THE SOLUTION OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH A REGULAR SINGULAR POINT

by

FRANK DAVID FAULKNER

B. S., Kansas State Teachers College, Emporia, 1940

A THESIS

submitted in partial fulfillment of the

requirements for the degree of

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE COLLEGE
OF AGRICULTURE AND APPLIED SCIENCE

1942

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INTRODUCTION

The present paper deals with the problem of finding fundamental sets of solutions for systems of linear differential equations which have a regular singular point. A fundamental set is a set of solutions such that every solution of the system can be expressed as a linear combination of the elements of that set. Any set of <u>n</u> linearly independent solutions forms a fundamental set. A singular point is called "regular" if the Laurent expansion of the solutions about that point contains a finite number of terms with negative exponents.

The problem was first solved by Horn about 1891. The first direct treatment was given by J. A. Nyswander in 1929. The methods employed in this paper are essentially those used by Nyswander. For the sake of simplicity the number of dependent variables, and hence the number of equations, is limited to two.

FORMAL SOLUTION OF THE SYSTEM

Consider the system of differential equations

$$(1) \qquad \begin{cases} t\dfrac{dx_1}{dt} = \theta_{11}(t)x_1 + \theta_{12}(t)x_2, \\ t\dfrac{dx_2}{dt} = \theta_{21}(t)x_1 + \theta_{22}(t)x_2, \end{cases}$$

where the θ's are analytic functions of t in the neighborhood of $t = 0$ and hence may be expanded in power series in t,

$$\theta_{ij}(t) = \sum_{\nu=0}^{\infty}\theta_{ij}^{(\nu)}t^{\nu}, \qquad (i,j = 1,2),$$

convergent for sufficiently restricted values of t, and at least one $\theta_{ij}^{(0)}$ is not zero. Assume a solution of the form

$$(2) \qquad x_i = t^{\lambda}\sum_{\nu=0}^{\infty}a_i^{(\nu)}t^{\nu}, \qquad (i = 1,2).$$

Then

$$\frac{dx_i}{dt} = t^{\lambda}\sum_{0}^{\infty}(\lambda+\nu)a_i^{(\nu)}t^{\nu-1};$$

and substitution of these values into (1) gives

$$\begin{cases} t^{\lambda}\sum_{\nu=0}^{\infty}(\lambda+\nu)a_1^{(\nu)}t^{\nu} = \theta_{11}t^{\lambda}\sum_{0}^{\infty}a_1^{(\nu)}t^{\nu} + \theta_{12}t^{\lambda}\sum_{0}^{\infty}a_2^{(\nu)}t^{\nu}, \\ t^{\lambda}\sum_{0}^{\infty}(\lambda+\nu)a_2^{(\nu)}t^{\nu} = \theta_{21}t^{\lambda}\sum_{0}^{\infty}a_1^{(\nu)}t^{\nu} + \theta_{22}t^{\lambda}\sum_{0}^{\infty}a_2^{(\nu)}t^{\nu}, \end{cases}$$

where $\theta_{ij} \equiv \theta_{ij}(t)$.

If the solution is to be of the form (2), the coefficients of the various powers of t must be equal on both sides of the equations, that is,

$$(3) \qquad \begin{cases} (\theta_{11}^{(0)}-\lambda)a_1^{(0)} + \theta_{12}^{(0)}a_2^{(0)} = 0, \\ \theta_{21}^{(0)}a_1^{(0)} + (\theta_{22}^{(0)}-\lambda)a_2^{(0)} = 0, \end{cases}$$

$$(4) \qquad \begin{cases} (\theta_{11}^{(0)}-\lambda-n)a_1^{(n)} + \theta_{12}^{(0)}a_2^{(n)} = -\sum_{\nu=0}^{n-1}(\theta_{11}^{(n-\nu)}a_1^{(\nu)} + \theta_{12}^{(n-\nu)}a_2^{(\nu)}) \\ \qquad\qquad \equiv -F_1^{(n)}(a), \\ \theta_{21}^{(0)}a_1^{(n)} + (\theta_{22}^{(0)}-\lambda-n)a_2^{(n)} = -\sum_{0}^{n-1}(\theta_{21}^{(n-\nu)}a_1^{(\nu)} + \theta_{22}^{(n-\nu)}a_2^{(\nu)}) \\ \qquad\qquad \equiv -F_2^{(n)}(a), \quad (n = 1,2,\cdots). \end{cases}$$

