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Median statistics cosmological parameter values

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ABSTRACT

We present median statistics central values and ranges for 12 cosmological parameters, using 582 measurements (published during 1990–2010) collected by [9]. On comparing to the recent *Planck* Collaboration [1] estimates of 11 of these parameters, we find good consistency in ten cases.

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1. Introduction

Recent cosmic microwave background anisotropy (see, e.g., [1, 16]), baryon acoustic oscillation peak length scale (see, e.g., [3, 12]), supernova Type Ia apparent magnitude versus redshift (see, e.g., [5,17]), and Hubble parameter as a function of redshift (see, e.g., [11,13,18]) measurements have small enough statistical error bars to encourage the belief that we will soon be in an era of precision cosmology. Of course, there have also been many earlier measurements, most having larger error bars, that have helped the field develop to the current position. In this paper we use statistical techniques to combine the results of the many earlier measurements, and so derive summary estimates of the corresponding cosmological parameters with much tighter error bars than any individual earlier measurement. We then compare these summary results to more precise recent measurements, largely those from the recent analysis of early *Planck* space mission cosmic microwave background (CMB) anisotropy data [1]. Using large-angle CMB anisotropy data to measure cosmological parameters is appealing because, once initial conditions and ionization history are established, it is possible to accurately compute cosmological model CMB anisotropy predictions as a function of cosmological parameter values.

Previous CMB anisotropy experiments, such as *WMAP*[1] and ground-based ones, along with data from other techniques discussed above, have focused attention on a "standard" cosmological model (for detailed discussions see [1,16]). This model, called the ΛCDM model [20], is a spatially-flat cosmological model with a current energy budget dominated by a time-independent dark energy density in the form of Einstein's cosmological constant, Λ, that contributes 68.3% of the current energy budget, non-relativistic cold dark matter (CDM) is the next largest contributor at 26.7%, followed by non-relativistic baryonic matter at 4.9% [1]. For recent reviews see [28,27,26].

A main goal of the *Planck* mission is to measure cosmological parameters accurately enough to check consistency with the ΛCDM model, as well as to possibly detect deviations. However, it is also of interest to find out if previous estimates of cosmological parameters are consistent with the *Planck* results. [1], and the references therein, have compared the *Planck* results to individual earlier measurements, most notably to the results from the *WMAP* experiment, from which they find small differences. However, it is also of interest to attempt to derive summary estimates for cosmological parameters from the many earlier measurements that are available, and to compare these summary estimates to the *Planck* results. This is what we do in this paper.

To derive our summary estimates of cosmological parameter values we use the very impressive compilation of data of [9]. We use 582 (of the 637) measurements for the dozen cosmological parameters collected by [9]. These values were published during 1990–2010, and, as estimated by [9], are approximately 60% of the measurements of the 12 cosmological parameters published during these two decades. The main focus of the [9] paper was to compare earlier and more recent measurements and analyze how measuring techniques and results evolve over time. In our paper we use two statistical techniques, namely weighted mean and median statistics, to find the best-fit summary measured value of each of the 12 cosmological parameters. We then compare our summary values to those found from the *Planck* data.

In the next section we briefly review the [9] data compilation. Sections 3 and 4 are brief summaries of the weighted mean and median statistics techniques we use to analyze the [9] data. Our analyses and results are described and discussed in Section 5, and we conclude in Section 6.

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[1] For more discussions on the use of *WMAP* data to estimate cosmological parameters see [16].

