A LAW OF THE ITERATED LOGARITHM FOR GENERAL LACUNARY SERIES

by

XIAOJING ZHANG

M.S., Xiamen University, China, 2006

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2012

Abstract

The main purpose of this thesis is to derive an upper bound and a lower bound in a law of the iterated logarithm for sums of the form $\sum_{k=1}^{N} a_k f(n_k x + c_k)$ where the n_k satisfy a Hadamard gap condition and $c_k \in \mathbb{R}^n$. Here we assume that f is a Dini continuous function on $\mathbb{R}^n$ which satisfies the property that for every cube Q of sidelength 1 with corners in the lattice $\mathbb{Z}^n$, f vanishes on ∂Q and has mean value zero on Q. And for the lower bound result, we need an extra condition that f has the property that there exists a number $c_0 > 0$ such that $\frac{1}{|Q|}\int_Q |f(u)|^2 du > c_0$ for all cubes of sidelength at least 1, so that we can keep f from becoming too "sparse" at infinity. We will introduce an important concept, dyadic martingales, and then proof of our theorems can be obtained by using a reduction to dyadic martingales.

A LAW OF THE ITERATED LOGARITHM FOR GENERAL LACUNARY SERIES

by

XIAOJING ZHANG

M.S., Xiamen University, China, 2006

A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
2012

Approved by:

Major Professor
Charles N. Moore

Abstract

The main purpose of this thesis is to derive an upper bound and a lower bound in a law of the iterated logarithm for sums of the form $\sum_{k=1}^{N} a_k f(n_k x + c_k)$ where the n_k satisfy a Hadamard gap condition and $c_k \in \mathbb{R}^n$. Here we assume that f is a Dini continuous function on $\mathbb{R}^n$ which satisfies the property that for every cube Q of sidelength 1 with corners in the lattice $\mathbb{Z}^n$, f vanishes on ∂Q and has mean value zero on Q. And for the lower bound result, we need an extra condition that f has the property that there exists a number $c_0 > 0$ such that $\frac{1}{|Q|}\int_Q |f(u)|^2 du > c_0$ for all cubes of sidelength at least 1, so that we can keep f from becoming too "sparse" at infinity. We will introduce an important concept, dyadic martingales, and then proof of our theorems can be obtained by using a reduction to dyadic martingales.

Table of Contents

List of Figures

Acknowledgments

First of all, I would like to thank my supervisor Professor Charles N. Moore for his patient guidance, valuable suggestions and enlightening comments during the period of my study in Kansas State University. His profound knowledge in mathematics has benefited me a lot. Working with him has always been an enjoyable and rewarding experience. This dissertation would have not come to fruition without his help.

Secondly, I would like to thank my committee members Professor Paul Nelson, Professor Diego Maldonado and Professor Pietro Poggi-Corradini, who were more than generous with their expertise and precious time. Thank you for agreeing to serve on my committee and for being so friendly, supportive and helpful throughout my study at Kansas State University. A special thanks to Professor Michael Babcock for serving as my committee chairman. I would like to also thank the Department of Mathematics, KSU, for providing me an opportunity to pursue my Ph.D. degree.

Finally, I would like to thank my family. I am very grateful for my parents Libing Zhu and Shangwei Zhang. Their understanding and unconditional love encouraged me to pursue the Ph.D. project and enjoy the beauty of math. Their firm and kind-hearted personalities have affected me to be steadfast and have an open heart. And I am greatly indebted to my devoted husband Hongzhou Huang, whose love and support has enabled me to complete this Ph.D. project.

Dedication

This dissertation is dedicated to:

My Heavenly Father, the **Source of all Wisdom**.

My grandparents **Jian Li** and **Anqing Zhu**,

and my parents **Libing Zhu** and **Shangwei Zhang**,

for their endless love, support and encouragement.

Chapter 1

Introduction

In this chapter we will recall the history of the Law of Iterated Logarithm and introduce useful definitions and notation which will be repeatedly used in later chapters. Also we will state some useful results.

1.1 History

In probability theory, the law of the iterated logarithm describes the magnitude of the fluctuations of a random walk, which comes from finding the rate of convergence in Borel's normal number theorem. So before we go into any theorems, let's take a look at the definition of normal numbers.

Definition 1.1.1 (Normal numbers)**.** For a real number $\omega \in [0, 1)$, consider its binary expansion, that is,

$$\omega = \sum_{i=1}^{\infty} c_i 2^{-i} \text{ where } c_i \in \{0, 1\}.$$

We say ω is simply normal if 0 and 1 each occur with frequency $\frac{1}{2}$. Precisely, let $N_n(\omega)$ denote the number of 1's in the first n places of the binary expansion of ω. Then $\frac{N_n(\omega)}{n}$ is the relative frequency of the digit 1 in the first n places, the limit $\lim_{n\to\infty} \frac{N_n(\omega)}{n}$ is the frequency of the digit 1 in the binary expansion of ω, and ω is simply normal if and only if $\lim_{n\to\infty} \frac{N_n(\omega)}{n} = \frac{1}{2}$. Similarly, for a real number $\omega \in [0,1)$, consider its decimal expansion, that is,

$$\omega = \sum_{i=1}^{\infty} c_i 10^{-i} \text{ where } c_i \in \{0,1,2,\ldots,9\}.$$

For a fixed number ω, $0 \le \omega \le 1$, let $N_n^{(j)}(\omega)$ denote the number of digits in the first n places of the decimal expansion of ω that are equal to j. Then ω is normal to the base 10 if the limit $\lim_{n\to\infty} \frac{N_n^{(j)}(\omega)}{n}$, representing the frequency of j in decimal depansion of ω, exists and equals $\frac{1}{10}$.

The concept of a normal number was introduced by Borel in 1909, and using the Borel-Cantelli lemma, he proved the normal number theorem:

Theorem 1.1.2 (Borel)**.** If $N_n(\omega)$ denotes the number of 1's in the first n places of the binary expansion of ω, then

$$\lim_{n\to\infty} \frac{N_n(\omega)}{n} = \frac{1}{2}$$

for almost every $\omega \in [0,1)$.

We write this as $N_n(\omega) \sim \frac{n}{2}$. And then, naturally, the next question to ask is, what can we say about the deviation $N_n(\omega) - \frac{n}{2}$? With efforts of Hausdorff (1913), Hardy and Littlewood (1914) and Khintchine (1923), the order bounds $O(n^{\frac{1}{2}+\epsilon})$, $O(\sqrt{n \log n})$ and $O(\sqrt{n \log\log n})$ were obtained. Then in 1924, Khintchine gave the definitive answer:

Theorem 1.1.3.

$$\limsup_{n\to\infty} \frac{N_n(\omega) - \frac{n}{2}}{\sqrt{\frac{1}{2} n \log\log n}} = 1$$

for almost every $\omega \in [0,1)$.

This result is known as the first Law of Iterated Logarithm(LIL). Note here if we let $f_j(\omega)$ denote the binary digit in the j^{th} place of ω and $N_n(\omega) = \sum_{j=1}^{n} f_j(\omega)$, then $\mathbb{E}(f_j) = \frac{1}{2}$, $\sigma^2(f_j) = \frac{1}{4}$ and thus $\frac{n}{2} = \mathbb{E}(N_n)$ and $\frac{1}{2}n = 2\sigma^2$. Now we consider the Rademacher functions

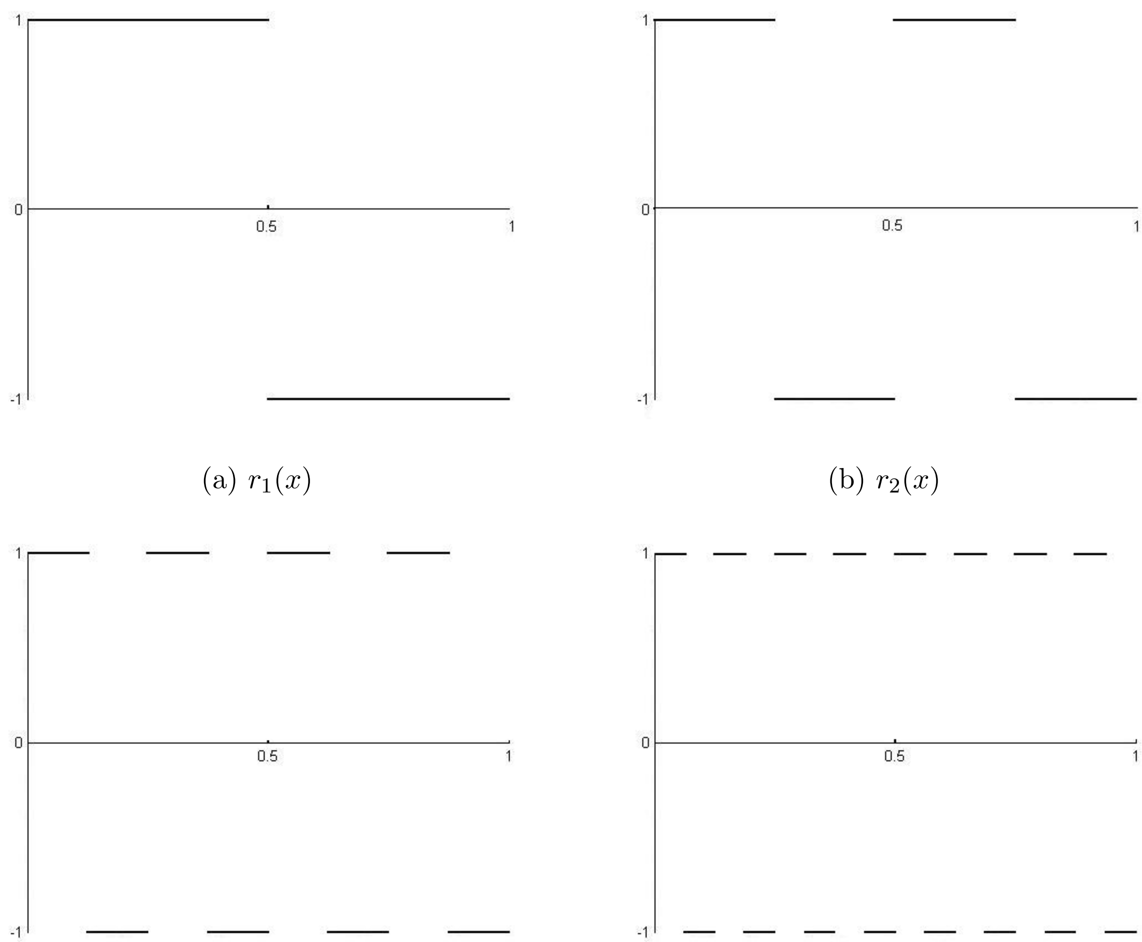
$$r_j(t) = \mathrm{sgn}(\sin(2^j \pi t)),\ \ j = 1, 2, 3, \ldots \ \text{for } t \in [0, 1], \tag{1.1.1}$$

where sgn is defined as

$$\mathrm{sgn}(t) = \begin{cases} 1 & \text{if } t \geq 0; \\ -1 & \text{if } t < 0. \end{cases}$$

The graphs of the first four Rademacher functions are as shown:

Figure 1.1: Rademacher functions



(a) $r_1(x)$ (b) $r_2(x)$

(c) $r_3(x)$ (d) $r_4(x)$

It is easy to see that $\mathbb{E}(r_j) = 0$, $\sigma^2(r_j) = 1$, and we get an equivalent assertion that

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} r_j(t)}{\sqrt{2n\log\log n}} = 1 \tag{1.1.2}$$

for almost every $t \in [0,1]$.

Rademacher functions can be used to represent random walks. Consider the integers $\{\ldots,-2,-1,0,1,2,\ldots\}$. Suppose you are standing at 0. Flip a fair coin. If the coin comes up heads, move to the right by one step. If it comes up tails, move to the left by one step. Repeat and continue this process. For any t that is not of the form $\frac{j}{2^m}$, the sequence $r_1(t),\ r_1(t)+r_2(t),\ r_1(t)+r_2(t)+r_3(t),\ldots$ is a random walk. According to the famous theorem of Pólya[10]:

Theorem 1.1.4. With probability one, the random walker will return to 0 in a finite number of steps.

This is a Markov process, which means, given the present state, the future and past states are independent; formally, for a sequence of random variables $\{X_1, X_2, X_3, \ldots\}$,

$$\mathbb{P}\left(X_{n+1}=x|X_1=x_1, X_2=x_2,\ldots,X_n=x_n\right) = \mathbb{P}(X_{n+1}=x|X_n=x_n).$$

Given any integer m, by the theorem of Khintchine (1.1.2), with probability one, we have $\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} r_j(t)}{\sqrt{2n\log\log n}} > \frac{1}{2}$. When n is sufficiently large, $\sum_{j=1}^{n} r_j(t) \geq \sqrt{2n\log\log n} \geq m$ infinitely often. Thus with probability one, the walker will land on m in a finite number of steps. Now assume we have m as the starting point, then with probability one, a random walker will return to this position in a finite number of steps. And after the walker returns, start once again a random walk; it will be just as if the walker is starting for the first time– there will be no memory of the past. With probability one, the walker will return again to m in finite number of steps. Continue and repeat this process; consequently we may conclude:

Theorem 1.1.5. With probability one, the random walker will visit every integer an infinite number of times.