Equations (3), being linear and homogeneous in $a_1^{(0)}$ and $a_2^{(0)}$, will have a non-trivial solution if and only if

$$(5) \qquad D(\lambda) = \begin{vmatrix} \theta_{11}^{(0)}-\lambda, & \theta_{12}^{(0)} \\ \theta_{21}^{(0)}, & \theta_{22}^{(0)}-\lambda \end{vmatrix} = 0.$$

Equation (5) is called the <u>indicial</u> <u>equation</u>. As a quadratic in λ it has two roots, λ_1 and λ_2. In general these values when substituted into (3) and (4) allow the <u>a</u>'s to be found successively in terms of the preceding <u>a</u>'s and λ. Difficulty arises, however, when the roots of the indicial equation are equal or when they differ by an integer. There are three cases to consider: Case I, say $\lambda_1 > \lambda_2$, $\lambda_1 = \lambda_2 + m$, <u>m</u> not an integer; Case II, $\lambda_1 = \lambda_2$; and Case III, say $\lambda_1 > \lambda_2$, $\lambda_1 = \lambda_2 + m$, <u>m</u> an integer.

Case I

There are the two solutions for the system of the form (2) where the <u>a</u>'s are found successively by (3) and (4),

$$(6) \qquad \begin{cases} x_{11} = t^{\lambda_1}\sum_{\nu=0}^{\infty} a_{11}^{(\nu)} t^{\nu}, \\ x_{21} = t^{\lambda_1}\sum_{\nu=0}^{\infty} a_{21}^{(\nu)} t^{\nu}, \end{cases}$$

$$\begin{cases} x_{12} = t^{\lambda_2}\sum_{0}^{\infty} a_{12}^{(\nu)} t^{\nu}, \\ x_{22} = t^{\lambda_2}\sum_{0}^{\infty} a_{22}^{(\nu)} t^{\nu}. \end{cases}$$

Case II

There are two subcases to be considered here.

(i) Let $\theta_{12}^{(0)} = \theta_{21}^{(0)} = 0$. Then the <u>a</u>'s are given as in Case I by (4) with $a_1^{(0)}$, $a_2^{(0)}$ both arbitrary.

(ii) Let either or both $\theta_{12}^{(0)}$, $\theta_{21}^{(0)}$ be different from zero.

Then substitution in (3) and (4) will give only one set of values for the a's; there is only one solution thus found. It will be shown that the second solution is

$$(7) \qquad x_{i2} = \left(\frac{\partial x_{i1}}{\partial \lambda}\right)_{\lambda_2} \qquad (i = 1,2),$$

$$= t^{\lambda}\sum_{\nu=0}^{\infty} a_i^{(\nu)} t^{\nu} \log t + t^{\lambda}\sum_{0}^{\infty}\left(\frac{\partial a_i^{(\nu)}}{\partial \lambda}\right)_{\lambda_2} t^{\nu}.$$

Assume that the second solution is of the form

$$(8) \qquad x_{i2} = t^{\lambda}\sum_{\nu=0}^{\infty} a_i^{(\nu)} t^{\nu} \log t + t^{\lambda}\sum_{0}^{\infty} b_i^{(\nu)} t^{\nu}.$$

Then

$$\frac{dx_{i2}}{dt} = t^{\lambda}\sum_{0}^{\infty}(\lambda+\nu)\, a_i^{\nu} t^{\nu-1} \log t + t^{\lambda}\sum_{0}^{\infty} a_i^{(\nu)} t^{\nu-1} + t^{\lambda}\sum_{0}^{\infty}(\lambda+\nu)\, b_i^{(\nu)} t^{\nu-1}.$$

