linguistic assessments from three types of sensors. Moses et al. (1999) developed a linguistic coordinate transformation algorithm for complex fuzzy sets. This makes the adaptive control may be implemented in one linguistic coordinate system and the linguistic outputs may then be transformed to another for further operations.

3.4.3 Fusion between Linguistic data

In linguistic approach, an important parameter to determine is the "granularity of uncertainty" (Herrera et al., 2000), i.e., the cardinality of the linguistic term set being used to express the information. According to the uncertainty degree that an expert qualifying a phenomenon has on it, the linguistic term set chosen to provide his/her knowledge will have more or less terms. When different experts have different uncertainty degrees on the phenomenon, then several linguistic term sets with a different granularity of uncertainty are necessary.

The use of more linguistic label sets gives decision makers more flexibility. Just like the use of crisp numbers, one expert likes to choose from 1 to 5 as his evaluation, while another expert prefers from 1 to 10 to make his opinion in more detail. The use of linguistic labels has the same problem. Some experts like to use five linguistic labels set such as S_1= {none, low, medium, high, perfect} and others may prefer more linguistic labels like S_2= {none, very low, low, medium, high, very high, perfect}. We can see here, S_2 specifies "low" to "very low" and "low", "high" to "high" and "very high". Using less linguistic labels makes the problem simple. Although the use of more linguistic labels may make computation a little complicated, it will benefit the accuracy of the decision.

A fusion algorithm is necessary to transform one set of linguistic labels to another. Suppose two experts use the two linguistic labels sets S_1 and S_2 respectively. Even both of them choose "low" as their score to an alternative, the same linguistic label "low" may have different definitions, i.e., the same label "low" may have different membership functions from different experts. Actually, the "low" from S_1 in the above example covers the "low" from S_2. The "low" in S_2 is a subset of the "low" in S_1. So, we cannot use this two "lows" equivalently in the process of the decision-making.

A fusion algorithm is used to convert multi-granularity linguistic term sets into a specific linguistic domain, which is a basic linguistic term set. The basic linguistic term set is chosen so as not to impose useless precision to the original evaluations and in order to allow an appropriate discrimination of the initial performance values. Herrera et al. (2000) propose an algorithm on transformation between linguistic sets.

3.4.3.1 The fusion method from Herrera et al. (2000)

The approach from Herrera et al. (2000) allows experts to use different sets of linguistic labels as their score sources. Then the author proposed a transformation function to unify the information. After aggregating all experts' opinions, the ranking methods are applied to choose the best alternative(s). Figure 3.2 taken from Herrera et al. (2000) shows the layer of the linguistic labels. The set can have 5, 7, 9, 11 or even more linguistic labels. But no matter how many linguistic labels a set has, it covers the real numbers interval [0, 1]. The more linguistic labels used, the less range one linguistic label covers. For instance, for S_1= {none, low, medium, high, perfect}, there are two linguistic labels "none" and "low" cover the range [0, 0.5). But if we use S_2= {none, almost none, very low, low, almost medium, medium, almost high, high, very high, almost perfect, perfect}, the same range [0, 0.5) is covered by five linguistic labels.

For different purposes, different experts can choose different linguistic sets as his/her evaluation pool. Here the linguistic label set S_j is defined by the expert E_j, he/she chooses from M_{j+1} linguistic labels to evaluate the alternatives. Problem now is to transform all sets S_j into a standard linguistic set S_T for later aggregation. The idea of the fusion method from Herrera et al. (2000) is to assign a membership to every linguistic label in the target set for each linguistic label being transformed. The membership is computed by finding the interaction of two linguistic labels, target and source. The multi-granularity transformation function τ_{S_T} is defined as,

$$\tau_{S_T} : S_M \to F(S_T) \tag{3.8}$$

$$\tau_{S_T}(l_i) = \{(s_j, \alpha_j^i) \mid j \in \{0,...,T\}\}, \forall l_i \in S_M \quad (3.9)$$

$$\alpha_j^i = \max_y \min\{\mu_{l_i}(y), \mu_{s_j}(y)\} \quad (3.10)$$

where $S_M = \{l_0, l_1, ..., l_M\}$ and $S_T = \{s_0, s_1, ..., s_T\}$ are source and target linguistic label sets, respectively, such that $T \geq M$.

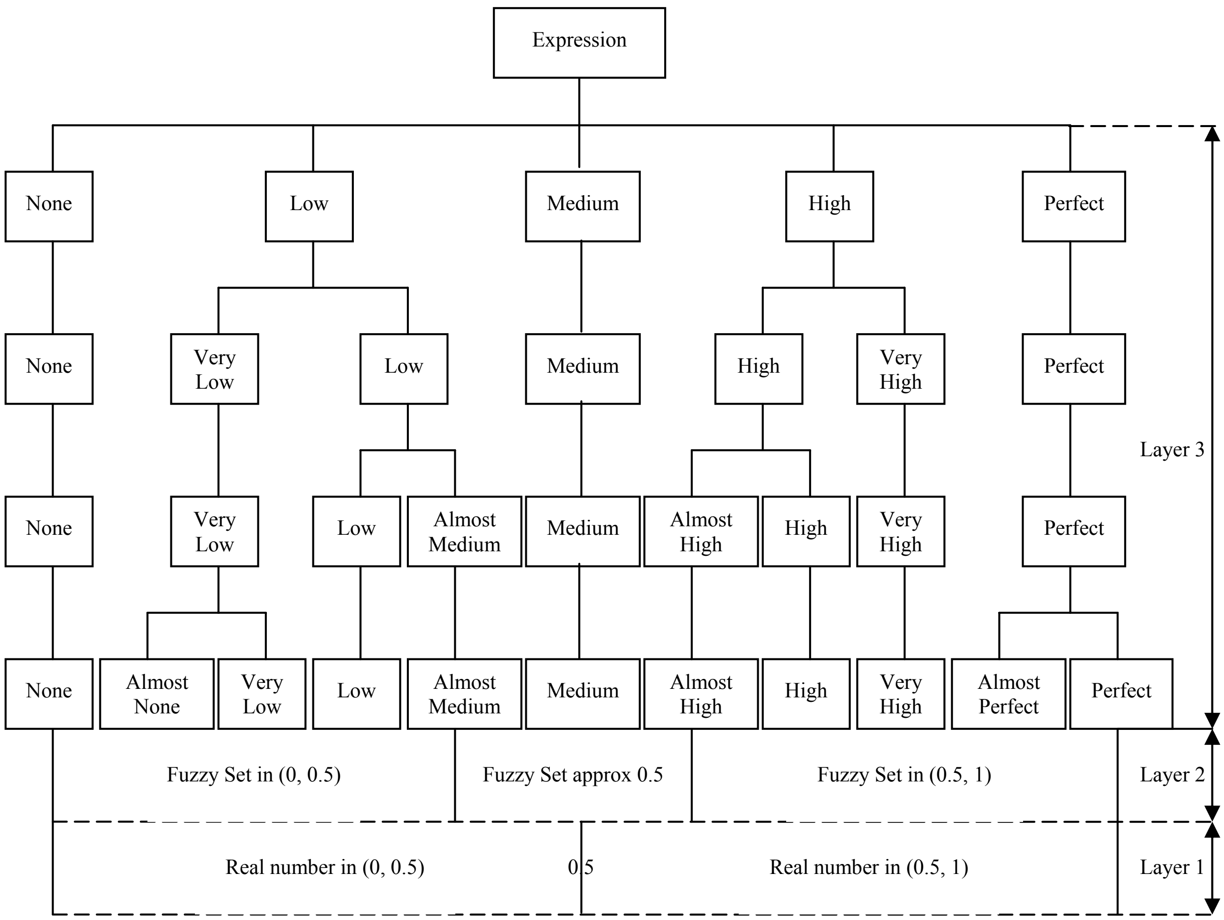


Figure 3.2 Layers of the linguistic labels (Herrera et al., 2000)

One question in this approach is how to choose the target set S_T (BLTS in Herera et al., 2000) from all the linguistic label sets. Fortunately, the author gives some rules. When there is only one term set with the maximum granularity, we choose the set with the maximum granularity as S_T, where $T = \max(M_1, M_2, \ldots, M_q)$. If we have two or more linguistic label sets with maximum granularity, then S_T is chosen depending on the semantics of these linguistic label sets, finding two possible situations to establish S_T:

1) If all the linguistic label sets have the same semantics, then S_T is any of them.
2) There are some linguistic label sets with different semantics. Then, S_T is a basic linguistic label set with a larger number of labels than the number of labels that a person is able to discriminate. (Normally 11 or 13).

Example 3.1

This example shows how this fusion approach works. Here, we have M=4 and T=6, $S_4 = \{l_0, l_1, ..., l_4\}$ and $S_6 = \{s_0, s_1, ..., s_6\}$ with the following membership functions:

$l_0 : (0, 0, 0.25)$,	$s_0 : (0, 0, 0.16)$,
$l_1 : (0, 0.25, 0.5)$,	$s_1 : (0, 0.16, 0.34)$,
$l_2 : (0.25, 0.5, 0.75)$,	$s_2 : (0.16, 0.34, 0.5)$,
$l_3 : (0.5, 0.75, 1.0)$,	$s_3 : (034, 0.5, 0.66)$,
$l_4 : (0.75, 1.0, 1.0)$	$s_4 : (0.5, 0.66, 0.84)$,
	$s_5 : (0.66, 0.84, 1.0)$,
	$s_6 : (0.84, 1.0, 1.0)$

Applying τ_{S_T}, for l_0 and l_1 are:

$$\tau_{S_T}(l_0) = \{(s_0,1), (s_1,0.58), (s_2,0.18), (s_3,0), (s_4,0), (s_5,0), (s_6,0)\}$$

$$\tau_{S_T}(l_1) = \{(s_0,0.39), (s_1,0.85), (s_2,0.85), (s_3,0.39), (s_4,0), (s_5,0), (s_6,0)\}$$

Figure 3.3 demonstrates the calculation of $\tau_{S_T}(l_1)$.

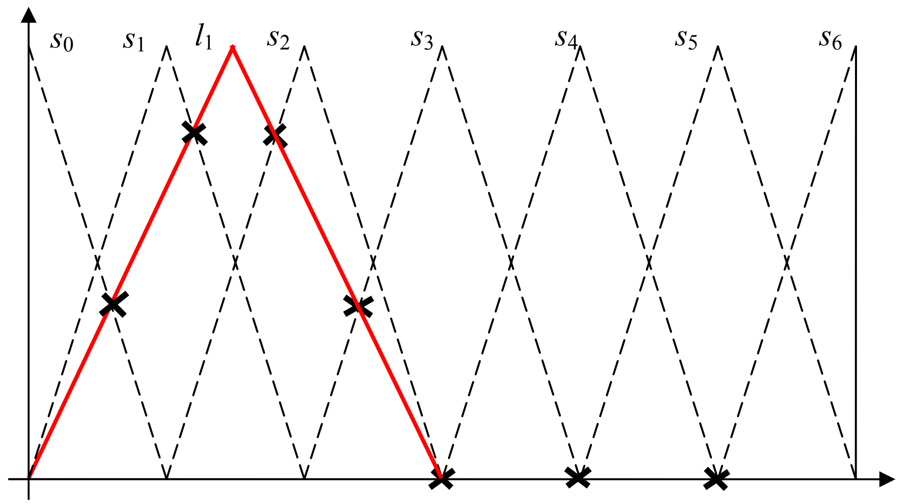


Figure 3.3 The transformation of l_1 by Herrera's method

3.5 A new fusion method

Herrera et al.'s (2000) fusion method enables us to unify the different linguistic labels sets. One of the problems of the method is that we need to assign a membership function to each linguistic label. Different membership function will result in different transformation results. Another problem is that we can only transfer a small linguistic label set into a larger one, but cannot do the inverse operation, i.e. we can use this method to transform a five linguistic labels set S_4 to another linguistic set with cardinality of 7, but we can not apply this method to map seven labels to five.

In this section we propose a new fusion method which computes the linguistic labels directly and with more attractive properties.

3.5.1 A new fusion approach to make the information uniform

The new fusion process assigns membership functions to all linguistic labels in the target set directly. Note that no matter how many linguistic labels a set has, it covers the real numbers interval [0, 1]. Figure 3.4 shows the basic idea of the new approach with

T=2, 4, 6, 8, 10, 12, 14. Then the interval a linguistic level covers overlaps to another from other sets. The ratio of the common interval they cover indicates the membership function should be assigned. Since usually membership functions of linguistic labels are symmetric, the transformation result only depends on how much the membership functions overlap with each other.

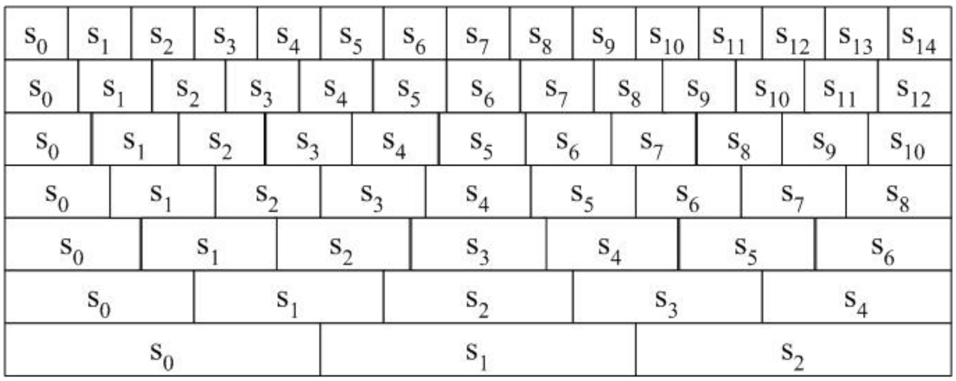


Figure 3.4 The tiers of the linguistic label sets

As we defined before, two linguistic label sets $S_T = \{s_0, s_1, \ldots, s_T\}$ and $S_M = \{l_0, l_1, \ldots, l_M\}$ are the target set and the source set respectively. So we transform S_M to S_T. In Figure 3.4, the linguistic label l_i in S_M covers the same range from part of S_{j-1}, the whole S_j, and part of S_{j+1}. The corresponding memberships $\mu_{S_{i,j-1}}$, $\mu_{S_{i,j}}$, $\mu_{S_{i,j+1}}$ to the three linguistic labels S_{j-1}, S_j and S_{j+1} in S_T overlapped with l_i are greater than 0. The membership function $\mu_{S_{i,j-1}} = a / w$ is defined by the ratio of the length covered by both linguistic labels S_{j-1} and l_i (the length *a* in the shaped area in Figure 3.5) to *w*, the width of the labels in S_T. Other labels in S_T get membership 0s since they have no common interval with the label l_i in S_M. Then the linguistic label l_i can be expressed by:

$$l_i = \{(s_j, \mu_{S_{ij}}(x)) \mid j \in [0, T]\} \tag{3.11}$$

Since all linguistic sets cover the range [0, 1], there are relations between the two sets S_T and S_M. We can use the following formulas to compute the membership functions $\mu_{S_{ij}}(x)$.

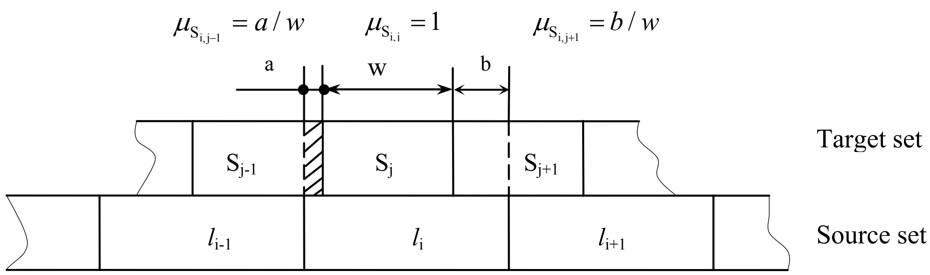


Figure 3.5 The idea of tier method on the linguistic label sets fusion

$$\tau_{S_T} : S_M \rightarrow F(S_T) \tag{3.12}$$

$$\tau_{S_T}(l_i) = \{(s_j, \mu_{S_{ij}}(x)) \mid \mathrm{j} \in [0, \mathrm{T}]\} \tag{3.13}$$

$$\mu_{S_{ij}}(x) = \begin{cases} 1 & \text{for } \mathrm{j}_{\min} < \mathrm{j} < j_{\max} \\ (\frac{j+1}{T+1} - \frac{i}{M+1})(T+1) & \text{for } j = j_{\min} \\ (\frac{i+1}{M+1} - \frac{j}{T+1})(T+1) & \text{for } j = j_{\max} \\ 0 & \text{others} \end{cases} \tag{3.14}$$

where, $j_{\min}$ and $j_{\max}$ are the indexes of the first and the last linguistic labels with nonzero membership functions in the target set S_T. In Figure 3.5, j_{min}=j-1 and j_{max}=j+1. For each i, the corresponding j_{min} and j_{max} are defined by:

$$\frac{j_{\min}}{T+1} \le \frac{i}{M+1} \le \frac{j_{\min}+1}{T+1}, \quad \mathrm{i} = 0, 1, ..., \ \mathrm{M} \tag{3.15}$$

$$\frac{j_{\max}}{T+1} \le \frac{i+1}{M+1} \le \frac{j_{\max}+1}{T+1}, \quad \mathrm{i} = 0, 1, ..., \ \mathrm{M} \tag{3.16}$$

Remark 1

The new fusion approach assumes that all linguistic labels in a set evenly cover the interval of [0, 1]. There are no overlaps between two adjacent linguistic labels. No matter how many linguistic labels there are in a set, they should cover the same interval of [0, 1].

Remark 2

The new fusion approach requires M<2T+1. This means the number of linguistic labels to be transformed (M) should not be larger than 2 times of the number of the linguistic labels will be presented (T). This can be easily proved by:

$$\frac{1}{T+1} < 2 \times \frac{1}{M+1} \tag{3.17}$$

This inequality comes from that no two source linguistic labels can be covered by the same linguistic label from the target set. If not, we have $\mathrm{M} \geq 2\mathrm{T}+1$, from the Formula 3.14 at least

$$\tau_{S_T}(l_0) = \tau_{S_T}(l_1) = \left\{(s_0, \frac{T+1}{M+1}), (s_1, 0), (s_2, 0), \ldots, (s_T, 0)\right\} \tag{3.18}$$

In this way, we cannot distinguish the two linguistic labels l_0 and l_1 after the fusion operation. For instance,

T=4, M=12, $\mathrm{M} > 2 \times 4 + 1 = 9$ doesn't satisfy the restriction, then l0 and l1 have the same number as:

$$\tau_{S_4}(l_0) = \tau_{S_4}(l_1) = \left\{(s_0, \frac{4+1}{12+1}), (s_1, 0), (s_2, 0), (s_3, 0), (s_4, 0)\right\} = \{(s_0, 0.38), (s_1, 0), (s_2, 0), (s_3, 0), (s_4, 0)\}$$

So, this rule guarantees we have different linguistic label sets after the fusion operation.

Example 3.2

This example (Figure 3.6) shows how to apply the new fusion algorithm to transform a larger linguistic labels set with cardinality M+1=11 to a small linguistic labels set with T=8.

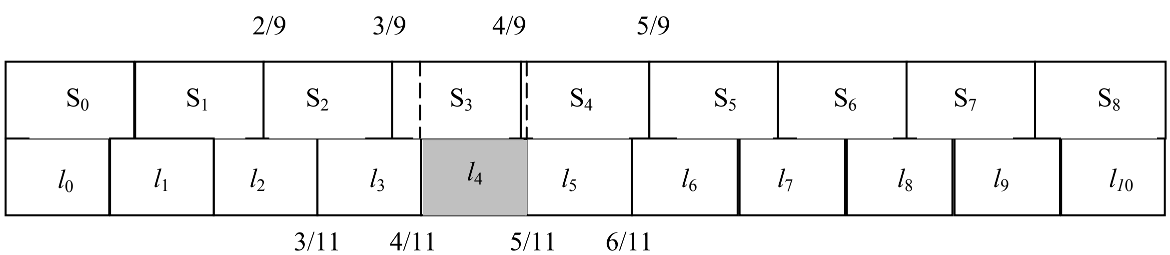


Figure 3.6 The transformation of large linguistic label set (M=10) to a small set (T=8)

For i=4, $\frac{i}{M+1} = \frac{4}{11}$, $\frac{i+1}{M+1} = \frac{5}{11}$

By $\frac{3}{8+1} \leq \frac{4}{10+1} \leq \frac{3+1}{8+1}$ and $\frac{4}{8+1} \leq \frac{5}{10+1} \leq \frac{4+1}{8+1}$, we can see that $j_{\min} = 3$ and $j_{\max} = 4$

So, $\mu_{S_{43}}(x) = (\frac{4}{8+1} - \frac{4}{10+1}) \times (8+1) = 0.73$,

$$\mu_{S_{44}}(x) = (\frac{5}{10+1} - \frac{4}{8+1}) \times (8+1) = 0.09,$$

$$\mu_{S_{4j}}(x) = 0, \textit{ for all } j = 0,1,...,8 \textit{ and } j \neq 3,4$$

Then we have the transformation result as:

$$\tau_{S_8}(l_4) = \{(s_0,0),(s_1,0),(s_2,0),(s_3,0.73),(s_4,0.09),(s_5,0),(s_6,0),(s_7,0),(s_8,0)\}$$

Example 3.3

This example (Figure 3.7) shows how to apply the new fusion algorithm to transform a small linguistic labels set with cardinality M+1=9 to a larger linguistic labels set with T+1=11.

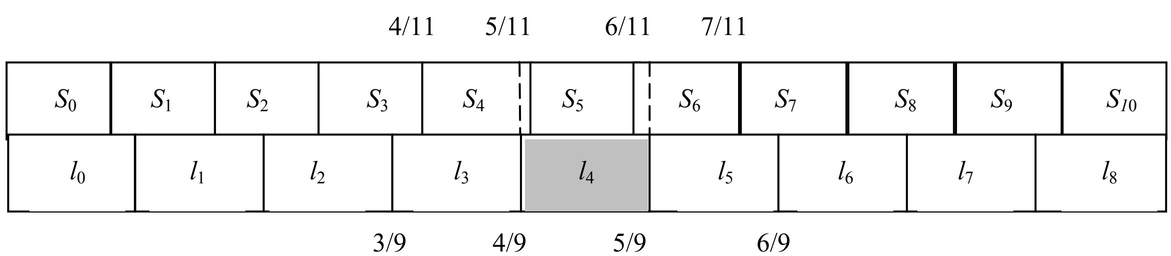


Figure 3.7 The transformation of small linguistic label set (M=8) to a large set (T=10)

For i=4, $\frac{i}{M+1} = \frac{4}{9}$, $\frac{i+1}{M+1} = \frac{5}{9}$

By $\frac{4}{10+1} \leq \frac{4}{8+1} \leq \frac{4+1}{10+1}$ and $\frac{6}{10+1} \leq \frac{5}{8+1} \leq \frac{6+1}{10+1}$, we have $j_{\min} = 4$ and $j_{\max} = 6$

So, $\mu_{S_{44}}(x) = (\frac{5}{10+1} - \frac{4}{8+1}) \times (10+1) = 0.11$, $\mu_{S_{45}}(x) = 1$,

$$\mu_{S_{46}}(x) = (\frac{5}{8+1} - \frac{6}{10+1}) \times (10+1) = 0.11,$$

$$\mu_{S_{4j}}(x) = 0, \textit{ for all } j = 0,1,...,10 \textit{ and } j \neq 4,5,6$$

Then we have the transformation result as:

$$\tau_{S_8}(l_4) = \{(s_0,0),(s_1,0),(s_2,0),(s_3,0),(s_4,0.11),(s_5,1),(s_6,0.11),(s_7,0),(s_8,0),(s_9,0),(s_{10},0)\}$$

3.5.2 Extension of the new fusion approach (Overlap allowed)

As we discussed earlier, we assume there are no overlaps between the intervals represented by two linguistic labels. This is not always true. One of the arts of fuzzy set theory is that there are no sharp differences in the boundaries of fuzzy sets. The former assumption is not always reasonable. There should be overlaps between membership functions of two linguistic labels as shown in Figure 3.8.

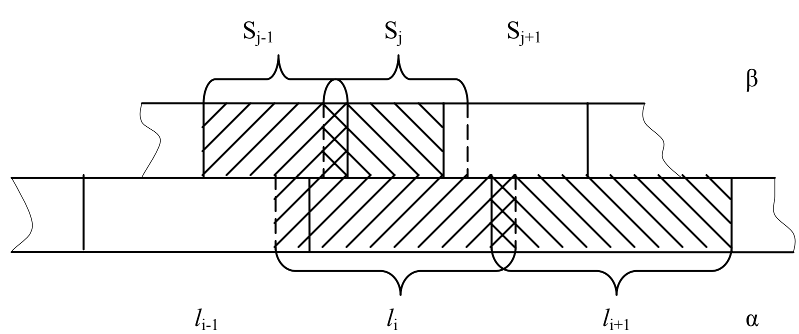


Figure 3.8 The tier method on the linguistic label sets fusion

Here we introduce a new parameter α to measure the degree of overlapping. α is defined as the increasing rate of the width of the interval the linguistic label represents without overlapping to the width of the interval the linguistic label covers with overlap. That means, if the width of the linguistic labels is w0 without overlapping, the new width is $w_1 = (1+\alpha)\cdot w_0$.For instance, the width without overlap (α=0) of 5 linguistic label set is w0=1.0/5=0.2. If α=0.5, the new width a linguistic label covers is 0.3, so the interval of one of the linguistic in this set will change from [0.2, 0.4] to [0.15, 0.45]. Here the increased width is expended to both directions. For the first and the last linguistic labels in linguistic sets, we just expend to one direction, i.e. the first linguistic label covers the interval of [0, 0.25] instead of [0, 0.2] with α=0.5.

Again, we assume the membership functions are symmetric. This is a reasonable assumption, since when we say a linguistic variable, most people will intuitionally have no bias on its two boundaries.

Given the desired parameter α and the original width of the interval the linguistic label covers, we can compute the new width we need for the linguistic label. Table 3.1 shows an example of a 5-linguistic label set:

Table 3.1 Comparison of the two linguistic sets with different α level for M=4

α	Original Width	New Width	s_0	s_1	s_2	s_3	s_4
0	0.20	0.20	[0, 0.20]	[0.20, 0.40]	[0.40, 0.60]	[0.60, 0.80]	[0.80, 1]
0.4	0.20	0.28	[0, 0.24]	[0.16, 0.44]	[0.36, 0.64]	[0.56, 0.84]	[0.76, 1]
0.5	0.20	0.30	[0, 0.25]	[0.15, 0.45]	[0.35, 0.65]	[0.55, 0.85]	[0.75, 1]

With the new parameter α, we should justify our Formula 3.12~3.14.

The new width of the linguistic label are $\frac{1+\alpha}{M+1}$ for all i=0,…,M. The interval of the ith label in the set SM changes from $\left[\frac{i}{M+1},\frac{i+1}{M+1}\right]$ to $\left[\frac{2i-\alpha}{2(M+1)},\frac{2i+\alpha+2}{2(M+1)}\right]$, for i=1,…,(M-1). While $\left[0,\frac{1+\alpha}{M+1}\right]$ when i=0 and $\left[\frac{M-\alpha}{M+1)},1\right]$ for i=M. To simplify the expressions, we introduce the definitions of $\gamma_{i,M,\alpha}^{L}$ and $\gamma_{i,M,\alpha}^{R}$ to represent the two boundaries of the intervals a linguistic label covers.

$$\gamma_{i,M,\alpha}^{L}=\begin{cases}0, & i=0\\ \frac{2i-\alpha}{2(M+1)}, & 0<i<M\\ \frac{2i-2\alpha}{2(M+1)}, & i=M\end{cases} \tag{3.19}$$

$$\gamma_{i,M,\alpha}^{R}=\begin{cases}\frac{2i+2\alpha+2}{2(M+1)}, & i=0\\ \frac{2i+\alpha+2}{2(M+1)}, & 0<i<M\\ 1, & i=M\end{cases} \tag{3.20}$$

Then the new function to compute $j_{\min}$ and $j_{\max}$ are defined by :

$$\frac{2j_{\min}-\beta}{2(T+1)}\leq\frac{2i-\alpha}{2(M+1)}\leq\frac{2j_{\min}+\beta+2}{2(T+1)},\quad \mathrm{i=0,1,...,\ \ M;\ j=0,1,...,\ \ T} \tag{3.21}$$

$$\frac{2j_{\max}-\beta}{2(T+1)}\leq\frac{2i+\alpha+2}{2(M+1)}\leq\frac{2j_{\max}+\beta+2}{2(T+1)},\quad \mathrm{i=0,1,...,\ \ M;\ j=0,1,...,\ \ T} \tag{3.22}$$

Where α and β are the width increasing rates of the source set and target set respectively.

It could happen that the left side of l_i lies between the overlapping area of two linguistic labels s_{j-1} and s_j (Figure 3.9). Actually, there are three cases for the location of the left sides of l_i. as well as the right side. Case one is the left side of l_i, $\frac{2i-\alpha}{2(M+1)}$ lies between the left side of S_{j-1} $\frac{2j-\beta}{2(T+1)}-\frac{1}{T+1}$ and the left side of S_j. Another case is that the left side of l_i $\frac{2i-\alpha}{2(M+1)}$ lies between the right side of S_{j-1} $\frac{2j+\beta}{2(T+1)}$ and the right side of S_j, $\frac{2j+\beta}{2(T+1)}+\frac{1}{T+1}$. The third case is it locates between the left side of S_j $\frac{2j-\beta}{2(T+1)}$ and the right side of S_{j-1}, $\frac{2j+\beta}{2(T+1)}$. We can still use the Formula 3.21 and 3.22 to define $j_{\min}$ and $j_{\max}$, but now the $j_{\min}$ and $j_{\max}$ are not single numbers, but sets of indexes of the linguistic labels. And they are the indexes of the linguistic labels with the membership functions greater than 0 and less than 1.

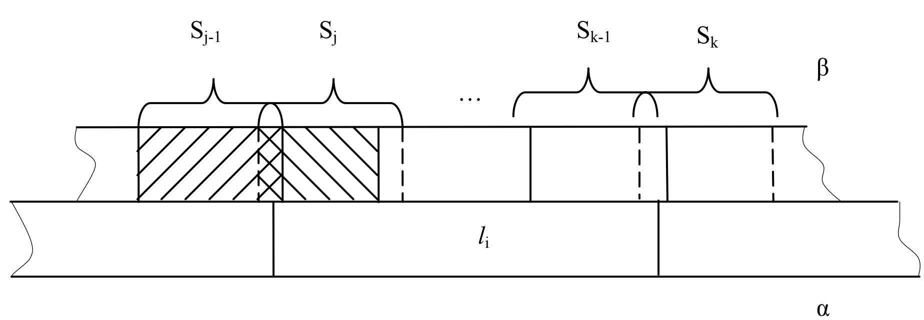


Figure 3.9 The j_{min} and j_{max} on the linguistic label sets fusion

Then there are more linguistic labels whose membership functions lie in the interval (0, 1). For example, in Figure 3.9, the membership functions assigned to s_{j-1} and s_j as well as s_{k-1} and s_k should be less than 1 and greater than 0. The membership function should be rewrite as:

$$\mu_{S_{ij}}(x)=\begin{cases} 1 & for\ \max\{j_{min}\}<\mathrm{j}<\min\{j_{max}\} \\ (\gamma^{R}_{j,T,\beta}-\gamma^{L}_{i,M,\alpha})(\dfrac{T+1}{1+\beta}) & for\ \ j\in j_{\min}\setminus\{j_{\min}\cap j_{\max}\} \\ (\gamma^{R}_{i,M,\alpha}-\gamma^{L}_{j,T,\beta})(\dfrac{T+1}{1+\beta}) & for\ \ j\in j_{\max}\setminus\{j_{\min}\cap j_{\max}\} \\ (\dfrac{1+\alpha}{M+1})(\dfrac{T+1}{1+\beta}) & for\ \ j\in\{j_{\min}\cap j_{\max}\} \\ 0 & \text{others} \end{cases} \quad (3.23)$$

Example 3.4

This example (Figure 3.10) shows how to apply the new fusion algorithm with the parameter α=0.5 and β=0.5 to transform a smaller linguistic labels set with cardinality M+1=7 to a large linguistic labels set with T=8.

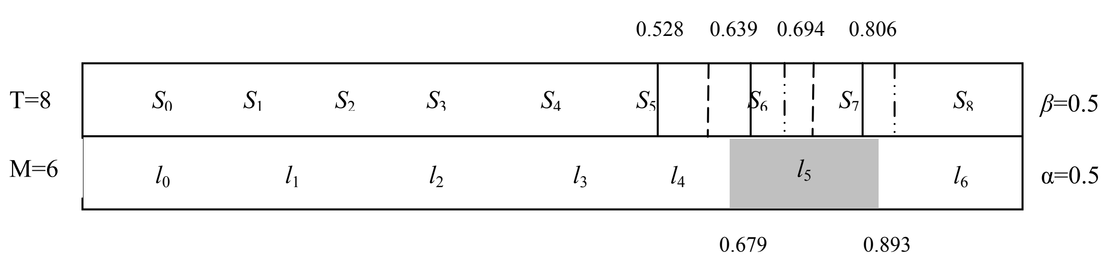


Figure 3.10 The transformation of small linguistic label set (M=6) to a large set (T=8) with extension

For i=5, $\alpha=\beta=0.50$, by Formula 3.21 and 3.22, we can define $j_{\min}=\{5,6\}$ and $j_{\max}=\{7,8\}$ with $j_{\min}\cap j_{\max}=\phi$.

For j=8, $j \notin \{j_{\min} \cap j_{\max}\}$ then $\mu_{S_{58}}(x) = (\gamma^R_{i,M,\alpha} - \gamma^L_{j,T,\beta})(\frac{T+1}{1+\beta}) = 0.359$, given i=5, j=8, M=6, T=8 and $\alpha = \beta = 0.50$.

Similarly, we have the transformation result as:

$$\tau_{S_8}(l_5) = \{(s_0,0),(s_1,0),(s_2,0),(s_3,0),(s_4,0),(s_5,0.090),(s_6,0.760),(s_7,0.856),(s_8,0.359)\}$$

Example 3.5

This example (Figure 3.11) shows the case when $j_{\min} \cap j_{\max} \neq \phi$.

