Fourth-order tensors

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Abstract

The focus of this report is to study fourth-order tensors and generalize some results from the linear algebra of second-order tensors to fourth-order tensors. A fourth-order tensor can be viewed as a linear map from second-order tensors to second-order tensors. We provide an orthonormal basis for the vector space of fourth-order tensors and use it to represent any fourth-order tensor by a fourth-dimensional array, which represents its component form. An inner product and norm are provided for this vector space. Composition of linear maps gives rise to multiplication of fourth-order tensors, which we present in component form. We study different kinds of symmetries for fourth-order tensors, in particular, major symmetry and minor symmetry. We provide an isomorphism between the vector space of fourth-order tensors and the vector space of second-order tensors of the same dimension. We use this isomorphism to prove a spectral theorem for fourth-order tensors that possess major symmetry.

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Chapter 1

Introduction

In this report, we study fourth-order tensors, certain notions of their symmetry, and provide a spectral theorem for fourth-order tensors. Higher-order tensors and, in particular fourth-order tensors, have been studied in the literature including[1–4]. Our approach to generalize some results from the linear algebra of second-order tensors to fourth-order tensors follows that given in[5]. In particular,[5] provides a spectral theorem for symmetric fourth-order tensors, but without providing a proof or reference in the literature where a proof can be found. In this report, we provide a proof for a spectral theorem for symmetric fourth-order tensors. Moreover, we present a spectral decomposition of symmetric fourth-order tensors of the form

$$\mathbb{A} = \mathbb{Q}\mathbb{D}\mathbb{Q}^T,$$

where $\mathbb{A}$ is a *symmetric* fourth-order tensor, $\mathbb{D}$ a *diagonal* fourth-order tensor, $\mathbb{Q}$ is an *orthogonal* fourth-order tensor, and ^{T} denotes the fourth-order tensor *transpose*.

This report is organized as follows. In Sections 1.1 and 1.2, we define second-order tensors as bilinear maps and provide an identification between second-order tensors and matrices for a fixed orthonormal basis. Symmetric second-order tensors are described in Section 1.3, where we present the spectral theorem for symmetric second-order tensors. In Section 1.4, we provide a bijection between second-order tensors and vectors. In Section 2.1, we define fourth-order tensors as multilinear maps and provide an identification between fourth-order tensors and four-dimensional arrays. On the vector space of fourth-order tensors, we provide an inner product and norm. Moreover, We define operations of multiplication, trace, and transpose of fourth-order tensors. In Section 2.2, we provide a representation for fourth-order tensors as second-order tensors and define the eigenvalues and eigentensors of fourth-order tensors. In Chapter 3, we provide certain notions of symmetry of fourth-order tensors, and present and prove a spectral theorem for major-symmetric fourth-order tensors.

1.1 Second-Order Tensors

Let $\mathcal{V} = \mathbb{R}^n$, the real coordinate space of dimension n, and $\mathfrak{B} := \{\mathbf{e}^i\}_{i=1,\cdots,n}$ be the standard orthonormal basis for $\mathcal{V}$. Then, any vector $\mathbf{u} \in \mathcal{V}$ can be written as $\mathbf{u} = u_i\mathbf{e}^i$, where the scalars u_i are the components of $\mathbf{u}$ with respect to the basis $\mathfrak{B}$. The dot product is defined by

$$\mathbf{u} \cdot \mathbf{v} := \mathbf{u}^T\mathbf{v} = u_i v_i.$$

In this report, we adopt the summation convention for repeated indices. In addition, we adopt the following notation: vectors are denoted by boldface lowercase letters (for example, $\mathbf{u}$), second-order tensors are denoted by a boldface uppercase letters (for example, $\mathbf{A}$), and we use script uppercase letters to denote fourth-order tensors (for example, $\mathbb{A}$). The outer product is denoted by $\otimes$ (for example, $\mathbf{u} \otimes \mathbf{v}$). Moreover, superscripts are used to label different tensors or vectors, while subscripts are only used to denote components of tensors or vectors.

1.2 Second-Order Tensors as Bilinear Maps

A bilinear map $\mathbf{T}$,

$$\mathbf{T} : \mathcal{V} \times \mathcal{V} \to \mathbb{R},$$

is called a second-order tensor. Associated with $\mathbf{T}$, there is a unique two-dimensional array (matrix) $\mathbf{A} = \mathbf{A}_\mathbf{T}$ such that

$$\mathbf{T}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{A}\mathbf{v},$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. The matrix $\mathbf{A}$ is given by

$$A_{ij} = \mathbf{T}(\mathbf{e}^i, \mathbf{e}^j).$$

To see this, we observe that

$$\begin{aligned}
\mathbf{T}(\mathbf{u}, \mathbf{v}) &= \mathbf{T}(u_i \mathbf{e}^i \ , \ v_j \mathbf{e}^j) \\
&= u_i v_j \mathbf{T}(\mathbf{e}^i, \mathbf{e}^j) \\
&=: u_i v_j A_{ij} \\
&= \mathbf{u} \cdot \mathbf{A}\mathbf{v}.
\end{aligned}$$

Conversely, given a matrix $\mathbf{A}$, define a bilinear map $\mathbf{T} = \mathbf{T}_\mathbf{A}$ by

$$\mathbf{T}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{A}\mathbf{v}.$$

With respect to a fixed basis of $\mathcal{V}$, the above provides a one-to-one identification between second-order tensors and two-dimensional arrays (matrices). Therefore, we interchangeably refer to a matrix as a second-order tensor and vice versa.