Substitution gives

$$t^{\lambda}\sum_{0}^{\infty}(\lambda+\nu) a_i^{(\nu)} t^{\nu} \log t + t^{\lambda}\sum_{0}^{\infty} a_i^{(\nu)} t^{\nu} + t^{\lambda}\sum_{0}^{\infty}(\lambda+\nu) b_i^{(\nu)} t^{\nu}$$
$$= \theta_{i1} t^{\lambda}\Big(\sum_{0}^{\infty} a_1^{(\nu)} t^{\nu} \log t + \sum_{0}^{\infty} b_1^{(\nu)} t^{\nu}\Big)$$
$$+ \theta_{i2} t^{\lambda}\Big(\sum_{0}^{\infty} a_2^{(\nu)} t^{\nu} \log t + \sum_{0}^{\infty} b_2^{(\nu)} t^{\nu}\Big), \qquad (i = 1,2).$$

Equating coefficients of log t and dividing by t gives (3), (4), and (5), hence the same values for λ are required and the a's for the second solution are identical to the a's for the first solution. Equating terms that do not involve log t gives, after dividing by t^{λ},

$$\sum_{0}^{\infty} a_i^{(\nu)} t^{\nu} + \sum_{0}^{\infty}(\lambda+\nu) b_i^{(\nu)} t^{\nu} = \theta_{i1}\sum_{0}^{\infty} b_1^{(\nu)} t^{\nu} + \theta_{i2}\sum_{0}^{\infty} b_2^{(\nu)} t^{\nu}, \quad (i = 1,2).$$

Equating coefficients of t^n gives the equations for the b's,

$$(10) \qquad \begin{cases} (\theta_{11}^{(0)}-\lambda-n) b_1^{(n)} + \theta_{12}^{(0)} b_2^{(n)} = a_1^{(n)} - F_1^{(n)}(b) \\ \theta_{21}^{(0)} b_1^{(n)} + (\theta_{22}^{(0)}-\lambda-n) b_2^{(n)} = a_2^{(n)} - F_2^{(n)}(b), \quad (n = 1,2,\cdots), \end{cases}$$

where the F's are the same functions as before, $\Big(F_i^{(n)}(b) = \sum_{\nu=0}^{n-1}\big(\theta_{i1}^{(n-\nu)} b_1^{(\nu)} + \theta_{i2}^{(n-\nu)} b_2^{(\nu)}\big)\Big)$. For n = 0 the equations are

$$\begin{cases} (\theta_{11}^{(0)}-\lambda) b_1^{(0)} + \theta_{12}^{(0)} b_2^{(0)} = a_1^{(0)} \\ \theta_{21}^{(0)} b_1^{(0)} + (\theta_{22}^{(0)}-\lambda) b_2^{(0)} = a_2^{(0)} \end{cases}$$

which leads to

$$\frac{\theta_{11}^{(0)}-\lambda}{\theta_{21}^{(0)}} = \frac{\theta_{12}^{(0)}}{\theta_{22}^{(0)}-\lambda} = \frac{a_1^{(0)}}{a_2^{(0)}}$$

since the determinant of the coefficients is zero. But the latter reduce to an identity under the given condition ($\lambda_1 = \lambda_2$). From (10) the b's may be solved for consecutively in terms of the a's and the preceding b's.

If equations (4) are differentiated with respect to λ, and $a_i^{(n)}$ is transferred to the right-hand side of the equations, one has

$$(11) \quad \begin{cases} (\theta_{11}^{(0)}-\lambda-n)\dfrac{\partial a_1}{\partial\lambda} + \theta_{12}^{(0)}\dfrac{\partial a_2^{(n)}}{\partial\lambda} = a_1^{(n)} - F_1^{(n)}\left(\dfrac{\partial a}{\partial\lambda}\right), \\ \theta_{21}^{(0)}\dfrac{\partial a_1^{(n)}}{\partial\lambda} + (\theta_{22}^{(0)}-\lambda-n)\dfrac{\partial a_2^{(n)}}{\partial\lambda} = a_2^{(n)} - F_2^{(n)}\left(\dfrac{\partial a}{\partial\lambda}\right), \quad (n = 1,2,\cdots.). \end{cases}$$

