164

A COMPARISON OF UICSM AND SMSG ALGEBRA UNITS WITH ACCOMPANYING SUGGESTIONS

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TABLE OF CONTENTS

PAGE

INTRODUCTION 1

Need for Modern Curriculum 1

Modern Curriculum Developed 2

Reasons for this Report 4

Definition of Terms 5

Methods of Research and Comparison 5

PHYSICAL MAKEUP OF MATERIALS 6

SMSG 6

UICSM 6

Suggestions 8

ORGANIZATION AND ARRANGEMENT OF TOPICS 8

Topics Covered by Both Programs 8

Topics Covered by SMSG Only 9

Topics Covered by UICSM Only 9

Suggestions 9

Sequence of Topics 11

Sequence of Topics in SMSG 11

Sequence of Topics in UICSM 12

Points of Sequence that Favor SMSG 13

Points of Sequence that Favor UICSM 13

Notation 14

Notation used by SMSG but not by UICSM 17

Notation used by UICSM but not by SMSG 20

Overall Organization and Arrangement 24
PRESENTATION OF REAL NUMBERS 27
Introduction of Real Numbers 27
SMSG 27
UICSM 31
Suggestions 35
Addition of Real Numbers 36
SMSG 36
UICSM 41
Suggestions 45
Multiplication of Real Numbers 48
SMSG 48
UICSM 53
Suggestions 58
Subtraction of Real Numbers 58
SMSG 58
UICSM 59
Suggestions 64
Division of Real Numbers 65
SMSG 65
UICSM 66
Suggestions 69
PRESENTATION OF OTHER BINARY OPERATIONS 70
Concept of Order 70
SMSG 70

UICSM 72

Suggestions 74

Ordered Pairs 75

SMSG 75

UICSM 75

Suggestions 77

PRESENTATION OF SINGULARY OPERATIONS 77

Squaring 77

SMSG 77

UICSM 79

Suggestions 81

Finding a Square Root 82

SMSG 82

UICSM 85

Suggestions 87

Absolute Value 87

SMSG 87

UICSM 88

Suggestions 89

VARIABLES 91

Concept of a Variable 91

SMSG 91

UICSM 93

Suggestions 99

Expressions Containing Variables 101

SMSG 101
UICSM 111
Suggestions 127
EQUATIONS AND INEQUATIONS 130
Equations 130
SMSG 130
UICSM 138
Suggestions 147
Inequations 149
SMSG . . . . 1[illegible]
UICSM 151
Suggestions 152
Quadratic Equations 152
SMSG 152
UICSM 154
Suggestions 154
ADMINISTRATIVE CONSIDERATIONS 155
Implementing the program 155
Educational Considerations 159
SUMMARY AND CONCLUSIONS 161

LIST OF FIGURES

FIGURE PAGE

1. Locus of an Inequality for SMSG 19
2. Locus of a Compound Inequality for SMSG 19
3. Locus of a Closed Inequality for SMSG 19
4. Locus of a Half-closed Compound Inequality for SMSG . . . 19
5. Locus of a Half-open Interval for SMSG 19
6. Locus of an Inequality for UICSM 23
7. The Number Line for SMSG 29
8. Associating Positive and Negative Real Numbers for SMSG . 29
9. Association of Opposites for SMSG 30
10. A Road with Milepost Markers for UICSM 32
11. An Exercise with Real Numbers for UICSM 34
12. Addition of Real Numbers on the Number Line for SMSG . . 37
13. Adding Real Numbers for UICSM 43
14. Multiplying Real Numbers for UICSM 55
15. Commutative Principle for Multiplying Real Numbers for UICSM 57
16. Using Ordered Pairs to Subtract Real Numbers in UICSM . . 63
17. Using Ordered Pairs to Divide Real Numbers in UICSM . . . 67
18. Dividing by Zero for UICSM 68
19. Absolute Valuing for UICSM 90
20. An Arithmetic Test for Al 94
21. Establishing a Pattern Sentence for UICSM 100

INTRODUCTION

Need for Modern Curriculum

The deficiency of the average high school student in mathematics was fully realized during World War II. The inadequacy was discovered among inductees during their training for that conflict. To meet the emergency of the situation, the U. S. Office of Education appointed two committees to probe the reasons for such a marked deficiency.[1]

The government study prompted interest in curriculum changes in schools and colleges. A few colleges decided to do something about it. The University of Illinois formed a committee to study the curriculum of the secondary school's mathematics program in 1951. Another group known as the School Mathematics Study Group received its initial financial support from the National Science Foundation in the summer and started writing at Yale University. Other groups developed plans through their colleges and with financial support from the National Science Foundation. Two of these groups were developing programs at Syracuse University and Ball State Teachers College.[2]

The schools that began to implement these modern programs realized that mathematics was being taught as though nothing new had been developed in the field for over three hundred years. This, of course, was far from the truth.

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[1] Recent Trends in the Teaching of Mathematics in Secondary Schools, A Report Prepared by Americana Institute. (New York: Americana Institute), p. 2.

[2] Ibid., pp. 2-3.

Mathematics is the fastest growing science today. It is the only field of learning in which all the major theories of the last two thousand years are still valid.[3] In addition to the many new branches of mathematics that have been developed, many old branches have heen applied to modern situations.

Modern Curriculum Developed

The educators realized that much had heen developed in the field in modern time and expected it to be introduced into the curriculum of the present day school. The programs developed by these colleges and universities, that have been mentioned, have been put to the test and found to be worthy. It is impossible to say that they are the final answer. This could hardly be true of any curriculum devised for a changing field. However, the innovations have updated the mathematics curriculum.

One teacher of the Illinois program spoke concerning what he was teaching to the Regional Orientation Conference of Administrators and Supervisors at Topeka, Kansas. In this speech he made the following comment:

> Our mathematics department wanted to try out the materials, and so did the administration. We were all of the opinion that the students couldn't learn any less mathematics than they were learning in the traditional classes, and perhaps they would learn more.[4]

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[3] Ibid., p. 3.

[4] Howard Maston, "A Teacher's View of UICSM," UICSM Newsletter, 3:20-24, January, 1961.

The time that they decided to become a pilot school for the Illinois program was in 1955. The first pilot schools were initiated about this time.

It is this paper's contention that nearly any of the present day modern experimental programs or the commercial texts that have evolved out of them, are better than what was being taught before their development. This paper's author feels, as do most teachers who have studied the modern approaches, that the students derive more from the modern classes than comparable students did under the traditional courses that were offered previous to the modern texts.

Look magazine had a general article on modern mathematics. They stated a comparison of the old and the new mathematics which follows:

> Think of the math student as something like a beginner in driving school. The traditional training produced a more or less competent driver--he knew what buttons to push to make the math machine go. The new text tries to produce a good driver who also understands the internal-combustion engine. He will know not only how to work the machine, but why it works as it does. In short, say curriculum reformers, school math will henceforth be a thinking man's subject.[5]

In the fall of 1957, Russie did more to accelerate curriculum revision in mathematics and science in the U. S. than the educators themselves had been able to accomplish for some time. The placing in orbit of Sputnik I set in motion a great clamor for more up-to-date curriculums to produce more and better qualified scientific personnel.

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[5]David R. Maxey, "Why Father Can't Do Johnny's Math," *Look*, 227:22, November 5, 1963.

Some of the programs were written before Sputnik I and the financial support furnished for the writing of others followed very soon. As the programs were organized, they were tested and soon the effect was felt over the whole country.

Today nearly all of the texts being written are incorporating to some degree the modern topics and methods developed by the first pilot programs. In just the first few years that they have been in practice, modern programs have enolved considerably. Most of the programs were not in hardbound books to allow changes from year to year. Now the experimental programs have developed a text that can be bound in the conventional manner. Nearly all of the publishing companies have come to issue hardback books in modern mathematics for all grade levels.

Reasons for this Report

From what this author has observed, the School Mathematics Study Group probably has gained more popularity than any other experimental program. The author had some firsthand experience with the study of the Illinois program. In his estimation it was the finest program that he could imagine and wished to compare the two programs on a very detailed basis to determine which was the better in general and what points might be stronger in each program.

The author feels that this paper could be valuable to the curriculum committees across the country that might be trying to decide on a modern math program, and even more valuable to the districts trying to decide between the two programs in particular.

The work has been of great benefit to the author and was a rewarding experience in general.

Definition of Terms

For the remainder of the paper the two programs will be referred to by the initials of their title. UICSM will refer to the University of Illinois Committee on School Mathematics high school mathematics program.[6] SMSG will mean School Mathematics Study Group algebra program.[7]

Methods of Research and Comparison

There has been a direct approach used in most of the research for this paper. Copies of the material written for the two plans' first year high school course were secured from the representative presses as a basis for comparison. These books have been read and the accompanying commentaries studied. The various parts of the programs have been separated and put together in a logical sequence of order for comparison by gathering the related topics and subsequent supporting points from among the various topics of each volume of copy. Movies of actual classroom presentation of pertinent points on the first year course of the UICSM work have been observed. Newsletters from both publishers have been read. These newsletters bring any late additions to the program,

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[6] University of Illinois Committee on School Mathematics, High School Mathematics Unit I (New Haven: Yale University Press, 1960, Revised Edition), title page.

[7] School Mathematics Study Group, First Course in Algebra (New Haven: Yale University Press, 1961), title page.

news of progress of the plan, successful methods that have been reported by participating teachers, discussion on ways of presenting various points, and other various items. Interviews of two teachers participating in each program were helpful to the research.

The author's personal preparation in the field of mathematics includes 24 semester hours of traditional undergraduate mathematics, 16 semester hours of modern graduate mathematics, a three-hour graduate credit course on UICSM first year program and five years experience teaching junior and senior high school mathematics.

PHYSICAL MAKEUP OF MATERIALS

SMSG

SMSG has two workbook-type units for the student and two for the teacher in the form of a commentary. The commentary serves the purpose of answer book and teaching guide. The commentary is a great help to the teacher. The commentary covers: methods of approach for most topics, an overview of the extent of the course, the organization of material, and a correlation of future topics with present topics.

UICSM

UICSM came in four separate units that cover a year's work. This was the format during the first years of experimentation. UICSM is now going to a hardback book put out by a commercial company. Because of the change in binding, the four volumes will probably be put into one book this coming school year.

UICSM has a commentary coordinated with their program also. It is constructed in a different fashion than the separate volume of the SMSG. The UICSM program has come to the student in a bundle of looseleaf sheets. These sheets were placed into a notebook and served as a very informal text. The teacher's units are constructed of the student's units plus different colored sheets of looseleaf paper that are inserted between the pages of the students' unit. These sheets vary from the student's in that they are green and do not have the upper corner to the outside edge of the page. This leaves a triangle, about two inches on each edge as a convenient way to turn from sheet to sheet of the students' work without thumbing through the many pages of commentary. The author feels this type of commentary may still be used even after the implementation of the hardback edition for the student.

A strength of the UICSM commentary arrangement arises from the freedom of movement it allows the teacher. Since the teacher may remove the sheets individually from the whole volume, he is no longer shackled to the book and is free to use the blackboard and other classroom facilities.

Although the informality of the make-up of both texts has weighty advantages, it is realized that there is a financial necessity to put the volumes into a hard-bound book that could be used for a normal period of time for a normal investment in materials. This author feels that UICSM will lose some of its attractiveness, particularly since the workbook type of construction provides space for the student to do his work right in the text.

Suggestions

The UICSM commentary should, if possible, remain as it was since it was a definite advantage to have the commentary for each page inserted in the book facing the page it covered. SMSG must surely lose something by having the commentary in another book. It is feared that many teachers would not use it as much as they should because it was not right there for immediate reference.

In either program the commentary should be utilized fully. They are planned to help the teacher to understand the underlying principles and the object of the particular lesson and its relation to the rest of the course. It would not be impossible for a teacher to miss the whole point of what the lesson was trying to accomplish if he ignored the commentary for even a short period of time.

A hard-bound book for the student and a looseleaf sheet type commentary for the teacher would be ideal in both cases. It would give the teacher the most utilitarian text he could have and would probably encourage the authors to include more mathematics background and teaching advice. The hardback book for the student would keep the cost within the range of all or at least most school districts.

ORGANIZATION AND ARRANGEMENT OF TOPICS

Topics Covered by Both Programs

The topics that are common to both courses of study come close to covering both programs. The sequence of presentation varies considerably and will be discussed later.

1. Use of numerals
2. Sets
3. The number line
4. Real numbers
5. Rational numbers
6. Properties of addition
7. Properties of multiplication
8. Properties of subtraction
9. Properties of division
10. Properties of order
11. Opposites
12. Variables or pronumerals
13. Equations
14. Inequations
15. Simplification of equations or inequations
16. Expressions
17. Simplification of expression
18. Graphs of equations
19. Graphs of inequations
20. Solving problems
21. Factors
22. Exponents
23. Factoring
24. Radicals
25. Simplification of radicals
26. Systems of equations
27. Solving of quadratic equations

Topics covered by SMSG only

1. Solving quadratic equations by completing the square
2. Functions

Topics covered by UICSM only

1. Isomorphism
2. Punctuation expressions
3. Lattices
4. Operations on sets

Suggestions

Since the topics covered by only one program cannot be directly compared, they shall be discussed as part of the suggestions for this section.

It was surprising to find that neither program used the quadratic formula for solving quadratic equations. Even though SMSG never introduced the quadratic equation, they accomplished the same thing by solving quadratic equations by completing the square. The omission has deprived the student of nothing since the quadratic equation is just a formula derived from solving a general quadratic equation by the mathod of completing the square. This topic is included in a later unit.

UICSM includes neither the quadratic equation nor completing the square. The reason for this is that UICSM is trying to develop a system for schools to use through the complete four-year program. It has developed eleven units of integrated study that are to be used as a group. Each of the units is to be started when the previous unit is completed. These eleven units are constructed to be a high school course in mathematics. They are also working on further units that will make it possible for the program to be started in the eighth grade.

SMSG's last chapter treats the concept of functions. This is covered by the UICSM program in a later unit. This is possibly a good unit for the SMSG work. It will give the student a better concept of a function for his later courses. Their feeling on this topic is "the idea is simple and is involved implicitly in our most elementary consideration."[8] This must be much the same thinking as UICSM since they include the topic in their next unit.

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[8]School Mathematics Study Group, Teacher's Commentary. (New Haven: Yale University Press, 1961), p. 573.

UICSM feels that the conventional or traditional method of teaching algebra, and real numbers in particular, is quite ambiguous. They have gone to some length to avoid any ambiguity until it is definitely understood what the ambiguity is and how it should be handled when it occurs. Because of this point in their philosophy, they introduce certain notation and topics, like isomorphism, to explain the ambiguity and what to do with it. Certain algebraic expressions are quite ambiguous if not properly punctuated with grouping symbols. They teach the student how to punctuate and how to handle the expression should grouping symbols not be present.

SMSG handles these two topics of punctuating expression and the ambiguity of some real humbers, but they do not develop them to the degree UICSM does.

UICSM develops the study of graphs by the number lattice. This is a logical way to approach the idea of a coordinate system for graphing. It is also more consistent with previous developments toward various topics that they use.

The operations on sets that UICSM includes are the conventional widely used operations that are part of many modern programs in use today:

Sequence of Topics

Sequence of Topics in SMSG

1. Sets
2. The number line
3. Numerals
4. Variables
5. Sentences

6. Properties of operations
7. Open sentences and English sentences
8. Real numbers
9. Properties of addition
10. Preperties of multiplication
11. Properties of order
12. Subtraction for real numbers
13. Division for real numbers
14. Factors.
15. Exponents
16. Radicals
17. Polynomial and rational expressions
18. Truth sets of open sentences
19. Graphs of open sentences in two variables
20. Systems of equations
21. Systems of inequations
22. Quadratic polynomials
23. Functions

Sequence of topics in UICSM

1. Numerals
2. Distance and direction (introduction of real numbers)
3. Addition of real numbers
4. Multiplication of real numbers
5. Numbers of arithmetic and real numbers (isomorphism)
6. Punctuating expressions (grouping)
7. Principles for number of arithmetic
8. Principles for real numbers
9. Inverse operations
10. Subtraction of real numbers
11. Opposites
12. Division of real numbers
13. Comparing numbers (order)
14. The number line
15. Sentences
16. Pronouns and pronumerals (variables)
17. Generalizations
18. Simplification of expressions
19. Theorems and basic principles
20. Oppositing and subtracting
21. Division
22. Comparing real numbers
23. Graphs and coordinates
24. Solution set of a sentence
25. Graph of a sentence
26. Equations
27. Equivalent equations
28. Transforming a formula

29. Solving problems
30. Quadratic equations
31. Solving inequations
32. Square roots
33. Lattices
34. The number line
35. Graphs of formulas
36. Factors
37. Exponents
38. Factoring

Points of sequence that favor SMSG. One point that makes SMSG sequence attractive is that the program more closely follows a traditional algebra course. This would make it more possible to handle transfers from a traditional algebra course. The author is not sure that this would be possible, but it is probably more feasible to put a transfer student into the SMSG program than into the UICSM plan. The close following of a traditional course would probably tend to attract those conservative teachers more firmly grounded in traditional methods. SMSG has developed individual units that are one year courses in themselves. This prompts many schools to use one section or sections and not others of the high school units.

Points of sequence that favor UICSM. The strong point for UICSM is that they are arranging a four year or more high school mathematics curriculum for the college capable student. They are thus allowed to put any topic into any unit that would best serve the course and the student's progress. It is the feeling of this author that the development of the real number as one of the earliest topics is a very strong point for UICSM. As has been mentioned previously, the author feels that a strong basis in the concept of real numbers is the most important point of any algebra program.

Notation

One of the big differences between the modern mathematics and the traditional mathematics which was being used nearly exclusively a few years ago, is the notation and terminology employed. Many algebra texts of the traditional courses mentioned terms like "associativity" and "commutativity," but did not pursue the use of them to any degree worth mentioning. The newer curriculums have included topics not widely covered before and in many cases needed notation different from what had been traditionally used to explain these topics. In addition to the explaining of new topics, many traditional points of study were given a new treatment. These more up-to-date approaches at times required different notations as well.

Most of the modern curriculums in math plan to lead the student to the same end that a traditional course would. The modern curriculums must equip the student to read and understand a book written by scholars before the use of modern notation. For these reasons the notation used many times will become a vehicle to implant an understanding firmly and then will be eliminated by showing that it is not really essential. A good example of this use and eventual elimination of the notation is the raised + and - symbols used in teaching real numbers.

Both programs adhere to the conventional notation in nearly all cases. The regular symbols such as $=$, $\neq$, $<$, $>$, $\leq$, $\geq$, $\nless$, $\ngtr$, $+$, $-$, X, and $\div$ still carry the same meaning. The two plans both use the same abbreviations for multiplication. The phrase "a time b" is still written "ab". The product of two and three is still written "2·3". Exponents and radicals are used in the same manner. Two vertical lines, one before an expression and one after, still are used as a symbol for absolute value. All of the traditional symbols remain in use and occasionally acquire an extended meaning.

A point that can be confusing and lead to much ambiguity is the discussing of numerals. To avoid confusion the programs use quotation marks to mean a numeral, and a numeral without quotations means a number. Here are some examples: "John" is a word. John is a boy. Hand me that large "4". This small "4" doesn't fit. UICSM goes to more length to put this point across. SMSG uses it but doesn't really go into the fact that they are using it or the reasons it is used.

Real numbers often prove ambiguous. Traditional courses indicate a positive seven with this symbol, +7. They would also write a negative four in a like manner, -4. It is quite apparent that a beginning algebra student might have trouble thinking clearly of the + and - symbols, as they must stand for two ideas now. First they indicate whether to add or subtract a number. Secondly, they indicate whether a number is positive or negative.

To avoid this confusing issue, UICSM and SMSG to a point use a raised small + to mean positive and a similar symbol to mean negative. But as has already been mentioned, this sort of peculiar notation must be eliminated eventually so that the student can pick up where the students of a traditional course are, or read meaningfully a book written previous to use of such notations. SMSG uses only the ($^{-}$)

symbol and only uses it for 12 pages. A treatment of opposites and a definition of the use in regular symbolism relieves the need of it. UICSM introduces the $(^{+})$ and $(^{-})$ symbols on page seven and treats the operation on real numbers with these as aids until page 88 where all of the ambiguity has been handled and conventional symbolism can be used with as much meaning.

The newer curricula developed for mathematics seem to be unified in giving reasons for steps that have been taken in solving a problem. These steps are usually furnished in the form of an abbreviation using the first letters of each word of the principle or property. CPA or cpa would mean commutative property of addition in SMSG. The same abbreviation in UICSM would be commutative principles of addition. The properties of SMSG and the principles of UICSM are very closely related. A list of the type of properties or principles that are developed by the two could be furnished, but it would not serve any significant purpose.

It is evident in the comparison that UICSM uses the principles for each step much more extensively and through a great deal more material than does SMSG. Both the plans develop the basic principles for numbers of arithmetic and then extend them to apply to the real number and finally to polynomial expressions. Then they both use the basic principle to prove the theorems that are used in general simplification of expression and equations.

This paper's author feels that these properties or principles are the most important aspects of the modern math approaches. It is the author's opinion that this logical step-by-step process and the subsequent listing of reasons for each step seems to give the student a better basis in mathematics. The formal proofs of geometry will come easier as will all problems that require logical and deductive reasoning. Since students are exposed to inductive reasoning in some of the proofs of basic principles, inductive reasoning should be improved also.

One other notation that is common to both programs is the notation used in graphing of inequations in two variables. The graphing of these general inequations involves regions of the number plane instead of a series of points that form the graph of an equation. In the case of a strict inequation in two variables the region to one side or above or below a particular line, the line that would be the graph of the corresponding equation, is composed of all the points that would satisfy the inequation. But the line itself is not included in the solution. The notation in each program is to use a dashed line for the graph of the corresponding equation and to shade the rest of the region. The use of the dashed line is to indicate that the line itself is not included as a part of the solution. In the case of greater than or equal to and less than or equal to inequations, the line of the corresponding equation is included as a solid line to indicate that it too is part of the solution. Both plans use the standard pair of braces as an indication of a set.

Notation used by SMSG but not by UICSM

SMSG uses only a few terms that UICSM does not use in the way of notation. SMSG uses standard notation and terminology in developing the coordinate axis. UICSM has used a more modern terminology. Because of this modern terminology there is a difference on this point.

In the graph of inequations in one variable, intervals, and the inequations less than or equal to and greator than or equal to, the programs go their separate ways for their notation. The graph of an inequation for SMSG is started with a small circle around the number that is not included in the solution. Figure No. 1 shows this type of notation for $x>2$. Figure No. 2 shows what an inequation is where the solution lies between two points, $3<x<6$. In tho case of the less than or equal to and the greater than or equal to inequation, the end-point is included in the solution. This is shown by notation as in figure No. 3 for $x\leq 4$. When an inequation or an interval is to be denoted and one endpoint is included but not the other, the result is similar to figure No. 4 for the inequation and figure No. 5 for the interval. And in the final interval and inequation that would include both endpoints, such as $5\leq x\leq 8$, each would have a dot at each endpoint to denote the inclusion of the same in the solution.

SMSG uses one symbol that this paper's author had never seen before. It may possibly be a standard notation that had never come to his attention. It is the notation for approximately equal to, "$\approx$".

Since SMSG does have a chapter on functions that UICSM does not have in their first four units, the notation would not be found in UICSM. The notation that they use is similar to the standard notation for functions. The notation for the function of x is "f(x)." It is read "f of x." Other similar notations are found for different kinds

Figure 1. Locus of the inequality for SMSG, $3<X$.*

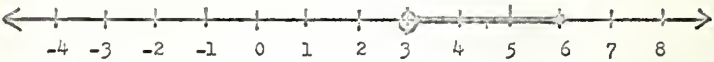

Figure 2. Locus of the compound inequality for SMSG, $3<X<6$.*

Figure 3. Locus of the closed inequality for SMSG $X\geq 4$.*

Figure 4. Locus of the half-closed compound inequality for SMSG, $-1<X\leq 4$.*

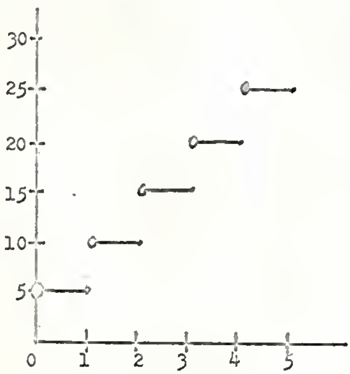

Figure 5. Locus of a half open interval for SMSG.**

*SMSG, p. 50.

