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Wave scattering by many small bodies and applications

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Wave scattering problem by many bodies is studied in the case when the bodies are small, $ka \ll 1$, where a is the characteristic size of a body. The limiting case when $a \to 0$ and the total number of the small bodies is $M = O(a^{-(2-\kappa)})$, where $\kappa \in (0, 1)$ is a number, are studied. © *2011 American Institute of Physics*. [doi:10.1063/1.3555192]

I. INTRODUCTION

Many-body scattering problem in the case of small scatterers embedded in an inhomogeneous medium has been solved in Refs. 3 and 4 under the following assumptions:

$$ka \ll 1, \quad d = O\big(a^{\frac{2-\kappa}{3}}\big), \quad \zeta_m = \frac{h(\mathbf{x}_m)}{a^\kappa}, \tag{1}$$

where a is the characteristic size of the small bodies, $k = 2\pi/\lambda = \omega/c_0$ is the wave number, and c_0 is the wave speed in free space, $\kappa \in (0, 1)$ is a parameter one can choose as one wishes, d is the distance between neighboring particles, $h(\mathbf{x})$ is a piecewise-continuous function in a bounded domain $D \subset \mathbb{R}^3$ with a smooth boundary S, $\Im h = h_2 \le 0$, $h = h_1 + ih_2$, $\mathbf{x}_m \in D_m$ is an arbitrary point, D_m is a small body, S_m is its surface, $\mathbf{N}$ is the unit normal to S_m, $1 \le m \le M$, M is the total number of the embedded small bodies in D, the unit normal $\mathbf{N}$ points out of D_m, ζ_m is the boundary impedance in the boundary condition,

$$\frac{\partial u}{\partial \mathbf{N}} = \zeta_m u \qquad \text{on} \quad S_m, \quad 1 \le m \le M; \qquad u = u_M, \tag{2}$$

and the distribution of small bodies in D is defined as

$$\mathcal{N}(\Delta) := \Sigma_{D_m \subset \Delta} 1 = \frac{1}{a^{2-\kappa}} \int_\Delta N(\mathbf{x}) d\mathbf{x} [1 + o(1)], \quad a \to 0, \tag{3}$$

where $\mathcal{N}(\Delta)$ is the number of small bodies in an arbitrary subdomain $\Delta \subset D$, $N(\mathbf{x}) \ge 0$ is a piecewise-continuous function, and for simplicity it is assumed that $D_m = B(\mathbf{x}_m, a)$ is a ball centered at the point $\mathbf{x}_m$, of radius a. The scattering problem, solved in Refs. 3 and 4, consisted of finding the solution to the equation,

$$[\nabla^2 + k^2 n_0^2(\mathbf{x})] u_m = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m, \tag{4}$$

satisfying boundary conditions (2) and the radiation condition,

$$u_M = u_0 + v_M, \quad \frac{\partial v_M}{\partial r} - ikv_M = o\Big(\frac{1}{r}\Big), \quad r := |\mathbf{x}| \to \infty. \tag{5}$$

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Here, u_0 is the solution to problem (4) and (5) in the absence of the embedded particles, i.e., the solution for the problem with $M = 0$,

$$[\nabla^2 + k^2 n_0^2(\mathbf{x})]u_0 = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{6}$$

where $n_0^2(x)$ is the refraction coefficient in the absence of embedded particles, $n_0^2(x) = 1$ in the region $D' := \mathbb{R}^3 \setminus D$, and

$$u_0 = e^{ik\alpha\cdot\mathbf{x}} + v_0, \quad \frac{\partial v_0}{\partial r} - ikv_0 = o\left(\frac{1}{r}\right), \quad r \to \infty, \tag{7}$$

where $\alpha \in S^2$ is the direction of propagation of the incident plane wave, and S^2 is the unit sphere in R^3.

We are interested in the behavior of the scattering solution as $a \to 0$ and the wavenumber $k > 0$ is arbitrary fixed. In other words, the physical assumption that the dimensionless parameter ka is very small, $ka << 1$, corresponds in our work to a study of the mathematical limiting procedure $a \to 0$.