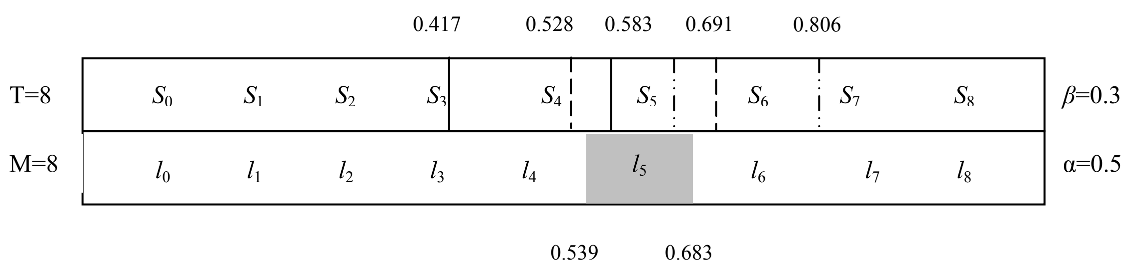


Figure 3.11 The transformation of two linguistic label sets with same cardinality (M=T=8)

For i=5, $\alpha = 0.50$, $\alpha = \beta = 0.30$, by Formula 3.21 and 3.22, we can define $j_{\min} = \{4,5\}$ and $j_{\max} = \{5,6\}$ with $j_{\min} \cap j_{\max} = \{5\}$. When j=5, $j \in \{j_{\min} \cap j_{\max}\}$, then $\mu_{S_{55}}(x) = \frac{1+\alpha}{M+1} \cdot \frac{T+1}{1+\beta} = \frac{1+0.5}{8+1} \cdot \frac{8+1}{1+0.3} = 0.862$. And the transformation result is:

$$\tau_{S_8}(l_5) = \{(s_0,0),(s_1,0),(s_2,0),(s_3,0),(s_4,0.263),(s_5,0.862),(s_6,0.263),(s_7,0),(s_8,0)\}$$

Remark 1

The overlapping degree α and β reflect the "semantics" of the linguistic labels. It is similar to the different definitions of membership functions from different experts.

Remark 2

With extension, the restriction of M<2T+1 should be changed too. By the same philosophy: "no two source linguistic labels can be covered by the same linguistic label from the target set", we have,

$\frac{1+\beta}{T+1} < 2 \times \frac{1+\alpha}{M+1}$, we obtain the new restriction: $\mathrm{M} < \frac{2(\mathrm{T}+1)(1+\alpha)}{(1+\beta)} - 1$.

3.5.3 Properties of the new fusion approach

The transformation result from Herrera et al.'s method (2000) depends on linguistic membership functions. Different membership function will result in different transformation result which will affect the choice of alternative(s). Also the approach can only transform smaller linguistic set to a larger one. The new approach works in both directions, thus suitable to different applications. We can transform a small set of linguistic labels into a large set of linguistic labels or transform a large set into a small one. The transformation direction depends on different applications. Both directions have advantages and disadvantages. The transformation from a large set into a small one will avoid useless precision, and then save the process time that will reduce the total decision cost. The transformation from a small set into a large one will keep more information from group members that have higher accuracy. For those do not need high accuracy applications, we can choose to transform form large linguistic sets to small set to save time and costs.

The new approach is easy to calculate. Since we do not compute the membership functions, it simplifies the calculations. And new fusion approach has the following interesting properties:

Property 1. The linguistic label set after transformation is ordered.

We have $l_i < l_j$, with i < j

Proof.

Since both linguistic sets S_T and L_M are ordered and cover the same increasing crisp number range [0, 1], the transformation function will not change the order of linguistic labels.

Property 2. Summation**.**

For each linguistic label l_i after transformation, its membership function satisfies,

$$\sum_{j=0}^{T} \mu_{S_{ij}}(x) = \frac{T+1}{M+1} \cdot \frac{1+\alpha}{1+\beta} + \frac{\beta}{1+\beta} p + \Delta$$

where $p = \min\{j_{\max}\} - \max\{j_{\min}\}$,

$$\Delta = \max(0, \frac{2\min\{j_{\min}\} + \beta + 2}{2(T+1)} - \frac{2i - \alpha}{2(M+1)}) + \max(0, \frac{2i + \alpha + 2}{2(M+1)} - \frac{2\max\{j_{\max}\} - \beta}{2(T+1)})$$

Property 3.

If we substitute $\alpha = 0$ and $\beta = 0$ into the Formula 3.23, we can get exactly Formula 3.14.

Property 4.

When M=T and $\alpha \neq \beta$, the two linguistic sets have the same cardinality, but different overlapping degree (semantics). This could happen when two experts use the same set of linguistic labels, but have different definitions of membership functions. With M=T, the Formula 3.23 will be,

$$\mu_{S_{ij}}(x) = \begin{cases} 1 & for\ \mathrm{j}_{\min} < \mathrm{j} < j_{\max} \\ \dfrac{(\alpha+\beta)+2(1-i+j)}{1+\beta} & for\ \ j = j_{\min} \\ \dfrac{(\alpha+\beta)+2(1+i-j)}{1+\beta} & for\ \ j = j_{\max} \\ 0 & \text{others} \end{cases} \tag{3.24}$$

Property 5.

When M=T and $\alpha = \beta$, the two linguistic sets have the same cardinality, and the same overlapping degree (semantics). Or we can say these two linguistic sets are identical. Formula 3.23 and 3.24 will give us the same value as $j_{min}=j_{max}$ and it is not hard to test that $\mu_{S_{ij}}(x) = 1$ for j= j_{min}.

In this context, a decision procedure is proposed with a view to obtaining the solution set of alternatives.

1) Each decision maker gives his/her opinions to all alternatives. They choose from their own linguistic set and give the cardinality of the set as well as the parameter α.

2) The fusion of the multi-granularity linguistic performance values is carried out in order to obtain collective performance evaluations. In this step, the multi-granularity linguistic information is made uniform using a linguistic term set as the uniform representation base, the basic linguistic term set.

3) An OWA based aggregation is applied.

4) Finally, the choice of the best alternative(s) from the collective performance evaluations is performed. To do that, a fuzzy preference relation is computed from the collective performance evaluations using a ranking method of pairs of fuzzy sets in the setting of Possibility Theory, applied to fuzzy sets on the basic linguistic term set.

5) Then, a consensus level may be set on the preference relation in order to rank the alternatives.

3.6 Summary

This chapter summarized five ways that can be used by experts to express their opinions. We then introduced several transformation functions that are used to transfer between these formats. Then a new fusion operator for linguistic labels was proposed. Comparing to the current existing methods, the new fusion approach is easy for computation and does not have to define the membership functions of all linguistic variables that are hard in reality.

CHAPTER 4

Aggregating Individual Preferences into a Group Decision

Aggregation is the third and the most important step in the group decision making process as shown in Figure 2.11. Generally, we aggregate all group members' opinions together in this step and get a group decision. In this chapter, we will summarize some aggregation operators suitable for fuzzy group decision-making, and categorize them into two groups: aggregation using associated membership functions and direct computing with linguistic labels. Then, we will introduce a new fuzzy linguistic OWA (FLOWA) operator, which has some interesting properties for linguistic labels aggregation.

4.1 Introduction

Multi-Criteria Decision Making (MCDM) and Multi-Expert Multi-Criteria Decision Making (ME-MCDM) are two rich and well-studied problem solving approaches usually aimed at ranking of alternatives (see for example, Triantaphyllou, 2000). Both approaches aggregate scores given by an expert to each alternative in correspondence with selected criteria into one score, which represents the overall performance of that alternative. This solution approach allows ranking of the alternatives, with the most preferred one ranked at the top.

4.1.1 Definitions

An aggregation operator F is a mapping of $X^n = (x_1, \ldots, x_n)$ with dimension n to one dimension of X, that is:

$$X^n \to X \tag{4.1}$$

The input vector X^n and the output result X here could be linguistic labels and/or crisp numbers. Usually, we also have a weighting vector $W=(w_1, \ldots w_n)$ associated with X^n. Also W could be either linguistic or numeric values.

4.1.2 Crisp aggregation approaches

This section gives a short review of some common used crisp aggregation approaches.

4.1.2.1 Average Aggregations

For crisp aggregation, the most used approach is average. Even just average, there are a lot of approaches. Bullen et al. (1988) summarized it in a general format as:

$$\mathrm{F}_{\alpha}(\mathrm{x}) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\alpha} \right)^{1/\alpha}, \; x \in I^n \tag{4.2}$$

We can see that with $\alpha = 1$, it is the most popular Arithmetic mean

$$\mathrm{F}_{1}(\mathrm{x}) = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{4.3}$$

Also given $\alpha \to 0$ or $\alpha \to -1$, we can get the Geometric mean and Harmonic mean

$$F_0(x) = \sqrt[n]{\prod_{i=1}^{n} x_i} \tag{4.4}$$

$$F_{-1}(x) = \frac{n}{\sum_{i=1}^{n} 1/x_i} \tag{4.5}$$

Klir (1995), Yuan (1991) and Zimmermann and Sebastian (1996) prove that the three averages have the following relation:

$$F_{-1}(x) \le F_0(x) \le F_1(x) \tag{4.6}$$

More general, Bullen et al., (1988) gives another aggregation formula

$$F_\alpha(x) = h^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} h(x_i)\right],\ x \in I^n \tag{4.7}$$

from which we can get (4.2) by substituted $h(x) = x^\alpha$, $\alpha \in R, \alpha \ne 0$. By adding weights, the formula can be:

$$Fw_\alpha(x) = h^{-1}\left[\sum_{i=1}^{n} w_i h(x_i)\right],\ x \in I^n \tag{4.8}$$

Then, the Arithmetic mean, Geometric mean and Harmonic mean changed to:

$$F_1(x) = \sum_{i=1}^{n} w_i x_i \tag{4.9}$$

$$Fw_0(x) = \prod_{i=1}^{n} x_i^{\ w_i} \tag{4.10}$$

$$Fw_{-1}(x) = \frac{1}{\sum_{i=1}^{n} w_i / x_i} \tag{4.11}$$

Quasi-linear cannot adequately model interaction (positive or negative) between criteria. To represent such interactions, weights are substituted with a non-additive set function that permits weights on each subset of criteria.

4.1.2.2 OWA Operator

Yager developed ordered weighted average (OWA) operator in 1983. The OWA operator is defined as:

$$F(x_1, x_2, ..., x_n) = w_1 b_1 + w_2 b_2 + ... + w_n b_n \tag{4.12}$$

Where b_i is the i^{th} largest element in the collection of $x_1, x_2, \ldots, x_n$.

By choosing different weighting vectors, OWA gets different aggregated result. The range the OWA covers varies from Min to Max (Figure 4.1). For example, the following three particular weight vectors generate Min, Arithmetic average, and Max operators respectively.

$W^*=(0, 0, \ldots, 0, 1)^T$

$$F_*(x_1, x_2, ..., x_n) = Min_j(x_j) \tag{4.13}$$

$W^*=(1/n, 1/n, \ldots, 1/n, 1/n)^T$

$$F'(x_1, x_2, ..., x_n) = \frac{1}{n}\sum_n x_j \tag{4.14}$$

$W^*=(1, 0, \ldots, 0, 0)^T$

$$F^*(x_1, x_2, ..., x_n) = Max_j(x_j) \tag{4.15}$$

Yager introduced another *orness* measure for OWA operator:

$$orness(W) = \frac{1}{n-1}\sum_{i=1}^{n}\left[(n-i)\times w_i\right] \tag{4.16}$$

The orness value reflects the degree of optimism of decision makers. The larger the orness, the more optimistic the decision makers are. OWA operators allow us through an appropriate selection of parameters, the so-called OWA weights, to model any degree of *orness* between 0 (corresponding to a pure and) and 1 (corresponding to a pure or). Since min and max evaluate the quantifiers $\forall$ (for all) and $\exists$ (at least one), respectively, the OWA operators essentially extend the space of quantifiers from the pair $\{\forall, \exists\}$ to the interval $[\forall, \exists]$. For example,

$$orness([0\,0\,...\,1]) = 0$$

$$orness([1/n \;\; 1/n \,...\, 1/n]) = 0.5$$

$$orness([1\,0\,...\,0]) = 1$$

Also, for symmetric OWA operators with property of $w_{n-j+1} = w_j$: α=0.5.

The OWA operator has four important properties, which makes it considered as a mean operator (Yager, 1988, 1996). These properties are: Commutativity, Monotonicity, Idempotency and Boundedness.

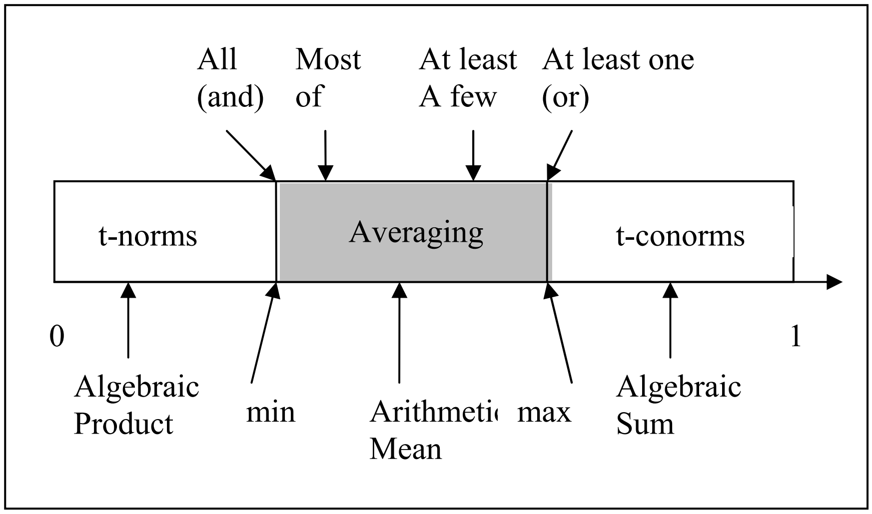


Figure 4.1 The OWA aggregation operators

A key step of this aggregation is the re-ordering of the arguments a_i in descending order so that the weight w_j is associated with the ordered position of the argument. The weight itself can represent the importance of the criteria or the effect of a linguistic quantifier as described next.

There are numerous aggregation operators based on OWA, *Neat* OWA (Marimin et al., 1998) is one of them. The main idea of the *Neat* OWA operator is that the weights are directly deduced from the values to be aggregated, such that:

$$w_i = \frac{x_i^{\alpha}}{\sum_i x_i^{\alpha}} \tag{4.17}$$

and then

$$F(x_1, x_2, ..., x_n) = \frac{\sum_i x_i^{\alpha+1}}{\sum_i x_i^{\alpha}} \tag{4.18}$$

Thus, by using this kind of weight function, the weights are directly deduced from the values to be aggregated. When α = 0.0 we get a simple average operator. When α approach infinity, we get a maximum operator. This is an *orlike* operator, which can also be used to aggregate the preference values.

4.1.2.3 T-norm, S-norm based Aggregation

As we summarized in chapter 2, T-norm and S-norm can be used as the aggregation operators. For instance, Calvo and Mesiar (2003) propose several weighted triangular norms-based aggregation operators.

1) Weighted T-conorm (S-norm) S_w aggregation operator:

$$S_w(x_1,...,x_n) = S(w_1 \cdot x_1,...,w_n \cdot x_n) \tag{4.19}$$

$$w \cdot x = \sup(y \in [0,1] \mid \exists \text{ i, j} \in \text{N}, \frac{i}{j} < \text{w and u} \in [0,1] \text{ such that } S(\overbrace{u,...,u}^{j-times}) < x \text{ } and \text{ y} = S(\overbrace{u,...,u}^{i-times}) \tag{4.20}$$

2) Weighted T-norm T_w aggregation operator (Dual operator to T-connorm):

$$T_w(x_1,...,x_n) = 1 - S_w(1-x_1,...,1-x_n) \tag{4.21}$$

3) Weighted triangular norms-based aggregation operators

$$A_w(x_1,...,x_n) = H(T_w(x_1,...,x_n), S_w(x_1,...,x_n)) \tag{4.22}$$

Where, T is a continuous t-norm, S is continuous (not necessarily dual of T) and H: $[0, 1]^2 \rightarrow [0, 1]$ is a binary aggregation operator.

4) Weighted *Uninorms*

Uninorms are associative, symmetric aggregation operators with neutral element e $\in$ [0, 1]. No *Uninorm* is continuous. Each *Uninorm* U can be related to a t-norm T and a t-conorm S such that ***e*** is an idempotent element of both T and S.

$$U_w(x_1,...,x_n) = U(T_w(\min(x_1,e),...,\min(x_n,e)), S_w(\max(x_1,e),...,\max(x_n,e))) \quad (4.23)$$

5) Weighted *Nullnorms* and some other t-norms-based aggregation operators

Nullnorms were associative symmetric aggregation operators with annihilator *a* $\in$[0, 1] such that 0 is neutral element for inputs from [0, a] and 1 is neutral element for inputs from [a, 1]. An aggregation operator V is a *Nullnorm* with annihilator a if and only if there is a t-norm T and a t-conorm S.

$$V_w(x_1,...,x_n) = med_a(T_w(x_1,...,x_n), S_w(x_1,...,x_n)) \quad (4.24)$$

$$V_w(x_1,...,x_n) = V(x_{m1},...,x_{mk}) = med(a, \min(x_i \mid w_i > 0), \max(x_i \mid w_i > 0)) \quad (4.25)$$

4.1.2.4 Weighted Median Aggregation

Weighted median aggregation was proposed by Yager in 1994. Assuming that $x \in [0,1]^n$ is sorted in some manner, the elements with a low importance are removed from the middle positions. Each term x_i is replaced by two elements x_i^+ and x_i^-.

$$x_i^+ = (1 - w_i) + w_i \cdot x_i \quad (4.26)$$

$$x_i^- = w_i \cdot x_i \quad (4.27)$$

Where $w_i \in [0,1]$ are weights associated with $x_i, i = 1,\ldots,n$ Hence, $x_i^+ - x_i^- = 1 - w_i$ and if the importance w_i is low then $(1-w_i)$ is higher and x_i^+ *and* x_i^- are more separated.

Hence, they are alternately on the top and bottom of the ordering, in order not to affect median selection.

The following general transformation can be also used:

$$x_i^+ = S(1 - w_i, x_i) \tag{4.28}$$

$$x_i^- = T(w_i, x_i) \tag{4.29}$$

Where S and T are t-conorm and t-norm, respectively. The weighted median is then computed as:

$$\text{med}_{\text{w}} = Median\,[x_1^+, x_1^-, x_2^+, x_2^-, \ldots, x_n^+, x_n^-] \tag{4.30}$$

4.1.2.5 Leximin Ordering Aggregation

Leximin ordering proposed by Dubois et al. (1996) provides a method to sort n-tuples of rankings. Leximin ordering, it can be expressed as follows:

$$\text{x} >_{\text{Leximin}} \text{y iff } \exists\, \text{k} \geq 1 \text{ such that } \forall\, \text{i} > \text{k}, \text{x}_\text{i} = \text{y}_\text{i} \text{ and } \text{x}_\text{k} > \text{y}_\text{k} \tag{4.31}$$

$$\text{x} =_{\text{Leximin}} \text{y if } \text{x}_\text{i} = \text{y}_\text{i}\ \forall\, \text{i} = 1,\ldots\text{n} \tag{4.32}$$

This method is a comparison between two alternatives with respect to their lowest scores on any criteria. The alternative with the highest “lowest” score is preferred. If the lowest scores are equal, the second lowest are compared, and so forth, until either one alternative is found to be superior, or until the criteria are exhausted. In this case, a “tie” is declared.

Yager (1997) proposed an analytic representation of Leximin ordering based on OWA weights. Let Δ denote a distinction threshold between the values being aggregated. That is, $|a - b| < \Delta$ is perceived as a = b. Then,

$$w_1 = \frac{\Delta^{(n-1)}}{(1+\Delta)^{(n-1)}} \tag{4.33}$$

$$w_j = \frac{\Delta^{(n-j)}}{(1+\Delta)^{(n+1-j)}} \quad \text{for } j = 2,...n \tag{4.34}$$

Thus,

$$F_{leximin}(x) = \sum_{i=1}^{n} w_i b_i, \quad x \in I^n \tag{4.35}$$

Where b is a sorted n-tuple of scores.

The aggregation in fuzzy domain is more complicated than that in crisp. Generally, there are two main approaches to aggregate linguistic labels in group decision making. Most methods use the associated membership functions. Among them, Baas and Kwakernaak's rating algorithm (1977) aggregates fuzzy scores and weights at differentα-cut levels with their associated membership functions. Chen and Hwang (1989) present a conversion scales approach to transform the linguistic expression into fuzzy numbers attribute by attribute. They give eight conversion scales and find a scale from the pool contains all linguistic terms with the principle that the scale should be as simple as possible. By assigning crisp scores to fuzzy numbers, they then apply classical MCDM method or TOPSIS method. After Yager introduced the OWA operator in (1988), a lot of aggregation researches have been done by applying OWA and linguistic quantifiers, such as (Yager, 1993a, 1994, 1995), Kacprzyk et al. (1992b).

The other category is to calculate linguistic labels directly. Defined in Herrera and Herrera-Viedma (1997) and Herrera et al. (1996a), Linguistic OWA (LOWA) is based on the OWA (Yager, 1983) and the convex combination of linguistic labels (Delgado et al., 1993). The idea is that the combination resulting from two linguistic labels should be itself an element in the set S. So, given si, sj∈S and i, j ∈[0, T], the LOWA method finds an index k in the set S representing a single resulting label. Another FLOWA method also based on OWA is from Ben-Arieh and Chen (2004). This model assigns membership functions to all linguistic labels in S by linearly spread the weights from the labels to be aggregated. The aggregating result is not a single label in S, but a fuzzy set with degrees to each label in S. A 2-tuple Fuzzy linguistic representation model based on the symbolic translation is introduced by Herrera and Martinez (2000). A linguistic 2-tuple (s, α) is used where s is a linguistic term and α is a numeric value representing the symbolic translation. A new approach to extend different classical aggregation operators with the 2-tuple linguistic model is developed. When both scores and weights are not crisp numbers, Yager (1998a) uses the fuzzy modeling technology to develop a model for the inclusion of importance in OWA aggregations. The so-called IOWA method suggested involves a transformation of the scores to be aggregated by their respective importance.

In the following two sections, we will introduce these aggregation approaches in detail.

4.2 Using associated membership functions

Most methods use the associated membership functions to aggregate linguistic labels. Such methods include Baas and Kwakernaak's *Rating and ranking algorithm* (1977), and the Fuzzy Compromise Programming (Prodanovic and Simonovic, 2003) among others. The vagueness of results increases step by step and the shape of membership functions do not keep when the linguistic labels are interactive.

4.2.1 Rating and Ranking Aggregation Method

Baas and Kwakernaak's rating algorithm (1977) assumed that all the alternatives in the choice set can be characterized by a number of properties (criteria), and that information is available to assign weights to these properties. The method basically consists of computing fuzzy weighted ratings for each alternative and comparing these ratings. This method aggregates fuzzy scores and fuzzy weights at different α-cut levels with their associated membership functions. For each α-cut, we determine the maximum and the minimum of the weighted scores. By gathering all α-cuts, one can get the final membership function of the aggregated result. More specifically the steps that are performed for each expert are:

1) For each pair of alternatives get the opinions x_{ij} and weights w_j

2) For α-cut from 0 to 1, calculate U as follows:

$$U = \frac{\sum_{j=1}^{n} w_j x_j}{\sum_{j=1}^{n} w_j} \tag{4.36}$$

3) For each α-cut, determine the maximum and the minimum values of U

4) For each alternative, gathering all α-cuts, we get the final membership function.

Example:

For an alternative, two experts have evaluations using linguistic labels (Table 4.1)

Table 4.1 : Evaluations from two experts using linguistic labels

Expert	Weighting of expert ($\tilde{w}_j$)	Rating of alternative ($\tilde{x}_j$)
E_1	Medium	Very Good
E_2	Very Important	Good

The membership functions associated with the four linguistic labels are defined as shown in Figures 4.2, and 4.3.

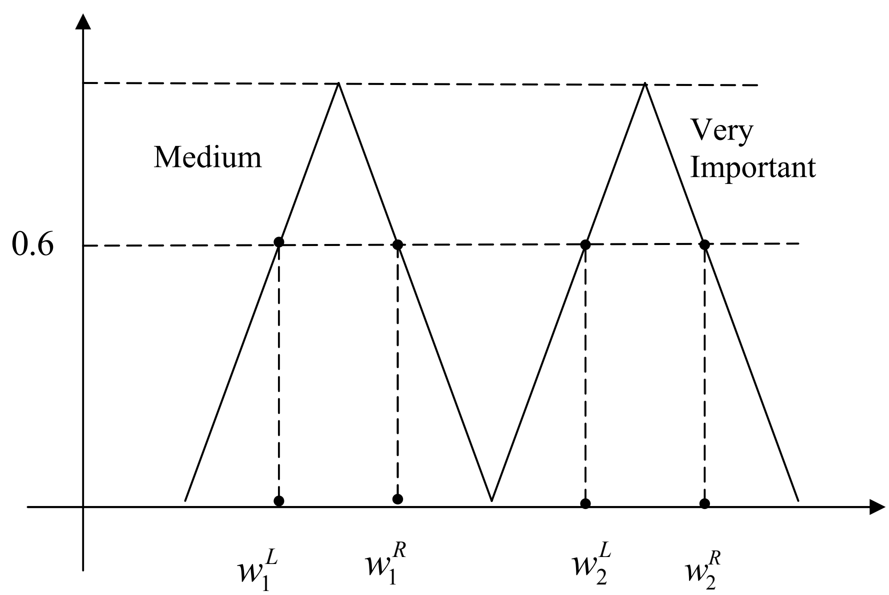


Figure 4.2 Membership values of weights

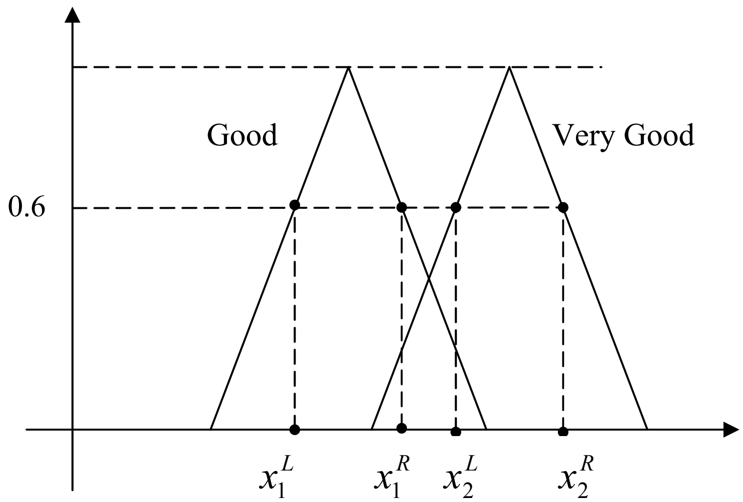


Figure 4.3 Membership values of opinions

Since there are two pairs of (w_1, x_1) and (w_2, x_2), we have $(w_1^L, w_1^R, x_1^L, x_1^R)$ and $(w_2^L, w_2^R, x_2^L, x_2^R)$.so there are 16 U values from:

$$U_1 = (w_1^L x_1^L + w_2^L x_2^L)/(w_1^L + w_2^L)$$

$$U_2 = (w_1^L x_1^L + w_2^R x_2^L)/(w_1^L + w_2^R)\ldots$$

$$U_{16} = (w_1^R x_1^R + w_2^R x_2^R)/(w_1^R + w_2^R)$$

The last step is to choose the minimum and the maximum from these values for each α-cut level.

The aggregated score of both experts considering their importance is depicted in Figure 4.4. The figure highlights the point of α =0.6.

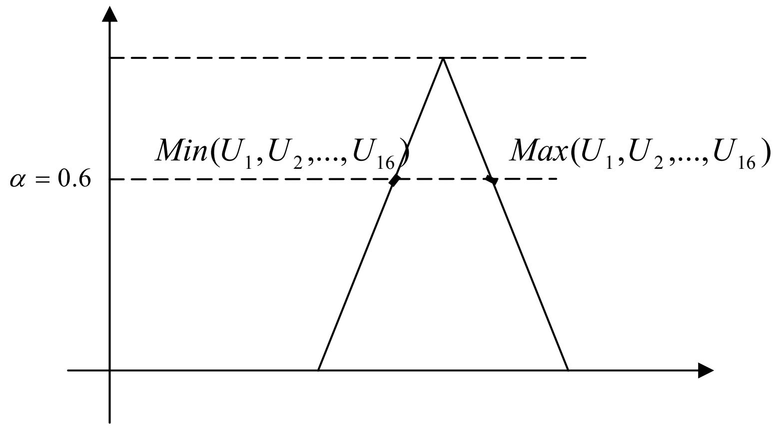


Figure 4.4 The aggregation result is a triangular fuzzy number

4.2.2 Fuzzy Compromise Programming

Using Fuzzy Compromise Programming, the decision maker evaluates each alternative according to its distance from an ideal value (Prodanovic and Simonovic, 2003). Thus, for each alternative, the decision maker sums the distances of each criterion from the ideal value, as shown in Figure 4.5. This sum represents the value of the alternative – and is used to compare the alternatives. The distance calculation can be Euclidean or more generally presented as:

$$D_j = \left[\sum_{z=1}^{t} \left\{ W_s^p \left(\frac{f_z^* - f_z}{f_z^* - f_z^-} \right)^p \right\} \right]^{\frac{1}{p}} \tag{4.37}$$

The equation represents the distance for alternative j using t criteria $z=1,..,t$. The value f* and f⁻ represent the positive and negative ideal values while f_z is the actual value of criterion z. The weight of each criterion is represented by w_z. while p is a parameter.

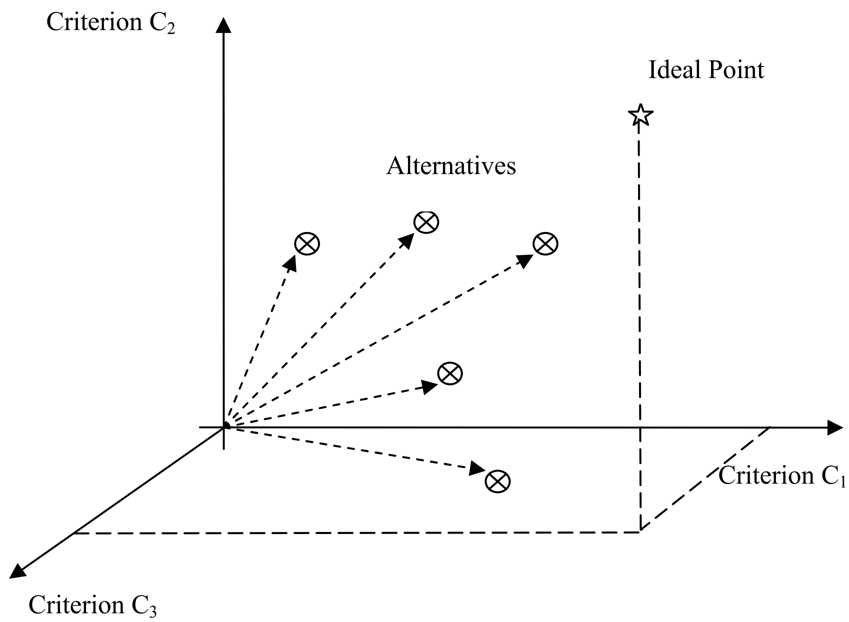


Figure 4.5 An illustration of compromise programming

Fuzzy Compromise Programming considers all input parameters as fuzzy sets, not just criteria values. This approach benefits from using fuzzy sets in representation of the various parameters which ensures that the model uses as much of the relevant information as possible. The more certain the expert is in a particular parameter value, the less fuzziness is assigned to the fuzzy number resulting in a more focused solution. The downside of this approach is that the distance measures are also fuzzy, also requiring a heavy computational load, again based on α-cut values.

4.2.3 Conversion Scales Approach

Chen and Hwang (1992) present a conversion scales approach to transform the linguistic expression into fuzzy numbers attribute by attribute. They give eight conversion scales and find a scale from the pool contains all linguistic terms with the principle that the scale should be as simple as possible.

First, the approach assigns crisp scores to fuzzy numbers.

$$\mu_T(M) = \frac{[\mu_R(M) + 1 - \mu_L(M)]}{2} \tag{4.38}$$

$$\mu_R(M) = \sup_x [\mu_M(x) \wedge \mu_{\max}(x)] \tag{4.39}$$

$$\mu_L(M) = \sup_x [\mu_M(x) \wedge \mu_{\min}(x)] \tag{4.40}$$

$$\mu_{\max}(x) = \begin{cases} x, & 0 \le \mathrm{x} \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.41}$$

$$\mu_{\min}(x) = \begin{cases} 1 - x, & 0 \le \mathrm{x} \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.42}$$

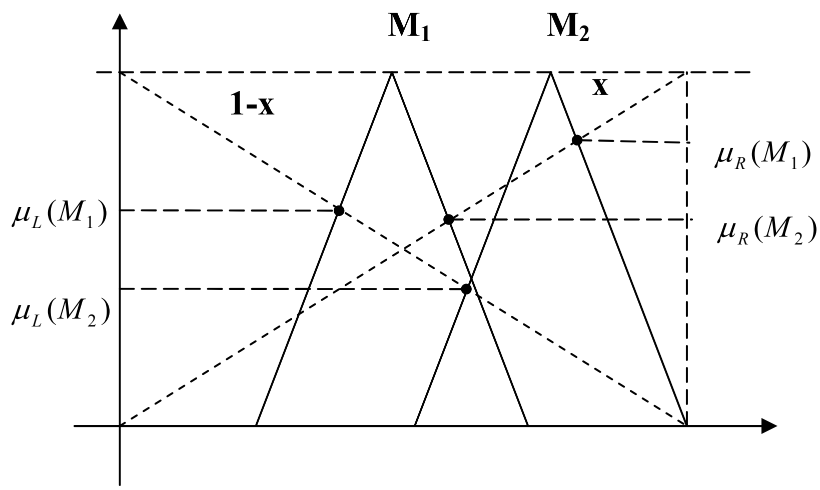


Figure 4.6 Conversion Scales Approach

Then apply classical MADA method or TOPSIS (Chen, 2000; Chu, 2002; Chu and Lin, 2003; Shih et al., 2001) method. The collective performance values are computed by:

$$\alpha_k^i = f(\alpha_k^{i1}, \alpha_k^{i2}, \ldots, \alpha_k^{im}) \tag{4.43}$$

where f is an "aggregation function", then

$$r_i = (\alpha_0^i, \alpha_1^i, ..., \alpha_T^i) \tag{4.44}$$

therefore, the scores for all alternatives are now:

$$\{r_1, r_2, ..., r_n) \tag{4.45}$$

4.3 Direct computations on labels

Aggregation using associated membership functions sometimes could be very complicated. For instance, even with triangular membership functions, aggregating two linguistic labels with one α-cut level, there are 16 combinations. In this section, we are going to introduce approaches aggregating linguistic labels directly, without considering their membership functions.