**Ibid., p. 512.

of functions. Sometimes the function is described in more than one part. When this occurs a notation of the following type is used:

$$g(x) \begin{cases} -1, & x \quad 0 \\ 0, & x = 0 \\ 1, & x \quad 0 \end{cases} \qquad \text{or,} \qquad h(x) \begin{cases} x, & x \quad 0 \\ -x, & x \quad 0 \end{cases}$$ [9]

Notation used by UICSM but not by SMSG

There are several notations that UICSM uses that SMSG does not. One of the earliest differences comes in the establishment of principles for number of arithmetic and carries on through real numbers and expressions. This difference is that UICSM distinguishes between the right and left distributive principles while SMSG does not. The notation for the right distributive principle for multiplication over addition is dpma. The left distributive principle for multiplication over addition is denoted by ℓdpma. The script ℓ is used so there will be no possibility of confusing it with the numeral "1."

The next notation used exclusively by UICSM is the opposite sign. It is an elongated minus sign. It is just what the name implies, a symbol to mean the opposite of a number. SMSG uses the standard minus sign for this. UICSM tries to distinguish between the symbol used for subtraction and the symbol used for the opposite of a number to reduce the ambiguity associated with using one symbol for both.

When developing the use of a variable, or pronumeral as UICSM calls it, the use of a paper with a hole in it is employed. Several second sheets are made that fit under the first sheets so that numerals

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[9] SMSG, op. cit., pp. 520-521.

appear through the holes in the paper. The students can then answer the question implied by the equations as true or false. Then another second sheet can be placed under the one with holes in it. After several second sheets have been used, the idea that the hole in the paper will allow any sort of numeral to take its place and that numeral makes the sentence or inequation or equation true or false becomes evident to the student. The second phase of this idea is realized when different shaped geometric symbols are used where the holes were before. This usage develops the concept the holes started to a greater degree. By this time the students are used to different kinds of figures being able to take the value of any number and that the same shape figure will have the same value no matter what the various substitutions might be. These steps using holes in the paper and then geometric shapes to contain numerals in equations lead the student to a very easy acceptance of a pronumeral.

The SMSG text always names the domain of all numbers in a particular set by the phrase "for every number." This usually means the set of all real numbers. UICSM introduces their readers to a different notation for the phrase "for each." This notation is the universal qualifier symbolized by "$\forall$." The phrase "for each x" is written $\forall x$.

Graphs of inequations in one variable, the less than or equal to and greater than or equal to inequations in one variable and intervals are a little different for UICSM. The notation for the single variate greater than or equal to and less than or equal to inequations are denoted the same as SMSG, using the dot to include the equal part of the sentence and an arrow in the proper direction. On a strict inequation the arrow to the proper part of the number is included as in SMSG, but instead of the small circle around the endpoint not included, UICSM uses a bracket facing away from the solution. Figure No. 6 gives an example of this type of notation.

Intervals are a little different also. UICSM uses a line segment over a pair of numbers for an interval that does not contain either endpoint. An interval that contains one endpoint is denoted by an ordered pair of numbers with a line segment over the pair and a dot on the left end of the line. The ordered pair is always listed with the included endpoint as the first component. An example of this follows: the interval $-4 < x \leq 5$ is written $\overline{5, -4}$.

The interval with both endpoints included is similar to the open interval with no endpoints. The two endpoints are written as in the other interval and a line segment with a dot on both ends is written over it, as "$-4 \leq x \leq 5$ written as an interval is $\overline{-4, 5}$."

The inequation written as an interval is denoted below, and is read "ray four zero," "ray four negative z," or "ray four" followed by any number less than four:

$x < 4$ written as an interval $\overrightarrow{4, 0}$, $\overrightarrow{4, -2}$, $\overrightarrow{4, -10}$, $\overrightarrow{4, 3}$
to name a few.

The less than or equal to or greater than or equal to inequations are written as an interval, as is illustrated below and is read the same as the strict inequation interval is:

$x \leq 4$ written as an interval is $\overrightarrow{4, 1}$, $\overrightarrow{4, -2}$.

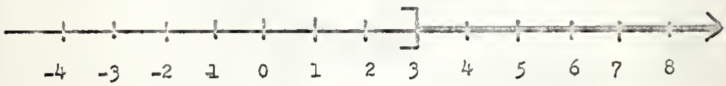

Figure 6. Locus of the inequality for UICSM $X > 3$.*

*UICSM 3-12.

When the entire number line is the solution, any ordered pair with a two-headed arrow over it is the notation for the answer.

When denoting a rational number, SMSG never says how to denote a rational number in terms of a repeating decimal. UICSM denotes it by placing a bar over the last full cycle written.

UICSM develops graphs from the topic of lattices. The ordered pair of the lattice gives birth to the terms of "first component" and "second component." Thus, a very logical choice for names of the axis of a graph, is first component axis and second component axis.

Since UICSM deals with set operation and SMSG does not, there would naturally be some notation that UICSM would need which would not be necessary for SMSG. The much used symbols for intersection of two sets and for union of two sets are used by UICSM.

UICSM also uses another set notation that SMSG does not. For the solution set of a sentence in the form of an equation or an inequation, they use a standard notation. The set of all x such that $x + 4 < 9$ is written $\{x: x + 4 < 9\}$.

As far as suggestions on notation are concerned, it is not for this author to say that either plan should use different notation. Both programs use good sound notation for what they present. The only suggestion that this paper would make is that SMSG incorporate more of the set operations that would require the adopting of more set notation.

Overall organization and arrangement

Both programs are quite good for what they try to accomplish and their philosophy of presentation.

SMSG leans toward the traditional mathematics in that it presents an algebra course for a first-year high school student. They cover the same material that would be covered in a traditional algebra course, plus a few other topics. The main difference is the method of presentation. This is good in that the student who has one year and no more of the course will be prepared to succeed in nearly any program.

The beginning of the year's work in algebra by discussing sets should stimulate a new interest in math by encountering a fresh topic. By introducing the use of a variable early, the student probably feels more assured that he is doing what an algebra course should do. Having proofs all through the work should help the student to build a deductive reasoning process that is necessary for modern mathematics. Establishing properties on the foundation of the numbers of arithmetic first is a sound idea. The real number properties come much easier with this previous experience. Constant application through worded exercises is a sound practice. The more practice in the word problems, the more proficiency the student will acquire in working them.

UICSM is arranged much different than SMSG for several reasons. UICSM does not try to complete algebra in the first year. They have a four-year sequence and the arrangement of the topics is made in that light. They have arranged the sequence in a way that they feel is best.

UICSM starts with a development of the real numbers. They first create a need for real numbers, then define them, and proceed to develop the operations for them. After they have developed the principles for numbers of arithmetic, they do the same for the real numbers. There is a pattern of creating a need for most major points before they are defined or introduced. This follows with variables after real numbers. UICSM develops the manipulation of variables to a great extent before presenting equations. Equations and inequations are handled very easily by the students, even on their first efforts. There was a statement made by a teacher, interviewed in connection with this paper, about her students' first effort with equations: "The students were able to solve equations the first time they tried them. We came to them and I gave the instruction to find the roots and they could do it."[10]

Set notation is developed later in the year by UICSM than by SMSG. By this placement the students are ready for a fuller development of the topic. UICSM uses more set notation and applies it to more material.

As has been pointed out, each program presents a different kind of approach. It would be hard to suggest either program change their format a great deal, for that would change the approach which is part of the basic philosophy of each program. But the author of this paper likes the organization and arrangement of the UICSM program. UICSM goes to more lengths to create need for the topics that they present and allowsthe student to discover for himself the underlying concepts of mathematics.

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[10]Comment by Paula Howard, O.B.S., personal interview.

PRESENTATION OF REAL NUMBERS

Introduction of a real number

SMSG. The first work with real numbers in SMSG is on the number line. The number line itself is introduced in tho first chapter. Its uses are many and varied up to the point of introduction of a real number. The number line is used to graph sets, show operations on numbers of arithmetic, graph the solution of inequations and other topics to this point. Although now the students are familiar with the number line, they have never used any more than the non-negative part of the number line. The next look the students have of the number line comes when they develop the negative part of it. At that time a line is put on the paper and the student is told to "choose two distinct points on the line and label the point on the left with the symbol 0 and on the right with 1."[11] Then that interval is used as a unit to develop the scale to the right of one. In the book the illustration of the zero is put in the middle of the paper to allow the student to imagine, possibly, that there may be a scale to the left of 0 also. At this time the number line is used to introduce non-negative rational numbers. The number line is used with the zero nearly to the left hand margin from then until the development of real numbers.

When real numbers are to be introduced, the picture of the number line is printed much as it was in the first illustration of it with the

[11] SMSG, op. cit., p. 7.

zero point in the middle of the page. The question is implanted in the student's mind about numbering to the left. The line extends to the left and the right without end, so how should the part to the left of zero be labeled? It is at this time that they give them the labeling. "...As before, we use the interval from 0 to 1 as the unit of measure, and locate points equally spaced along the line to the left."[12] (Figure No. 7) It is at this point that the raised (⁻) negative sign is introduced by SMSG. The student is now told these labels will be negative numbers. It is explained that numbers the same distance from zero would be associated. (Figure No. 8)

At this point several sets are defined. The set of all numbers is defined as the set of real numbers. The numbers to the left of zero are defined as the negative real numbers. The numbers to the right of zero are given to be the positive real numbers. The set that had been called the numbers of arithmetic are called non-negative real numbers. And the set of all non-negative whole numbers together with the negative whole numbers is the set of integers. Rational and irrational numbers are also discussed at this time.

After a section on order on the number line, SMSG introduces the idea of opposites. The number line is drawn again and the successive units to each side of zero are connected with a two headed arrow in the shape of a semicircle. (Figure No. 9) Since these numbers are an equal distance from zero and opposite sides of zero, the natural name

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[12] _Ibid._, p. 97.

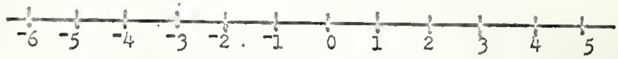

Figure 7. The number line for SMSG.*

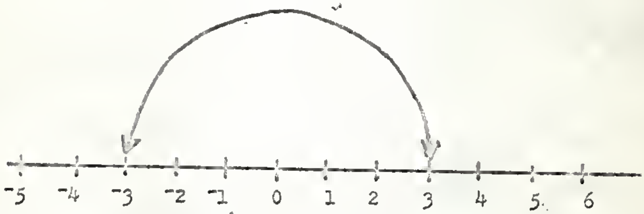

Figure 8. Associating positive and negative real numbers in SMSG.**

*SMSG, op. cit., p. 97. **Ibid., p. 98.

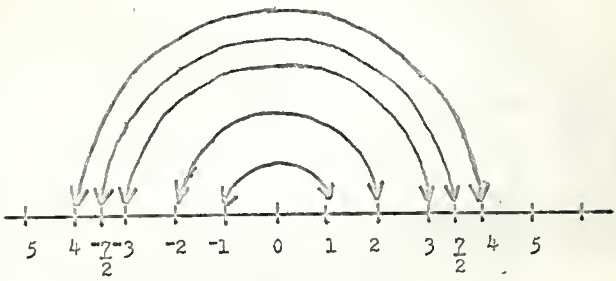

Figure 9. Association of opposite for SMSG.*

*<u>Ibid</u>., p. 108.

of opposites is given them. "$^-2$ is the opposite of 2. $\frac{1}{2}$ is the opposite of $^-\frac{1}{2}$. 0 is the opposite of 0."[13] These statements are then rewritten with symbols replacing the words. "Is" is replaced with an equal sign. "The opposite of" is replaced with the lower dash, similar to a subtraction symbol. The same sentences are then written as follows: "$^-2 = -2$. $\frac{1}{2} = -^-\frac{1}{2}$. $0 = -0$."[14] These sentences are read the same as the earlier quoted sentences were. It is then pointed out to the student that since $^-2$ and -2 are equal there is no need to write the raised dash for negative, and the regular dash they had been used to using could be used all the time. It is to be read as "the opposite of."

Since the positive real numbers and the numbers of arithmetic were, for all practical purposes, defined to be the same, there is no need to worry about positive symbols. These were never introduced.

The student is now using the same symbols for real numbers that a traditional course would use. For only twelve pages is the different notation used for a negative number.

UICSM. The very first day in the regular text, UICSM introduces real numbers. The student is given the illustration of a road with sign posts every mile. (Figure No. 10) .Trips from one sign post to another are given the names of trips-to-the-east, or trips-to-the-west. An example of this would be as follows. A trip from Q to M is a 2-mile-trip-to-the-east. A trip from W to A is a 6-mile-trip-to-the-west.

The set of the numbers of arithmetic is mentioned here as the

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[13] Ibid., p. 108. [14] Ibid., p. 109.

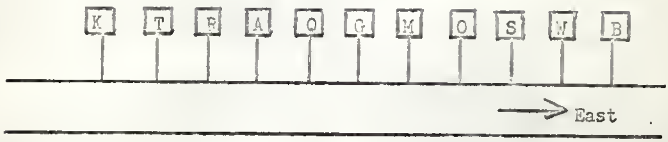


Figure 10. A road with milepost marker for UICSM.*

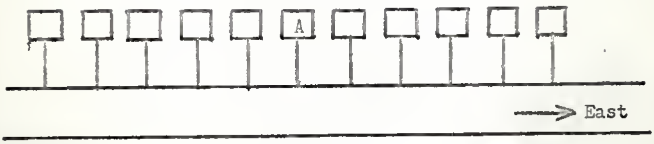
*UICSM 1-2.

numbers they had used previously and it is shown that they are not adequate for this kind of a situation. The number should show a distance as before, but it should also show the direction that the trip is to be made.

The names of trips-to-the-east and trips-to-the-west are then shortened to a number with the appropriately directioned arrow over it, "$\overrightarrow{2}$" for a 2-mile-trip-to-the-east, and a "$\overleftarrow{3}$" for a 3-mile-to-the-west-trip. These are then called a right 2 and a left 3. The students are then asked to use the road with the sign post in writing several real numbers for trips. The same idea is developed along a north and south road with the same arrowed numerals used to mean a specified distance and direction.

The next exercise given is one of the most intriguing that this author has ever seen. The students are given a road with blank sign post markers except for one. They are given a list of starting points and ending points of several trips and the real number that would measure that particular trip. They are then to fill the markers to make the proper map of the road. (Figure No. 11) This author started over at least twice, but had no thought of giving up, only that of making sure it was correct. Parents of students who have tried this have remarked at how interesting it was. The students like it too.

After a section on addition, using the arrowed numbers, the student is introduced to the more standard notation. They are told that the symbols they have been using are fine, but other people probably don't use them. The student played a part in selecting a symbol to



1. A trip from A to G is measured by $\overleftarrow{4}$.

2. A trip from J to G is measured by $\overleftarrow{2}$.

3. J to B, $\overleftarrow{3}$ 4. I to E, $\overleftarrow{2}$

5. F to C, $\overleftarrow{2}$ 6. G to D, $\overrightarrow{5}$

7. L to H, $\overleftarrow{2}$ 8. B to A, $\overrightarrow{5}$

9. A to F, $\overrightarrow{5}$ 10. E to C, $\overrightarrow{6}$

11. L to F, $\overrightarrow{1}$ 12. K to E, $\overleftarrow{9}$

Figure 11. An exercise with real numbers for UICSM.*

*UICSM 1-4.

use. From the left direction arrow comes the raised ($^-$) negative sign and from the right direction arrow comes the raised ($^+$) positive sign. These symbols are then used for about eighty pages of material.

The positive symbol is eliminated before the negative symbol. Several pages of work on tables that contain either an addition isomorphism or a multiplication isomorphism are given. The student becomes interested, as he uses each table, to see if it will be an addition or a multiplication isomorphism. The student is never given the term isomorphism, but realizes what he is doing. Soon they want to find one that will work for both addition and multiplication. The table that has the number of arithmetic and the positive real numbers in it meets this requirement. Since these two sets of numbers act alike they are able to use either. Thus the positive symbol is dropped, but the ambiguity of the resulting numeral has been taken care of. This is done on page 31. On page 86 new names for negative numbers are developed. It is mentioned that -9 is the opposite of a positive nine. But since negative nine, written $^-9$, is also the opposite of a positive nine the two must be the same. $^-9$ was the name that they had been using up to that time. The name that is to be used now is -9. They do not completely discard the raised symbols. These are kept for stressing points throughout the rest of the course.

Suggestions. SMSG gives a good clear definition of the real number. UICSM lets the student explore and develop for himself the idea before definitely telling what it is. This writer feels that the longer use and added attention to the removing of the raised notation gives UICSM the edge in their presentation of the standard symbols. This will be pointed out again in the discussion of various operations of the real numbers.

Addition of real numbers

SMSG. Once again the SMSG authors use the number line in presenting a topic. This time it is used in presenting addition of real numbers. The uses of the number line are: first, to show the addition of two positive real numbers; secondly, the addition of a positive number and a negative number when the absolute value of the positive number is greater than the absolute value; third, the addition of a negative number to zero; fourth, the addition of a negative number and a positive number, when the absolute value of the negative number is greater than the absolute value of the positive number; and finally, the addition of two negative numbers. Figure No. 12 shows the five cases just listed.

It is explained to the student just how these illustrations apply to the addition of real numbers. Each problem starts at zero and moves a certain number of units in the proper direction according to the real number given first. The second real number will have the same treatment with one exception, its progression will start where the first real number movement ended. This is carried on until each real number to be added has had its movement from the end of the previous real number's move. The final end point will be the sum of the real numbers. All negative numbers are to have a movement to the left and all positive numbers to the right. This is explained to the student in terms of

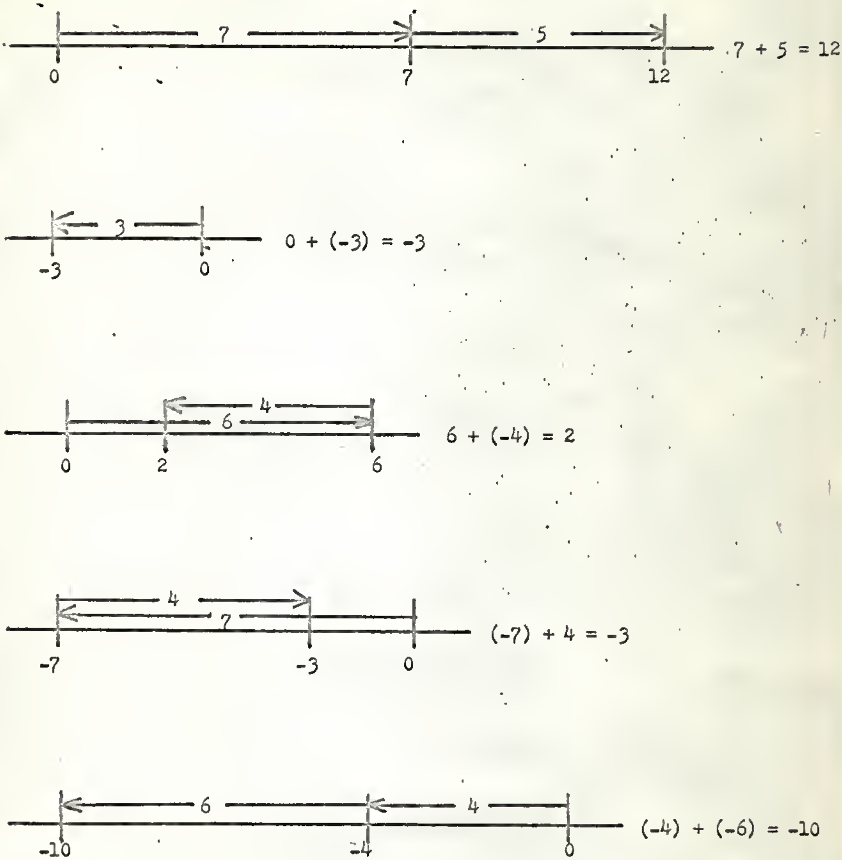


Figure 12. Addition of real numbers on the number line of SMSG.*

*UICSM, op. cit., 1-4.

absolute value. After locating the endpoint of the first move, the result of the second move can be found by going from the endpoint of the first number the distance represented by the absolute value of the second. They move that distance to the right if the second number is positive and to the left if it is negative.

At this point it is mentioned that the addition of two non-negative real numbers is already known from previous experience.

Next comes a discussion of the last case of figure No. 12. This case was the addition of two negative real numbers. The student knows that the results of the number line movements gave a result of $^{-}10$. A discussion follows that indicates that the opposite of the sum of the absolute values of those two numbers would give the same results. Since $|-4|$ would be $|+4|$, by earlier definition, and $|-6|$ would be +6 and the sum of +4 and +6 is a +10, then the opposite of this sum would yield the same results as the number line movements. This is indicated in the following notation $(-4) + (-6) = -(|-4| + |-6|)$. After trying another example of this nature the definition for the addition of two negative real numbers is given. "If a and b are both negative numbers, then $a + b = -(|a| + |b|)$.[15]

After a few problems to serve as an illustration of using absolute value to find sum of two real numbers, when one is positive and one is negative, there is a worded statement made about the sum of each case. These cases occur when the positive number has the greater

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[15]*Ibid*., p. 125.

absolute value, when the negative number has the greater absolute value and when the absolute values are the same.

The next statement made is a formal algebraic statement for the sum of a positive and a negative number.

If $a \geq 0$ and $b < 0$, then:
$a + b = |a| - |b|$, if $|a| \geq |b|$
and $a + b = -(|b| - |a|)$, if $|b| > |a|$

If $b \geq 0$ and $a < 0$, then:
$a + b = |b| - |a|$, if $|b| \geq |a|$
and $a + b = -(|a| - |b|)$, if $|a| > |b|$.[16]

After SMSG introduced the number line and defined rational numbers for labeling the number lines, the program then begins to develop properties for addition of number of arithmetic and the rational numbers in general. They developed these properties mostly by experimentation to develop the idea and then the definition in algebraic terms. After stating addition to be a binary operation, the addition of three numbers was indicated but grouped in different ways. The first indicated sum had the first two numbers grouped with the third number added to their sum. The second indicated sum has the last two numbers grouped with their sum added to the first. These two groupings proved to have the same result. After experimenting with a few other examples of a similar nature, the property is named as the associative property of addition. Thirty-eight pages later after the introduction of a variable, the formal algebraic statement for the property is made: "For every number a, for every number b, for every

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[16] _Ibid_., p. 127.

number c, (a + b) + c = a + (b+ c)."[17] Then in chapter 5, after the introduction of real numbers and addition of real numbers, a few more problems indicating the associative property applied to real numbers are shown. This is followed immediately by the statement: "for all real numbers a, b, and c, (a + b) + c = a + (b + c)."[18]

The property of commutativity is developed in almost the exact same way that the property of associativity is. The experimentation of two rational numbers added in a different order yielded the same results. The property was named and was not formally stated until several pages later and after much use in between. This statement of the commutative property of addition of rational numbers was "For every number a and every number b, a + b = b + a."[19]

After the introduction of real numbers, the addition of real numbers, and a few experimental applications, the formal statement of commutativity of real numbers was made: "For any two real numbers a and b, a + b = b +a."[20]

Other properties for addition are defined by SMSG. After a short discussion they give this definition for the closure property of addition: "For every number a and every number b, a + b is a number."[21] In a short discussion later each of the following properties are defined. "Addition Property of Opposites: For every real number a, a + (-a) = 0. Addition property of 0: For every real number a, a + 0 = a."[22]

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[17]Ibid., p. 62. [18]Ibid., p. 130. [19]Ibid., p. 62.

[20]Ibid., p. 130. [21]Ibid., p. 61. [22]Ibid., p. 131.