It was proved in Refs. 3 and 4, that, as $a \to 0$, the limiting field u does exist and solves the equation,

$$[\nabla^2 + k^2 n^2(\mathbf{x})]u = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{8}$$

where

$$n^2(\mathbf{x}) \equiv n_0^2(\mathbf{x}) - 4\pi k^{-2} h(\mathbf{x})N(\mathbf{x}). \tag{9}$$

Therefore, in the limit $a \to 0$, under the constraints (1)–(3), the limiting medium, obtained by the embedding of many small particles, has the refraction coefficient $n^2(\mathbf{x})$, given by (9). Since the functions $h(\mathbf{x})$ and $N(\mathbf{x})$ are at our disposal, subject to the restrictions $N(\mathbf{x}) \geq 0$, $\Im h(\mathbf{x}) \leq 0$, it is possible to create any desired refraction coefficient $n^2(\mathbf{x})$, $\Im n^2(\mathbf{x}) \geq 0$, by choosing $h(\mathbf{x})$ and $N(\mathbf{x})$ suitably.

It is assumed that the term "piecewise-continuous" function f in this paper means that the set $\mathcal{M}$ of discontinuities of f is of Lebesgue's measure zero and, if $\mathcal{S}$ is a subset of this set such that f is unbounded on $\mathcal{S}$, $f|_{\mathcal{S}} = \infty$, then f grows not too fast as $\mathbf{x}$ tends to $\mathcal{S}$

$$|f(\mathbf{x})| \leq \frac{c}{[\text{dist}(\mathbf{x}, \mathcal{S})]^{\nu}}, \quad 0 \leq \nu < 3, \quad c = \text{const} \geq 0, \tag{10}$$

so that the integral $\int_D f(\mathbf{x})d\mathbf{x}$ exists as an improper integral.

This paper is related to:[3] its goal is to develop a theory, similar to the one in Ref. 3, for a different governing equation

$$L_0 u_0 := \nabla \cdot (c^2(\mathbf{x})\nabla u_0) + \omega^2 u_0 = 0 \quad \text{in} \quad R^3, \tag{11}$$

where the wave speed $c(\mathbf{x}) = c_0 = \text{const}$ in $D' := R^3 \backslash D$, the complement of D in R^3, and $c(\mathbf{x})$ is a smooth and strictly positive function in D. The speed $c(\mathbf{x})$, in general, has S as its discontinuity surface. In this case, Eq. (11) is understood in the distributional sense as an integral identity,

$$\int_{R^3} (-c^2(\mathbf{x})\nabla\phi\nabla u_0 + \omega^2 \phi u_0)d\mathbf{x} = 0 \quad \forall \phi \in C_0^\infty(R^3). \tag{12}$$

Alternatively, one may understand Eq. (11) as the following transmission problem:

$$L_0 u_0^+ = 0 \quad \text{in} \quad D, \quad u_0^+ = u_0 \quad \text{in} \quad D, \tag{13}$$

$$L_0 u_0^- = 0 \quad \text{in} \quad D', \quad u_0^- = u_0 \quad \text{in} \quad D', \tag{14}$$

$$u_0^+ = u_0^-, \quad c_+^2(\mathbf{x})\frac{\partial u_0}{\partial \mathbf{N}^+} = c_-^2(\mathbf{x})\frac{\partial u_0}{\partial \mathbf{N}^-} \quad \text{on} \quad S. \tag{15}$$

The transmission conditions (15) together with Eqs. (13) and (14) are equivalent to problem (12). Existence and uniqueness of the solution to (13)–(15) was proved in Ref. 6.

The scattering problem, we are interested in, can be stated as follows:

$$L_0 u = 0 \quad \text{in} \quad R^3 \setminus \bigcup_{m=1}^{M} D_m; \quad u = u_M, \tag{16}$$

$$\frac{\partial u}{\partial \mathbf{N}} = \zeta_m u \quad \text{on} \quad S_m, \quad 1 \le m \le M, \tag{17}$$

$$u = u_0 + v, \quad v_r - ikv = o(\frac{1}{r}), \quad r \to \infty. \tag{18}$$

In Sec. II problem (16)–(18) is investigated and the limiting behavior of u as $a \to 0$ is found. We conclude this Introduction by a brief derivation of the governing Eq. (11).