4.3.1 Linguistic OWA (LOWA) Operator

Defined in Herrera and Herrera-Viedma (1997) and Herrera et al. (1996a), this method is based on the OWA (Yager, 1988) and the *convex combination of linguistic labels* (Delgado et. al., 1993). The idea is that the combination resulting from two linguistic labels should be itself an element in the set S. So, given s_i, $s_j \in S$ and i, j $\in$[0, T], the LOWA method finds an index k in the set S representing a single resulting label.

$$\begin{aligned} &\varphi\{a_1, ..., a_m\} \\ &\quad = W \cdot B^T = C^m\{w_k, b_k, k = 1, ..., m\} \\ &\quad = w_1 \otimes b_1 \oplus (1 - w_1) \otimes C^{m-1}\{\beta_h, b_h, h = 2, ..., m\} \end{aligned} \tag{4.46}$$

Where $W = [w_1, ..., w_m]$, is a weighting vector, such that,

- $w_i \in [0,1]$

- $\sum_i w_i = 1$

$\beta_h = w_h / \sum_2^m w_k, h = 2,...,m$ and $B = \{b_1,...,b_m\}$ is a vector associated to X, such that,

$$B = \sigma(X) = \{a_{\sigma(1)},..., a_{\sigma(m)}\} \tag{4.47}$$

Where,

$$a_{\sigma(j)} \leq a_{\sigma(i)} \quad \forall i \leq j \tag{4.48}$$

with σ being a permutation over the set of labels X. C^m is the convex combination operator of m labels and if m=2, then it is defined as:

$$\begin{aligned} &C^2\{w_i, b_i, i = 1,2\} \\ &\quad = w_1 \otimes s_j \oplus (1 - w_1) \otimes s_i = s_k \quad \mathrm{s_i, s_j \in S, (j \geq i)} \end{aligned} \tag{4.49}$$

$$\mathrm{k = \min\{T, i + round(w_1 \cdot (j\text{-}i))\}} \tag{4.50}$$

Where *round* is the usual round operation, and $b_1=s_j$, $b_2=s_i$.

As an example of this approach three experts E_1, E_2 and E_3 are evaluating an alternative A. Each chooses a linguistic label from the set S to express his/her opinion. Let us use the same nine linguistic labels set defined above S= {I, EU, VLC, SC, IM, MC, ML, EL, C}. Suppose the labels that the experts choose are X= {s_1, s_5, s_7}. The aggregate value of these three linguistic labels is the score of the alternative under considerations. Also, the experts have weights of $w_5 = 0.5$, $w_1 = 0.125$, and $w_7 = 0.375$. Using the LOWA algorithm the aggregated opinion of the three experts is simply s_4.

The LOWA operator does not consider membership functions associated with the linguistic labels and the combination result is a single element in the linguistic label set S. Thus, it is easy to order the result. The LOWA has another obvious problem by using the round w_1*(j-i). When this value is around 0.5, varying the experts' weights even slightly can result in a totally different linguistic label thus a different solution. For example, assume that there are nine linguistic labels $S = \{s_0, s_1, s_2, \ldots, s_8\}$ where T=8, and we need to combine two linguistic labels s_1 and s_5 with the weights given as 0.6252 and 0.3748 respectively. Applying the LOWA method, with i=1, j=5, w_i=0.6252 and w_j=0.3748 produces: $k = \min\{8, 1 + \text{round}(0.3748 \times (5 - 1))\} = \min\{8, 1 + \text{round}(1.499)\} = 2$. So the result is s_2.

Now if we change the weights of w_j from 0.3748 to 0.3753. The result is: $k = \min\{8, 1 + \text{round}(0.3753 \times (5 - 1))\} = \min\{8, 1 + \text{round}(1.501)\} = 3$. Thus, the aggregation result is s_3. The example shows that the LOWA method loses a great amount of useful information in the aggregation process.

4.3.2 2-tuple OWA

Problem of the LOWA method is the loss of information caused by the need to express the results in the initial expression domain that is discrete via an approximate process. This loss of information implies a lack of precision in the final results from the fusion of linguistic information.

A 2-tuple Fuzzy linguistic representation model based on the symbolic translation is introduced by Herrera and Martinez (2000). A linguistic 2-tuple (s, α) is used where s is a linguistic term and α is a numeric value representing the symbolic translation.

Let $S = \{s_0, s_1, ..., s_T\}$ be a linguistic term set, and β be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set S, i.e., the result of a symbolic aggregation operation $\beta \in [0, T]$, being T+1 the cardinality of S. Let i=round(β) and $\alpha = \beta - i$ be two values such that $i \in [0, T]$ and $\alpha \in [-0.5, 0.5]$ then α is called a *symbolic translation*.

The symbolic translation of a linguistic term, s_i , is a numerical value assessed in [-0.5, 0.5) that supports the "difference of information" between a counting of information $\beta \in [0, T]$ obtained after a symbolic aggregation operation and the closest value in {0, 1, …,T} that indicates the index of the closest linguistic term in S.

From this concept, a linguistic representation model which represents the linguistic information by means of 2-tuples (s_i, α_i), and $\alpha_i \in [-0.5, 0.5)$: s_i represents the linguistic label center of the information; α_i is a numerical value expressing the value of the translation from the original result β to the closest index label, i, in the linguistic term set (s_i,), i.e., the symbolic translation.

The linguistic information will be expressed by means of 2-tuples, which are composed by a linguistic term and a numeric value assessed in [0.5, 0.5). Using the same example in LOWA, which three experts E_1, E_2 and E_3 evaluate an alternative A. The linguistic labels set is defined as S= {I, EU, VLC, SC, IM, MC, ML, EL, C}. The experts choose X= {s_1, s_5, s_7} as their evaluations. Experts have weights $w_1 = 0.125$, $w_5 = 0.5$, and $w_7 = 0.375$. Then similar to LOWA β=5.125, i=5, and α=0.125. So the aggregation result is (s_5, 0.125).

This model allows a continuous representation of the linguistic information on its domain, therefore, it can represent any counting of information obtained in a aggregation process.

Together with the 2-tuple representation model, the authors introduce several 2-tuple aggregation operators that are based on classical aggregation operators without any loss of information.

4.3.2.1 Arithmetic Mean

Let $\mathrm{x} = \{(\mathrm{s}_1, \alpha_1),...,(\mathrm{s}_\mathrm{n}, \alpha_\mathrm{n})\}$ be a set of 2-tuples, the 2-tuple arithmetic mean $\overline{\mathrm{x}}^\mathrm{e}$ is computed as:

$$\overline{\mathrm{x}}^\mathrm{e} = \Delta(\sum_{i=1}^{n} \frac{1}{n} \Delta^{-1}(s_i, \alpha_i)) = \Delta(\frac{1}{n} \sum_{i=1}^{n} \beta_i) \tag{4.51}$$

Where $\Delta \, and \, \Delta^{-1}$ are defined as:

$$\Delta : [0, g] \rightarrow s \times [-0.5, 0.5) \tag{4.52}$$

$$\Delta(\beta) = (s_i, \alpha), with \begin{cases} s_i, & i = round(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5) \end{cases} \tag{4.53}$$

$$\Delta : s \times [-0.5, 0.5) \rightarrow [0, g] \tag{4.54}$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta \tag{4.55}$$

The arithmetic mean for 2-tuples allows us to compute the mean of a set of linguistic values without any loss of information.

4.3.2.2 Weighted Average Operator

The weighted average allows different values x_i have a different importance in the nature of the variable x. To do so, each value $x_i = (s_i, \alpha_i)$ has a weight associated w_i indicating its importance in the nature of the variable. The equivalent operator for linguistic 2-tuples is defined as:

$$\overline{x}^e = \Delta\left(\frac{\sum_{i=1}^{n} \Delta^{-1}(s_i, \alpha_i) \cdot w_i}{\sum_{i=1}^{n} w_i}\right) = \Delta\left(\frac{\sum_{i=1}^{n} \beta_i \cdot w_i}{\sum_{i=1}^{n} w_i}\right) \tag{4.56}$$

4.3.2.3 Ordered Weighted Aggregation (OWA) Operator

Combined with Yager's OWA operator (1988), the 2-tuple OWA operator for linguistic 2-tuples is computed as:

$$F^e = \Delta\left(\sum_{j=1}^{n} \beta_j^* \cdot w_j\right) \tag{4.57}$$

Where β_j^* is the jth largest of the β_j values. We can see that the result of the aggregation of a set of 2-tuples is also a 2-tuple. And aggregation operators to deal with the 2-tuple linguistic models are extended.

Example:

Three experts E_1, E_2 and E_3 evaluate an alternative A. The linguistic labels set is defined as S= {I, EU, VLC, SC, IM, MC, ML, EL, C}. The experts evaluations are X= {(s_1, 0.12), (s_5, -0.1), (s_7, 0.2)}. Experts have weights $w_1 = 0.125$, $w_5 = 0.5$,and $w_7 = 0.375$. Then the α values are 0.12, -0.1, and 0.2. β values are 1.12, 4.9, and 7.2 respectively. Substitute the β values into (4.51), we get the 2-tuple with arithmetic mean:

$$\overline{x}^e = \Delta(\frac{1}{n}\sum_{i=1}^{n}\beta_i) = \Delta(\frac{1}{3}(1.12 + 4.9 + 7.2)) = \Delta(4.03)$$

Similarly, 2-tuple with weighted average, and 2-tuple with OWA can be calculated as the followings. We summarize the three 2-tuple aggregation results in Table 4.2.

$$\overline{x}^e = \Delta\left(\frac{\sum_{i=1}^{n}\beta_i \cdot w_i}{\sum_{i=1}^{n} w_i}\right) = \Delta\left(\frac{1.12 \times 0.125 + 4.9 \times 0.5 + 7.2 \times 0.375}{0.125 + 0.5 + 0.375}\right) = \Delta(5.29)$$

$$F^e = \Delta\left(\sum_{j=1}^{n}\beta_j^* \cdot w_j\right) = \Delta\left(7.2 \times 0.125 + 4.9 \times 0.5 + 1.12 \times 0.375\right) = \Delta(3.77)$$

Table 4.2 Aggregation results with different 2-tuple functions

Method	2-tuple with Arithmetic Mean	2-tuple with Weighted Average	2-tuple with OWA
Aggregation Result	(s_4, 0.03)	(s_5, 0.29)	(s_4, -0.23)

4.3.3 IOWA

When both scores and weights are not crisp numbers, Yager (1998a, 2003) uses the fuzzy modeling technology to develop a model for the inclusion of importance in OWA aggregations. The so-called IOWA method suggested involves a transformation of the scores to be aggregated by their respective importance. This is another approach to the inclusion of importance in the OWA operator aggregation technique. The IOWA is defined as:

$$a^* = F(b_1, b_2, \dots b_m) \tag{4.58}$$

Where

$$a^* = F(w_j, b_j) \tag{4.59}$$

$$b_j = G(u_j, x_j) \tag{4.60}$$

$$G(u,x) = \frac{high(\alpha)G_{\max}(u,x) + medium(\alpha)G_{\max}(u,x) + low(\alpha)G_{\min}(u,x)}{high(\alpha) + medium(\alpha) + low(\alpha)} \tag{4.61}$$

Where x_i is the scores and u_i is the importance of the i^{th} objective.

In this case, a_j is the scores and u_j is the importance of the j^{th} criteria. α is the *orness* used to measure the degree of optimism of a decision maker (Yager, 1988). G*(u, a*) is a transformation function that depends on the following rules defined by the decision maker or analyst.

If the degree of *orness* is *high,* G(*u, a*) is $G_{max}(u, a)$,

If the degree of *orness* is *medium,* G(*u, a*) is $G_{avg}(u, a)$,

If the degree of *orness* is *low,* G(*u, a*) is $G_{min}(u, a)$,

Example (Yager, 1998a)

Given the *high, medium and low* as the functions of *orness* α, and assume *High(*α *)+Medium(*α *)+Low(*α *)=1,*we have

$$G(u,a) = \frac{high(\alpha)G_{\max}(u,a) + medium(\alpha)G_{avgx}(u,a) + low(\alpha)G_{\min}(u,a)}{high(\alpha) + medium(\alpha) + low(\alpha)}$$
$$= high(\alpha)G_{\max}(u,a) + medium(\alpha)G_{avg}(u,a) + low(\alpha)G_{\min}(u,a)$$

By defining

$$G_{max} = u \cdot a, G_{min} = \bar{u} + u \cdot a, G_{avg} = \frac{n}{\sum_{j=1}^{n} u_j} u \cdot a \, ,$$

the final transformation function is:

$$G(u, a) = \begin{cases} (-2\alpha + 1)(1 - u + ua) + 2\alpha(\frac{n}{T} ua) & \alpha \le 0.5 \\ (2\alpha - 1)ua + (2 - 2\alpha)(\frac{n}{T} ua) & \alpha \ge 0.5 \end{cases}$$

Given four criteria with the following score and importance:

(u,a)= (0.7, 0.8), (1, 0.7), (0.5, 1.0), (0.3, 0.9). The weights required for linguistic quantification are: W=(0.4, 0.3, 0.2, 0.1). We can calculate the *orness* to be α=0.67>0.5, then, the final aggregated result is:

a*=F(0.78, 0.98, 0.7, 0.38)= F_W(0.98, 0.78, 0.7, 0.38) =0.98*0.4+0.78*0.3+0.7*0.2+0.38*0.1=0.8

The steps to apply IOWA are:

1) Given weights w_i find Orness α
2) By Orness, Choose G (u, α)
3) By b=G (u, α) and w, find a*

For instance, if we know the definitions of *High, Low* and *Medium*,

$$High(\alpha) = \begin{cases} 2\alpha - 1 & \alpha \ge 0.5 \\ 0 & \alpha \le 0.5 \end{cases}$$

$$\text{Low}(\alpha) = \begin{cases} -2\alpha + 1 & \alpha \le 0.5 \\ 0 & \alpha \ge 0.5 \end{cases}$$

$$\text{Meidum}(\alpha) = \begin{cases} 2\alpha & \alpha \le 0.5 \\ 2 - 2\alpha & \alpha \ge 0.5 \end{cases}$$

as well as the function of G,

$$\text{G(u, x)} = \begin{cases} (-2\alpha + 1)(1 - u + ux) + 2\alpha(\frac{n}{T}ux) & \alpha \le 0.5 \\ (2\alpha - 1)ux + (2 - 2\alpha)(\frac{n}{T}ux) & \alpha \ge 0.5 \end{cases}$$

For example, the vector of (u, a) is given as: (0.7, 0.8), (1, 0.7), (0.5, 1.0), (0.3, 0.9). And W=(0.4, 0.3, 0.2, 0.1), then, we calculate the *orness* level α=0.67>0.4. Then substitute u and α into the formula above, we get G (u, α)=(0.78, 0.98, 0.7, 0.38). So, finally, the aggregation result a*=F(0.78, 0.98, 0.7, 0.38)=F_W(0.98, 0.78, 0.7, 0.38)=0.98*0.4+0.78*0.3+0.7*0.2+0.38*0.1=0.8.

The IOWA method allows aggregation of the scores considering their respective importance. Using this approach and the OWA terminology, the aggregated value of each alternative is defined as the OWA value using modified weights as presented below.

4.3.4 Yager's all/and/min Aggregation

Yager's ME-MCDM evaluation process (1993b) is a two-stage process:

In the first stage, individual experts are asked to provide an evaluation of the alternatives. The evaluation consists of a rating for each alternative on each of the criteria. Each of the criteria may have a different level of importance. The values to be used for the evaluation of the ratings and importance will be drawn from a linguistic scale which makes it easier for the evaluator to provide the information. Second stage, to aggregate the individual expert's evaluations to obtain an overall linguistic value for each object. This overall evaluation can then be used by the decision maker as an aid in the selection process.

Example:

1) Each expert chooses a linguistic term from the following scale S as an evaluation to alternative A_i on the criteria C_j.

Perfect (P)	S_7
Very High (VH)	S_6
High (H)	S_5
Medium (M)	S_4
Low (L)	S_3
Very Low (VL)	S_2
None (N)	S_1

The use of such a scale provides a natural ordering, $S_i>S_j$ if $i>j$. Of primary significance is that the use of such a scale doesn't impose undue burden on the evaluator in that it doesn't impose the meaningless precision of numbers. The scale is essentially a linear ordering and just implies that one score is better than another. The use of linguistic terms associated with these scales makes it easier for the evaluator to manipulate.

2) Aggregation operator

The unit score of each alternative by each expert is defined as:

$$A_{ik} = Min_j[NEG(I(q_j)) \vee A_{ik}(q_j)] \tag{4.62}$$

Where $\vee$ indicates the max operation.

Example:

Given the data in the Table 4.3 as the evaluation from expert E_k to alternative A_i on six criteria.

Table 4.3 Example 1 for Yager's all/and/min Aggregation

Criteria	C_1	C_2	C_3	C_4	C_5	C_6
Importance	P	VH	VH	M	L	L
Score	H	M	L	P	VH	P

The aggregation result is as followings:

$$\begin{aligned} A_{ik} &= Min[NEG(P) \vee H, NEG(VH) \vee M, NEG(VH) \vee L, ..., NEG(L) \vee P] \\ &= Min[N \vee H, VL \vee M, VL \vee L, M \vee P, H \vee VH, H \vee P] \\ &= Min[H, M, L, P, VH, P] \\ &= L \end{aligned}$$

This method has a problem that the least score plays the central role in determining the combination score. Any number of good scores can not make up for a bad score. For example, given the following evaluation in Table 4.4:

Table 4.4 Example 2 for Yager's all/and/min Aggregation

Criteria	C_1	C_2	C_3	C_4	C_5	C_6
Importance	P	P	H	P	P	P
Score	P	P	L	P	P	P

Although all evaluations to five of six criteria are "P", the aggregate result is still determined by the lowest score which is "L" here.

$$
\begin{aligned}
A_{ik} &= Min[NEG(P) \vee P, NEG(P) \vee P, NEG(H) \vee L, ..., NEG(P) \vee P] \\
&= Min[N \vee P, N \vee P, L \vee L, N \vee P, N \vee P, N \vee P] \\
&= Min[P, P, L, P, P, P] \\
&= L
\end{aligned}
$$

Combining experts' opinions

We have q experts and the collection of evaluations $A_{i1}, A_{i2}, ..., A_{iq}$ where A_{ik} is the unit evaluation of the ith alternative by the kth expert.

The first step for the decision maker is to provide an aggregation function which Yager denotes as Q. This function can be seen as a generalization of the idea of how many experts he or she feels need to agree on a project for it to e acceptable. Then $Q(i)$ indicates how satisfied he/she would be in selecting a proposal with which I of the experts were satisfied.

Characteristics of the function Q:

As more experts agree, the decision maker's satisfaction or confidence should increase. $Q(i) \geq Q(j),\ i > j$

If all the experts are satisfied, then his/her satisfaction should be the highest possible. $Q(q) = perfect$

If no experts are satisfied then the satisfaction to Q should be lowest. $Q(0) = none$

One way to calculate the Q is defined as,

$$Q_A(k) = S_b(k) \tag{4.63}$$

Where $b_{(k)} = Int[1 + (k * \frac{l-1}{q})]$, q is the number of experts, l is the number of linguistic labels.

The overall evaluation for the ith alternative is calculated by:

$$A_i = Max_{j=1,\ldots q}[Q(j) \wedge B_j] \quad (4.64)$$

Where B_j is the jth highest score among the experts' unit scores for the alternative and $\wedge$ indicates the max operation.

Yager's Group MCDM evaluation process (1993b) is a two-stage process: In the first stage, individual experts are asked to provide an evaluation of the alternatives. The evaluation consists of a rating for each alternative on each of the criteria. Each of the criteria may have a different level of importance. The values to be used for the evaluation of the ratings and importance will be drawn from a linguistic scale, which makes it easier for the evaluator to provide the information. The second stage performs the aggregation of the individual evaluations to obtain an overall linguistic value for each alternative. Implicit in this linguistic scale are two operators, the maximum and minimum of any two scores as discussed earlier.

The aggregated score of each alternative with *j* criteria is simply defined as:

$$A_{ik} = Min_j[NEG(I(q_j)) \vee A_{ik}(q_j)] \quad (4.65)$$

Where $I(q_j)$ is the importance of criteria *j*, A_{ik} is the opinion of expert k, $\vee$ indicates the max operation. Negation of a linguistic term is calculated as:

$$Neg(S_i) = S_{l-i+1} \quad \text{if } we \text{ have a scale of } l \text{ items} \tag{4.66}$$

4.3.5 Xu's Linguistic Average

Xu (2004) introduced another way to present linguistic set:

$$S = \{s_\alpha \mid \alpha = -t,...,t\} \tag{4.67}$$

1) α is an even number, thus the linguistic set has a odd cardinality.

   $s_\alpha < s_\beta$, $iff\ \alpha < \beta$

2) The negation operator: neg $(s_\alpha) = s_{-\alpha}$, with neg $(s_0) = s_0$, the following aggregation rules are proposed:

$$s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha = s_{\alpha+\beta} \tag{4.68}$$

$$\lambda s_\alpha = s_{\lambda\alpha},\ \lambda \in [0,1] \tag{4.69}$$

$$\lambda(s_\alpha \oplus s_\beta) = \lambda s_\alpha \oplus \lambda s_\beta,\ \lambda \in [0,1] \tag{4.70}$$

Other direct linguistic aggregation methods include Cheng's Adjusted fuzzy rating method(1999), Herrera and Herrera-Viedma's linguistic weighted disjunction (LWD) operator, linguistic weighted conjunction (LWC) operator, and linguistic weighted averaging (LWA) operator (1997). Also, there are some derivative OWA aggregation processes that utilize linguistic quantifiers to generate linguistic rules for aggregation. Such methods include Yager's Weighted Goals (1994), Sugeno's Ordered weighted maximum (OWMAX) and Ordered weighted minimum operators (OWMIN) (1974), and Herrera et al.'s direct approach (1996a) etc.

4.4 Fuzzy Linguistic OWA (FLOWA) Operator

Based on the LOWA method, Ben-Arieh and Chen (2004) present a new aggregation operator denoted as *Fuzzy Linguistic OWA (FLOWA)*.

4.4.1 FLOWA Definition

Figure 4.7 shows the basic concept of the FLOWA. Assume that there are *m* linguistic labels to be aggregated, X= $\{s_i \ldots s_l \ldots s_j\}$ ($i<l<j$), where s_i is the smallest label in X and s_j is the largest one and $X \subseteq S$. Also a weighting vector W= $[w_i \ldots w_l \ldots w_j]$ ($i<l<j$) is associated with the linguistic labels representing the experts' weight. Thus w_l represents the weight of the expert who chooses label *l* as the linguistic representation of his/her preference. And: $w_l \in [0,1]$, and $\sum_l w_l = 1$

In the FLOWA approach the final result should lie between s_i and s_j (including s_i and s_j). Instead of only choosing the s_k in the set of linguistic labels S, we assign membership functions to all the linguistic labels between s_i and s_j. We decreasingly spread the original weight w_i on s_i to the linguistic labels from s_i to s_j. Similarly, we increasingly spread the original weight on s_j to the linguistic labels from s_i to $s_{j.}$. The weight of label *l* which lies between labels i and j is spread decreasingly to the both directions.

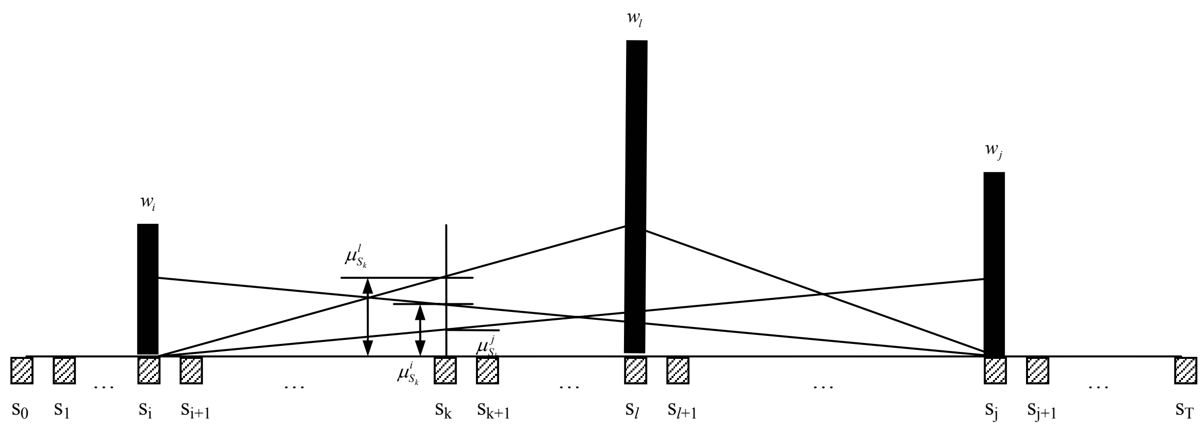


Figure 4.7 The concept of FLOWA

The membership function of the k^th^ label between s_i and s_j in the FLOWA operator F is define as:

$$F : \mu_{s_k} = \sum_{l=0}^{T} \mu_{s_k}^l \tag{4.71}$$

Where, μ_{s_k} is the fuzzy membership assigned to the k^{th} linguistic label s_k after aggregating the weights on label set X= {s_i… s_j} and $\mu_{s_k}^l$ is the membership function of the k^{th} linguistic label s_k, $s_k \in S$ generated from the weighted linguistic label s_l , $s_l \in X$. The $\mu_{s_k}^l$ is defined by:

1) l=i

$$\mu_{s_k}^l = \frac{2(j-k)}{(j-i)(j-i+1)} w_l \tag{4.72}$$

2) l=j

$$\mu_{s_k}^{l} = \frac{2(k-i)}{(j-i)(j-i+1)} w_l \quad (4.73)$$

3) i<l<j

$$\mu_{s_k}^{l} = \begin{cases} \frac{2(k-i)}{(j-i)(l-i)} w_l & \text{for, } i < k \le l \\ \frac{2(j-k)}{(j-i)(j-l)} w_l & \text{for, } l \le k < j \end{cases} \quad (4.74)$$

4) l<i or l>j

$$\mu_{s_k}^{l} = 0 \quad (4.75)$$

Notice that if w_i, w_j>0 and w_l=0, for all $l \neq$ i, j, then equation (4.66) has only two parts for $\mu_{s_k}^{i}$ and $\mu_{s_k}^{j}$ from (4.67) and (4.68) respectively.

Example:

Assume three experts E_1, E_2 and E_3 evaluating an alternative A. Each chooses a linguistic label from the set S to express his/her opinion. Let us use the same nine linguistic labels set defined in (Herrera et al., 1996a) as S= {I, EV, VLC, SC, IM, MC, ML, EL, C}, where s_i<s_j, given i<j and s_i, s_j $\in$S. Suppose the labels that the experts choose are X= {s_1, s_5, s_7}. The aggregate value of these three linguistic labels is the score of the alternative under considerations. Also, the experts have the weights of $w_l = 0.5$, $w_i = 0.125$, and $w_j = 0.375$.

As an example, for k=1:

$$\mu_{s_1} = \mu_{s_1}^{1} + \mu_{s_1}^{5} + \mu_{s_1}^{7} = \frac{2\times(7-1)}{(7-1)\times(7-1+1)}\times 0.125 + \frac{2\times(1-1)}{(7-1)\times(5-1)}\times 0.5 + \frac{2\times(1-1)}{(7-1)\times(7-1+1)}\times 0.375 = 0.0357$$

The final result shown in Figure 4.8 is a fuzzy set {$0/s_0$, $0.0357/s_1$, $0.0893/s_2$, $0.1429/s_3$, $0.1964/s_4$, $0.25/s_5$, $0.1706/s_6$, $0.1071/s_7$, $0/s_8$}. We can see that after the aggregation, the linguistic label s_5=MC has the highest possibility as the aggregation result. In contrast, the aggregation result of the LOWA algorithm is simply s_4.

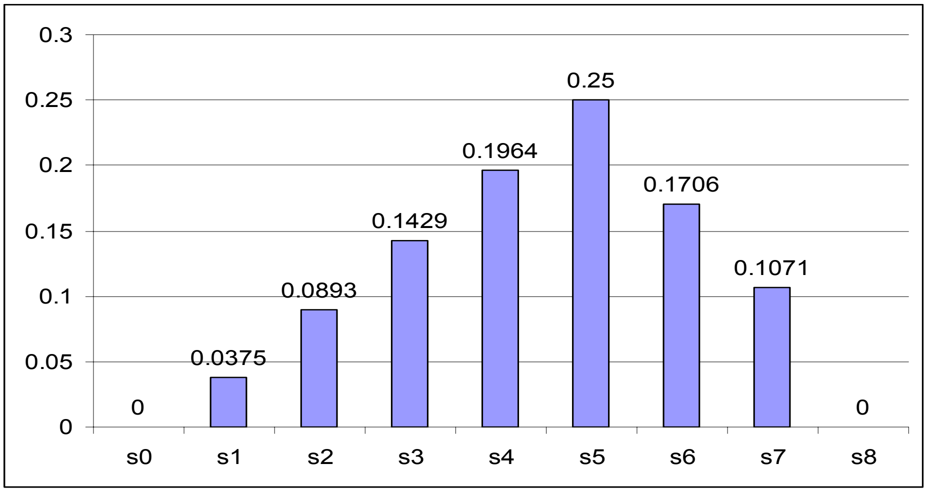


Figure 4.8 Example of aggregating three linguistic labels

4.4.2 Properties of the FLOWA operator

The FLOWA algorithm has several interesting properties:

1) Property 1. The aggregation result is normalized. (Meaning that the membership functions of all the labels in the aggregate sum to one.)

$$\sum_{k=0}^{T} \mu_{s_k} = 1$$

Proof.

We have $\sum_{l=0}^{T} \mu_{s_k}^{l} = \sum_{l=i}^{j} \mu_{s_k}^{l} = w_l$, then

$$\sum_{k=0}^{T}\mu_{s_k} = \sum_{k=0}^{T}\sum_{l=0}^{T}\mu_{s_k}^{l} = \sum_{l=0}^{T}\sum_{k=0}^{T}\mu_{s_k}^{l} = \sum_{l=0}^{T}\sum_{k=i}^{j}\mu_{s_k}^{l} = \sum_{l=0}^{T} w_l = 1$$

Since all the weights $w_l, (0 \le l \le T, s_l \in X)$ are distributed among all the linguistic labels that participate in the solution, and $\sum_{l=0}^{T} w_l = 1, s_l \in X)$, so all the fuzzy memberships should sum to 1.

2) Property 2: Linearity. The aggregation result is a linearly distributed weight between any two linguistic labels to be aggregated, either increasing, decreasing or constant.

Proof.

Without lose of generality, suppose we are going to aggregate three linguistic labels X= {s_i, s_l, s_j} with i<l<j. Then, when k<l, the k[th] linguistic label in S gets the weight by,

$$\mu_{s_k} = \left[\frac{2j}{(j-i)(j-i+1)}w_i - \frac{2i}{(j-i)(l-i)}w_l - \frac{2i}{(j-i)(j-i+1)}w_j\right] + \left[\frac{2(l-i)(w_j - w_i) + 2(j-i+1)w_l}{(j-i)(l-i)(j-i+1)}\right]k$$

Whereas when k>l,

$$\mu_{s_k} = \left[\frac{2j}{(j-i)(j-i+1)}w_i + \frac{2j}{(j-i)(l-i)}w_l - \frac{2i}{(j-i)(j-i+1)}w_j\right] + \left[\frac{2(l-i)(w_j - w_i) - 2(j-i+1)w_l}{(j-i)(l-i)(j-i+1)}\right]k$$

In both cases after aggregation the membership function μ_{s_k} of the k[th] linguistic label in S is a linear function of k.

3) Property 3: The maximum membership after aggregation could only happen in a linguistic label s_k, where $s_k \in X$

Proof.

From property 2, it is obvious that the maximum membership never happens between s_i, s_l and s_j. If X= {s_i, s_l, s_j}, the maximum membership could only happen in s_i, s_j or s_l with the corresponding value of:

$$\mu_{s_i} = \frac{2}{j-i+1} w_i$$

$$\mu_{s_j} = \frac{2}{j-i+1} w_j$$

$$\mu_{s_l} = \frac{2}{j-i} w_l + \frac{2(j-l)w_i + 2(l-i)w_j}{(j-i)(j-i+1)}$$

4) Property 4. The FLOWA operator is commutative in a limited sense.

$$F(s_i ... s_k, s_l ... s_j) = F(F(s_i, s_j), \pi(s_k, s_l))$$

Where π is any permutation over the set of arguments. This property implies that the aggregation process has to start with the two extreme labels. Once this is accomplished, the order of integrating the other labels is immaterial.

<u>Proof:</u>

From equations (4.66), (4.67), and (4.68) it is easy to see that once labels *i* and *j* (the two extremes) are aggregated the contribution of any label *l* and *k* is independent of the order of aggregation.

An example is provided in Figure 4.9 and Table 4.5. This example shows that the FLOWA operator is order dependent in this limited sense. When labels are aggregated without the two extreme labels the results are erroneous as shown. Thus, when we aggregate linguistic labels, the min and max labels need to be aggregated first.