Next the addition property of equations is defined after a few examples to illustrate the property. "Addition Property of Equality: "For any real number a, b, c, if a = b then a + c = b + c."[23]

The remaining point that should be made, concerning the addition of real numbers for SMSG, is the additive inverse. The existence of an additive inverse is proved in the manner in which a theorem would be proved. They suppose any number x has an additive inverse. Thus $x + y = 0$. Then by applying the addition property of equality they arrive at $(-x) + (x + y) = (-x) + 0$. By associativity of addition $(-x) + x \quad + y = -x$. Then by addition property of opposites $0 + y = -x$. Because of the addition property of 0, $y = -x$. Therefore, they can make the statement "any real number has exactly one additive inverse, namely -x."[24]

UICSM. Even before UICSM introduces the raised positive and negative sign, they introduce addition of real numbers. The first time the student adds real numbers is six pages after the introduction of the unit. The only preceeding material is the concept of a real number. It has already been mentioned that this has not reached a full development.

The first explanation given the student in adding real numbers is on "the road" much like what was used for establishing the concept of a real number earlier. This road resembles the number line more in its appearance, but is used a great deal like the road used earlier in

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[23]Ibid., p. 133. [24]Ibid., p. 138.

UICSM's first unit. A point is located on an east-west road and marked by the letter A. From A the first real number is marked off the appropriate distance and direction. (Figure No. 13) The endpoint of this trip is marked, and instructions given to measure the next trip from the endpoint of the first. When this point has been reached by marking off units from the endpoint of the previous trip, the endpoint is marked. The next trip then begins at that point and is marked off the appropriate number of units in the proper direction and the endpoint of that trip marked. When the final endpoint has been reached, the student is asked what real number would measure a direct trip from the beginning of the first trip to the endpoint of the final trip. With the trips along an east-west road as a basis, the students are given problems to work. The problems are not as yet stated as addition. The problems are given to them as a series of real numbers still using the arrow. The addition is stated in terms of a trip followed by another trip. The "followed by" being the addition sign and the sum represented by the direct trip from the beginning of the first trip to the end of the last trip. The student doesn't realize that he is adding. The instruction to find the real number that would measure a direct trip from the beginning to the end of the problem is supplemented with the instruction to use trips along the road as long as needed, but to try to develop short cuts so that the answer can be found immediately. The problems listed in the latter part of the exercise make it almost necessary to find some kind of short cut because of the size of the numbers.

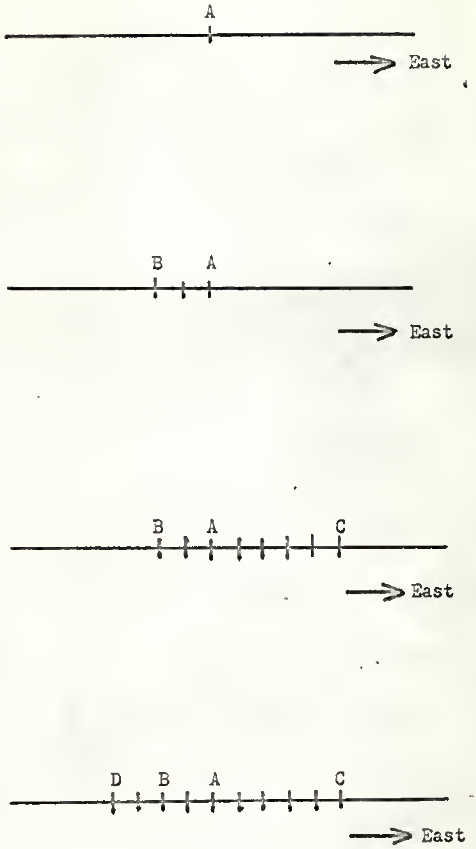

Figure 13. Adding Real Numbers for UICSM.*

*UICSM, op. cit., 1-6.

On the next page the directing arrows, that are used as a sign in the previous exercises, are changed to a more common notation. The right arrow is replaced by the raised plus sign ($^+$) to mean positive or to the right. The left arrow is replaced with the raised minus sign ($^-$) to mean negative or to the left. Then a short exercise, finding a direct trip on the road after a series of trips measured by the real numbers using the positive and negative signs, is employed. On the next page the addition is given to the student. The term following one trip with another is suggested to be like adding one number to another. Then the trip followed by a trip becomes a real number added to another real number. The exercise that follows simply states that the following numerals should be simplified. These numerals appear in this manner "$^+6 + ^-2$, $^+3 + ^+8$, $^-2 + ^-3$."[25] The teacher supplements the instruction given with the ideas to use the trips along the road if trouble arises and to look for short cuts when possible. Page nine is the first formal exercise in actual adding of real numbers as they are most commonly written. The student continues adding without any rule being given to them for the next 170 pages of material.

At the time rules are to be introduced, the teacher doesn't put the rules on the board. The rules for addition and multiplication are solicited from the student at the same time. During the interim between the introduction of addition and the actual soliciting of a rule, the teacher occasionally will ask if they are developing a rule in their

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[25]UICSM, op. cit., pp. 1-9.

own minds. He will not ask for it and encourages the students to develop their own rule and not ask the other students what theirs is. A concise way of stating the addition of two negative numbers is discussed in class, then the students are asked to state, in a concise manner, a rule for adding two positive numbers and several rules for multiplication. Then a concise statement for the addition of a negative number to a positive number is stated formally. The student is then to furnish the statement for adding a positive to a negative.

Suggestions. This author feels that the addition of real numbers is the most important single factor in algebra. The main tie to arithmetic is broken here. The truths of algebraic statements and manipulations never violate the laws learned in arithmetic, but the student may think that they have. The laws of arithmetic are not sufficient for algebra and have to be extended. If this extension is successful for addition, it will be successful for the other operations with a great deal more ease. The addition of real numbers sets the format for the students study of mathematics. A solid foundation here is absolutely necessary. The UICSM presentation of real numbers recognizes this fact and goes to the proper extent to lay this foundation with great care.

The method used by UICSM in the addition of real numbers and many other topics is called the discovery method. In October 1962 The Mathematics Teacher published an article by Ross Taylor who had this to say about the discovery method.

> The UICSM program has the discovery method built into it. I have found that with the UICSM material class time is

> saved because the students are able to discover, while doing their homework, many things that would normally have to be presented in class. Very often, reviewing the homework and presenting the new material for the day are one and the same process. I have a hunch that information that is originally "discovered" by the student will remain with them longer and will be easier to rediscover if forgotten. Practice at making discoveries should be a virtue in itself in this age when our progress and even our very survival depends upon the creativeness of man. The concept of nonverbal awareness appears to have a large influence upon the UICSM material. Students are able to gain concepts and use them long before they are able to explain them. For example, UICSM students begin adding real numbers on page nine of Unit I, but they don't get around to stating or even seeing a rule for the addition of real numbers until page 29 or Unit II (170 pages later).[26]

The commutative principle of addition is the first principle of addition developed. It comes after the establishment of a commutative principle for multiplication. Once the development of the same principle for multiplication has been accomplished, the commutative principle for addition takes only an example and then the statement of the principle.

When the development of this principle for numbers of arithmetic is completed, several pages of practice with it and other principles follow. The first application to real numbers comes in the form of an exercise. This exercise employs the same principles that have been used for numbers of arithmetic, but applies them to real numbers. The students are asked to decide which principle would apply to each of the expressions. With this exercise as background the principle of

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[26]Ross Taylor, "First Course in Algebra - UICSM and SMSG - a comparison," The Mathematics Teacher, 55:479, October, 1962.

commutativity for addition of real numbers is made in the form of an example. After the introduction to pronumerals, variables in SMSG, UICSM develops the principles in algebraic language. The device used here is a series of letters between two boys. One of the boys lives in an inaccessible area that has no mathematics teachers and the second boy in the United States is teaching him arithmetic by mail. There is a long exchange of letters which point up the difficulties of trying to state a principle of mathematics without using pronumerals. The boys follow what is probably a very close facsimile to what a student would do himself in first trying to state a mathematical principle. When the problem is finally resolved it is in a clear concise algebraic statement. Then, as an exercise, the student writes his own statement for ten different principles. The commutative principle of addition is one of these.

The associative principle of addition in both the UICSM and SMSG programs is developed through usefulness. The addition of the second two numbers is very often more convenient to add than are the first two numbers. "$(47 + 75) + 25 = 122 + 25 = 147$ is not as simple as $47 + (75 + 25) = 47 + 100 = 147$."[27] An illustration of this sort of example shows the usefulness as well as the truth of the grouping. The principle is then named as the associative principle of addition. As in the case of commutativity, associativity is stated as an example for real numbers after the students have had a chance to try to

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[27]UICSM, op. cit., pp. 1-46.

identify the principle applied to a set of problems dealing with real numbers. The formal statement again comes from the student in the same exercise in which they had to state the ten principles.

The principle for adding 0 is developed in a similar manner. Some open equations of the form 4 + ____ = 4 are listed and the principle implied is named on the same page. The principle is stated to hold true for real numbers after their introduction and the student asked to furnish examples of the principle as a problem in an exercise. When the students have to state the ten principles, after the introduction of pronumerals and the letters by the two boys, the principle for adding 0 is formally stated.

The principle of opposites is first implied after an exercise that contains several opportunities to see what happens when a positive real number and its corresponding negative real number are added to the same number. The principle is named a few pages later and a short discussion of what the principle is accompanies the naming. The principle of opposites gets its formal statement at the time the ten principles are stated after the letters by the two boys.

Multiplication of Real Numbers

SMSG. The definition of multiplication of real numbers is approached somewhat differently than is the addition of real numbers. Since some properties of multiplication have been developed, those properties are used to define multiplication of real numbers.

The properties that SMSG uses are the properties for numbers of arithmetic. The first of these is the associative property. It is developed through usefulness as was the addition property of associativity. Since it is easier to simplify 7 x (6 x 1/3) than it is to simplify (7 x 6) x 1/3, the usefulness is established. After a few examples it is named the associative property of multiplication. In the next chapter, after the introduction of variables, the property is discussed and named at the same time as the commutative property of multiplication. "Associative Property of Multiplication: For every number a and every number b and every number c, (ab)c = a(bc)."[28]

The commutative property is illustrated by some examples and named in much the same way as the property of associativity. After the introduction of a variable, it is again discussed and then stated formally as "Commutative Property of Multiplication: For every number a and every number b, ab = ba."[29]

The identity element for multiplication is illustrated through examples and then names as the multiplication property of one. It is then formally stated, as it wasn't introduced until after variables were introduced. "For every number a, a(1) = a."[30]

The next paragraph indicates the multiplication property of zero. The formal statement follows directly. "For every number a, a(0) = 0."[31]

The distributive property is seen to be useful through illustration in the form of examples like 15(7) + 15(3) and 15(7 + 3). The labor involved in each is pointed out to lead the student to understand

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[28]SMSG, op. cit., p. 71. [29]Ibid., p. 62.

[30]Ibid., p. 58. [31]Ibid.

the advantage of this method of evaluating the expression. To contrast this the example of 150(1/2) + 150(1/3) and 150(1/2 + 1/3) is offered. The application of this type of property to long multiplication is then made. Of course, 62 x 23 is not as easy to work as 62(20 + 3) written 62(20) + 62(3). This discussion on the properties' usefulness is preceded by a discussion on problems that lend themselves to this type of statement. Later, after the introduction of a variable and two illustrations, this formal statement is made: "For every number a, for every number b, for every number c, a(b + c) = ab + ac."[32]

SMSG starts to develop the multiplication of real numbers by referring to the mathematical structure they have developed for the numbers of arithmetic. This set is also defined by SMSG to be the non-negative real numbers and is referred to as such in the same paragraph. Since they want the negative real numbers to have the same properties as the non-negative real numbers, the following properties, that were developed for the numbers of arithmetic, will still have to hold true:

ab = ba (Commutative property of multiplication)
(ab)c = a(bc) (Associative property of multiplication)
a·1 = a (Multiplication property of one)
a·0 = 0 (Multiplication property of zero)
a(b + c) = ab + ac (Distributive properties).[33]

After listing some examples of what the multiplication of two real numbers must look like, they consider one that is familiar to the student to use in developing the multiplication of the real numbers.

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[32] Ibid., p. 66. [33] Ibid., p. 145.

After selecting one of these products, they apply the properties to it to transform it into something that could be used to define multiplication of the real numbers.

0 = (3)0
0 = (3)(2 + (-2)), by writing 0 = 2 + (-2); (notice how this introduces a negative number into the discussion.)
0 = (3)(2) + (3)(-2), if the distributive property is to hold for real numbers:
0 = 6 + (3)(-2), since (3)(2) = 6

We know from uniqueness of the additive inverse that the only real number which yields 0 when added to six is the number -6. Therefore, if the properties of numbers are expected to hold, the only possible value for (3)(-2) which we can accept is -6.

Next, we take a similar course to answer the second question.

0 = (-2)(0), if the <u>multiplication property</u> of 0 is to hold for real numbers;
0 = (-2)(3 + (-3)), by writing 0 = 3 + (-3);
0 = (-2)(3) + (-2)(-3), if the <u>distributive property</u> is to hold for real numbers,
0 = (3)(-2) + (-2)(-3), if the <u>commutative property</u> is to hold for real numbers,
0 = (-6) + (-2)(-3), by the previous result, which was (3)(-2) = -6.[34]

After comparing a few examples that lead to the formal definitions of multiplication of real numbers, the definition is formally stated on the next page.

For every two real numbers a and b,
If a and b are both negative or both non-negative, then ab = |a| |b|.
If one of the numbers a and b is non-negative and the other is negative, then ab = -(|a| |b|).[35]

SMSG then proceeds to develop the other properties of

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[34] *Ibid.*, p. 146.

[35] *Ibid.*, p. 147.

multiplication for the real numbers. Since they are already stated for positive numbers, they need only be shown for the negative numbers. Most of these are developod through the use of the definition multiplication of real numbers. The statements are changed to read for every real number instead of for every number.

Other properties of multiplication are then developed. One of the first of these additional properties is the property of multiplication of -1. It is stated that "for every real number a, $(-1)a = -a$."[36] This point and most of the properties after this point, are proven by using the properties that have been accepted as true before this proof. To prove this point the statement is made that if $(-1)a$ is equal to $-a$, which is read opposite of a, then $a + (-1)a$ must equal zero. $a + (-1)a = 1(a) + (-1)a$ by the multiplication property of one. Then $1(a) + (-1)a = (1 + (-1))a$ by the distributive property. Next $(1 + (-1))a = 0 \cdot a$ by the sum of opposite property. And $0 \cdot a = 0$ by the multiplication property of zero. Thus since $(-1)a$ will meet the requirement for the unique number the opposite of a, it can be written $-a$.

The multiplicative inverse is defined mostly by examining examples. It is shown that the number 0 has no multiplicative inverse by the example $0 \cdot b = 1$. What value could b have to make this a true statement? Obviously there isn't any such number because of the property for multiplication of zero. The existence of a multiplicative inverse

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[36] *Ibid.*, p. 155.

is stated and the uniqueness of the inverse is assumed. "Existence of multiplicative inverses. For every real number c different from 0, there exists a real number d such that cd = 1."[37]

The multiplication property of equality is shown by some examples and then stated formally. "Multiplication property of equality: For any real numbers a, b, and c, if a = b then ac = bc."[38]

UICSM. Once again the element of discovery is seen coming to use in UICSM's presentation of multiplication of real numbers. The method of developing an awareness of what the results should be is unique. The image of a movie camera taking a picture of a tank in which the water is being pumped into and out of a tank by a pump is employed. The first real number is associated with the movie camera taking a picture of the water being lowered or raised in the tank. When the film is shown it can be run forward or backward. If the film is shown backward, it is represented by a negative real number. If it is shown forward, it is represented by a positive real number. The pump and the tank are represented in terms of real numbers by the change in the water level. If the water is being pumped into the tank, then the rate per minute is a positive real number. If the water is going out, then a negative real number is used to show the rate per minute that the water is being pumped out. Thus, the picture of water being pumped into the tank at three gallons per minute, but shown with the film running backward for four minutes would be represented by the

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[37] Ibid., p. 163. [38] Ibid., p. 165.

multiplication problem $^{-}4 \times {}^{+}3 = {}^{-}12$.

Figure No. 14 shows the first exercise for the students for multiplication of real numbers. As in the addition problems, the student is encouraged to find his own short cut to the corresponding multiplication problem. The next exercise approximates the reverse of this exercise. The table is given with the real numbers and problem furnished. The student has to tell what the projection of the film is to be, what the pump is doing to the water level, and the observed change as well as the answer to the problem.

The student continues to use his own mental rule or method of multiplying these real numbers from page 21 in the first unit until page 29 of the second unit. At the same time the student is asked to state formal rules for addition of real numbers, he is asked to do the same for multiplication. Some of the rules for addition were discussed in the book, but all of the rules for multiplication of real numbers are furnished by the student.

Most of the principles that were developed for addition are developed for multiplication also. The commutative principle for multiplication was the first principle developed by UICSM. The other principles were expressed mostly as examples after this principle was introduced. Some of the exercises leading up to these principles for addition and multiplication contained problems that would start the student thinking about using these principles before they were ever developed.

Figure No. 15 shows the device used by UICSM to lead the student

Pump	Movie	Observed Change in Volume
Filling 4 gal. per minute	Running backward 2 minutes	Decrease of 8 gallons
$^{+}4$	$^{-}2$	$^{-}8$
Corresponding multiplication statement:		$^{+}4 \times {}^{-}2 = {}^{-}8$
Emptying 4 gal. per minute	Running forward 2 minutes	
Corresponding multiplication statement:		
Filling 4 gal. per minute	Running forward 2 minutes	
Corresponding multiplication statement:		
Emptying 4 gal. per minute	Running backward 2 minutes	
Corresponding multiplication statement:		
Filling 8 gal. per minute	Running forward 3 minutes	
Corresponding multiplication statement:		
Emptying 8 gal. per minute	Running forward 3 minutes	
Corresponding multiplication statement:		

Figure 14. Adding real numbers for UICSM*

*Ibid., p. 1-21.

to an awareness of commutativity. The students sometimes never see the principle from the table, but many see it right away. This is the introduction to the principles for numbers of arithmetic. The principle is stated in example form a few pages later after some practice using it. Then after a few more pages of practice it is stated in example form for real numbers. In unit II after the introduction of pronumerals, the student gives a formal statement of the principle in the exercise that was mentioned in the addition of real numbers section of this paper.

The associative principle for multiplication, and the distributive principle for multiplication follows nearly an identical pattern of development to the commutative principle for multiplication. In each case the opportunity to identify instances of the principles in a group of exercises is given before the example statement of the principle for real numbers. This conforms to the pattern of development for addition of real number principles. The usefulness of each of these problems is stressed in the example in much the same manner as in the development by SMSG.

The principle for multiplying by the real number $^{+}1$ and the principle for multiplying by the real number 0 are stated and then developed through example and exercise. These problems are not formally stated until the second unit when the student lists the statement of the ten principles himself.

UICSM does a great deal with inverses. The first statement of a multiplicative inverse comes in a discussion of reciprocals. Reciprocals are defined as numbers whose products are 1. The fact that

X	$\frac{2}{3}$	1200	$\frac{3}{4}$	87	21
21	14		$\frac{63}{4}$	1827	441
87	58	104,400		7569	
$\frac{2}{3}$	$\frac{4}{9}$		$\frac{1}{2}$		
$\frac{3}{4}$		900	$\frac{9}{16}$	$\frac{261}{4}$	
1200	800	1,440,000			25,200

Figure 15. Commutative Principle for Multiplying Real Numbers for UICSM.*

*<u>Ibid</u>., pp. 1-44.

the multiplicative inverse is another name for a reciprocal is not mentioned. The inverse of multiplying a number is dividing by that same number is a statement made in connection with division.

Suggestions. Once again it is felt that discovery by UICSM is a more worthwhile type of experience for the student than is the development by use of the properties by SMSG. SMSG develops multiplication in a very logical way. They make good use of the properties in their introduction of the rule as well as good use of absolute value in the definition. Even though they have a fine development, the discovery by the student will probably be grasped more firmly and remain with the student for a longer period of time. The author does think that SMSG's early definition will have to rely more on example than by the formal statement that is made in SMSG text. Frequent reference to the formal rule will probably help the student to a better understanding of the rule as the class progresses.

Subtraction of Real Numbers

SMSG. SMSG starts their discussion of subtraction of real numbers by citing the change making process involved in many purchases. It is pointed out that in making change the process is to find what number has to be added to the amount of purchase to equal the amount given in payment. This is the same as stating the corresponding subtraction problem that is stated as the amount given in payment less the amount of purchase is the amount of change to be given. To illustrate this point an 83 cent purchase with a dollar offered in payment is given as the problem. The amount of change is denoted as a variable x. "100 - 83 = x" is one way of stating the problem. "83 + x = 100" is the first way stated above. When the opposite of 83 is added to both sides of the latter equation, the results read as follows: "x = 100 + (-83)." Thus the parallel between "100 - 83 = x" and "x = 100 + (-83)" is established. Then equations similar to "100 - 83 = x" are listed with the student to complete the equivalent expression, "100 + (   ) = x," that is listed across from it. After some discussion and examples the definition is then given to the student. "To subtract the real number b from the real number a, add the opposite of b to a. Thus, for real number a and b, a - b = a + (-b)."[39]

The properties that have been developed for addition and multiplication are then tested for subtraction. It takes only an example like "8 - 6 = 6 - 8" to show that commutativity doesn't hold, "(8 - 6) - 4 = 8 - (6 - 4)" or some similar example shows that associativity is not a property of subtraction. It is then shown that the application of the definition can make these true statements by writing the subtraction as the addition of the opposite of the number being subtracted. Thus, "8 + (-6) = (-6) + 8" is a true statement. "(8 + (-6)) + (-4) = 8 + ((-6) + (-4)" becomes a true statement also.

UICSM. The development of subtraction of real numbers begins their work with inverses for UICSM. Using numbers of arithmetic the problem "13 - 4 = 9" is stated. It is stated that subtracting four undoes what adding four did. The problem is then stated "(9+4) - 4 = 9."

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[39] Ibid., p. 210.

The statement "subtracting four is the inverse of adding four" is then made.

The next device employed is that of making a list of ordered pairs all of which indicates the same operation. The first number in the pair has some operation performed on it to arrive at the second number. In the example presently under consideration, the first component would be thirteen, the second would be nine and the ordered pair would be (13, 9). This ordered pair would be put with a group of ordered pairs all of which indicate the subtraction of four. These ordered pairs are enclosed by a figure of some kind, usually of no regular shape. Once this list contains a sufficient number of ordered pairs, it is used to indicate the inverse operation by writing the ordered pairs in reverse order. The teacher, after establishing the first list, starts a second list in another irregular shaped figure and asks for the inverse of the first operation. In the case under consideration in the earlier part of this section the first list would illustrate adding by four. The teacher asks then for some examples of subtracting by four. The students after practicing with this device once or twice, recognize the ordered pairs of the inverse operation as being the reverse of the first list's pairs. They read them off quite rapidly. The students see that the ordered pairs indicate subtracting by four. Thus subtracting four and the inverse of adding four are the same thing.

After the use of these lists is familiar to the student, a list is given and a list for the inverse requested. When the second list is completed then the student is to name both operations. After some practice at this, the two lists and their operations are given and the student is asked to use the list to find a solution to problems that indicate the same operation as one of the lists.

A few pages later an exercise is given in which the student uses the idea of opposites. In the first part of the exercise the student adds corresponding positive and negative numbers and has an answer that is the same as the number he started with. In the middle of the exercise a note is inserted that states the negative real number is the opposite of the corresponding positive number and that the positive real number is the opposite of the corresponding negative real number. This is followed by questions on opposites. "If you add ___ to a number, you get back the number by adding the opposite of -3 to the sum"[40] is a good example of the problems in the latter part of the exercise.