The starting point is the Euler equation,

$$\dot{\mathbf{v}} + (\mathbf{v}, \nabla)\mathbf{v} = -\frac{\nabla p}{\rho}, \tag{19}$$

where $\mathbf{v}$ is the velocity vector of the sound wave, $p = p(\rho)$ is the static pressure, ρ is the density, and

$$\nabla p = c^2(\mathbf{x}) \nabla \rho, \tag{20}$$

where $c(\mathbf{x})$ is the sound speed.

Let the material in D be initially at rest, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ be a small perturbation of the equilibrium zero velocity, the density be of the form $\rho = \rho_0 + \psi(\mathbf{x}, t)$, where ρ_0 is the equilibrium density of the material, which is assumed to be constant, and ψ and $\mathbf{v}$ are small quantities of the same order of smallness.

The continuity equation is

$$\dot{\psi} = -\nabla \cdot (\rho_0 \mathbf{v}), \tag{21}$$

where the term $\nabla \cdot (\psi \mathbf{v})$ of the higher order of smallness is neglected. Differentiating (21) with respect to time yields

$$\ddot{\psi} = -\nabla \cdot (\rho_0 \dot{\mathbf{v}}). \tag{22}$$

Under the same assumptions about $\rho = \rho_0 + \psi(\mathbf{x}, t)$ and $\mathbf{v}$, the term $(\mathbf{v}, \nabla)\mathbf{v}$ in (19) is of the higher order of smallness and is, therefore, neglected. Multiplying (19) by ρ and neglecting the term $\psi \dot{\mathbf{v}}$ of higher order of smallness yields the acoustic momentum equation,

$$\rho_0 \dot{\mathbf{v}} = -\nabla p. \tag{23}$$

Substituting (20) in (23) gives

$$\rho_0 \dot{\mathbf{v}} = -c^2(\mathbf{x}) \nabla \psi, \tag{24}$$

where the relation $\nabla \rho = \nabla \psi$ was used. This relation is exact for a constant ρ_0.

Substituting (24) in (22) yields

$$\ddot{\psi} - \nabla \cdot (c^2(\mathbf{x}) \nabla \psi) = 0. \tag{25}$$

If $\psi = e^{-i\omega t} u$, then (25) reduces to Eq. (11).

II. THE SCATTERING PROBLEM

In this section, problem (16)–(18) is studied. Assumptions (1) and (3) are still valid.

Let G be the Green's function for the operator L_0,

$$L_0 G(\mathbf{x}, \mathbf{y}) = -\delta(\mathbf{x} - \mathbf{y}) \quad \text{in} \quad \mathbb{R}^3. \tag{26}$$

G satisfies the radiation condition

$$\frac{\partial G}{\partial |\mathbf{x}|} - ikG = o(\frac{1}{|\mathbf{x}|}) \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{27}$$

The following result from Ref. 5 will be used.

Theorem 1: *In a neighborhood of a point of smoothness of* $c(\mathbf{x})$, *one has*

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|c(\mathbf{x})}(1+o(1)), \quad |\mathbf{x}-\mathbf{y}| \to 0. \tag{28}$$

In a neighborhood of the point $x \in S$, *where* S *is a smooth discontinuity surface of* $c(\mathbf{x})$, *one has*

$$G(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{1}{4\pi c_+(\mathbf{x})}[r_{\mathbf{xy}}^{-1} + bR^{-1} + o(1)], & \mathbf{y} \in D, \\ \frac{1}{4\pi c_-(\mathbf{x})}[r_{\mathbf{xy}}^{-1} - bR^{-1} + o(1)], & \mathbf{y} \in D'. \end{cases} \tag{29}$$

where

$$b := \frac{c_+(\mathbf{x}) - c_-(\mathbf{x})}{c_+(\mathbf{x}) + c_-(\mathbf{x})}, \quad r_{\mathbf{xy}} := |\mathbf{x}-\mathbf{y}|, \quad R = \sqrt{\rho^2 + (|x_3| + |y_3|)^2}, \tag{30}$$

$$\rho = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}. \tag{31}$$

The origin of the local coordinate system lies on S, *the plane* $x_3 = 0$ *is tangent to* S, $c_+(\mathbf{x})$ *and* $c_-(\mathbf{x})$ *are the limiting values of* $c(\mathbf{x})$ *when* $\mathbf{x} \to S$ *from inside (and outside) of* D.