Obviously for a random walk, after n steps, the distance from the starting point will be be bounded by

$$-n \leq \sum_{j=1}^{n} r_j \leq n,$$

The Law of the Iterated Logarithm gives more precise estimates: given $\epsilon > 0$, then eventually,

$$-(1+\epsilon)\sqrt{2n\log\log n} \leq \sum_{j=1}^{n} r_j \leq (1+\epsilon)\sqrt{2n\log\log n}.$$

Thus, the LIL describes the magnitude of the fluctuation of a random walk. In 1929, Kolmogorov generalized the result to the class of independent random variables, which is considered the classical LIL[7]:

Theorem 1.1.6. Let $S_m = \sum_{k=1}^{m} X_k$ where $\{X_k\}$ is a sequence of real-valued independent random variables. Let s_m be the variance of S_m. Suppose $s_m \to \infty$ and $|X_m|^2 \leq \dfrac{K_m s_m^2}{\log\log(e^e + s_m^2)}$ for some sequence of constants $K_m \to 0$. Then, almost surely,

$$\limsup_{m\to\infty} \frac{S_m}{\sqrt{2s_m^2 \log\log s_m}} = 1.$$

An examination of the graphs of the Rademacher functions and the functions $\cos(2^{k-1}x)$ (see Figure 1.2) leads to the conjecture that, even though these functions are not independent, there may be similar results in this setting. Over years there have been many studies to obtain similar results in other situations in analysis for more general cases. In 1950, Salem and Zygmund[13] considered the case when the X_k are replaced by functions $a_k \cos n_k x$ on $[-\pi, \pi]$ and gave an upper bound result:

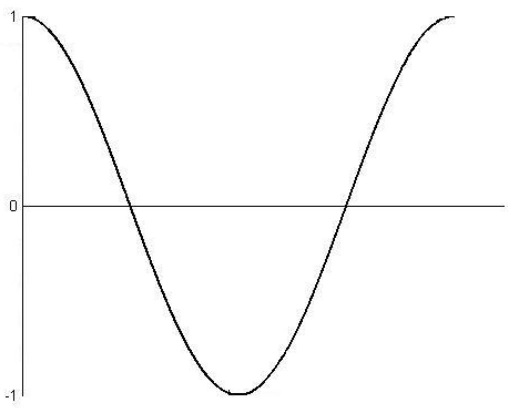
Theorem 1.1.7. Let $S_m(\theta) = \sum_{k=1}^{m} a_k \cos n_k\theta$ where n_k is a sequence of positive integers satisfying $\frac{n_{k+1}}{n_k} > q > 1$. Let $B_m = (\frac{1}{2}\sum_{k=1}^{m} |a_k|^2)^{1/2}$ and $M_m = \max_{1\leq k\leq m} |a_k|$. Suppose $B_m \to \infty$ as $m \to \infty$ and $|M_m|^2 \leq \dfrac{K_m B_m^2}{\log\log(e^e + B_m)}$ for some sequence of constants $K_m \to 0$. Then

$$\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \log\log B_m}} \leq 1.$$

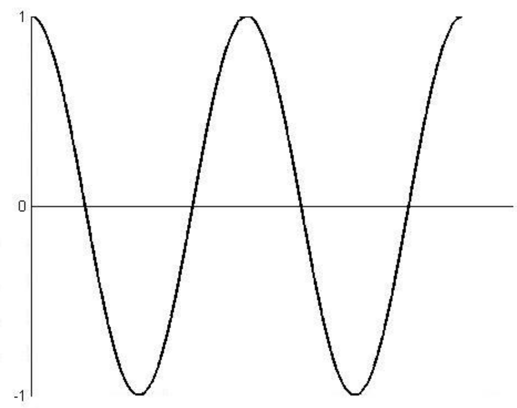
for almost every $\theta \in [-\pi, \pi]$.

A typical example of Salum and Zygmund's LIL is when we take $a_k = 1$ and $n_k = 2^{k-1}$ for each $k = 1, 2, \ldots,$.

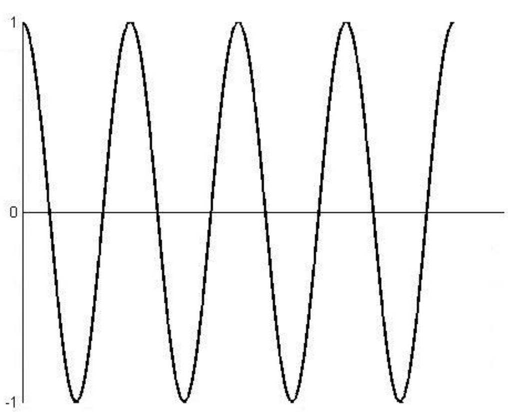
Figure 1.2: $\cos(2^{k-1}x)$ functions



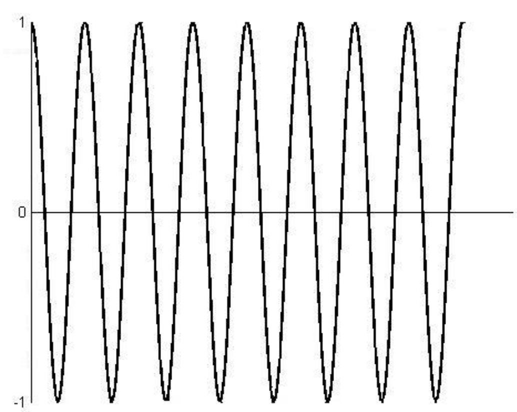
(a) $\cos(x)$



(b) $\cos(2x)$



(c) $\cos(4x)$



(d) $\cos(8x)$

This was extended to the full upper and lower bound by Erdös and Gál[5] in a specific case:

Theorem 1.1.8. Let n_k be an infinite sequence of positive integers, satisfying the lacunarity condition $\frac{n_{k+1}}{n_k} \geq q > 1$. Then

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} \exp 2\pi i n_k x\right|}{\sqrt{N \log\log N}} = 1.$$

for almost all x.

In 1959, M. Weiss[17] gave the complete answer for lacunary trigonometric series:

Theorem 1.1.9. Let

$$S(x) = \sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x)$$

be a lacunary trigonometric series, that is to say, one such that $n_{k+1}/n_k > q > 1$ for all k. We write

$$B_N = \left(\frac{1}{2}\sum_{k=1}^{N}(a_k^2 + b_k^2)\right)^{1/2}, M_N = \max_{1\leq k\leq N}(a_k^2 + b_k^2)^{1/2}, S_N(x) = \sum_{k=1}^{N}(a_k \cos n_k x + b_k \sin n_k x).$$

If, for $N \to +\infty$,$B_N \to \infty$ and $M_N = o\left(\frac{B_N}{(\log\log B_N)^{1/2}}\right)$, then we have, for almost all x,

$$\limsup_{N\to+\infty} \frac{S_N(x)}{(2B_N^2 \log\log B_N)^{1/2}} = 1.$$

Notice here that we do not need to assume that the n_k's are integers. In 1963, Takahashi[15] extended the result of Salem and Zygmund beyond trigonometric functions:

Theorem 1.1.10. Consider a real measurable function f satisfying $f(x+1) = f(x)$, $\int_0^1 f(x)dx = 0$, and suppose n_k is a lacunary sequence of integers, that is, there is a number q so that

$$\frac{n_{k+1}}{n_k} > q > 1 \tag{1.1.3}$$

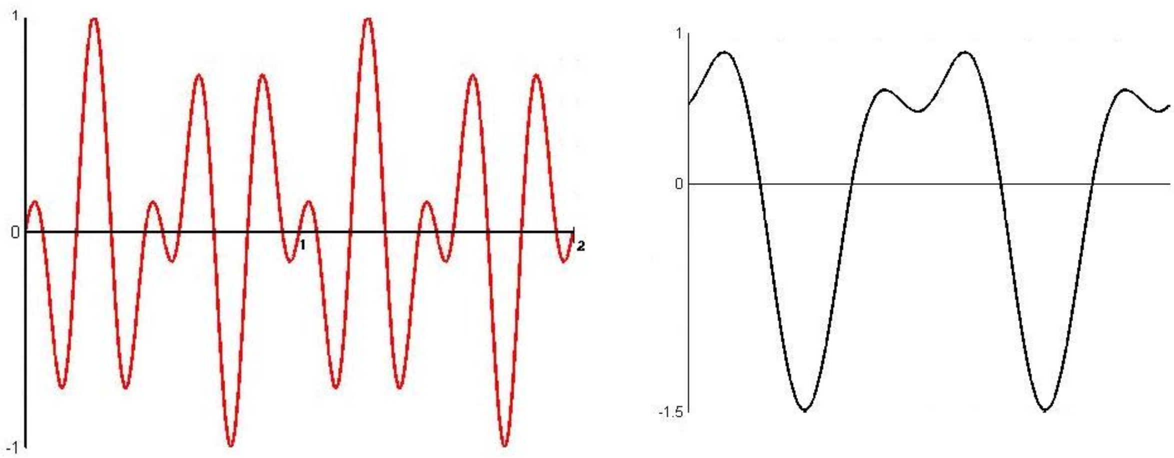
for every $k = 1, 2, \ldots$. Suppose that $f \in \text{Lip } \alpha$, $0 < \alpha \leq 1$. Then

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} f(n_k t)}{\sqrt{N \log\log N}} \leq C \quad \text{a.e.} \tag{1.1.4}$$

where C is a constant depending on q and α.

Here are examples of functions that satisfy conditions of Takahashi's theorem:

Figure 1.3: functions for Takahashi's theorem



(a) function 1

(b) function 2

Several authors – Dhompongsa[4], Takahashi[16], and Peter[9], have considered versions of this with a gap condition weaker than (1.1.3).

In 1986 Dhompongsa[4] showed:

Theorem 1.1.11. Let $\{[0,1], \mathcal{F}, P\}$ be the unit interval with Lebesgue measurable sets $\mathcal{F}$ and Lebesgue measure P. For $\frac{1}{2} < \alpha$, let Λ_α be the class of real-valued functions f on [0,1] with $f(0) = f(1)$, $\int_0^1 f(x)dx = 0$ and satisfying a Lipschitz condition

$$|f(x) - f(y)| \leq |x - y|^\alpha,\ 0 \leq x, y \leq 1.$$

Extend the functions of Λ_α to have period 1 on $\mathbb{R}$. Let $\{n_k, k \geq 1\}$ be a sequence of integers satisfying

$$\frac{n_{k+1}}{n_k} \geq 1 + \frac{c}{k^\delta}\ (c > 0)$$

for some $0 < \delta < 1$, and suppose there is a constant A such that the number of solutions of the equation $n_k \pm n_l = v$ does not exceed A for any $v \geq 0$. Then for each α with $\frac{1}{2} + \frac{\delta}{2} < \alpha$,

$$\limsup_{N\to\infty} \sup_{f\in\Lambda_\alpha} \frac{\left|\sum_{k\le N} f(n_k x)\right|}{\sqrt{N\log\log N}} \le C$$

for almost all $x \in [0,1]$, where C is a constant depending on α, δ and A.

In 1988, Takshashi[16] showed:

Theorem 1.1.12. Let $f(t)$ be a real valued Lebesgue measurable on $(-\infty, +\infty)$ satisfying $f(t+1) = f(t)$, $\int_0^1 f(t)dt = 0$, and $\int_0^1 f^2(t)\,dt < +\infty$, and n_k be an increasing sequence of positive integers. If $f \in \text{Lip }\delta(\delta > \frac{1}{2})$ and n_k satisfies $\frac{n_{k+1}}{n_k} > 1 + ck^{-\alpha}(c > 0, 0 < \alpha < \frac{1}{2}$ and $k \ge 1)$, then

$$\limsup_{N\to\infty} \frac{\sum\limits_{k\le N} f(n_k t)}{\sqrt{N\log\log N}} \le \|f\|, \text{ for a.e. t},$$

where $f \sim \sum_{k=1}^\infty a_h \cos 2\pi h(t+\alpha_h), a_h \ge 0$, and $\|f\| = \sum_{h=1}^\infty a_h$.

In 2000 Erika Péter[9] showed:

Theorem 1.1.13. Let $f \sim \sum_{k=1}^\infty (a_k \cos 2\pi kx + b_k \sin 2\pi kx)$ satisfy

$$\sum_{k=1}^\infty (|a_k| + |b_k|) < +\infty$$

and

$$\sum_{k\ge n} (a_k^2 + b_k^2) = O(n^{-\beta}) \text{ for some } \beta > 0.$$

Let (n_k) be a sequence of positive integers satisfying

$$\frac{n_{k+1}}{n_k} \ge 1 + k^{-\delta},\ \delta < \frac{1}{2}.$$

Then we have

$$\limsup_{N\to\infty} \frac{\left|\sum_{k\le N} f(n_k x)\right|}{\sqrt{N\log\log N}} \le \|f\|_A \text{ a.e.}$$

where $\|f\|_A = \sum_{k=1}^\infty (|a_k| + |b_k|)$.

Closely related is the central limit theorem for trigonometric series due to Salem and Zygmund[12] and central limit theorems for more general lacunary sequences of Gapoškin[6] and Aistleitner and Berkes[1].

In 1947, Salem and Zygmund showed:

Theorem 1.1.14. Consider a lacunary trigonometric series

$$\sum_{k=1}^{\infty}(a_k \cos n_k x + b_k \sin n_k x), \text{ with } \frac{n_{k+1}}{n_k} > q > 1, \tag{1.1.5}$$

Let $S_N(x)$ denote the Nth partial sum of (1.1.5), that is, $S_N(x) = \sum_{k=1}^{N}(a_k \cos n_k x + b_k \sin n_k x)$. Let $C_N = \sqrt{\frac{1}{2}\left(a_1^2 + b_1^2 + \ldots + a_N^2 + b_N^2\right)}$ and $c_k = \sqrt{a_k^2 + b_k^2}$. Let $Z_n(y)$ be the set of points x from $(0,\ 2\pi)$ at which $S_N(x)/C_N \leq y$. Let $F_N(y) = |Z_N(y)|/2\pi$, so that F_N is the distribution function of S_N/C_N.

(i) If $F_N(y)$ tends to a distribution function $F(y)$ such that either $F(y) > 0$ or $F(y) < 1$ for all finite y, then

$$c_n/C_n \to 0. \tag{1.1.6}$$

(ii) If (1.1.6) is satisfied and if $C_n \to \infty$, then $F_N(y)$ tends to the Gaussian distribution with mean value 0 and dispersion 1.

(iii) Let E be a point set on $(0,\ 2\pi)$, with $|E| > 0$, and let $F_N(y; E) = |Z_N(y)E|/|E|$. If $C_n \to \infty$ and if (1.1.6) holds, then $F_N(y; E)$ tends to the Gaussian distribution with mean value 0 and dispersion 1.

In 1970, Gaposhkin showed:

Theorem 1.1.15. Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers satisfying the Hadamard gap condition $n_{k+1}/n_k \geq q > 1$ and assume that

$$\sigma_N^2 := \int_0^1 \left(\sum_{k=1}^{N} f(n_k x)\right)^2 dx \geq CN \tag{1.1.7}$$

holds for a positive constant $C > 0$. Assume further that for any fixed positive integers a, b, μ the number of solutions of the Diophantine equation

$$an_k - bn_l = \mu \ (k, l \geq 1)$$

is bounded by a constant $C(a, b)$, independent of μ. Then

$$\lim_{N\to\infty} \mathbb{P}\left\{x \in (0,1) : \sum_{k=1}^{N} f(n_k x) \leq t\sigma_N\right\} = \Phi(t). \tag{1.1.8}$$

Definition 1.1.16. Given a sequence n_k of positive integers, define for any $d \geq 1, \upsilon \in \mathbb{Z}$,

$$L(N, d, \upsilon) = \#\{1 \leq a, b \leq d, 1 \leq k, l \leq N : an_k - bn_l = \upsilon\}$$

$$L(N, d) = \sup_{\upsilon > 0} L(N, d, \upsilon).$$

Recently, in 2010, C. Aistleitner and I. Berkes showed:

Theorem 1.1.17. [1] Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition and let f be a function of bounded variation satisfying $f(x+1) = f(x)$, $\int_0^1 f(x)\, dx = 0$ and (1.1.7). Assume that for any fixed $d \geq 1$ we have

$$L(N, d) = o(N) \text{ as } N \to \infty.$$

Then the central limit theorem (1.1.8) holds. If f is a trigonometric polynomial of order r, it suffices to assume (1.1.8) for $d = r$.