With these preceding sections as background the student is then confronted with a problem that involves subtracting a negative number. The solution is put in terms of the student's preparation. He has been using lists of problems to find inverses. He has learned that subtracting is the inverse of adding. He has used lists to solve problems. All these points are put together in one problem, "$^{+}9 - {}^{-}5 = ?$"[41] Two lists are furnished: One for adding negative five and one for the inverse operation subtracting negative five. The student looks at the list for adding negative five and confirms in his own mind that it is a true list. He then goes to the list for subtracting negative five to find the

[40] UICSM, op. cit., pp. 1-74. [41] Ibid., pp. 1-75.

positive nine he is subtracting negative five from. When he finds an ordered pair with the first component that matches the positive nine in his problem, he knows the second component is the answer to the problem. See figure No. 16.

Exercises that follow show that subtracting a real number is the same as adding its opposite. These discussions and exercises lead to the naming of an operation called oppositing. This operation is denoted by an elongated minus sign. The principle application for this point is in subtracting a polynomial. The opposite sign in this case gives more meaning than would a subtraction sign or a negative sign. It would also have more meaning in the case of the opposite of five squared. Normally this expression is written -5^2. The student has trouble deciding if he is to square a negative five or if the expression means the negative of five squared. When it is read the opposite of five squared it is less ambiguous by its definition.

The student has now had three different meanings attached to the dash. The raised dash has meant negative, the regular dash in the middle of a numeral means subtraction and the elongated dash means opposite. These three symbols are all one in a traditional course and only two of them are used in SMSG. The student can handle an expression such as the following with no trouble: —7 - — —8.

The above problem was listed on page 87 of the first unit. On page 71 of the second unit the principle for subtraction is listed as a theorem for the student to prove. "The principle for subtraction:

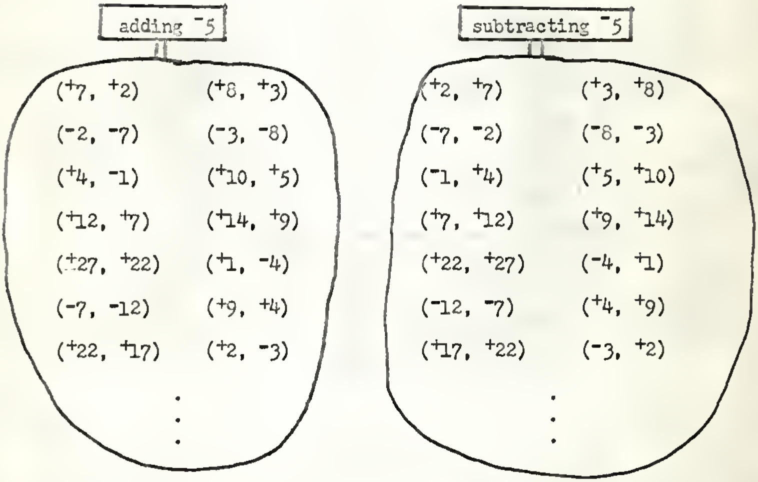

Figure 16. Using ordered pairs to subtract real numbers in UICSM*

*Ibid., pp. 1-75.

$\forall x \forall y, x - y = x + -y$."[42]

SMSG proves the additive inverse for the student in the same section as their development for addition of real numbers. Since UICSM relates the additive inverse to subtraction, this paper has waited until covering subtraction to cover the addition inverse. The same procedure will be followed for UICSM's development of the multiplicative inverse.

UICSM doesn't call the additive inverse by the same name that SMSG does. UICSM uses the name 0-sum theorem. The 0-sum theorem is not applied to equations as much as SMSG's additive inverse. UICSM develops another principle called the cancellation principle for addition which is used transforming equations. The statements of the 0-sum theorem and the cancellation principle of addition follow below:

"0 - Sum Theorem: $x\ y$ if $x + y = 0$ then $-x = y$"[43]

"Cancellation Principle for Addition: If $x + z = y$ then $x = y$."[44]

Suggestions. The element of discovery is not as pronounced in UICSM's development of subtraction as in the development of addition and multiplication of real numbers. The UICSM concept of subtraction is carried out over a long period of time and a great deal of material. The preparation for the first time the student subtracts a negative number is time well spent. This has been a trouble spot for some time in the traditional algebra. It is felt that the shock will be greater in SMSG's presentation of this point than in UICSM. SMSG's presentation

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[42] Ibid., pp. 2-71. [43] Ibid., pp. 2-68. [44] Ibid., pp. 2-65.

should give them a good clear definition of subtraction, but it probably will not be as easily implemented as UICSM's presentation of the concept.

Division of Real Numbers

SMSG. SMSG starts their discussion of division by bringing to the student's mind the definition of subtraction. The statement is made since division is related to multiplication in the same way that subtraction is related to addition, that division's definition could be similar. The additive inverse was used in exploring subtraction, and the multiplicative inverse is to be used to define division. With this short introduction the definition is given. "For any real numbers a and b ($b \neq 0$), 'a divided by b' means 'a multiplied by the reciprocal of b'."[45] The use of reciprocal for multiplicative inverse began soon after the multiplicative inverse was introduced.

A few pages later the proof of the following theorem is given. "For $b \neq 0$, $a = cb$ if and only if $\frac{a}{b} = c$. This amounts to saying that a divided by b is the number which multiplied by b gives a."[46] This helps the student to broaden his concept of division. Other small points are made on division before leaving the operation. The following two points are listed and the student asked to prove them. "For any real number a, $\frac{a}{1} = a$."[47] "For any non-zero real number a, $\frac{a}{a} = 1$."[48]

Simplification of fractions and operations on fractions follow these theorems and definitions for the student's application of what he has learned.

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[45]SMSG, op. cit., p. 223. [46]Ibid., p. 225. [47]Ibid. [48]Ibid.

UICSM. The work that UICSM does with inverses is carried on in this section. The point that division is the inverse of multiplication is made in much the same way that subtraction is the inverse of addition was made. There is little hesitation, at this point, in introducing the student to the lists in the irregular shapes. The work with this type of device has made the student familiar with what the function of the list is.

The list for multiplying by $^-3$ is given and the inverse operation is listed beside it. The student has been lead to the idea that division is the inverse of multiplication and second list is the inverse of the multiplying by $^-3$ list, so the second list must be dividing by $^-3$ (Figure No. 17). When referring to this set it is referred to as the inverse of multiplying by $^-3$. The exercise that is given at the end of this discussion is a set of division problems. The first few of these problems can be solved by using the ordered pairs of the second list.

After the introduction of pronumerals and the formal statement of the basic principles, the further development of division can be pursued in the second unit. One of the first points made at this time is that dividing by zero can't be done. The use of the two lists with the inverse of the first list being displayed in the second list is employed again. The first list is multiplying by 0 and the second includes all the reversed pairs of the first list and is labeled dividing by zero. (Figure No. 18) The problems are then discussed. $93 \div 0$ is offered and a solution sought in the second list but with no results. Then $0 \div 0$ is examined. It is discovered that all of the pairs

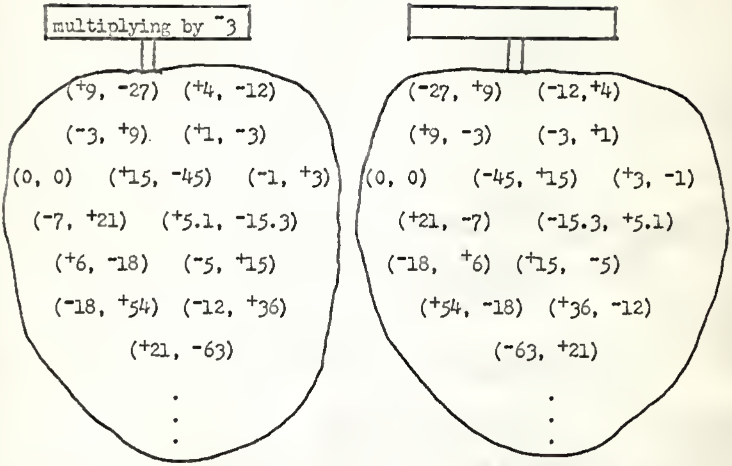
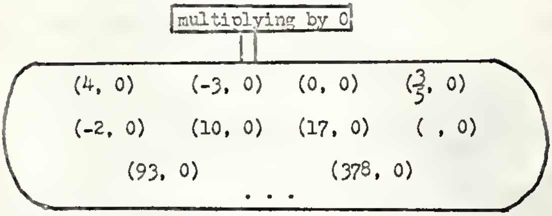


Figure 17. Using ordered pairs to divide real numbers in UICSM*

*Ibid., pp. 1-93.



So, some of the pairs in "dividing by 0" would be

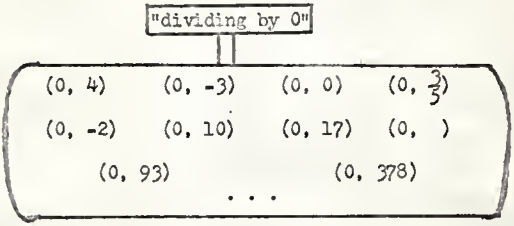


Figure 18. Dividng by zero for UICSM.*

*Ibid., pp. 2-82.

could be the solution. The conclusion is then drawn that $93 \div 0$ and $0 \div 0$ can't be numerals for numbers.

The formal statements for division follow shortly after the section on division by zero. These statements are made after a short discussion and some examples. "The Principle of Quotients: $\forall x \forall y \neq 0 \ (x \div y) \cdot y = x$."[49] "The Division Theorem: $\forall x \forall y \neq 0 \ z$ if $z \cdot y = x$ then $z = x \div y$."[50] The division theorem is later proven in the text.

Suggestions. The sections on introduction of division of real numbers in both programs is shorter than the introduction for other operations on the real numbers. In each case the material proceeding has contributed to the section. Much of the material in developing the other operations has either developed an understanding, concept or device that is applied in the development of division of real numbers.

The device that UICSM uses for inverse operations, listing two sets of ordered pairs, is not only novel but quite useful in developing the concepts of these sections. It is probably best applied to division by zero. It is quite obvious to the student and leaves a lasting impression when no consistent result can be found for this operation.

SMSG is consistent in their presentation of this topic. They follow the pattern of defining and proving by use of properties for the operation that they have used before.

The author is once more impressed by the UICSM development. They pay more attention to detail and go to greater extents in making their point. This is necessary in allowing the student to come to his

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[49]UICSM, op. cit., pp. 2-86. [50]Ibid.

own awareness of the rules for dividing.

PRESENTATION OF OTHER BINARY OPERATIONS

Concept of Order

SMSG. Before fifty pages of material have gone by SMSG is using the symbols $<$ and $>$. These are binary operations in that to be larger than or less than a number, there has to be a second number to be compared to. SMSG gives these symbols their traditional definition on page 49. Greater than and less than symbols are needed in discussion sentences that involve inequalities. Quite often the solution of an inequality is shown in pictorial form by means of a graph. Illustrations of these graphs were furnished under the notation section of this paper.

Open sentences involving inequalities are introduced. These open sentences when written in mathematical notation are called inequations. The compound inequations of less than or equal to and greater than or equal to are introduced almost immediately after the strict inequation is introduced.

These inequations are used for a time before any properties are developed for them. In chapter five, when the concept of a real number is being introduced, order on the number line is considered. Since this is the first time the student has encountered negative numbers, it is important he realize how the new numbers compare in terms of order. It is agreed that to the left is to mean less than and to the right is to mean greater than. The symbols for less than and greater than lend themselves to this sort of description. When the arrow is pointing toward the first number, it is to the left of the second number and is also less than the second number. When the arrow is pointing toward the second number, the first number is to the right of the second number and is also greater than the second number.

After some work toward the basic understanding of order with negative and positive real numbers, the comparison property is stated. "If a is a real number and b is a real number, then exactly one of the following is true: $a<b$, $a = b$, $b<a$."[51] As an exercise the student is asked to state a comparison property for $\leq$.

After some examples the Transitive Property for inequality of real numbers is stated. "If a, b, c, are real numbers and if $a<b$ and $b<c$, then $a<c$."[52]

In the same chapter five the concept of opposites is introduced as part of the work with real numbers. When the student has had some work with opposites then the general property of opposites is given. "For real numbers a and b, is $a<b$ then $-b<-a$."[53]

Following chapter five comes two chapters which deal with addition and multiplication of real numbers. Chapter eight then gives a more formal treatment to order. The three properties developed in chapter five that have to do with inequations are again mentioned. The next property to be developed is the addition property of order. SMSG uses a very good illustration for the students. They liken this property to two men packing a ladder. No matter how far the men walk or in what

[51] SMSG, *op. cit.*, p. 105. [52] *Ibid.*, p. 106. [53] *Ibid.*, p. 111.

direction they walk, the distance between them will be the same and the order will be unchanged. The property is then stated. "Addition Property of Order. If a, b, c, are real numbers and if $a<b$, then $a + c<b + c$."[54]

After some exercises with the addition property, the following theorem is proved in the text. "If $z = x + y$ and y is a positive number, then $x<z$."[55] The proof of the converse follows next. "If x and z are two real numbers such that x z, then there is a positive real number y such that $z = x + y$."[56]

After practice applying these two theorems the multiplication property of order is developed. Some experimenting and examples are given, and then the property itself is listed as a theorem. Multiplication of the positive real numbers to both sides of the inequation is furnished in the book. The student is left the proof of the multiplication by a negative real number. "Multiplication Property of Order. If a, b, and c are real numbers and if $a<b$, then $ac<bc$ if c is positive and $bc<ac$ if c is negative."[57]

Then points of order involved with the solving of inequations and will be discussed in that section of this paper.

UICSM. In the first unit, before the introduction of the pronumeral, UICSM explains the concept of order in terms of addition. If one number of arithmetic is smaller or less than another number of arithmetic, then there is a number of arithmetic, not 0, that can be added to the first number such that their sum would equal the second

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[54]Ibid., p. 188. [55]Ibid., p. 190. [56]Ibid., p. 192. [57]Ibid., p. 196.

number. This is the discussion for numbers of arithmetic.

The next section on real numbers is developed much the same. The student is asked if there is a real number that can be added to the smaller of the two real numbers to make their sum equal to the second real number. The answer is yes. They are then asked if there is any real number that could be added to the larger of the two that would make their sum the same as the smaller real number. The answer is yes once more. The fact is then brought out that since there is always a real number that can be added to any given real number to make their sum equal to the second real number, this is not a valid test for order. The student is then asked what test would be valid for finding the smaller number. It is then stated in the text that adding a positive number would be sufficient.

At this point the number line is introduced. UICSM states that when all the real numbers are "lined up in order" the line of real numbers, on the number line for short, is the result. The number on the right of the two numbers is the larger. During these sections on order, the standard symbols of order are introduced and the student begins to work with them in the problems.

Near the end of the second unit which is planned to be finished at the end of the first semester, the concept of order is defined in terms of subtraction. The generalizations that are stated at this time are "$\forall x \forall y$ (a) if $x - y$ is a positive number then $x > y$, and (b) if $x - y$ is not a positive number then $x \not> y$."[58]

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[58]UICSM, _op. cit._, pp. 2-109.

Near the beginning of unit three, the solution set of a sentence is considered. When the notation for solution sets has been developed, the graph of the set is discussed. The graph of an inequation shows the solution with more meaning than an English or verbal description of the solution set can. The student experiences graphing the solution set and writing inequations from graphs pictured in the book. Then UICSM introduces the locus of a sentence. In their discussion of the number line UICSM introduces intervals. The notation for these two topics was discussed in the section dealing with notation. The locus of an inequation is said to be an interval of the number line. The student is then given instruction and practice in writing the solutions of inequations as intervals and sketching the graph of various intervals.

Suggestions. Locus of a sentence and the resulting interval are not discussed in SMSG. SMSG's graphing of inequations and inequalities is quite extensive. The student should be solidly founded in graphing of inequations from SMSG. They use the graphing of the solution set of a sentence from the early part of the year and throughout much of the remaining material. UICSM continues to use the graphing of these solutions but to a lesser degree than SMSG.

SMSG's notation for the graph of an inequation is possibly a little clearer and more easily understood. They use the graphing of the solution of the inequation a little more. The presentations do not vary a great deal other than the points the UICSM introduces that SMSG doesn't. Taking the factors just mentioned into consideration, the edge would need be given to SMSG on this topic.

Ordered Pairs

SMSG. The only time that SMSG uses the ordered pair is in the graphing of points and the solution of sets of sentences involving two variables. This point has been used by the traditional algebra courses for many years. SMSG seems to stay with traditional algebra in their use of the ordered pair. Possibly SMSG makes more point of the fact that these pairs are ordered and must be treated as such than would an algebra text written in the traditional methods. Since graphing of solution sets is the main point in SMSG's graphing, further discussion will come in the section treating equations.

UICSM. UICSM makes use of the ordered pair to a great extent. One of the main uses of the ordered pair in UICSM's program, is in inverse operations. These ordered pairs are actually a list of problems involving the same operation with the same real number. The development of the ordered pair for use in inverse operations comes in class. The teacher solicits several problems all involving the same operation for a particular number. As these problems are given to the teacher, they are written down in a column. The first component is listed, followed by the operation symbol, then the real number that is the object of the operation in each case, the equality symbol and finally the result of the problem. When the teacher has completed the list, he comments that a great deal of writing is required to make such a list. He asks for suggestions on ways the writing could be reduced.

The students see that there is much repetition in what has been written. Suggestions are made to remove various symbols and numbers until the operation symbol, the equality symbol, and the number which is the object of each operation have all been erased from the blackboard. This leaves the first component, or number with which the problem started, and the result of the problem. These are then put into a parenthesis with a comma separating them and the ordered pair is ready to be used in developing inverse operations. The use of these ordered pairs in developing the operations for real numbers has been discussed in the sections dealing with the development of the real number operations.

Intervals are written as an ordered pair in certain situations. An interval is always written as an ordered pair, but in only a few cases does the order matter. When a half open interval is written the closed endpoint is always listed first. In the discussion of notation it was brought out how this is to be denoted.

UICSM develops the lattice before graphing of equations in two variables. The ordered pair that has been employed in the development of real numbers is somewhat different than the ordered pair of the lattice. The main point in the ordered pairs is that they are two components that are ordered. The first component is to be the horizontal distance and the second component the vertical distance. The lattice itself is the points that correspond to the integers of the coordinate graph used for real numbers. If the intersection of all the lines of the coordinate graph, were to remain on the paper and the rest of the lines removed, then the resulting points would be a lattice.

When the student becomes familiar with the use of the ordered pairs of the lattice, sets and operations on sets are introduced. The ordered pair then becomes an ordered pair of pronumerals. These two pronumeral components are then further described by equations, inequations, compound inequations and polynomial expressions.

The transition is made to the number plane, coordinate graph, at this time. This extension uses the real number line as its basis instead of only the integers. The same ordered pairs that were utilized in the work with the lattice are then employed on the number plans.

Suggestions. There is really little to compare on the subject of ordered pairs. The use that SMSG makes of graphs is sound and rather extensive, but their use of the ordered pair is very limited outside of graphing. UICSM's use of the ordered pair is very helpful to them in developing many of their concepts. They have given a new look to the development of graphs of equations in two variables as well as the graphs of inequations in two variables. They have used set operations as part of this development. It is felt that SMSG could employ a more modern approach to their program by making more use of this device and the topics it can lead to.

PRESENTATION OF SINGULAR OPERATIONS

Squaring

SMSG. SMSG introduces variables quite early in their program. Since variables are an early topic, they can be used for many other discussions nearly any place in the text.

The definition that SMSG gives to squaring for convenience of the student in writing expressions when a variable is to be multiplied by itself. The operation takes on greater proportions later in the course. "If a variable occurs in an open sentence in the form 'a·a' meaning 'a multiplied by a', it is convenient to write 'a·a' as 'a^2', read 'a squared."[59]

A discussion on exponents in general begins in the second semester. The need for exponents is developed when the desire for a short way of writing the prime factors of a composite number is sought. Some composite numbers have several factors of the same prime number. The base with its exponent is defined as the number that is to be multiplied and how many of them that are going to be multiplied. A general definition is then given in algebraic terms. "...$a^n = \underbrace{a \times a \times a \times a \times \ldots \times a}_{n \text{ factors}}$."[60]

When the introduction of a polynomial being squared is introduced, it is done by a double use of the distributive property over the indicated product of the polynomial to be squared times itself. This method makes use of the original definition but doesn't make a point of that fact.

SMSG extends the operation squaring to other exponents. When the basic definitive is given to the exponent, the topic is then developed to include all the basic operations.

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[59]SMSG, op. cit., p. 44. [60]Ibid., p. 267.

The basis for the extension of squaring to other exponents is laid in the laws of exponents. SMSG calls these properties of exponents. The product of the same base with two different exponents is given as "$a^m a^n = a^{m+n}$."[61] The division of same numbers with two different bases is given as "...if $m > n$, $\frac{a^m}{a^n} = a^{m-n}$."[62] This is proven in the text by use of properties. The same property but with m = n is given as "If m = n, $\frac{a^m}{a^n} = \frac{a^m}{a^m} = 1$."[63] Then the case is given where $m < n$ is listed and properties are used to prove it. "If $m < n$, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$."[64]

All of these properties of exponents that involve division stipulate that the base cannot be zero. Two other properties of exponents are defined. "... for m and n positive integers and $a \neq 0$, $a^0 = 1$, and $a^{-n} = \frac{1}{a^n}$."[65]

The use of these properties for exponents are illustrated in each case as they are developed and defined. The student is given exercises involving their use to solidify understanding of the properties.

UICSM. In the second unit, after the introduction of a pronumeral, the squaring of a number is introduced for the first time. The student is asked to consider the generalization "For each x, x(2x + 3) = 2(xx) + 3x."[66] The same generalization is set up using seven as a substitution for x. This can be checked by computation, but it is used to set up a pattern that could be used for testing any substitution for x. The

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[61] Ibid., p. 268. [62] Ibid., p. 270. [63] Ibid. [64] Ibid., p. 271.

[65] Ibid., p. 273. [66] UICSM, op. cit., pp. 2-33.

sevon is placed inside a square. The use of geometric shape was introduced oarly in unit two as part of the development of a pronumeral. Then, using the principles that were developed for tho various operations, the pattern is established for transforming the expression from $x(2x + 3)$ to $2(xx) + 3x$.

This is the first time that the concept of a number multiplied by itself is used. The term "x^2" is not used hore nor is it used for some time. Multiplication problems that involve powers larger than the second power are integrated into the student's work at this time also. These products are written by repeatod factors of pronumeral itself. On page eighty-eight of the second unit, 166 pages later, the abbreviation "x^2" is introduced to the student. This is followed by an exercise to familiarize the student with the use of the new notation.

On the following page the squaring of a polynomial is first encountered. As does SMSG, UICSM uses the double application of the distributive principle to the indicated product of the polynomial times itself to work this problem.

Near the end of the fourth unit's work the exponent is introduced. The raised numeral is simply named as an exponent and several examples are given to illustrate how the exponent is to be used.

A few pages later when the student is familiar with the new notation, the process of discovery is started for the student to acquire knowledge of multiplying and dividing two different powers of

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[66]UICSM, *op. cit.*, pp. 2-33.

the same number. An example of a multiplication problem is given in which the two powers of the same pronumeral are written out in repeated factor form. Each is enclosed by a grouping symbol of its own in the original repeated factor form and the regrouping is made into a single repeated factor expression. The definition of the exponent can then be applied to the whole group of factors and written as a single pronumeral with an exponent. The division is written in the same repeated factor form and the associative and commutative principles applied to group the common factors of the numerator over the common factors of the denominator. The student is equipped with the knowledge to reduce the part common to both the numerator and denominator. This leaves two expressions that can be written with a pronumeral base and an exponent in both the numerator and denominator. Neither term of the fraction contains a factor contained in the other term. With these two examples as background the student is given a set of exercises to work and is told to look for short cuts. The rules are never given to the student for these principles of exponents in the first four units. For each kind of problem they encounter there is an example given, but never a rule. The teacher's commentary lists the rules and asks that the precise rule be developed if there is an insistance by the class to develop written generalizations for what they have become nonverbally aware. The proofs are given in the later unit after the students have been introduced to mathematical induction.