In Ref. 5, the operator L_0 corresponds to the case $\omega = 0$. However, in Ref. 3 it is proved that adding to L_0 a term $q(\mathbf{x})G(\mathbf{x},\mathbf{y})$ with a bounded function q does not change the main term of the asymptotic of G as $\mathbf{x} \to \mathbf{y}$.

The solution to problem (16)–(15) is sought in the form

$$u = u_0 + \sum_{m=1}^{M} \int_{S_m} G(\mathbf{x},t)\sigma_m(t)dt. \tag{32}$$

For any $\sigma_m \in L^2(S_m)$, the function u, defined in (32), solves Eq. (16) and satisfies the radiation condition (15), since G does. Therefore, (32) will be the solution to problem (16)–(15) if σ_m are such that the boundary conditions (17) are satisfied. Uniqueness of the solution to problem (16)–(15) follows from essentially the same arguments as in Ref. 3, see the proof of Theorem 1 in Ref. 3.

The boundary conditions (15) imply

$$u_{eN} + \frac{A_m\sigma_m - \sigma_m c_m^{-1}}{2} = \zeta_m u_e + \zeta_m \int_{S_m} G(s,t)\sigma_m(t)dt, \tag{33}$$

where

$$c_m := c(\mathbf{x}_m), \qquad \zeta_m = h(\mathbf{x}_m)/a^\kappa,$$

and

$$u_e(\mathbf{x}) := u_e^{(m)} := u_0(\mathbf{x}) + \sum_{m' \neq m} \int_{S_{m'}} G(\mathbf{x},t)\sigma_{m'}(t)dt. \tag{34}$$

The field $u_e^{(m)}$ is called the effective (self-consistent) field. It is the field acting on the mth particle from all other particles and from the incident field u_0.

The operator A_m is the operator of the normal derivative of the single-layer potential,

$$T\sigma_m := \int_{S_m} G(\mathbf{x},t)\sigma_m(t)dt,$$

at the boundary S, and

$$\frac{\partial T\sigma_m}{\partial \mathbf{N}^-} = \frac{A\sigma_m - \sigma_m c^{-1}(\mathbf{x}_m)}{2}, \quad A\sigma_m = 2\int_{S_m} \frac{\partial G(\mathbf{s},t)}{\partial \mathbf{N_s}}\sigma_m(t)dt, \quad \mathbf{s} \in S_m, \tag{35}$$

In Eq. (35), $\frac{\partial T\sigma_m}{\partial \mathbf{N}^-}$ is the limiting value of the normal derivative on S_m from outside of D_m. Equation (35) is well known from the potential theory in the case $c(\mathbf{x}) = 1$ and $G(\mathbf{x}, \mathbf{y}) = \frac{e^{i\omega r_{\mathbf{xy}}}}{4\pi r_{\mathbf{xy}}}$, $r_{\mathbf{xy}} := |\mathbf{x} - \mathbf{y}|$.

If $c(\mathbf{x}) \neq 1$, then by Theorem 1, one may consider T and A as the operators, corresponding to $c(\mathbf{x}) = 1$, divided by $c(\mathbf{x}_m)$, because $c(\mathbf{s})$ is assumed smooth in D, and, therefore, it varies negligibly on the small distances of the order a.

The basic idea of solving the scattering problem (16)–(18) is to use a representation of the scattered field as a sum of single layer potentials and transform this representation to a sum of two terms of which one is much larger than the other asymptotically, as $a \to 0$, (cf. Ref. 3).

The approach is to reduce the solution of the many-body scattering problem by small bodies to finding some numbers rather than the unknown functions σ_m, $1 \leq m \leq M$. If M is very large, it is practically impossible to use the usual system of boundary integral equations for finding the unknown σ_m.

Let us rewrite Eq. (32) as follows:

$$u = u_0(\mathbf{x}) + \sum_{m=1}^{M} G(\mathbf{x}, \mathbf{x}_m)Q_m + \sum_{m=1}^{M} \int_{S_m} [G(\mathbf{x}, t) - G(\mathbf{x}, \mathbf{x}_m)]\sigma_m(t)dt, \tag{36}$$

where

$$Q_m := \int_{S_m} \sigma_m(t)dt, \tag{37}$$

and prove that

$$\left|G(\mathbf{x}, \mathbf{x}_m)Q_m\right| \gg \left|\int_{S_m} [G(\mathbf{x}, t) - G(\mathbf{x}, \mathbf{x}_m)]\sigma_m(t)dt\right|, \quad a \to 0; \qquad d_1 < |\mathbf{x} - \mathbf{x}_m|, \tag{38}$$

where $a \ll d_1 \ll d$.