In this thesis we will generalize the LIL of Takahashi, Theorem 1.1.10. We will retain the gap condition 1.1.3 but broaden the class of functions f.

1.2 Martingales

We need to introduce some notation and definitions.

Definition 1.2.1. Let $(\Omega,\ \mathcal{F},\ P)$ be a probability space. A *martingale* sequence of length n is a chain $X_1, X_2, \ldots, X_n$ of random variables and corresponding sub σ-fields $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ that satisfy the following relations:

1. Each X_i is an integrable random variable which is measurable with respect to the corresponding σ-field $\mathcal{F}_i$.

2. The σ-fields $\mathcal{F}_i$ are increasing i.e. $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for every i.

3. For every $i \in [1, 2, \ldots, n-1]$, we have the relation

$$X_i = \mathbb{E}\{X_{i+1}|\mathcal{F}_i\} \text{ a.e. } P.$$

Throughout, a cube $Q \subseteq \mathbb{R}^n$ will be called *dyadic* if it has the form

$$Q = [k_1 2^l, (k_1+1)2^l) \times \ldots \times [k_n 2^l, (k_n+1)2^l)$$

for some $l, k_1, \ldots, k_n \in \mathbb{Z}$; for such a cube Q we say that Q has *sidelength* 2^l and denote this as $\ell(Q) = 2^l$. We will use the notation $|Q|$ to denote the Lebesgue measure of Q.

For $m \in \mathbb{Z}$ we let $\mathcal{F}_m$ denote the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength 2^{-m} and we will let $\mathcal{F}$ denote the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength ≤ 1. By a slight abuse of notation, we will also use $\mathcal{F}_m$ to denote the σ-field generated by the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength 2^{-m}. (The usage will be clear from the context.) For $x \in \mathbb{R}^n$ we also define $\mathcal{F}^x = \{Q + x : Q \in \mathcal{F}\}$ and $\mathcal{F}_m^x = \{Q + x : Q \in \mathcal{F}_m\}$.

Definition 1.2.2. Suppose $Q \in \mathcal{F}_0$. A *dyadic martingale* on Q is a sequence of integrable functions $\{g_m\}_{m=0}^{\infty}$ on Q such that each g_m is $\mathcal{F}_m$ measurable and $g_m = E(g_{m+1}|\mathcal{F}_m)$ for every m. Here $E(g_{m+1}|\mathcal{F}_m)$ denotes the conditional expectation: $E(g_{m+1}|\mathcal{F}_m)(x) = \frac{1}{|Q|}\int_Q g_{m+1} dy$, if $x \in Q \in \mathcal{F}_m$. For $k \geq 1$, set $d_k = g_k - g_{k-1}$, and we also define the square function $Sf_m = (\sum_{k=1}^{m} E(d_k^2|\mathcal{F}_{k-1}))^{1/2}$.

Inspired by the LIL's for sums of independent random variables, in 1970 W. Stout [14] extended these results to martingles.

Theorem 1.2.3 (LIL for martingales)**.** [14] Let $(X_n, \mathcal{F}_n, n \geq 1)$ be a martingale defined on a probability space $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}(X_1) = 0$. Let $Y_n = X_n - X_{n-1}$ for $n \geq 1, X_0 = 0, \mathcal{F}_0 = (\emptyset, \Omega), s_n^2 = \sum_{i=1}^{n} \mathbb{E}[Y_i^2|\mathcal{F}_{i-1}]$, and $u_n = (2 \log\log s_n^2)^{\frac{1}{2}}$. If $s_n^2 \to \infty$ and

$$|Y_n| \leq \frac{K_n s_n}{u_n} \text{ for } n \geq 1$$

where K_n are $\mathcal{F}_{n-1}$ measurable with $K_n \to 0$, then

$$\limsup_{n\to\infty} \frac{X_n}{s_n u_n} \leq 1.$$

Throughout this dissertation we will make use of many of the ideas and techniques found in its proof which we accordingly reproduce here.

Proof. Denote the indicator function of a set A by $I(A)$. Let $k > 0$ be a constant to be specified later. Let $Y'_n = Y_n I(K_n \leq k)$. $(Y'_n, \mathcal{F}, \geq 1)$ is easier to work with because it is a martingale difference sequence such that $|Y'_n| \leq k s_n/u_n$.

Let $X'_n = \sum_{i=1}^{n} Y'_i$. Since $P[Y'_n \neq Y_n \text{ i.o.}] = P[K_n > k \text{ i.o.}] = P[\limsup_{n\to\infty} K_n > k] = 0$, it suffices to show that $\limsup_{n\to\infty} \frac{X'_n}{s_n u_n} \leq 1$. To this end, we show that $P[X'_n > (1+\delta) s_n u_n \text{ i.o.}] = 0$ for all $\delta > 0$. Let $(X'_n)^* = \max_{j\leq n} X'_j$.

$$P[X'_n > (1+\delta)s_n u_n \text{ i.o.}] \leq P[(X'_{t_k})^* > (1+\delta)s_{t_{k-1}+1} u_{t_{k-1}+1} \text{ i.o.}]$$

$$\frac{s^2_{t_{k-1}+1} u^2_{t_{k-1}+1}}{s^2_{t_k} u^2_{t_k}} \geq \frac{p^{-2} \log\log p^{2(k-1)}}{\log\log p^{2k}} \approx p^{-2}.$$

Thus choosing $\delta' > 0$ and $p > 1$ such that $(1+\delta) > p(1+\delta')$, it follows that $P[X'_n > (1+\delta)s_n u_n \text{ i.o.}] \leq P[(X'_{t_k})^* > (1+\delta')s_{t_k} u_{t_k} \text{ i.o.}]$. Thus it suffices to show that $P[(X'_{t_k})^* > (1+\delta')s_{t_k} u_{t_k} \text{ i.o.}] = 0$.

We now establish a conditional Levy inequality. On $n \leq t_k$, define

$$I(B_n) = I(X'_{t_k} - X'_n + (2\mathbb{E}[(X'_{t_k} - X'_n)^2 | \mathcal{F}_n])^{\frac{1}{2}} \geq 0)$$

and

$$I(A_n) = I(X^*_{n-1} < \epsilon, X'_n - (2\mathbb{E}[(X'_{t_k} - X'_n)^2 | \mathcal{F}_n])^{\frac{1}{2}} \geq \epsilon)$$

where

$$X^*_n = \max_{j\leq n} \left((X'_j - (2\mathbb{E}[(X'_{t_k} - X'_j)^2 | \mathcal{F}])^{\frac{1}{2}} \right).$$

Now

$$
\begin{aligned}
E[I(X'_{t_k} > \epsilon)] &\geq E\left[\sum_{n=1}^{t_k} I(A_n)I(B_n)\right] = E\left[\sum_{n=1}^{t_k} I(t_k \geq n)I(A_n)I(B_n)\right] \\
&= E\left[\sum_{n=1}^{t_k} I(t_k \geq n)I(A_n)\mathbb{E}[I(B_n)|\mathcal{F}_n]\right].
\end{aligned}
$$

On $t_k \geq n$, an application of the conditional Chebychev inequality yields

$$
\mathbb{E}[I(B_n)|\mathcal{F}_n] \geq \frac{1}{2}\mathbb{P}\left(I(X'_{t_k} - X'_n + (2\mathbb{E}[(X'_{t_k} - X'_n)^2|\mathcal{F}_n])^{\frac{1}{2}} \geq 0) \geq \frac{1}{2}|\mathcal{F}_n\right) = \frac{1}{2}.
$$

Thus

$$
\mathbb{E}[I(X'_{t_k} \geq \epsilon)] \geq (\frac{1}{2})\mathbb{E}\left[\sum_{n=1}^{\infty} I(t_k \geq n)I(A_n)\right] = (\frac{1}{2})\mathbb{E}[I(X^*_{t_k} \geq \epsilon)].
$$

On $t_k \geq n$,

$$
\mathbb{E}\left[\sum_{i=n+1}^{t_k} \mathbb{E}[(Y'_i)^2|\mathcal{F}_{i-1}]|\mathcal{F}_n\right] = \mathbb{E}[(X'_{t_k} - X'_n)^2|\mathcal{F}_n].
$$

Since

$$
\sum_{i=n+1}^{t_k} \mathbb{E}[(Y'_i)^2|\mathcal{F}_{i-1}] \leq p^{2k} \quad \text{for all } n \leq t_k,
$$

$$
\mathbb{E}[I\left((X'_{t_k})^* \geq \epsilon\right)] \leq \mathbb{E}[I(X^*_{t_k} \geq \epsilon - 2^{\frac{1}{2}}p^k)].
$$

Thus for $0 < \delta'' < \delta'$ and k sufficiently large,

$$
\begin{aligned}
2\mathbb{E}[I(X'_{t_k} &\geq (1+\delta'')(2p^{2k}\log\log p^{2k})^{\frac{1}{2}})] \\
&\geq 2\mathbb{E}[I(X'_{t_k} \geq (1+\delta')(2p^{2k}\log\log p^{2k})^{\frac{1}{2}} - 2^{\frac{1}{2}}p^k)] \\
&\geq \mathbb{E}[I(X^*_{t_k} \geq (1+\delta')(2p^{2k}\log\log p^{2k})^{\frac{1}{2}} - 2^{\frac{1}{2}}p^k)] \\
&\geq \mathbb{E}[I((X'_{t_k})^* \geq (1+\delta')(2p^{2k}\log\log p^{2k})^{\frac{1}{2}})].
\end{aligned}
$$

Thus for k sufficiently large,

$$
\mathbb{E}[I(X'_{t_k} \geq (1+\delta'')(2p^{2k}\log\log p^{2k})^{\frac{1}{2}})] \leq \exp -(1+\delta'')^2\log\log p^{2k}(1-k(1+\delta'')/2)
$$

where $c = k(2\log\log p^{2k})^{-\frac{1}{2}}$ and $\epsilon = (1+\delta'')(2\log_2 p^{2k})^{\frac{1}{2}}$ with k chosen such that $(1+\delta'')k \leq 1$. Combining, it follows for k sufficiently large that $\mathbb{E}[I((X'_{t_k})^* > (1+\delta')(2p^{2k}\log\log p^{2k})^{\frac{1}{2}})] \leq$

$2(2k\log p)^{-\eta}$ for some $\eta > 1$ by choosing $k > 0$ such that $(1+\delta'')^2(1-k(1+\delta'')/2) > 1$. Thus

$$\sum_{k=1}^{\infty}\mathbb{E}[I((X'_{t_k})^* > (1+\delta')(2p^{2k}\log\log p^{2k})^{\frac{1}{2}})] < \infty \text{ for all } \delta' > 0.$$

Since $s_{t_k}u_{t_k} \approx (2p^{2k}\log\log p^{2k})^{\frac{1}{2}}$, it follows that

$$\sum_{k=1}^{\infty}\mathbb{E}[I((X'_{t_k})^* > (1+\delta')s_{t_k}u_{t_k})] < \infty \text{ for all } \delta' > 0.$$

It follows by the Borel Cantelli lemma that $\mathbb{P}[(X'_{t_k})^* > (1+\delta')s_{t_k}u_{t_k} \text{ i.o.}] = 0$ for all $\delta' > 0$, establishing the theorem. □

1.3 Examples

Examples of dyadic martingales.

Example 1.3.1. With Rademacher functions defined in (1.1.1), if we define functions $s_n = \sum_{j=1}^{n} a_j r_j$ where a_j is a sequence of real numbers. Then $\{s_n\}$ is a dyadic martingale.

Example 1.3.2. Let μ be a finite signed measure on [0,1] and we define

$$f_n(x) = \sum_{i=1}^{2^n} 2^n \mu\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) \chi_{i,n}(x),$$

where $\chi_{i,n}$ is the characteristic function of the interval $[\frac{i-1}{2^n}, \frac{i}{2^n})$. Then f_n is a dyadic martingale.

Example 1.3.3. Define functions $f_n,\ n = 1, 2, \ldots$ as

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [0, 1-\frac{1}{2^n}); \\ -(2^n - 1) & \text{if } x \in [1-\frac{1}{2^n}, 1). \end{cases}$$

Then f_n is a dyadic martingale. This is an interesting example as for any f_n, $\int_0^1 f_n(x)\,dx = 0$, but $\lim_{n\to\infty} f_n(x) = 1$ a.e. and obviously $\int_0^1 1\,dx = 1$.

Chapter 2

Law of the iterated logarithm

In this Chapter we will give our main results, which are extensions of the LIL of Takahashi, and after introducing useful lemmas, we will derive the proof of our main theorem.

2.1 Upper bound in the law of iterated logarithm

Our main result is an extension of Takahashi's theorem. Here we retain the gap condition of lacunary sequence n_k, but broaden the class of function f:

Theorem 2.1.1. Suppose f is a Dini continuous function on $\mathbb{R}^n$ with the property that $f(x) = 0$ whenever any coordinate of x is an integer, and $\int_Q f(x)dx = 0$ whenever $Q \in \mathcal{F}_0$. Let (n_k) be a sequence of positive numbers satisfying the lacunarity condition $\frac{n_{k+1}}{n_k} \geq q > 1$ and (c_k) be a sequence in $\mathbb{R}^n$. Then there exists a constant C, depending only on n, q, and the quantity $\int_0^1 \omega(\delta)/\delta \, d\delta$, such that for any sequence of numbers (a_k) with $A_m = \sqrt{\sum_{k=1}^m |a_k|^2} \to \infty$ as $m \to \infty$, we have

$$\limsup_{m\to\infty} \frac{|\sum_{k=1}^m a_k f(n_k x + c_k)|}{\sqrt{A_m^2 \log\log A_m^2}} \leq C \quad a.e.$$

Notice that we do not assume the n_k are integers, nor do we assume any periodicity of f, and Dini continuity is a weaker condition than Lipschitz continuity, which will be shown in the next section.

Corollary 2.1.2. Suppose $f(x)$ is a Dini continuous function on $\mathbb{R}$ satisfying $f(x+1) = f(x)$

and $\int_0^1 f(x)dx = 0$. Then with n_k, a_k and c_k as in the Theorem,

$$\limsup_{m\to\infty} \frac{|\sum_{k=1}^{m} a_k f(n_k x + c_k)|}{\sqrt{A_m^2 \log\log A_m^2}} \leq C \ a.e.$$

Proof of the corollary. The conditions on f imply that there exists a $c \in [0,1]$ with $f(c) = 0$. Then $f(c+m) = 0$ for every integer m. Consider $g(x) = f(x+c)$; this satisfies the hypotheses of the Theorem. □

An example of functions that satisfy our theorem is shown as below.