Let $\mathrm{Lin}(\mathcal{V})$ be the set of all second-order tensors. This set forms a vector space of dimension n^2. The inner product on this vector space is given by

$$\langle \mathbf{A}, \mathbf{B} \rangle := \mathbf{A} : \mathbf{B} = A_{ij} B_{ij},$$

and the norm on $\mathrm{Lin}(\mathcal{V})$ is given by the induced norm

$$\|\mathbf{A}\| = \sqrt{\mathbf{A} : \mathbf{A}}.$$

For any $\mathbf{A} \in \mathrm{Lin}(\mathcal{V})$, let

$$A_{ij} := \mathbf{e}^i \cdot \mathbf{A}\mathbf{e}^j = \mathbf{A} : \mathbf{e}^i \otimes \mathbf{e}^j = \langle \mathbf{A}, \mathbf{e}^i \otimes \mathbf{e}^j \rangle.$$

We note that the set $\{\mathbf{e}^i \otimes \mathbf{e}^j\}_{i,j=1,\ldots,n}$ forms an orthonormal set in $\mathrm{Lin}(\mathcal{V})$, since

$$\mathbf{e}^i \otimes \mathbf{e}^j : \mathbf{e}^k \otimes \mathbf{e}^l = \begin{cases} 1, & \text{if } i = k, j = l, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $\mathbf{A}$ has the representation

$$\mathbf{A} = A_{ij} \mathbf{e}^i \otimes \mathbf{e}^j,$$

and therefore the set $\{\mathbf{e}^i \otimes \mathbf{e}^j\}_{i,j=1,\ldots,n}$ forms an orthonormal basis for second-order tensors.

The product $\mathbf{AB}$ of two second-order tensors, is a second order tensor, defined by composition

$$(\mathbf{AB})\mathbf{u} = \mathbf{A}(\mathbf{B}\mathbf{u}),$$

for all $\mathbf{u} \in \mathcal{V}$. Viewing $\mathbf{AB}$ as a bilinear map from $\mathcal{V} \times \mathcal{V}$ to $\mathbb{R}$ given by $\langle (\mathbf{AB})\mathbf{u}, \mathbf{v} \rangle$, we conclude that the product of two second-order tensors is also a second-order tensor. In components, we have

$$
\begin{aligned}
(\mathbf{AB})_{ij} &= \mathbf{e}^i \cdot (\mathbf{AB})\mathbf{e}^j \\
&= \mathbf{e}^i \cdot \mathbf{A}(\mathbf{B}\mathbf{e}^j) \\
&= e^i_k(\mathbf{A}(\mathbf{B}\mathbf{e}^j))_k \\
&= e^i_k A_{kl}(\mathbf{B}\mathbf{e}^j)_l \\
&= e^i_k A_{kl} B_{lp} e^j_p \\
&= A_{il} B_{lj}.
\end{aligned}
$$

The transpose of $\mathbf{A} \in \mathrm{Lin}(\mathcal{V})$ is the second-order tensor denoted $\mathbf{A}^T$ and defined by

$$\mathbf{u} \cdot \mathbf{A}^T\mathbf{v} = \mathbf{v} \cdot \mathbf{A}\mathbf{u},$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. In components,

$$
\begin{aligned}
(\mathbf{A}^T)_{ij} &= \mathbf{e}^i \cdot \mathbf{A}^T\mathbf{e}^j \\
&= \mathbf{e}^j \cdot \mathbf{A}\mathbf{e}^i \\
&= A_{ji}.
\end{aligned}
$$

Hence, the transpose of second order tensor

$$\mathbf{A} = A_{ij}\mathbf{e}^i \otimes \mathbf{e}^j,$$

is given by

$$\mathbf{A}^T = A_{ji}\mathbf{e}^i \otimes \mathbf{e}^j.$$

1.3 Symmetric Second-Order Tensors

Let $\mathrm{Sym}(\mathcal{V}) = \{\mathbf{A} \in \mathrm{Lin}(\mathcal{V}) \mid \mathbf{A} = \mathbf{A}^T\}$, be the set of all symmetric second-order tensors. A second-order tensor $\mathbf{A}$ is said to be antisymmetric if $\mathbf{A} = -\mathbf{A}^T$, or in components, $A_{ij} = -A_{ji}$. Every second-order tensor $\mathbf{A}$ can be decomposed uniquely as $\mathbf{A} = \mathbf{A}^s + \mathbf{A}^a$, where

$$\mathbf{A}^s = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T),$$

is the symmetric part of $\mathbf{A}$ and

$$\mathbf{A}^a = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T),$$

is the antisymmetric part of $\mathbf{A}$.

The set of all symmetric second-order tensors $\mathrm{Sym}(\mathcal{V})$ is a subspace of $\mathrm{Lin}(\mathcal{V})$ of dimension $n(n+1)/2$. The symmetric second-order tensors

$$\{2^{-\frac{1}{2}(1+\delta_{ij})}(\mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^i)\}_{1 \leq i \leq j \leq n},$$

where δ_{ij} is the Kronecker delta, form an orthonormal basis of $\mathrm{Sym}(\mathcal{V})$.