The region $d_1 < |\mathbf{x} - \mathbf{x}_m|$ clearly tends to D as $a \to 0$. The main term of the asymptotics of the function $\sigma_m(t)$, as $a \to 0$, does not depend on $t \in S_m$, it is a constant with respect to t depending on a, equal to $\frac{Q_m}{4\pi a^2}$. (see Ref. 3).

Let us explain why the above inequality (38) holds. Its left-hand side is $O(|Q_m|/|x - x_m|)$, while its right-hand side does not exceed $\max_{z\in B(x_m,a)} |\nabla_z G(x, z)|a|Q_m|$.

One has

$$\max_{z\in B(x_m,a)} |\nabla_z G(x, z)| = O(\max\{\frac{1}{|x - x_m|^2}, \frac{k}{|x - x_m|}\}).$$

If $|x - x_m| \gg a$ and $ka \ll 1$, then $\frac{1}{|x-x_m|} \gg a \max\{\frac{1}{|x-x_m|^2}, \frac{k}{|x-x_m|}\}$. Therefore, inequality (38) is valid.

Let us choose the region $|x - x_m| \gg a$ to be $|x - x_m| \geq d_1$, where $d_1 = O\left(a^{\frac{2-0.5\kappa}{3}}\right)$, so one has $a \ll d_1 \ll d$ as $a \to 0$. We now want to prove that the input into the scattering solution u of the terms in the second sum in Eq. (36), which lie in the region $|x - x_m| \leq d_1$ is negligible as $a \to 0$.

Since the distance between neighboring particles is $O(d)$, one concludes that there is one particle of radius a, centered at x_m, and there are no other particles in the region $a < |x - x_m| \leq d_1$ because the distance between small particles is $O(d) \gg d_1$. The input of one particle to the second sum in (36) is the quantity of the order $O(aa^{-2}a^{2-\kappa})$ as $a \to 0$, i.e., $O(a^{1-\kappa})$. This quantity tends to zero as $a \to 0$. Here the term $O(a^{-2})$ is the order of $|\nabla_z G(x, z)|$ when $|z - x_m| = O(a)$, and the term $O(a^{2-\kappa})$ is the order of $|Q_m|$ [see (46) below]. This explains the order of the magnitude $O(a^{1-\kappa})$ of the input of the particles which lie in the region $a < |x - x_m| \leq d_1$ to the second sum in (36). Since $O(a^{1-\kappa})$ is negligible as $a \to 0$, one can neglect the second sum in (36) as $a \to 0$.

Thus, the solution u of the many-body scattering problem can be written as

$$u = u_0(\mathbf{x}) + \sum_{m=1}^{M} G(\mathbf{x}, \mathbf{x}_m)Q_m, \quad |\mathbf{x} - \mathbf{x}_m| \gg a, \tag{39}$$

with the error that tends to zero as $a \to 0$.

Consequently, the scattering problem is solved if the numbers Q_m, $1 \le m \le M$, are found. This simplifies the solution of the many-body scattering problem drastically because Eq. (32) requires the knowledge of the functions $\sigma_m(t)$, $1 \le m \le M$, rather than the numbers Q_m, in order to find the solution u of the scattering problem.

The next step is to derive the main term of the asymptotics of Q_m as $a \to 0$.

To do this, we integrate (33) over S_m and neglect the terms of the higher order of smallness as $a \to 0$.

One has

$$\int_{S_m} u_{eN} d\mathbf{s} = \int_{D_m} \nabla^2 u_e d\mathbf{x} = (\nabla^2 u_e)(\mathbf{x}_m)|D_m|, \quad |D_m| = \frac{4\pi a^3}{3}, \tag{40}$$

where the Gauss divergence theorem was applied and a mean value formula for the integral over D_m was used.