Suggestion. Once again the comparison of discovery to definition becomes apparent. As before it is felt by the author of the paper that discovery is the better method. Any time a student finds something through his own thinking it should remain longer and be better understood than when it is defined for him.

One suggestion that can be made here, that wouldn't require a great deal of change in program, is that SMSG include more work on squaring of a polynomial. It is implied through their work with factoring more than it is approached directly. It is felt that the student would grasp the factoring of perfect square trinomials better if he were more familiar with squaring of a binomial.

Finding the Square Root

SMSG. SMSG and UICSM both find square roots by approximation. SMSG starts their discussion by explaining how it is easy to find the square root of numbers that have rational roots, but the irrational roots are not as easy to recognize. They start a quest for two rational numbers that are each approximately equal to the irrational root and that contain the root between them.

One way of finding these two roots is shown by deciding what two integers it would fall between and then taking the squares of all the tenths between the two numbers. This list will show two approximations for the root. These two will be the number whose square is largest without exceeding the number whose root is to be found and the next larger number. If this isn't a good enough approximation, the two closest tenths can be selected, and the square of all the hundredths between them calculated. There will be two similar roots in this list. If these are not a sufficient answer, the same thing can be done for thousands, then ten thousands and on until the desired accuracy is attained.

It is easy to see that this could take a great deal of computation to get even a rough approximation of the root. Since this is such a laborious process, there should be a better method of finding the root. Then the standard notation is introduced. If the number is not less than one hundred, it is put into standard notation as the product of a number between one and one hundred and an even power of ten. The students have already had work with simplifying radicals so they recognize the even powers of ten as being rational numbers and can recognize what their roots are. Then the root of the number between one and one hundred is all that needs to be found. The root of a number between one and one hundred can be easily approximated to be between two integers. The integer that is the closer of two is selected and multiplied by the root obtained from the even power of ten and an estimate formed.

When the student is acquainted with the estimation process for the approximation of the root the process of refining the approximation begins. The largest integer smaller than the root is selected and divided to two decimal places into the number for which the number is being sought. The answer will be larger than the root and the quotient and divisor are averaged. This is a fair approximation when rounded to one decimal point. The average is then divided to three decimal places into the number whose root is being sought. Once more the two numbers are averaged and an approximation to two decimal places is obtained. This process can be carried on until the desired accuracy is obtained.

Earlier in the chapter on square roots the general topic of roots is discussed. The opening discussion is summarized by these sentences.

> If b is a positive real number, and $a^2 = b$, then a is a square root of b. If a is a square root of b, so is $-a$. The positive square root of b is denoted by $\sqrt{b}$ and is commonly called "the" square root of b. The negative square root of b is then $-\sqrt{b}$."[67]

It is then mentioned that if $a^3 = b$ then a was the cube root of b and was written " $\sqrt[3]{b}$." The student later writes a description of fourth roots and "nth" roots.

Next the basic parts of a radical are defined and the fact that perfect squares have rational roots. A proof that the square root of two is irrational follows.

The topic discussed after the proof of the existance of the irrational number, the square root of two, is simplication of radicals. The underlying concept is listed as a theorem and proved in the text. "Theorem 11-3 $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$, for any non-negative number a and b."[68] The next point that is made takes advantage of this theorem by finding two factors for the number whose square root is being sought. One of these numbers must be a perfect square. Then the square root of a perfect square can be found and a simpler radical is the result.

"Theorem 11-4 of $a \geq 0$ and $b > 0$, then $\frac{a}{b} = \frac{a}{b}$,"[69] is the theorem

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[67]SMSG, op. cit., p. 284. [68]Ibid., p. 290. [69]Ibid., p. 294.

that provides the basis for simplification of fractional radicals. Rationalizing the denominator is then discussed and the process of accomplishing the rational denominator shown.

A short section on addition of radicals follows. The machinery is given to the student for this process and a short exercise is listed.

UICSM. The two programs have a nearly identical pattern for introducing the concept of an irrational square root. UICSM introduces the idea when looking for solutions to difference of two squares quadratics. When the quadratic equation is of the form $x^2 - a^2 = 0$, the factoring method of solving the equation is adequate. When the quadratic equation is of the form $x^2 - a = 0$, the factoring method breaks down. But there should still be a number that would give a solution. Then, by methods similar to the one used by SMSG, a squaring process is begun for the tenths between the two integers closest to the square root of the number. This is repeated for the hundredths, thousandths, ten thousandths and indicates that this could be continued to obtain numbers as close to square root as desired. The existence of the proof of this number is delayed to a later unit.

Examples of a variety of different problems are given and the student given exercises in changing radicals to simplest form, adding radicals, multiplying radicals and radical expression.

Following this introduction to the work with square roots and square root radicals, UICSM includes a few pages on approximation of error. This is an interesting section that strengthens the concepts presented in the adjoining sections. Without the work on approximating errors, the values at which the student arrives for the approximate root would not have as much meaning.

UICSM then starts the dividing and averaging method used by SMSG. The main difference in their method is the constant check that is kept on the error involved in each average. The same closing in on the irrational root is accomplished by successive dividing of approximate roots and averaging the quotient with the divisor for an even closer approximation for the root.

More work with radicals and square roots follows this section on dividing and averaging.

The principle square roots are discussed next. The student is lead to the awareness of some numbers having square roots while others do not. UICSM then lists the following generalization.

> "(1) For each number $x > 0$ there is just one positive number, $\sqrt{x}$, whose square is x, and there is just one negative number, $-\sqrt{x}$, whose square is x.
> (2) 0 has just one square root, 0.
> (3) Negative real numbers do not have real number square roots.[70]

The principle square roots is then defined to be the non-negative square root of the non-negative real number.

Next there are a few examples listed to prompt the student to accept the following generalization "For each x, $\sqrt{x^2} = |x|$."[71] Some exercises follow that provide experience in working this type of a

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[70]UICSM, op. cit., pp. 3-131. [71]Ibid., pp. 3-132.

problem.

Suggestions. The two methods of iterative approximation that are used by the different programs are a big change from what has been accomplished in square roots in traditional algebra. The methods used in this section of the two programs have more mathematical justification for the student than has the method used traditionally in most algebra texts. The student of a traditional algebra class was told to do it this way and he aped the actions and procedures of the instructor to get the answer. The student should be able to understand both of the methods used in these programs.

SMSG has one very clear cut advantage in this section. They first bring the square root to product of an even power of ten and a number between one and one hundred. This estimation gives the student a very good place to start. When the method is applied by UICSM, students may have some difficulty finding the two integers that the root falls between. If the standard notation would be found first in UICSM, the student wouldn't have as much trouble.

SMSG doesn't work so closely with the error involved in their answer as does UICSM. Possibly it isn't really essential that the error be determined. But the student has a much better idea of the accuracy of his answer when he can calculate its error.

Absolute Value

SMSG. SMSG starts with a definition for its development of absolute value. This is standard procedure for most of their topics.

The absolute value of a non-zero real number is the greater of that number and its opposite. The absolute value of 0 is 0.[72]

The following material discusses this definition and gives examples of what is stated. Another definition is then given.

The distance between a real number and 0 on the real number line is the absolute value of that number.[73]

After more examples the following definitions are given for absolute value. "For every real number x which is 0 or positive, $|x| = x$. For every negativo real number x, $|x| = -x$."[74] More problems follow these definitions.

The concept developed in these pages are used in other parts of the text in exercises and occasionally in developing other topics.

UICSM. After UICSM introduces the number line and develops the concept of order on the number line, it is ready to present the concept of absolute value. The idea of the distance between points on the number line is presented. The student is reminded these distances are numbers of arithmetic and is then asked to find the distance between several real numbers. These points are given as ordered pairs.

After working an exercise of finding the distance between points, expressed as an ordered pair, the same situation is established by subtraction. Two numbers are subtracted in different order and the results are similar except for the sign. The number of arithmetic which corresponds to these corresponding real numbers is the same for

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[72]SMSG, op. cit., p. 113. [73]Ibid., p. 114. [74]Ibid., p. 115.

the two real numbers. It is then stated that the distance between the two points is the number of arithmetic that corresponded to the two real numbers. The next statement defines what this number of arithmetic is. "We call the number of arithmetic which corresponds with a real number the absolute value of the real number."[75] A list of examples follows the definition. The student is next asked to list some pairs written in the same fashion as an ordered pair, to show the operation absolute valuing. Figure No. 19 shows another way UICSM gives to visualize absolute valuing.

The standard notation for absolute value is then given and some problems listed for the student to become acquainted with and to apply the new notation.

The next topic explores the possibility of an inverse for absolute valuing. The fact that the inverse does not produce a unique answer is shown. The term unique is also discussed with the student at this time. The ordered pairs are again brought to use by showing what the reverse order of the set for absolute valuing would look like. It contains some pairs which have alike first pairs but unlike second elements. Therefore, the inverse of absolute valuing is not an operation since it does not produce a unique result. Further discussion shows that absolute valuing is to be usod for real numbers only.

Suggestions. Both programs have an adequate development of absolute value. They both develop the topic along the lines the y

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[75]UICSM, op. cit., pp. 1-104.

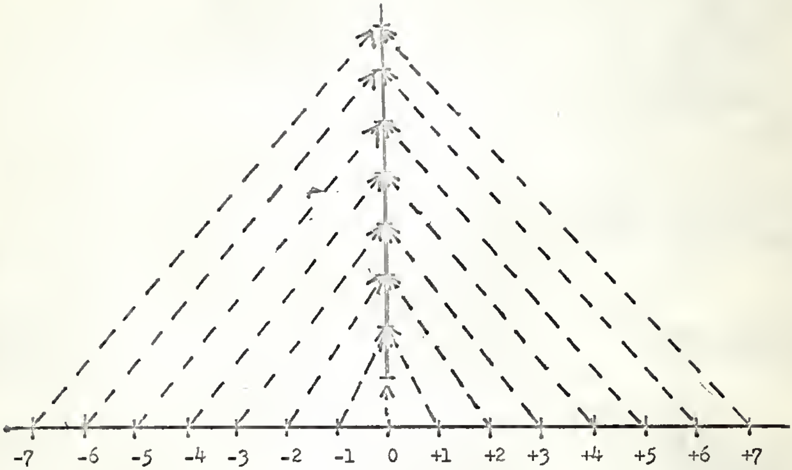

Figure 19. Absolute Valuing for UICSM*

UICSM 1-105.

generally use in developing their individual programs. UICSM uses more of a definition in this section than it usually applies. UICSM pursues the topic a little further than does SMSG. This is due to the examination of an inverse for absolute valuing. In addition to the consideration given to an inverse for absolute valuing, UICSM gives a further treatment to another small point at this time. They use the absolute value in connection with affixing of signs to numbers of arithmetic.

Even though both programs have a rather short treatment of absolute value, they apply the use of the concept in several other topics throughout the remainder of the text.

VARIABLE

Concept of a Variable

SMSG. SMSG begins their discussion on variables by discussing names for numbers. SMSG makes statements pointing up the use of numerals. These statements are to show that numerals are names for numbers and not the number itself. When a statement is made about a numeral, quotation marks are used to indicate the numeral. The point is made that there are many names, or numerals, for the same number. The simplification of names for numbers and preference of multiplication over addition in these simplifications is then discussed. A list of problems is then given for practice in simplifying numerals.

After the simplification of numerals has been practiced, the properties for addition and multiplication are then developed for the numbers of arithmetic.

The concept of a variable is first introduced on page thirty-five. The use of a variable is first thought of in terms of arithmetic games. Operations are performed on a number with a certain result being produced. In the game given in the text the same result is obtained as the number that was originally selected. An example is selected and the operations performed on the number that was chosen. In the text seventeen was selected as an example. First, seventeen was multiplied by three; then twelve was added to the product of seventeen times three; this sum was then divided by four and the difference found between the quotient and four. This series of operations once again produced a seventeen. The same pattern is then written in mathematical notation for seventeen. Next the same pattern is to be developed for any number. The word number is used and the pattern developed.

number

3 (number)

3 (number) + 12

$$\frac{3\,(\text{number}) + 12}{3}$$

$$\frac{3\,(\text{number}) + 12 - 4}{3}$$ [76]

The statement is then made that just the letter "n", used for the word number, would save a lot of writing. The pattern is then recopied using n instead of the word number. A discussion follows on how n would take on any value. It is then names as a variable. The number that a variable represents is said to be the value of the variable. The

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[76] SMSG, *op. cit.*, p. 36.

set of values of the variable is called the domain of the variable. Some problems are then given using variables.

As soon as the introduction of a variable has been made the chapter on sentences begins. When an assertation is made about numbers, this assertation is called as sentence. These sentences can be false or true. When the sentence is written that contains a variable, the sentence is said to be an open sentence. This definition is used since the point of whether a sentence is true or not must remain open until further information is available about the variable. An open phrase is defined in much the same way that an open sentence is defined.

<u>UICSM</u>. As an introduction to the first unit of work, UICSM discussed the use of numerals for numbers. Some of the pitfalls of discussing numbers and numerals are pointed up in a letter between two boys. A boy in one of the adjacent states had a penpal in a remote area of Alaska. Since there was no one to teach the Alaskan boy arithmetic the agreement was made for the boy in one of the adjacent states to teach the other boy arithmetic by mail. The boy being taught was called Al, and the boy doing the teaching was named Stan.

Stan decided if he was going to teach Al arithmetic he needed to know what Al already knew about math. He sent a test to Al and was shocked by the reply. Figure No. 20 shows why he was surprised.

The point is then made that Stan wrote symbols to mean numbers and Al considered only the symbols. The letters then continue and the difference is made between things and names of things. The class

1.	Take 2 away from 21.	1
2.	What is half of 3?	2
3.	Add 5 to 7.	57
4.	Does 2 x $4\frac{1}{2}$ equal 9?	no
5.	Which is larger, .000065 or .25?	.000065
6.	How many times does 3 go into 8?	twice
7.	How many times does 9 go into 99?	twice
8.	Which is larger, 3 or 23?	23
9.	What is a number smaller than 4?	4
10.	What is a number larger than 4?	4

Figure 20. An arithmetic test for A1*

*UICSM, 1-B

has a discussion on names and names of things also, and it is decided to use a single quote mark to enclose numerals. When the number itself is to be named, the numeral is written without the quote mark to enclose it. The student is given an exercise which involves placing the quote mark in English sentences to make them meaningful.

The discussion then centers on names for numbers whereas it had been centered on a variety of different names. Names for numbers are once more named as numerals or in some cases numeral expressions. The student is given a group of numeral expressions and asked to put the expressions in groups that name the same number. A variety of simplifications is necessary but all of the numeral expressions fall into three groups that each represent a number.

The next exercise is one in which the student must give a true or false answer to the mathematical sentences that are listed. The next exercise gives a list of open sentences. The sentences are open in that they leave a blank space to hold a place for a number to be written. The sentences are written in identical pairs and the instruction given to make one true and one false. These open sentences are used frequently throughout the first unit whenever an open sentence is needed.

The development of a variable for UICSM begins in earnest in the introduction of their second unit. In discussing notation used by the different programs, it was mentioned that UICSM used papers with holes in them. These papers with holes in them are portrayed by having a second colored sheet behind the sheet with the holes in them. The use made of these holes in paper is applied to a true-or-false test. A true statement is stated on the paper and a false statement is also stated on the papers. There is a blank for the answer before all of the test questions. The last eight questions are stated with a hole where one of the numbers would normally be. There are several second sheets duplicated and handed out. The second sheet has numerals in positions to match the holes of the first sheet. The student then answers the resulting questions true or false. The sheets are exchanged and the same exercise repeated for the different second sheet.

The use of this kind of a test is dramatized by a ninth grade class in a typical junior high. The teacher is said to do this quite often. But one day when he is sick, the principal gives the class the test. Instructions for doing the test are no talking and no questions. The list is to be collected in five minutes. The principal forgets to handout the second sheets. The students reading this story are asked why the test couldn't be finished. If they are thinking well, they should realize that the statement is neither true nor false until a numeral is put into the hole.

The students of the UICSM class are then given an exercise that is much like the test given in the story. The first experience they have with the paper with holes is to make their own second sheet and exchange it with another student. After the test has been answered by the students who had exchanged, they are returned and exchanged with another student and the test taken again. This second exercise is similar to the first. They are given pages with the second sheet under it and circles on them to take the place of the holes. This test is a series of questions about numbers and operations the student is quite familiar with. The number is to be the same for all blanks within a problem and answers a simple question in each case. Because there is sometimes more than one blank in a problem, some trial and error may be necessary for the student to find the proper numeral to make true statements.

At this point the UICSM class returns to the story of the math class. One of the students has an older brother who has had algebra. He talks about the use of letters in algebra. The student hasn't seen this type of symbol used yet and wonders about it. He would like to put the question to some of the smarter students to see if they might know something about them. Rather than approaching them directly he used one of the tests with holes in it. His second sheet had letters instead of numerals that are familiar to the students. His paper is then exchanged with the smartest boy in class. The boy realized he couldn't do the test and started to tell the teacher that he couldn't do the test when he stopped and did some thinking. He then realized that this type of a test is similar to the test having no second sheet. The student can't know if the question is true or false until there is a meaningful numeral to take the place of the hole. At this point the story is left.

The use of the hole in the paper is a forerunner to using geometric shapes in mathematical sentences. The holes in the paper were round, and the circles used in one of the exercises were round. The student is given an exercise in which he is to place numerals into geometric shapes, called frames, to make true statements. The exercise is used after some discussion on the significance the holes in the paper had in relation to numerals. The holes held a place for a numeral; the frames in the exercise do the same thing. But the frames thomselves, like the hole, are not numerals. The exercise is a true or false test and the students are given the numeral to use in the frame.

The next exercise is similar but has two or more different kinds of frames in the same sentence. The substitutions are indicated by a numeral for each different shaped frame in each problem. Some of the sentences are written with only frames and indicate principles that have already been learned. The associative principle is an example of this type of indicated principle. Some of the numbers that are to be substituted into the frames are somewhat hard to work. These numbers make it more conducive to the evaluation of the truth of the statement then the actual computation of the problem.

After the students place numerals in each frame for a while, then they are told to use substitution. Soon they are substituting expressions of several numerals with different operations indicated in the expression. The frames soon have expressions containing other frames substituted for them. One of the unique features of the use of frames is in an exercise in which the student has to produce his own pattern sentence using frames. This sentence is generated by several examples and the student writes a similar sentence using only frames. Some of the generating sentences are true and some of them are false.

No matter what situation the student produces, the same type. Figure No. 21 shows some of these problems. Most of the problems dealing with frames are constructed to lead up to the use of the principle on pronumeral expressions.

The similarity of pronouns in English statements to the frames of the mathematical statements is then made. It is stated and illustrated that a pronoun holds a place for a name. It is stated and illustrated also that a pronumeral holds a place for a numeral. Neither of the statements are true or false until a name is substituted in place of the pronoun or a numeral in place of the pronumeral. The fact that frames could be used in English statements with as much meaning as pronouns is shown. After this type of discussion is pursued for a short time, standard pronumerals are introduced.

It is stated that frames have been used, but it is more common to use letters. Open sentences are then written using letters instead of frames, the standard abbreviations for multiplication are introduced, and the student is ready to start using letters as pronumerals.

Suggestions. Once more it is possible to see the use of discovery and the application of a definition. Four pages are used to introduce the concept of a variable by SMSG. Twenty-seven pages are used by UICSM. SMSG goes directly to the variable in their four pages. The only thing they try to accomplish in those four pages is to define a variable. UICSM not only lets the student develop an awareness of what a pronumeral is but also lays a great deal of background work for principles

$$3 + 9 = 9 + 3$$

$$^{-}8 + 0 = 0 + {}^{-}8$$

$$1 + 1 = 1 + 1$$

$$2 \times 3 + 6 \times 5 = 6 \times 5 + 2 \times 3$$

Pattern sentence:________________________________

$$1 \times 57 = 57$$

$$1 \times (8 + 3) = 8 + 3$$

$$1 \times (\square \times \triangle) = \square \times \triangle$$

$$1 \times -(3 - 7) = -(3 - 7)$$

Pattern sentence:________________________________

Figure 21. Establishing a Pattern Sentence for UICSM*

*UICSM, 2-12, 2-13

of addition and multiplication at the same time. It is felt that more understanding will result from the development by UICSM.

One point that is easily made by UICSM that SMSG never mentions is the use of upper and lower case letters. The point is made that a 'A' and a 'a' are as much different as a square frame and a triangle frame. SMSG does use capital letters as coefficients when writing general forms of equations.

Again it is hard to make any suggestions that wouldn't require a basic change in the structure of a text. But the use of discovery for variables would be useful in developing student understanding.

Ross Taylor made a statement concerning the use of letters for numbers in his article in the Mathematics Teacher.

> Perhaps the greatest single difference between the texts is concerned with the concept of the meaning of letters as used in algebra. The SMSG text uses the approach that the variable is the name of the definite though often unspecified number, while the UICSM text states that a variable (or as they say, pronumeral) holds the place for a name of a number. I can't pretend to be impartial or unprejudiced on this issue for I have been teaching in the UICSM program for four years, while this is my first year's experience with the SMSG program. However, I feel that the use of the placeholder concept is far superior to the name of a number approach. In the first place, the placeholder definition is logically sound. Second, the use of the placeholder definition allows for an extremely affective teaching device with the use of frames as placeholders for numbers and algebraic expressions.[77]

UICSM uses the frames occasionally throughout the remainder of the unit as a pronumeral.

Expression Containing Variables

SMSG. SMSG starts working with monomial and polynomial expressions

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[77] Ross Taylor, op. cit., pp. 479-480.

as early as they introduce variables themselves. Many of the simplifications of expressions containing variables are never formally defined but are an outgrowth of the work with the distributive, associative and commutative properties. While developing properties for real numbers, some theorems for simplification of expressions are listed in the section on reciprocals. Additional theorems and properties, other than the basic properties, have been discussed in the development of real numbers.

In developing reciprocals the following theorems are listed in the text and proven for the student. "The number 0 has no reciprocal."[78] "The reciprocal of a positive number is positive and the reciprocal of a negative number is negative."[79] "The reciprocal of the reciprocal of a non-zero real number a is a itself."[80] At this point the student is given some practice in applying the theorems just mentioned. The next theorem on reciprocals is "For any non-zero real numbers a and b $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$."[81] More practice is then given for the student. The final theorem in the section is "For real numbers a and b, ab = 0 if and only if a = 0 or b = 0."[82] Once again the student is given practice to apply what he has learned.

SMSG does not name a polynomial or a monomial until the second semester. After some discussion on their concept of a polynomial they give a formal definition: "A phrase formed from integers and

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[78] SMSG, op. cit., p. 174. [79] Ibid., p. 175. [80] Ibid.

[81] Ibid., p. 177. [82] Ibid., p. 179.

variables, with no indicated operations other than addition, subtraction, multiplication or taking opposites, is called a polynomial over the integers."[83]

The point that SMSG is pursuing, when the definition of polynomial is listed, is factoring of polynomial expressions. When some of the factored forms of polynomials are expressed, the statement is made that some of the factors are polynomials as well. The factors that include only indicated products or possibly the taking of opposites on the indicated products are called monomials.

The first factoring that is given as a formal procedure follows after some practice in identifying factorings for polynomials over the integers. This first procedure is the distributive property. The students first were introduced to the distributive property for numbers of arithmetic five pages before the variable was introduced. The distributive property was given a formal definition on page sixty-six. So the use of the distributive property is familiar to the students when it arises as a procedure for factoring. Several different applications of the distributive property are shown for one particular polynomial. Since only one of these is a complete factoring, it is indicated that the complete factoring is the form desired in applying the distributive property to polynomials. Factoring of polynomials of the form $a\ c + b\ c = (a + b)c$ are shown where c is a polynomial itself. Thus, $5(x - 2) + (x^2 - 2x)$ could be written as

[83] Ibid., p. 314.

$5(x - 2) + x(x - 2)$ by applying the distributive property to the second polynomial. Then by using the polynomial $x - 2$ as c and applying the above mentioned factoring form, the final factoring would be $(5 + x)$ $(x - 2)$.