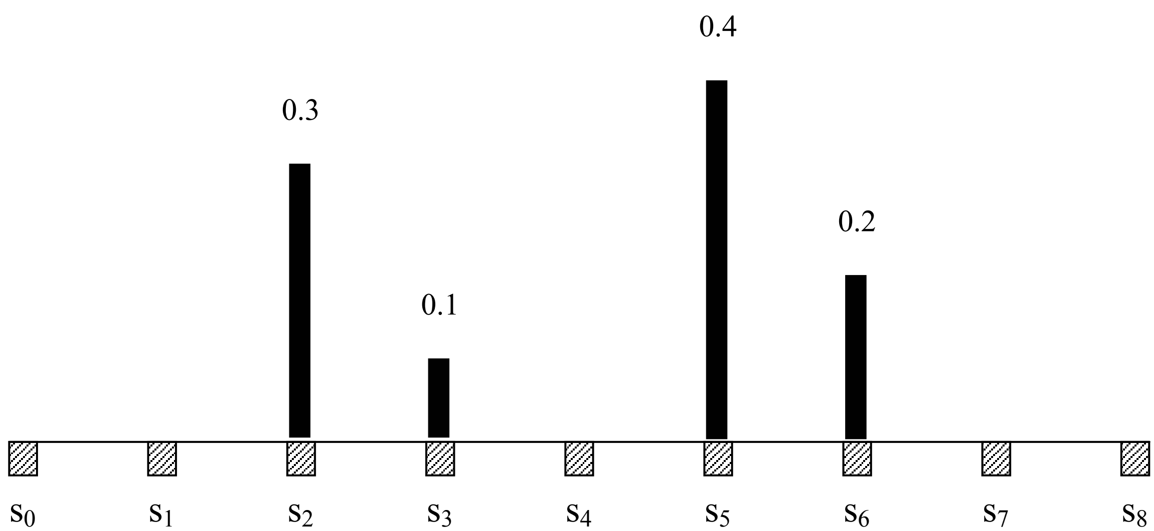


Figure 4.9 An example of FLOWA property 4

Table 4.5 An example of FLOWA property 4

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
$F(s_2, F(s_3, s_5, s_6))$	0	0	0.12	0.14	0.26	0.38	0.1	0	0
$F(s_6, F(s_2, s_3, s_5))$	0	0	0.15	0.25	0.26	0.26	0.08	0	0

5) Property 5: The result of aggregating two adjacent labels is the same two labels.

Proof:

As presented in Figure 4.10 , if j=i+1, by formula (5) ~ (6), $\mu_{s_i}^i = w_i$, $\mu_{s_j}^j = w_j$.

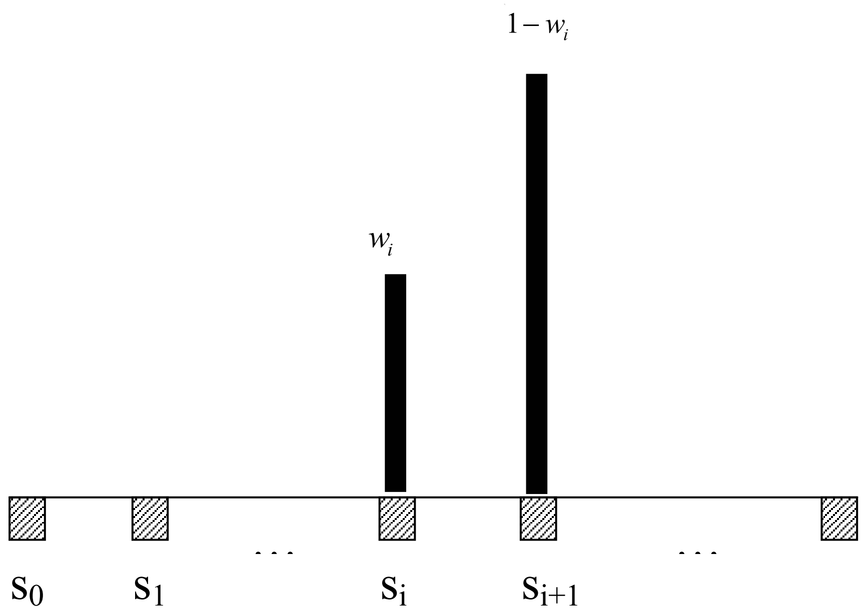


Figure 4.10 The result of aggregating two adjacent labels

4.4.3 Comparison of LOWA and FLOWA

After aggregating *m* linguistic labels X= {s_i… s_j}, any linguistic label between i and j gets a weight as the aggregated result, while the linguistic labels in S with k<i and k>j get weights of 0. This shows that the aggregated result is not another linguistic label, but a set of labels between s_i and s_j each with a membership functions. This membership value represents the degree of confidence in the label.

4.5 Summary

This chapter summarizes the current use of linguistic labels in decision-making and groups them into two categories: computing with associated membership functions and directly with linguistic labels. Then we present a new direct linguistic labels aggregation operator FLOWA for the fuzzy group decision-making problem. The FLOWA method is more detailed and includes more information about the aggregate than existing direct methods.

CHAPTER 5

Consensus and Contribution Measure

There is no one widely accepted definition on consensus. Even dictionary definitions of consensus vary. But all definitions agree that consensus has two common meanings (Web: *wikipedia.org*):

"One is a general agreement among the members of a given group or community. The other is as a theory and practice of getting such agreements. "

Many discussions focus on whether agreement needs to be unanimous; These discussions miss the point of consensus, which is not a voting system but a taking seriously of everyone's input, and a trust in each person's discretion in follow up action. In consensus, people who wish to take up some action want to hear those who oppose it, because they do not wish to impose, and they trust that the ensuing conversation will benefit everyone. Action despite opposition will be rare, and done with attention to minimize damage to relationships. In a sense, consensus simply refers to how any group of people who value liberty might work together.

In this chapter, we will explain the meaning of consensus and introduce state-of-the-art literature review on current existing consensus measure methods which are categorized into two groups, hard and soft consensus. We also present two new consensus measure methods based on the preference order of alternatives and Markov chain theory. We will discuss the method of optimizing the group consensus in this chapter followed by the general guide on how to improve consensus level.

5.1 Introduction

Consensus is traditionally meant as a strict and unanimous agreement of all the experts regarding all possible alternatives. Ness and Hoffman define consensus in (1998) as "*Consensus is a decision that has been reached when most members of the team agree* *on a clear option and the few who oppose it think they have had a reasonable opportunity to influence that choice. All team members agree to support the decision.*"

The expression of concerns and conflicting ideas is considered desirable and important. When a group creates an atmosphere which nurtures and supports disagreement without hostility and fear, it builds a foundation for stronger, more creative decisions. Consensus is viewed as a pathway to a true group decision. Sharing opinions prior to reaching a decision, as is done in jury setting, clearly reduces the effective number of independent voices. In some situations, discussion is disallowed. For example, during figure skating competitions, judges are expressly forbidden from interacting.

It is very rare when all individuals in a group share the same opinion about the alternatives, since a diversity of opinions commonly exists. Consensus makes it possible for a group to reach a final decision that all group members can support among these differing opinions.

5.1.1 Understanding Consensus

Consensus is a process of nonviolent conflict resolution. The expression of concerns and conflicting ideas is considered desirable and important. When a group creates an atmosphere which nurtures and supports disagreement without hostility and fear, it builds a foundation for stronger, more creative decisions.

It would be illustrative here to quote Lower and Laddaga (1985): "…It can correctly be said that there is a consensus among biologists that Darwinian natural selection is an important cause of evolution though there is currently no consensus concerning Gould's hypothesis of specification. This means that there is a widespread agreement among biologists concerning the first matter but disagreement concerning the second…".

An immediate consequence is therefore that from a pragmatic point of view it makes sense to speak about a degree of consensus, or a distance from (ideal) consensus.

Consensus is not a simple matter. It is both a methodology and an aim (Web: *innatenonviolence.org*). It has to be seen as a creative process and not a deadener of initiative, something which stifles projects. It does not have to be the death of initiative, and indeed if it is then something is woefully wrong. Where there is respect then birds of a feather within a larger flock can go ahead with a project without everyone agreeing, or, if need be, under a different hat. How this process happens has to be seen as creative and organic - part of developing the best in humanity and not a power struggle for the heart and soul of an organization. Because if it comes to the latter then the heart is already lost and the soul will likely follow.

Once a decision has been adopted by consensus, it cannot be changed without reaching a new consensus. If a new consensus cannot be reached, the old decision stands.

Each individual is responsible for expressing one's own concerns. It is best if each concern is expressed as if it will be resolved. The group then responds by trying to resolve the concern through group discussion. If the concern remains unresolved after a full and open discussion, then the facilitator asks how the concern is based upon the foundation of the group. If it is, then the group accepts that the proposal is blocked.

From this perspective, it is not decided by the individual alone if a particular concern is blocking consensus; it is determined in cooperation with the whole group. The group determines a concern's legitimacy. A concern is legitimate if it is based upon the principles of the group and therefore relevant to the group as a whole. If the concern is determined to be unprincipled or not of consequence, the group can decide the concern is inappropriate and drop it from discussion. If a reasonable solution offered is not accepted by the individual, the group may decide the concern has been resolved and the individual is out of order for failure to recognize it.

Herein lies a subtle pitfall. For consensus to work well, it is helpful for individuals to recognize the group's involvement in determining which concerns are able to be resolved, which need more attention, and, ultimately, which are blocking consensus. The pitfall is failure to accept the limit on an individual's power to determine which concerns are principled or based upon the foundation of the group and which ones are resolved.

After discussion, if the concern is valid and unresolved, it again falls up on the individual to choose whether to stand aside or block consensus.

The individual is responsible for expressing concerns; the group is responsible for resolving them. The group decides whether a concern is legitimate; the individual decides whether to block or stand aside.

All concerns are important and need to be resolved. It is not appropriate for a person to come to a meeting planning to block a proposal or, during discussion, to express their concerns as major objections or blocking concerns. Often, during discussion, the person learns additional information which resolves the concern. Sometimes, after expressing the concern, someone is able to creatively resolve it by thinking of something new. It often happens that a concern which seems to be extremely problematic when it is first mentioned turns out to be easily resolved. Sometimes the reverse happens and a seemingly minor concern brings forth much larger concerns. Here are three consensus sufficient conditions: (Web: *NIH Consensus Statements*)

- I have had the opportunity to voice my opinions
- I believe the group has heard me
- I can actively support the group's decision as the best possible at this time, even if it is not my first choice" and

The following is a description of different types of concerns and how they affect individuals and the group.

Concerns which can be addressed and resolved by making small changes in the proposal can be called minor concerns. The person supports the proposal, but has an idea for improvement.

When a person disagrees with the proposal in part, but consents to the overall idea, the person has a reservation. The person is not completely satisfied with the proposal, but is generally supportive. This kind of concern can usually be resolved through discussion.

Sometimes, it is enough for the person to express the concern and feel that it was heard, without any actual resolution.

When a person does not agree with the proposal, the group allows that person to try and persuade it to see the wisdom of the disagreement. If the group is not persuaded or the disagreement cannot be resolved, the person might choose to stand aside and allow the group to go forward. The person and the group are agreeing to disagree, regarding each point of view with mutual respect. Occasionally, it is a concern which has no resolution; the person does not feel the need to block the decision, but wants to express the concern and lack of support for the proposal.

So, we can summarize that consensus means that every member can say...... (Web: *Consensus Decision Making*)

- “I believe that you understand my point of view and that I understand yours.”
- “Whether or not I prefer the group decision, I support them because they were reached fairly and openly, and they are the best decisions at this time."
- "We all share in the final decision."

Consensus decision making requires:

- Sufficient time to explore all the information and opinions
- Strong facilitative leadership
- Members willing to contribute their views and discuss their reasons
- Commitment and effort to develop an atmosphere of honesty and openness in the group
- Willingness to confront and resolve controversy and conflict"

There are some common misunderstandings with consensus.

5.1.1.1 Time in Consensus

It is often said that consensus is time-consuming and difficult (Web: *A Guide to Formal Consensus*). Making complex, difficult decisions is time-consuming, no matter what the process. Many different methods can be efficient, if every participant shares a common understanding of the rules of the game. Like any process, Consensus can be inefficient if a group does not first assent to follow a particular structure.

But, consensus is not inherently time-consuming. Decisions are not an end in themselves. Decision making is a process which starts with an idea and ends with the actual implementation of the decision. While it may be true in an autocratic process that decisions can be made quickly, the actual implementation will take time. When one person or as mall group of people makes a decision for a larger group, the decision not only has to be communicated to the others, but it also has to be acceptable to them or its implementation will need to be forced upon them. This will certainly take time, perhaps a considerable amount of time. On the other hand, if everyone participates in the decision making, the decision does not need to be communicated and its implementation does not need to be forced upon the participants. The decision may take longer to make, but once it is made, implementation can happen in a timely manner. The amount of time a decision takes to make from start to finish is not a factor of the process used; rather, it is a factor of the complexity of the proposal itself. An easy decision takes less time than a difficult, complex decision, regardless of the process used or the number of people involved. Of course, Consensus works better if one practices patience, but any process is improved with a generous amount of patience.

5.1.1.2 Group Size

Consensus works better when more people participate (Web: Consensus Consulting Group (CCG)). Consensus is more than the sum total of ideas of the individuals in the group. During discussion, ideas build one upon the next, generating new ideas, until the best decision emerges. This dynamic is called the creative interplay of ideas. Creativity plays a major part as everyone strives to discover what is best for the group. The more people involved in this cooperative process, the more ideas and possibilities are generated. Consensus works best with everyone participating.

If the structure is vague, decisions can be difficult to achieve. They will become increasingly more difficult in larger groups. Consensus is designed for large groups. It is a highly structured model. It has guidelines and formats for managing meetings, facilitating discussions, resolving conflict, and reaching decisions. Smaller groups may need less structure, so they may choose from the many techniques and roles.

5.1.1.3 Consensus and Voting

While voting enables a group to determine who is in favor or opposed to a proposal, voting does not always indicate what people can or will support (Arrow, 1986). Generally speaking, when a group votes using majority rule, a competitive dynamic is created within the group because it is being asked to choose between two (or more) possibilities. It is just as acceptable to attack and diminish another's point of view as it is top remote and endorse your own ideas. Often, voting occurs before one side reveals anything about itself, but spends time solely attacking the opponent! In this adversarial environment, one's ideas are owned and often defended in the face of improvements.

5.1.1.4 Consensus and Group Thinking

A group, by definition, is a number of individuals having some unifying relationship. Consensus strives to take into account everyone's concerns and resolve them before any decision is made. Most importantly, this process encourages an environment in which everyone is respected and all contributions are valued.

Groups which desire to involve as many people as possible need to use an inclusive process. To attract and involve large numbers, it is important that the process encourages participation, allows equal access to power, develops cooperation, promotes empowerment, and creates a sense of individual responsibility for the group's actions. All of these are cornerstones of Consensus. The goal of consensus is not the selection of several options, but the development of one decision which is the best for the whole group. It is synthesis and evolution, not competition and attrition.

If a group drives for consensus so strong that dissent is (intentionally and unintentionally) suppressed, it is group thinking (Web: *Group Think*). Group thinking produces lack of judgement. People are not voluntarily but pressured to agree. But the group members have the belief that the group is impervious to threats and false perceptions of unanimity.

5.1.2 Difficulties in generating consensus

Hypothetically, agreement with other experts is a necessary characteristic of an expert. Common wisdom claims that experts in a given field should agree with each other. If opinions do not match, then some of the members of this set of experts must not be functioning at the appropriate level. However, in practice, a consensus among experts implies that the expert community has largely solved the problems of the domain. In that case, each individual expert is getting the correct answer, usually with the aid of well-developed technology, therefore their answers agree. (Weiss and Shanteua, in press).

In reality, however, disagreement among experts is inevitable and even useful. Moreover, one might argue that too much inter-individual agreement is a signal that the problem is trivial, and scarcely worthy of an expert.

Consensus makes it possible for a group to reach a final decision that all group members can support among these differing opinions. True expertise is characterized by the following properties:

- The domains where experts work is very complex. Single optimal solution does not exist.
- A distinction can be made between the different levels of decisions made by experts. Experts might disagree at one level, but agree at another.
- Despite the assumption made by many researchers, experts are seldom asked to make single-outcome decisions. The job of the expert is to clarify alternatives and describe possible outcomes for clients.

- Experts generally work in dynamic situations with frequent updating. Thus, the problems faced by experts are unpredictable, with evolving constraints.

- Experts work in realms where the basic science is still evolving.

5.2 Consensus Measure Methods

The ultimate goal of a procedure in group decision making, in the context considered here, is to obtain an agreement between the experts as to the choice of a proper decision, i. e. to reach consensus (Kacprzyk et al., 1992a). Initially, the group may be far from consensus. However, it can be expected that during the decision-making process, opinions of its members will converge. Consensus is not to be enforced nor obtained through some negotiations or bargaining-like process but is expected to emerge after some exchange of opinions among the experts.

Any group decision-making process is basically aimed at reaching a "consensus". Consensus has become a major area of research in group decision making (Bordogna et al., 1997; Herrera et al., 1997; Herrera-Viedma et al., 2002; Kacprzyk et al., 1997). Generally, the approaches towards consensus in the literature can be divided into two groups. The first treats consensus as a "mathematical aggregated consensus" (Ng and Abramson, 1992). This type of consensus requires some kind of a binding arbitration so the contributing experts do not need to converge in their opinions. In most cases, the consensus is achieved by changing the weights of the experts (e.g. Lee, 2002). In the other type, the experts are encouraged to modify their opinion to reach a closer agreement in opinions (e.g. Hsu and Chen, 1996).

Generally, we have two ways to represent consensus (Bordogna et. al., 1997): Hard consensus measure and soft consensus measure.

5.2.1 Hard Consensus Measure

In this category, consensus is measured in the interval [0, 1]. In more detail, we can still divided these methods into the following groups based on the ways to calculate the consensus level:

1. Count number of experts

The simplest consensus measure method is to count the number of experts with the group. Usually, the ratio of the number counted to the total group is taken as the consensus.

2. Distance

The method measures the distances between experts and even expert and the group opinion. The consensus is a function of the distance.

3. Similarity/Dissimilarity

Similar to the distance measure, similarities or dissimilarities between experts can be measured. Some methods measure the similarity or dissimilarity between experts and the aggregated group opinion. Thus consensus is a function of the similarity/dissimilarity. Of course, consensus is the increasing function of similarity and decreasing function of dissimilarity.

4. Order-Based

Based on the evaluations from experts, the preference orders of all alternatives from each expert can be calculated. By comparing the order difference from expert and the aggregated group, the consensus is then measured.

5.2.1.1 Count Number of Experts

Fairhurst and Rahman (2000) believe a consensus occurs when at least k of the experts agrees, where k can be defined as,

$$k = \begin{cases} \frac{q}{2}+1, & \text{if } q \text{ is even} \\ \frac{q+1}{2}, & \text{if } q \text{ is odd} \end{cases} \tag{5.1}$$

As we defined before, q is the total number of experts in the decision making group.

Bryson and Mobolurin (1997) recognize the group members into learning mode for those who are still uncertain about their own scores and interested in having discourse with other members. Those members who are fairly certain about their evaluations are in strategic mode. Decision makers use the AHP to reach a consensus decision with regard to a set of alternatives. The consensus decision is represented by a normalized numeric preference vector where the ratio of elements in the vector reflects the group's belief in the relative importance of alternatives.

Tan et al. (1995) used a modified version of the fuzzy model of consensus to produce a consensus level for the group for each alternative. The acceptance of and commitment to the decision are dependent upon the consensus level achieved. If group remembers do not agree with each other and with the decision, it is unlikely that they will accept and be committed to the decision. A group decision reached as a result of conformance pressure or domination by a minority is likely to be no better than an individual decision.

The overall decision combination algorithm therefore reflects a hybridization of the concepts of both 'decision consensus' and 'best expert, best decision' scenarios.

5.2.1.2 Distance

Kuncheva (1994) proposes five measures for degree of consensus based on the distance metric.

$$c^1 = 1 - \min_{i \neq j} \left| w_i x_i - w_j x_j \right|, i, j = 1, \ldots, q \tag{5.2}$$

$$c^2 = 1 - \max_{i \neq j} \left| w_i x_i - w_j x_j \right|, i, j = 1, \ldots, q \tag{5.3}$$

$$c^3 = 1 - \frac{1}{q} \sum_{i=1}^{q} \left| w_i x_i - \frac{1}{q} u \right| \tag{5.4}$$

$$c^4 = 1 - \frac{2}{q(q-1)} \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \left| w_i x_i - w_j x_j \right| \tag{5.5}$$

$$c^5 = 1 - \max\left| w_i x_i - \frac{1}{q} u \right|, i = 1, ..., q \tag{5.6}$$

$$u = \frac{1}{q}\sum_{i=1}^{q} w_i x_i$$

where q is the number of decision makers, x_i is the distance metric for decision maker i, w_i provides parametric control and possible weighting of decision makers, and c is in [0, 1] is the degree of consensus measure for an alternative.

Also, Bezdek et al. (1978) compute a distance as consensus from a difference between an average preference matrix and preference matrices from each expert.

5.2.1.3 Similarity/Dissimilarity

A dissimilarity based consensus is from Fedrizzi (1990). First, experts give pairwise comparison of alternative A_i to alternative A_j from expert E_k on criterion C_h:

$$A^{kh} = [a_{ij}^{kh}] \tag{5.7}$$

where $a_{ij}^{kh} = \dfrac{1}{a_{ji}^{kh}}$ uses a ration scale which from 1 to 9. Then the pairwise comparison of criterion C_i to Criterion C_j from expert E_k are given as:

$$B^k = [b_{ij}^k] \tag{5.8}$$

The formula used to aggregate all q experts' opinions is:

$$A^h = [a_{ij}^h] = \left[\left(\prod_{k=1}^{q} a_{ij}^{hk} \right)^{\frac{1}{q}} \right] \tag{5.9}$$

$$b_{ij} = \left[\left(\prod_{k=1}^{q} b_{ij}^{k} \right)^{\frac{1}{q}} \right] \tag{5.10}$$

The dissimilarity measure is defined as:

$$d(x,y) = \frac{1}{2}\left|\log_9 x - \log_9 y\right| \tag{5.11}$$

or:

$$d(x,y) = \left|g(x) - g(y)\right| \tag{5.12}$$

where $g(x) = \frac{1}{2}(1 + \log_9 x)$

A consensus measure based on dissimilarity follows the procedure:

1. The degree of agreement between expert E_k and the group between alternative A_i and A_j

$$v_{ij}^{h}(k) = 1 - d(a_{ij}^{kh}, a_{ij}^{h}) \tag{5.13}$$

2. The degree of agreement between expert E_k and the group to all the relevant pairs of alternatives

$$v^{h}(k) = \frac{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} v_{ij}^{h}(k) * \beta_{ij}^{h}}{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \beta_{ij}^{h}} \tag{5.14}$$

Where $\beta_{ij}^{h} = \dfrac{w_i^h + w_j^h}{2}$ and * is any t-norm

3. The degree of agreement between expert E_k and the group to their preferences between Q_1 relevant pairs of alternatives

$$v_{Q1}^{h}(k) = \mu_{Q1}(v^{h}(k)) \tag{5.15}$$

4. The degree of agreement of all experts with the group as to their preferences between Q_1 relevant pairs of alternatives

$$v_{Q1}^{h} = \frac{\sum_{k=1}^{m} v_{Q1}^{h}(k)}{m} \tag{5.16}$$

5. The degree of agreement of Q_2 experts with the group as to their preferences between Q_1 relevant pairs of alternatives

$$E_h = \mu_{Q2}(v_{Q1}^{h}) \tag{5.17}$$

5.1.2.4 Order-Based

Herrera-Viedma, et. al.'s consensus model (Herrera-Viedma et al. 2002) compares the positions of the alternatives based on the individual solutions and the group solution. Based on the consensus level and the offset of individual solutions, the model gives feedback suggesting the direction in which the individual experts should change their opinion.

$$C_G = (1-\beta)\cdot\sum_{t=1}^{\nu}\frac{C_t}{\nu} + \beta\cdot\sum_{s=1}^{\gamma}\frac{C_s}{\gamma} \tag{5.18}$$

$$C_i = 1 - \frac{1}{q}\sum_{k=1}^{q} p_k \tag{5.19}$$

$$p_k = \left(\frac{\left|O_{A_i}^{G} - O_{A_i}^{E_k}\right|}{n-1}\right)^{b} \tag{5.20}$$

Where, C_G is the group consensus of the final solution.

C_i is the consensus level of the i^{th} alternative achieved by the group.

$O_{A_i}^{G}$ is the index of the i^{th} alternative in the group's selection order.

$O_{A_i}^{E_k}$ is the order of the i^{th} alternative based on the k^{th} expert

q is the number of experts and *n* is the number of alternatives.

$\beta \in [0,1]$ is a parameter to control the OR-LIKE behaviour of the aggregation operator.

γ is the cardinal of the set X_{sol}, the means the number of alternatives selected in the solution set. ν is the cardinal of the set X- $X_{sol.}$

The parameter b controls the rigorousness of the consensus process, in such a way, that values of close to one decrease the rigorousness and thus the number of rounds to develop in the group discussion process, and values of close to zero increase the rigorousness and thus, the number of rounds. Appropriate values for are: 0.5, 0.7, 0.9, 1.

Fedrizzi et al. (1995) uses a so called Opinion Changing Aversion (OCA) function estimated for each expert to represent expert's resistance to opinion changing. Based on this, the author gives an alternatives ranking from each expert and consensus ranking from the group.

This "hard" interpretation of consensus is sometimes counterintuitive, since one may be fully satisfied (consensus = 1) even in case of agreement only among most of the experts, but not all (Bordogna et. al., 1997). In the next subsection, we will introduce approaches use linguistic labels instead of just a numeric number to represent the consensus level.

5.2.2 Soft Consensus Measure

In this category, the consensus is not measured by a crisp number but a linguistic label, like "most". This soft consensus is actually a linguistic quantifier used helping to aggregate the evaluations from experts. Fedrizzi (1990) propose a consensus measure based on dissimilarity between the preference relations. The procedure computes a 'soft" degree of consensus, which is a numeric value assessing the truth of a statement like "most pairs of experts agree on almost all the alternatives".

5.2.2.1 Bordogna et al.'s Method

Bordogna et al. (1997) uses the statement "*Most of the experts agree on alternative Ax*" which is interpreted as "Most of the experts agree with most of the other experts on alternative Ax".

For example, in Table 5.1, *q* experts express their overall evaluations to *n* alternatives as:

Table 5.1 *q* experts express their evaluations to *n* alternatives using linguistic labels

Overall performance values	$\mathbf{E_1}$	$\mathbf{E_2}$	**…**	$\mathbf{E_q}$
A_1	high	…	…	…
…	…	…	O_{ix}	perfect
A_n	…	low	…	…

A linguistic degree of consensus among the experts' overall performances is computed for each alternative. A procedure to evaluate the consensus degree among Q experts for each alternative: (Q is a quantifier identifying a fuzzy majority) is as follows:

For each alternative, pair-wise comparisons of experts' overall performance labels produce the degree of agreement between pairs of experts.

A matrix of qxq is then constructed for each alternative. An element $\mathrm{Ag}(E_i, E_j)$ is the linguistic label, which express the closeness between the overall performance labels of expert E_i and E_j.

$$\mathrm{Ag}(E_i, E_j) = Neg(\mathrm{d}(O_{ix}, O_{jx})) \quad (5.21)$$

Where O_{ix} denotes the linguistic overall performance label of expert E_i on alternative Ax. And the **d** function is a difference operator of linguistic labels in the same scale S.

$$\mathrm{d}(s_i, s_j) = s_r \text{ with } r = |i - j| \quad (5.22)$$

$$\neg(s_i) = s_{T-i} \tag{5.23}$$

This process is depicted in Table 5.2.

Table 5.2 Using linguistic quantifier Q to aggregate *q* experts' evaluations

A_i	E_1	E_2	…	E_q	$Ag(E_i)$ from OWA_Q
E_1	high	…	…	…	-
E_2	…	…	…	medium	-
…	…	…	$Ag(E_i, E_j)$	…	-
E_q	…	low	…	…	-
Q_2 E_i agree on alternative Ax			**Final**		

For each expert E_i (a row of the matrix qxq), $Ag(E_i, E_j), i \neq j$ are pooled to obtain an indication of the agreement $Ag(E_i)$ of expert E_i with respect to Q_1 other experts. The degree of consensus among Q1 experts is shown in the last column pf Table 5.2.

$$OWA_{Q1}(i^{th} \text{ row}) \tag{5.24}$$

The values $Ag(E_i)$ are finally aggregated to compute the truth of the sentence "Q_2 experts agree on alternative Ax". This value is stored in the bottom-right cell. Consensus among Q_2 experts

$$OWA_{Q_2}(\text{last column}) \tag{5.25}$$

5.2.2.2 Herrera et al. 's Method

Herrera et al. 's rational consensus model (1997) is a two phases consensus seeking process: *counting* process and *coincidence* process.

1. Counting Process

In this process, the method counts the number of experts using the same linguistic label.

First, $V_{ij}[s_t]$ represents the set of expert numbers, who use linguistic s_t as the preference on the alternative pair (x_i, x_j)

$$V_{ij}[s_t] = \{k \mid P_{ij}^k = s_t, \quad k = 1,...q\} \tag{5.26}$$

And use $V_{ij}^C[s_t]$ as the cardinal of the set $V_{ij}[s_t]$. Then the aggregated experts' importance degree

$$V_{ij}^G[s_t] = \begin{cases} \Phi_{Q^1}\left(u_G(z_1), u_G(z_2),...u_G(z_m)\right) & \text{if } V_{ij}^G[s_t] > 1 \\ s_0 & \text{Otherwise} \end{cases} \tag{5.27}$$

where $z_k \in V_{ij}[s_t]$, $m = V_{ij}^C[s_t]$

Example 5.1:

Given four alternatives, four experts give their pairwise comparison evaluations as the following:

$$P^1 = \begin{bmatrix} - & SC & EL & VLC \\ MC & - & ML & EL \\ SC & SC & - & VLC \\ EL & IM & ML & - \end{bmatrix} \qquad P^2 = \begin{bmatrix} - & MC & IM & VLC \\ IM & - & ML & IM \\ IM & SC & - & VLC \\ ML & MC & EL & - \end{bmatrix}$$

$$P^3 = \begin{bmatrix} - & EL & C & I \\ IM & - & MC & SC \\ EU & IM & - & VLC \\ C & EL & ML & - \end{bmatrix} \qquad P^4 = \begin{bmatrix} - & SC & MC & SC \\ EL & - & IM & SC \\ IM & ML & - & VLC \\ C & MC & C & - \end{bmatrix}$$

The, apply formula 5.26, $V_{13}[SC] = \{4\}$, since only $P_{13}^4 = MC$, and $V_{24}[SC] = \{3,4\}$ because $P_{24}^3 = P_{24}^4 = SC$. But, $V_{23}[C] = \{\Phi\}$, since nobody uses linguistic label "C" to compare alternative A_2 to A_3.

So, $V_{13}^C[MC] = 1$, $V_{23}^C[C] = 0$, $V_{24}^C[SC] = 2$.

For $V_{13}^G[MC] = I$, because $V_{13}^C[MC] = 1$, $V_{13}^G[MC] = S_0 = I$

For $V_{23}^G[C] = I$, because $V_{23}^C[C] = 0$, $V_{23}^G[C] = S_0 = I$

Using $Q = "At\,Least\,half"\ with\ (0.0, 0.5)$, we get the membership function of the linguistic quantifier Q.

$$Q(r) = \begin{cases} 0 & r < 0 \\ \dfrac{r}{0.5} & 0 \le r \le 0.5 \\ 1 & r > 0.5 \end{cases}$$

As described in Chapter 2, we apply the formula $\omega_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)$ to compute the weighing vector, so,

$$\omega_1 = Q\left(\frac{1}{2}\right) - Q\left(\frac{0}{2}\right) = 1,\ \omega_2 = Q(1) - Q\left(\frac{1}{2}\right) = 0$$

$$V_{24}^G[SC] = \Phi_{Q^1}\left(u_G(3), u_G(4)\right) = \Phi_{Q^1}\left(SC, EU\right)$$

$k = \min\{8,\ 1 + round(1 \times (3-1))\} = 3\ \Rightarrow S_k = S_3 = SC$

so, $V_{24}^{G}[SC] = SC$, similarly, $V_{41}^{G}[C] = SC$

2. Coincidence Process

In this process, we first find the linguistic label with the highest frequency measure, LCR_{ij}.

If the two labels have the same frequency, we choose the one with the highest expert importance.

Then we use $n_{ij} = MAX_{s_t \in S}\{V_{ij}^{C}[s_t]\}$ to represent max number of experts, who choose s_t, and linguistic label $M_{ij} = \{s_y \mid V_{ij}^{C}[s_y] = n_{ij},\ s_y \in S\}$ whose cardinal $V_{ij}^{C}[s_t]$ equal to n_{ij}.

The proportional number of expert can be calculated by:

$$ICR_{ij}^{1} = \begin{cases} V_{ij}^{C}[s_{ij}] \big/ m & if\ \ V_{ij}^{C}[s_{ij}] > 1 \\ 0 & otherwise \end{cases} \tag{5.28}$$

And the respective average importance degree:

$$ICR_{ij}^{2} = \begin{cases} V_{ij}^{G}[s_{ij}] & if\ \ V_{ij}^{C}[s_{ij}] > 1 \\ s_0 & otherwise \end{cases} \tag{5.29}$$

Example 5.1 (Continue):

Continuing the Example 5.1, we have

$$LCR = \begin{bmatrix} - & SC & ? & VLC \\ IM & - & ML & SC \\ IM & SC & - & VLC \\ C & MC & ML & - \end{bmatrix}$$

$$\begin{cases} LCR_{12} = SC & because\ P_{12}^1 = SC,\ P_{12}^2 = MC,\ P_{12}^3 = EL,\ P_{12}^4 = SC \\ LCR_{13} = ? & because\ P_{13}^1 = EL,\ P_{13}^2 = IM,\ P_{13}^3 = C,\ P_{13}^4 = MC \end{cases}$$

$$\begin{cases} n_{12} = 2, & n_{13} = 0 \\ n_{34} = 4, & because\ P_{34}^1 = VLC,\ P_{34}^2 = VLC,\ P_{34}^3 = VLC,\ P_{34}^4 = VLC \end{cases}$$

$$\begin{cases} M_{12} = \{SC\} & M_{13} = \{\phi\} \\ M_{34} = \{VLC\} \end{cases}$$

The ICR[1] and ICR[1] can be calculated as:

$$ICR^1 = \begin{bmatrix} - & 0.5 & 0 & 0.5 \\ 0.5 & - & 0.5 & 0.5 \\ 0.5 & 0.5 & - & 1 \\ 0.5 & 0.5 & 0.5 & - \end{bmatrix} \text{ where } \begin{array}{ll} m = 4 & \\ V_{12}^C[S_{12}] = 2 & (SC, MC, EL, SC) \\ V_{13}^C[S_{13}] = 0 & (EL, IM, C, MC) \\ V_{34}^C[S_{34}] = 4 & (VLC, VLC, VLC, VLC) \end{array}$$

$$ICR^2 = \begin{bmatrix} - & EL & I & C \\ C & - & C & SC \\ C & C & - & C \\ SC & C & EL & - \end{bmatrix}$$

For example,

$$V_{12}^{G}[S_{12}] = MAX_{S_y \in \{SC\}} \left\{ V_{12}^{G}(SC) \right\}$$

$$= MAX \left\{ \Phi_{Q^1} \left(u_G(1),\, u_G(4) \right) \right\}$$

$$= MAX \left\{ \Phi_{Q^1} \left(EL,\, EU \right) \right\}$$

$$= S_7 = EL$$

3. Consensus Degree

The consensus degree is the distance between current consensus and ideal consensus of s_T.