For $\lambda = \lambda_2$ these equations are identical to (10), hence

$$\left(\frac{\partial a_i^{(n)}}{\partial\lambda}\right)_{\lambda_2} = b_i^{(n)}$$

Thus the two solutions are

$$\begin{aligned} x_{i1} &= t^{\lambda_2}\sum_0^\infty a_i^{(\nu)}t^\nu, \qquad\qquad (i = 1,2), \\ x_{i2} &= t^{\lambda_2}\sum_0^\infty a_i^{(\nu)}t^\nu \log t + t^{\lambda_2}\sum_0^\infty b_i^{(\nu)}t^\nu \\ &= t^{\lambda_2}\sum_0^\infty a_i^{(\nu)}t^\nu \log t + t^{\lambda}\sum_0^\infty \left(\frac{\partial a_i^{(\nu)}}{\partial\lambda}\right)_{\lambda_2} t^\nu \\ &= \left(\frac{\partial x_{i1}}{\partial\lambda}\right)_{\lambda_2}. \end{aligned}$$

Case III

The roots of the indicial equation differ by an integer: $D(\lambda_1) = 0$, $D(\lambda_1-m) = 0$, m an integer. Substitution of $\lambda = \lambda_1$ into equations (3) and (4) gives one solution of the form (2). However when λ_2 is substituted into equations (3) and (4), for

$n = m$ the determinant of the coefficients vanishes while in general the determinants of the numerators do not. Setting $a_i^{(0)} = a_i(\lambda - \lambda_1)$ will make the first $\underline{m}$ of the $\underline{a}$'s zero and the remaining will be finite (if the series converges), but an inspection of the solutions shows that the second solution is identical to the first except for an arbitrary constant coefficient.

Again assume the second solution to be of the form

$$\text{(8)} \qquad x_{i2} = t^{\lambda}\sum_0^{\infty} a_i^{(\nu)} t^{\nu} \log t + t^{\lambda}\sum_0^{\infty} b_i^{(\nu)} t^{\nu}, \quad (i = 1,2).$$

Substitution into (1) gives

$$t^{\lambda}\sum_0^{\infty}(\lambda+\nu)a_i^{(\nu)}t^{\nu}\log t + t^{\lambda}\sum_0^{\infty}a_i^{(\nu)}t^{\nu} + t^{\lambda}\sum_0^{\infty}(\lambda+\nu)b_i^{(\nu)}t^{\nu}$$
$$= \theta_{i1}t^{\lambda}\sum_0^{\infty}a_1^{(\nu)}t^{\nu}\log t + \theta_{i1}t^{\lambda}\sum_0^{\infty}b_1^{(\nu)}t^{\nu}$$
$$+ \theta_{i2}t^{\lambda}\sum_0^{\infty}a_2^{(\nu)}t^{\nu}\log t + \theta_{i2}t^{\lambda}\sum_0^{\infty}b_2^{(\nu)}t^{\nu},$$

whence, equating coefficients of log t and dividing by t^{λ},

$$\sum_0^{\infty}(\lambda+\nu)a_i^{(\nu)}t^{\nu} = \theta_{i1}\sum_0^{\infty}a_1^{(\nu)}t^{\nu} + \theta_{i2}\sum_0^{\infty}a_2^{(\nu)}t^{\nu}.$$

This gives the same equations for the $\underline{a}$'s, (3) and (4), and the same indicial equation (5), so the same values for λ are required and the same values for the $\underline{a}$'s. The equations for the terms not involving log t are

$$\text{(10)} \qquad \begin{cases} (\theta_{11}^{(0)}-\lambda-n)b_1^{(n)} + \theta_{12}^{(0)}b_1^{(n)} = a_1^{(n)} - F_1^{(n)}(b), \\ \theta_{21}^{(0)}b_1^{(n)} + (\theta_{22}^{(0)}-\lambda-n)b_2^{(n)} = a_2^{(n)} - F_2^{(n)}(b). \end{cases}$$