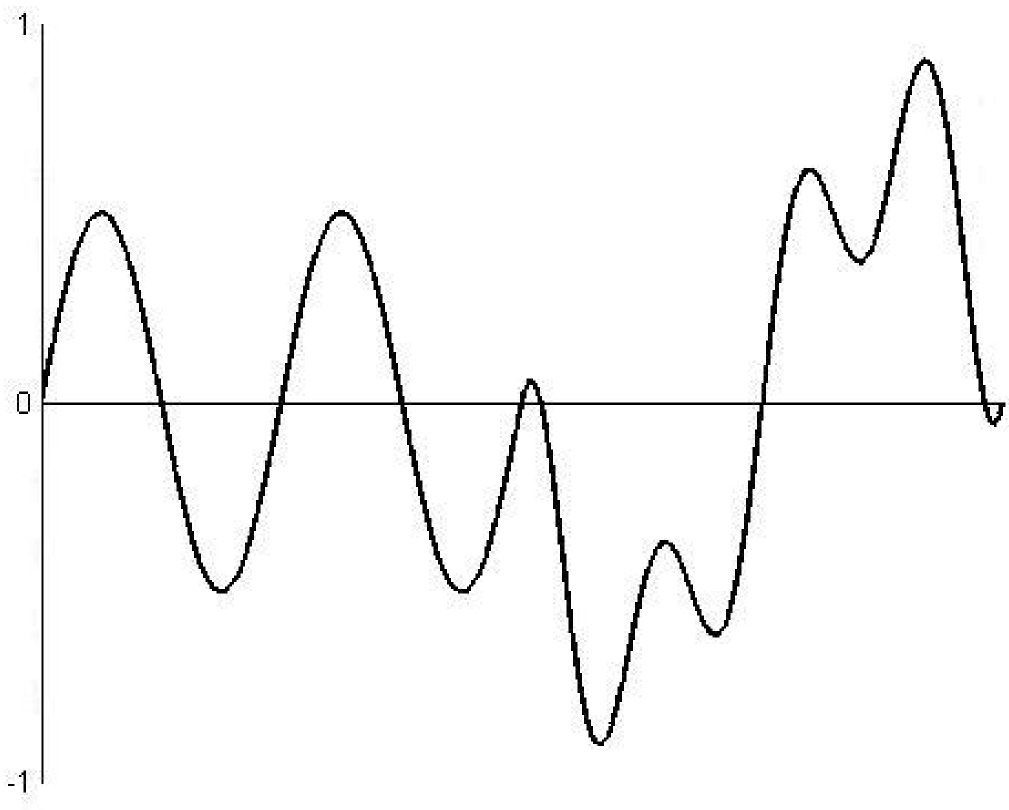


Figure 2.1: A Dini continuous function that satisfies the conditions of Theorem (2.1.1)

The proof of the Theorem will use a reduction to dyadic martingales. This is not the first time such a theorem has been proved using martingale techniques (e.g. see Peter[9]), but the approach here is very different.

2.2 Lemmas

In this section we will collect some lemmas which will be used to prove the theorems in chapter 2 and 3.

Lemma 2.2.1. Let $n_1 < n_2 < \dots$ be an infinite sequence of positive numbers satisfying the lacunarity condition $\frac{n_{k+1}}{n_k} \geq q > 1$, $k = 1, 2, \dots$. If $0 < \alpha < \beta$ then

$$\sum_{\alpha \leq n_k \leq \beta} 1 \leq \frac{\log(\beta q/\alpha)}{\log q}, \tag{2.2.1}$$

Proof. Let k_0 be defined by the inequality $n_{k_0} < \alpha \leq n_{k_0+1}$ (put $n_0 = 0$) and $i \geq 0$ be defined by the inequality $n_{k_0+i} \leq \beta < n_{k_0+i+1}$. If $i = 0$ then (2.2.1) is true. If $i \geq 1$ then we have $\beta \geq n_{k_0+i} \geq q^{i-1} n_{k_0+1} \geq q^{i-1}\alpha$. Hence $\beta q/\alpha \geq q^i$ and (2.2.1) follows immediately. □

Lemma 2.2.2. Suppose $k \geq 1$ and $2^{k-1} \leq n_k < 2^k$. For any cube $J \subset \mathbb{R}^n$ with $\ell(J) = \frac{1}{n_k}$, there exists a unique dyadic cube Q of sidelength $\frac{1}{2^k}$, which contains the center of J. Consequently, $J \subseteq \tilde{Q}$ where $\tilde{Q}$ is concentric with Q and $\ell(\tilde{Q}) = 3\ell(Q)$.

Proof. Because the dyadic cubes of sidelength $\frac{1}{2^k}$ are disjoint and cover $\mathbb{R}^n$, there is a unique cube Q with $\ell(Q) = \frac{1}{2^k}$ containing the center of J. Let c_J and c_Q denote the centers of J and Q respectively. Then if $x \in J$, $|x - c_Q| \leq |x - c_J| + |c_J - c_Q| \leq \frac{\sqrt{n}}{2 \cdot 2^{k-1}} + \frac{\sqrt{n}}{2 \cdot 2^k} = \frac{3\sqrt{n}}{2 \cdot 2^k}$, and hence $J \subset \tilde{Q}$. □

The following is from Chang, Wilson and Wolff[3], where we refer the reader for the proof.

Lemma 2.2.3. There is a positive integer N, $x_1, \dots, x_N \in \mathbb{R}^n$ and disjoint subsets B^j of $\mathcal{F}$ such that

$$\left\{ Q \in \mathcal{F} : \ \ell(Q) \leq \frac{1}{8} \right\} = \bigcup_{j=1}^{N} B^j,$$

$$\textit{if } Q \in B^j, \textit{ then } \tilde{Q} \subseteq Q' \textit{ for a unique } Q' \in \mathcal{F}^{x_j} \textit{ with } \ell(Q') = 8\ell(Q),$$

$$\textit{and if } Q_1, Q_2 \in B^j \textit{ and } Q_1 \neq Q_2, \textit{ then } Q_1' \neq Q_2'.$$

Definition 2.2.4. If f is a function on $\mathbb{R}^n$ we define the modulus of continuity ω of f as $\omega(f,\delta) = \sup\{|f(x) - f(y)| : |x-y| < \delta\}$. When f is clear from context, we will write $\omega(f,\delta) = \omega(\delta)$. We say that f is Dini continuous if

$$\int_0^1 \frac{\omega(\delta)}{\delta} d\delta < \infty. \tag{2.2.2}$$

It is easy to see if the integral in (2.2.2) is finite, then $\int_0^c \omega(\delta)/\delta \, d\delta$ is finite for any $c > 0$.

Fact 2.2.5. Every Lipschitz continuous function is Dini continuous, but not vice versa.

Lemma 2.2.6. Let J be a cube in $\mathbb{R}^n$ and let $\chi_J(x)$ denote the indicator function of J. Suppose f is a function which vanishes on ∂J. Then $\sup_{|x-y|\le\delta} |f(x)\chi_J(x) - f(y)\chi_J(y)| \le \sup_{|x-y|\le\delta} |f(x) - f(y)|$. Consequently, $\omega(\chi_J f, \delta) \le \omega(f,\delta)$ and $\chi_J f$ is Dini continuous if f is.

Proof. Suppose $x, y \in \mathbb{R}^n$ with $|x-y| \le \delta$. If $x \notin J$ and $y \notin J$, or if both $x, y \in J$, then we easily obtain $|f(x)\chi_J(x) - f(y)\chi_J(y)| \le \omega(f,\delta)$. If $x \in J$ but $y \notin J$, then choose $z = tx + (1-t)y$, $t \in [0,1]$ with $z \in \partial J$. Then $f(z) = 0$, $|z - x| \le \delta$, and so $|f(x)\chi_J(x) - f(y)\chi_J(y)| = |f(x) - 0| = |f(x) - f(z)| \le \omega(f,\delta)$. $\square$

Lemma 2.2.7. If f is Dini continuous then for any $c > 0$, $\displaystyle\sum_{l=1}^{\infty} \omega(c2^{-l}) \le 2 \int_0^c \frac{\omega(\delta)}{\delta} d\delta$.

Proof.

$$\int_0^c \frac{\omega(\delta)}{\delta} d\delta = \sum_{l=0}^{\infty} \int_{\frac{c}{2^{l+1}}}^{\frac{c}{2^l}} \frac{\omega(\delta)}{\delta} d\delta \ge \sum_{l=0}^{\infty} \int_{\frac{c}{2^{l+1}}}^{\frac{c}{2^l}} \frac{\omega\left(\frac{c}{2^{l+1}}\right)}{\frac{c}{2^l}} d\delta = \frac{1}{2} \sum_{l=1}^{\infty} \omega(c2^{-l}).$$

$\square$

Lemma 2.2.8. Let Q be a dyadic cube in $\mathbb{R}^n$ and let $Q(l)$, $l = 1, 2, \ldots, 2^n$ be the dyadic subcubes of Q obtained by bisecting the edges of Q. Suppose f is Dini continuous on Q with modulus of continuity ω. Then for each l,

$$\left| \frac{1}{|Q(l)|} \int_{Q(l)} f(y) dy - \frac{1}{|Q|} \int_Q f(y) dy \right| \le \omega(\sqrt{n}\ell(Q)).$$

Proof. Without loss of generality take $l = 1$. Then

$$\left|\frac{1}{|Q(1)|}\int_{Q(1)} f(y)dy - \frac{1}{|Q|}\int_Q f(y)dy\right| =$$

$$\left|\frac{1}{|Q(1)|}\int_{Q(1)} f(y)dy - \sum_{k=1}^{2^n}\frac{1}{2^n|Q(k)|}\int_{Q(k)} f(y)dy\right| =$$

$$\left|\frac{1}{2^n|Q(1)|}\sum_{k=1}^{2^n}\int_{Q(1)} f(y)dy - \int_{Q(k)} f(y)dy\right| \leq \omega(\sqrt{n}\ell(Q)).$$

□

Lemma 2.2.9. (Upper half LIL for dyadic martingales.) If f_m is a dyadic martingale on Q_0 then

$$\limsup_{m\to\infty}\frac{|f_m|}{\sqrt{2(Sf_m)^2\log\log(Sf_m)}} \leq 1$$

almost surely on the set where $S(f_m) \to \infty$.

Lemma 2.2.9 is a special case of a much more general martingale LIL due to Stout[14]. We only need this version, which is much simpler to show. (See[3],Corollary 3.2)

2.3 The proof of the theorem

Proof. According to Lemma 2.2.1, we can assume that for each $k \geq 1$, there exists exactly one n_k with $2^{k-1} \leq n_k < 2^k$. We may also assume that $a_1 = a_2 = 0$. For $m \geq 1$, let $f_m(x) := \sum_{k=3}^{m+2} a_k f(n_k x + c_k)$.

For $k = 1, 2, \ldots$, define $\mathcal{G}_k$ as the set of cubes in $\mathbb{R}^n$ of the form

$$[\frac{-c_{k1}+l_1}{n_k}, \frac{-c_{k1}+l_1+1}{n_k}) \times \cdots \times [\frac{-c_{kn}+l_n}{n_k}, \frac{-c_{kn}+l_n+1}{n_k}),$$

where $c_k = (c_{k1}, \ldots, c_{kn})$, and $l_1, \ldots, l_n$ are in $\mathbb{Z}$. Then $f(n_k x + c_k)$ vanishes on ∂J for each $J \in \mathcal{G}_k$. Note that $\mathbb{R}^n$ is covered by a disjoint union of the cubes in $\mathcal{G}_k$.

For a cube $Q \in \mathcal{F}_k$, of sidelength $\ell(Q) = \frac{1}{2^k}$, define

$$\lambda_Q(x) = \begin{cases} a_k f(n_k x + c_k)\chi_J(x) & \text{if } Q \text{ contains the center of a cube } J \in \mathcal{G}_k; \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.1}$$

Note that each $Q \in \mathcal{F}_k$ contains the center of at most one $J \in \mathcal{G}_k$ and that some cubes $Q \in \mathcal{F}_k$ may not contain the center of any cube in $\mathcal{G}_k$, in which case $\lambda_Q = 0$. By Lemma 2.2.2, supp $\lambda_Q \subseteq \tilde{Q}$. Apply Lemma 2.2.3 to decompose $\mathcal{F}$ into the disjoint families B^j.

For $1 \leq j \leq N$, and for each $Q \in \mathcal{F}^{x_j}$, let

$$f_Q^{(j)}(x) = \begin{cases} \lambda_{Q_0}(x) & \text{if } Q = Q_0' \text{ for some } Q_0 \in B^j; \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $Q \in \mathcal{F}^{x_j}$

$$supp f_Q^{(j)} \subseteq Q \tag{2.3.2}$$

and

$$\int_Q f_Q^{(j)}(x)dx = 0. \tag{2.3.3}$$

We then define

$$\Lambda_m^{(j)}(x) = \sum_{\substack{Q \in B^j \\ 2^{-m-2} \leq \ell(Q) \leq 2^{-3}}} \lambda_Q(x) = \sum_{\substack{Q \in \mathcal{F}^{x_j} \\ 2^{-m+1} \leq \ell(Q) \leq 1}} f_Q^{(j)}(x), \tag{2.3.4}$$

so that with this notation

$$f_m(x) = \sum_{j=1}^N \Lambda_m^{(j)}(x) = \sum_{j=1}^N \sum_{\substack{Q \in B^j \\ 2^{-m-2} \leq \ell(Q) \leq 2^{-3}}} \lambda_Q(x). \tag{2.3.5}$$

Define dyadic martingales $g^{(j)} = \{g_m^{(j)}\}_{m=0}^\infty$ by $g_m^{(j)} = E(\Lambda_m^{(j)} | \mathcal{F}_m^{x_j}), m \geq 1$ and $g_0^{(j)} = 0$. To see that $g^{(j)}$ is a martingale, note that

$$E(g_{m+1}^{(j)} | \mathcal{F}_m^{x_j}) = E(\Lambda_{m+1}^{(j)} | \mathcal{F}_m^{x_j}) = E(\Lambda_m^{(j)} | \mathcal{F}_m^{x_j}) + \sum_{Q \in \mathcal{F}^{x_j} : \ell(Q) = 2^{-m}} E(f_Q^{(j)} | \mathcal{F}_m^{x_j})$$

and the terms in the sum vanish due to (2.3.2) and (2.3.3). This is a small abuse of terminology, because the $g^{(j)}$ are defined on all of $\mathbb{R}^n$ which is not a probability space. However, the restriction of $g^{(j)}$ to each cube $Q \in \mathcal{F}^{x_j}$ of sidelength 1 is a martingale on the probability space Q, and $\mathbb{R}^n$ can be exhausted by a countable number of such cubes.