- Level of preference

$$PCR_{ij} = Q^2(ICR_{ij}^1) \wedge U_I(x_{ij}) \tag{5.30}$$

where $U_I(x_{ij}) = ICR_{ij}^2$

- Level of alternative

$$PCR_i = \Phi_{Q^1}(PCR_{ij}), \quad j = 1, \ldots .n \tag{5.31}$$

- Level of relation

$$RC_i = \Phi_{Q^1}(PCR_{ij}), \quad i, j = 1, \ldots .n \tag{5.32}$$

From the same example:

$$PCR = \begin{bmatrix} - & EL & I & C \\ C & - & C & SC \\ C & C & - & C \\ SC & C & EL & - \end{bmatrix}$$

Given the linguistic quantifier $Q^2 = "At\, least\, half"$

$$PCR_1 = \Phi_{Q^1}(EL, I, C) = \Phi_{Q^1}(EL, C) = C$$

Similarity, $PCR_2 = C$, $PCR_3 = C$, $PCR_4 = C$.

Then, $RC = C$

- Linguistic Distance

The linguistic distance is the distance between experts' opinions and the current consensus:

$$D_{ij}^k = \begin{cases} P_{ij}^k - LCR_{ij} & if \;\; P_{ij}^k > LCR_{ij} \\ LCR_{ij} - P_{ij}^k & if \;\; LCR_{ij} \geq P_{ij}^k \\ S_T & otherwise \end{cases} \tag{5.33}$$

Then the for alternative A_i

$$D_i^k = \Phi_{Q^1}(D_{ij}^k) \qquad (j = 1, ...n) \tag{5.34}$$

with relation

$$D_R^k = \Phi_{Q^1}(D_{ij}^k) \qquad (i, j = 1, ...n) \tag{5.35}$$

For instance,

$$D^1 = \begin{bmatrix} - & I & C & I \\ EU & - & I & IM \\ EU & I & - & I \\ EU & EU & I & - \end{bmatrix}$$

with $D_1^1 = \Phi_{Q^1}(I,C,I) = \Phi_{Q^1}(I,C) = MC = S_5$

Similarity, $D_R^1 = SC$, $D_R^2 = VLC$, $D_R^3 = SC$, $D_R^4 = SC$.

5.2.2.3 Kacprzyk and Zadrozny's Method

Kacprzyk and Zadrozny (1997) also introduce importance on experts in evaluating consensus. And for the first time they propose the evaluations of experts by measuring the contribution to consensus (CTC) and contribution to consensus for alternatives (Option), CTCO.

The contribution to consensus of a given expert E_k, $CTC(E_k) \in [-1, 1]$, is determined as a difference between the consensus degree calculated for the whole group and the consensus degree for the group without the export E_k.

The contribution to Consensus of Option A_i, $CTCO(A_i) \in [-1, 1]$, is determined as a difference between the consensus degree for the whole set of options and that for the set without option A_i.

5.2.2.4 Kacprzyk et al.'s Fuzzy majority Method

A fuzzy majority forms a natural generalization of majority. It may be directly related to and expressed by a linguistic quantifier, exemplified by 'most", 'almost all' etc. There are a couple of approaches dealing with the formalization of linguistic quantifiers including classical Zadeh's linguistic quantifiers and Yager's OWA operators.

Based on Zadeh's linguistic quantifiers (1983b, 1996), Kacprzyk et al. (1992a) use the statement "Most of the individuals agree on almost all (of the relevant) issues (options)" with the following procedure:

For each pair of individuals we derive a degree of agreement as to their preferences r_{ij}^k between all the pair of options. For instance,

$$E_2: \ R^2 = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ 0.0 & 0.1 & 0.7 \\ 0.9 & 0.0 & 0.7 \\ 0.3 & 0.3 & 0.0 \end{bmatrix} \quad E_1: \ R^1 = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ 0.0 & 0.1 & 0.6 \\ 0.9 & 0.0 & 0.7 \\ 0.4 & 0.3 & 0.0 \end{bmatrix}$$

$$E_3: \ R^3 = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ 0.0 & 0.2 & 0.6 \\ 0.8 & 0.0 & 0.7 \\ 0.4 & 0.3 & 0.0 \end{bmatrix}$$

Aggregate these degrees to obtain a degree of agreement of each pair of individuals as to their preferences between Q_1 pairs of relevant options.

$$v_{ij}(k_1, k_2) = \begin{cases} 1 & \text{if } r_{ij}^{k_1} = r_{ij}^{k_2} \\ 0 & \text{otherwise} \end{cases} \tag{5.36}$$

Then, for the same example $V^{12} = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ - & 1 & 0 \\ 1 & - & 1 \\ 0 & 1 & - \end{bmatrix} \quad V^{13} = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ - & 0 & 1 \\ 0 & - & 1 \\ 1 & 1 & - \end{bmatrix}$

$$V^{23} = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ - & 0 & 0 \\ 0 & - & 1 \\ 0 & 1 & - \end{bmatrix}$$

Aggregate these degrees to obtain a degree of agreement of Q_2 pairs of important individuals to their preferences between Q_1 pairs of relevant options.

Define the relevance of a pair of options

$$b_{ij}^{B} = \frac{1}{2}(\mu_B(A_i) + \mu_B(A_j)) \tag{5.37}$$

$$b_{ij}^{I} = \frac{1}{2}(\mu_I(A_i) + \mu_I(A_j)) \tag{5.38}$$

Given $b_i^B = \frac{1}{A_1} + \frac{0.6}{A_2} + \frac{0.2}{A_3}$, so, $b_{12}^B = 0.8,\ b_{13}^B = 0.6,\ b_{23}^B = 0.4$

Given $b_i^I = \frac{0.8}{A_1} + \frac{1}{A_2} + \frac{0.4}{A_3}$, so, $b_{12}^I = 0.9,\ b_{13}^I = 0.6,\ b_{23}^I = 0.7$

The degree of agreement between individuals k_1 and k_2 as to their preference between all the relevant pairs of options is:

$$v_B(k_1,k_2) = \frac{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(v_{ij}(k_1,k_2) \wedge b_{ij}^B)}{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} b_{ij}^B} \tag{5.39}$$

In this example, $v_B(1,2) = \frac{(1 \wedge 0.8) + (0 \wedge 0.6) + (1 \wedge 0.4)}{0.8 + 0.6 + 0.4} = \frac{2}{3}$, $v_B(1,3) = \frac{5}{9}$, $v_B(2,3) = \frac{2}{9}$

And then the degree of agreement between individuals k_1 and k_2 as to their preference between Q_1 relevant pairs of options is:

$$v_{Q1}^B(k_1,k_2) = \mu_{Q1}(v_B(k_1,k_2)) \tag{5.40}$$

If Q_1='most', then by $\mu_{most}(x) = \begin{cases} 1, & \text{for x} \geq 0.8 \\ 2x - 0.6, & \text{for } 0.3 < \text{x} < 0.8 \\ 0, & \text{for x} \leq 0.3 \end{cases}$,

While the degree of agreement of all the pairs of individuals as to their preferences between Q1 relevant pairs of options is:

$$v_{Q1}^{I,B} = \frac{2}{q(q-1)} \frac{\sum_{k_1=1}^{q-1}\sum_{k_2=k_1+1}^{q}(v_{Q1}^B(k_1,k_2) \wedge b_{k_1,k_2}^I)}{\sum_{k_1=1}^{q-1}\sum_{k_2=k_1+1}^{q} b_{k_1,k_2}^I} \tag{5.41}$$

Here, $v_{Q1}^{B}(1,2) = \mu_{Q1}(\frac{2}{3}) = \frac{11}{15}$, $v_{Q1}^{B}(1,3) = \mu_{Q1}(\frac{5}{9}) = \frac{23}{45}$, $v_{Q1}^{B}(2,3) = \mu_{Q1}(\frac{2}{9}) = 0$

$$v_{Q1}^{I,B} = \frac{2}{3(3-1)} \frac{(\frac{11}{15} \wedge 0.9) + (\frac{23}{45} \wedge 0.6) + (0 \wedge 0.7)}{0.9 + 0.6 + 0.7} = 0.19$$

Finally, the degree of agreement of Q2 pairs of individuals as to their preferences between Q_1 relevant pairs of options is:

$$con(Q_1, Q_2, I, B) = \mu_B(v_{Q1}^{I,B}) \tag{5.42}$$

So, $con(Q_1, Q_2, I, B) = \mu_B(0.19)$

5.2.2.5 Dimitrov 's Method

Dimitrov's (1994) Fuzzy Symplectic System (FSS) is a Multi-Stakeholder Decision-Making System (MSDMS) with fuzzy motion that satisfies the fundamental condition for consensus seeking. A fuzzy motion of a Multi-Stakeholder Decision-Making System (MSDMS) is described as:

$$d\tilde{x} / dt = F(\tilde{x}, t) \tag{5.43}$$

$$\tilde{x} = \begin{bmatrix} p \\ q \end{bmatrix}$$

p and q specifying the fuzzy state of the system: p is its momentum and q is its position. When changing in time, the pair (p(t), q(t)) depicts the trajectory of MSDMS in the phase space defining by p and q.

In any instant of time each expert must have his (her) own decision space available for making choices about the values of p_i and q_j. This means that the freedom for making decisions of experts to move and make choice must be guaranteed in MSDMS independently of time during a consensus seeking process. This requirement is referred as a fundamental condition for seeking consensus in MSDMS. Its fulfillment endows the system with a unique structural property characterizing it as a symplectic.

For each individual expert, the area to move is described by the integral:

$$\oint_{\Gamma} p_i dq_i \tag{5.44}$$

taken around a closed path Γ projected on (p_i, q_i) coordinate plane. The group areas build a fuzzy symplectic space to move and make choice about p and q:

$$\sum_{i=1}^{q} \oint_{\Gamma} p_i dq_i = \oint_{\Gamma} pdq \tag{5.45}$$

where $\oint_{\Gamma} pdq$ is Poincare's integral invariant taken around a close path Γ in (p, q) space.

Consensus happens when Poincare's integral invariant round the path Γ_1 has the same value for any other closed path Γ_2, that is:

$$\oint_{\Gamma_1} pdq = \oint_{\Gamma_2} pdq \tag{5.46}$$

Poincare’s integral invariant guarantees that the freedom of expert is preserved independently of time during consensus seeking process. The freedom for making decisions of experts is something like the variance or STD.

Table 5.3 Summaries current existing consensus measuring approaches

Method	Expert Input	Experts with weights	Aggregation Method	Consensus
Ben-Arieh and Chen (2004)	Linguistic labels	Yes	OWA	[0,1]
Bezdek et al. (1978)	preference matrix	No	Distance from a difference between an average preference matrix and preference matrices from each expert.	[0, 1]
Bordogna et al., (1997)	Pairwise comparison using linguistic labels.	No	OWA with Linguistic quantifier	"Most of the experts agree on alternative Ax"
Bryson and Mobolurin (1997)	-	No	Derivate the group mean and the deviation of each individual score from this mean.	Normalized numeric preference vector
Cheng et al. (2000)	Pairwise comparisons [0, 1]	No	Linguistic Quantifiers	Linguistic
Dimitrov (1994)	linguistic labels	No	Choose the evaluation with ratio of the consensus degree to its value is near from the golden mean value	[0, 1] numeric
Fedrizzi (1990)	Pairwise comparison using 1 to 9 scale	No	Geometric mean	most pairs of experts agree on almost all the alternatives. (Linguistic quantifier based on dissimilarity between the preference relations)

Method	Expert Input	Experts with weights	Aggregation Method	Consensus
Fedrizzi (1995)	Linguistic values with trapezoidal membership function	No	Tong and Bonissone (1980)	Numeric
Herrera et al. (1997)	Pairwise comparison using linguistic labels.	Yes	OWA with Linguistic quantifier	Linguistic label
Herrera-Viedma et al. (2002)	Preference Ordering, Fuzzy preference relation, Multiple preference relation and/or Utility function	No	OWA with Linguistic quantifier	[0, 1] numeric
Kacprzyk (1996)	Fuzzy preference relation	No	OWA with Linguistic quantifier	Linguistic label
Kacprzyk et al. (1992)	Fuzzy preference relation with values [0, 1]	Yes	OWA with Linguistic quantifier	Linguistic label
Kacprzyk and Fedrizzi (1988)	Fuzzy preference relation with values [0, 1]	Yes	OWA with Linguistic quantifier	Linguistic label

Method	Expert Input	Experts with weights	Aggregation Method	Consensus
Kacprzyk and Zadrozny (1997)	Fuzzy preference relation	Yes	Linguistic quantifier	***Most*** pairs of the ***important*** individuals agree as to ***most*** pairs of ***important*** options
Lee (2002)	Trapezoidal fuzzy number	Yes	Similarity measure, Weighted average	Get weights to experts, then aggregate fuzzy numbers
Mak and Bui (1996)	1 or 2 with linguistic meaning	No	t-statistics	Probability distribution
Ng and Abramson (1992)	Probability distribution	Yes	1. Weighted average 2. Logarithmic Opinion Pool (French, 1985) 3. Conjugate Method (Winkler, 1968) 4. Bordley's approach (1982)	Numeric probability
Spillman et al. (1980)	Fuzzy preference matrices	No	α-cuts on the respective individual fuzzy preference matrices.	[0, 1]
Tan et al. (1995)	Fuzzy preference relation	No	SAGE software	[0, 1] numeric
Zahedi (1986)	Confidence Intervals	Yes	Weighted average of mean from each confidence interval	Number

5.2.3 Choosing Consensus Model

There is no one right way to make decisions. The best style of decision making is determined by the situation. When and how to make decisions based on an understanding of the environment, the people and the priorities.

So does the consensus measure. Effective decision making is not a mysterious process. Decisions can be made by a variety of methods which take into consideration such issues as time constraints and information availability. Another consideration is the question of who is expected to execute the decision.

Kacprzyk et al.(1997) give one criterion to evaluate consensus, that is "The willingness of working together again". This means group members were satisfied with the decision making process and think their opinions were heard.

5.3 Process of Improving Consensus

Consensus is not to be enforced nor obtained through some negotiations or bargaining-like process but is expected to emerge after some exchange of opinions among the experts. What is needed is a tool which would make it easier to reach consensus.

Here is a general procedure helping to reach consensus among experts.

1. Identify Areas of Agreement.

2. Clearly State Differences.

- State positions and perspectives as neutrally as possible.
- Do not associate positions with people. The differences are between alternative valid solutions or ideas, not between people.
- Summarize concerns and list them.

3. Fully Explore Differences.

- Explore each perspective and clarify.
- Involve everyone in the discussion - avoid a one-on-one debate.
- Look for the "third way": make suggestions or modifications, or create a new solution.

4. Reach Closure.
5. Articulate the Decision.

- Ask people if they feel they have had the opportunity to fully express their opinions.
- Obtain a sense of the group. (Possible approaches include "go rounds" and "straw polls," or the Consensus Indicator tool. When using the Consensus Indicator, if people respond with two or less, then repeat steps one through three until you can take another poll.)
- At this point, poll each person, asking, "Do you agree with and will you support this decision?"

5.3.1 Some guidelines for reaching consensus: (Web: *Crow, NPD Solution***)**

Make sure everyone is heard and feels listened to. Avoid arguing for ones own position. Present each expert's position as clearly as possible. Listen to other team members' reactions and comments to assess their understanding of each other's position. Consider their reactions and comments carefully before pressing ones own point of view further.

Do not assume that someone must win and someone must lose when a discussion reaches a stalemate. Instead, look for the next most acceptable alternatives for all parties. Try to think creatively. Explore what possibilities exist if certain constraints were removed.

Do not change mind simply to avoid conflict, to reach agreement, or maintain harmony. When agreement seems to come too quickly or easily, be suspicious. Explore the reasons and be sure that everyone accepts the solution for basically similar or complementary reasons. Yield only to positions that have objective or logically sound foundations or merits.

Differences of opinion are natural and expected. Seek them out, value them, and try to involve everyone in the decision process. Disagreements can improve the group's decision. With a wider range of information and opinions, there is a greater chance of that the group will hit upon a more feasible or satisfactory solution.

5.4 Three New Consensus Measure Methods

In this section, we will present three new consensus measure methods. One is order-based method, in which we consider the weights of experts' opinion and the alternatives in the solution set. The second method is to measure the similarity between the opinion from individual expert and the group. The difference from other similarity based method is that the data is from the FLOWA method. The other method is based on the Markov theory, which takes advantage of the steady state of the Markov chain.

5.4.1 Order-Based

Ben-Arieh and Chen (2004) assume that experts do not have to change their opinions in order to reach a consensus. This assumption is well grounded in research and an excellent review of this phenomenon of expert disagreement in different domains can be found in (Shanteau, 2001). An additional example for such an expert decision is, again, judging figure skating. In this case, the judges, who are carefully trained experts, evaluate the performance using very well defined performance guidelines using uniform criteria. In such judging there is no expectations that all experts will eventually converge to an agreement. On the contrary – the experts are expected to produce diversified opinion and the usual procedure is to eliminate the high and low extreme opinions (assign a weight of zero) and average the rest (assign a weight of n-2).

In the model presented by Ben-Arieh and Chen (2004), the degree of importance of each expert is being considered in calculating the consensus. Moreover, once the consensus is calculated the experts with a more extreme opinion will lose some of their weight (credibility, influence, etc.). The experts however need not modify their opinions to achieve consensus.

$$C_i = \sum_{k=1}^{q} \left[\left(1 - \frac{\left| O_{A_i}^{G} - O_{A_i}^{E_k} \right|}{n-1} \right) \times u_k \right] \tag{5.47}$$

$$C_G = \frac{1}{p} \sum_{i=1}^{P} C_{[i]} \qquad \text{where } [i] \text{ represents the alternative ranked in } i\text{th position} \tag{5.48}$$

Where, C_G is the group consensus of the final solution, C_i is the consensus level of the i^{th} alternative achieved by the group. $O_{A_i}^{G}$ is the index of the i^{th} alternative in the group's selection order. $O_{A_i}^{E_k}$ is the order of the i^{th} alternative based on the k^{th} expert. u_k is the importance of the k^{th} expert's opinion. q is the number of experts and n is the number of alternatives.

This definition compares each individual solution presented by an expert with the group solution generated separately.

Based on this, we propose a consensus seeking process as shown in Figure 5.1. In this process we measure the contribution of the individual members to the group decision. Experts who contribute more to the group decision improve their importance while individual that are contrary to the group lose some of their weight. The process continues and calculates a new group decision with a new consensus level. The process continues until the desired consensus level is reached.

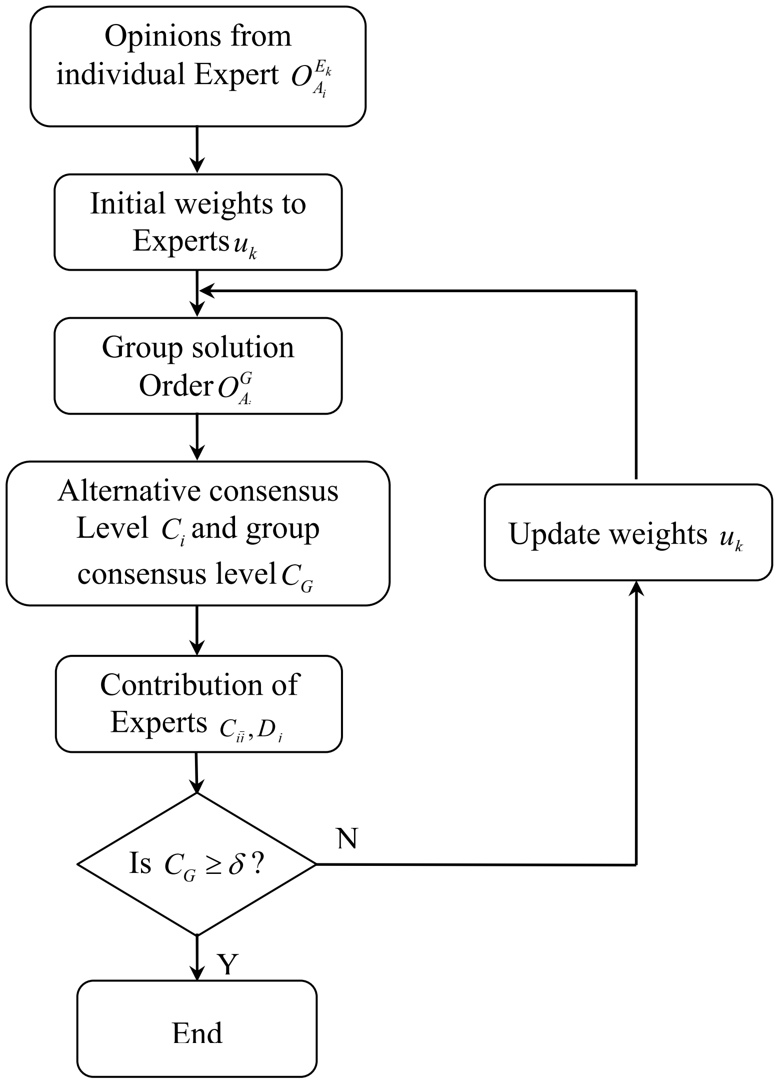


Figure 5.1 Flowchart of the group decision process

Description of the Group Decision Process

- Calculate the current consensus level CG and compare it with the desired consensus level δ. If CG>δ, we accept the group's solution.
- If not, we measure the contributions of group members D_j. D_j is calculated by the following formulas.

$$C_{i\bar{j}} = \sum_{k \in E \setminus \{j\}} \left[\left(1 - \frac{\left| O_{A_i}^{G} - O_{A_i}^{E_k} \right|}{n-1} \right) \times \beta_k \right], \text{where } \beta_k = u_k / \sum_{i \in E \setminus \{j\}} u_i \quad (5.49)$$

$$D_{ij} = C_i - C_{i\bar{j}} \quad (5.50)$$

$$D_j = \sum_{i=1}^{n} D_{ij} \quad (5.51)$$

Here, $C_{i\bar{j}}$ is the group consensus level on alternative *i* without expert *j*. D_{ij} is the contribution of the j^{th} expert on the i^{th} alternative, which is the difference between the group consensus on alternative i with and without expert j. D_j is the cumulative contribution of the j^{th} expert to all alternatives. The higher D_j it is, the higher contribution the j^{th} expert makes to the group.

- Update the weights of the experts. If the current consensus level is lower than a specified threshold, which means that there is enough discrepancy between the experts' opinions, we need to update the weights of the experts and recalculate the group decision solution. The following equations show how to update the weights.

$$t_k^{r+1} = u_k^r \cdot (1 + D_k)^{\beta} \quad (5.52)$$

$$u_k^{r+1} = \frac{t_k^{r+1}}{\sum_k t_k^{r+1}} \quad (5.53)$$

Here u_k^r is the importance of expert k in the r^{th} iteration. Parameter β represents the influence of the contribution of the expert on his/her weight. The higher the value of the parameter β, the faster the process converges to the desired consensus level.

- After updating the weights, we recalculate the group solution and consensus level. The process continues until the desired consensus level is reached.

Example 5.2:

In this example, we demonstrate the entire process of generating a group decision and converging towards consensus. The example is adopted from Herrera-Viedma et al. (2002) with the purpose of showing how to apply the FLOWA aggregation method and the procedure of reaching the desired consensus level. Let the nine linguistic labels set S be S = {I, SW, WO, SI, EQ, SB, SU, SS, CS}, where,

s_8=CS	Certainly Superior
s_7=SS	Significantly Superior
s_6=SU	Superior
s_5=SB	Somewhat Better
s_4=EQ	Equivalent
s_3=SI	Somewhat Inferior
s_2=WO	Worse
s_1=SW	Significantly Worse
s_0=I	Incomparable

A set of four alternatives A= {A_1, A_2, A_3, A_4} as well as a set of four experts E= {E_1, E_2, E_3, E_4} whose opinions are expressed by the following four linguistic preference relation matrices. Here P_{ij}^k is the degree of the preference of i[th] alternative A_i over j[th] alternative A_j derived from expert k. For example, expert E_1 thinks that alternative A_1 has a small chance to be better than alternative A_2. Thus he/she chooses s_3 as his preference, therefore, the element P_{12}^1 is s_3. To keep the matrices consistent they satisfy $P_{ij}^k + P_{ji}^k = s_T$.

$$P^1 = \begin{bmatrix} - & s_3 & s_5 & s_1 \\ s_5 & - & s_7 & s_3 \\ s_3 & s_1 & - & s_0 \\ s_7 & s_5 & s_8 & - \end{bmatrix} P^2 = \begin{bmatrix} - & s_1 & s_3 & s_3 \\ s_7 & - & s_6 & s_5 \\ s_5 & s_2 & - & s_3 \\ s_5 & s_3 & s_5 & - \end{bmatrix} P^3 = \begin{bmatrix} - & s_5 & s_7 & s_3 \\ s_3 & - & s_6 & s_2 \\ s_1 & s_2 & - & s_0 \\ s_5 & s_6 & s_8 & - \end{bmatrix} P^4 = \begin{bmatrix} - & s_6 & s_7 & s_5 \\ s_2 & - & s_4 & s_3 \\ s_1 & s_4 & - & s_0 \\ s_3 & s_5 & s_8 & - \end{bmatrix}$$

Step 1. Aggregate each expert's opinion

We apply a FLOWA operator guided by a linguistic quantifier $Q_1 = "\textit{As many} \text{ as possible}"$ to aggregate each expert's evaluation. The associated weights are calculated as $W = [0, 1/3, 2/3]$. For alternative A_1 from expert E_1, we aggregate {s_3, s_5, s_1} or ordered linguistic labels {s_5, s_3, s_1} with associated weights [0, 1/3, 2/3]. Then we apply formulae (4.66) to (4.69) to calculate the membership of the linguistic labels. The aggregating result matrix for expert E_1 is shown in Table 5.4.

Table 5.4 The aggregation result for expert E1

E_1	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
A_1	0	0.444	0.333	0.222	0	0	0	0	0
A_2	0	0	0	0.444	0.333	0.222	0	0	0
A_3	0.667	0.333	0	0	0	0	0	0	0
A_4	0	0	0	0	0	0.444	0.333	0.222	0

Step 2. Aggregate all experts' opinions

In this step, we aggregate the four experts' opinions to find the final score for each alternative. We assume that all experts have the same importance meaning that $u = [0.25, 0.25, 0.25, 0.25]$. Applying the classic aggregation method $\mu_{ij}^{G} = \sum_{k=1}^{q} u_k \mu_{ij}^{E_k}$, where $\mu_{ij}^{E_k}$ is the membership of the j[th] linguistic label on alternative A_i from expert E_k and μ_{ij}^{G} is the aggregated membership from all experts. Table 5.5 shows the calculation result in this example.

Table 5.5 The aggregated result from the group

G	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
A_1	0	0.222	0.167	0.222	0.083	0.222	0.083	0	0

A_2	0	0	0.334	0.278	0.083	0.222	0.083	0	0
A_3	0	0.250	0.167	0.083	0	0	0	0	0
A_4	0	0	0	0.222	0.167	0.389	0.167	0.056	0

Step 3. Ranking

At this point the decision of each expert and the group are represented as fuzzy sets, thus, we can apply Lee and Li's (1988) mean and standard deviation based fuzzy sets outranking approach equations to rank these fuzzy sets by their fuzzy means and standard deviations. Tables 5.6 and 5.7 present the ranking results from expert E_1 and the whole group, respectively. By comparing the fuzzy means and the standard deviations, we derive the order of the alternatives from each expert and the group. The orders are shown in Table 5.8. Thus Table 5.8 shows that the group preferred alternative A_4, followed by A_1.

Table 5.6 Ranking result from expert E1

E_1	A_1	A_2	A_3	A_4
$\bar{x}(A_i)$	1.7778	3.7778	0.3333	5.7778
$\sigma(A_i)$	0.7857	0.7857	0.4714	0.7857

Table 5.7 Ranking result from the whole group

G	A_1	A_2	A_3	A_4
$\bar{x}(A_i)$	3.6296	3.3333	1.0000	4.2963
$\sigma(A_i)$	1.6136	1.4907	1.0541	1.0116

Table 5.8 Alternative Orders from individual experts and the whole group

Expert	Order	O_{A1}	O_{A2}	O_{A3}	O_{A4}
E_1	A_4, A_2, A_1, A_3	3	2	4	1
E_2	A_2, A_4, A_3, A_1	4	1	3	2
E_3	A_4, A_1, A_2, A_3	2	3	4	1

E_4	A_1, A_4, A_2, A_3	1	3	4	2
Group	A_4, A_1, A_2, A_3	2	3	4	1

Step 4. Consensus Measure

Based on the order of the alternatives from each expert and the group, we can apply equations (5.47) and (5.48) to measure the consensus levels. Table 5.9 shows the group consensus level for a single alternative solution i.e. p=1 and for p=2 (the decision maker needs to choose two best alternatives).

Table 5.9 Consensus level to alternatives (r=0)

	C_1	C_2	C_3	C_4
Alternative Consensus	0.667	0.750	0.917	0.833
Group Consensus (p=1)	$C_G=C_4=0.833$			
Group Consensus (p=2)	$C_G=(C_4+C_1)/2=0.750$			

To measure the contribution of individual group members, we aggregate the new group without one individual expert. By equations (5.49), (5.50), and (5.51), we can find the new consensus level for the alternatives without a particular expert. Tables 5.10, 5.11, and 5.12 show the new group orders excluding each expert, the various partial consensus levels and the contributions of the four experts towards the group decision.

Table 5.10 New group orders without particular expert

Without Expert	New Group Order
E_1	A_1, A_4, A_2, A_3
E_2	A_4, A_1, A_2, A_3
E_3	A_4, A_2, A_1, A_3
E_4	A_4, A_2, A_1, A_3

Table 5.11 New alternative consensus level without one particular expert

Group	A_1	A_2	A_3	A_4

$C_{i\bar{1}}$	0.6667	0.7778	0.8889	0.7778
$C_{i\bar{2}}$	0.7778	0.8889	1.0000	0.8889
$C_{i\bar{3}}$	0.6667	0.7778	0.8889	0.7778
$C_{i\bar{4}}$	0.7778	0.7778	0.8889	0.8889

Table 5.12 Contribution of the experts

i	E_1	E_2	E_3	E_4
D_j	0.0556	-0.3889	0.0556	-0.1667

Step 5. Update weights to experts

We assign the original weighting vector of the experts (r=0) to be u= [0.25, 0.25, 0.25, 0.25]. After calculating the contribution of individual expert D_i, we apply equations (5.52) and (5.53) to update the weights. Table 5.13 shows the change of weights to experts during the first two iterations. It indicates that in this case, the weights to experts E_1 and E_3 increase while the weights to experts E_2 and E_4 decrease. This is the result of experts E_1 and E_3 contributing more to the final group decision.

Table 5.13 Weights of experts

i	E_1	E_2	E_3	E_4
Original Weights (u^0), r=0	0.2500	0.2500	0.2500	0.2500
New Weights (u^1), r=1	0.2969	0.1719	0.2969	0.2344
New Weights (u^2), r=2	0.3189	0.1308	0.3383	0.2120

With the change of the weights to experts, the consensus level increases. When the process continues gradually the weights of the experts change while the consensus level increases. The process is depicted in Figure 5.2. The figure shows the change of the weights of the four experts as the consensus changes from 0.833 to 0.99. The figure shows that to increase consensus initially the importance of experts E_1 and E_3 increase.

Ultimately, to reach a perfect consensus with experts having differing opinions, eventually all experts but one are completely discounted.

One important thing in this procedure is that the alternative order from the four-member group is always A_4, A_1, A_2, A_3 regardless of the weights to experts changed in this case.

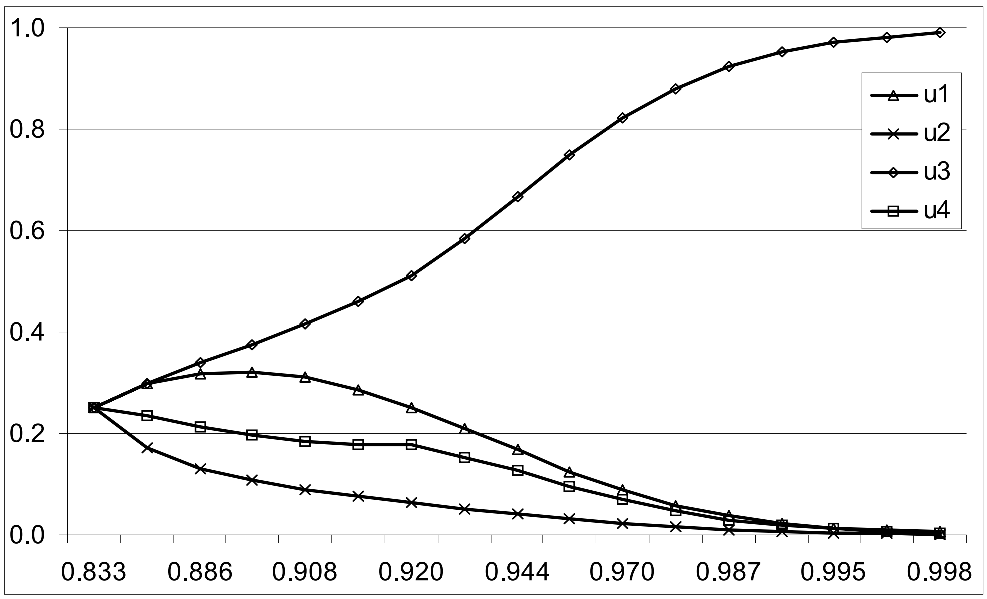


Figure 5.2 Group consensus levels with the changing of weights to experts

5.4.2 Similarity-Based

The new similarity based consensus measure works with the FLOWA aggregation operator. After we aggregate group members' opinions, we have a distribution along the linguistic labels for each expert. Also, we have a distribution for the group. The approach is to measure the distance between individual expert and the group, and then similarity as well as the consensus.