For $n = 0$, $D(\lambda) = 0$, so either the $\underline{a}$'s are zero and the equations are homogeneous or

$$\frac{\theta_{11}^{(0)}-\lambda}{\theta_{21}^{(0)}} = \frac{\theta_{12}^{(0)}}{\theta_{22}^{(0)}-\lambda} = \frac{a_1^{(0)}}{a_2^{(0)}}.$$

The latter relation does not hold in this case, holding only when the roots of the indicial equation are equal. However, if $a_i^{(0)} = a_i(\lambda - \lambda_2)$, for $\lambda = \lambda_2$ the equations are homogeneous and all conditions are satisfied except possibly for $n = m$ when the determinant of the coefficients is zero. The values of $a_1^{(m)}$, $a_2^{(m)}$ may be found from equations (3) by taking the limit as $\lambda \to \lambda_2$.

Let

$$a_i^{(\nu)\prime} = \lim_{\lambda \to \lambda_2} \frac{a_i^{\nu}}{\lambda - \lambda_2} .$$

Then

$$a_1^{(m)} = \lim_{\lambda \to \lambda_2} \frac{-(\theta_{22}^{(0)} - \lambda - m)F_1^{(m)}(a) + \theta_{12}^{(0)}F_2^{(m)}(a)}{D(\lambda - m)}$$

$$= \frac{(\theta_{22}^{(0)} - \lambda - m)F_1^{(m)}(a') - \theta_{12}^{(0)}F_2^{(m)}(a')}{m} ,$$

$$a_2^{(m)} = \lim_{\lambda \to \lambda_2} \frac{-(\theta_{11}^{(0)} - \lambda - m)F_2^{(m)}(a) + \theta_{21}^{(0)}F_1^{(m)}(a)}{D(\lambda - m)}$$

$$= \frac{(\theta_{11}^{(0)} - \lambda - m)F_2^{(m)}(a') - \theta_{21}^{(0)}F_1^{(m)}(a')}{m} .$$

Equations (8) must be satisfied for $n = m$, and since the equations are not homogeneous,

$$\frac{\theta_{11}^{(0)} - \lambda - m}{\theta_{21}^{(0)}} = \frac{\theta_{12}^{(0)}}{\theta_{22}^{(0)} - \lambda - m} = \frac{a_1^{(m)} - F_1^{(m)}(b)}{a_2^{(m)} - F_2^{(m)}(b)} .$$

If $a_i^{(\nu)\prime} = b_i^{(\nu)}$, these reduce to an identity. If equations (3) are differentiated with respect to λ as above, then, for $\lambda = \lambda_2$ and $a_i^{(0)} = a_i(\lambda - \lambda_2)$, equations (11) are identical to (10) with $\frac{\partial a_i^{(\nu)}}{\partial \lambda}$ substituted for $b_i^{(\nu)}$. Since the first <u>m</u> <u>a</u>'s have the factor $\lambda - \lambda_2$, $\frac{\partial a_i^{(\nu)}}{\partial \lambda} = a_i^{(\nu)\prime}$ and all conditions are satisfied. The two solutions are

$$x_{i1} = t^{\lambda_2} \sum_0^{\infty} a_i^{(\nu)} t^{\nu} \qquad (i = 1,2)$$

$$x_{i2} = t^{\lambda_2} \sum_0^{\infty} a_i^{(\nu)} t^{\nu} \log t + t^{\lambda_2} \sum_0^{\infty} b_i^{(\nu)} t^{\nu}$$

$$= t^{\lambda} \sum_0^{\infty} a_i^{(\nu)} t^{\nu} \log t + t^{\lambda} \sum_0^{\infty} \left(\frac{\partial a_i^{(\nu)}}{\partial \lambda}\right)_{\lambda_2} t^{\nu}$$

$$= \left(\frac{\partial x_{i1}}{\partial \lambda}\right)_{\lambda_2} ,$$

where $a_i^{(0)} = a_i(\lambda - \lambda_i)$.

