Bounding the largest inhomogeneous Diophantine approximation constant

by

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B.A., Tribhuvan University, 2013

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AN ABSTRACT OF A DISSERTATION

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Abstract

For an irrational real α and real $\gamma \notin \alpha\mathbb{Z} + \mathbb{Z}$, one defines the two-sided inhomogeneous approximation constant

$$M(\alpha, \gamma) := \liminf_{|n| \to \infty} |n| ||n\alpha - \gamma||,$$

and the worst-case of inhomogeneous approximation

$$\rho(\alpha) := \sup_{\gamma \notin \alpha\mathbb{Z}+\mathbb{Z}} M(\alpha, \gamma).$$

By a well-known theorem of Minkowski, we have

$$\rho(\alpha) \leq \frac{1}{4}.$$

This dissertation focuses on bounding $\rho(\alpha)$ in terms of $R := \liminf_{i \to \infty} a_i$, where the a_i are the partial quotients in the negative (i.e. the 'round-up') continued fraction expansion of α. We prove that if R is odd, then the upper bound $1/4$ can be replaced by

$$\frac{1}{4}\left(1 - \frac{1}{R}\right)\left(1 - \frac{1}{R^2}\right),$$

which is optimal. The optimal upper bound for even $R \geq 4$ was already known.

We also obtain bounds of the form $\rho(\alpha) \geq C(R)$ for any $R \geq 3$ which are best possible when R is even (and asymptotically precise when R is odd). In particular,

$$\rho(\alpha) \geq \begin{cases} \dfrac{1}{6\sqrt{3}+8} = \dfrac{1}{18.3923\ldots}, & \text{when } R = 3, \\[2ex] \dfrac{1}{4\sqrt{3}+2} = \dfrac{1}{8.9282\ldots}, & \text{when } R \geq 4. \end{cases}$$

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Major Professor
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Dedication

To my beloved family

Chapter 1

Introduction and Main Results

1.1 Introduction

We begin with the review of inhomogeneous Diophantine approximation. For a real irrational number α and a real number γ, we define the two-sided inhomogenous Diophantine approximation constant

$$M(\alpha,\gamma) := \liminf_{|n|\to\infty} |n| ||n\alpha - \gamma||, \tag{1.1}$$

where $||x||$ denotes the distance from x to the nearest integer. Since $||m + x|| = ||x||$ for any integer m, we may assume that $\alpha, \gamma \in (0, 1)$. When $\gamma = m\alpha + l$ for some $m, l \in \mathbb{Z}$, the constant in (1.1) becomes

$$M(\alpha,\gamma) = \liminf_{|n|\to\infty} |n| ||(n-m)\alpha - l|| = \liminf_{q\to\infty} q||q\alpha||,$$

which is the classical homogeneous Diophantine approximation problem where $\gamma = 0$.

Theorem 1.1.1. *(Hurwitz) For every irrational number α, there are infinitely many rational numbers $\frac{p}{q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

Theorem 1.1.1 tells that there are infinitely many rational numbers $\frac{p}{q}$ with

$$q|q\alpha - p| < \frac{1}{\sqrt{5}},$$

implying

$$q||q\alpha|| < \frac{1}{\sqrt{5}}.$$

This gives an upper bound on the homogeneous approximation constant

$$M(\alpha, 0) \leq \frac{1}{\sqrt{5}}. \tag{1.2}$$

The homogeneous problem is a well understood problem, and the constant $M(\alpha, 0)$ can be determined from the regular continued fraction expansion of α,

$$M(\alpha, 0) = \frac{1}{\limsup_{i\to\infty}(a_i + [0; a_{i+1}, a_{i+2}, a_{i+3}, \ldots] + [0; a_{i-1}, a_{i-2}, a_{i-3}, \ldots])}, \tag{1.3}$$

where

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} =: [a_0; a_1, a_2, a_3, \ldots]. \tag{1.4}$$

Notice that $M(\alpha, 0)$ depends on the largest partial quotient in the expansion (1.4) of α. Define

$$r := \limsup_{i\to\infty} a_i.$$

Then, (1.3) gives

$$\frac{1}{\sqrt{r^2 + 4r}} \leq M(\alpha, 0) \leq \frac{1}{\sqrt{r^2 + 4}},$$

with equality for $\alpha = [0; \overline{1, r}] = \frac{1}{2}(\sqrt{r^2 + 4r} - r)$ and $\alpha = [0; \overline{r}] = \frac{1}{2}(\sqrt{r^2 + 4} - r)$, respectively. Clearly, when $r = 1$, for example $\alpha = [0; \overline{1}] = \frac{\sqrt{5}-1}{2}$, we get $M(\alpha, 0) = \frac{1}{\sqrt{5}}$. Hence, the upper bound (1.2) is the best possible.

This dissertation focuses on the inhomogeneous case where $\gamma \notin \alpha\mathbb{Z}+\mathbb{Z}$ and the worst-case inhomogeneous approximation

$$\rho(\alpha) := \sup_{\gamma \notin \alpha\mathbb{Z}+\mathbb{Z}} M(\alpha, \gamma).$$

For a quadratic number field $\mathbb{Q}(\sqrt{D})$, let

$$\alpha = \begin{cases} \sqrt{D}, & \text{if } D \equiv 2 \text{ or } 3 \pmod 4, \\ \frac{1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

Then, the size of $\rho(\alpha)$ is closely related to whether $\mathbb{Q}(\sqrt{D})$ is Euclidean with respect to norm. In particular,

$$\rho(\alpha) < \frac{1}{|\alpha - \bar{\alpha}|} \;\Rightarrow\; \mathbb{Z}[\alpha] \text{ is norm-Euclidean},$$

where $\bar{\beta}$ denotes the conjugate $a - b\sqrt{m}$ of $\beta = a + b\sqrt{m}$. In the middle of the twentieth century, there was substantial work related to this inhomogeneous approximation constant and the associated inhomogeneous Markoff values. There is a conjecture of Barnes and Swinnerton-Dyer [1, 2] that for a quadratic α, the value of $\rho(\alpha)$ is always isolated, although the second largest value can be a limit point [7]. We are here interested in optimally bounding $\rho(\alpha)$, both from above and below. When discussing such a problem, it is essential to understand an upper bound arising from a well-known theorem of Minkowski, see Chap III [3].