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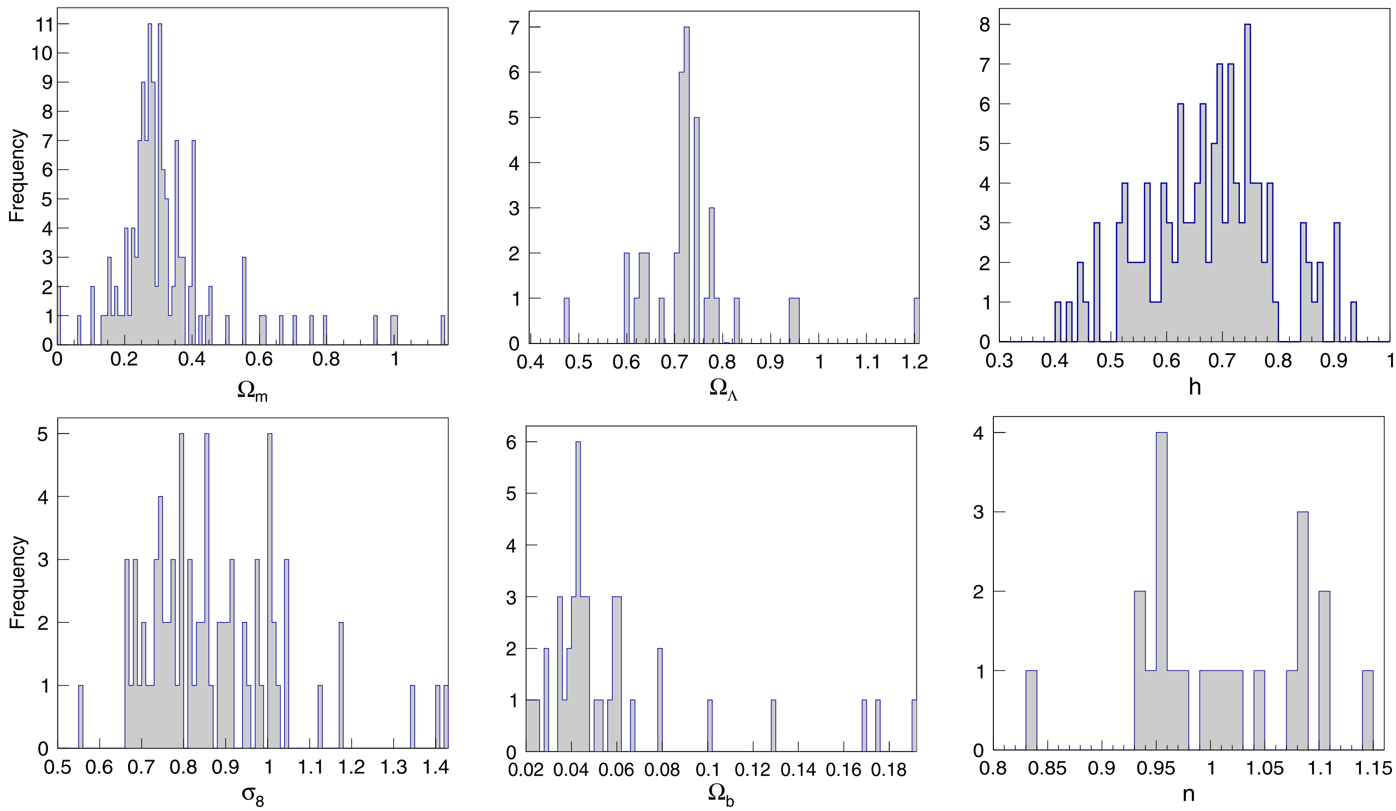

Fig. 1. Histograms of Ω_m, Ω_Λ, & h (top row, from left to right), and σ_8, Ω_b, & n (bottom row, from left to right). Although used in our analyses, values of 39 for Ω_m and -1.5 for n are not plotted. The bin size is 0.01 for all cases except for Ω_b, where it is 0.001.

2. Data compilation

The data we use in our analyses here were compiled by [9]. These data were collected from the abstracts of papers listed on the NASA Astrophysics Data System (ADS).[2] They estimate that by searching abstracts only, about 40% of available measurements were missed. Nevertheless, a great deal of data were collected. [9] searched papers published in a 20 year period (1990–2010) and tabulated 637 measurements. Of the 637 measurements, 582 were listed with a central value and 1σ error bars (these are the data we use in this paper[3,4]) while 55 were upper or lower limits with no central value.