Theorem 1.3.1 (The spectral theorem for second-order tensors)**.** *If $\mathbf{A} \in Sym(\mathcal{V})$, then there exists $\{\lambda^k\}_{1\leq k \leq n} \in \mathbb{R}$ and $\{\mathbf{u}^k\}_{1 \leq k \leq n}$ an orthonormal basis of $\mathcal{V}$ such that*

$$\mathbf{A}\mathbf{u}^k = \lambda^k \mathbf{u}^k, \quad k = 1, \ldots, n.$$

Moreover, let $\mathbf{Q} = [\mathbf{u}^1 \ldots \mathbf{u}^n]$ be the matrix with $\mathbf{u}^k$ as the k-th column, and let $\mathbf{D}$ be the diagonal matrix with $D_{kk} = \lambda^k$, for $k = 1, \ldots, n$. Then

$$\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T. \tag{1.1}$$

The real number λ^k is the eigenvalue of $\mathbf{A}$ associated with the eigenvector $\mathbf{u}^k$. Such a representation is called the spectral decomposition of $\mathbf{A}$.

Proposition 1.3.2. *The spectral decomposition of* $\mathbf{A} \in Sym(\mathcal{V})$ *is given by*

$$\mathbf{A} = \sum_{k=1}^{n} \lambda^k \mathbf{u}^k \otimes \mathbf{u}^k. \tag{1.2}$$

Proof. Writing (1.1) in components

$$\begin{aligned}
A_{ij} &= (\mathbf{Q}\mathbf{D}\mathbf{Q}^T)_{ij} \\
&= \sum_{k=1}^{n} Q_{ik}(\mathbf{D}\mathbf{Q}^T)_{kj} \\
&= \sum_{l,k=1}^{n} Q_{ik} D_{kl} (\mathbf{Q}^T)_{lj} \\
&= \sum_{k=1}^{n} Q_{ik} \lambda^k (\mathbf{Q}^T)_{kj} \\
&= \sum_{k=1}^{n} (\mathbf{u}^k)_i \lambda^k (\mathbf{u}^k)_j \\
&= \sum_{k=1}^{n} \lambda^k (\mathbf{u}^k \otimes \mathbf{u}^k)_{ij}.
\end{aligned}$$

Thus,

$$\mathbf{A} = \sum_{k=1}^{n} \lambda^k \mathbf{u}^k \otimes \mathbf{u}^k.$$

□

Using (1.2), we have

$$\begin{aligned}
\mathrm{tr}(\mathbf{A}) &= \sum_{k=1}^{n} \lambda^k \mathrm{tr}(\mathbf{u}^k \otimes \mathbf{u}^k) \\
&= \sum_{k=1}^{n} \lambda^k (\mathbf{u}^k \cdot \mathbf{u}^k) \\
&= \sum_{k=1}^{n} \lambda^k.
\end{aligned}$$

And by using (1.1), we have

$$\begin{aligned}
\det(\mathbf{A}) &= \det(\mathbf{Q}\mathbf{D}\mathbf{Q}^T) \\
&= \det(\mathbf{Q})\det(\mathbf{D})\det(\mathbf{Q}^T) \\
&= \det(\mathbf{D}) \\
&= \prod_{k=1}^{n} \lambda^k.
\end{aligned}$$

1.4 Second-Order Tensors as Vectors

We consider a bijection map ψ that assigns to each pair of indices $(i,j), 1 \leq i,j \leq n$, a single index $\psi(i,j)$ that ranges from 1 to n^2:

$$\psi : [[1,n]] \times [[1,n]] \to [[1,n^2]], \quad (i,j) \to \psi(i,j),$$

with the notation $[[1,n]] = \{1,2,\ldots,n\}$. We refer to such map ψ as a correspondence map. For $n = 3$, an example of such maps is

(i,j)	$(1,1)$	$(2,2)$	$(3,3)$	$(2,3)$	$(1,3)$	$(1,2)$	$(3,2)$	$(3,1)$	$(2,1)$
$\psi(i,j)$	1	2	3	4	5	6	7	8	9

Let $\tilde{\mathcal{V}} := \mathbb{R}^{n^2}$. Then, given $\mathbf{A} \in \mathrm{Lin}(\mathcal{V})$, define $\tilde{\mathbf{a}} \in \tilde{\mathcal{V}}$ by

$$\tilde{a}_{\psi(i,j)} := A_{ij}. \tag{1.3}$$

Equivalently,

$$
\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{\psi} \tilde{\mathbf{a}} = \begin{pmatrix} a_{11} \\ a_{22} \\ a_{33} \\ a_{23} \\ a_{13} \\ a_{12} \\ a_{32} \\ a_{31} \\ a_{21} \end{pmatrix}.
$$

In particular, let $\mathbf{e}^i \otimes \mathbf{e}^j \xrightarrow{\psi} \tilde{\mathbf{e}}^{\psi(i,j)}$. For example,

$$
\mathbf{e}^1 \otimes \mathbf{e}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\psi} \tilde{\mathbf{e}}^{\psi(1,2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