For $x \in \mathbb{R}^n$, let us denote by $Q_m^{x_j}(x)$ the unique dyadic cube of sidelength 2^{-m} in $\mathcal{F}^{x_j}$ containing x. Then, using (2.3.5), the definition of the $g^{(j)}$, and (2.3.4), we have

$$\left| f_m(x) - \sum_{j=1}^{N} g_m^{(j)}(x) \right| \leq \sum_{j=1}^{N} \sum_{\substack{Q \in \mathcal{F}^{x_j} \\ 2^{-m+1} \leq \ell(Q) \leq 1}} \left| f_Q^{(j)}(x) - E(f_Q^{(j)} | \mathcal{F}_m^{x_j})(x) \right|$$

$$\leq \sum_{j=1}^{N} \sum_{\substack{Q \in \mathcal{F}^{x_j} \\ 2^{-m+1} \leq \ell(Q) \leq 1}} \frac{1}{|Q_m^{x_j}(x)|} \int_{Q_m^{x_j}(x)} \left| f_Q^{(j)}(x) - f_Q^{(j)}(y) \right| dy.$$

If $\ell(Q) = 2^{-k}$, $k \leq m-1$, and $y \in Q_m^{x_j}(x)$, then by the definition of $f_Q^{(j)}$, λ_Q (2.3.1), and Lemma 2.2.6, $|f_Q^{(j)}(x) - f_Q^{(j)}(y)| \leq |a_{k+3}| \omega(n_{k+3}\sqrt{n}\ell(Q_m^{x_j}(x)))$. Thus,

$$\begin{aligned}
\left| f_m(x) - \sum_{j=1}^{N} g_m^{(j)}(x) \right| &\leq \sum_{j=1}^{N} \sum_{k=0}^{m-1} |a_{k+3}|\, \omega(n_{k+3}\sqrt{n}\ell(Q_m^{x_j}(x))) \\
&\leq \sum_{j=1}^{N} \sum_{k=3}^{m+2} |a_k|\, \omega\big(\sqrt{n}\frac{2^k}{2^m}\big) \\
&= N \sum_{k=3}^{m+2} |a_k|\, \omega\big(8\sqrt{n}\frac{2^{k-3}}{2^m}\big) \\
&\leq N \left(\sum_{k=3}^{m+2} |a_k|^2 \right)^{1/2} \left(\sum_{k=3}^{m+2} \omega\big(8\sqrt{n}\frac{2^{k-3}}{2^m}\big)^2 \right)^{1/2} \\
&= CA_{m+2},
\end{aligned}$$

where for the last inequality we have used Lemma 2.2.7.

We now estimate the square functions of the martingales $g_k^{(j)}$. For $1 \leq j \leq N$, let

$d_k^{(j)} = |g_k^{(j)} - g_{k-1}^{(j)}|$, $k = 1, 2, \ldots$. Then, using Lemma 2.2.8,

$$
\begin{aligned}
|d_k^{(j)}(x)| &= \left| E(\Lambda_k^{(j)}|\mathcal{F}_k^{x_j})(x) - E(\Lambda_{k-1}^{(j)}|\mathcal{F}_{k-1}^{x_j})(x) \right| \\
&= \left| E(\Lambda_k^{(j)}|\mathcal{F}_k^{x_j})(x) - E(\Lambda_k^{(j)}|\mathcal{F}_{k-1}^{x_j})(x) \right| \\
&= \left| \frac{1}{|Q_k^{x_j}(x)|} \int_{Q_k^{x_j}(x)} \Lambda_k^{(j)}(y)dy - \frac{1}{|Q_{k-1}^{x_j}(x)|} \int_{Q_{k-1}^{x_j}(x)} \Lambda_k^{(j)}(y)dy \right| \\
&\le \sum_{\substack{Q\in\mathcal{F}^{x_j} \\ 2^{-k+1}\le \ell(Q)\le 1}} \left| \frac{1}{|Q_k^{x_j}(x)|} \int_{Q_k^{x_j}(x)} f_Q^{(j)}(y)dy - \frac{1}{|Q_{k-1}^{x_j}(x)|} \int_{Q_{k-1}^{x_j}(x)} f_Q^{(j)}(y)dy \right| \\
&\le \sum_{l=0}^{k-1} |a_{l+3}|\, \omega\big(n_{l+3}\sqrt{n}\, \ell(Q_k^{x_j}(x))\big) \\
&\le \sum_{l=3}^{k+2} |a_l|\, \omega\big(\sqrt{n}\frac{2^l}{2^k}\big) \\
&\le \left(\sum_{l=3}^{k+2} |a_l|^2 \omega\big(8\sqrt{n}\frac{2^{l-3}}{2^k}\big) \right)^{1/2} \left(\sum_{l=3}^{k+2} \omega\big(8\sqrt{n}\frac{2^{l-3}}{2^k}\big) \right)^{1/2} \\
&\le M \left(\sum_{l=3}^{k+2} |a_l|^2 \omega\big(8\sqrt{n}\frac{2^{l-3}}{2^k}\big) \right)^{1/2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
(Sg_m^{(j)}(x))^2 &= \sum_{k=1}^{m} E((d_k^{(j)})^2|\mathcal{F}_{k-1}) \le M^2 \sum_{k=1}^{m} \sum_{l=3}^{k+2} |a_l|^2 \omega\big(8\sqrt{n}\frac{2^{l-3}}{2^k}\big) \\
&\le M^2 \sum_{l=3}^{m+2} |a_l|^2 \sum_{k=l-2}^{m} \omega\big(8\sqrt{n}\frac{2^{l-3}}{2^k}\big) \\
&\le M^2 M \sum_{l=3}^{m+2} |a_l|^2 \\
&= M^3 A_{m+2}^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\limsup_{m\to\infty} & \frac{\left|\sum_{k=1}^{m+2} a_k f(n_k x + c_k)\right|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}} \\
&\le \limsup_{m\to\infty} \frac{|f_m(x) - \sum_{j=1}^{N} g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}} + \limsup_{m\to\infty} \frac{\sum_{j=1}^{N} |g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}} \\
&\le \limsup_{m\to\infty} \frac{C}{\sqrt{\log\log A_{m+2}^2}} + \sum_{j=1}^{N} \limsup_{m\to\infty} \frac{|g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}} \\
&= \sum_{j=1}^{N} \limsup_{m\to\infty} \frac{|g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}}.
\end{aligned}
$$

For j fixed, $\limsup_{m\to\infty} \frac{|g_m^{(j)}(x)|}{\sqrt{(Sg_m^{(j)}(x))^2 \log\log(Sg_m^{(j)}(x))^2}} \le \sqrt{2}$ almost surely on the set $\{Sg_m^{(j)}(x) \to \infty\}$ by Lemma 2.8. But then for such x, $(Sg_m^{(j)}(x))^2 \le M^3 A_{m+2}^2$ and hence

$$
\limsup_{m\to\infty} \frac{|g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}} \le C
$$

almost surely on this set. Because $\{Sg_m^{(j)}(x)$ is bounded$\}$ $=$ $\{|g_m^{(j)}(x)|$is bounded$\}$ almost surely (see[2]),

$$
\limsup_{m\to\infty} \frac{|g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}} = 0
$$

almost surely on the set $\{Sg_m^{(j)}(x)$ is bounded$\}$ and we obtain the conclusion of the theorem.

$\square$

Chapter 3

Lower bound result

In this chapter we will provide a lower bound in the result of the previous chapter.

3.1 Lower bound in the law of iterated logarithm

Theorem 3.1.1. Assume that f, n_k, a_k, A_m, and c_k are as in the previous theorem, again with $A_m \to \infty$ as $m \to \infty$. Suppose also that f has the property that there exists a number $c_0 > 0$ such that $\dfrac{1}{|Q|}\displaystyle\int_Q |f(u)|^2 du > c_0$ for all cubes of sidelength at least 1. Set $M_n = \max_{1 \le k \le n} |a_k|$ and suppose that $M_n^2 \le \dfrac{K_n A_n^2}{\log\log A_n^2}$ for some sequence of numbers $K_n \to 0$ as $n \to \infty$. Then, if q is sufficiently large, there exists a constant c, depending only on n, q, c_0 and the quantity $\int_0^1 \omega(\delta)/\delta \, d\delta$, such that

$$\limsup_{m\to\infty} \frac{|\sum_{k=1}^m a_k f(n_k x + c_k)|}{\sqrt{A_m^2 \log\log A_m^2}} \ge c \quad a.e.$$

Notice that in both of these theorems we do not assume the n_k are integers, nor do we assume any periodicity of f. We do not know the best possible values of C and c in these inequalities. In the classical LILs, $C = c = 1$, but it seems difficult to obtain such precision here. In the lower bound the so called "Kolmogorov condition" $M_n^2 \le \dfrac{K_n A_n^2}{\log\log A_n^2}$ is an essential hypothesis, even in the trigonometric case. (See[8], pg. 81.) The property that $\dfrac{1}{|Q|}\displaystyle\int_Q |f(u)|^2 du > c_0$ is also necessary and keeps f from becoming too "sparse" at infinity. For example, consider a function f on $\mathbb{R}$ given by $f(x) = \varepsilon_n \sin(2\pi x)$ for $x \in$

$(-n-1,-n]\cup[n,n+1)$, where $\varepsilon_n \to 0$, say montonically. By Theorem 2.1.1 (or Salem and Zygmund[13]),

$$\limsup_{m\to\infty} \frac{\left|\sum_{k=1}^{m} \sin 2\pi(2^k x)\right|}{\sqrt{m\log\log m}} \leq C \quad a.e.$$

and thus,

$$\limsup_{m\to\infty} \frac{\left|\sum_{k=1}^{m} f(2^k x)\right|}{\sqrt{m\log\log m}} = 0 \quad a.e.$$

The latter can be seen by breaking the sum in the numerator as $\sum_{k=1}^{N} + \sum_{k=N+1}^{m}$ which gives that the limsup is bounded by $\varepsilon_{2^{N+1}}$ on $(-\infty, -\frac{1}{2^N}] \cup [\frac{1}{2^N}, \infty)$.

The proof of the Theorem will involve a mix of ideas and techniques from previous chapter, the study of dyadic martingales, and classical probability theory. In Section 2 we will collect some definitions and lemmas which will be used in the course of the proof. Throughout we will use the convention that C and c represent absolute constants, depending only on q, n and the quantity (2.2.2), whose value may change from line to line. Sometimes we will need to temporarily track constants and these will be labeled as C_1, C_2, etc.

3.2 Preliminaries

We record some lemmas.

Lemma 3.2.1. Suppose k is a positive integer, $c > 0$. Then

1. $\sum_{j=k+1}^{\infty} \omega\left(\frac{n_k}{n_j}c\right) \leq \max\{\frac{1}{\log 2}, \frac{1}{\log q}\} \int_0^{\frac{2c}{q}} \frac{\omega(\delta)}{\delta} d\delta$
2. $\sum_{k=1}^{j-1} \omega\left(\frac{n_k}{n_j}c\right) \leq \max\{\frac{1}{\log 2}, \frac{1}{\log q}\} \int_0^{\frac{2}{q}c} \frac{\omega(\delta)}{\delta} d\delta$
3. $\sum_{j=k+1}^{\infty} \frac{1}{n_j} \leq \frac{1}{n_k}\frac{1}{q-1}$
4. $\sum_{k=1}^{j-1} \frac{1}{n_k} \leq \frac{1}{n_1}\frac{q}{q-1}$

Proof.

$$\int_0^{\frac{2}{q}c} \frac{\omega(\delta)}{\delta} d\delta = \int_0^{\frac{2}{q}} \frac{\omega(cs)}{s} ds = \int_{\frac{1}{q}}^{\frac{2}{q}} \frac{\omega(cs)}{s} ds + \sum_{k=1}^{\infty} \int_{\frac{1}{q^{k+1}}}^{\frac{1}{q^k}} \frac{\omega(cs)}{s} ds$$

$$\geq \log 2\, \omega(\frac{1}{q}c) + \sum_{k=1}^{\infty} \log q\, \omega\left(\frac{1}{q^{k+1}}c\right) \geq \min\{\log 2, \log q\} \sum_{k=1}^{\infty} \omega\left(\frac{1}{q^k}c\right).$$

Then

$$\sum_{j=k+1}^{\infty} \omega\left(\frac{n_k}{n_j}c\right) \leq \sum_{k=1}^{\infty} \omega\left(\frac{1}{q^k}c\right) \leq \max\{\frac{1}{\log 2}, \frac{1}{\log q}\} \int_0^{\frac{2}{q}c} \frac{\omega(\delta)}{\delta} d\delta \quad \text{and}$$
$$\sum_{k=1}^{j-1} \omega\left(c\frac{n_k}{n_j}\right) \leq \sum_{k=1}^{j-1} \omega\left(\frac{1}{q^k}c\right) \leq \max\{\frac{1}{\log 2}, \frac{1}{\log q}\} \int_0^{\frac{2}{q}c} \frac{\omega(\delta)}{\delta} d\delta$$

which gives (1) and (2). For (3) we have

$$\sum_{j=k+1}^{\infty} \frac{1}{n_j} = \frac{1}{n_k} \sum_{j=k+1}^{\infty} \frac{n_k}{n_j} \leq \frac{1}{n_k} \sum_{j=1}^{\infty} \frac{1}{q^j} = \frac{1}{n_k} \frac{1}{q-1}.$$

The proof of (4) is similar. $\square$

In what follows, we will need a lower bound for $\| \sum_{k=1}^{N} a_k f(n_k x + c_k) \|_2$ on $[0,1]^n$. This will be done simply by squaring and estimating the terms $a_k a_j \int_{[0,1]^n} f(n_k x + c_k) f(n_j x + c_j) dx$. We will use the well-established principle that if, say n_j is much larger than n_k, then $f(n_k x + c_k)$ is roughly constant on cubes where $f(n_j x + c_j)$ has mean value zero, which leads to a small value for the integral.

Lemma 3.2.2. If $j > k$, then

$$\int_{[0,1]^n} |f(n_j x + c_j) f(n_k x + c_k)| \, dx \leq \left(\int_{[0,1]^n} |f(n_j x + c_j)|^2 dx\right)^{\frac{1}{2}} \left(\omega(\frac{\sqrt{n} n_k}{n_j}) + \frac{\sqrt{2n}\|f\|_\infty}{\sqrt{n_j}}\right).$$

Proof. Recall that $\mathcal{F}_0$ denotes the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength 1. Consider the family of cubes of the form $Q_{j,m} = \frac{1}{n_j} Q_m - \frac{1}{n_j} c_j$, where $Q_m \in \mathcal{F}_0$. Note that $\int_{Q_{j,m}} f(n_j x + c_j) dx = 0$. We say $Q_{j,m}$ is of type I if $Q_{j,m} \subset [0,1]^n$, and $Q_{j,m}$ is of type II if $Q_{j,m} \cap [0,1]^n \neq \emptyset$ and $Q_{j,m} \cap ([0,1]^n)^c \neq \emptyset$. Let $R = (\cup Q_{j,m}) \cap [0,1]^n$, where the union is taken over all type II cubes. Then $|R| \leq 1 - \left(1 - \frac{2}{n_j}\right)^n \leq \frac{2n}{n_j}$. For each type I $Q_{j,m}$, let $a_{j,m}$ denote its center.