The 12 cosmological parameters [9] considered are:

1. Ω_m, the non-relativistic matter density parameter.
2. Ω_Λ, the cosmological constant density parameter.
3. h, the Hubble constant in units of 100 $\mathrm{km\,s^{-1}\,Mpc^{-1}}$.
4. σ_8, the rms amplitude of (linear) density perturbations averaged over $8h^{-1}$ Mpc spheres.
5. Ω_b, the baryonic matter density parameter.
6. n, the primordial spectral index.
7. $\beta = \Omega_m^{0.6}/b$, where b is the galaxy bias.
8. m_ν, the sum of neutrino masses.
9. $\Gamma = \Omega_m h$.
10. $\Omega_m^{0.6}\sigma_8$.
11. Ω_k, the space curvature density parameter.
12. ω_0, the dark energy equation of state parameter in a simplified, incomplete, XCDM-like parameterization.

Figs. 1 and 2 show the 12 histograms of the 582 [9] measurements. The histograms for parameters Ω_k, Ω_m, m_ν, and n have outlying values of 0.7, 39, 2.48 eV, and -1.5, respectively, omitted from their plots, though these values were used in our analyses.

3. Weighted mean statistics

In analyzing data with known errors it is conventional to first consider a weighted mean statistic. This method yields a goodness of fit criterion that can be a valuable diagnostic tool.

The standard formula (see, e.g., [22]) for the weighted mean of cosmological parameter q is

$$q_{\mathrm{wm}} = \frac{\sum_{i=1}^{N} q_i/\sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}, \tag{1}$$

where $q_i \pm \sigma_i$ are the central values and one standard deviation errors of the $i = 1, 2, \ldots, N$ measurements. The weighted mean standard deviation of cosmological parameter q is

$$\sigma_{\mathrm{wm}} = \left(\sum_{i=1}^{N} 1/\sigma_i^2\right)^{-1/2}. \tag{2}$$

One can also compute the goodness of fit χ^2,

$$\chi^2 = \frac{1}{N-1}\sum_{i=1}^{N}\frac{(q_i - q_{\mathrm{wm}})^2}{\sigma_i^2}. \tag{3}$$

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[2] adsabs.harvard.edu.

[3] While some measurements in Ref. [9] collection also use *WMAP* results in combination with their own, we assume, because of the number of measurements, that the induced correlations are not significant enough to influence the results.

[4] Most of these measurements were listed with two significant figures, so results of our analyses are tabulated to two significant figures (except for ω_0, which consisted mostly of three significant figure measurements and were so tabulated here). The error bar we use in our analyses is the average of the 1σ upper and lower error bars of [9].