FUNDAMENTAL SETS OF SOLUTIONS

Case I

The roots of the indicial equation differ by a number not an integer. To prove linear independence it is necessary and sufficient to show that there are no values for the c's other than $c_1 = c_2 = 0$ such that

$$(12) \qquad \begin{cases} c_1 x_{11} + c_2 x_{21} = 0, \\ c_1 x_{12} + c_2 x_{22} = 0. \end{cases}$$

Assume that there are c's such that (12) is true. Then it is necessary that

$$(13) \qquad \begin{vmatrix} x_{11}, & x_{21} \\ x_{12}, & x_{22} \end{vmatrix} = 0.$$

Substituting the values of the x's from (6) into (13) and dividing by $t^{\lambda_1 + \lambda_2}$ makes (13) equivalent to

$$(14) \qquad \begin{vmatrix} \sum_0^\infty a_{11}^{(\nu)} t^\nu, & \sum_0^\infty a_{21}^{(\nu)} t^\nu \\ \sum_0^\infty a_{12}^{(\nu)} t^\nu, & \sum_0^\infty a_{22}^{(\nu)} t^\nu \end{vmatrix} = 0$$

except possibly for $t = 0$. For $t = 0$, (14) becomes

$$(15) \qquad \begin{vmatrix} a_{11}^{(0)}, & a_{21}^{(0)} \\ a_{12}^{(0)}, & a_{22}^{(0)} \end{vmatrix} = 0.$$

Assuming that $\theta_{12}^{(0)}$ is not zero (there is always some relationship of this form that holds), (15) becomes

$$\begin{vmatrix} a_{11}^{(0)}, & a_{11}^{(0)} \dfrac{(\theta_{11}^{(0)} - \lambda_1)}{\theta_{12}^{(0)}} \\ a_{12}^{(0)}, & a_{12}^{(0)} \dfrac{(\theta_{11}^{(0)} - \lambda_2)}{\theta_{12}^{(0)}} \end{vmatrix} =$$

$$\frac{(a_{11}^{(0)}a_{12}^{(0)})^2}{\theta_{12}^{(0)}}\begin{vmatrix}1, & 0\\ 1, & \lambda_1-\lambda_2\end{vmatrix} \neq 0.$$

Since (14) is not zero for $t = 0$, it does not vanish in some neighborhood of $t = 0$, and (13) is not identically true. But (13) is the necessary and sufficient condition that (12) be satisfied for other values for the c's other than $c_1 = c_2 = 0$. Thus the solutions are linearly independent.

Case II

(i) The roots of the indicial equation are equal and $\theta_{12}^{(0)} = \theta_{21}^{(0)} = 0$. The a's are solved for uniquely in terms of $a_1^{(0)}$ and $a_2^{(0)}$ which are both arbitrary and may be chosen so that

$$\begin{vmatrix}a_{11}^{(0)}, & a_{21}^{(0)}\\ a_{12}^{(0)}, & a_{22}^{(0)}\end{vmatrix} \neq 0.$$

(ii) If either of the pair $\theta_{12}^{(0)}$, $\theta_{21}^{(0)}$ is not zero, then the determinant for the c's, by the procedure for Case I, and eliminating log t from the determinant, is

$$(15)\qquad \begin{vmatrix}\sum_0^\infty a_1^{(\nu)}t^\nu, & \sum_0^\infty a_2^{(\nu)}t^\nu\\ \sum_0^\infty b_1^{(\nu)}t^\nu, & \sum_0^\infty b_2^{(\nu)}t^\nu\end{vmatrix} = 0.$$

For $t = 0$ this becomes

$$(16)\qquad \begin{vmatrix}a_1^{(0)}, & a_2^{(0)}\\ b_1^{(0)}, & b_2^{(0)}\end{vmatrix} = 0.$$

One of the pair $\theta_{12}^{(0)}$, $\theta_{21}^{(0)}$ is not zero, say $\theta_{12}^{(0)}$. The determinant (16) becomes

$$(17)\qquad \begin{vmatrix}a_1^{(0)}, & \dfrac{(\theta_{11}^{(0)}-\theta_{22}^{(0)})a_1^{(0)}}{\theta_{12}^{(0)}}\\ b_1^{(0)}, & \dfrac{(\theta_{11}^{(0)}-\theta_{22}^{(0)})b_1^{(0)}}{\theta_{12}^{(0)}} - \dfrac{a_1^{(0)}}{\theta_{12}^{(0)}}\end{vmatrix} \neq 0.$$

Since $a_i^{(0)} \neq 0$, (17) does not vanish, hence there is a neighborhood of the origin where the first determinant is not zero and so the c's must be zero. Thus the solutions are linearly independent.