Furthermore,

$$\int_{S_m} A\sigma_m d\mathbf{s} = -\frac{1}{c_m}\int_{S_m} \sigma_m d\mathbf{s} = -\frac{Q_m}{c_m}, \tag{41}$$

where (cf. Ref. 3)

$$\int_{S_m} A\sigma\, d\mathbf{s} := \frac{1}{c_m}\int_{S_m} d\mathbf{s} \int_{S_m} \frac{\partial}{\partial N_s}\frac{1}{4\pi r_{\mathbf{st}}}\sigma(t)dt = -\frac{1}{c_m}\int_{S_m}\sigma(t)dt. \tag{42}$$

Thus, integrating (33) over S_m yields

$$\nabla^2 u_e(\mathbf{x}_m)|D_m| - c_m^{-1}Q_m = \zeta_m u_e(x_m)|S_m| + \frac{\zeta_m}{c_m}\int_{S_m} dt\sigma_m(t)\int_{S_m} d\mathbf{s}\frac{1}{4\pi r_{st}}, \tag{43}$$

where $|S_m| = 4\pi a^2$ is the surface area of the sphere S_m and formula (28) was used, namely, we have replaced $G(\mathbf{s}, t)$ by $\frac{1}{4\pi r_{st}}\frac{1}{c(\mathbf{s})}$ using the smallness of D_m, and we have replaced $c(\mathbf{s})$ by $c(\mathbf{x}_m) = c_m$ because $|x_m - s| \le a$ and a is small.

Using the identity

$$\int_{S_m} \frac{d\mathbf{s}}{4\pi r_{\mathbf{st}}} = a \quad \text{if} \quad |\mathbf{s} - \mathbf{x}_m| = a \quad \text{and} \quad |t - \mathrm{x}_m| = a, \tag{44}$$

one gets from (43) the following relation:

$$Q_m(c_m^{-1} + c_m^{-1}\zeta_m a) = -4\pi\zeta_m u_e(\mathbf{x}_m)a^2 + O(a^3). \tag{45}$$

If $a \to 0$ and $\kappa \in (0, 1)$, then

$$\zeta_m a = h(\mathbf{x}_m)a^{1-\kappa} = o(1), \qquad a \to 0,$$

the term $O(a^3)$ in (45) can be neglected, and one gets the main term of the asymptotics of Q_m as $a \to 0$, namely,

$$Q_m = -4\pi h(\mathbf{x}_m)u_e(\mathbf{x}_m)c(\mathbf{x}_m)a^{2-\kappa}[1 + o(1)], \qquad a \to 0. \tag{46}$$

Therefore, (34), (39), and (46) yield

$$u_e(\mathbf{x}) = u_0(\mathbf{x}) - 4\pi \sum_{m' \ne m} G(\mathbf{x}, \mathbf{x}_{m'})h(\mathbf{x}_{m'})u_e(\mathbf{x}_{m'})c(\mathbf{x}_{m'})a^{2-\kappa}[1 + o(1)]. \tag{47}$$

Taking $\mathbf{x} = \mathbf{x}_m$ and neglecting $o(1)$ term in (47), one gets a linear algebraic system for the unknown quantities $u_m := u_e(\mathbf{x}_m)$, $1 \le m \le M$,

$$u_m = u_{0m} - 4\pi \sum_{m' \ne m} G(\mathbf{x}_m, \mathbf{x}_{m'})h(\mathbf{x}_{m'})c(\mathbf{x}_{m'})u_{m'}a^{2-\kappa}. \tag{48}$$

Let us now derive and use a generalization of the result proved originally in Ref. 3. This generalization is formulated as Theorem 2 below.

Consider the sum

$$I = \lim_{a\to 0} a^{2-\kappa} \sum_{m=1}^{M} f(\mathbf{x}_m), \tag{49}$$

where the points $\mathbf{x}_m$ are distributed in D according to (3).

Assume that $f(\mathbf{x})$ is piecewise-continuous and (10) holds. If f is unbounded, that is, the set $\mathcal{S}$ is not empty, then the sum (49) is understood as follows:

$$I := \lim_{\delta\to 0} \lim_{a\to 0} a^{2-\kappa} \sum_{m=1, dist(\mathbf{x}_m,\mathcal{S})\geq\delta}^{M} f(\mathbf{x}_m). \tag{50}$$

Theorem 2: *Under the above assumptions, there exists the limit* (49) *and*

$$\lim_{a\to 0} a^{2-\kappa} \sum_{m=1}^{M} f(\mathbf{x}_m) = \int_D f(\mathbf{x}) N(\mathbf{x}) d\mathbf{x}. \tag{51}$$

Proof of Theorem 2 is given at the end of this paper.