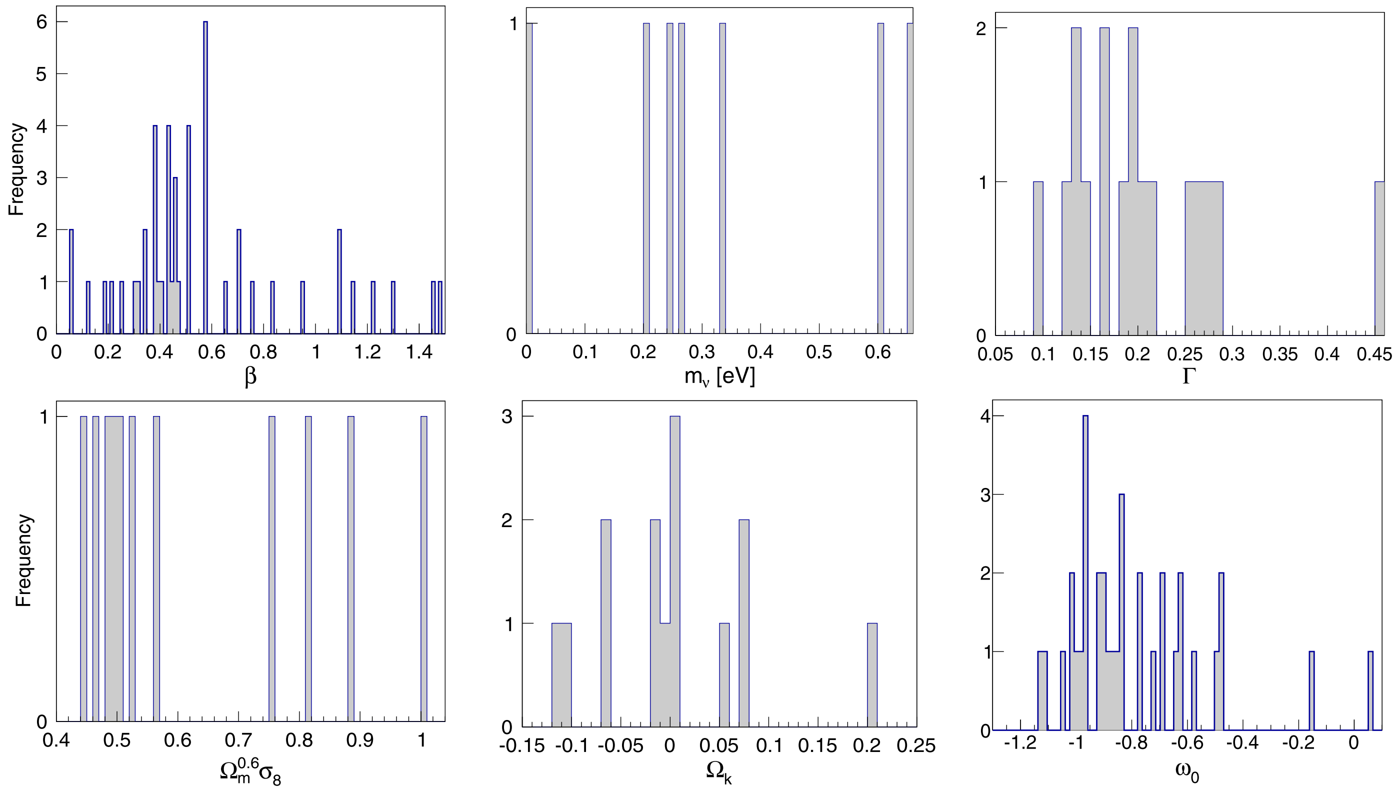

Fig. 2. Histograms of β, m_ν, & Γ (top row, from left to right), and $\Omega_m^{0.6}\sigma_8$, Ω_k, & ω_0 (bottom row, from left to right). Although used in our analyses, values of 2.48 eV for m_ν and 0.7 for Ω_k are not plotted. All of the above plots have a bin size of 0.01.

Since this method assumes Gaussian errors, χ has expected value unity and error $1/\sqrt{2(N-1)}$. Hence, the number of standard deviations that χ deviates from unity is a measure of good-fit and is given as

$$N_\sigma = |\chi - 1|\sqrt{2(N-1)}. \quad (4)$$

A large value of N_σ could be an indication of unaccounted-for systematic error, the presence of correlations between the measurements, or the invalidity of the Gaussian assumption.

4. Median statistics

The second statistical method we use is median statistics. This method makes fewer assumptions than the weighted mean method, and so can be used in cases when the weighted mean technique cannot. For a detailed description of the median statistics technique see [14].[5] In summary, if we assume that the given measurements are: (1) statistically independent; and (2) have no systematic error for the data set as a whole (as we also assume for weighted mean statistics), then as the number of measurements, N, increases to infinity, the median will reveal itself as a true value. This median is independent of measurement error [14], which is an advantage if the errors are suspect. This is also a disadvantage that results in a larger uncertainty for the median than for the weighted mean, because the information in the error bar is not used.