Then, the set $\{\tilde{\mathbf{e}}^{\psi(i,j)}\}_{1\leq i,j\leq n}$ forms an orthonormal basis for $\tilde{\mathcal{V}}$. Therefore, any $\mathbf{A} \in \mathrm{Lin}(\mathcal{V})$ can be viewed as an n-dimensional second-order tensor $\mathbf{A} = A_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$ or as an n^2-dimensional vector

$$
\tilde{\mathbf{a}} = \tilde{a}_{\psi(i,j)}\tilde{\mathbf{e}}^{\psi(i,j)}.
$$

We observe that

$$\mathrm{tr}(\mathbf{A}^T\mathbf{B}) = \langle \mathbf{A}, \mathbf{B} \rangle = \tilde{\mathbf{a}} \cdot \tilde{\mathbf{b}} \tag{1.4}$$

and, consequently, that $\|\mathbf{A}\| = \|\tilde{\mathbf{a}}\|$. Therefore, the two-way map $\mathbf{A} \leftrightarrow \tilde{\mathbf{a}}$ is an isometry between $\mathrm{Lin}(\mathcal{V})$ and $\tilde{\mathcal{V}}$.

Chapter 2

Fourth-Order Tensors

2.1 Fourth-Order Tensors as Linear Maps

A fourth-order tensor is a map

$$\mathbf{T} : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R},$$

which is linear in each component. Associated with $\mathbf{T}$, there is a unique four-dimensional array $\mathbb{A} = \mathbb{A}_{\mathbf{T}}$ such that

$$\mathbf{T}(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{m}) = \mathbb{A} : \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{m}.$$

The four-dimensional array $\mathbb{A}$ is given by

$$A_{ijkl} = \mathbf{T}(\mathbf{e}^i, \mathbf{e}^j, \mathbf{e}^k, \mathbf{e}^l).$$

To show this, we have

$$\begin{aligned}
\mathbf{T}(\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{m}) &= \mathbf{T}(u_i\mathbf{e}^i, v_j\mathbf{e}^j, w_k\mathbf{e}^k, m_l\mathbf{e}^l) \\
&= u_i v_j w_k m_l \mathbf{T}(\mathbf{e}^i,\mathbf{e}^j,\mathbf{e}^k,\mathbf{e}^l) \\
&= A_{ijkl}(\mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\otimes\mathbf{m})_{ijkl} \\
&= \mathbb{A} : \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\otimes\mathbf{m}.
\end{aligned}$$

Conversely, given a four-dimensional array $\mathbb{A}$, define a multilinear map $\mathbf{T} = \mathbf{T}_{\mathbb{A}}$ by

$$\mathbf{T}(\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{m}) = \mathbb{A} : \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\otimes\mathbf{m},$$

for all $\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{m} \in \mathcal{V}$.

A fourth-order tensor $\mathbb{A}$ can also be viewed as a linear map

$$\mathbb{A} : \mathrm{Lin}(\mathcal{V}) \to \mathrm{Lin}(\mathcal{V}).$$

In components,

$$(\mathbb{A} : \mathbf{U})_{ij} = A_{ijkl}U_{kl},$$

for any $\mathbf{U} \in \mathrm{Lin}(\mathcal{V})$.

The set of all fourth-order tensor (the set of all linear maps from $\mathrm{Lin}(\mathcal{V})$ to $\mathrm{Lin}(\mathcal{V})$), is a linear space of dimension n^4 and is denoted by $\mathbb{L}(\mathcal{V})$.

On $\mathbb{L}(\mathcal{V})$, we define the inner product

$$\langle \mathbb{A}, \mathbb{B} \rangle := \mathbb{A} : \mathbb{B} = A_{ijkl}B_{ijkl}, \tag{2.1}$$

and the induced norm

$$\|\mathbb{A}\| = (\mathbb{A} : \mathbb{A})^{\frac{1}{2}}.$$

Any fourth-order tensor $\mathbb{A}$ has the representation

$$\mathbb{A} = A_{ijkl}\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}^l,$$

where,

$$A_{ijkl} = \mathbb{A} : \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}^l = \mathbf{e}^i \otimes \mathbf{e}^j : (\mathbb{A} : (\mathbf{e}^k \otimes \mathbf{e}^l)).$$

With respect to the inner product (2.1), the set $\{\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}^l\}_{1\leq i,j,k,l\leq n}$ forms an orthonormal basis of $\mathbb{L}(\mathcal{V})$.

The product $\mathbb{A}\mathbb{B}$ of two fourth-order tensors is defined by composition

$$(\mathbb{A}\mathbb{B}) : \mathbf{U} = \mathbb{A} : (\mathbb{B} : \mathbf{U}), \quad \text{for all } \ \mathbf{U} \in \mathrm{Lin}(\mathcal{V}).$$