After some practice applying the distributive property to polynomials that are already in a form that lends itself to using the distributive property, the form $x^2 + bx + ax + ab$ is considered. This is factored by the distributive property also, but the terms must be grouped first. When the first two monomials are grouped and the last two monomials grouped, the distributive property can be applied to each. The result is $x(x + b)$ for the first group and $a(x + b)$ for the second group. The distributive property is then applied to the two groups as a whole and the final factoring is $(x + a)(x + b)$. After some examples of regrouping monomials within the polynomial, the student is given some problems to factor. Since the simplest factoring must be obtained in each case, the student may have to regroup in a different way if his first grouping doesn't produce the simplest form.

The difference of two squares is factored next. This is accomplished by first finding the product of the indicated sum and the indicated difference of the same two monomials. Then the equality is reversed so that the product, which can be written as a perfect square minus a perfect square, is equal to the indicated product of the sum and difference of the same two monomials. In the exercise that follows an example is given of factoring the difference of two cubes and an example for the sum of two cubes. A few of each are listed for the student to factor in addition to the factoring of the difference of two squares.

When SMSG finds the product of two polynomials, it is done by repeated application of the distributive property. They use this method of expansion on the indicated sum and difference of the same two monomials in developing the factoring of the difference of two squares. This multiplication of polynomials is also used in developing other factoring forms.

The square of a polynomial consisting of an indicated sum or difference of two monomials is found by multiplying the polynomial times itself and then applying the distributive property until the indicated sum or difference of only monomials remains. Since there will be two monomials that are the same, the final expanded product can be written in the form $a^2 + 2(ab) + b^2$. It is then shown that all polynomials that are the sum or difference of two monomials can be put into this form. The next step is to reverse the equality and write the expanded product equal to the indicated sum or difference as a polynomial squared. In doing this the first and third monomial of the product must be able to be written as a monomial squared. The second monomial must be able to take the form of the product of the first and third monomials and two. If the second monomial has an opposite symbol involved in it, the polynomial squared to which it is equal will be an indicated difference.

After these special products and special forms for factoring have been completed, the general polynomials that are considered for factoring, are names as quadratic polynomials. Many of the special forms that were factored previous to the section labeled as quadratic polynomials were quadratic polynomials themselves.

SMSG expands the indicated product of $(x + m)(x + n)$ by repeated application of the distributive property. It is mentioned that m and n are integers in the product $x^2 + (m + n)x + mn$. Since m and n are specified integers, the process can be reversed and the polynomial $x^2 + (m + n)x + mn$ is equal to the indicated product $(x + m)(x + n)$. Examples of this factoring are then shown for specific polynomials. The integer factor of the monomial involving the first power of x is written as a sum of the two factors of the integer monomial. This is indicated in the general form by the $m + n$ factor and the product of the same two integers mn for the integer monomial. Thus $x^2 + 22x + 72$ is written $x^2 + (4 + 18)x + (4 \cdot 18)$ and then factored to the indicated product $(x + 4)(x + 18)$.

The factorization of the polynomial consisting of the indicated sum of three monomials, where the quadratic factor of the variable has an integer factor associated with it to form an indicated product, is somewhat different. The development of it is quite similar to the development of other quadratic polynomials. The product of general polynomials is found from the indicated product $(ax + b)(cx + d)$. This yields $(ac)x^2 + (ad + bc)x + bd$. The factoring comes from writing the polynomial to be factored in this form so the indicated product can be written. This involves a good deal of trial and error. The text then shows ways of reducing this trial and error process as much as possible through wise selection of factors.

Polynomials over the rational and real numbers are considered next. Tho definition is given for polynomial that involve rational numbers. "A phrasc formed from rational numbers and variables, with no indicated operations other than addition, subtraction, multiplication and taking opposites, is called a polynomial over the rational numbers."[84] The student is left the task of defining a polynomial over the real numbers.

Examples are used to show that a polynomial over the rationals can be expressed as the product of a rational number and a polynomial over the integers.

Other examples are used to show that an expression like $x^2 - 2$, which is not factorable over the integers or rationals, can be factored over the real numbers to $(x + \sqrt{2})\ (x - \sqrt{2})$. These examples lay the ground work for completing the square. When the square is completed, there is a number that is added and subtracted from the expression so that the value is not changed but the form of the expression is altered. When four is added and subtracted from $x^2 + 4x - 2$, the resulting expression is $(x^2 + 4x + 4) - 2 - 4$. This result can be expressed as a difference of two squares $(x + 2)^2 - (\sqrt{6})^2$. The difference of the two squares is then factored into $(x + 2 + \sqrt{6})\ (x + 2 - \sqrt{6})$.

The next section that SMSG develops is the algebra of rational expressions. This is a treatment of what is commonly called algebraic

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[84] Ibid., p. 347.

fractions. After a definition of rational expressions and some discussion about the definition, the domain of the variable is considered. It is shown that the domain of the variable will never include any value that would cause the denominator of a rational expression to be zero. For every value of the variable the rational expression must equal a real number. Rational expressions are then given the following properties.

"For real numbers a, b, c, d we have the properties:

(1) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

(2) $\frac{b}{b} = 1$ Where neither b nor d is zero.

(3) $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$"[85]

The text then proceeds to list examples for application of the above mentioned properties to actual rational expressions. The students then are given exercises that employ these properties. Most of the problems use a combination of properties (1) and (2). The two fractions are multiplied, the common factors of the numerator and denominator are arranged over each other, and property number two is applied. The third property is applied in the addition of fractions. The denominators are indicated as being common in the statement of the property. The finding of the lowest common multiple for the denominator is then shown. The expression of the denominator are expressed as prime factors. A rational

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[85] Ibid., p. 352.

expression, that is composed of the factors a certain denominator lacks to have the least common multiple, is written in the form of property two and multiplied as in property one. The result is a rational expression with the least common multiple as a denominator. After all of the rational expressions to be added have been given the least common multiple for a denominator, property three is applied to produce a single fraction.

There is never a time when SMSG considers addition and subtraction of polynomial expressions as a topic in itself. The addition and subtraction of polynomials is more or less implied in the development of these operations on real numbers. When the properties of addition and subtraction are developed, the use of polynomials over the integers is employed even though the student hasn't been given the name of the expression as that time. The associative and commutative properties for addition are the main properties used when polynomials are added or subtracted.

As it has been shown earlier in this report, multiplication is used a great deal in developing phrases of polynomials. One of the most direct applications of the use of multiplication of polynomials is shown in the development of factoring. Multiplication of polynomials was first used by SMSG on page 160. This first usage came soon after the properties for multiplication of real numbers was developed. In nearly all cases it had been shown as an application of the distributive property or repeated application of the same.

Division of the polynomial expressions grows out of the section on algebra of the rational expressions. It could not be shown before that time, since the rational expression is the first type of expression that indicates dividing a polynomial by a polynomial. The definition of division of polynomial is given after some examples are studied. First the division of an improper fraction to produce a mixed number is shown. This is expressed as the sum of the product of the whole number and the divisor with the remainder all over the divisor. An example using polynomials is written in much the same way. A monomial is factored from a portion of the polynomial to form a product of the monomial and the divisor. This product has the remainder of the polynomial added to it and is all written over the divisor. This form is then rewritten as a mixed expression of the monomial plus the indicated quotient of the remainder of the polynomial over the divisor. The definition is then stated "Let N and D be two polynomials in one variable. Then to divide N by D means to obtain polynomials Q and R, with R of lower degree than D, such that $\frac{N}{D} = Q + \frac{R}{D}$."[86] SMSG then develops a process for dividing a polynomial into a polynomial that parallels traditional algebra courses' division of polynomials process. In this division vertical arrangement of subtraction is used. Therefore this vertical arrangement is introduced at this time.

Exponents and radicals are never developed in connection with polynomials. A few exercises involve the use of radicals, but they are used as an extension of the students' knowledge of radicals of

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[86] *Ibid.*, p. 359.

monomials. Exponents have been used on polynomials in squaring the polynomials but never are developed any further. A few exercises indicate a higher power than two, but it is not in direct application to finding a higher power of the polynomial.

UICSM. UICSM uses the phrase "pronumeral expression" instead of polynomial expression in their program. As soon as the standard pronumerals are introduced, they are used to generate pronumeral expressions. Even before the introduction of standard pronumerals, when frames were still being used as pronumerals, the frames were used to generate pronumeral expressions.

One of the first points developed concerning pronumeral expressions is a value of a pronumeral. This is concerned with substitution of a real number for the pronumeral and finding the resulting value of the expression. The student, who is given a group of equations is told to generate some true statements and some false statements from these equations. If the student generates a true statement, he has found the root of the equation, but this fact is not pointed out at the time. Some of the equations are so constructed a true statement is generated by any correct substitution; some are false for any correct substitution.

The next two exercises are complementary to each other. The first lists a group of pronumeral expressions and gives a group of numeral sentences as problems. The object is to see which of the pronumeral expressions could be used to express each numeral sentence. As an example $2 + 8 + 3 + 5$ could have $x + y$ as a pronumeral expression by using 2 + 8 + 3 for x and 5 for y. It could also have a + b + c as a pronumeral expression by using 2 + 8 as a, 3 for b and 5 for c. The second exercise lists four numeral sentences in a problem. The object is to write a pronumeral expression that can generate the first three sentences but not the fourth. An example might be 3 - 2 + 3, 9 - 2 + 9, and 3x - 2 + 3x for the first three and 3 - 6 + 3 for the fourth. The pronumeral expression could be x - 2 + x or x - 2 + y.

With a basis for evaluation of a pronumeral expression laid and some experience in writing pronumeral expressions from numeral sentences, UICSM starts to develop the means for writing concise pronumeral statements about principles they have stated only in example form. The means they employ for this point has been used before when at the very beginning of the introduction to unit one two boys started to correspond and Stan was to teach Al arithmetic by mail. Their correspondence dramatized the difference between numbers and numerals. At this point the boys correspond again. This time the object is to point out the difficulty in stating principles accurately and concisely.

The discussion begins with Al's request to know what the left distributive principle is. Stan replied with some examples of the principle called instances by UICSM. Al's reply points out that these are only instances and certainly not all the instances of the principle. He needs to know what the principle itself is so that he can recognize it. He reminds Stan he has no one to ask where he lives. So Stan replies with this statement.

First, there is a numeral, followed by a left parenthesis, then a numeral, then a plus sign, then a numeral then a right

> parenthesis, then an equality sign, then a numeral, then a multiplication dot, then a numeral, then a plus sign, then a numeral, then a multiplication dot, then finally a numeral.[87]

Al quickly points out that $5\ (9 + 17) = 8 \cdot 6 + 41 \cdot 7$ would be an instance of the left distributive principle from Stan's description. Stan replies with an open sentence $x\ (y + z) = xy + xz$. He supplements the open sentence with the statement that numerals should be substituted for x, y, and z. Al's reply points out that the left distributive principle is supposed to be something about numbers. All that Stan gave him had to do with numerals and open sentences. Stan then tries to state what he means in terms of numbers.

> For each real number you take, if you multiplied it by the sum of your second real number and your third real number, you get the same number as you would get if you added the numbers you get by multiplying your first number by your second and by multiplying your first number by your third number. Whew![88]

Al's reply points out that $5\ (3 + 4) = 5 \cdot 4 + 5 \cdot 3$ could be an instance of what Stan wrote. Stan wearily replies that he meant to imply that third number was added to the second and the product of the first and third numbers was to be added to the product of the first and second numbers, but he forgot to. Al then makes the first entirely correct statement of the principle.

> For each first real number I pick, for each second real number and for each third real number, it turns out that the first number X (the 2nd number + the 3rd number) equals the first number X the 2nd number and the 1st number X the 3rd number.[89]

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[87] UICSM, _op_. _cit_., pp. 2-24. [88] _Ibid_., pp. 2-25. [89] _Ibid_., pp. 2-26.

Stan sees this is correct and writes it a shorter way, "For each x, for each y, for each z, $x(y + z) = xy + xz$."[90] Al then points out that different pronumerals used in the same pattern would make no difference.

It is at this point the student writes his own statement for many of the principles for real numbers. Examples of these principles for real numbers have been stated before, but the formal algebraic statement is furnished by the student. The students write the algebraic statement for rules of addition and multiplication of real numbers as the next exercise. Two of the rules for addition are shown in the text and the student writes the other two and all of the multiplication principles.

The text of UICSM then goes into the section on generalizations. A generalization is a general statement about numbers. Since it usually contains a pronumeral, it nearly always is a pronumeral expression. When the student is confronted with a generalization, they should be equipped to justify their belief about its truth. If the generalization stated is true, the student should be able to justify the answer by the principles for real numbers which he has learned. If the generalization is false, the student should be able to find a substitution for the pronumeral that would prove it false for at least that one case. Substitution that proves a false generalization is called a counter-example. For each x, $2 + 3x = 5x$. When five is used as a value for pronumeral $2 + 3 \cdot 5 = 17$ and $5 \cdot 5 = 25$, but $17 \neq 25$. Therefore 5 is a counter-example

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[90] *Ibid.*

for the generalization $2 + 3x = 5x$. The counter-example is a sufficient negative proof for a generalization, but what about proving a statement true? The development of a method of positive proof must be established as well. Tho students are given a generalization and shown three instances for it with different substitutions. The three instances are verified by computation of each side of the equation and found to be true instances of the generalization. The student is then asked for a substitution for the pronumeral that would yield a false statement. The question is asked of the student how he could be certain that there were no numbers that could be used to generate a false statement. It is pointed out that the student could not verify all instances of the generalization. UICSM then suggests that the left hand side of the equation and the right hand side be made identical. The first instance of the generalization is then used and the principles applied to it so that the left hand side is transformed to be identical to the right side. The same pattern of application of principles is then developed for the second instance, but a frame is used around the substitution for the pronumeral. Since the student is accustomed to using frames for pronumerals, the point can be readily made that this same frame could take on any value for the pronumeral. Thus the generalization is written using a frame for a pronumeral and the statement is made that this is a testing pattern for any instance of the given generalization. One more generalization is shown with the frame around the first substitution for pronumeral. When the testing pattern has been established the use of a standard pronumeral is employed and a second testing pattern established from the first pattern. In the second test pattern, the letter pronumeral takes the place of the frame in the first pattern. For each step in any of the testing patterns, there is a principle stated that justifies that step. At this point UICSM is quite particular about every change of any nature being justified. The student is then given a good deal of practice in writing test patterns.

The next step in the student's development of pronumeral expressions is the listing of several logical simplifications for a given generalization. The student is instructed to select the phrase that would complete the generalization and to be able to justify his answer. Some of the different choices could be found by different test patterns and still be correct. Some of the answers could be obtained with a faulty step in the test pattern. The student is beginning to perform simplification mentally, but he must always be prepared to produce a test pattern for his answer. The students develop the defense mechanisms of a counter-example to prove to someone who does not agree with them that they are wrong and test patterns to prove their answer correct for all those who claim to have a counter-example. Many times students will find their own error when writing their test pattern to try to refute a disbeliever.

UICSM discusses perimeters of geometric figures next. The simplification of perimeter of these figures is employed to establish a need to simplify expressions. The simplification of these figures brings the term of "equivalent expressions" into the students vocabulary. The beginning and end of generalizations are equivalent expressions. The definition of equivalent expressions is given after it has been illustrated through example.

> Equivalent numerical expressions are numerals for the same number. Equivalent pronumeral expressions are expressions such that for each substitution both expressions have the same value.[91]

After the geometric perimeter exercise, which shows a need for further simplification of expressions and the introduction of the term equivalent expression, the student is ready to begin to handle simplification of pronumeral expressions. An exercise is listed in which the students are given two expressions and asked if they are equivalent. The student is to state simply that they are equivalent or not equivalent. Since the student probably will need to do some simplification on the first expression before he can decide, an example is shown of the simplification process for showing two expressions equivalent. The reasoning behind the belief is given as well as the test pattern for the simplification. In the problems that list expressions that are not equivalent, the student finds a counter-example of them. The next several exercises give the students an expression and they are asked to simplify it. There are several simplifications that might be acceptable in some cases, but the student is to try for the simplest.

Up to this point UICSM has developed simplification of pronumeral expressions through application of the basic principles for real numbers. They point out at this time that there are many principles that would be

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[91] *Ibid.*, pp. 2-49.

quite helpful if they could use them. The example of the one times theorem is cited. The fact is stated that all the generalizations are actually theorems when expressed in their original for equal to their simplification, and the test pattern for the generalization is the proof of the theorem. The fact is stated that theorems can be used in justifying a step in a proof in the same way a basic principle is used. The left distributive principle is then proven as a theorem, using the basic principles for real numbers to serve as an example.

The basic principles, for which the student provided the algebraic statement as an exercise on page 2-27, are then stated for reference on page 2-61. The "universal qualifier" symbol, ∀ , is introduced at this time and is used from this point on as a substitute for the words "for each."

The student is next given the following theorems to prove as an exercise.

1. ∀x, 1·x = x (The one times theorem)
2. ∀x∀z∀b∀c, ax + bx + cx = (a + b + c)x (Extended distributive theorem)
3. ∀x∀y∀a∀b, (ax) (by) = (ab) (xy) (Product rearrangement theorem)
4. ∀x∀y∀a∀b, (a + x) + (b + y) = (a + b) + (x + y) (Sum rearrangement theorem)[92]

Addition principles are then considered by UICSM. The first principle UICSM solves is the uniqueness principle for addition. This parallels the addition property of equality in SMSG. It is proven and stated as "∀x∀y∀z, if x = y then x + z = y + z."[93]

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[92] Ibid., pp. 2-61.

[93] Ibid., pp. 2-64.

Then by use of the basic principles and the uniqueness principle for addition, the cancellation principle for addition is proven. It is stated as "VxVyVz, if a + z = y +z then x = y."[94] As an exercise the student is to prove the following theorems.

1. VxVyVz, if x = y then z + x = z + y (Left uniqueness principle for addition)
2. VxVyVz, if z + x = z + y then x = y (Left cancellation principle for addition)
3. VxVy, if x = y then -x = -y (Uniqueness principle for opposition)
4. VxVyVz, if z = y then xz = yz (Uniqueness principle for multiplication)
5. VxVyVz, if x = y then zx = zy (Left uniqueness principle for multiplication)
6. VuVvVxVy, if u = v and x = y then u + x = v +y
7. VuVvVxVy, if u = v and u + x = v + y then x = y.[95]

As the student solves these theorems he is to list a reason for each step.

Principles for opposites are then considered. The principle of opposites itself is stated as "Vx, x + -x = 0."[96] This principle corresponds to the addition property of opposites in SMSG. The 0-sum theorem is stated following some discussion of the principles of opposites. "VxVy, if x + y = 0 then y = -x."[97] The SMSG additive inverse corresponds to this theorem. UICSM then uses the 0-sum theorem to solve the theorem "Va, --a = a."[98] The student is given the following theorems to prove as an exercise.

1. VaVb, - (a + b) = -a + -b (Distributive theorem for opposition over addition)

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[94]Ibid., pp. 2-65. [95]Ibid., pp. 2-66. [96]Ibid., pp. 2-67.

[97]Ibid., pp. 2-68. [98]Ibid., pp. 2-69.

2. $\forall c \forall d, -(c + -d) = d + -c$
3. $\forall p \forall q\ -(pq) = p \cdot -q$.
4. $\forall x \forall y\ -(xy) = -xy$.
5. $\forall x \forall y$ if $x = -y$ then $-x = y$.[99]

After the proof of x, $---x = -x$ as an example, the following are proofs listed as another exercise.

1. $\forall x \forall y\ -xy = x(-y)$
2. $\forall x \forall y \forall z, -x(y + z) = -(xy) + -(xz)$
3. $\forall x \forall y \forall z, -x(-y + -z) = xy + xz$
4. $\forall x, x \cdot -1 = -x$
5. $\forall x, -x = -1 \cdot x$ (The -1 times theorem)[100]

Subtraction principles follow with a rather short development. Only two theorems are given for the student to write a proof. These two generalizations point out the fact that the inverse of adding a real number is adding the opposite of that number, and that adding the opposite of a real number is the same operation as subtracting a real number. The latter of these two facts is the principle for subtraction and is stated as, "$\forall x \forall y, x - y = x + -y$."[101]

Following the principle of subtraction comes an exercise that involves two operations. First is a group of open sentences in which the student must fill in a blank. Secondly the group of open sentences imply a theorem when they are all completed, and the student is to state the theorem and be ready to prove it. This type of exercise introduces many theorems about subtraction and is a diversion from the tedious routine of continual proving of theorems. Another exercise, used as a complement to the exercise just mentioned, contains multiple-choice completion theorems. The left side of the theorem is stated and

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[99] *Ibid.* [100] *Ibid.*, pp. 2-70. [101] *Ibid.*, pp. 2-71.

the student selects from four choices one that could equal the expression on the left hand side of the equal sign. This may require some proving on the student's part, but is a diversion from the routine proof of theorems. The student is then given a large group of pronumeral expressions that are to be simplified. These expressions involve the use of the many subtraction generalizations and theorems the students have been proving and expressing in their proofs. Many of the theorems and principles that have been introduced before the subtraction principles' development are used as well.

Division principles are the next topic considered. The first of these concerns an inverse for multiplying by zero. Since this point was covered in this section dealing with division of real numbers, it will not be discussed further at this time. UICSM then begins to look for an inverse of multiplying by a non-zero number. After some discussion, the statement is made that the inverse of multiplying by a nonzero number has an inverse if "$\forall x \forall y \neq 0$ there is just one z such that $z \cdot y = z$."[102] UICSM then continues their discussion and brings out the principle of quotients and the division theorem which have already been listed and discussed in an earlier section of this paper.

UICSM then applies the principle of quotients and the division theorem to prove the cancellation principle of multiplication. The student is given several other basic theorems to prove as an exercise.

1. $\forall x \forall y \neq 0\ \forall z$ if $zy = z$ then $z = \frac{x}{y}$ (Division Theroem)

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[102] Ibid., pp. 2-85.

2. $\forall x, \frac{x}{1} = x$
3. $\forall x \neq 0, \frac{x}{x} = 1$
4. $\forall x, \frac{-x}{-1} = -x$
5. $\forall x \neq 0, \frac{0}{x} = 0$
6. $\forall x \forall y \neq 0$, if $\frac{x}{y} = 0$ then $x = 0$[103]

The 0-product theorem is then discussed and stated as "$\forall x \forall y$ if $xy = 0$ then $x = 0$ or $y = 0$."[104] Throughout the development of division, there are short exercises listed to apply what the students have been learning.

Next simplification of expression containing fractions are developed. Two basic generalizations for the simplification of fractions are then listed and first of the two proven in the text. These generalizations are:

"$\forall x \forall y \neq 0\ \forall u \forall v \neq 0, \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}$

$\forall x \forall y \neq 0\ \forall u \forall v \neq 0, \frac{x}{y} - \frac{u}{v} = \frac{xv - uy}{yv}$ "[105]

The basic theorem for multiplying fractions is then introduced with examples and exercises to implement its use. The theorem is "$\forall x \forall y \neq 0\ \forall u \forall v \neq 0\ \frac{x}{y} \cdot \frac{u}{v} = \frac{xu}{yv}$"[106]

The two theorems for reducing fractions are stated as "$\forall x \forall y \neq 0\ \forall z \neq 0, \frac{xz}{yz} = \frac{x}{y}$" [107] and "$\forall x \forall y \neq 0\ \forall z = 0, \frac{x}{y} = \frac{x + z}{y + z}$."[108] To help facilitate the work in the reducing of fractions, the

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[103] Ibid., pp. 2-85. [104] Ibid. [105] Ibid., pp. 2-92.

[106] Ibid., pp. 2-93. [107] Ibid., pp. 2-94. [108] Ibid., pp. 2-95.

generalization "$\forall x \forall y \forall z \neq 0,\ \frac{xy}{z} = \frac{x}{z}y$"[109] is introduced. Several other generalizations follow from this point. Three are listed for the student and they are asked to prove them. "$\forall x \forall y \neq 0,\ \frac{x}{y} = x \cdot \frac{1}{y}$; $\forall x \forall y \neq 0\ \frac{xy}{y} = x$ and $\forall x \neq 0\ \forall y \forall z\ \frac{xy + xz}{x} = y + z$."[110] The student is then given a long exercise to practice reducing fractions.