Then

$$
\begin{aligned}
&\int_{[0,1]^n} |f(n_k x + c_k) f(n_j x + c_j)| \, dx \\
&= \sum_{Q_{j,m} \text{ of type I}} \int_{Q_{j,m}} |f(n_k x + c_k) f(n_j x + c_j)| \, dx + \int_R |f(n_k x + c_k) f(n_j x + c_j)| dx \\
&\leq \sum_{Q_{j,m} \text{ of type I}} \int_{Q_{j,m}} |(f(n_k x + c_k) - f(n_k a_{j,m} + c_k)) f(n_j x + c_j)| \, dx \\
&\quad + \left(\int_R |f(n_k x + c_k)|^2 dx \right)^{\frac{1}{2}} \left(\int_R |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}} \\
&\leq \sum_{Q_{j,m} \text{ of type I}} \omega\left(\frac{\sqrt{n} n_k}{2n_j}\right) \int_{Q_{j,m}} |f(n_j x + c_j)| dx + \frac{\sqrt{2n} \|f\|_\infty}{\sqrt{n_j}} \left(\int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}} \\
&\leq \omega\left(\frac{\sqrt{n} n_k}{2n_j}\right) \left(\int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}} + \frac{\sqrt{2n} \|f\|_\infty}{\sqrt{n_j}} \left(\int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}}.
\end{aligned}
$$

□

Lemma 3.2.3. $|\int_{[0,1]^n} f(n_j x + c_j) dx| \leq 2n \frac{\|f\|_\infty}{n_j}$. More generally, if Q is a dyadic cube of sidelength $\frac{1}{2^L}$ where $2^L \leq n_N < 2^{L+1}$ then for $j \geq N$, $\frac{1}{|Q|} |\int_Q f(n_j x + c_j) dx| \leq 2n \frac{2^L \|f\|_\infty}{n_j}$

Proof. Using the notation of the previous proof we have:

$$
\begin{aligned}
\left| \int_{[0,1]^n} f(n_j x + c_j) dx \right| &\leq \left| \sum_{\text{type I } Q_{j,m}} \int_{Q_{j,m}} f(n_j x + c_j) dx \right| + \int_R |f(n_j x + c_j)| dx \\
&= 0 + \int_R |f(n_j x + c_j)| dx \leq |R| \|f\|_\infty \leq 2n \frac{\|f\|_\infty}{n_j}.
\end{aligned}
$$

The second statement follows from this by a change of variables. □

Lemma 3.2.4. If q is sufficiently large, then

$$
\int_{[0,1]^n} |\sum_{k=1}^N a_k f(n_k x + c_k)|^2 dx \geq c A_N^2
$$

for some constant $c > 0$ depending only on n, q and the quantity in (2.2.2).

Proof.

$$\int_{[0,1]^n} \left(\sum_{k=1}^{N} a_k f(n_k x + c_k) \right)^2 dx = \sum_{k=1}^{N} a_k^2 \int_{[0,1]^n} |f(n_k x + c_k)|^2 dx$$
$$+ 2 \sum_{k=1}^{N} \sum_{j=k+1}^{N} a_k a_j \int_{[0,1]^n} f(n_k x + c_k) f(n_j x + c_j) dx$$

For typographical convenience in what follows, set $m_q = \max\{\frac{1}{\log 2}, \frac{1}{\log q}\}$. We estimate the second term, using Lemma 3.2.2 and all parts of Lemma 3.2.1

$$\begin{aligned}
&\sum_{k=1}^{N} \sum_{j=k+1}^{N} |a_k a_j| \int_{[0,1]^n} |f(n_k x + c_k) f(n_j x + c_j)| dx \\
&\leq \sum_{k=1}^{N} \sum_{j=k+1}^{N} |a_k a_j| \left(\int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}} \left(\omega(\frac{\sqrt{n} n_k}{2 n_j}) + \frac{\sqrt{2n} \|f\|_\infty}{\sqrt{n_j}} \right) \\
&\leq \sum_{k=1}^{N} |a_k| \left(\sum_{j=k+1}^{N} a_j^2 \omega(\frac{\sqrt{n} n_k}{2 n_j}) \int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}} \left(\sum_{j=k+1}^{N} \omega(\frac{\sqrt{n} n_k}{2 n_j}) \right)^{\frac{1}{2}} \\
&\quad + \sqrt{2n} \|f\|_\infty \sum_{k=1}^{N} |a_k| \left(\sum_{j=k+1}^{N} a_j^2 \int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}} \left(\sum_{j=k+1}^{N} \frac{1}{n_j} \right)^{\frac{1}{2}} \\
&\leq \left(m_q \int_0^{\sqrt{n}/q} \frac{\omega(\delta)}{\delta} d\delta \right)^{\frac{1}{2}} \sum_{k=1}^{N} |a_k| \left(\sum_{j=k+1}^{N} a_j^2 \omega(\frac{\sqrt{n} n_k}{2 n_j}) \int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}} \\
&\quad + \left(\sqrt{2n} \|f\|_\infty \frac{1}{\sqrt{q-1}} \right) \sum_{k=1}^{N} |a_k| \left(\sum_{j=k+1}^{N} \frac{a_j^2}{n_k} \int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}} \\
&\leq \left(m_q \int_0^{\sqrt{n}/q} \frac{\omega(\delta)}{\delta} d\delta \right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} a_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} \sum_{j=k+1}^{N} a_j^2 \omega(\frac{\sqrt{n} n_k}{2 n_j}) \int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}} \\
&\quad + \left(\sqrt{2n} \|f\|_\infty \frac{1}{\sqrt{q-1}} \right) \left(\sum_{k=1}^{N} a_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} \sum_{j=k+1}^{N} \frac{a_j^2}{n_k} \int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}}
\end{aligned}$$

$$= \left(m_q \int_0^{\sqrt{n}/q} \frac{\omega(\delta)}{\delta} d\delta \right)^{\frac{1}{2}} \left(\sum_{k=1}^N a_k^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^N a_j^2 \int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \sum_{k=1}^{j-1} \omega(\frac{\sqrt{n} n_k}{2 n_j}) \right)^{\frac{1}{2}}$$

$$+ \left(\sqrt{2n} \|f\|_\infty \frac{1}{\sqrt{q-1}} \right) \left(\sum_{k=1}^N a_k^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^N a_j^2 \int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \sum_{k=1}^{j-1} \frac{1}{n_k} \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=1}^N a_k^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^N a_j^2 \int_{[0,1]^n} |f(n_j x + c_j)|^2 dx \right)^{\frac{1}{2}} \left(m_q \int_0^{\sqrt{n}/q} \frac{\omega(\delta)}{\delta} d\delta + \frac{\sqrt{2nq} \|f\|_\infty}{\sqrt{n_1}(q-1)} \right)$$

Therefore,

$$\int_{[0,1]^n} \left| \sum_{k=1}^N a_k f(n_k x + c_k) \right|^2 dx$$

$$\geq \sum_{k=1}^N a_k^2 \int_{[0,1]^n} |f(n_k x + c_k)|^2 dx - c_q A_N \left(\sum_{k=1}^N a_k^2 \int_{[0,1]^n} |f(n_k x + c_k)|^2 dx \right)^{\frac{1}{2}}$$

where $c_q = \left(m_q \int_0^{\sqrt{n}/q} \frac{\omega(\delta)}{\delta} d\delta + \frac{\sqrt{2nq}\|f\|_\infty}{\sqrt{n_1}(q-1)} \right)$. By hypothesis, $\int_{[0,1]^n} |f(n_k x + c_k)|^2 dx > c_0$ for every k, and the lemma follows by taking q sufficiently large (and hence c_q sufficiently small). $\square$

We will need the following subgaussian estimate for dyadic martingales (see Chang, Wilson and Wolff[3]).

Lemma 3.2.5. If g_m is a dyadic martingale on Q then for each m and every $\lambda > 0$,

$$|\{x \in Q : |g_m(x)| \geq \lambda\}| \leq \exp\left(-\frac{\lambda^2}{2\|Sg_m\|_\infty^2} \right)$$

We would like a similar estimate for sums of the form $\sum_{k=1}^m a_k f(n_k x + c_k)$.

Lemma 3.2.6. Put $f_m(x) = \sum_{k=1}^m a_k f(n_k x + c_k)$ where f is as in the hypotheses of Theorem 3.1.1. Then there exists constants C and c depending only on q, n and the quantity (2.2.2) such that

$$|\{x \in [0,1]^n : |f_m(x)| \geq \lambda\}| \leq C \exp\left(-c \frac{\lambda^2}{A_m^2} \right).$$

Proof. By Lemma 2.2.1 we can break up the sequence n_k into a finite number of sequences each of which has the property that for each $k \geq 1$ there exists exactly one n_k with $2^{k-1} \leq n_k < 2^k$. That is, we may write $f_m = f_{m1} + \cdots + f_{mK}$ for some positive integer K and each f_{mj} has at most one n_k in each dyadic block $[2^k, 2^{k+1})$. Then since $|\{x \in [0,1]^n : f_m(x) > \lambda\}| \leq \sum_{j=1}^{K} |\{x \in [0,1]^n : f_{mj} > \frac{\lambda}{K}\}|$, the desired estimate follows if we can get such an estimate for each f_{mj}. In other words, we may assume, without loss of generality, that f_m has only one n_k in each dyadic block $[2^k, 2^{k+1})$. We first also assume that $a_1 = a_2 = 0$. For $m \geq 1$, let $f_m(x) := \sum_{k=3}^{m+2} a_k f(n_k x + c_k)$. Under these conditions, it is shown in Chapter 2 that there exists a family of dyadic martingales $\{g_m^{(j)}\}$, $j = 1, \ldots, N$, and an absolute constant C_1 such that

$$\left| f_{m+2}(x) - \sum_{j=1}^{N} g_m^{(j)}(x) \right| \leq C_1 A_{m+2}$$

$$\text{and for each } \ j, \quad (Sg_m^{(j)}(x))^2 \leq C_1 A_{m+2}^2.$$

Here C_1 and N depend only on the dimension n. Thus, for $\lambda > C_1 A_{m+2}$,

$$\begin{aligned}
&|\{x \in [0,1]^n : |f_{m+2}(x)| \geq \lambda\}| \leq \left| \{x \in [0,1] : |\sum_{j=1}^{N} g_m^{(j)}(x)| \geq \lambda - C_1 A_{m+2}\} \right| \\
&\leq \sum_{j=1}^{N} \left| \{x \in [0,1] : |g_m^{(j)}(x)| \geq \frac{\lambda - C_1 A_{m+2}}{N}\} \right| \leq \sum_{j=1}^{N} \exp\left(-c \frac{(\lambda - C_1 A_{m+2})^2}{(Sg_m^{(j)}(x))^2} \right) \\
&\leq N \exp\left(-c \frac{(\lambda - C_1 A_{m+2})^2}{A_{m+2}^2} \right) \leq C \exp\left(-c \frac{\lambda^2}{A_{m+2}^2} \right).
\end{aligned}$$

By taking C large enough so that $C \exp(-cC_1^2) \geq 1$, this remains valid for $\lambda \leq C_1 A_{m+2}$. Finally, to remove the assumption that $a_1 = a_2 = 0$, set $\tilde{f}_m(x) = f_m(x) - a_1 f(n_1 x + c_1) - a_2 f(n_2 x + c_2)$, so that $\tilde{f}_m$ satisfies the above inequality. Noting that $\|f\|_\infty \leq C$, where C depends on the quantity in (2.2.2), and using the inequality $\exp(-c(\alpha - \beta)^2) \leq \exp(-\frac{3c}{4}\alpha^2 + 3c\beta^2)$, valid for $\alpha, \beta > 0$, we have

$$\begin{aligned}
&|\{x \in [0,1]^n : |f_m(x)| > \lambda\}| \leq \left| \{x \in [0,1]^n : \tilde{f}_m(x) > \lambda - (|a_1| + |a_2|)\|f\|_\infty\} \right| \\
&\leq C \exp\left(-c \frac{(\lambda - (|a_1| + |a_2|)\|f\|_\infty)^2}{A_m^2} \right) \leq C \exp\left(-c \frac{\lambda^2}{A_m^2} \right).
\end{aligned}$$

$\square$

The following is adapted from part of the proof of Proposition 5 in Bañuelos, Klemeš, and Moore[11].

Lemma 3.2.7. Suppose that $g(x)$ is a real valued function defined on a set E, $|E| > 0$, and that

$$\left|\frac{1}{|E|}\int_E g(x)dx\right| \leq \varepsilon A \quad \text{and} \quad \frac{1}{|E|}\int_E g(x)^2 dx \geq c_0 A^2$$

for some constants $A > 0$, $0 < \varepsilon < 1$, $c_0 > 0$. Suppose also that g satisfies

$$|\{x \in E : |g(x)| > \lambda\}| \leq Ce^{-c\frac{\lambda^2}{A^2}}|E| \quad \text{for all} \quad \lambda > 0,$$

where C, c are constants. Then if ε is sufficiently small, there exists a $\delta > 0$, depending only on ε, c_0, C, and c such that

$$|\{x \in E : g(x) \geq \delta A\}| \geq \delta |E|.$$

Proof. Let $0 < \delta < L$ to be chosen momentarily. Then

$$\begin{aligned}
c_0 A^2 &\leq \frac{1}{|E|}\int_E |g(x)|^2 dx \\
&= \frac{1}{|E|}\int_{\{x\in E:|g(x)|>LA\}} |g(x)|^2 dx + \frac{1}{|E|}\int_{\{x\in E:|g(x)|\leq LA\}} |g(x)|^2 dx \\
&\leq C(LA)^2 e^{-cL^2} + C\int_{LA}^{\infty} 2\lambda e^{-c\frac{\lambda^2}{A^2}} d\lambda + \frac{LA}{|E|}\int_E |g(x)|dx \\
&\leq CA^2(L^2 + \frac{1}{c})e^{-cL^2} + \frac{LA}{|E|}\int_E |g(x)|dx
\end{aligned}$$

By choosing L sufficiently large, depending on c, C, and c_0, we have

$$C'A \leq \frac{1}{|E|}\int_E |g(x)|dx.$$

But then

$$\frac{1}{|E|}\int_E g^+(x)dx = \frac{1}{2|E|}\int_E |g(x)| + g(x)dx \geq \frac{C'}{2}A - \frac{\varepsilon}{2}A = CA.$$

Thus,

$$CA \leq \frac{1}{|E|}\int_{\{x\in E:g^+\leq\delta A\}} g^+(x)dx + \frac{1}{|E|}\int_{\{x\in E:\delta A<g^+\leq L'A\}} g^+dx + \frac{1}{|E|}\int_{\{x\in E:g^+\geq L'A\}} g^+dx$$
$$\leq \delta A + \frac{L'A}{|E|}|\{x\in E: g^+(x)\geq \delta A\}| + CA(L')^2e^{-c(L')^2}$$

By choosing δ sufficiently small, and L' sufficiently large, the conclusion follows. □

As to be expected, we will need a Borel-Cantelli type lemma for independent, or at least weakly dependent random variables. This is provided by the following, whose proof can be found in Bañuelos and Moore[8], pg. 79:

Lemma 3.2.8. For $k = 1, 2, \ldots$, suppose F_k is a collection of dyadic cubes whose union is $[0,1]^n$ such that F_{k+1} is a refinement of F_k. Suppose that the maximum length of the elements of F_k tends to zero. Suppose $\mathcal{E}_k \subset F_k$ has the property:

$$\forall Q \in F_k, \qquad \left| Q \cap \bigcup_{J\in\mathcal{E}_{k+1}} J \right| > |Q|\frac{C}{k}.$$

Set $E_k = \bigcup_{J\in\mathcal{E}_k} J$. Then for a.e. x, $x \in E_k$ i.o.