In components, we have

$$(\mathbb{A}\mathbb{B})_{ijkl} = A_{ijpq}B_{pqkl}.$$

To see this, we observe that

$$\begin{aligned}
(\mathbb{A}\mathbb{B})_{ijkl} &= \mathbf{e}^i \otimes \mathbf{e}^j : (\mathbb{A}\mathbb{B} : (\mathbf{e}^k \otimes \mathbf{e}^l)) \\
&= \mathbf{e}^i \otimes \mathbf{e}^j : (\mathbb{A} : (\mathbb{B} : (\mathbf{e}^k \otimes \mathbf{e}^l))) \\
&= (\mathbf{e}^i \otimes \mathbf{e}^j)_{rs}(\mathbb{A} : (\mathbb{B} : (\mathbf{e}^k \otimes \mathbf{e}^l)))_{rs} \\
&= e^i_r e^j_s A_{rspq}(\mathbb{B} : (\mathbf{e}^k \otimes \mathbf{e}^l))_{pq} \\
&= A_{ijpq}(\mathbb{B} : (\mathbf{e}^k \otimes \mathbf{e}^l))_{pq} \\
&= A_{ijpq}B_{pqrs}(\mathbf{e}^k \otimes \mathbf{e}^l)_{rs} \\
&= A_{ijpq}B_{pqrs}e^k_r e^l_s \\
&= A_{ijpq}B_{pqkl}.
\end{aligned}$$

The transpose of $\mathbb{A} \in \mathbb{L}(\mathcal{V})$ is a fourth-order tensor denoted by $\mathbb{A}^T$ and is defined by

$$\langle \mathbf{U}, \mathbb{A}^T : \mathbf{V} \rangle = \langle \mathbf{V}, \mathbb{A} : \mathbf{U} \rangle, \text{ for all } \mathbf{U}, \mathbf{V} \in \mathrm{Lin}(\mathcal{V}).$$

We observe that

$$\begin{aligned} \langle \mathbf{U}, \mathbb{A}^T : \mathbf{V} \rangle &= \mathbf{U} : (\mathbb{A}^T : \mathbf{V}) \\ &= U_{ij}(\mathbb{A}^T : \mathbf{V})_{ij} \\ &= U_{ij}(\mathbb{A}^T)_{ijkl} V_{kl}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle \mathbf{V}, \mathbb{A} : \mathbf{U} \rangle &= \mathbf{V} : (\mathbb{A} : \mathbf{U}) \\ &= V_{kl}(\mathbb{A} : \mathbf{U})_{kl} \\ &= V_{kl} A_{klij} U_{ij}. \end{aligned}$$

Therefore, the transpose of a fourth-order tensor, is given in component form by

$$(\mathbb{A}^T)_{ijkl} = (\mathbb{A})_{klij}.$$

Equivalently, the transpose of a fourth-order tensor

$$\mathbb{A} = A_{ijkl} \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}^l,$$

is given by

$$\mathbb{A}^T = A_{klij} \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}^l.$$

The trace of a fourth-order tensor $\mathbb{A}$ is defined by

$$\mathrm{tr}\mathbb{A} := A_{ijij}.$$

We observe that the inner product can also be represented by

$$\langle \mathbb{A}, \mathbb{B} \rangle = \mathrm{tr}(\mathbb{A}^T \mathbb{B}).$$

2.2 Fourth-order tensors as second-order tensors

Let ψ be a correspondence map that assigns to each pair of indices $(i, j), 1 \leq i, j \leq n$, a single index $\psi(i, j)$ that ranges from 1 to n^2, as in Section 1.4. Then, given a fourth-order tensor $\mathbb{A} \in \mathbb{L}(\mathcal{V})$, define a second-order tensor $\tilde{\mathbf{A}} \in \mathrm{Lin}(\tilde{\mathcal{V}})$ by

$$\tilde{\mathbf{A}}_{\psi(i,j)\psi(k,l)} := (\mathbb{A})_{ijkl} = A_{ijkl}. \tag{2.2}$$

We say that the fourth-order tensor $\mathbb{A}$ corresponds to the second-order tensor $\tilde{\mathbf{A}}$ under the map ψ.

Lemma 2.2.1. *Let $\mathbb{A}$ and $\mathbb{B}$ be fourth-order tensors in $\mathbb{L}(\mathcal{V})$ and $\mathbf{M}$ a second-order tensor in Lin$(\tilde{\mathcal{V}})$. Let $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ be the corresponding second-order tensors to $\mathbb{A}$ and $\mathbb{B}$, respectively, as defined in (2.2) and let $\tilde{\mathbf{m}}$ be the corresponding vector to $\mathbf{M}$ as defined in (1.3). Then, under the map ψ:*

1. *The fourth-order tensor $\alpha\,\mathbb{A} + \beta\,\mathbb{B}$ corresponds to the second-order tensor $\alpha\tilde{\mathbf{A}} + \beta\tilde{\mathbf{B}}$, for any $\alpha, \beta \in \mathbb{R}$.*

2. *The second-order tensor $\mathbb{A} : \mathbf{M}$ corresponds to the vector $\tilde{\mathbf{A}}\tilde{\mathbf{m}}$.*

3. *The fourth-order tensor $\mathbb{A}\mathbb{B}$ corresponds to the second-order tensor $\tilde{\mathbf{A}}\tilde{\mathbf{B}}$.*

4. $\langle \mathbb{A}, \mathbb{B} \rangle = \langle \tilde{\mathbf{A}}, \tilde{\mathbf{B}} \rangle$.

5. $\|\mathbb{A}\| = \|\tilde{\mathbf{A}}\|$.

6. $tr\mathbb{A} = tr\tilde{\mathbf{A}}$.

Proof. 1. Follows from linearity.