If (1) is true then any value in the data set has a 50% chance of being above or below the true median value. As described in [14], if N independent measurements M_i, where $i = 1, \ldots, N$, are taken then the probability of exactly n measurements being higher (or lower) than the true median is

$$P_n = \frac{2^{-N} N!}{n!(N-n)!}. \quad (5)$$

It is interesting to note that for large N the expectation value of the distribution width, x, of the true median is $\langle x \rangle = 0.5$, with a standard deviation $\langle x^2 - \langle x \rangle^2 \rangle^{1/2} = 1/(4N)^{1/2}$ [14]. Of course, as N increases to infinity, a Gaussian distribution is reached and median statistics recovers the usual standard deviation proportionality to $1/N^{1/2}$.

5. Analysis

Since both weighted mean and median statistics techniques have individual benefits, we analyze the compilation of data for 12 parameters from [9] using both methods. Our results are shown in Table 1. Among other things, the table lists our computed weighted mean and corresponding standard deviation σ_{wm} value for the cosmological parameters, as well as the computed median value and the 1σ and 2σ intervals around the median.

Column 5 of Table 1 lists N_σ, the number of standard deviations the weighted mean goodness-of-fit parameter χ deviates from unity, see Eq. (4). In all cases N_σ is much greater than unity, indicating that the weighted mean results cannot be trusted. In the case of the Hubble constant this is likely due to the fact that the observed error distribution is non-Gaussian, see [6].[6] Perhaps a similar effect explains the large N_σ values for some of the other

[5] For recent applications of median statistics see, e.g., [25,2,24,19,4,10].

[6] The weighted mean technique also could not be used to combine different Ω_m measurements [7] or different cosmic microwave background temperature anisotropy observations [22].

Table 1
Weighted mean and median statistics results.

Parameter	N [a]	WM [b]	$\sigma_{\rm wm}$ [c]	N_σ [d]	MS [e]	1σ MS range [f]	2σ MS range [f]	ECV [g]	1σ or 2σ range [h]
Ω_m	138	0.28	3.8×10^{-4}	140	0.29	(0.21, 0.41)	(0.053, 0.76)	0.315	(0.297, 0.331)
Ω_Λ	38	0.72	9.1×10^{-4}	30	0.72	(0.63, 0.77)	(0.47, 0.81)	0.685	(0.669, 0.703)
h	124	0.63	4.3×10^{-4}	160	0.68	(0.54, 0.76)	(0.41, 0.88)	0.673	(0.661, 0.685)
σ_8	80	0.86	1.1×10^{-3}	130	0.84	(0.72, 1.0)	(0.56, 1.3)	0.829	(0.817, 0.841)
Ω_b	43	0.042	1.8×10^{-4}	110	0.046	(0.031, 0.066)	(0.020, 0.17)	0.049	(0.048, 0.049)
n	24	0.96	9.2×10^{-4}	41	0.98	(0.94, 1.1)	(−1.5, 1.1)	0.960	(0.953, 0.968)
β	48	0.34	2.9×10^{-3}	87	0.52	(0.39, 0.75)	(0.20, 1.2)		
m_ν [eV]	8	0.014	4.4×10^{-3}	16	0.26	(0.0070, 0.60)	(0.0, 0.65)		< 0.933
Γ	17	0.18	4.1×10^{-3}	9.8	0.19	(0.13, 0.27)	(0.090, 0.45)	0.212	(0.199, 0.223) [i]
$\Omega_m^{0.6}\sigma_8$	11	0.56	1.1×10^{-2}	13	0.52	(0.46, 0.56)	(0.45, 0.57)	0.415	(0.400, 0.427) [i]
Ω_k	15	5.0×10^{-3}	9.2×10^{-4}	23	0.0	(−0.091, 0.081)	(−1.1, 0.21)	−0.037	(−0.086, 0.006)
ω_0	36	−0.968	4.73×10^{-4}	51.9	−0.986	(−1.07, −0.808)	(−1.25, −0.419)	−1.49	(−2.06, −0.840)