Several modifications on theorems that have been listed are then shown for fractions with common denominators.

The student is given a generalization to prove that it will justify the invert-and-multiply rule that was given to them in grade school math classes. Theis generalization is "$\forall x \forall y \neq 0\ \forall z \neq 0,$ $x \div \frac{y}{z} = x \cdot \frac{z}{y}$."[111] Three other theorems are listed as part of the exercise given for application.

1. $\forall x \forall y \neq 0\ \forall u \neq 0\ \forall v \neq 0\ \frac{x}{y} \div \frac{u}{v} = \frac{xv}{yu}$
2. $\forall x = 0 \forall y \neq 0\ \frac{1}{x} = \frac{y}{x}$
3. $\forall x \forall y \neq 0\ \forall z \neq 0\ \frac{x}{y} \div z = \frac{x}{yz}$ [112]

A section follows in which several small but important points are made concerning opposite signs and how they can be used with division. These theorems on opposition and division are followed by a long list of exercises that make use of all the points the student has learned during the section on division as well as nearly all the principles and theorems that have been introduced up to this time.

At the beginning of the development of generalization, the student

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[109] Ibid., pp. 2-96. [110] Ibid., pp. 2-97. [111] Ibid., pp. 2-101.

[112] Ibid.

gave a principle to justify every step in a test-pattern or simplification of an expression. This listing is gradually diminished to where the student now lists the justifications for theorems only. If the student wishes to continue the practice, it is not particularly discouraged.

At the end of the second unit, UICSM leaves their development of pronumeral expressions and develops equations. The development of equations is a later section so it will not be discussed at this time. After tho development of equations, UICSM returns to their work with pronumeral expressions. They lead up to the expansion of an indicated product of two binomials through the multiplication of positive real numbers. There is an exercise given in which the student is asked to mentally simplify expressions like 22 X 24. Then several more problems ask which is the larger of a pair of products such as 23 X 25 or 22 X 26. It is quite easy for one who knows how to find the product of two binomials to mentally consider (20 + 3) (20 + 5) and (20 + 2) (20 +6) when checking to find the larger. In multiplying tho two pairs of binomials, it is found that the first product is the same in each and that the sum of the second two products would be the same for each and only the last product would give a different result. If the student recognizes this method, the exercise is quite simple. If not, the exercise is a laborious task. Whether tho student recognizes the method or not, a need is created for such knowledge. One section of the exercises indicates tho product of two binomials but still uses real numbers. The last few problems of the exercise make it necessary for the student to find the product of two binomials each containing a pronumeral to get the answer. UICSM then goes into an explanation of "expanding" pronumeral expressions. The method is a consequence of repeated application of the distributive property and the use of the distributive property once more to combine the like center terms. When the student has been given some practice at expanding pronumeral expressions, the expanded expression is given to the student with one of the binomials and the student is to supply the second binomial. This requires some thinking in terms of factoring. At this point the notation of an exponent is introduced and more practice given in writing the product of two binomials using this new notation.

The next point in the expansion of pronumeral expression is the squaring of a binomial. This point is approached with little extra concern. It is developed as another kind of multiplication and not as a special product.

Factoring is brought to the attention of the student next. The student has been using the distributive and left distributive principles for factoring and this fact is pointed out. The statement is made that $x^2 + 6x + 8$ and $(x + 4)\ (x + 2)$ are equivalent. Then a list of problems is given. In some of the problems one of the binomials is given and the other to be furnished by the students. In other problems one term of each binomial is given and the student is to supply the other terms and sign. On the next page an explanation for factoring is given. Frames are used as a placeholder for the numbers of each binomial. The product of two such binomials is found by expansion. This produces a pronumeral expression in letters and frames. The frames indicate that tho real number factor of tho pronumeral of the middle term will be the sum of tho same two numbers as the product for the real number term. After a pair of examples the student is given an exercise in factoring trinomial expressions. Some of theso aro perfect squares, but receive no different treatment than the regular trinomial expressions. An example is then shown using frames to factor a trinomial expression where the quadratic term has a real number factor. The frames indicate that tho products of tho first term and the factors of the last term must be added in the proper order using the proper factors to produce the real number factor for the middle term. A trial and error process is shown for finding the correct number. The factoring of the difference of two squares is shown by an example. The dovelopment of factoring from the first use of frames to show the method of factoring to the application of the factoring of the difference of two squares comes in one exercise with examples interrupting the exercise to show new types of problems as they appear.

Pronumeral expressions are used throughout the remainder of the year, but receive a further treatment after the introduction of exponent notation. The squaring notation was introduced during the development of expansion of binomials products, in tho third unit, but the exponents higher than two are introduced in the fourth unit. There are really no new theorems introduced after the introduction of exponent notations. The theorems that were introduced earlier are applied here to the larger powers of the pronumerals.

UICSM introduces radicals in their approximation of a square root. They extend the use of radicals to square roots of numbers that contain a factor which is a perfect square. The number under the radical is factored to two factors, one of which is a perfect square, and each put under a radical sign. The square root of the perfect square is then found and the approximation for the irrational root of the number that is not a perfect square is found and their product is an approximation for the square root of the original number. When the student has had some practice finding approximation for the square roots of numbers by this method, pronumeral expressions receive their development. The development and exercise for this section are quite brief. A generalization gives the basis for the simplification as follows: "...for each x, $\sqrt{x^2} = x$."[113] The exercise is only a few problems in length and contains mostly radicals that have perfect square roots. The trinomial pronumeral expressions that are listed can be factored as a binomial squared with only one exception. Occasional problems involving pronumerals under radicals are found throughout the remainder of the four units.

Suggestions. UICSM very meticulously develops pronumeral expressions. Nearly the entire second unit is devoted to reducing ambiguities in signs and manipulations. UICSM leaves little to be questioned in handling simplifications and transforming of expressions. The proofs that are used in developing the generalizations and theorems are not too long or difficult. The student becomes quite adept at

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[113] Ibid., pp. 2-132.

working them. It is felt that this sort of training is of great value to students in mathematics. A suggestion of the author to the teacher presenting the UICSM material would be to watch the student closely during the period in which so many proofs are boing required of them and not to make them do so many that they become bored. It is felt that students, while gaining a great deal from the experience of formal proofs, might lose interest bocause of the repetition of the exercises.

The variation of the two approaches to variables is probably more pronounced than on any other single topic. SMSG introduces the variable in their second chapter. This would probably be the second or third week of school. UICSM begins their presentation of pronumerals about tho ninth or tenth week of school. Once the topic is introduced in SMSG there are several other topics pursued in which the variable plays a part in the development. Use of the variable is part of the development of the basic principles for real numbers. Equations are being solved by SMSG before a full treatment of variable expressions is considered. Property of order, exponents and radicals are developed before the chapter on polynomial and rational expressions. SMSG gives their chapter on polynomial and radical expressions about the fifth or sixth week of the second semester. UICSM pursues their topic of pronumerals from its introduction at the beginning of the second unit, throughout the entire second unit. This would cover about the second quarter of the school year. The factoring of each unit would come at approximately the same time. UICSM presents factoring after equations in their third unit near the end of the third quarter's work. Both programs make considerable use of the variable or pronumeral after the introduction. UICSM's later introduction of the variable is primarily due to their development of the real number, which is the concern of the first unit.

SMSG's early introduction to variables gains certain advantages for their program. It allows them to develop formal statements and definitions of topics like real numbers as they encounter them. Since this is one of the underlying distinctions of the program, the early introduction of the variable is a necessity. They can also integrate solving of equations and find truth sets as they progress.

UICSM's development of pronumerals starts later than most other programs, but concentrates on the development and utilization of the concept once it is introduced. From the variation of the two programs in presenting factoring, it is felt that the student must gain a better understanding of variables from UICSM's presentation. The factoring by UICSM is a short topic consisting of a few examples and a short explanation of procedure while SMSG uses several pages of material to explain the procedure of factoring.

SMSG presents one topic that UICSM doesn't in connection with their work in polynomials at the first year level. This topic is division of a polynomial by a polynomial.

The author's philosophy of organization of material seems to be more in line with UICSM's presentation in most cases. Variables and pronumerals show a difference in method of presentation as well as organization and arrangement of the topics. This author would recommend the development used by UICSM. He would also suggest that SMSG use more proofs in development of simplification of expressions and algebraic manipulation. It is felt that the student might lack the knowledge to handle some manipulations that arise.

EQUATIONS AND INEQUATIONS

Equations

SMSG. Quite early in the first semester SMSG begins to find the truth set of an open sentence. At the start of chapter three, sentences are introduced. These sentences are defined as assertations about numbers. Some examples of sentences are shown in the form of example and the point made that sentences can be true or false and the student should be able to recognize that fact. A short exercise in selecting the true and false assertations out of a group of problems is then given. On the next page open sentences are introduced. The open sentence is defined as a sentence that contains a variable. The sentence is said to be open since it cannot be known whether the sentence is true or not without more information. The truth of some open sentences is then determined by the student when a value is suggested for them. There is some experimentation shown the student for the purpose of finding a number that would make them true sentences.

Truth sets of open sentences is the next subject. An open sentence is listed and its domain given as the set of all numbers of arithmetic. After several substitutions it is shown that only one of these numbers produces a true sentence. It is then mentioned that the open sentence is a sorter. It sorts the domain of the variable into two subsets, one of which will make the sentence true and one that will make the sontence false. The truth set of the open sentonce is the set of all numbers which make the sentence true. An exercise follows in which a number is given as a substitution for the variable and the question asked if it belongs to the truth set of the open sentence for which it is given. Another exercise gives an open sentence and a set of numbers which include the truth set. The student is to find the truth set from the given set.

Graphs of truth sets follow. These graphs for single variate open sentences consist of a number line with a dot on the number line to indicate the truth set of the open sentences.

Tho next topic is open sentences involving inequalities. Open sentences of this type will be discussed as a separate section of this paper, but need to be mentioned here since compound sentences involving an inequality and equality are the following topic. A simple open sentence is shown for which the truth set could easily be found. Then the corresponding open sentence involving an inequality is shown and the fact stated that any number larger than the truth set of the corresponding open sentence would be a solution. This solution is then shown geometrically by means of a graph. The notation for this type of graphic representation was shown in the section on notation. After some practice in graphing truth sets and writing open sentences involving inequalities from graphs of solutions, SMSG goes on to compound inequalities.

When a sentence has more than one clause it can be connected by either of the conjunctions "or" or "and." "Or" is used as the conjunction between these two because either can be used. The same is true for a mathematical statement. "And" is not used because both of the conjunctions can't be used at the same place. The "and" indicates that both clauses are to be considered in the same way that both conjunctions would need to be used if they had been connected by the conjunction and. For a compound mathematical statement to be true, both of the clauses it connects must be true. Two pages of discussion about clauses connected by "and," or "or" is given and some exercises in determining if the resulting sentences are true or false.

The graphs of these compound open sentences are then developed. These have been discussed in the notation section of this report.

The identity elements are then developed by use of the variable and equations. It is shown that zero must be added to a number to get the same number back. A similar quality for one is shown for multiplication property of one.

The use of variable and equations is employed as SMSG then develops the associative, commutative and distributive properties.

The next chapter involves writing open sentences from English sentences and writing English sentences from open sentences. SMSG doesn't go into any solutions at this point. They are mainly concerned in developing the ability to write these sentences and phrases.

The next chapters for SMSG introduce real numbers and the properties for addition and multiplication of real numbers. In the properties that SMSG introduces for addition are the additive inverse, addition property of equality, and the property of opposites. These properties are basic in the solution of equations. Properties that are developed in the chapter on multiplication are the multiplicative inverse and the multiplication property of equality.

As soon as the student has been equipped with the properties of addition and multiplication just named, SMSG starts a systematic method of finding roots. They start with the equation $3x + 7 = x + 15$. They add the additive inverse of x and the additive inverse of seven to both sides of the equation by the addition property of equality. After the associative and commutative principles have been applied, tho property of opposites is employed to make the equation read $2x + 8$. Using the multiplication property of equality and the multiplicative inverse of 2, the equation can then be written as $x = 4$. It is mentioned with each step that a value for x in one equation would also be a value of x in tho equation preceding it. This is true because only the addition property of equality, and the multiplication property of equality are usod.

After a reconstruction of the equation to show that the solution for $x + 4$ must be a solution for $3x + 7 = x + 15$, a statement is made about $x + 4$ and $3x + 7 = x + 15$ that is used in other equations. "...x is a solution of '$3x + 7 = x = 15$' if and only if x is a solution of '$x = 4$'."[114] There is a statement made as a result of the if and only if statement and a reconstruction of the original equation by reversing the steps in the solving of the equation. The rosulting statement shows

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[114]SMSG, op. cit., p. 168.

that "x = 4" and "3x + 7 = x + 15" are equivalent sentences since they have the same truth sets. The student is then given a group of exercises to work after one more example is shown in the text. As part of the exercise the students are to complete some English statement of problems. For the first time the student is told to find the truth set for the worded problems after they are translated into open sentences.

Six chapters and 205 pages later, the topic of truth sets for open sentences is once more revived. A quick two page summation of the earlier presentation opens the chapter. The intervening material has included the section on rational expressions. The equations that are presented, as a result of simplifying rational expressions, form quadratic equations. A discussion of quadratic equations will follow in a later section and will not be discussed at this time.

The next chapter deals with graphs of open sentences in two variables. After the student has been introduced to the standard notation and terminology of the coordinate axis and to the graphing and naming ordered pairs on the coordinate axis, the development of the system of graphing an equation in two variables begins in earnest. It is established that for each number associated with the first variable, there will be a second number to be associated with the second variable. In this way the open sentence is to act as a sorter. It sorts the set of all real numbers into two subsets. These subsets are the set of ordered pairs which make the sentence true and the set of ordered pairs that make the sentence false. Another point stressed in this section is the meaning of an equation like "x = 6" or "y = 9." The first equation is an abbreviation for "(1)x + (0)y = 6" and the second an abbreviation for "(0)x + (1)y = 9." The student is then given some problems in expressing equations of this type in the manner just described and in finding some solutions for equation with two variables.

The open sentence not only sorts the set of all real numbers into two subsets, but also sorts the points of the number plane into two subsets as well. The two subsets consist of all the points which when coordinated satisfy the sentence and all the other points. The coordinates of the subset which satisfy the sentence is called the graph of the sentence.

An equation is then transformed into what is called y-form. The y- form is the equation solved for the variable y. From the y- form of the equation a table of values if formed and the points located on a coordinate axis. The fact is then stated, after a short discussion, that a line connecting the points would contain only points that satisfy the given equation. In fact, every point on the line will satisfy the equation. It is then stated if a specified line is the graph of a particular open sentence, two questions are answered affirmatively. "(1) if two ordered numbers satisfy the sentence, they are the coordinates of a point on the line; (2) if a point is on the line, its coordinates satisfy the open sentence."[115] The following general statement is then made. "If an open sentence is of the form Ax + By + C = 0, where A, B, C, are real numbers with A and B not

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[115] Ibid., p. 416.

both zero, then its graph is a line; every line in the plane is the graph of an open sentence of this form."[116] The student is then given some practice in graphing equations.

The next section concerning graphs develops the concept of graphing by the slope-intercept method. After the text provides discussion and exercises on the concept of the slope of a line, this definition is given: "The slope of a line is the coefficient of x in the corresponding sentence written in the y- form. It is the number which determines the direction of the line."[117] A discussion then follows on determining the slope by a ratio of vertical change to horizontal change. The theorem for this point is given as follows: "Given two points P and Q on a non-vertical line, the ratio of the vertical change from P to Q is the slope of the line."[118] During the discussion before tho first statement about the slope of the line, the y- intercept is defined for the student as the number that gave the position of the line. It is pointed out to be the real number of the equation in its y- form. The student then receives practice in writing equations from information about slopes and intercepts as well as drawing graphs using this method.

Graphs of open sentences when only integers are considered is then shown. Their graphs are similar to the others, but only points appear instead of lines. Graphs of absolute values are also shown.

With this background in graphs for open sentences with two

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[116] Ibid. [117] Ibid., p. 427. [118] Ibid., p. 430.

variables, the topic of systems of equations is then begun. The first item shown is the graph of two different equations on the same graph. The fact is stated that the graph of these two equations could be connected by "and" or they could be connected by "or." The difference in what would be the result of the two considerations shown and the statement made that the connective and would be used when talking about the truth set of a system of equations. The student is then given a few systems of equations for which he is to find the truth set by graphing. It is then pointed out that some of the roots are difficult to find, since the graphs of the two lines do not intersect in such a way that integers are determined for roots. This necessitates approximation. The fact is pointed out that many lines go through the same point. Some examples are shown to illustrate this fact. A system of equations, for which the roots are familiar to the student, is selected and both equations multiplied by different numbers. Then the roots for the original system are substituted into the equations obtained from multiplying the original system of equations and the roots satisfy the equations.

SMSG writes each equation in a system of equations equal to zero. If both of the equations equal zero, then the sum of the two would also equal zero. Since any multiple of an expression that equals zero would also equal zero, the multiples of the systems of equations can be added and set equal to zero. SMSG then makes a formal general statement about systems of equations.

> If $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are equations of two lines which intersect in exactly one point, and

> if a and b are real numbers, then a(Ax + By + C) + b(Dx + Ey + F) = 0 is the equation of a line which passes through the point of intersection of the first two lines.[119]

The information in this statement is then applied to eliminate one variable from a system of equations by selecting a and b so that sum of the two monomials in x or the two monomials in y will be zero. The resulting equation will be an equation in one variable which the student is equipped to solve. SMSG first shows this process to find the root of each variable by selecting a and b twice, once to eliminate the monomials that contain x and the other time to eliminate the monomials containing y. When the student is ready to work exercises for himself, an example is shown for finding the second variable by substituting into one of the original equations. Up to the point where the student begins to solve systems of equations for himself, the sum of the two equations has been shown horizontally. After the students have solved a few problems, the vertical arrangement is shown. Later the substitution method of eliminating one variable is developed in the conventional manner of traditional algebra.

UICSM. Equations are so much a part of the presentation of both programs that each uses them from the very beginning. Both programs use equations before they introduce variables. UICSM uses the equation much longer before pronumerals are introduced because of their belated introduction of the pronumeral itself. Two units and the

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[119] *Ibid.*, p. 470.

first semester are planned to be covered before formal solution of equations begins for UICSM classes. As it has been shown, unit two deals with generalizations and theorems. Each of the theorems has used equations to show the relationship implied by the statement of the generalization. There are three different places that the student is actually required to solve equations before any formal method of proof has been taught.

During the first unit, when real numbers are being developed, several situations are present in which the student must fill in the missing number. He will actually have to solve the equation in some cases to do this. On page seventy-two of the first unit, the student finds statements of the following nature. "Add $^{+}3$ to it, and you get $^{+}7$. Add $^{-}7$ to it, add $^{+}7$ to the sum, and you get $^{+}4$."[120] This type of a problem actually requires a mental solution of an equation. In the introduction to the second unit, when frames are being developed for use as pronumerals, an exercise is listed that involves solving of equations. The problems of the exercise are stated in English with a circle in place of the numeral that would make the sentence true. An example would be, "If I subtract -3 from ◯, I get +4."[121] A few pages later in the second unit, the student has been given letters as standard pronumerals, and has been shown how to evaluate an expression when given a value. An exercise follows, stated in standard equation notation for the first time. The equations are listed and the student

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[120]UICSM, op. cit., pp. 1-72. [121]Ibid., pp. 2-E

is told to generate some true statements from the equations by finding a value for the pronumeral that would make it true and to generate some false statements in the same fashion. Each time a true statement is generated, the equation has been solved. The fact that the student has solved the equation is not mentioned at this time.

The introduction to chapter three, whose title is equations and inequations, is the next time the student becomes involved in solving equations. The introduction to unit three is only one page. This is unusual when comparing it with the other unit's introductions. The need for finding a way to solve equations is established on this page and the course's teacher can assure those who may have been wondering, that they are going to learn to solve equations.

The solution set of a sentence is the next topic that involves solving of equations. An inequation is shown; several replacements for the pronumeral that would make it true; and several substitutions for the pronumeral that would make it false are shown. The statement is then made that the numbers that make the sentence true are said to satisfy the sentence. The set of numbers that satisfy the open sentence is said to be the solution set of the sentence. The student is then asked to describe the solution sets of several equations and inequations. The second part of the exercise to give an open sentence for a solution set of both equations and inequations. Set notation is then introduced for writing solution sets. Instead of the verbose expressions the set of all numbers such that each of them plus six is less than nine, the set notation produces $\{x:x + 6<9\}$ for the same sentence. The two sentences are read the same. The student then continues to find the solution sets. As the next part of the exercise, the student is to find which of a group of equations would have (5, -5) as a solution set.

The next section deals with graphing of solution sets on the number line. Since the notation section has shown how this is denoted, it will not be covered at this time. The development of graphing of solution sets is mostly shown by examples.

The following section deals with the locus of inequations, compound inequations, and compound inequations and equations. The notations for writing the interval produced by such inequations and equations have been shown previously.

Equations are then pursued in their own section. A definition of an equation is listed and a definition of an inequation follows. "To solve an equation is to find all of its roots,"[122] is another statement made. The student is then given a long list of equations to solve. The only immediate explanation that is given is an example. An equation is written and the statement made that any number that could be added to the indicated real number to equal the other indicated real number would be a solution. The number that would satisfy those conditions is stated and is named as a root. The question is then asked if the root already found is the only root. A bit of help is then given by suggesting that the cancellation principle for addition might help to

[122] Ibid., pp. 3-19.

answer the question. Some of the equations that the student is asked to solve in the group of problems involve two and three steps. The next part of the exercise orients the student to applying his knowledge of simplification. From a list of simpler equations that might be the same, the student is to pick the proper equation. In the next section of the exercise the student begins to simplify equations on their own after some examples are shown to guide them toward the proper thinking. This exercise starts on page 3-19 and runs to page 3-25. The first problem on the first page is $x + 9 = 15$. The last problem on the last page is $5(bb - 3) - 8(bb - 3) + 3(bb - 3) = 6$. On the last section of the exercise, the student has some examples for simplification for which the roots are listed, but not more than the simplification is shown to them.

The next page lists the notation for subsets. The symbol is $\subseteq$. The problems are true or false problems. An example is $\{x : x + 4 > 6\} \subseteq \{x : x + 4 > 5\}$. The example is read as, the solution set of $x + 4 > 6$ is a subset of the solution set of $x + 4 > 5$. The student is shown the thinking that goes into the solution of the problem and then a list of similar problems involving sets of numbers, solution sets of inequations and solution sets of equations.

The next exercise leads the student toward what is called the addition transformation principle. A set is given and a group of sets that are possibly equal are listed below it. The list of sets is such that the student should recognize them as having the given set as a subset. The transformation principle for addition or the transformations principle for multiplication is implied in most of the sets listed. These principles are actually and uniqueness and cancellation principles for each operation that were developed in the second unit. The next section of the exercise is similar, but the equations that have the same roots are to be found. The following section of the exercise lists an equation and requests three other equations with the same roots. The next two sections show that the equation has the first equation's solution set as a subset of its own solution set and shows that the sets will be identical. The same thing is shown for multiplying each side of the equation by the same number. These last two sections are to end with a precise statement of the addition transformation principle and the multiplication transformation principle. Each of these principles combines the uniqueness principle and the cancellation principle for each operation into a two-part principle named by the single transformation title. These principles aren't stated in the student's text at this time, but they are listed in the commentary.