2. In component form, we have

$$\begin{aligned}
(\mathbb{A} : \mathbf{M})_{ij} &= A_{ijkl} M_{kl} \\
&= \tilde{\mathbf{A}}_{\psi(i,j)\psi(k,l)} \tilde{\mathbf{m}}_{\psi(k,l)} \\
&= (\tilde{\mathbf{A}} \tilde{\mathbf{m}})_{\psi(i,j)}.
\end{aligned}$$

3. In component form, we have

$$\begin{aligned}
(\mathbb{A}\mathbb{B})_{ijkl} &= A_{ijpq} B_{pqkl} \\
&= \tilde{\mathbf{A}}_{\psi(i,j)\psi(p,q)} \tilde{\mathbf{B}}_{\psi(p,q)\psi(k,l)} \\
&= (\tilde{\mathbf{A}} \tilde{\mathbf{B}})_{\psi(i,j)\psi(k,l)}.
\end{aligned}$$

4. We observe that

$$\begin{aligned}
\langle \mathbb{A}, \mathbb{B} \rangle &= A_{ijkl} B_{ijkl} \\
&= \tilde{\mathbf{A}}_{\psi(i,j)\psi(k,l)} \tilde{\mathbf{B}}_{\psi(i,j)\psi(k,l)} \\
&= \tilde{\mathbf{A}} : \tilde{\mathbf{B}} = \langle \tilde{\mathbf{A}}, \tilde{\mathbf{B}} \rangle.
\end{aligned}$$

5. Follows directly from 4.

6. Using the definition of the trace of a fourth-order tensor, we obtain

$$\begin{aligned}
\mathrm{tr}\mathbb{A} &= A_{ijij} \\
&= \tilde{\mathbf{A}}_{\psi(i,j)\psi(i,j)} \\
&= \mathrm{tr}\tilde{\mathbf{A}}.
\end{aligned}$$

$\square$

It follows from Lemma 2.2.1 that the bijection map $\mathbb{A} \leftrightarrow \tilde{\mathbf{A}}$ defined by (2.2) is an isometry.

Lemma 2.2.2. *Given a correspondence map ψ and a fourth-order tensor $\mathbb{A} \in \mathbb{L}(\mathcal{V})$ with the corresponding second-order tensor $\tilde{\mathbf{A}} \in Lin(\tilde{V})$. Then under the map ψ, the fourth-order tensor $\mathbb{A}^T$ corresponds to the second-order tensor $\tilde{\mathbf{A}}^T$. Consequently, if $\mathbb{A} = \mathbb{A}^T$, then $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^T$.*

Proof. We observe that

$$
\begin{aligned}
(\mathbb{A}^T)_{ijkl} &= A_{klij} \\
&= \tilde{\mathbf{A}}_{\psi(k,l)\psi(i,j)} \\
&= \tilde{\mathbf{A}}^T_{\psi(i,j)\psi(k,l)}.
\end{aligned}
$$

$\square$

Lemma 2.2.3. *Let $\lambda \in \mathbb{C}$, $\mathbb{A} \in \mathbb{L}(\mathcal{V})$, and $\mathbf{U} \in Lin(\mathcal{V})$, with the corresponding $\tilde{\mathbf{A}} \in Lin(\tilde{\mathcal{V}})$ and $\tilde{\mathbf{u}} \in \tilde{\mathcal{V}}$, respectively. Then, $\mathbb{A} : \mathbf{U} = \lambda \mathbf{U}$ if and only if $\tilde{\mathbf{A}}\tilde{\mathbf{u}} = \lambda \tilde{\mathbf{u}}$.*

Proof. Observe that

$$
\begin{aligned}
(\lambda \mathbf{U})_{ij} &= \lambda U_{ij} \\
&= \lambda (\mathbb{A} : \mathbf{U})_{ij} \\
&= \lambda A_{ijkl} U_{kl} \\
&= \lambda \tilde{\mathbf{A}}_{\psi(i,j)\psi(k,l)} \tilde{\mathbf{u}}_{\psi(k,l)} \\
&= \lambda (\tilde{\mathbf{A}}\tilde{\mathbf{u}})_{\psi(i,j)}.
\end{aligned}
$$

$\square$

For $\mathbb{A} : \mathbf{U} = \lambda \mathbf{U}$, the scalar λ is called the eigenvalue of $\mathbb{A}$ associated with the eigentensor $\mathbf{U}$.

Chapter 3

Symmetries of Fourth-Order Tensors

Symmetry of tensors can be described by the invariance of the tensor components (or components of its multi-dimensional array representation) under a change of order of indices. Since second-order tensors are represented by two-dimensional arrays, there is only one type of such symmetry, which is given by $A_{ij} = A_{ji}$. However, for fourth-order tensors, there are different types of symmetries in which the components are invariant under certain permutations of four indices.

For fourth-order tensors, there are several notions of symmetry such as major, minor and total symmetries that represent constant under exchanging pairs of indices. On the other hand, For second-order tensors, there is one notion of symmetry that represents constant under changing the order of the two indices. These notions of symmetries are generally used in elasticity theory.