Case III

The roots of the indicial equation differ by an integer, say $\lambda_1 = \lambda_2 + m$, m an integer. There are the two solutions for the system

$$\begin{cases} x_{i1} = t^{\lambda_2}\sum_0^{\infty} a_i^{(\nu)} t^{\nu}, & (i = 1,2), \\ x_{i2} = t^{\lambda_2}\sum_0^{\infty} a_i^{(\nu)} t^{\nu} \log t + t^{\lambda_2}\sum_0^{\infty} b_i^{(\nu)} t^{\nu} \end{cases}$$

$$= \left(\frac{\partial x_{ii}}{\partial \lambda}\right)_{\lambda_2},$$

where $a_i^{(0)} = a_i(\lambda - \lambda_2)$. The first m a's are zero. As before, linear independence is equivalent to the nonvanishing of the determinant (if either $\theta_{12}^{(0)}$ or $\theta_{21}^{(0)}$, say $\theta_{12}^{(0)}$, is not zero)

$$\begin{vmatrix} a_1^{(m)}, & a_2^{(m)} \\ b_1^{(0)}, & b_2^{(0)} \end{vmatrix} =$$

$$(a_1^{(m)} b_2^{(0)})^2 \begin{vmatrix} 1, & \dfrac{\theta_{11}^{(0)} - \lambda_2 - m}{\theta_{12}^{(0)}} \\ 1, & \dfrac{\theta_{11}^{(0)} - \lambda_2}{\theta_{12}^{(0)}} \end{vmatrix} \neq 0.$$

If $\theta_{12}^{(0)} = \theta_{21}^{(0)} = 0$, then the determinant becomes

$$\begin{vmatrix} F_1^{(m)}(b), & 0 \\ 0, & b_2^{(0)} \end{vmatrix} \neq 0.$$

By the reasoning of the previous cases, the two solutions are linearly independent.

CONVERGENCE OF THE SOLUTIONS

Let it be assumed that the $\theta_{ij}(t)$ of (1) converge for $|t| < R$. The solutions found may or may not converge. However at any regular point there is a set of solutions that converge. For some point near $t = 0$ there exists a fundamental set,

$$\begin{cases} x_{11}, & x_{21} \\ x_{12}, & x_{22}. \end{cases}$$

Describe a circle about the origin in the t-plane and let $\bar{x}_{ij}$ be the analytic continuation of x_{ij} after the circuit is completed. Then

$$\bar{x}_{ij} = c_{j1} x_{i1} + c_{j2} x_{i2}, \qquad (i, j = 1, 2). \tag{18}$$

Pick

$$X_i = \alpha x_{i1} + \beta x_{i2}$$

such that

$$\bar{X}_i = \rho X_i$$

Where ρ is a constant. Combining these two values for $\bar{X}_i$ and equating coefficients of x_{i1}, x_{i2} gives

$$\begin{cases} \alpha c_{11} + \beta c_{12} = \alpha\rho \\ \alpha c_{21} + \beta c_{22} = \beta\rho. \end{cases} \tag{19}$$

Thus the <u>fundamental</u> <u>equation</u>,

$$\begin{vmatrix} c_{11} - \rho, & c_{12} \\ c_{21}, & c_{22} - \rho \end{vmatrix} = 0, \tag{20}$$

is obtained. From this two values for ρ are obtained and each determines a ratio of α to β. There are two cases to be considered: Case I, $\rho_1 \neq \rho_2$; and Case II, $\rho_1 = \rho_2$.