Step 1. Aggregation results from FLOWA

As we introduced in Chapter 4, the FLOWA aggregation operator results in a possibility distribution along the linguistic label set. That is, after distribution of the evaluation from each expert, the possibility of the j^{th} linguistic label from k^{th} expert to i^{th} alternative is: μ_{ijk}^{F}. Then the group opinion to alternative A_i on the j^{th} linguistic label represented by a possibility distribution is:

$$\mu_{ij} = \sum_{k=1}^{q} \mu_{ijk}^{F} \tag{5.54}$$

Step 2. Distance from the group

Assume each expert uses one linguistic label as his/her evaluation to each alternative, we define the original membership function to linguistic label s_j from expert E_k on alternative A_i as following:

$$D_{ik} = \left| x_{ik} - \bar{x}_i \right| \tag{5.55}$$

Where x_{ik} is the initial evaluation to alternative Ai from expert E_k, $\bar{x}_i$ is the mean value calculated from aggregated group opinion by FLOWA.

Step 3. Similarity/Dissimilarity Measure

The similarity is a function of the distance. Currently, there are many similarity measures as we summarized in Chapter 2. One possible similarity measure is to compare the ratio of the total distance.

$$s_{ik} = 1 - \frac{D_{ik}}{T} \tag{5.56}$$

$$s_k = \frac{1}{n}\sum_{i=1}^{n} s_{ik} \tag{5.57}$$

Where S_{ik} is the similarity between expert E_k and the group on alternative A_i, s_k is the average similarity between expert E_k and the group.

Step 4. Group Consensus

Thus, the consensus is the average of the similarities from all experts.

$$C = \frac{1}{q}\sum_{k=1}^{q} s_k \tag{5.58}$$

Again, q is the number of expert in the group.

Example 5.3:

A group with four experts and three alternatives, the Table 5.14 gives the evaluations from the group. Experts choose their evaluations form a linguistic set with nine labels S={s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8}.

Table 5.14 Experts' evaluations

	A_1	A_2	A_3	Weight
E_1	s_1	s_2	s_0	0.0625
E_2	s_4	s_1	s_3	0.1875
E_3	s_0	s_3	s_4	0.3125
E_4	s_2	s_2	s_3	0.4375

For alternative A_1, we have the following membership functions (Table 5.15), in which the group opinion is calculated from FLOWA.

Table 5.15 FLOWA aggregation result

A_1	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8

E_1	0	1	0	0	0	0	0	0	0
E_2	0	0	0	0	1	0	0	0	0
E_3	1	0	0	0	0	0	0	0	0
E_4	0	0	1	0	0	0	0	0	0
Group	**0.125**	**0.253**	**0.340**	**0.207**	**0.075**	**0.000**	**0.000**	**0.000**	**0.000**

Apply the distance and the similarity formula (5.55)~(5.58):

Table 5.16 Similarity based consensus

Expert	Similarity
E_1	0.8585
E_2	0.8429
E_3	0.8203
E_4	0.9679
Consensus	**0.8724**

5.4.3 Markov Chain Based Consensus Measure

Based on Degoot and Morris (1974), experts give peer evaluation by giving weights to other members' opinions. Consensus happens when the Markov process reaches the steady state. Our method to measure the current consensus level is to compare the current peer-evaluation matrix to the steady state matrix. The similarity between these two matrix can be used as the consensus level. This is exactly the same idea with TOPSIS, Technique for Order Preference by Similarity to Ideal Solution. Several similarity measure approaches from the multi-dimensional category can handle the problem like this. One of them is from Di Nola et al. (1994) as we introduced in Chapter 2. We will use the following example to demonstrate the procedure of the consensus measure method.

Example 5.4:

1. Experts evaluate alternatives by giving pairwise comparison matrix, totally 4 alternatives

$$P^1 = \begin{bmatrix} - & s_3 & s_5 & s_1 \\ s_5 & - & s_7 & s_3 \\ s_3 & s_1 & - & s_0 \\ s_7 & s_5 & s_8 & - \end{bmatrix} \quad P^2 = \begin{bmatrix} - & s_1 & s_3 & s_3 \\ s_7 & - & s_6 & s_5 \\ s_5 & s_2 & - & s_3 \\ s_5 & s_3 & s_5 & - \end{bmatrix} \quad P^3 = \begin{bmatrix} - & s_5 & s_7 & s_3 \\ s_3 & - & s_6 & s_2 \\ s_1 & s_2 & - & s_0 \\ s_5 & s_6 & s_8 & - \end{bmatrix} \quad P^4 = \begin{bmatrix} - & s_6 & s_7 & s_5 \\ s_2 & - & s_4 & s_3 \\ s_1 & s_4 & - & s_0 \\ s_3 & s_5 & s_8 & - \end{bmatrix}$$

2. Apply FLOWA to get evaluation of each alternative from each expert. Totally, we have 4 similar tables from 4 experts in this case.

Table 5.17 The evaluation from expert E1

E_1	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
A_1	0	0.44	0.33	0.22	0	0	0	0	0
A_2	0	0	0	0.44	0.33	0.22	0	0	0
A_3	0.67	0.33	0	0	0	0	0	0	0
A_4	0	0	0	0	0	0.44	0.33	0.22	0

3. Experts evaluate peer group member's opinion by giving weighting vectors

$$\begin{matrix} E1 \\ E2 \\ E3 \\ E4 \end{matrix} \begin{bmatrix} 0.60 & 0.10 & 0.10 & 0.20 \\ 0.05 & 0.90 & 0.05 & 0 \\ 0.40 & 0.20 & 0.30 & 0.10 \\ 0.20 & 0.10 & 0 & 0.70 \end{bmatrix} \tag{5.59}$$

We have now 4 weighting vectors (each row) indicating 4 experts' opinions to other members' opinions. From the 4 weighting vectors, we generate one weighting vector as the group's weighting vector. Use OWA to aggregate the weighting vectors from all group members, column by column. For example, use "*most*" experts' opinions.

$Q(r) = \begin{cases} 0, & if\ r < a \\ \dfrac{r-a}{b-a}, & if\ a \le r \le b \\ 1, & if\ r > b \end{cases}$, with (a, b)=(0.3, 0.8), we generate aggregating vector (0, 0.4, 0.5, 0.1).

For the weights to E1, using $(0.6, 0.05, 0.40, 0.20)^T$, applying OWA method, F(0.6, 0.05, 0.40, 0.20)=0*0.6+0.4*0.40+0.5*0.20+0.1*0.05=0.265. For E2, F(0.10, 0.90, 0.20, 0.10)=0*0.9+0.4*0.20+0.5*0.10+0.1*0.10=0.140. Similarly, For E3, F(0.10, 0.05, 0.30, 0)=0*0.30+0.4*0.10+0.5*0.05+0.1*0=0.065, and for E4, F(0.20, 0, 0.10, 0.70)=0*0.7+0.4*0.20+0.5*0.10+0.1*0 =0.130.

Now, normalize the weights (0.265, 0.140, 0.065, 0.130), we get, ***w***=(0.442, 0.233, 0.108, 0.217).

4. Aggregate all experts' opinions together get a group opinions

$$\mu_{ij}^{G} = \sum_{k=1}^{q} w_k \cdot \mu_{ijk} \tag{5.60}$$

Where w_k is the normalized weight to expert E_k, μ_{ijk} is the membership value to j^{th} linguistic label with alternative A_i from the FLOWA aggregation, q is the number of experts. Table 5.18 shows the group evaluation aggregated using weighted average from four experts' FLOWA outputs.

Table 5.18 Group evaluation aggregated by group's weighting vector

G	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
A_1	0	0.10	0.20	0.30	0.25	0.15	0	0	0
A_2	0	0	0	0.45	0.27	0.15	0.13	0	0
A_3	0.27	0.18	0.16	0.14	0.10	0.10	0.05	0	0
A_4	0	0	0.30	0.25	0.20	0.15	0.10	0	0

5. Measure consensus

Similar to Degoot and Morris's method (1974), find steady-state weights using Markov chain approach (5.60). The steady-state weights indicate the ideal weighting vector the group can get, that means, using this weighting vector, the group gets the best consensus.

$$\pi = \pi P \quad (5.61)$$

$$\sum \pi = 1 \quad (5.62)$$

Where π is called the *stationary distribution*. In the example, the steady state matrix is calculated as:

$$\begin{matrix} E1 \\ E2 \\ E3 \\ E4 \end{matrix} \begin{bmatrix} 0.098 & 0.458 & 0.047 & 0.397 \\ 0.098 & 0.458 & 0.047 & 0.397 \\ 0.098 & 0.458 & 0.047 & 0.397 \\ 0.098 & 0.458 & 0.047 & 0.397 \end{bmatrix}$$

Measure similarity between the steady-state matrix and the matrix from experts in step 3.

(0.442, 0.233, 0.108, 0.217) and (0.098, 0.458, 0.047, 0.397)

$$s = 1 - \sum_{k=1}^{q} (u_k^S - u_k^G)^2 = 1 - 0.205 = 0.795$$

5.5 Changing Opinions Based on Consensus

Consensus is a tool used to measure the quality of the group decision making. The goal of the group decision making is to get an acceptable group solution with all group members' support. Based on the consensus level and the contribution of each group member, we should be able to give suggestions on which direction the group should move for a better consensus, and thus better group solution.

In this section, we shall present a solution to give the best moving direction for the group in using consensus.

5.5.1 A Feedback Mechanism from Herrera-Viedma et al. (2002)

When the consensus measure has not reached the consensus level required, then the experts' opinions must be modified. Herrera-Viedma et al. (2002) use the proximity measures to build a feedback mechanism so that experts can change their opinions in order to get closer opinions between them. This feedback mechanism will be applied when the consensus level is not satisfactory, and will be ceased when a satisfactory consensus level is reached.

The rules of this feedback mechanism will be easy to understand and to apply, and will be expressed in the following form: "If the expert's position in the ranking is high (first, second, etc.) then that expert does not change his opinion much, but if it is low (last) then that expert has to change his opinion substantially. In other words, the first experts to change their opinions are those whose individual solutions are furthest from the collective temporary solution." Herrera-Viedma et al. (2002) use a threshold to calculate how many experts have to change their opinions.

The opinions will be changed using the following three rules:

- If $O_{A_i}^{G} - O_{A_i}^{E_k} < 0$, then increase evaluations associated with alternative A_i.
- If $O_{A_i}^{G} - O_{A_i}^{E_k} = 0$, do not change evaluations associated with alternative A_i.
- If $O_{A_i}^{G} - O_{A_i}^{E_k} > 0$, then decrease evaluations associated with alternative A_i.

$O_{A_i}^{G}$ is the index of the i^{th} alternative in the group's selection order.

$O_{A_i}^{E_k}$ is the order of the i^{th} alternative based on the k^{th} expert

The consensus reaching process will depend on the size of the group of experts as well as on the size of the set of alternatives, so that when these sizes are small and when opinions are homogeneous, the consensus level required is easier to obtain. On the other hand, the change of opinion can produce a change in the temporary collective solution, especially when the experts opinions are quite different, i.e., in the early stages of the consensus process. In fact, when experts opinions are close, i.e., when the consensus measure approaches the consensus level required, changes in experts' opinions will not affect the temporary collective solution; it will only affect the consensus measure. This is a convergent process to the collective solution, once the consensus measure is high "enough". This will be illustrated with a practical example in the next section.

5.5.2 A New Changing Direction Approach

Since the final solution is one or several alternatives, we do not need to spend time on some alternatives that are obviously "bad solutions". The idea here is that we divided all alternatives into four areas: green zone, red zone, and two gray zones. We use two thresholds: the threshold r_δ is for the ranking order of alternatives, while the threshold c_δ is for the consensus. Then the following rules can be applied to deicide which zone an alternative belongs:

- If $O_{A_i}^G \geq r_\delta$ and $c_i \geq c_\delta$, then alternative A_i should be assigned into the green zone.
- If $O_{A_i}^G < r_\delta$ and $c_i < c_\delta$, then alternative A_i should be assigned into the red zone.
- If $O_{A_i}^G \geq r_\delta$ and $c_i < c_\delta$, or $O_{A_i}^G < r_\delta$ and $c_i \geq c_\delta$, then alternative A_i should be assigned into the gray zone.

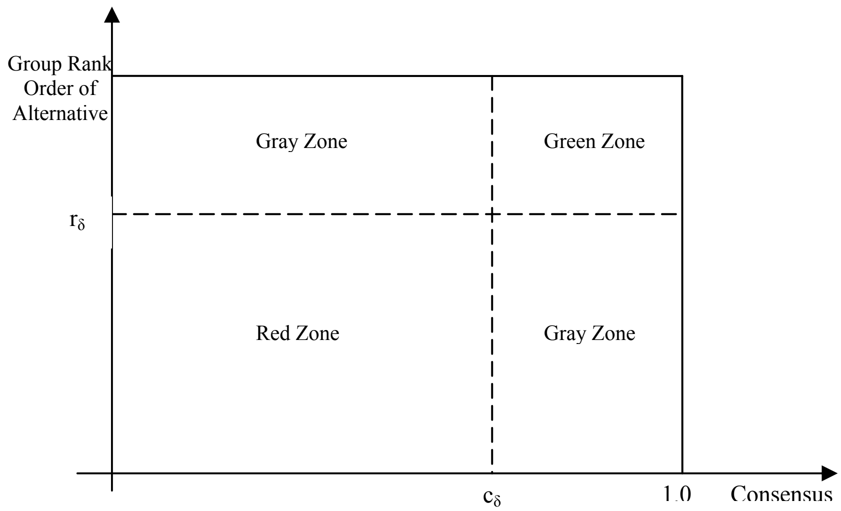


Figure 5.3 All alternatives are assigned into four zones for further consideration

We then apply the following rules to make choices:

- If the number of alternatives in green zone is greater or equal to p (number of alternatives need for the solution), we stop here and give the alternatives in green zone as solution.
- If the number of alternatives in green zone is smaller to p, we should try to increase the consensus and/or ranking of alternatives in the gray areas as candidates.
- For the alternatives in the red area, both consensus and the ranking are not promising, they should not be evaluated anymore.

5.6 Summary

In this chapter, we explained the meaning of consensus and introduced the state-of-the-art literature review on current existing consensus measure methods which are categorized into two groups, hard and soft consensus. We also presented three new consensus measure methods based on similarity measure, the preference order of alternatives, and Markov chain theory. In a consensus-based approach for achieving sustainability, the decision process becomes iterative, using feedback to evaluate progress in discussions among decision makers. Degree of consensus indicates the level of agreement with the ordinal ranking of each alternative.

CHAPTER 6

Cost and Utility of Consensus Analysis

Ephrati and Rosenschein (1992) proposed the Clarke Tax mechanism as a plausible group decision procedure where experts choose their own utility functions for the evaluations of alternatives. The basic idea is to make sure that each expert has only one dominant strategy. This guarantees that the group as a whole will choose an outcome that maximize the sum of the members' individual utilities.

In this chapter, we analyze the cost in group decision making and generates a least cost of group consensus. We develop optimization models to maximize two types of consensus under a budget constraint. Finally considering utilization of the consensus provides a practical recommendation to the desired level of consensus, considering its cost benefits.

6.1 Introduction

Time is a major factor of risk. Risk event will always occur within a time interval. Hansson (2002) writes this about the meaning of the word "risk". The most common ones are the following:

- Risk = an *unwanted event* which may or may not occur.
- Risk = the *cause* of an unwanted event which may or may not occur.
- Risk = the *probability* of an unwanted event which may or may not occur.

- Risk = the statistical *expectation value* of unwanted events which may or may not occur.

- Risk = the fact that a decision is made under conditions of known probabilities

6.2 Utility Theory Review

Traditionally, the desires or wants that can be ordered on a single scale are called *utility.* There are a number of axioms of rational choice, e.g. *completeness,* which says any two bundles can always be compared and ranked; *transitivity* states that when A is preferred to B, and B is preferred to C, then A should be preferred to C.

Utility is replaced with `value' in prospect theory (Kahneman and Tversky, 1979). 'Utility' usually thought of in terms of net wealth - total assets; 'Value' is considered in terms of gains and losses from some reference point. Also the value function for gains and losses is different. The value function for losses (the curve lying below the horizontal axis) is convex and relatively steep, whereas that for gains is concave and less steep.

People seem to differ in their tolerance for uncertainty and their willingness to take risks. People tend to be interested in the *potential* that uncertainty offers, seeking to maximise the possible gains available (Lopes, 1987). These she labels risk-seekers. Others are more concerned with security and motivated to avoid extreme losses by seeking safe choices. These are labelled risk averse. Kahneman and Tversky (1979) have conducted lots of experiments to show that risk-seeking in losses is a robust effect, particularly when the probabilities of loss are substantial. e.g., if forced to choose between an 85% chance to lose $1000 (with a 15% chance to lose nothing) and a sure loss of $800, a large majority of people will choose the gamble over the sure loss. This is a risk-seeking choice because the expectation of the gamble (-$850) is inferior to the expectation of the sure loss (-$800). This type of behaviour has also been observed in non-monetary situations such as hours of pain, loss of lives.

So, in summary, you have to look at the rationality of choices under uncertainty or risk in terms of the context within which the choice is made, and it may also be necessary to take individual differences in risk preference also into account. The context is one factor that enters into the perception stage and becomes part of the calculation in the decision process.

The standard decision theoretical solution to the utilitarian causal dilution problem is the maximization of expected utility. One way to maximize expected utility is to choose among a set of alternatives one of those that have the highest expected, i.e. probability-weighted utility. This decision rule is based on a precise method for dealing with probabilistic mixtures.

6.2.1 The Attitude Towards Risk

Every day we face risks; when we take a shower, walk across the street or make an investment. The effort that the decision makers make for example buying insurance, depends on their preferences. An individual might be very conservative and choose to purchase a lot of insurance, or she might like to take risks and not purchase any insurance at all. A question that many people ask is: how much effort is the good worth? So, how much is the individual willing to risk in order to receive the good or award? The answer is: it depends on the individual's attitude towards risk. The decisions are made upon the amount of risk that she is willing to face.

An individual's attitude towards risk makes it possible to categorize her into one of the three following groups (Varian, 1999).

6.2.1.1 Risk aversion

If the individual is risk-averse, she prefers to have the expected value of wealth rather than to face the gamble. For a risk-averse customer the utility of the expected value of wealth is greater than the expected utility of wealth.   That is, for any prospect (x, f(x)), $u(E(x)) > E(u(x))$.

6.2.1.2 Risk Prone

A risk-loving individual makes her decisions by preferring to face a gamble (i.e. take the risk), rather than to obtain the so-called expected value of wealth. For a risk-loving customer the expected utility of wealth is greater than the utility of the expected value wealth.

The certainty income is the income received for certain that has the same utility as a risky prospect. $u(CE) = E(u(x))$. A comparison of the certainty equivalent and the expected income gives a measure of risk called the risk premium, defined in this context as $RP = E(x) – CE$ or implicitly by the relationships $u(CE) = u(E(x)\text{-}RP) = E(u(x))$.

6.2.1.3 Risk Neutral

A risk-neutral person does not care about the risking of wealth at all- only about its expected value.

6.2.1.4 Concavity, convexity and the attitude to risk

Strict concavity of the utility function implies $u(p_1x_1 + p_2x_2) > p_1u(x_1)+p_2u(x_2)$ and therefore implies strict risk aversion. Strict convexity implies risk loving. If the utility function is linear, the individual is risk neutral. Figure 6.1 shows the three types of risk attitudes.

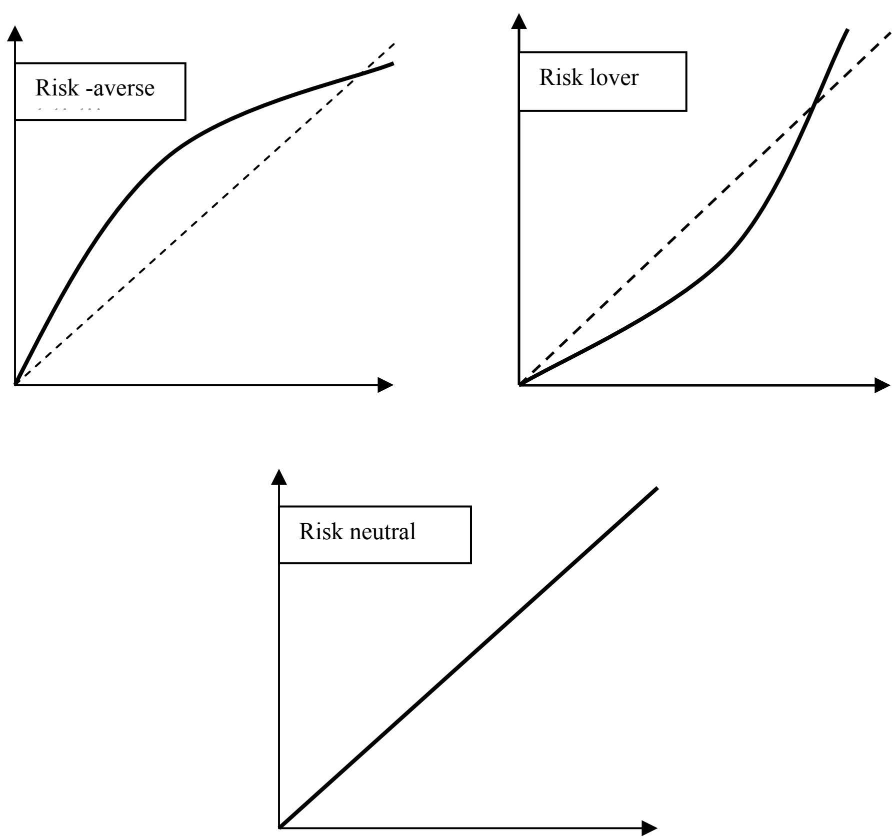


Figure 6.1 Three types of risk attitudes

6.2.2 Risk Measures in Linguistic Group Decision Making

We introduce a parameter α in FLOWA to represent the risk attitude from expert. α_k, is the power for the kth expert's aggregation function. For expert E_k's evaluation, instead of using a linear distribution (y=x) in FLOWA, we use two parameter functions. $y = x^{\alpha}$ and $y = x^{\frac{1}{\alpha}}$ to control pessimism and optimism as explained in Table 6.1.

Table 6.1 Expert's risk attitude

	α
Pessimism	$0 < \alpha \le 1$
Optimism	$\alpha \ge 1$

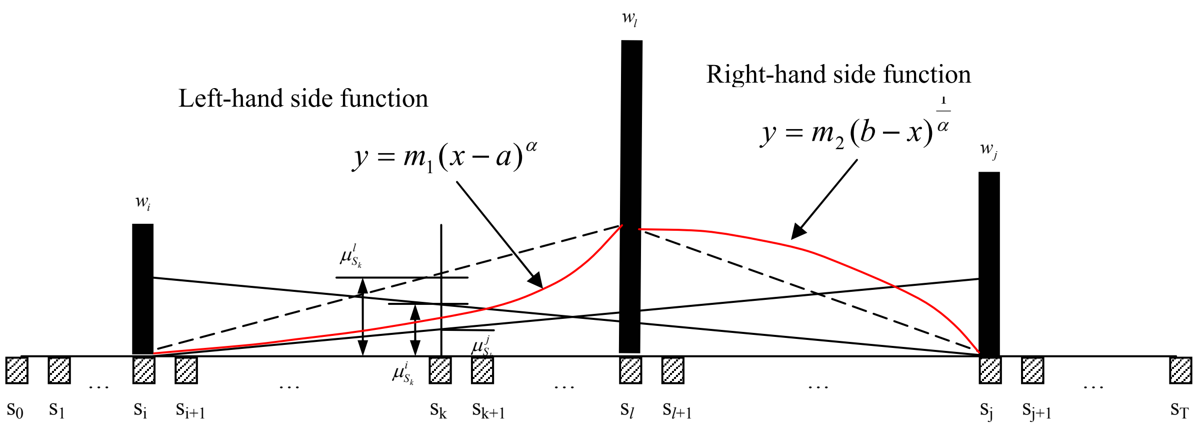


Figure 6.2 FLOWA with experts' preference parameter α

As we stated in Chapter 4, we define $a = j_{\min} = \min_k(x_{ik})$, $b = j_{\max} = \max_k(x_{ik})$, then the left-hand side function as shown in Figure 6.2 is

$$y = m_1(x-a)^{\alpha}, x = a, \ldots, x_{ik} \tag{6.1}$$

The right-hand side function is:

$$\mathrm{y} = m_2 (b - x)^{\frac{1}{\alpha}}, x = x_{ik},..., b \tag{6.2}$$

To solve for m_1 and m_2, we add two constraints:

$$\sum_{j=a}^{k} m_1 (j - a)^{\alpha} + \sum_{j=k+1}^{b} m_2 (b - j)^{\frac{1}{\alpha}} = w_k \tag{6.3}$$

$$m_1 (x_{ik} - a)^{\alpha} - m_2 (b - x_{ik})^{\frac{1}{\alpha}} = 0 \tag{6.4}$$

The constraint 6.3 is used to make all the weights the linguistic labels between s_a and s_b sum to the original weight w_k on the linguistic label s_k. The constrain 6.4 guarantees the weight to the linguistic label s_k calculated from left-hand side function and right-hand side function is the same. Then,

$$m_1 = \frac{w_k}{\sum_{j=a}^{x_{ik}} (j - a)^{\alpha} + \frac{(x_{ik} - a)^{\alpha}}{(b - x_{ik})^{\frac{1}{\alpha}}} \sum_{j=x_{ik}+1}^{b} (b - j)^{\frac{1}{\alpha}}} \tag{6.5}$$

$$m_2 = \frac{w_k}{\frac{(b - x_{ik})^{\frac{1}{\alpha}}}{(x_{ik} - a)^{\alpha}} \sum_{j=a}^{x_{ik}} (j - a)^{\alpha} + \sum_{j=x_{ik}+1}^{b} (b - j)^{\frac{1}{\alpha}}} \tag{6.6}$$

Thus,

$$\mu_{ijk}^{F} = \begin{cases} \dfrac{w_k}{\sum_{l=a_i}^{x_{ik}} (l-a_i)^{\alpha_k} + \dfrac{(x_{ik}-a_i)^{\alpha_k}}{(b_i - x_{ik})^{\frac{1}{\alpha_k}}} \sum_{l=x_{ik}+1}^{b_i} (b_i - l)^{\frac{1}{\alpha_k}}} (j-a_i)^{\alpha_k}, & a_i \le j \le x_{ik} \\ \dfrac{w_k}{\dfrac{(b_i - x_{ik})^{\frac{1}{\alpha_k}}}{(x_{ik}-a_i)^{\alpha_k}} \sum_{l=a_i}^{x_{ik}} (l-a_i)^{\alpha_k} + \sum_{l=x_{ik}+1}^{b_i} (b_i - l)^{\frac{1}{\alpha_k}}} (b_i - j)^{\frac{1}{\alpha_k}}, & x_{ik} \le j \le b_i \end{cases} \tag{6.7}$$

The aggregated group opinion is:

$$\mu_{ij} = \sum_{k=1}^{q} \mu_{ijk}^{F} \tag{6.8}$$

The parameter α represents the risk attitude of expert (Table 6.1). For instance, in Figure 6.2 an optimistic expert uses $\alpha \ge 1$, the higher index of linguistic labels get more weights than the lower ones. In this way, we combine experts' risk attitudes in the FLOWA aggregation process.

6.3 Consensus Analysis with Cost Optimization

6.3.1 Literature Review

Subramanian and Venkataraman (1998) develop a cost-based algorithm that takes a decision support processing query plan as input and generates an optimal "covering plan", by minimizing redundancies such as repeated access of same data source and multiple execution of similar processing sequences in the original plan. Minimizing these redundancies would significantly reduce the query processing cost.

Pednault et al. (2002) propose a novel approach to sequential decision making based on the reinforcement learning framework. The approach attempts to learn decision rules that optimize a sequence of cost-sensitive decisions so as to maximize the total benefits accrued over time.

Trull (1966) investigates 100 case examples to understand the mechanics involved in heuristic decisions, or decisions where the lack of a measure or index of effectiveness. Examination of the cases showed that although the decision process was not explicit, a certain clustering of key variables appeared as a common feature of the decision-reaching process. One cluster of variables was found to surround the quality of the decision itself. These included Compatibility with Existing Operating Constraints, Nearness to Optimum Time for Decision related to Proximity to Optimum Amount of Information, and the Problem Solver's Influence on the Decision. The second major factor, implementation of the decision, was affected by Avoidance of Conflict of Interests, Reward-Risk Factors, and the Degree of Understanding Achieved.

Lee's (2002) similarity based optimal consensus model is an improvement of Hsu's (1996) similarity aggregation method (SAM). The new method can deal with the situation where the supports do not intersect and tell whether the aggregation weights of individual opinions derived from SAM are optimal or not. Lee chooses Tong's distance metric(1980):

$$d_p(A,B) = \left(\sum_{i=1}^{4} (|a_i - b_i|)^p \right)^{1/p} \tag{6.9}$$

And the similarity between A and B is defined as:

$$S_p(A,B) = 1 - \frac{1}{4u^p} (d_p(A,B))^p \tag{6.10}$$

Where U is the universe of discourse and u=max (U)-min (U)

Also, the dissimilarity between A and B is defined as:

$$c - S_2(A,B) \tag{6.11}$$

Where c is a constant and c>1

Then an optimization model is constructed to minimize the sum of weighted dissimilarity between aggregated opinion and each individual opinion. So, the optimization model is as followings. (Note: this analytical problem is quite similar to fuzzy c-means problem.)

$$\underset{M \times R^4}{Minimize} : Z_{m,c}(W,R) = \sum_{j=1}^{q} (w_j)^m (c - S_2(R_j, R)) \tag{6.12}$$

Subject to:

$$\sum_{j=1}^{q} w_j = 1 \tag{6.13}$$

$$0 \le w_j \le 1 \tag{6.14}$$

m is an integer and m>1.

Lee (2002) proposes an OAM algorithm to solve the optimization model with the degree of importance of experts. The consensus is a fuzzy set.

6.3.2 Cost Analysis

As we discussed before, it is very rare when all individuals in a group share the same opinion about the alternatives, since a diversity of opinions commonly exists. There are several explanations that allow for experts not to converge to a uniform opinion. It is well accepted that experts are not necessarily the decision makers but provide an advice (Shanteau et al., 2002). Weiss and Shanteau (in press) also describe five structural and five functional factors that explain this necessary lack of agreement. Consensus makes it possible for a group to reach a final decision that all group members can support among these differing opinions. But consensus could be very time consuming. Cost analysis can be applied to improve decision making. A cost analysis can be the critical process needed to solve group decision making problems. In an expanding-resource environment, cost analysis can be used to encourage efficiency by associating inputs (costs) with outputs (group solution with desired consensus).

Cost analysis is a management process which can be used to provide information useful to effective decision-making (Dopuch et al., 1982). Knowledge about costs is an essential ingredient in effective decision-making and contributes to improved planning, implementation, and analysis of every aspect of endeavor. Cost analysis provides a tool for understanding what services are being provided; what they cost; why they cost what they do; and what can, and should, be changed.

There are two kinds of costs: direct cost and indirect cost. Direct costs are usually subdivided into personnel costs consisting of wages and benefits, and other direct costs consisting of equipment used and supplies consumed in the delivery of a service. Usually, there are two types of indirect costs: indirect service costs and indirect administrative costs. Indirect service costs are those that might be performed by a service unit by and for itself, but which are centrally controlled. Indirect administrative costs are associated with activities that must be incurred by the organization, but which do not directly benefit any service delivery function.

In group decision making, the main cost is associated with time, which connects to direct costs such as the experts' resources utilization and indirect costs as administration. Here, we simply assume the cost is a function of the time for improving consensus. We use a resistance coefficient ε associated with each expert to measure efforts needed to persuade the expert to make change. This resistance coefficient represents the time, thus cost for consensus. We also assume that the resistance coefficient ε to each expert is known. In practice, this value can be learned from history data, or estimated based on expertise or personality. A further assumption is that the resistance coefficient ε is a fixed value. Future research can give it a function of the changing opinion. We do not consider the budget constraint, since we are trying to get the cost of consensus. In this way we should be able to find which consensus gives us the best value of investment.

6.3.3 Optimization Model

In this section, we shall build an optimization model for the following procedure describes the original data format from experts, how to add aggregation constraint, order calculation, and consensus measure.

6.3.3.1 Step 1. Expert evaluation

Expert E_k choose one linguistic label s_j as the evaluation x_{ki} to alternative A_i (Table 6.2).

Table 6.2 Experts' evaluations represented by linguistic labels

	A_1	A_2	……	A_n
E_1	x_{11}	x_{12}	……	x_{1n}
E_2	x_{21}	x_{12}	……	x_{2n}
……	……	……	x_{ki}	……
E_q	x_{q1}	x_{12}	……	x_{qn}

Where $x_{ki} = j,\ wher\, s_j \in S,\quad S = \{s_0, s_1, ..., s_T\}$

6.3.3.2 Step 2. FLOWA to aggregate all experts' opinions

Define ε_k, ($\varepsilon_k \geq 0$, k=1, 2,…,q) as a resistance coefficient to E_k. It stands for the willingness of the expert E_k to be persuaded on changing his opinion. ε_k =0 means the expert is very easy to be persuade to change his evaluation with no additional cost.

$$\mu_{ij} = \sum_{k=1}^{q} \mu_{ij}^{E_k} \tag{6.15}$$

μ_{ij} is the aggregated membership from all experts to the linguistic label s_j on alternative A_i, this is calculated from FLOWA (Table 6.3).