3.1 Major Symmetry

In elasticity theory, the symmetry of a fourth-order tensor is called major symmetry, when this symmetry represents invariance of $\mathbb{A}_{ijkl}$ under changing the pair of indices (i, j) and (k, l). Equivalently, for $\mathbb{A} \in \mathrm{Lin}(\mathcal{V})$, we say that $\mathbb{A}$ is symmetric (or possesses major symmetry) if

$\mathbb{A} = \mathbb{A}^T$. In components,

$$A_{ijkl} = A_{klij}, 1 \leq i, j, k, l \leq n.$$

A fourth-order tensor $\mathbb{A}$ is skew-symmetric if $\mathbb{A} = -\mathbb{A}^T$. In components

$$A_{ijkl} = -A_{klij}, 1 \leq i, j, k, l \leq n.$$

Any fourth-order tensor can be decomposed uniquely into a symmetric part and a skew-symmetric part given by

$$\mathbb{A} = \frac{1}{2}(\mathbb{A} + \mathbb{A}^T) + \frac{1}{2}(\mathbb{A} - \mathbb{A}^T).$$

The set of all major-symmetric fourth-order tensors

$$\mathbb{S}(\mathcal{V}) := \{\mathbb{A} \in \mathbb{L}(\mathcal{V}) \mid \mathbb{A} = \mathbb{A}^T\},$$

is a subspace of $\mathbb{L}(\mathcal{V})$ of dimension $n^2(n^2+1)/2$.

Using the results of Chapter 2 on the correspondence between fourth-order tensors and second-order tensors, we present and prove a spectral theorem for major-symmetric fourth-order tensors.

Theorem 3.1.1. *Let $\mathbb{A} = \mathbb{A}^T$. Then, there exist $\{\lambda^m\}_{m=1,\cdots,n^2}$, and $\{\mathbf{U}^m\}_{m=1,\cdots,n^2}$ an orthonormal basis for Lin$(\mathcal{V})$ such that*

$$\mathbb{A} : \mathbf{U}^m = \lambda^m \, \mathbf{U}^m.$$

Moreover,

$$\mathbb{A} = \sum_{m=1}^{n^2} \lambda^m \, \mathbf{U}^m \otimes \mathbf{U}^m.$$

Proof. Using Lemma (2.2.2), since $\mathbb{A} = \mathbb{A}^T$ then $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^T$. By the spectral theorem, Theorem

1.1, there exist $\{\lambda^m\}_{m=1,\cdots,n^2}$, and $\{\tilde{\mathbf{u}}^m\}_{m=1,\cdots,n^2}$ an orthonormal basis for $\tilde{\mathcal{V}} = \mathbb{R}^{n^2}$ such that

$$\tilde{\mathbf{A}}\tilde{\mathbf{u}}^m = \lambda^m \tilde{\mathbf{u}}^m,$$

and

$$\tilde{\mathbf{A}} = \sum_{m=1}^{n^2} \lambda^m \tilde{\mathbf{u}}^m \otimes \tilde{\mathbf{u}}^m.$$

Define $\{\mathbf{U}^m\}_{m=1,\cdots,n^2}$ in $\mathrm{Lin}(\mathcal{V})$ by

$$U^m_{ij} = \tilde{\mathbf{u}}^m_{\psi(i,j)} \quad , \quad i, j = 1, \cdots, n.$$

Then, by (1.4), $\{\mathbf{U}^m\}_{m=1,\cdots,n^2}$ forms an orthonormal basis for $\mathrm{Lin}(\mathcal{V})$. In addition, by Lemma (2.2.3)

$$\mathbb{A} : \mathbf{U}^m = \lambda^m \mathbf{U}^m, \;\; m = 1, \cdots, n^2.$$

Moreover,

$$\tilde{\mathbf{A}} = \sum_{m=1}^{n^2} \lambda^m \tilde{\mathbf{u}}^m \otimes \tilde{\mathbf{u}}^m,$$

which in component form is given by

$$\begin{aligned} A_{ijkl} &= \tilde{\mathbf{A}}_{\psi(i,j)\psi(k,l)} \\ &= \sum_{m=1}^{n^2} \lambda^m (\tilde{\mathbf{u}}^m \otimes \tilde{\mathbf{u}}^m)_{\psi(i,j)\psi(k,l)} \\ &= \sum_{m=1}^{n^2} \lambda^m \tilde{\mathbf{u}}^m_{\psi(i,j)} \tilde{\mathbf{u}}^m_{\psi(k,l)} \\ &= \sum_{m=1}^{n^2} \lambda^m U^m_{ij} U^m_{kl} \\ &= \sum_{m=1}^{n^2} \lambda^m (\mathbf{U}^m \otimes \mathbf{U}^m)_{ijkl}. \end{aligned}$$

□

The spectral theorem for fourth-order tensors can also be written in terms of fourth-order tensor multiplication as follows.