3.3 The proof of the theorem

Let M be a fixed large positive number. Define $N_1 \leq N_2 \leq \cdots$ by

$$N_l = \min\left\{N : \sum_{k=1}^{N} a_k^2 > M^l\right\}.$$

Let $\varepsilon > 0$ and assume $\varepsilon << 1$.

Consider a large positive integer l. Using the definition of N_l and the fact that $|a_{N_l}|^2 < \varepsilon A_{N_l}^2$, for N_l sufficiently large, we can assume that $A_{N_l}^2 = A_{N_l-1}^2 + a_{N_l}^2 < M^l + \varepsilon A_{N_l}^2$ and hence

$$M^l < A_{N_l}^2 < \frac{M^l}{1-\varepsilon}. \tag{3.3.1}$$

Consequently,

$$(1-\varepsilon)M < \frac{A_{N_{l+1}}^2}{A_{N_l}^2} < \frac{M}{1-\varepsilon}. \tag{3.3.2}$$

Then by Lemma 3.2.6 and (3.3.2) we obtain

$$
\begin{aligned}
&\left|\{x \in [0,1]^n : |\sum_{k=1}^{N_l} a_k f(n_k x + c_k)| \geq \sqrt{\frac{1+\varepsilon}{cM(1-\varepsilon)}}\sqrt{A_{N_{l+1}}^2 \log\log A_{N_{l+1}}^2}\}\right| \\
&\leq C \exp\left(-c\frac{1+\varepsilon}{cM(1-\varepsilon)}\frac{A_{N_{l+1}}^2 \log\log A_{N_{l+1}}^2}{A_{N_l}^2}\right) \\
&\leq C \exp\left(-\frac{1+\varepsilon}{M(1-\varepsilon)}(1-\varepsilon)M \log\log A_{N_{l+1}}^2\right) \\
&\leq C \exp\left(-(1+\varepsilon)\log\log M^{l+1}\right) = C((l+1)\log M)^{-(1+\varepsilon)}.
\end{aligned}
$$

So by the Borel-Cantelli lemma, for almost every $x \in [0,1]^n$,

$$
\left|\sum_{k=1}^{N_l} a_k f(n_k x + c_k)\right| < \sqrt{\frac{1+\varepsilon}{cM(1-\varepsilon)}}\sqrt{A_{N_{l+1}}^2 \log\log A_{N_{l+1}}^2}. \tag{3.3.3}
$$

for all sufficiently large l (depending on x).

The definition of N_l and (3.3.1) yields:

$$
\sum_{k=N_l+1}^{N_{l+1}} a_k^2 = A_{N_{l+1}}^2 - A_{N_l}^2 > M^{l+1} - \frac{M^l}{1-\varepsilon} = M^{l+1}\left[1 - \frac{1}{M(1-\varepsilon)}\right] \geq A_{N_{l+1}}^2(1-\varepsilon-\frac{1}{M}). \tag{3.3.4}
$$

By hypotheses, for all sufficiently large l,

$$
\max_{1\leq k\leq N_{l+1}} a_k^2 \leq K_{N_{l+1}}^2\left(\frac{A_{N_{l+1}}^2}{\log\log A_{N_{l+1}}^2}\right) \leq \frac{\varepsilon}{2}\left(\frac{A_{N_{l+1}}^2}{\log\log A_{N_{l+1}}^2}\right),
$$

which, by (3.3.4) and the definition of $A_{N_{l+1}}$ implies that

$$
\max_{1\leq k\leq N_{l+1}} a_k^2 \leq \frac{K_{N_{l+1}}^2}{1-\varepsilon-\frac{1}{M}}\frac{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}{\log\log A_{N_{l+1}}^2} < \frac{\varepsilon/2}{(1-\varepsilon-\frac{1}{M})}\frac{1}{\log l}\sum_{k=N_l+1}^{N_{l+1}} a_k^2.
$$

We may assume that ε is small enough and M large enough so that $1-\varepsilon-\frac{1}{M} > \frac{1}{2}$. Thus,

$$
\max_{1\leq k\leq N_{l+1}} \frac{|a_k|}{\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}} \leq \sqrt{\frac{\varepsilon}{\log l}}. \tag{3.3.5}
$$

Let $0 < \mu < 1$. Suppose l is large so that $\mu \log l >> 1$. We define a sequence of positive integers $l_1, l_2 \cdots, l_{\lfloor\rfloor}$, where for simplicity we write $\lfloor\rfloor = \left\lfloor\frac{\mu\log l}{1+\varepsilon}\right\rfloor$ ($\lfloor\ \rfloor$ represents the greatest integer function) as follows:

Let l_1 be the first time such that

$$\sum_{k=N_l+1}^{N_l+l_1} a_k^2 \geq \frac{1}{\mu \log l} \sum_{k=N_l+1}^{N_{l+1}} a_k^2,$$

so that

$$\sum_{k=N_l+1}^{N_l+l_1-1} a_k^2 < \frac{1}{\mu \log l} \sum_{k=N_l+1}^{N_{l+1}} a_k^2. \tag{3.3.6}$$

Likewise, let l_2 be the first time such that

$$\sum_{k=N_l+l_1+1}^{N_l+l_2} a_k^2 \geq \frac{1}{\mu \log l} \sum_{k=N_l+1}^{N_{l+1}} a_k^2,$$

so that

$$\sum_{k=N_l+l_1+1}^{N_l+l_2-1} a_k^2 < \frac{1}{\mu \log l} \sum_{k=N_l+1}^{N_{l+1}} a_k^2. \tag{3.3.7}$$

Similarly we define $l_3, \ldots, l_{\lfloor\rfloor}$.

Because of (3.3.6), $N_l + l_1 \leq N_{l+1}$ and hence by (3.3.6) and (3.3.5)

$$\sum_{k=N_l+1}^{N_l+l_1} a_k^2 = \sum_{k=N_l+1}^{N_l+l_1-1} a_k^2 + a_{N_l+l_1}^2 \leq \frac{1+\varepsilon}{\mu \log l} \sum_{k=N_l+1}^{N_{l+l}} a_k^2.$$

Combining this and (3.3.7) yields

$$\sum_{k=N_l+1}^{N_l+l_2-1} a_k^2 \leq \left(\frac{1+\varepsilon}{\mu \log l} + \frac{1}{\mu \log l}\right) \sum_{k=N_l+1}^{N_{l+l}} a_k^2 < \sum_{k=N_l+1}^{N_{l+1}} a_k^2, \tag{3.3.8}$$

the last inequality being a consequence of the fact that

$$r\left(\frac{1+\varepsilon}{\mu \log l}\right) + \frac{1}{\mu \log l} < 1 \text{ for positive integers } r \text{ with } r \leq \left\lfloor \frac{\mu \log l}{1+\varepsilon} \right\rfloor - 1. \tag{3.3.9}$$

Thus, $N_l + l_2 \leq N_{l+1}$, so by (3.3.8) and again using (3.3.5), we have

$$\begin{aligned} \sum_{k=N_l+1}^{N_l+l_2} a_k^2 &= \sum_{k=N_l+1}^{N_l+l_2-1} a_k^2 + a_{N_l+l_2}^2 \leq \left(\frac{1+\varepsilon}{\mu \log l} + \frac{1}{\mu \log l} + \frac{\varepsilon}{\mu \log l}\right) \sum_{k=N_l+1}^{N_{l+l}} a_k^2 \\ &= 2\left(\frac{1+\varepsilon}{\mu \log l}\right) \sum_{k=N_l+1}^{N_{l+l}} a_k^2. \end{aligned} \tag{3.3.10}$$

Continuing in the same fashion, using (3.3.5) and (3.3.9) we have

$$\sum_{k=N_l+1}^{N_l+l_3-1} a_k^2 \leq \left(2\left(\frac{1+\varepsilon}{\mu \log l}\right) + \frac{1}{\mu \log l}\right) \sum_{k=N_l+1}^{N_{l+1}} a_k^2 < \sum_{k=N_l+1}^{N_{l+1}} a_k^2, \tag{3.3.11}$$

which implies that $N_l + l_3 \leq N_{l+1}$. We continue this process, repeatedly using (3.3.5) and (3.3.9) to conclude $N_l + l_{\lfloor\rfloor} \leq N_{l+1}$.

Consider a dyadic cube Q such that $|Q| = 2^{-L}$ where L is chosen so that $2^L \leq n_{N_l} < 2^{L+1}$. By rescaling to Q, Lemma 3.2.4 implies that

$$\int_Q \left| \sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) \right|^2 dx \geq c|Q| \sum_{k=N_l+1}^{N_l+l_1} a_k^2.$$

Similarly, again by rescaling to Q, Lemma 3.2.6 implies that

$$\left| \{x \in Q : |\sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k)| \geq \lambda\} \right| \leq C \exp\left(-c\frac{\lambda^2}{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}\right) |Q|.$$

Finally, notice that for k with $N_l + 1 \leq k \leq N_l + l_1$, (3.3.5) yields

$$|a_k| \leq \sqrt{\frac{\varepsilon}{\log l}} \sqrt{\sum_{k=N_l+1}^{N_{l+l}} a_k^2} \leq \sqrt{\frac{\varepsilon}{\log l}} \sqrt{\mu \log l} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} = \sqrt{\mu\varepsilon} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}.$$

Consequently by Lemma 3.2.3, and Lemma 3.2.1 (4),

$$\left| \frac{1}{|Q|} \int_Q \sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) dx \right| \leq \sum_{k=N_l+1}^{N_l+l_1} |a_k| \frac{2n2^L \|f\|_\infty}{n_k}$$
$$\leq \|f\|_\infty \sqrt{\mu\varepsilon} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \sum_{k=N_l+1}^{N_l+l_1} \frac{2n2^L}{n_k} \leq C\sqrt{\varepsilon} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}.$$

Then Lemma 3.2.7 applies to give $\delta > 0$ (which depends only on ε and constants which themselves depend only on q and n) so that

$$\begin{aligned} &\left| \left\{ x \in Q : \sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) > \frac{\delta}{\sqrt{\mu \log l}} \sqrt{\sum_{k=N_l+1}^{N_{l+l_1}} a_k^2} \right\} \right| \\ &\geq \left| \left\{ x \in Q : \sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) > \delta \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \right\} \right| \geq \delta |Q|. \end{aligned} \tag{3.3.12}$$

Set $h(x) = \sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k)$. Choose L_1 so that $2^{L_1} \le n_{N_l+l_1} < 2^{L_1+1}$. Fix x, y and suppose $|x-y| < \frac{\sqrt{n}}{2^{L_1}}$. Then using the hypotheses of the theorem, the definition of $A_{N_{l+1}}$, Lemma 3.2.1 (2) and (3.3.4), and again assuming that $1 - \varepsilon - \frac{1}{M} > \frac{1}{2}$, we have

$$\begin{aligned}|h(x) - h(y)| &\le \sum_{k=N_l+1}^{N_l+l_1} |a_k|\,|f(n_k x + c_k) - f(n_k y + c_k)| \le \sum_{k=N_l+1}^{N_l+l_1} |a_k| \omega\left(\frac{\sqrt{n} n_k}{2^{L_1}}\right) \\ &\le \frac{K_{N_{l+1}} A_{N_{l+1}}}{\sqrt{\log\log A^2_{N_{l+1}}}} \sum_{k=N_l+1}^{N_l+l_1} \omega\left(\frac{\sqrt{n} n_k}{n_{N_l+l_1}} 2\right) \le C K_{N_{l+1}} \frac{\sqrt{2\sum_{k=N_l+1}^{N_{l+1}} a_k^2}}{\sqrt{\log l}}.\end{aligned} \tag{3.3.13}$$

Thus, if $h(x) > \frac{\delta}{\sqrt{\mu \log l}} \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}$, then

$$|h(y)| \ge |h(x)| - C\frac{K_{N_{l+1}}}{\sqrt{\log l}} \sqrt{\sum_{k=N_l+1}^{N_{l+l}} a_k^2} \ge \left(\frac{\delta - C\sqrt{\mu} K_{N_{l+1}}}{\sqrt{\mu \log l}}\right) \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}.$$

From (3.3.12) we conclude that there exists a collection of dyadic subcubes $\{Q'\}$ of Q with each $|Q'| = 2^{-L_1}$ such that $\forall x \in Q'$,

$$\sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) \ge \left(\frac{\delta - C\sqrt{\mu} K_{N_{l+1}}}{\sqrt{\mu \log l}}\right) \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2},$$

and with $\left|\bigcup_{Q' \subset Q} Q'\right| > \delta |Q|$.