Applying Theorem 2 to the sum (47), one obtains the following result:

Theorem 3: *There exists the limit,*

$$\lim_{a\to 0} u_e(\mathbf{x}) := u(\mathbf{x}),$$

and the limiting function solves the equation,

$$u(\mathbf{x}) = u_0(\mathbf{x}) - 4\pi \int_D G(\mathbf{x}, \mathbf{y}) h(\mathbf{y}) c(\mathbf{y}) N(\mathbf{y}) u(\mathbf{y}) d\mathbf{y}. \tag{52}$$

Applying operator L_0, defined in (11), to (52) and using the relations

$$L_0 G = -\delta(\mathbf{x} - \mathbf{y}), \quad L_0 u_0 = 0, \tag{53}$$

one obtains the following new equation for the limiting effective field u:

$$L_0 u = 4\pi h(\mathbf{x}) c(\mathbf{x}) N(\mathbf{x}) u. \tag{54}$$

This equation can be written as

$$Lu := \nabla \cdot (c^2(\mathbf{x}) \nabla u) + \omega^2 u - 4\pi h(\mathbf{x}) c(\mathbf{x}) N(\mathbf{x}) u = 0. \tag{55}$$

Therefore, embedding many small particles into D and assuming (1)–(3) , one obtains in the limit $a \to 0$ a medium with essentially different properties described by the new equation (55).

Let us now prove Theorem 2.

Proof of Theorem 2:

Let $\mathcal{S}$ be the subset of the set of discontinuities of f on which f is unbounded, let the assumption (10) hold, and let

$$D_\delta := \{\mathbf{x} : \mathbf{x} \in D, dist(\mathbf{x}, \mathcal{S}) \geq \delta\}. \tag{56}$$

Consider a partition of D_δ into a union of small cubes Δ_p, centered at the points $\mathbf{y}_p$, with the side $b = a^{1/3}$. One has

$$\begin{aligned} a^{2-\kappa} \sum_{m=1, dist(\mathbf{x}_m,\mathcal{M})\geq\delta}^{M} f(\mathbf{x}_m) &= \sum_p f(\mathbf{y}_p)[1 + o(1)] a^{2-\kappa} \sum_{\mathbf{x}_m\in\Delta_p} 1 \\ &= \sum_p f(\mathbf{y}_p) N(\mathbf{y}_p) |\Delta_p| [1 + o(1)] \\ &\to \int_{D_\delta} f(\mathbf{y}) N(\mathbf{y}) d\mathbf{y} \quad \text{as} \quad a \to 0. \end{aligned} \tag{57}$$

Here in the second sum we replaced $f(\mathbf{x}_m)$ by $f(\mathbf{y}_p)$ for all points $\mathbf{x}_m \in \Delta_p$. This is done with the error $o(1)$ as $a \to 0$ because f is continuous in D_δ. In the third sum, we have used formula (3) for $\Delta = \Delta_p$. The last conclusion, namely, the existence of the limit as $a \to 0$, follows from the known result: the Riemannian sum of a piecewise-continuous bounded in D_δ function $f(\mathbf{x})N(\mathbf{x})$ converges to the integral $\int_{D_\delta} f(\mathbf{x})N(\mathbf{x})d\mathbf{x}$ if $\max_p diam\Delta_p \to 0$. In our case,

$$diam\Delta_p = \sqrt{3}a^{1/3} \to 0 \quad \text{as} \quad a \to 0, \tag{58}$$

so formula (57) follows.

From the assumption (10) with $\nu < 3$, one concludes that

$$\lim_{\delta \to 0} \int_{D_\delta} f(\mathbf{x})N(\mathbf{x})d\mathbf{x} = \int_D f(\mathbf{x})N(\mathbf{x})d\mathbf{x}. \tag{59}$$

The integral on the right in (59) exists as an improper integral if ν is less than the dimension of the space, i.e., $\nu < 3$. Therefore, formula (51) is established.

Theorem 2 is proved. □

While this paper has been under consideration (from October 2009), some related papers have been published, see Refs. 1, 2, 7 and 8.

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