Equivalent equations are presented as the next section. This section applies the use of the transformation principle just learned and points out what the procedures are in solving an equation. The student should have a pretty good nonverbal awareness of these from the earlier solving of equations that he has done. The first example worked in this section is $3x + 4 = 19$. The statement is made that fifteen must be added to four to equal nineteen. Thus, the product of three and some number must be fifteen. The division theorem is applied to get $\frac{15}{3}$, showing five is a root. Then $5x + 9 = 13 - 2x$

is given. Since this equation is more involved, guesses are tried and the explanation is made that there is a better procedure. If an equation that has the same roots can be found that is also easier to solve, the solution would be much more simple. The transformation principle is shown in several different ways and the fact mentioned that all of these equations would have the same roots and would be equivalent equations. Since an equation with a pronumeral on just one side would be easier to solve, the addition transformation principle is then applied using 2x. The equation produced is $7x + 9 = 13$. The addition transformation principle is applied once more using -9 and $7x = 4$ is the result. Then the multiplication transformation principle is applied using $\frac{1}{7}$, and $x = \frac{4}{7}$ is the result. The steps are all put together to give the student a model of an equation and its solution through use of equivalent equation.

Various kinds of equations are then developed using transformation principles and equivalent equations. When equations that contain pronumerals in the denominator of a fraction are considered, it is evident that no value that would produce a value of zero for the denominator of the fraction can be allowed as a root.

The next two sections apply the use of equations: The first sections apply the knowledge of equations to the transforming of formulas; the second section to solving of worded problems.

In the fourth unit UICSM develops their treatment of systems of equations. Since the development of systems of equations comes as a consequence of the ability to graph, graphs must be considered first.

In the introduction to unit four the number lattice is introduced. Since the lattice has been discussed previously in this paper, it will not be discussed at this point. The lattice and work with the lattice gives birth to terminology that is then applied to the number plane. Since the coordinate axis are called first component axis and second component axis in the work with lattices, the terminology carries over to the number plane. Many other facets of the lattice are used again in the number plane.

While still considering the number lattice, UICSM considers intersection of sets and union of sets. Examples of an intersection of two sets and the union of two sets are shown. They are also given a verbal description. An exercise is then given for the student. Set notation is used to describe the solution set for sentences under consideration. One set called A is described as "$\{(x,y)$; x and y integer: $-2<x<3$ and $3<y<6\}$."[123] Another, set B, is described in a similar fashion. Questions are asked about how many points would be in the set A and in the set B; how many points in the intersection of sets A and B and how many points would be in the union of sets A and B. During the exercise the standard notation for intersection, $\cap$, and the standard notation for union, $\cup$, are introduced. The phrase "the number of points" is replaced by the letter n written in front of the set's name in parenthesis. "The number of points in set A: is then written "n(A)". "The number of points in the intersection of sets A and B: is then written "$n(A\cap B)$". The exercises progress to where equations containing pronumeral expressions are being considered.

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[123]<u>Ibid</u>., pp. 4-12.

The last problem of the exercise is as follows.

A = (x,y), x and y integers : $2y + 2x = 1$
B = (x,y), x and y integers : $3y - 3x = 1$
$n(A) = \_\_\_$, $n(B) = \_\_\_$, $n(A \cap B) = \_\_\_$, $n(A \cup B) = \_\_\_$.[124]

The $n(A \cap B)$ in this type of a problem would be one if the roots of the equation were integers or zero if they were not. The $n(A \cup B)$ would be infinite. The n(A) and n(B) would also be infinite. The method of finding the number of points would be to graph the open sentences on a number lattice and then to count the points for each question when possible.

The number plane is then introduced. The same sort of a plane is considered as in the number lattices except all real numbers are considered instead of only integral real numbers. After the terminology is set the student is given practice locating points and some numbers games to get the student working with pronumeral expression in relation to the number plane. The student is then given a set of problems which involve the graphing of an equation in two variables. Some of the graphs contain inequations and the greater than or equal to or less than or equal to inequations as well. When a strict inequation is graphed, a dotted line is used to show that the line is not included in the graph. When it is to be included in the greater than or equal to or less than or equal to inequation, the line solid line is used. The student is then given practice writing equations and inequations from graphs.

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[124]*Ibid.*, pp. 4-15.

At this point the term locus is given the same meaning as the solution set of an open sentence. The locus of a pronumeral expression is discussed and shown by example. The order in which the pronumerals are used makes a difference. This has been pointed out to the student in the use of ordered pairs. If the pronumerals used are not x and y, then whichever pronumeral that is selected as the first component for graphing can make a difference on the appearance of the locus.

After some practice in drawing the locus or graphing the solution set of some open sentences, the standard terminology of x - axis, and y - axis are given to the coordinate axis and the number plane is abbreviated to (x - y)-plane. More practice is then given for graphing, The quadrants of the graph are then named and established. The student is given some practice in stating where the quadrants points of a solution set of a graph will fall. The next exercise is the culmination of the study of solution of two equations with two pronumerals for the first four units. The student is given a pair of equations and the instructions to graph the equations and give the ordered pairs which are the intersection of their solution sets. No algebraic solution of systems of equations appears in the first four units. That topic falls in unit five which is planned for the fall of the next year.

Suggestions. SMSG has a better development for equations than most traditional texts. They apply the additive inverse and multiplicative inverse technique which is not generally found in a traditional algebra text. This one point is their main variance from what a traditional text would give in developing equations. The use of the properties in connection with solution of equations will produce a better understanding than would the axioms used by any traditional texts. It certainly is much better than such rules as "change the sign when you take it across the equals sign" that have been used not many years ago and probably are still being used in classes today.

In relation to UICSM's development of equations, SMSG has the lack of discovery found in UICSM. The student in UICSM is led toward and encouraged to find his own method of solving through the exercises that are listed. The student also states his own transformation principles that are used in solution of equations. It is felt that anytime the students have an opportunity to try something on their own before a full explanation is given, he will be more oriented to the problems he is facing and have more interest in what is being presented than when a definition is given and then a list of problems given with which to apply the definition. Until trying some problems the student doesn't know where his trouble lies. If the student tries to work the problems first with some small guides to start him, he may understand on his own, but he will at least know what he needs to learn to accomplish tasks of a similar nature.

UICSM goes to greater lengths to prepare the student for upcoming topics. The material is so written that the material to be covered later is being developed in most of the earlier topics. Equations are used from the beginning of the material and some solutions are actually required in much of the material previous to the introduction of solution sets themselves. As was stated earlier in this report, a UICSM participating teacher said that the students were able to solve equations the very first time they attempted them with no real explanation or help from the teacher.

The programs vary widely as to the approach on the subject. Both employ principles or properties to the solution of equations. These principles or properties help the student to understand why he is able to solve an equation by the steps that are taken. Either program should give the student a good background in solutions of equations.

In the case of systems of equations, UICSM has a more modern approach through the use of set notation. The intersection of sets for the solutions of the system of equations should be more understandable for the student. SMSG does go into the algebraic solution of systems in the first year. This is a point they must consider by the sequence of their work. UICSM puts the algebraic process in the next year and can do so by virtue of their organization of material.

It is felt that UICSM has the better overall program for equations through their more modern approach and utilization of discovery.

Inequations

SMSG. The development of inequations is unified with the study of equations in both programs to a degree. As each program develops notation for equations, they develop similar notations that apply to inequations. Several times both programs will develop a point for inequations at the same time they develop the point for equations.

SMSG introduces the notation for inequations quite early. When chapter four is developing open sentences, the open inequation is considered at the same time. As has been mentioned, chapter eight develops the proporties of order. When the binary operation of ordering was considered by this report these properties were listed. Solution of inequations was developed in chapter eight as well. After the introduction of the addition property of order, the solution of inequations that can be solved by use of that property are developed. The solution of these inequations is nearly identical to the solution of equations. The answer is expressed as an inequality, but the solution is obtained in the same manner the equality is obtained as the solution in an equation. The addition property of order is used in much the same way as the addition property of equality and the additive inverse are used on equations.

The multiplication property of order is then introduced. It is used to solve inequations that have variables with real number products. The two properties combined are used to solve all inequations that arise. The multiplication property of order is used for inequations in the same manner the multiplication property of equality and the multiplicative inverse are used for equations.

The solution of systems of inequations for SMSG is approached entirely by graphing. Since the solution of the systems of equations or inequations is the graph of the two clauses when considering the connective "and," only the solution that overlaps is correct. In graphing of inequations, regions of the graph are considered instead of lines. The solution of two inequations in a system of inequations is the region of the graph that is shaded by both inequations.

UICSM. Solution of inequations is somewhat combined with the solution of equations. The algebraic solution of the inequation is a separate section. After stating the transformation principles of equations as a reminder, UICSM solves an inequation for the student and asks if they believe the solution to be a valid solution. There must be transformation principles similar to the transformation principles for equations if the solution of an inequation is to be possible. The next few pages are spent developing these transformation principles for inequations and showing examples of their application. The addition transformation principle for inequations is stated "$\forall x \forall y \forall z$, ($x + z > y + z$ if and only if $x > y$)."[125] The multiplication transformation principle for inequations is stated as the following two parts, "(a) $\forall x \forall y \forall z > 0$ ($xz > yz$ if and only if $x > y$) (b) $\forall x \forall y \forall z < 0$ ($xz < yz$ if and only if $x > y$)."[126] The factoring transformation principle for inequations is stated as "(a) $\forall x \forall y$ ($xy > 0$ if and only if (($x > 0$ and $y > 0$) or ($x < 0$ and $y < 0$)) (b) $\forall x \forall y$ ($xy < 0$ if and only if (($x > 0$ and $y < 0$) or ($x < 0$ and $y > 0$))."[127] Examples are given on application of the principles to actual inequations, and then an exercise is given for the student to apply his understanding. Some of the examples show the graphic representation as well as the algebraic statement of the solution.

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[125] *Ibid.*, pp. 3-101. [126] *Ibid.*, pp. 3-102. [127] *Ibid.*, pp. 3-103.

In unit four when the set notation has been developed and the intersection and union of sets is being found, inequations are developed with equations. In the sections on lattices, where the elements of the solution set must be examined in terms of integers, most of the inequations were of the type $2 \leq x \leq 4$. This type of inequation makes graphing much easier. When the number plane is introduced, the inequation is present to receive application of its solution set to the new device. The standard notation for graphing the solution set of an inequation applies to UICSM's graphing as well as SMSG's. UICSM never mentions the solution of systems of inequations. Even the graphic solution is left to a later unit.

Suggestions. The solution of inequations in each program follows the pattern that has been established in the solution of equations. Most of the suggestions made for the equation section would apply to inequations as well. The programs do not differ a great deal in their presentation of this point. The main difference is that SMSG solves systems of inequations by graphing and UICSM doesn't in their first four units.

Quadratic Equations

SMSG. In the section dealing with factoring of polynomial expressions, SMSG introduced completing the square for polynomial expressions. If an expression is not a perfect square trinomial, it can often be made into such a trinomial by adding and subtracting the same number to the expression. The number that is added completes the expression to be a perfect square trinomial. This perfect square trinomial is then factored as a binomial squared. The indicated difference of the binomial and the real number is then factored as a difference of two squares. Not always is the real number a perfect square. If it isn't,,then the real number is written as a square root radical squared to make an indicated difference of two squares. When the expression has been written as the indicated difference of two squares, it is said to be in standard form.

When a quadratic equation is to be solved by SMSG, it is written first in standard form equal to zero. After the equation is written in standard form, it is factored and each factor is then set equal to zero and the resulting equations solved. This setting of each factor equal to zero is justified by the theorem if ab = 0 then a = 0 or b = 0. This same theorem is used to solve factorable quadratic equations that are not perfect squares.

In the chapter on functions the graphic solution of a quadratic equation is shown. The roots of a quadratic equation can be found by graphing its function and determining the values at which the graph crosses the x-axis. The abscissa value or values when the graph intersects the x-axis are the roots of the equation.

The very last problem in the SMSG text is the solution of the general quadratic equation for x. This solution produces the quadratic formula. This quadratic formula is named but never used in the solution of quadratic equations in the first year algebra course.

UICSM. Quadratic equations for UICSM are treated largely in a later unit not included in the first year's work. The only solution of quadratic equations that UICSM develops in the first four units is by a factoring method. The equation is written as a pronumeral expression equal to zero. The pronumeral expression is then factored and each factor set equal to zero. Each of these resulting equations is solved to obtain the roots of the quadratic equation. As usual UICSM tries to prompt some interest in the topic before it is introduced. On the page preceding the explanation of the solution of quadratic equations the student is asked to solve some equations of the same type. The sample the student receives before the explanation is a part of a section on factoring. Quadratic equations that contain unfactorable expressions are never considered in the first four units.

Suggestions. Once more UICSM's unconventional arrangement makes the comparison of the two programs somewhat difficult. Since quadratic equations are never really given a full treatment by UICSM the subject can hardly be evaluated. SMSG approaches the solution of quadratic equations from quite a different angle than this author has encountered before. Adding and subtracting a number to complete the square has been a topic for second year algebra in traditional algebra texts. This method of completing the square followed by expressing the equation in standard form of the indicated difference of two squares to allow the expression to be factored is unique in the experience of this author. It is felt that the method is sound and should produce good results.

ADMINISTRATIVE CONSIDERATIONS

Implementing the Program

A good deal of consideration must go into picking either of these programs. Several items would have to be taken care of before either program could be implemented into various school systems. One of the main points that must be considered is teacher preparation. If the math department is current in their preparation with veteran teachers schooled in modern math as well as the new teachers, there should be no problem in their implementing the program of SMSG. Most strong math departments could handle UICSM as well, but schooling in their specific objectives is a great help to the teacher in realizing the organization of the material and how it accomplishes these objectives.

The National Science Foundation has provided financial support for both programs in training teachers. The individual teacher can obtain a stipend to defray expenses to obtain the background in these programs through summer institutes when the right qualifications are met. Usually an institution offering the program will select those persons that have the opportunity to teach the program the following fall or those that may already be teaching the program. A system wishing to get one of the programs started should be able to train their teachers as a group. If teachers from each of the junior high schools that feed the same high school and teachers from that high school all submit their applications as a unit with the explanation of what they are trying to accomplish, the institution offering the institute in the program would be much more inclined to accept the group than they would individuals.

William T. Hale, Assistant Director of the UICSM project, made the following statement about teacher training in the Mathematics Teacher.

> Because we are convinced beyond doubt of the desirability (we might even say, necessity) of giving practical realistic training to teachers who are interested in using the newer curriculum materials in mathematics, we have formulated a plan for such training. We hope to continue our summer institute programs and, as a concomitant part of the training program, maintain a supervisory follow-up during the academic year following each institute. Such supervision would, as in the past, include visits to cooperating schools by staff members for the purpose of assisting teachers (through conference and demonstration teaching) and informing school patrons of the changes taking place, assuring them of the care with which these changes are being made.[128]

This quotation shows the concern of the project staff for preparation as well as pointing up some of the procedures in carrying out their plans.

Scheduling of the classes in the programs would vary somewhat. As it has been mentioned, SMSG courses are free-standing units in themselves. It would be quite difficult for a school to select only one year's program from UICSM's material in the manner some schools do from the SMSG work. UICSM would need provision to continue from year to year and from junior high to high school. Homogeneous grouping would be an aid to either program. They are both intended for the college capable student. Transfers during the course of the year will have trouble unless they are transferring from the same program. It

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[128] William T. Hale, "UICSM's Decade of Experimentation," The Mathematics Teacher, 54:8, December, 1961.

is felt that most students that have had a modern math program for the part of the year before transfering could handle SMSG's program as a transfer. It is felt that this would be more of a problem with UICSM. The following statement is made by Howard Marston about transfers into the UICSM program.

> The real problem is the student who transfers to our school from other school systems. We have kept conventional classes in the 10th, 11th, and 12th grades for these students. The first few years we integrated a few of these transfer students who wanted to study, or whose parents wanted them to study, the UICSM math. This was not easy to do, and, as the UICSM took bolder steps in revising their program, we stopped doing this altogether.
>
> This year, though, we are trying something new. We have taken the best of those 10th grade students who had a year of conventional algebra at other schools and formed a special class in mathematics for them. To these students from September until now, I have been teaching selected topics from the first four units, which constitute a year and a third of the UICSM program at our school. These are topics they did not have or had very little of in algebra. Now for the rest of the year this class will study the UICSM Second Course. Next year they can be integrated with our other Third Course students.[129]

Cost is always a big factor in considering any program. The two programs were fairly comparable in price until UICSM decided to bind their text in a hard bound edition. It is not known what the price will be for the new materials. Usually hard bound books that can be used for several years are more economical than a lower priced set of texts that must be replenished each year.

When implementing either program, the prime considerations would be at what ability level(s) to offer the program, what sort of sequence

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[129]Howard Marston, <u>op</u>. <u>cit</u>., p. 22.

sequence should be used from year to year, what should the policy be toward transfers and at what grade level should the program be started. In connection with the last consideration we find several alternatives. Both programs have some arithmetic units that can be used in connection with the high school programs. In some cases accelerated classes have utilized the two programs in the eighth grade. Occasionally the first unit of UICSM will be taught as the final part of the eighth grade year. In some cases the sequence varies from that recommended by the project committee for UICSM. Some schools teach only the first three units the first year and start the fourth the second year. The arrangement and organization of the materials have made this flexibility possible.

Probably any modern program will have its troubles with transfers. This point should not prevent school systems from adopting one of these two texts or some other modern text. There will probably be a problem with transfers until the modern math becomes universal. At that time the adjustment to a new text will be somewhat similar to what has been experienced with traditional texts in the past.

The ideal situation for UICSM materials to be used is in a small farm community or any other community that has few transfers. Of course, large districts that can amply provide for transfers can implement any program. The medium sized district that has several transfers and can not make separate provisions for transfers is the district that will have trouble on this point.

Educational Considerations

The student should be the primary concern in adopting any program. What the student might accomplish both quantitatively and qualitatively should be examined carefully. Both programs have literature available on what their program offers and how it compares with traditional math. The amount of learning should be improved over traditional math by either of the programs under consideration here. More understanding of mathematic principles should be the product of teaching any modern program. Both of the programs of this report should align themselves with this point. The following statement helps to point up what students derive from a modern course compared to a traditional math course.

> What about those students who have had four years of the UICSM math and have gone on to college? We have had two such groups. These students report that they are being exempted from certain math courses and are in some instances being allowed to study honors courses. We teach our students no calculus. I would not call our course an accelerated course: it is more enrichment, giving the students solid background and real understanding. I asked one of my former students how well she thought she was prepared for her college math--having studied the UICSM course--compared to her classmates at college. She replied, "Better than any of them" The class was using Taylor's analytic geometry and calculus text. She said that there were so many questions about proving this and deriving that which the other students found quite difficult, but which caused her no trouble at all since she had been doing just that for four years in high school.[130]

Another point to consider in selecting a modern program, while keeping the student in mind, is how much interest will it generate in the students? In interviews of a UICSM teacher and a SMSG teacher,

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[130] Ibid.

comments were made concerning the interest of the students in the material. The UICSM teacher commented that there was high interest. She said that the long sections of proofs for generalizations bored the students to a degree, but interest was generally high. The following is a comment from another UICSM teacher.

> You should have seen the skeptical looks on the faces of these students the first week as they saw they were studying ninth grade material; since then they have become so excited about their math that there are times when they can hardly stay in their seats. One girl after two weeks said, "I used to hate math. The teacher would do some problems and then we would do more like them for homework. It was boring. But I just love this class. I really have to think." Others have decided now to major in mathematics. The parents of several of these students have expressed appreciation for the program because of the enthusiasm of their children for it.[131]

The SMSG teacher stated that he was having trouble getting the students to absorb the enthusiasm he had for modern math.

Any curriculum device that will instill more interest in the student's study and present more material at the same time surely is educationally sound. Either of the programs in this report should accomplish this point. The student surely must learn more since all the topics of traditional math are covered by the programs and many additional topics are covered and additional skills acquired enroute. It is this author's opinion that the thinking and logical proofs that are necessary in the course of study will help the student's work in other classes as well. From this author's point of view the UICSM material accomplishes more toward these points than SMSG.

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[131] Ibid.

When considering a new modern math program for a school system, or when considering new texts for a traditional math program, all possibilities should be considered. Reports like this report might be of help to curriculum committees. Reports by periodicals on various strengths and weaknesses in different programs could be examined. Results of tests on students participating in various programs should be compared. But in each case there should be an examination of several programs as well as a variety of programs before any selection is made. If the best text for the situation is in use, it should assert itself. It there is a better method available, it will probably be discovered. The needs of the student who in turn must meet the needs and demands of society, must be met with the utmost in materials and instruction.

SUMMARY AND CONCLUSIONS

SMSG presents an approach that follows the pattern of definition, example and exercises. Most of the material is explained for the student. The proofs listed are usually an additional part of a theorem that is a result of a theorem or property just proved or defined in the text.

UICSM uses discovery in most of its topics. The student is directed toward what he should know, but is expected to develop his own awareness of the concept. In many cases the student is asked to verbalize his own concept. The student performs a large number of proofs and if he is successful acquires much understanding from these proofs. Even in cases where the concept is to come as an explanation by the text, there are usually problems previous to the explanation that will set the student's thinking and establish a need for this type of principle in the student's mind.

UICSM develops arithmetic for real numbers before any concept of a variable is formalized. SMSG uses variables nearly from the first. Both approaches have certain advantages. This author's point of view aligns itself to the intensive study of real numbers before the formalization of variable concepts. This author feels a solid foundation in real numbers is a definite necessity in any study of algebra and UICSM material accomplishes a better understanding of real numbers than does SMSG.

SMSG has an individual year's work in each of its courses. UICSM has a continuing program that does not lend itself to selecting just one year's work from their program. It is felt that this gives rise to a more unified program and offers more elasticity in the placement of topics. Because of these points UICSM seems to have an advantage in organization and arrangement.

SMSG gives rise to more discussions of topics not included in the text. This statement by Ross Taylor points out that fact.

> I find that I have many more discussions about the abstract nature of mathematics with my SMSG class than I have with my UICSM classes. In my SMSG classes we get off on such topics as fields, modula arithmetic, comparing the number of integers with the number of even integers, etc.[132]

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[132] Ross Taylor, op. cit., p. 480.

UICSM offers a great volume of supplementary and miscellaneous exercises for the students. It also lists frequent quizzes on the green sheets for the teacher's use.

The overall conclusion on the part of this report is that the UICSM material is superior to that of SMSG. It is realized that the authors of the SMSG material must feel that they have produced what is the best program available at this time. There are surely countless numbers that would agree with them. UICSM authors probably feel the same way about their material. The author of this comparison definitely chooses UICSM material as the better of the two.

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A COMPARISON OF UICSM AND SMSG ALGEBRA UNITS WITH ACCOMPANYING SUGGESTIONS

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AN ABSTRACT OF A MASTER'S REPORT

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The purpose of this paper was primarily to try to determine which of the two programs, University of Illinois Committee on School Mathematics (UICSM) or School Mathematics Study Group (SMSG), had the best presentation of first-year algebra. The paper was written for the consideration of school districts trying to decide on a modern mathematics algebra text and more particularly for those districts trying to decide between these two programs.

The method used in compiling this paper was direct comparison. Both texts were studied carefully and related articles were read in support of the opinions of the author. The various sections and topics were broken down and related to the supporting points throughout the rest of the text and then directly compared in a logical sequence.

The SMSG text was somewhat similar to a traditional algebra text. It has applied modern terminology and notation to what has been taught in the past. The sequence of topics was quite similar to traditional texts. The UICSM has taken more strides toward a truly modern algebra program. In addition to using modern terminology and notation, UICSM has utilized the discovery process in most of the topics presented. The presentation unfolds in such a way that the student is to develop a nonverbal awareness of the principle and is later asked to verbalize his concept when he has had time to develop it more completely. UICSM has a four-year sequence of topics. This sequence presents more modern topics in the first year than does the SMSG program.

SMSG began their algebra text with a discussion of sets. They introduced variables quite early. This arrangement allowed use of these mathematical tools throughout the text. UICSM began their first unit with an intensive study of real numbers. Either approach has advantages, but the author feels that real numbers need a solid development before variables should be considered.

SMSG's program gives rise to more discussions about the abstract nature of mathematics than does UICSM's program.

UICSM has a more utilitarian teacher's commentary than SMSG has. UICSM also offers a great volume of supplementary and miscellaneous exercises for the students. The UICSM teacher's commentary has frequent quizzes for the teacher's use.

It was the author's opinion that the UICSM program was superior to the SMSG program. The main reasons in support of this conclusion were the arrangement of topics and the discovery method employed by UICSM.