[a] Number of measurements.
[b] Weighted mean central value.
[c] Standard deviation of weighted mean.
[d] Number of standard deviations χ deviates from unity, Eq. (4).
[e] Median statistics central value.
[f] Median statistics range. In several cases for the 2σ range there were not enough measurements to determine a 2σ lower limit. In these cases, the lowest data point was used to represent the 2σ lower limit. This is the case for Ω_Λ, Ω_b, n, β, m_ν, Γ, $\Omega_m^{0.6}\sigma_8$, and Ω_k.
[g] Estimated constrained value using *Planck* + WP (*WMAP* polarization) data. These are from the last column of Table 2 of [1], except for m_ν, Ω_k, and ω_0 which are from the third column of Table 10 in [1]. For m_ν there was no central value listed and so a 2σ upper limit is given.
[h] Values are taken from tables listed in the previous footnote. A 1σ range was given for all parameters except for m_ν, Ω_k, and ω_0 where a 2σ upper limit or range is given.
[i] Here we have added in quadrature the errors on Ω_m and h to get the range of Γ. To get the range for $\Omega_m^{0.6}\sigma_8$ we have taken the error on $\Omega_m^{0.6}$ which is given as $0.6\Omega_m^{0.4}\sigma_{\Omega_m}$ and added it in quadrature with the error on σ_8.

parameters here. In any case, for our purpose here, the important point is that the weighted mean technique cannot be used to derive a summary estimate by combining together the different measurements tabulated by [9] for each cosmological parameter.[7]

In a situation like this the median statistic technique can be used to combine together the measurements to derive an effective summary value of the cosmological quantity of interest (e.g., [7, 22]). Column 6 of Table 1 lists the computed medians of the 12 cosmological parameters; the corresponding 1σ and 2σ ranges of these parameters are listed in columns 7 and 8.[8]

The median statistics estimate for the Hubble parameter here, $h = 0.68^{+0.08}_{-0.14}$, is consistent with that estimated earlier by [8] from 553 measurements of h tabulated by Huchra, $h = 0.68 \pm 0.028$ (with understandably much tighter error bars as a consequence of the many more measurements than the 124 we have used here).[9] Interestingly, from many fewer Ω_m measurements than considered here, [7] determine consistent, but somewhat tighter median statistics constraints on Ω_m by discarding the most discrepant, ∼5%, of the measurements (those which contribute the most to χ^2).

Also of interest, the median statistics estimates in Table 1 of $\Omega_m = 0.29$ and $\sigma_8 = 0.84$ result in $\Omega_m^{0.6}\sigma_8 = 0.40$, which is significantly smaller than the median statistics estimate $\Omega_m^{0.6}\sigma_8 = 0.52$ listed in Table 1 that was determined directly from the 11 measurements of [9].[10] On the other hand, $\Gamma = \Omega_m h$ computed using the median statistics estimates of $\Omega_m = 0.29$ and $h = 0.68$ is $\Gamma = 0.20$, and is in very good agreement with Table 1 median statistics value of $\Gamma = 0.19$ from the 17 measurements of [9].

In most cases the median statistics results of Table 1 provide reasonable (2010) summary estimates for the cosmological parameters. The one exception, perhaps, is that for h, which is estimated to be $h = 0.68 \pm 0.028$ by [8] from very many more measurements than the 124 used to derive the h value in Table 1. Perhaps the best current estimate of cosmological parameter values are those determined from the initial cosmic microwave background anisotropy measurements made by the *Planck* satellite [1]. The last two columns of Table 1 lists the *Planck* estimates for most of these parameters. Here, the estimated cosmological constrained value and 1σ standard deviation range (with the exception of Ω_k and ω_0 that have 2σ ranges, and m_ν that has a 2σ upper limit) are listed.[11]