$\mu_{ij}^{E_k}$ is the weight distributed from the weight u_k of expert E_k to the j^{th} linguistic label on alternative A_i. The weight u_k of expert E_k can be either by assignment or calculating form OWA.

Table 6.1 FLOWA aggregation results

	A_1	**……**	**A_n**	Resistance Coefficient
E_1	x_{11}	……	x_{1n}	ε_1
E_2	x_{21}	……	x_{2n}	ε_2
……	……	x_{ki}	……	ε_k
E_q	x_{q1}	……	x_{qn}	ε_q
G	$(\mu_{10}, \mu_{11}, ..., \mu_{1T})$	$(\mu_{i0}, \mu_{i1}, ..., \mu_{iT})$	$(\mu_{n0}, \mu_{n1}, ..., \mu_{nT})$	

The aggregated group opinion is:

$$\mu_{ij} = \sum_{k=1}^{q} \mu_{ijk}^{F} \tag{6.16}$$

6.3.3.3 Step 3. Ranking the alternatives

Based on the aggregated group opinion, we calculate the mean (6.17) and the standard deviation (6.18) of the aggregated group opinion to each alternative (Ben-Arieh and Chen, in press). The mean $\bar{x}_i(G)$ is as the group opinion of the i[th] alternative.

$$\bar{x}_i(G) = \frac{\sum_{j=0}^{T} j \cdot \mu_{ij}}{\sum_{j=0}^{T} \mu_{ij}} = \sum_{k=0}^{T} j \cdot \mu_{ij} \tag{6.17}$$

$$\sigma_i(G) = \left[\frac{\sum_{j=0}^{T} j^2 \mu_{ij}}{\sum_{j=0}^{T} \mu_{ij}} - [\bar{x}_i]^2 \right]^{1/2} = \left[\sum_{j=0}^{T} j^2 \mu_{ij} - [\bar{x}_i]^2 \right]^{1/2} \tag{6.18}$$

Compute the order of the alternatives from each expert and the whole group. $O_{A_i}^{E_k}$ represents the order of the i^th^ alternative from k^th^ expert. The order is from the original expert evaluation x_{ik}.

$$O_{A_i}^{E_k} = Order(A_i \mid x_{ik}) \tag{6.19}$$

The order of the i^th^ alternative from the group $O_{A_i}^{G}$ is based on the calculated mean and the standard deviation. The larger mean value an alternative has, the smaller ranking order it gets. If two alternatives have the same mean value, the one with smaller standard deviation will get the smaller ranking order.

$$O_{A_i}^{G} = Order(A_i \mid (\bar{x}_i, \sigma_i)) \tag{6.20}$$

6.3.3.4 Step 4. Consensus Measure

As we presented in chapter 5, there are several consensus measure methods. Here, we use two approaches: one is order based, the other is mean based.

Method 1: Order based

The consensus of the ith alternative from the group C_i is based on the order difference cumulated from each group member and the group. (Ben-Arieh and Chen, in press). The total group consensus C is the averaging value of the alternative consensus.

$$C_i = \sum_{k=1}^{q}\left[\left(1 - \frac{\left|O_{A_i}^{G} - O_{A_i}^{E_k}\right|}{n-1}\right) \times w_k\right] \quad (6.21)$$

$$C = \frac{1}{n}\sum_{i=1}^{n} C_i \quad (6.22)$$

Method 2: Mean based

The mean based consensus measure is based on the distance between the opinion from individual expert and from the group, then measure the similarity with the formula:

$$s_{ik} = 1 - \frac{\left|x_{ik} - \overline{x}_i\right|}{T} \quad (6.23)$$

$$s_k = \frac{1}{n}\sum_{i=1}^{n} s_{ik} \quad (6.24)$$

Where s_{ik} is a linguistic label as the evaluation from expert E_k to alternative $A_{i.}$ $\overline{x}_i$ is the group opinion from formula (6.17). *T+1* is the cardinality of the linguistic label set, and *n* is the number of alternatives.

Thus, the consensus is the average of the similarities from all experts.

$$C = \frac{1}{q}\sum_{k=1}^{q} s_k \quad (6.25)$$

6.3.3.5 Step 5. Changing Direction

Based on the consensus level, we could give suggestions on how to change opinions for better consensus. We give three similar rules from Herrara et al. (1996b).

Method 1: Order based

For order based consensus, $O_{A_i}^{E_k} > O_{A_j}^{E_k}$ means expert E_k thinks that alternative A_j is more important than alternative A_i. The opinions will be changed using the following three rules:

- If $O_{A_i}^{G} - O_{A_i}^{E_k} < 0$, then increase evaluations associated with alternative A_i.
- If $O_{A_i}^{G} - O_{A_i}^{E_k} = 0$, do not change evaluations associated with alternative A_i.
- If $O_{A_i}^{G} - O_{A_i}^{E_k} > 0$, then decrease evaluations associated with alternative A_i.

$O_{A_i}^{G}$ is the index of the i^{th} alternative in the group's selection order.

$O_{A_i}^{E_k}$ is the order of the i^{th} alternative based on the k^{th} expert

<u>Method 2: Mean based</u>

The opinions will be changed using the following three rules:

- If $\bar{x}_i - x_{ik} > 0$, then increase evaluations associated with alternative A_i.
- If $\bar{x}_i - x_{ik} = 0$, do not change evaluations associated with alternative A_i.
- If $\bar{x}_i - x_{ik} < 0$, then decrease evaluations associated with alternative A_i.

$\bar{x}_i$ is the mean value of the i^{th} alternative from the group opinion

x_{ik} is the evaluation to the i^{th} alternative from the k^{th} expert

6.3.3.6 Step 6. Opinion Change Costs

Define ε_k , ($\varepsilon_k \geq 0$, k=1, 2,…,q) as a resistance coefficient to E_k (Table 6.3). It stands for the willingness of the expert E_k to be persuaded on changing his opinion. ε_k =0 means the expert is very easy to be persuade to change his evaluation with no additional cost. The total cost of changing opinions can be calculated by:

$$Cost = \sum_{k=1}^{q}\sum_{i=1}^{n}\varepsilon_k \cdot | x_{ik} - x_{ik}' | \qquad (6.26)$$

Where x_{ik}' is the initial evaluation from expert E_k to alternative A_i, while x_{ik} is the final evaluation to be solved, q is the number of experts, and n is the number of alternatives.

6.3.3.7 Step 7. Optimization model with budget constraint δ.

Given the cost constraint δ , the total optimization model is summarized in the followings.

Order-based consensus model

Objective: max $C = \frac{1}{n}\sum_{i=1}^{n} C_i$

Subject to:

$$Cost = \sum_{k=1}^{q}\sum_{i=1}^{n}\varepsilon_k \cdot | x_{ik} - x_{ik}' | \leq \delta \qquad (6.27)$$

$$C_i = \sum_{k=1}^{q}\left[\left(1 - \frac{\left|O_{A_i}^{G} - O_{A_i}^{E_k}\right|}{n-1}\right) \times w_k\right] \qquad (6.28)$$

$$O_{A_i}^{G} = Order(A_i \mid (\bar{x}_i, \sigma_i)) \qquad (6.29)$$

$$O_{A_i}^{E_k} = Order(A_i \mid x_{ik}) \tag{6.30}$$

$$\bar{x}_i(G) = \frac{\sum_{j=0}^{T} j\mu_{ij}}{\sum_{j=0}^{T} \mu_{ij}} = \sum_{k=0}^{T} j\mu_{ij} \tag{6.31}$$

$$\sigma_i(G) = \left[\frac{\sum_{j=0}^{T} j^2 \mu_{ij}}{\sum_{j=0}^{T} \mu_{ij}} - [\bar{x}_i]^2 \right]^{1/2} = \left[\sum_{j=0}^{T} j^2 \mu_{ij} - [\bar{x}_i]^2 \right]^{1/2} \tag{6.32}$$

$$\mu_{ij} = \sum_{k=1}^{q} \mu_{ijk}^F \tag{6.33}$$

$$\mu_{ijk}^F = \begin{cases} \dfrac{w_k}{\sum_{l=a_i}^{x_{ik}} (l - a_i) + \dfrac{(x_{ik} - a_i)}{(b_i - x_{ik})} \sum_{l=x_{ik}+1}^{b_i} (b_i - l)} (j - a_i), & a_i \le j \le x_{ik} \\ \dfrac{w_k}{\dfrac{(b_i - x_{ik})}{(x_{ik} - a_i)} \sum_{l=a_i}^{x_{ik}} (l - a_i) + \sum_{l=x_{ik}+1}^{b_i} (b_i - l)} (b_i - j), & x_{ik} \le j \le b_i \end{cases} \tag{6.34}$$

$$\mathrm{a_i} = \mathrm{j_{min}} = \min_k(\mathrm{x_{ki}}) \tag{6.35}$$

$$\mathrm{b_i} = \mathrm{j_{max}} = \max_k(\mathrm{x_{ki}}) \tag{6.36}$$

$$\alpha_k > 0$$

$$\mathrm{x_{ik}} \in \{0,1,\ldots,T\}, \quad 1 \le i \le n; 1 \le k \le q \tag{6.37}$$

$$0 \le j \le T \tag{6.38}$$

Mean-based consensus model

Objective: max $C = \frac{1}{q}\sum_{k=1}^{q} s_k$

Subject to:

$$Cost = \sum_{k=1}^{q}\sum_{i=1}^{n} \varepsilon_k \cdot | x_{ik} - x_{ik}' | \leq \delta \tag{6.39}$$

$$s_{ik} = 1 - \frac{|x_{ik} - \bar{x}_i|}{T} \tag{6.40}$$

$$s_k = \frac{1}{n}\sum_{i=1}^{n} s_{ik} \tag{6.41}$$

$$\bar{x}_i(G) = \frac{\sum_{j=0}^{T} j\mu_{ij}}{\sum_{j=0}^{T} \mu_{ij}} = \sum_{k=0}^{T} j\mu_{ij} \tag{6.42}$$

$$\mu_{ij} = \sum_{k=1}^{q} \mu_{ijk}^{F} \tag{6.43}$$

$$\mu_{ijk}^{F} = \begin{cases} \dfrac{w_k}{\sum_{l=a_i}^{x_{ik}}(l - a_i) + \dfrac{(x_{ik} - a_i)}{(b_i - x_{ik})}\sum_{l=x_{ik}+1}^{b_i}(b_i - l)}(j - a_i), & a_i \leq j \leq x_{ik} \\ \dfrac{w_k}{\dfrac{(b_i - x_{ik})}{(x_{ik} - a_i)}\sum_{l=a_i}^{x_{ik}}(l - a_i) + \sum_{l=x_{ik}+1}^{b_i}(b_i - l)}(b_i - j), & x_{ik} \leq j \leq b_i \end{cases} \tag{6.44}$$

$$\mathrm{a_i} = \mathrm{j_{min}} = \min_k(\mathrm{x_{ki}}) \tag{6.45}$$

$$\mathrm{b_i} = \mathrm{j_{max}} = \max_k(\mathrm{x_{ki}}) \tag{6.46}$$

$$\alpha_k > 0$$

$$x_{ik} \in \{0,1,...,T\}, \quad 1 \le i \le n; 1 \le k \le q \tag{6.47}$$

$$0 \le j \le T \tag{6.48}$$

6.3.4 Solution Methodology

It is clear that the models are nonlinear problems (NLP). For a small problem with small number of group number and alternatives, we can enumerate the solution space and find the optimal solution quickly. But, for a larger problem, the solution space increases exponentially as:

$$(T+1)^{nq} \tag{6.49}$$

For instance, as shown in Table 6.4, even for a small 5 linguistic labels set with T=4, three alternatives and 3 experts, the solution space is $(4+1)^{3x3}=5^9$=1,953,125. For a PC computer with 1.0 GHz CPU power, the computation time of one feasible solution is about 1/15 second. So searching the total solution space will take about 36 hours.

Table 6.4 Calculation time comparison

T+1	n	q	Solution Space	Seconds	Hours	Days
5	3	3	1,953,125	130,208	36.17	1.5
7	3	3	40,353,607	2,690,240	747.29	31.1
5	4	4	1.52588E+11	10,172,526,042	2,825,701.68	117,737.6
7	4	4	3.32329E+13	2.21553E+12	615,424,640.18	25,642,693.3

For a large problem, it is not efficient to use enumeration. The response time is critical for the consensus optimization problem. It is meaningless if it takes too long to solve the problem. Since the decision cost is a function of time. It is necessary to develop a more efficient solution methodology.

In this section, a heuristic-based search algorithm is developed for the consensus optimization problem (Figure 6.3). A "generate-and-test" approach is adopted, in which one or a few possible solutions are generated, and the status of the problem is tested. Based on the result, the nest step is decided. The breadth-first search procedure is used as the search engine. A natural question is how efficient the algorithm will be, since the search space may be large. That is where heuristics come in. The key here is to compare the current node in the searching tree. If the next node will give positive effect on higher consensus with cost not exceed the budget, we continue searching to the next node, otherwise, return to the last node.

6.3.4.1 Branch and Bound

Figure 6.3 shows the breadth-first branch-and-bound search algorithm for the consensus optimization problem. At each node, the problem is represented by set of evaluations from the group members with the calculated consensus and the cost to move to the current values.

The Breadth First Search (BFS) algorithm starts at a specific node (Initial experts' evaluation), which becomes current node. Then algorithm traverses by any edge incident to the current node. If the edge leads to an already visited node, then we backtrack to current node. If, on other hand, edge leads to an unvisited node, then we go to this node and it becomes our new current node. We proceed in this manner until we can fathom a node by comparing the cost and the budget constraint. The process terminates when all nodes are fathomed.

The best solution is saved during the search process and is updated whenever a new and better solution is found. There are three important components that deserve separate discussions. The first is the preprocessing. Pre-processing defines the order of the branching nodes. By comparing the cost of changing opinions, the expert with the least cost should be branched first. Another criterion for the order of branching is to do the outlier analysis. The expert farthest from the group should be considered first, since changing his/her evaluation will benefit the consensus most.

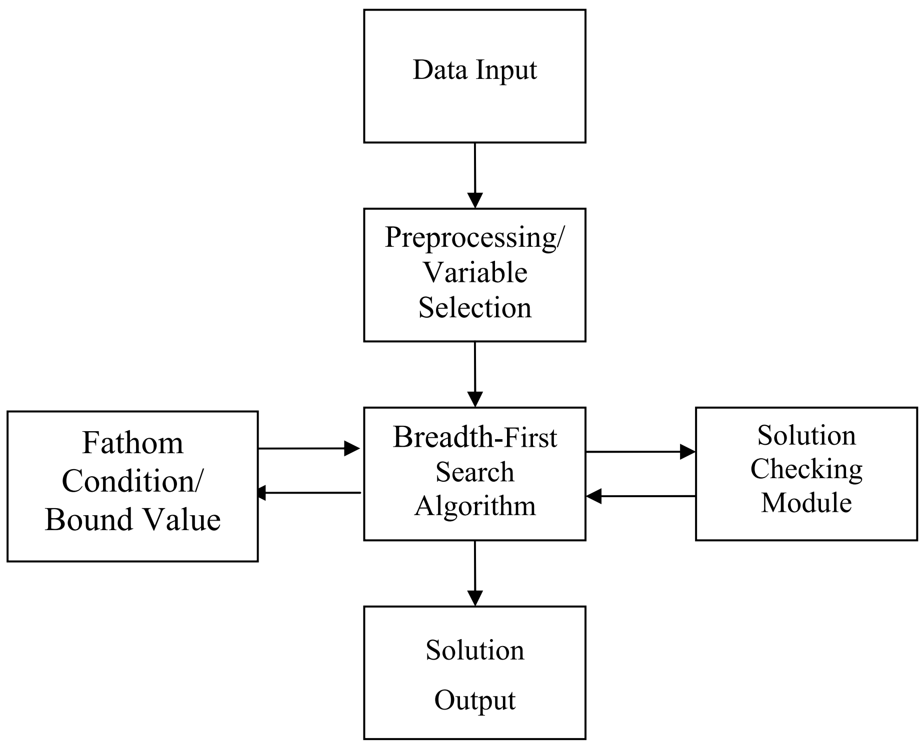


Figure 6.3 Schematic flow of the search algorithm for the consensus optimization

To improve the efficiency of the algorithm, a pruning scheme is necessary to avoid searching for branches that will not lead to better solutions than those already found. There are some criteria that can be used to evaluate the quality of the solutions generated. One of the most important criteria is of course the current cost value. Since we are using the BFS to search for the solution, the first node with cost to the budget threshold is the one with better consensus. Branching deeper may get better consensus, but will increase the cost. Another criterion is to compare the consensus. If the current node gives the same cost value, but worse consensus, it may not be the feasible solution even for the children of the node. These criteria are used to prune branches of the breadth-first search tree that are lead to inferior solutions in the algorithm.

So, at each node, we check the current solution on the three criteria: (Figure 6.4)

- Choose which value (changing expert) to change
- Choose which direction (increase/decrease) to move
- Calculate and compare with current consensus and cost value

6.3.4.2 Fathom

Fathoming is to determine that it is not necessary to explore the descendants of a particular node in the search tree. In the consensus optimization problem, a live node *k* is fathomed if the following condition is observed.

$$\text{Cost}^{k} > \delta \text{ (by cost bound)}$$

With respect to the fathoming by bounds, the efficiency of implicit enumeration depends on several things, including

1. The quality of the upper bound (for the consensus maximization problem, the smaller the bound the better);
2. The computational difficulty in obtaining the upper bound (the easier the better);
3. The quality of the lower bound (the larger the better);

4. The computational difficulty in obtaining the lower bound;
5. The sensitivity of the objective function value to the decision variables in **x** (the more sensitive the better; many "near-optimal" solutions can cause difficulties).

Then, if at some node k, $\bar{z}^k$ =(Consensus, Cost)^k satisfies this inequality, the node is fathomed.

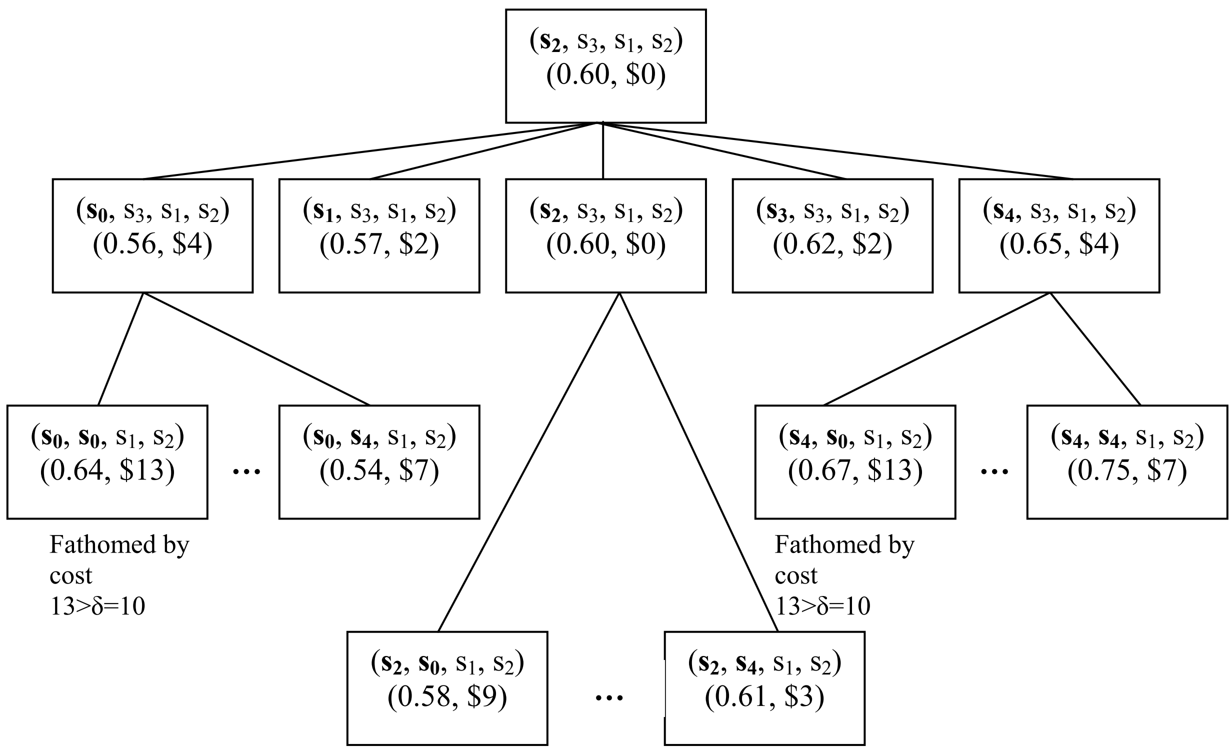


Figure 6.4 Branching examples in BFS for the consensus optimization problem

6.3.4.3 Pseudo Code

Algorithm consensus optimization

Begin

OPEN: Queue;

Cost Threshold: int;

Preprocessing and get the start node;

Put start node on OPEN;

While OPEN is not empty

Begin

Remove the bottom most node from OPEN and label it Current;

Calculate the Cost and Consensus of the Current node s;

If node s is a solution node (Consensus>δ, Cost improved)

Fathom node s;

Update solution list with s;

Else

Begin

Select a new expert with the least cost to change opinions;

Expand node *s*, generating all its successors;

Push these successors in the top of OPEN, in the order in which they are to be explored;

End If

End While

End Algorithm

6.3.4.4 Computation Time Comparison

For a large problem, it is not efficient to use enumeration. The response time is critical for the computation on finding an optimal solution. The efficiency of the rule-based search algorithm depends on the problem, especially the initial experts' evaluations, since this is used for the cost calculation. For instance, the problem with 5 linguistic labels, 3 alternatives, and 3 experts, the solution space is 1,953,125. As we stated earlier, it takes 130,208 seconds or 2,170 minutes to enumerate all the solution space. But using the branch and bound search algorithm, it takes only 44 minutes for worst case, which is given the budget as the smallest cost for a consensus to 1.

6.4 Utility of Consensus

By solving the optimization model, we could plot the relationship between cost and consensus. That is, given a budget constraint, the best consensus we could get. It is common sense that the more money we spend, the better consensus we could get. Now the question is how much money we should spend on a problem to get an acceptable consensus?

At this point, we utilize the utility theory for selecting the desired consensus level with best money value. To do this, we make two assumptions.

- We assume the decision making group gives their utility functions of the cost
- We assume the utility function of the desired consensus is also given.

6.4.1 Utility of Cost (Monetary Outcome)

The utility of the cost represents the money value of the budget, i.e., the expected return the decision makers are expecting. The utility function can be formulated as:

$$u = 1 - e^{-\alpha_1 x} \tag{6.50}$$

Where x is the given budget level, and the parameter α_1 represent the risk attitude the decision makers have (Figure 6.5).

6.4.2 Utility of Consensus

The utility of the consensus represents the benefit of the consensus the decision makers are expecting. Similarly, the utility function can be formulated as:

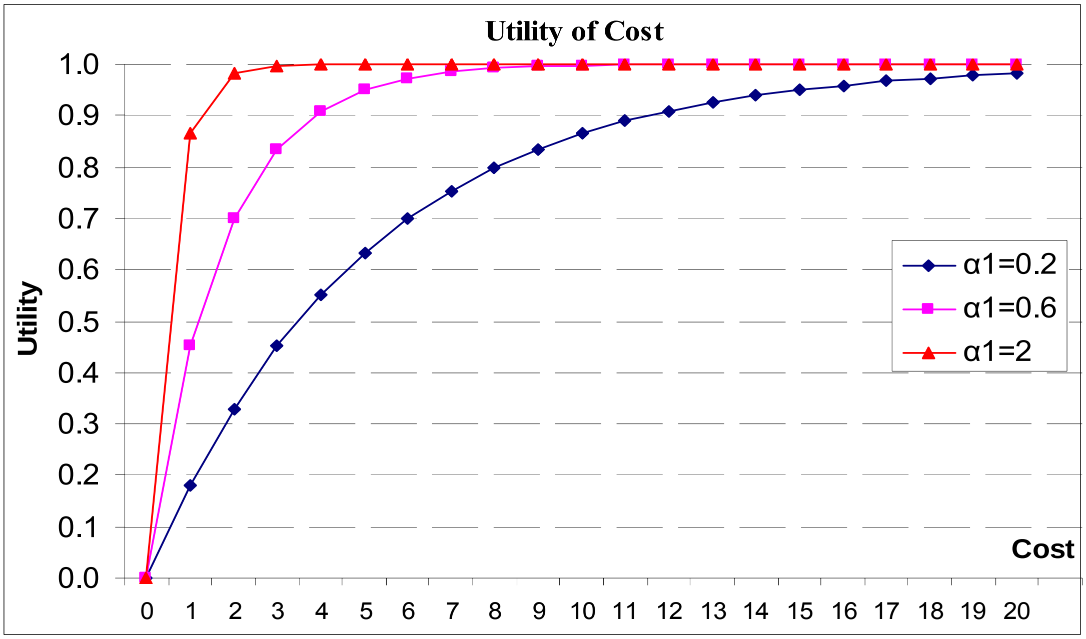


Figure 6.5 Utility of Cost

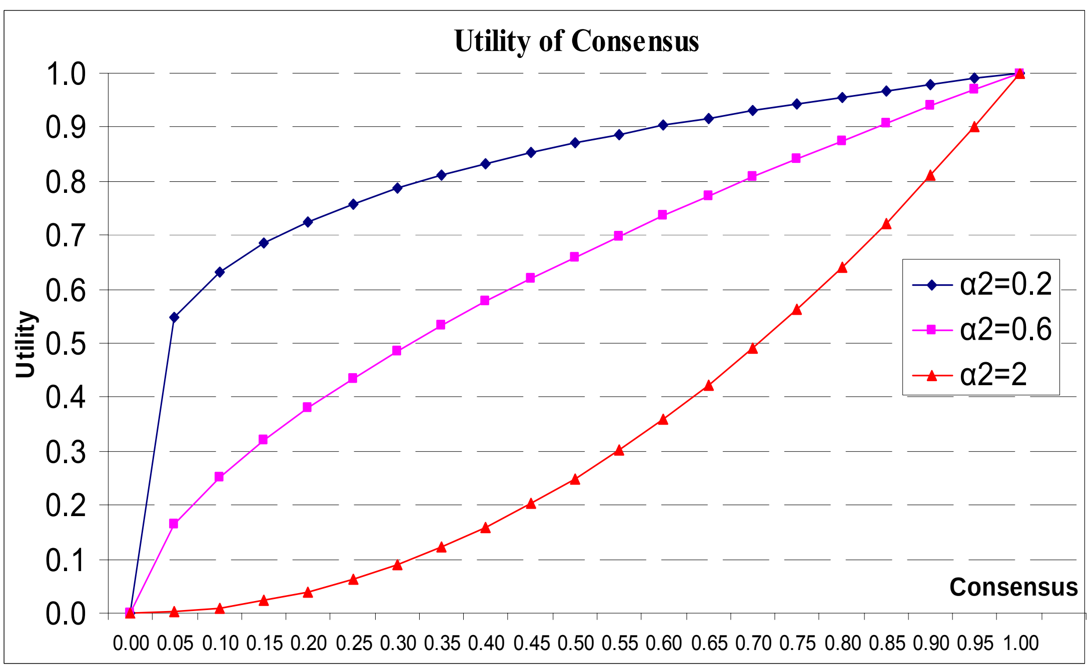


Figure 6.6 Utility of consensus

$$u = y^{\alpha_2} \tag{6.51}$$

Where y is the consensus level, and the parameter α_2 represent the risk attitude the decision makers have (Figure 6.6).

6.4.3 Cost vs. Consensus

Given two utility functions of cost and consensus, we are able to generate the desired consensus level with given budget constraint. From formula (6.50) and (6.51), we get,

$$y = \left[1 - e^{-\alpha_1 x}\right]^{1/\alpha_2} \tag{6.52}$$

This shows the ideal relationship between cost and the consensus. That is, given a budget constraint, the desired consensus level is also available. The curves with different α_1 and α_2 values are given in Figure 6.7. By comparing the desired consensus and the best consensus we could get for a given problem, we should be able to choose which cost gives the best money value that is the best budget constraint we should use.

As shown in Figure 6.8, two curves represent the theoretical desired consensus for given budget constraint and practical best consensus we can get by solving the optimization model respectively. For any budget between 0 and x_2, we can always get a consensus from the optimization model better than the desired level. That means any budget between 0 and x_2 gives us good money value. On the other hand, any budget greater than x_2 gives us worse money value, since the best consensus we can get by solving the optimization model is smaller than the desired consensus.

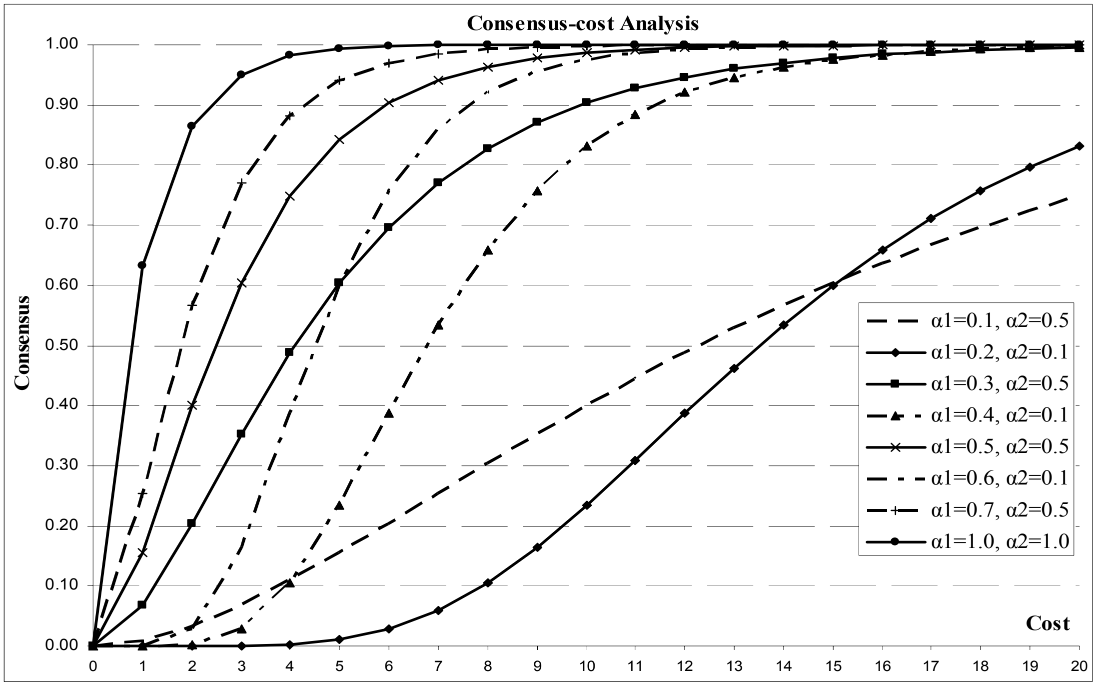


Figure 6.7 Consensus-Cost analysis with different risk attitudes

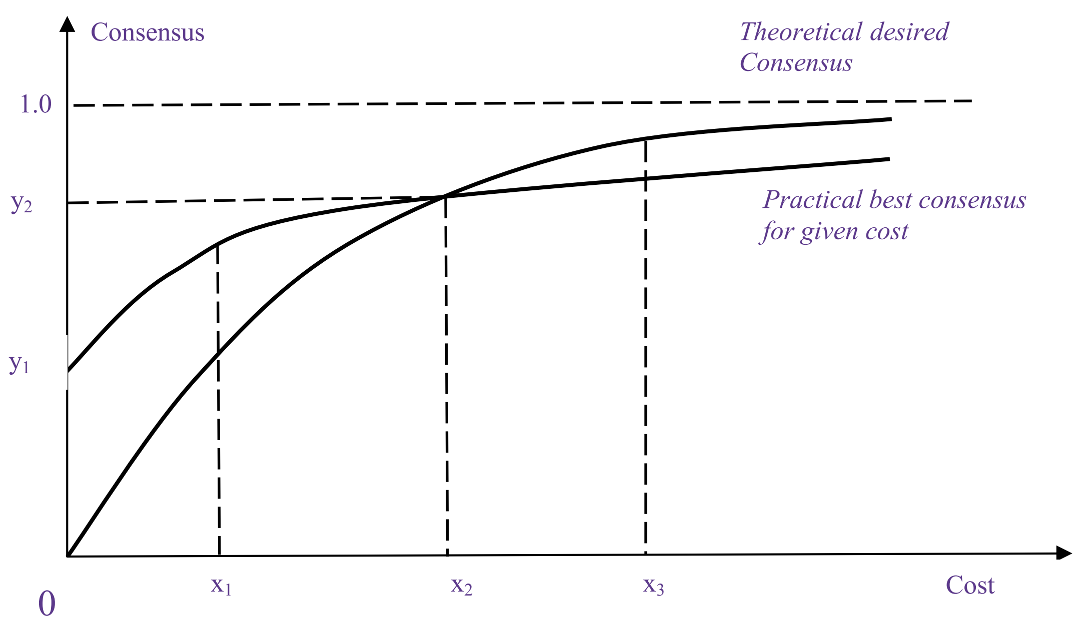


Figure 6.8 Choosing the best cost constraint

6.5 A Numeric Example

In this section, we demonstrate the entire process of generating a group decision and converging towards consensus. The example is designed with the purpose of showing how to apply the FLOWA aggregation method and the procedure of reaching the desired consensus level.

6.5.1 Problem Description

Let the five linguistic labels set S be S = {W, BD, IM, GD, EX}, where,

s_4=EX	Excellent
s_3=GD	Good
s_2=IM	Impartial
s_1=BD	Bad
s_0=W	Worse

A set of three alternatives A= {A_1, A_2, A_3} as well as a set of three experts E= {E_1, E_2, E_3} whose opinions are expressed by the following linguistic preference relation matrices. Experts have an associated weighting vector ***W***=(0.2627, 0.3875, 0.3498) representing their importance or expertise in solving the problem. Expert uses a linguistic label x_{ik} as the evaluation to the ith alternative A_i derived from expert k, $x_{ik} = j, j \in S$ is the jth linguistic label in the set S. For example, expert E_2 choose s_3 as his/her evaluation to the alternative A_1, then $x_{12} = 3$. The resistance coefficients reflect the costs for experts to change their opinions (Table 6.5).