Case I

There are two distinct values for X_i such that

$$\overline{X}_{ij} = \rho_j X_{ij}, \qquad (i,j = 1,2).$$

Let

$$(21) \qquad \lambda_j = \frac{1}{2\pi i}\log \rho_j .$$

If

$$(22) \qquad \phi_j = t^{\lambda_j}$$

then

$$\overline{\phi}_j = \rho_j \phi_j$$

and

$$\frac{\overline{X}_{ij}}{\overline{\phi}_j} = \psi_{ij}(t)$$

where $\psi_{ij}(t)$ is uniform near $t = 0$. Hence, in view of the treatment in the earlier part of the paper,

$$X_{ij} = t^{\lambda_j}\psi_{ij}(t)$$
$$= t^{\lambda_j}\sum_0^\infty a_{ij}^{(\nu)} t^\nu .$$

As a consequence of the unique existence of the solutions of analytic differential equations, it follows that the $\psi_{ij}(t)$ converge for any value of t such that $|t| < R$.

Case II

(i) Let $\rho_1 = \rho_2$, $c_{12} = c_{21} = 0$.

Then

$$\overline{X}_{ij} = \rho X_{ij}, \qquad (i,j = 1,2)$$

where

$$X_{ij} = t^{\lambda}\psi_{ij}(t)$$

where ψ_{ij} is uniform near $t = 0$ and $\lambda = \frac{1}{2\pi i}\log\rho$.

(ii) $\rho_1 = \rho_2$, and at least one of the pair c_{12}, c_{21}, is not zero. Let X_i be any integral of the system such that

$$\overline{X}_i = \rho X_i, \qquad (i = 1,2),$$

and let X_{i2} be any linearly independent integral. The equations then become

$$\begin{cases}\overline{X}_i = \sigma X_i = \rho X_i, \\ \overline{X}_{i2} = \sigma X_{i2} = c_1 X_i + c_2 X_{i2},\end{cases}$$

where σ is the unknown. The characteristic equation becomes

$$\begin{vmatrix}\rho-\sigma, & 0 \\ c_1, & c_2-\sigma\end{vmatrix} = 0;$$

or, since ρ is a double root (the roots are independent of the particular integrals chosen), $c_2 = \rho$,

$$\overline{X}_{i2} = c_1 X_i + \rho X_{i2}.$$

Let

$$X_{i1} = \frac{c_1 X_i}{\rho}.$$

Then the X_{i1}, X_{i2} form a fundamental set such that

$$\begin{cases}\overline{X}_{i1} = \rho X_{i1}, \\ \overline{X}_{i2} = \rho(X_{i1} + X_{i2}).\end{cases}$$

As before,

$$X_{i1} = t^{\lambda}\psi_{i1},$$

where ψ_{i1} is a uniform function near $t = 0$, and

$$\frac{\overline{X}_{i2}}{\overline{X}_{i1}} = \frac{X_{i2}}{X_{i1}} - 1.$$

If

$$\phi = \frac{1}{2\pi i} \log t,$$

then

$$\bar{\phi} = \phi - 1,$$

and

$$\frac{\bar{X}_{12}}{\bar{X}_{11}} - \bar{\phi} = \frac{X_{12}}{X_{11}} - \phi$$

so that $\frac{X_{12}}{X_{11}} - \phi$ is uniform near $t = 0$.

Thus

$$X_{12} = X_{11}\left(\frac{1}{2\pi i} \log t + \psi_1(t)\right)$$

$$= t^{\lambda}\left(\psi_{12}(t) + \frac{1}{2\pi i} \log t \, \psi_{11}(t)\right).$$

As before, these solutions converge for $|t| < R$ as a result of the unique existence of solutions of the system (1) at an ordinary point.

For any root $\rho = \rho_\sigma$ of the fundamental equation it is observed that the corresponding λ as given by (21) is determined only to an additive integer. Thus the above convergence proof for the case $\rho_1 = \rho_2$ includes the case where $\lambda_1 = \lambda_2 + m$, m an integer.

ACKNOWLEDGMENT

The author takes this opportunity to express a sincere appreciation to Dr. G. C. Munro for his helpful suggestions in the development of this problem.

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