Consider such a Q'. Arguing as above we have

$$\begin{aligned}&\left|\left\{x \in Q' : \sum_{k=N_l+l_1+1}^{N_l+l_2} a_k f(n_k + c_k) > \frac{\delta}{\sqrt{\mu \log l}} \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}\right\}\right| \\ &\ge \left|\left\{x \in Q' : \sum_{k=N_l+l_1+1}^{N_l+l_2} a_k f(n_k + c_k) > \delta \sqrt{\sum_{k=N_l+l_1+1}^{N_l+l_2} a_k^2}\right\}\right| \ge \delta |Q'|.\end{aligned}$$

As previously, this leads us to a collection of dyadic subcubes $\{Q''\}$ of Q' with $|Q''| = 2^{-L_2}$, where L_2 satisfies $2^{L_2} \le n_{N_l+l_2} < 2^{L_2+1}$, such that $\forall x \in Q''$,

$$\sum_{k=N_l+l_1+1}^{N_l+l_2} a_k f(n_k x + c_k) \ge \left(\frac{\delta - C\sqrt{\mu} K_{N_{l+1}}}{\sqrt{\mu \log l}}\right) \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}$$

and with $\left|\bigcup_{Q''\subset Q'} Q''\right| > \delta|Q'|$. We continue this process. Eventually we come to a subcollection of cubes $\{I\}$ with $|I| = 2^{-L_{\lfloor\rceil}}$, where $\lfloor\rfloor = \left\lfloor \dfrac{\mu \log l}{1+\varepsilon}\right\rfloor$, and $L_{\lfloor\rfloor}$ is the number satisfying $2^{L_{\lfloor\rfloor}} \leq n_{N_l+l_{\lfloor\rfloor}} < 2^{L_{\lfloor\rfloor}+1}$, such that $\forall x \in I$,

$$\sum_{k=N_l+l_{\lfloor\rfloor-1}+1}^{N_l+l_{\lfloor\rfloor}} a_k f(n_k x + c_k) \geq \left(\frac{\delta - C\sqrt{\mu}K_{N_{l+1}}}{\sqrt{\mu}\log l}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}.$$

Moreover, $\left|\bigcup_{I\subset\tilde{Q}} I\right| > \delta|\tilde{Q}|$ where $\tilde{Q}$ is the previous generation cube. On each I, we need to estimate the remaining terms $\displaystyle\sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} a_k f(n_k x + c_k)$. Using (3.3.13) and Lemma 3.2.3 we have:

$$\left|\frac{1}{|I|}\int_I \sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} a_k f(n_k x + c_k)dx\right| \leq \sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} |a_k| \left|\frac{1}{|I|}\int_I f(n_k x + c_k)dx\right|$$

$$\leq C\sqrt{\frac{\varepsilon}{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}\sum_{k=N_l+l_{\lfloor\rceil}+1}^{N_{l+1}} \frac{2^{L_{\lfloor\rceil}}\|f\|_\infty}{n_k} \leq C_1\sqrt{\frac{\varepsilon}{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}$$

By Chebyshev,

$$\left|\left\{x \in I : \left|\sum_{k=N_l+l_{\lfloor\rceil}+1}^{N_{l+1}} a_k f(n_k x + c_k)\right| > 2C_1\sqrt{\frac{\varepsilon}{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}\right\}\right| \leq \frac{1}{2}|I|,$$

so that in particular,

$$\sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} a_k f(n_k x + c_k) > -2C_1\sqrt{\frac{\varepsilon}{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2} \tag{3.3.14}$$

on at least $\frac{1}{2}$ of the measure of I. Choose $\tilde{L}$ so that $2^{\tilde{L}} \leq n_{N_{l+1}} < 2^{\tilde{L}+1}$. Let $h(x) = \sum_{k=N_l+l_{\lfloor\rceil}+1}^{N_{l+1}} a_k f(n_k x + c_k)$.

Let x be a point at which (3.3.14) holds and suppose $|x-y| \leq 2^{-\tilde{L}}$. Estimating as before (as in (3.3.13)) we have:

$$|h(x) - h(y)| \leq CK_{N_{l+1}}\frac{\sqrt{2\sum_{k=N_l+1}^{N_{l+1}} a_k^2}}{\sqrt{\log l}}$$

Thus, if

$$h(x) > -2C_1\sqrt{\frac{\varepsilon}{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}$$

then

$$h(y) > \left(-2C_1\sqrt{\frac{\varepsilon}{\log l}} - \frac{CK_{N_{l+1}}}{\sqrt{\log l}}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2} = -C\left(\frac{\sqrt{\varepsilon}+K_{N_{l+1}}}{\sqrt{\log l}}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}.$$

Consequently, there exists a collection of dyadic subcubes $\{J\}$ of I with $|J| = 2^{-\tilde{L}}$ such that for every $x \in J$,

$$\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k f(n_k x + c_k) > -C\left(\frac{\sqrt{\varepsilon}+K_{N_{l+1}}}{\sqrt{\log l}}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2},$$

and with $|\cup_{J\subset I} J| \geq \frac{1}{2}|I|$.

Finally, adding the estimates from all of the above generations, we have

$$\begin{aligned}&\sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) + \cdots + \sum_{k=N_l+l_{\lfloor\rfloor}-1}^{N_l+l_{\lfloor\rfloor}} a_k f(n_k x + c_k) + \sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} a_k f(n_k x + c_k)\\ &> \left[\left\lfloor\frac{\mu\log l}{1+\varepsilon}\right\rfloor\left(\frac{\delta - C\sqrt{\mu}K_{N_{l+1}}}{\sqrt{\mu\log l}}\right) - C\left(\frac{\sqrt{\varepsilon}+K_{N_{l+1}}}{\sqrt{\log l}}\right)\right]\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}.\end{aligned}$$

on a subcollection $\{J\}$ of dyadic subcubes of Q with

$$|Q\cap\bigcup J| > |Q|\delta^{\left\lfloor\frac{\mu\log l}{1+\varepsilon}\right\rfloor}\frac{1}{2} \geq \frac{1}{2}|Q|\delta^{\frac{\mu\log l}{1+\varepsilon}} = \frac{1}{2}|Q|e^{(\log\delta)\frac{\mu\log l}{1+\varepsilon}} = \frac{1}{2}|Q|l^{\frac{\mu\log(\delta)}{1+\varepsilon}} \geq \frac{1}{2}\frac{|Q|}{l},$$

where the latter inequality holds if μ is chosen sufficiently small. We remark that neither δ nor ε depend on μ so this is possible.

We may also assume that l is large enough so that

$$\left\lfloor\frac{\mu\log l}{1+\varepsilon}\right\rfloor \Big/ \left(\frac{\mu\log l}{1+\varepsilon}\right) > \frac{1}{1+\varepsilon}. \tag{3.3.15}$$

Thus, on the subcubes J, if l is sufficiently large, we can estimate

$$\begin{aligned}
&\sum_{k=N_l+1}^{N_{l+1}} a_k f(n_k x + c_k) \\
&> \left\lfloor \left[\frac{\mu \log l}{1+\varepsilon}\right] \left(\frac{\delta - C\sqrt{\mu}K_{N_{l+1}}}{\sqrt{\mu \log l}}\right) - C\left(\frac{\sqrt{\varepsilon} + K_{N_{l+1}}}{\sqrt{\log l}}\right)\right] \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2} \\
&\geq \left[\frac{1}{1+\varepsilon}\frac{\mu \log l}{1+\varepsilon} \left(\frac{\delta - C\sqrt{\mu}K_{N_{l+1}}}{\sqrt{\mu \log l}}\right) - C\left(\frac{\sqrt{\varepsilon} + K_{N_{l+1}}}{\sqrt{\log l}}\right)\right] \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2} \\
&> \eta \sqrt{\log l} \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2},
\end{aligned}$$

where η depends only on μ, ε, and δ, but can be taken as a fixed positive number for all l sufficiently large. Thus, if we let F_l denote the family of dyadic cubes Q in $[0, 1]$ of sidelength 2^{-L} (recall $2^L \leq n_{N_l} < 2^{L+1}$) and let $\mathcal{E}_{l+1}$ denote the union of those cubes J of sidelength $2^{-\tilde{L}}$ (recall $2^{\tilde{L}} \leq n_{N_{l+1}} < 2^{\tilde{L}}$) found in all of the Q using the above argument, then, for large enough l (depending only on ε and M), the hypotheses of Lemma 3.2.8 are satisfied, so that there exits $\eta > 0$ such that for a.e. x there exists a subsequence of $\{N_l\}_{l=1}^{\infty}$, (depending on x) such that for each l in this subsequence we have

$$\frac{\sum_{k=N_l+1}^{N_{l+1}} a_k f(n_k x + c_k)}{\sqrt{\log l \sum_{k=N_l+1}^{N_{l+1}} a_k^2}} > \eta.$$

For such an x, then by (3.3.4), and again assuming that $1 - \varepsilon - \frac{1}{M} > \frac{1}{2}$, for an infinite subsequence of the N_l we have

$$\frac{\sum_{k=N_l+1}^{N_{l+1}} a_k f(n_k x + c_k)}{\sqrt{\log l \sum_{k=1}^{N_{l+1}} a_k^2}} > \frac{\eta}{2}.$$

By (3.3.1),

$$\log\log A_{N_{l+1}}^2 \leq \log((l+1)\log M - \log(1-\varepsilon)) \leq 2\log l,$$

the latter inequality holding for l sufficiently large. Consequently,

$$\frac{\sum_{k=1}^{N_{l+1}} a_k f(n_k x + c_k) - \sum_{k=1}^{N_l} a_k f(n_k x + c_k)}{\sqrt{\sum_{k=1}^{N_{l+1}} a_k^2 \log\log\left(\sum_{k=1}^{N_{l+1}} a_k^2\right)}} \geq \frac{\eta}{2\sqrt{2}}$$

But from (3.3.3) for a.e. x we have,

$$\frac{\left|\sum_{k=1}^{N_l} a_k f(n_k x + c_k)\right|}{\sqrt{\sum_{k=1}^{N_l+1} a_k^2 \log\log \sum_{k=1}^{N_l+1} a_k^2}} \leq \sqrt{\frac{1+\varepsilon}{cM(1-\varepsilon)}}$$

for sufficiently all large l (depending on x).

Hence for a.e. x there is an infinite subsequence of sufficiently large enough l so that,

$$\frac{\left|\sum_{k=1}^{N_{l+1}} a_k f(n_k x + c_k)\right|}{\sqrt{\sum_{k=1}^{N_{l+1}} a_k^2 \log\log \sum_{k=1}^{N_{l+1}} a_k^2}} \geq \frac{\eta}{2\sqrt{2}} - \sqrt{\frac{1+\varepsilon}{cM(1-\varepsilon)}}.$$

Thus, for a.e. x,

$$\limsup_{n\to\infty} \frac{\left|\sum_{k=1}^{n} a_k f(n_k x + c_k)\right|}{\sqrt{\sum_{k=1}^{n} a_k^2 \log\log \sum_{k=1}^{n} a_k^2}} \geq \frac{\eta}{2\sqrt{2}} - \sqrt{\frac{1+\varepsilon}{cM(1-\varepsilon)}}.$$

We can let $M \nearrow \infty$ and obtain the desired result.

Chapter 4

Future work

It has long been appreciated that the partial sums of lacunary series exhibit many of the properties of sums of independent random variables. This is evidenced by many results in analysis which give central limit theorem type behavior or laws of the iterated logarithm (LILs) for lacunary series. The classical LIL of Kolmogorov[7] was first proved for Bernoulli random variables by Khintchine, then in 1950 Salem and Zygmund[13] considered the case for trigonometric functions $a_k \cos n_k x$ on $[-\pi, \pi]$ and gave an upper bound result, which was extended to the full upper and lower bound by Erdös and Gál[5]. Later on Takahashi[15] extends the result of Salem and Zygmund and derives a LIL for lacunary series, and in this paper we extended the results of Takahashi by broadening the class of functions f. It would be worth of study to improve our result according to the followings:

Mentioned in Chapters 2 and 3, it would be interesting to see if the hypotheses of Dini continuity is necessary or if a weaker hypothesis would suffice.

Mentioned in Chapter 3, we needed q sufficiently large. It would be interesting to see if that condition is necessary.

In both the upper and lower bound results, it would be interesting to determine the best possible values of the bounds. That is, to find the best possible values of C and c respectively in inequalities

$$\limsup_{m\to\infty} \frac{|\sum_{k=1}^{m} a_k f(n_k x + c_k)|}{\sqrt{A_m^2 \log\log A_m^2}} \leq C \quad a.e.$$

and

$$\limsup_{m\to\infty} \frac{|\sum_{k=1}^{m} a_k f(n_k x + c_k)|}{\sqrt{A_m^2 \log\log A_m^2}} \geq c \quad a.e.$$

Bibliography

[1] C. Aistleitner and I. Berkes. On the central limit theorem for $f(n_k x)$. *Probab. Theory Related Fields*, 146 no. 1-2:267–289, 2010.

[2] D.L. Burkholder and R.F. Gundy. Extrapolation and interpolation of quasilinear operators on martingales. *Acta Math.*, 124:249–304, 1970.

[3] S.Y. Chang, J.M. Wilson, and T.H. Wolff. Some weighted norm inequalities concerning the Schrödinger operator. *Comment. Math. Helv.*, 60:217–246, 1985.

[4] S. Dhompongsa. Uniform laws of the iterated logarithm for Lipschitz classes of functions. *Acta Sci. Math. (Szeged)*, 50:105–124, 1986.

[5] P. Erdös and L.S. Gál. On the Law of the Iterated Logarithm I, II. *Nederl. Akad. Wetensch. Proc. Ser.*, A 58:65–76 and 77–84, 1955.

[6] V.F. Gaposkin. The central limit theorem for some weakly dependent sequences. *Theory Probab. Appl. XV*, no. 4:649–666, 1970.

[7] N. Kolmogorov. Über des Gesetz des iterierten Logarithmus. *Math. Ann.*, 101:126–139, 1929.

[8] R. Ba nuelos and C. N. Moore. *Probabilistic Behavior of Harmonic Functions.* Birkhäuser Verlag, 1999.

[9] E. Peter. A probability limit theorem for $\{f(n_k x)\}$. *Acta Math. Hungar.*, 87 (1-2):23–31, 2000.

[10] G. Pólya. Über eine Aufgabe des Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Strassennetz. *Math. Ann.*, 84:149–160, 1921.

[11] I. Klemeˇ s R. Bañuelos and C. N. Moore. The lower bound in the law of the iterated logarithm for harmonic functions. *Duke Math. J.*, 60:689–715, 1990.

[12] R. Salem and A. Zygmund. On lacunary trigonometric series I, II. *Proc. Natl. Acad. Sci. USA*, 33/34:333–338/ 54–62, 1947/ 1948.

[13] R. Salem and A. Zygmund. La loi du logarithme itéré pour les séries trigonométriques lacunaire. *Bull. Sci. Math.*, 74:209–224, 1950.

[14] W. Stout. A martingale analogue of Kolomogorov's law of the iterated logarithm. *Z. Wahrscheinlichskeitstheorie Verw. Geb.*, 15:279–290, 1970.

[15] S. Takahashi. The Law of the Iterated Logarithm for a Gap Sequence with Infinite Gaps. *Tohoku Math. J.*, (2)15, no. 3:281–288, 1963.

[16] S. Takahashi. An asymptotic behavior of $\{f(n_k t)\}$. *Sci. Rep. Kanazawa Univ.*, 33:27–36, 1988.

[17] M. Weiss. The Law of Iterated Logarithm for Lacunary Trigonometric Series. *Trans. Amer. Math. Soc.*, 91, No.3:444–469, 1959.