Table 6.5 Experts' evaluations and their resistance for changing

	A_1	A_2	A_3	Resistance Coefficient (Pseudocost)
E_1	s_1	s_2	s_3	\$2
E_2	s_3	s_1	s_2	\$3
E_3	s_2	s_3	s_1	\$4

6.5.2 Mathematic Models

The objective function to the order-based consensus optimization is to maximize

$$C = \frac{1}{n}\sum_{i=1}^{n} C_i \tag{6.53}$$

The objective function to the mean-based consensus model is to maximize

$$C = \frac{1}{3}(s_1 + s_2 + s_3) \tag{6.54}$$

Where s_k is the similarity between k^{th} expert and the group.

For both cases, the cost constraint can be written as:

$$\begin{aligned} Cost = {} & 2\times|x_{11}-1| + 2\times|x_{21}-2| + 2\times|x_{31}-3| \\ & + 3\times|x_{12}-3| + 3\times|x_{22}-1| + 3\times|x_{32}-2| \\ & + 4\times|x_{11}-2| + 4\times|x_{11}-3| + 4\times|x_{11}-1| \end{aligned} \tag{6.55}$$

6.5.3 Consensus

By solving the optimization model with the heuristic based branch and bound, we plot the mean-based and order-based consensus-cost analysis in Figure 6.9 and 6.10 respectively. Figure 6.11 compares the methods in one figure.

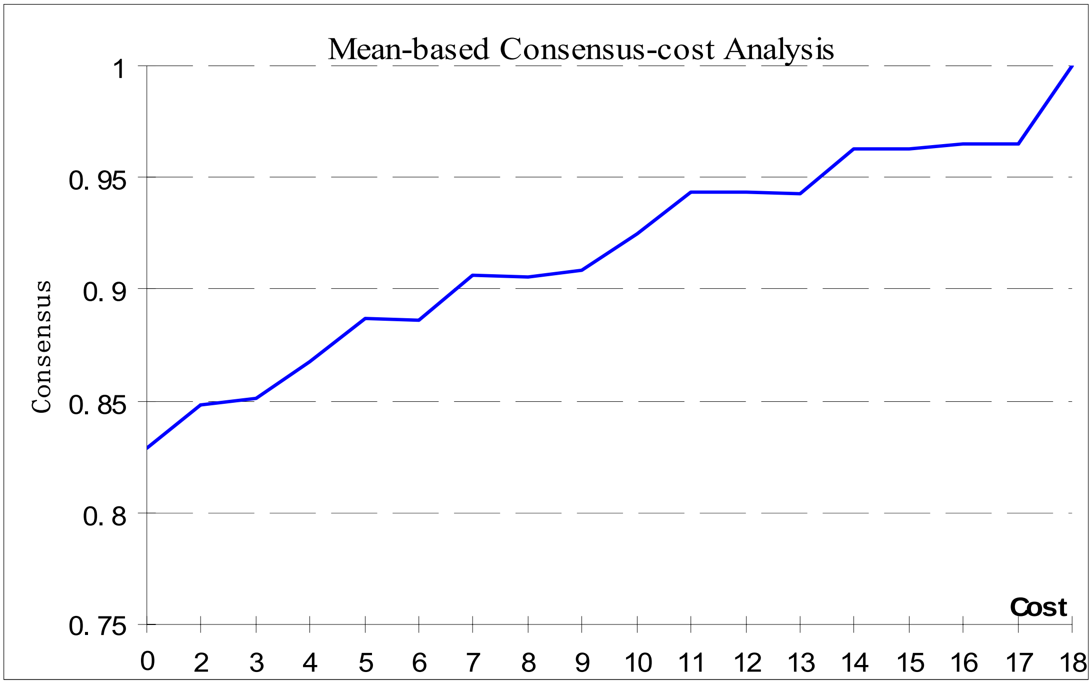


Figure 6.9 Mean-based consensus-cost analysis

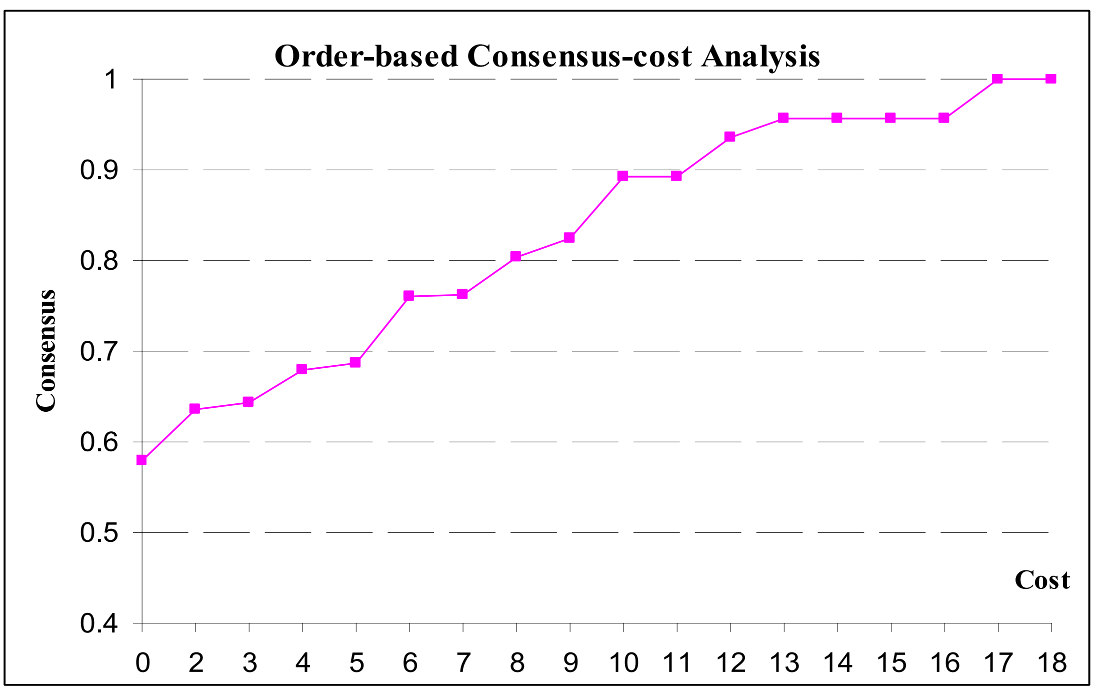


Figure 6.10 Order-based consensus-cost analysis

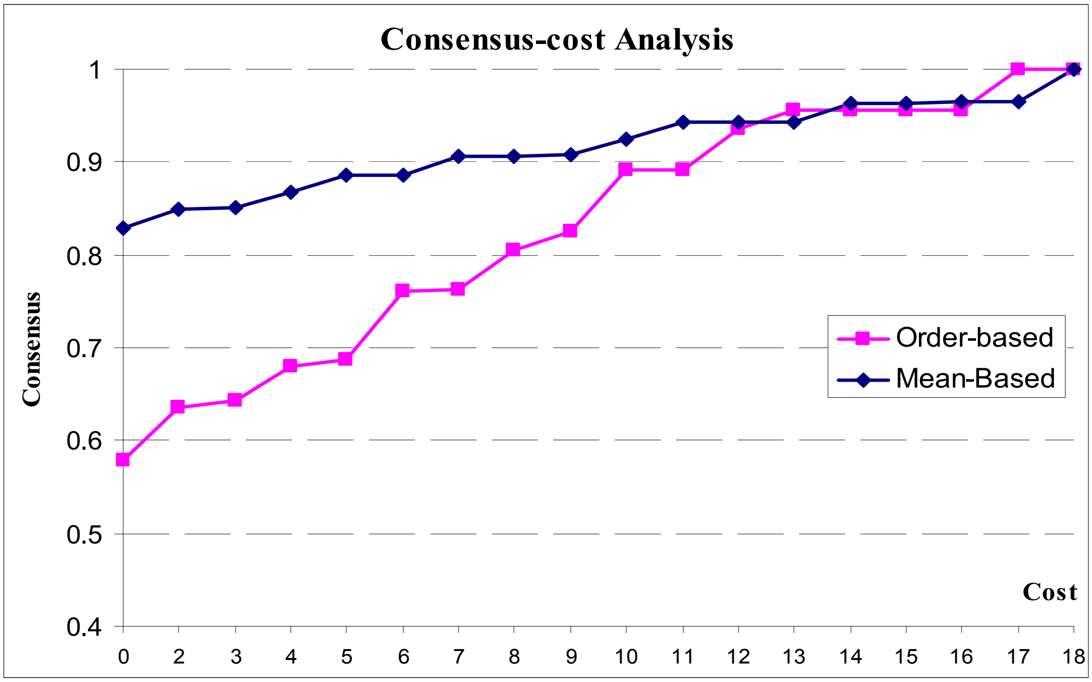


Figure 6.11 Comparing the two consensus-cost analysis methods

6.5.4 Moving Directions Based on Consensus Feedback

Method 1: Order based

The opinions will be changed using the following three rules:

- If $O_{A_i}^{G} - O_{A_i}^{E_k} < 0$, then increase evaluations associated with alternative A_i.
- If $O_{A_i}^{G} - O_{A_i}^{E_k} = 0$, do not change evaluations associated with alternative A_i.
- If $O_{A_i}^{G} - O_{A_i}^{E_k} > 0$, then decrease evaluations associated with alternative A_i.

$O_{A_i}^{G}$ is the index of the i^{th} alternative in the group's selection order.

$O_{A_i}^{E_k}$ is the order of the i^{th} alternative based on the k^{th} expert

Table 6.6 Experts' evaluations and their resistance for changing

	A_1	A_2	A_3	Resistance Coefficient	Cost
E_1	s_1	s_2	s_3	2	0
E_2	s_3	s_1	s_2	3	0
E_3	s_2	s_3	s_1	4	0
Consensus					**(0.5791, $0)**

From FLOWA, we get the orders of alternatives, from individual expert and the group: (Table 6.7).

Table 6.7 Orders from individual expert and the group

Order	A_1	A_2	A_3
E_1	***3***	2	1
E_2	1	3	2
E_3	2	1	3
Group	**1**	**2**	**3**
Mean from FLOWA	***2.0832***	1.9749	1.9419

Expert's changing direction is based on the order difference from the group's. For instance, the order of alternative A1 from Expert E1 is $O_{A_1}^{E_1} = 3$, but the order form the group is $O_{A_1}^{G} = 1$, so we have the relation: $O_{A_1}^{G} - O_{A_1}^{E_1} = 1 - 3 < 0$. This means, expert E1 should increase his/her evaluation to alternative A_1.

$$\begin{matrix} E1 \\ E2 \\ E3 \end{matrix} \begin{bmatrix} s_1 & s_2 & s_3 \\ s_3 & s_1 & s_2 \\ s_2 & s_3 & s_1 \end{bmatrix} \Rightarrow \begin{matrix} E1 \\ E2 \\ E3 \end{matrix} \begin{bmatrix} s_1 & s_1 & s_3 \\ s_3 & s_1 & s_2 \\ s_2 & s_3 & s_1 \end{bmatrix}$$

The new consensus is as the following Table 6.8:

Table 6.8 Expert E1 increases the evaluation to alternative A1

	A_1	A_2	A_3	Resistance Coefficient	Cost
E_1	s_1	$\mathbf{s_1}$	s_3	2	2
E_2	s_3	s_1	s_2	3	0
E_3	s_2	s_3	s_1	4	0
Consensus					**(0.6355, $2)**

Method 2: Mean based

The opinions will be changed using the following three rules:

- If $\bar{x}_i - x_{ik} > 0$, then increase evaluations associated with alternative A_i.
- If $\bar{x}_i - x_{ik} > 0$, do not change evaluations associated with alternative A_i.
- If $\bar{x}_i - x_{ik} > 0$, then decrease evaluations associated with alternative A_i.

$\bar{x}_i$ is the mean value of the i^{th} alternative from the group opinion

x_{ik} is the evaluation to the i^{th} alternative from the k^{th} expert

Table 6.9 Experts' evaluations and their resistance for changing

	A_1	A_2	A_3	Resistance Coefficient	Cost
E_1	s_1	s_2	s_3	2	0
E_2	s_3	s_1	s_2	3	0
E_3	s_2	s_3	s_1	4	0
Consensus					**(0.8287, $0)**

Table 6.10 shows the mean values to alternatives from individual expert and the group:

Table 6.10 Mean values from individual expert and the group

Mean	A_1	A_2	A_3
E_1	1	2	***3***
E_2	3	1	2
E_3	2	3	1
Group	**2.0832**	**1.9749**	***1.9419***

Expert's changing direction is based on the mean difference from the group's. For instance, the mean of alternative A1 from Expert E1 is $x_{31} = 3$, but the mean form the group is $\bar{x}_3 = 1.9419$, so we have the relation: $\bar{x}_3 - x_{31} = 1.9419 - 3 < 0$. This means, expert E1 should decrease his/her evaluation to alternative A_3.

$$\begin{matrix} E1 \\ E2 \\ E3 \end{matrix} \begin{bmatrix} s_1 & s_2 & s_3 \\ s_3 & s_1 & s_2 \\ s_2 & s_3 & s_1 \end{bmatrix} \Rightarrow \begin{matrix} E1 \\ E2 \\ E3 \end{matrix} \begin{bmatrix} s_1 & s_2 & s_2 \\ s_3 & s_1 & s_2 \\ s_2 & s_3 & s_1 \end{bmatrix}$$

The new consensus is as the following Table 6.11:

Table 6.11 Expert E1 decreases the evaluation to alternative A3

	A_1	A_2	A_3	Resistance Coefficient	Cost
E_1	s_1	s_2	$\mathbf{s_2}$	2	2
E_2	s_3	s_1	s_2	3	0
E_3	s_2	s_3	s_1	4	0
Consensus					**(0.8484, $2)**

Table 6.12 and Table 6.13 list the optimal consensus values with the corresponding cost constraints from order-based and mean-based methods respectively.

Table 6.12 All maximum consensus for giving costs (order-based)

E1			E2			E3			Consensus	Cost
A1	A2	A3	A1	A2	A3	A1	A2	A3		
S1	**S2**	**S3**	**S3**	**S1**	**S2**	**S2**	**S3**	**S1**	**0.5791**	**$0**
S1	**S1**	**S3**	**S3**	**S1**	**S2**	**S2**	**S3**	**S1**	**0.6355**	**$2**
1	**S2**	**S3**	**S3**	**S1**	**S1**	**S2**	**S3**	**S1**	**0.6437**	**$3**
S1	**S0**	**S3**	**S3**	**S1**	**S2**	**S2**	**S3**	**S1**	**0.6792**	**$4**
S2	**S1**	**S3**	**S3**	**S1**	**S2**	**S2**	**S3**	**S1**	**0.6792**	**$4**
S1	**S1**	**S3**	**S3**	**S1**	**S1**	**S2**	**S3**	**S1**	**0.6875**	**$5**
S1	**S1**	**S3**	**S3**	**S2**	**S2**	**S2**	**S3**	**S1**	**0.6875**	**$5**
S2	**S2**	**S3**	**S3**	**S1**	**S1**	**S2**	**S3**	**S1**	**0.6875**	**$5**
S2	**S2**	**S3**	**S3**	**S2**	**S2**	**S2**	**S3**	**S1**	**0.6875**	**$5**
S1	**S2**	**S3**	**S3**	**S3**	**S2**	**S2**	**S3**	**S1**	**0.7603**	**$6**
S1	**S2**	**S1**	**S3**	**S2**	**S2**	**S2**	**S3**	**S1**	**0.7625**	**$7**
S1	**S1**	**S3**	**S3**	**S3**	**S2**	**S2**	**S3**	**S1**	**0.8041**	**$8**
S1	**S2**	**S2**	**S3**	**S3**	**S2**	**S2**	**S3**	**S1**	**0.8041**	**$8**
S2	**S2**	**S3**	**S3**	**S3**	**S2**	**S2**	**S3**	**S1**	**0.8041**	**$8**
S1	**S2**	**S3**	**S3**	**S4**	**S2**	**S2**	**S3**	**S1**	**0.8249**	**$9**
S1	**S2**	**S1**	**S3**	**S3**	**S2**	**S2**	**S3**	**S1**	**0.8916**	**$10**
S1	**S2**	**S0**	**S3**	**S3**	**S2**	**S2**	**S3**	**S1**	**0.9354**	**$12**
S1	**S2**	**S1**	**S3**	**S4**	**S2**	**S2**	**S3**	**S1**	**0.9562**	**$13**
S1	**S2**	**S1**	**S2**	**S3**	**S2**	**S2**	**S3**	**S2**	**1**	**$17**
S2	**S2**	**S2**	**S2**	**S2**	**S2**	**S2**	**S2**	**S2**	**1**	**$18**

Table 6.13 All maximum consensus for giving costs (mean-based)

E1			E2			E3			Consensus	Cost
A1	A2	A3	A1	A2	A3	A1	A2	A3		
S1	S2	S3	S3	S1	S2	S2	S3	S1	0. 8287	$0
S1	S2	S2	S3	S1	S2	S2	S3	S1	0. 8484	$2
S1	S2	S3	S2	S1	S2	S2	S3	S1	0. 8515	$3
S2	S2	S2	S3	S1	S2	S2	S3	S1	0. 8677	$4
S1	S2	S3	S2	S1	S2	S2	S3	S1	0. 8866	$5
S2	S2	S2	S3	S1	S2	S2	S3	S2	0. 8859	$6
S2	S2	S2	S2	S1	S2	S2	S3	S1	0. 9063	$7
S1	S2	S3	S2	S2	S2	S2	S3	S1	0. 9053	$8
S2	S2	S2	S2	S1	S2	S2	S3	S2	0. 9087	$9
S2	S2	S3	S2	S1	S2	S2	S3	S2	0. 9087	$9
S2	S2	S2	S2	S2	S2	S2	S3	S1	0. 925	$10
S2	S2	S2	S2	S1	S2	S2	S3	S2	0. 9437	$11
S2	S2	S1	S2	S1	S1	S2	S3	S1	0. 9437	$12

S2	S3	S3	S2	S3	S2	S2	S3	S1	0.9428	$13
S2	S2	S2	S2	S2	S2	S2	S2	S1	0.9625	$14
S2	S2	S2	S2	S2	S2	S2	S3	S2	0.9625	$14
S2	S2	S1	S2	S2	S1	S2	S3	S1	0.9625	$15
S2	S3	S2	S2	S3	S2	S2	S3	S1	0.9625	$15
S1	S2	S2	S2	S2	S2	S2	S2	S2	0.9649	$16
S2	S2	S3	S2	S2	S2	S2	S2	S2	0.9649	$16
S1	S2	S1	S2	S2	S1	S2	S2	S1	0.9649	$17
S1	S3	S2	S2	S3	S2	S2	S3	S2	0.9649	$17
S2	S2	S2	S2	S2	S1	S2	S2	S1	0.9649	$17
S2	S2	S2	S2	S3	S2	S2	S3	S2	0.9649	$17
S2	S3	S3	S2	S3	S2	S2	S3	S2	0.9649	$17
S2	S2	S2	S2	S2	S2	S2	S2	S2	1	$18

6.5.5 Choose Consensus Level Based on Utility

We have a general pattern that with the increasing of budget available, the consensus we could get increases.

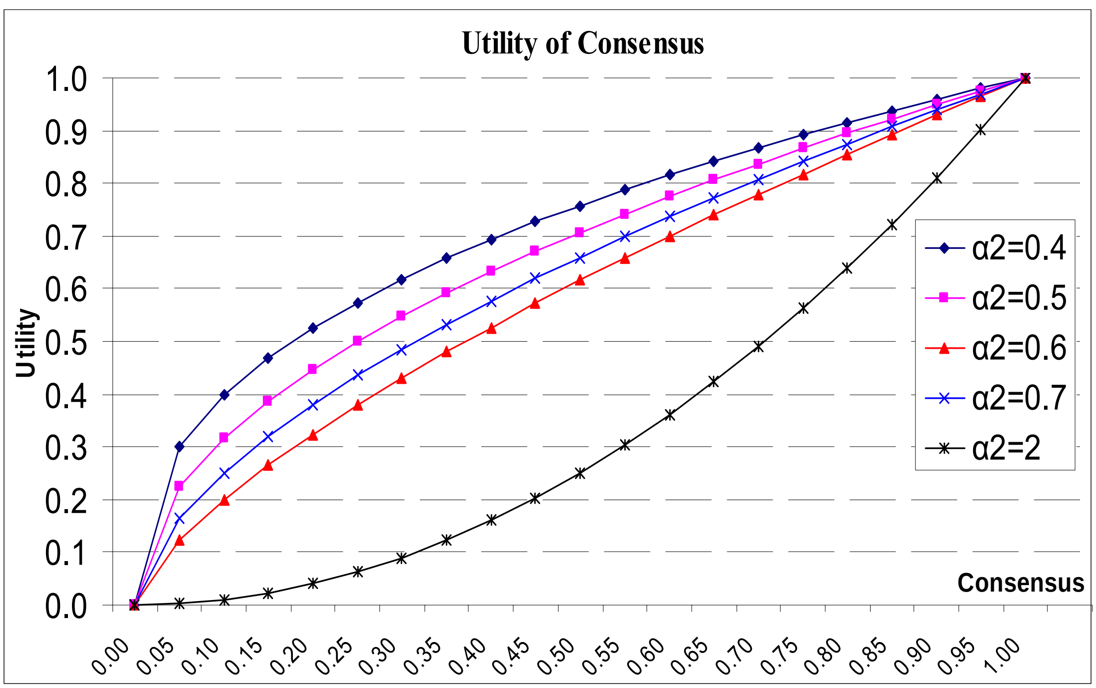


Figure 6.12 Utility of consensus with different levels of α_2

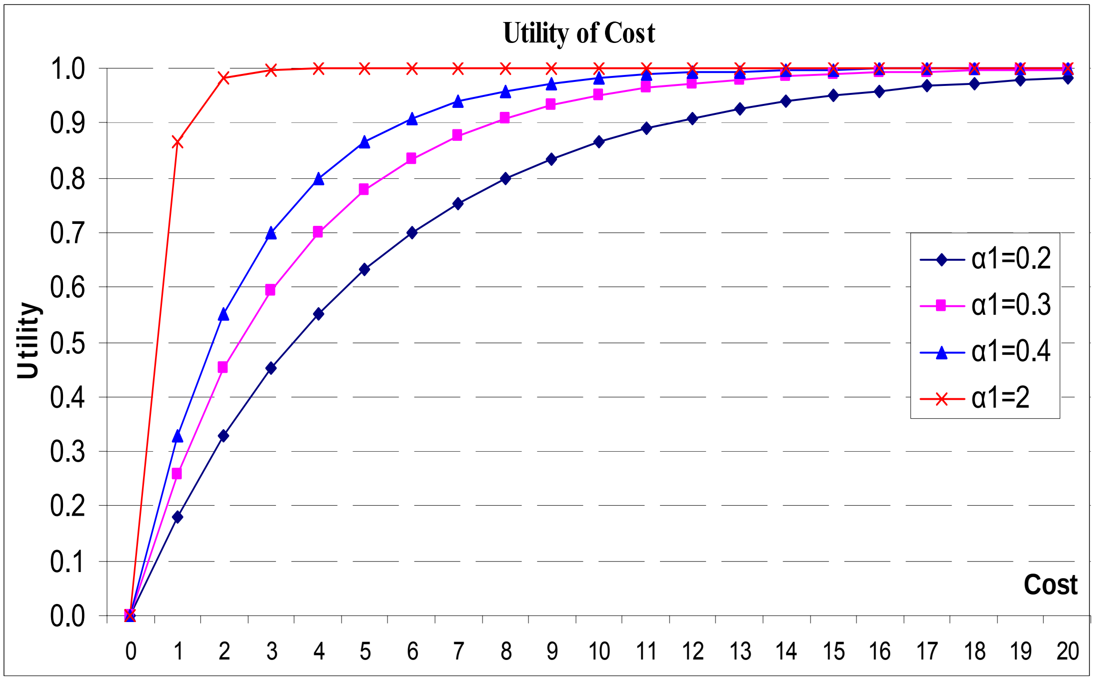


Figure 6.13 Utility of cost with different levels of α_1

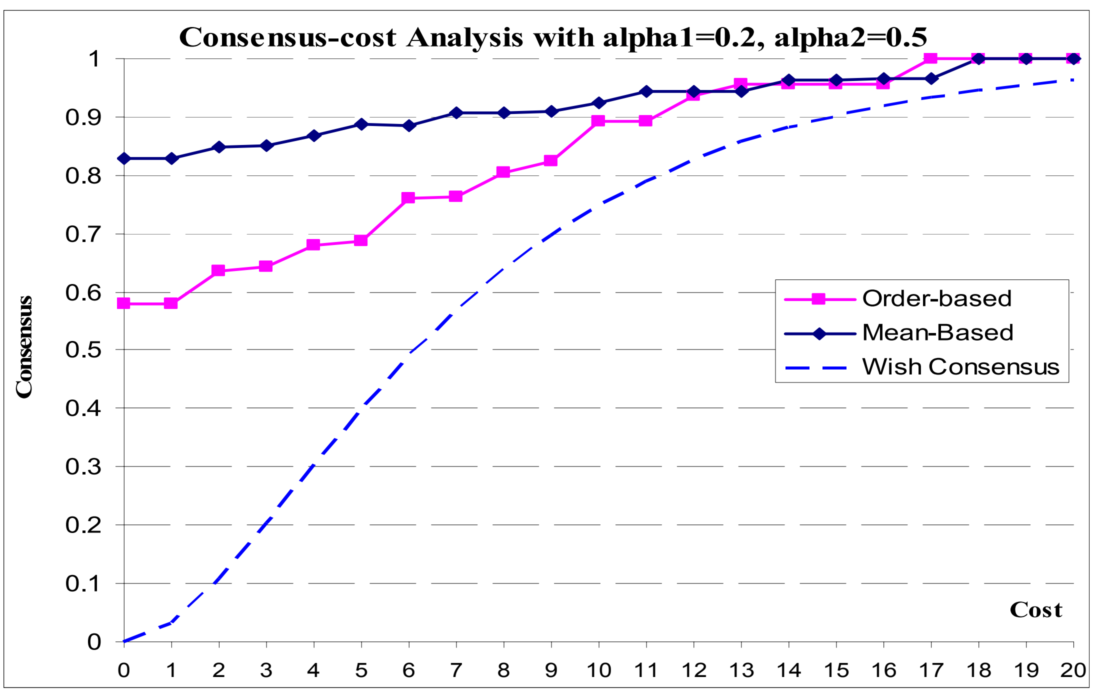


Figure 6.14 Consensus-cost analysis with α_1=0.2, α_2=0.5

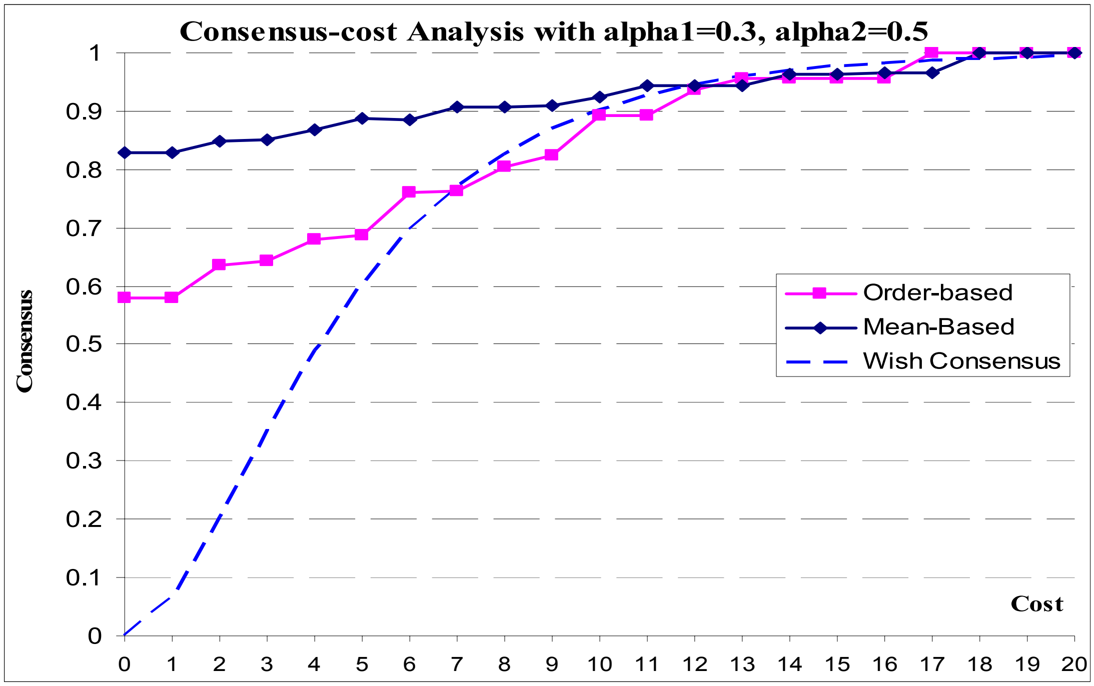


Figure 6.15 Consensus-cost analysis with α_1=0.3, α_2=0.5

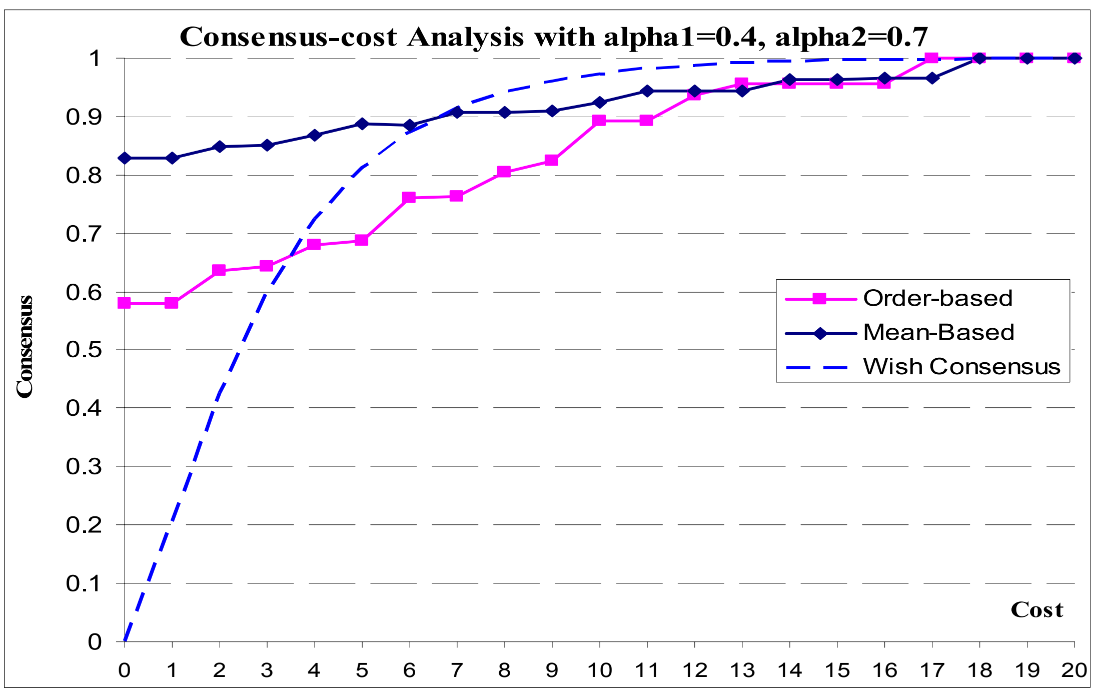


Figure 6.16 Consensus-cost analysis with α_1=0.4, α_2=0.7

Table 6.14 Choose the best budget and consensus with different risk attitudes

α value		Order-based		Mean-based	
α1	α2	Cost	Consensus	Cost	Consensus
0.2	0.4	-	-	-	-
0.2	0.5	-	-	-	-
0.2	0.6	-	-	-	-
0.2	0.7	-	-	-	-
0.3	0.4	$8	0.8041	$12	0.9460
0.3	0.5	$7	0.7625	$12	0.9460
0.3	0.6	$6	0.7603	$10	0.9250
0.3	0.7	$5	0.6875	$9	0.9087
0.4	0.4	$5	0.6875	$8	0.9063
0.4	0.5	$4	0.6792	$7	0.9063
0.4	0.6	$4	0.6792	$7	0.9063
0.4	0.7	$3	0.6437	$6	0.8859

6.6 Summary

In this chapter, we analyzed the cost in group decision making, and generated a least cost of group consensus. We developed optimization models to maximize two types of consensus under a budget constraint. Finally considering utilization of the consensus provides a practical recommendation to the desired level of consensus, considering its cost benefits.

CHAPTER 7

Conclusion and Future Research

7.1 Conclusion of the Study

In this study, we used linguistic variables to handle uncertainties in group decision making. We developed a 5-step group decision making procedure that aims at a desired consensus: heterogeneous information fusion, group evaluation aggregation, alternative ranking and selection consensus measure, and consensus and cost utility analysis. New approaches have been developed in this group decision making process, as shown in the following table:

Heterogeneous Information Fusion	Fusion operator for mapping linguistic label sets
Group Evaluation Aggregation	FLOWA aggregation operator
Alternative Ranking and Selection	Area based fuzzy similarity measure method
Consensus Measure	1. Position based consensus measure 2. Similarity based consensus measure 3. Markov chain based consensus measure
Consensus and Cost Utility Analysis	1. Extended FLOWA with expert's risk attitude 2. Consensus optimization model for least cost analysis

7.2 Future Research

In the last decade, fuzzy set theory has provided a new research direction on both concepts and methodologies to formulate and solve mathematical programming and group decision making problems. However, consensus in group decision making is a relatively new research area, and a number of problems still remain to be solved in future research. These problems are summarized in the followings:

In this study, we assume that the resistant coefficients to experts (represents the cost of changing opinion) are reasonably given. However, in practice, we should generate and obtain these values from decision makers and/or historical resources. These values should be a function of experts' expertise, personality, decision making history etc.

In defining cost for consensus, a further assumption was made in our model that the resistance coefficient ε is a fixed value. Future research can give it a function of the changing opinion. Then the objective function will also be nonlinear.

As indicated before, ranking approaches are very important to resolve fuzzy/linguistic constraints. However, most of the existing ranking approaches are not perfect. Searching for better ranking methods is urgent and momentous to resolve especially linguistic group decision making problems.

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