Theorem 3.1.2. *Given the spectral decomposition*

$$\mathbb{A} = \sum_{m=1}^{n^2} \lambda^m \mathbf{U}^m \otimes \mathbf{U}^m.$$

Define $\mathbb{D} \in \mathbb{L}(\mathcal{V}),$ *by*

$$D_{ijkl} = \begin{cases} \lambda_{\psi(i,j)}, & \textit{if } i = k \textit{ and } j = l, \\ 0, & \textit{otherwise.} \end{cases}$$

and define $\mathbb{Q} \in \mathbb{L}(\mathcal{V}),$ *by*

$$Q_{ijkl} = U_{ij}^{\psi(k,l)}.$$

Then,

$$\mathbb{A} = \mathbb{Q}\mathbb{D}\mathbb{Q}^T.$$

Proof.

$$\begin{aligned}
(\mathbb{Q}\mathbb{D}\mathbb{Q}^T)_{ijkl} &= Q_{ijpq}(\mathbb{D}\mathbb{Q}^T)_{pqkl} \\
&= Q_{ijpq}D_{pqrs}Q_{klrs} \\
&= Q_{ijpq}D_{pqpq}Q_{klpq} \\
&= U_{ij}^{\psi(p,q)}\lambda^{\psi(p,q)}U_{kl}^{\psi(p,q)} \\
&= \sum_{m=1}^{n^2} \lambda^m(\mathbf{U}^m \otimes \mathbf{U}^m)_{ijkl} \\
&= (\mathbb{A})_{ijkl}.
\end{aligned}$$

□

Such a representation is called the spectral decomposition of $\mathbb{A}$. Since $\mathrm{tr}\mathbb{A} = \mathrm{tr}\tilde{\mathbf{A}}$, then

$$\mathrm{tr}(\mathbb{A}) = \sum_{m=1}^{n^2} \lambda^m.$$

The spectral theorem for symmetric fourth-order tensors can be used to define a notion of *determinant* for such tensors as follows

$$\det(\mathbb{A}) := \prod_{m=1}^{n^2} \lambda^m. \tag{3.1}$$

Using Lemma 2.2.1 and (3.1), it follows that this notion of determinant satisfies

$$\det(\mathbb{A}\mathbb{B}) = \det(\mathbb{A})\det(\mathbb{B}).$$

3.2 Minor Symmetry

Minor symmetry is the second type of symmetry of fourth-order tensors and is defined by

$$\langle \mathbf{U}, \mathbb{A}\mathbf{V} \rangle = \langle \mathbf{U}^T, \mathbb{A}\mathbf{V} \rangle = \langle \mathbf{U}, \mathbb{A}\mathbf{V}^T \rangle, \quad \text{for all } \ \mathbf{U}, \mathbf{V} \in \mathrm{Lin}(\mathcal{V}).$$

In component form, we have

$$\begin{aligned} A_{ijkl} &= A_{jikl} && (3.2) \\ &= A_{ijlk}, && (3.3) \end{aligned}$$

for $1 \leq i, j, k, l \leq n$. To see this, we observe that

$$\begin{aligned} \langle \mathbf{U}, \mathbb{A} : \mathbf{V} \rangle &= \mathbf{U} : (\mathbb{A} : \mathbf{V}) \\ &= U_{ij}(\mathbb{A} : \mathbf{V})_{ij} \\ &= U_{ij} A_{ijkl} V_{kl}. \end{aligned}$$

and on the other hand,

$$\begin{aligned}
\langle \mathbf{U}^T, \mathbb{A} : \mathbf{V} \rangle &= \mathbf{U}^T : (\mathbb{A} : \mathbf{V}) \\
&= (\mathbf{U}^T)_{ji} (\mathbb{A} : \mathbf{V})_{ji} \\
&= U_{ij} A_{jikl} V_{kl}.
\end{aligned}$$

Thus,

$$U_{ij}(A_{ijkl} - A_{jikl})V_{kl} = 0,$$

for all $\mathbf{U}, \mathbf{V} \in \mathrm{Lin}(\mathcal{V})$, from which (3.2) follows. Moreover,

$$\begin{aligned}
\langle \mathbf{U}, \mathbb{A} : \mathbf{V}^T \rangle &= \mathbf{U} : (\mathbb{A} : \mathbf{V}^T) \\
&= U_{ij} : (\mathbb{A} : \mathbf{V}^T)_{ij} \\
&= U_{ij} \mathbb{A}_{ijlk} \mathbf{V}^T_{lk} \\
&= U_{ij} \mathbb{A}_{ijlk} V_{kl}.
\end{aligned}$$

Thus,

$$U_{ij}(A_{ijkl} - A_{ijlk})V_{kl} = 0,$$

for all $\mathbf{U}, \mathbf{V} \in \mathrm{Lin}(\mathcal{V})$, from which (3.3) follows.

The first minor symmetry is the invariance under exchange of the first pair of indices, and the second minor symmetry is the invariance under exchange of the second pair of indices.

Bibliography

[1] Aurobrata Ghosh and Rachid Deriche. 4th order symmetric tensors and positive adc modelling. In *Visualization and Processing of Tensors and Higher Order Descriptors for Multi-Valued Data (Dagstuhl Seminar 11501)*. Springer, 2013.

[2] Mikhail Itskov. *Tensor algebra and tensor analysis for engineers*. Springer, 2007.

[3] John Henry Heinbockel. *Introduction to tensor calculus and continuum mechanics*, volume 52. Trafford Victoria, Canada, 2001.

[4] Maher Moakher. The algebra of fourth-order tensors with application to diffusion mri. In *Visualization and Processing of Tensor Fields*, pages 57–80. Springer, 2009.

[5] Maher Moakher. Fourth-order cartesian tensors: old and new facts, notions and applications. *The Quarterly Journal of Mechanics & Applied Mathematics*, 61(2):181–203, 2008.