Comparing our computed median results to the recent *Planck* values, one finds that almost all of the *Planck* central value results fall within the 1σ range of our median results. One exception is $\Omega_m^{0.6}\sigma_8$, possibly because of reasons discussed above; our estimates of $\Omega_m = 0.29$ and $\sigma_8 = 0.84$ result in a $\Omega_m^{0.6}\sigma_8$ value which is very consistent with the *Planck* estimate of $\Omega_m^{0.6}\sigma_8 = 0.415$. The other exception is ω_0 which *Planck* estimates to be −1.49. Our median statistics 2σ range is $-1.25 \leqslant \omega_0 \leqslant -0.808$ computed from the 36 measurements of [9]. [9] note that the number of measurements for ω_0 are still increasing with time,[12] unlike the case for the other parameters. Also, as stated in [1], the *Planck* + *WMAP* constraint on ω_0 is not very significant. However, when combined with other data tighter constraints result; for instance, including BAO data

[7] Table 1 shows that the $\sigma_{\rm wm}$ for each cosmological parameter is very small; a related reason to not trust the weighted average analysis.

[8] The median statistics ranges tabulated are purely statistical. As discussed in, e.g., Ref. [11], it is possible to use median statistics to also estimate the systematic uncertainties, however, there are not enough measurements for any of the 12 parameters for us to be able to properly use this technique. As a consequence, the ranges tabulated are slightly underestimated.

[9] For earlier, very consistent, estimates of h using median statistics see [14] and [6].

[10] It is likely that the larger $\Omega_m^{0.6}\sigma_8 = 0.52$ found here is mostly a consequence of the higher $\Omega_m^{0.6}\sigma_8$ values of a number of earlier analyses based on large-scale peculiar velocity measurements. While there are not enough measurements tabulated for us to more carefully examine this, it might be relevant that [9] in the fifth paragraph of their Section 3.4, when discussing their Fig. 13, note that peculiar velocity measurements have not had a great track record when used to measure cosmological parameters.

[11] The variance for parameters Γ and $\Omega_m^{0.6}\sigma_8$ were not given in [1], but were calculated by adding their component's errors in quadrature (see the last footnote in Table 1). All parameter estimates use both *Planck* temperature power spectrum data as well as *WMAP* polarization measurements at low multipoles. [1] do not provide a *Planck* estimate for β.

[12] In fact, only around the time of *WMAP*1 were measurements, instead of limits, being published [9].

they find $\omega_0 = -1.13^{+0.24}_{-0.25}$ at 2σ, Table 10 of [1]. As such, the estimation of ω_0 is an area still under development and so we should not give much weight to the difference in our estimate from that of *Planck*, and we emphasize that our median statistics value for ω_0 is reasonably consistent with the *Planck* + *WMAP* estimate.

More provocatively, it is instructive to compare our median statistics central estimates to the 1σ (or 2σ) *Planck* ranges. As expected, we see that our estimate of Ω_m (Ω_Λ) lies somewhat above (below) the corresponding *Planck* 1σ range. Our estimates of Ω_b and Γ are below the corresponding *Planck* 1σ ranges. Our estimate of n is well above the *Planck* 1σ range, being quite consistent with the simplest scale-invariant spectrum [15,21,29] while *Planck* data strongly favors a non-scale-invariant spectrum, also readily generated by quantum fluctuations during inflation (see, e.g., [23]). And as might have been anticipated, our median statistics central $\Omega_m^{0.6}\sigma_8$ value is well above the *Planck* 1σ range.

6. Conclusion

From the measurements compiled by [9], the median statistics technique can be used to compute summary estimates of 12 cosmological parameters. On comparing 11 of these values to those recently estimated by the *Planck* Collaboration, we find good consistency in ten cases. The exception is the parameter $\Omega_m^{0.6}\sigma_8$, and it is likely that the *Planck* estimate of this cosmological parameter is more accurate. We also note that the ω_0 estimation is still in its infancy and so one should not give much significance to this current mild discrepancy.

It is very reassuring that summary estimates for a majority of cosmological parameters considered by [9] are very consistent with corresponding values estimated from the almost completely independent *Planck* + *WMAP* polarization data. This provides strong support for the idea that we are now converging on a "standard" cosmological model.

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