Theorem 1.1.2. *(Minkowski) Let $L_1(x,y) = \lambda_1 x + \mu_1 y, L_2(x,y) = \lambda_2 x + \mu_2 y$ be a pair of linear forms over $\mathbb{R}$ and let $\Delta = \lambda_1\mu_2 - \lambda_2\mu_1 \neq 0$. Then, for any real numbers ρ_1, ρ_2, there are integers x, y such that*

$$|L_1 + \rho_1||L_2 + \rho_2| \leq \frac{1}{4}|\Delta|. \tag{1.5}$$

Moreover, if $\frac{\mu_1}{\lambda_1}$ is irrational and $\epsilon > 0$, then there exists a solution of (1.5) with $|L_1+\rho_1| < \epsilon$.

Corollary 1.1.3. *For any irrational number α and real number $\gamma \neq s\alpha + t$, there are infinitely many integers n such that*

$$|n|||n\alpha - \gamma|| \leq \frac{1}{4}.$$

Proof. See Appendix A. ☐

Notice that, from Corollary 1.1.3, we have $M(\alpha, \gamma) \leq \frac{1}{4}$ and hence

$$\rho(\alpha) \leq \frac{1}{4}. \tag{1.6}$$

The upper bound $\frac{1}{4}$ in (1.6) is optimal; for example, Grace [8] showed that for any irrational number $\alpha = [0; a_1, a_2, a_3, \ldots]$, where a_1 is odd and the other a_i are even and $a_2 < a_3 < a_4 \cdots$, we have $\rho(\alpha) = \frac{1}{4}$. On the other hand, absolute bounds of the form

$$\rho(\alpha) \geq C \tag{1.7}$$

have some history. Davenport [4] obtained (1.7) with $C = \frac{1}{128}$, Ennola [5]

$$C = \frac{1}{16 + 6\sqrt{6}} = \frac{1}{30.69\cdots},$$

and Pinner [13] improved the absolute lower bound to

$$C = \frac{(\sqrt{10} - 3)(7 - \sqrt{13})}{31 - 2\sqrt{10} - 3\sqrt{13}} = \frac{1}{25.1592\cdots}. \tag{1.8}$$

For a simpler proof with $C = \frac{1}{32}$, see Rocket and Szüsz [16]. The smallest known value of $\rho(\alpha)$ is still an example of Pitman [15]

$$\rho\left(\frac{\sqrt{3122285} - 1097}{1094}\right) = \frac{547}{4\sqrt{3122285}} = \frac{1}{12.9213\cdots},$$

and hence this is an upper bound on the optimal absolute lower bound C.

To deal with $\rho(\alpha)$, we found it more convenient to use the negative 'round-up' expansion of α. We obtain optimal upper and lower bounds on $\rho(\alpha)$ in terms of the partial quotients a_i in the negative continued fraction expansion of α,

$$\alpha = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots}}} =: [0; a_1, a_2, a_3, \cdots]^-, \tag{1.9}$$

where the integers $a_i \geq 2$ are generated by the algorithm

$$\alpha_0 := \{\alpha\} = \alpha, \quad a_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil, \quad \alpha_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil - \frac{1}{\alpha_n}.$$

We write α_i and $\bar{\alpha}_i$ for the forwards and backwards expansions from the ith point:

$$\alpha_i := [0; a_{i+1}, a_{i+2}, \ldots]^-, \qquad \bar{\alpha}_i := [0; a_i, a_{i-1}, \ldots, a_1]^-. \tag{1.10}$$

A method to evaluate $M(\alpha, \gamma)$ by using (1.9) and a sequence of integers t_i obtained from an appropriate α-expansion of γ,

$$\gamma = \sum_{i=1}^{\infty} \frac{1}{2}(a_i - 2 + t_i) D_{i-1}, \qquad D_{i-1} := \alpha_0 \alpha_1 \cdots \alpha_{i-1}, \tag{1.11}$$

was given in [12]. We review this in Chapter 2; see [9] for alternative algorithms to compute $M(\alpha, \gamma)$.

By contrast with the homogeneous case, the inhomogeneous constant $\rho(\alpha)$ will be affected by the smallest partial quotients

$$R := \liminf_{i \to \infty} a_i. \tag{1.12}$$

It was also shown in Corollary 1 [12] that if $R \geq 3$ in the negative continued fraction expansion, then

$$\rho(\alpha) \leq \frac{1}{4}\left(1 - \frac{1}{R}\right). \tag{1.13}$$

The restriction $R \geq 3$ is needed here; the examples of Grace with $\rho(\alpha) = \frac{1}{4}$ will have long strings of 2's in their negative expansion. This can be seen by switching from regular expansions to negative expansions using the following identity:

$$[0; a_1, a_2, a_3, a_4, \cdots] = [0; a_1 + 1, \underbrace{2, 2, \cdots, 2}_{a_2 - 1 \text{ times}}, a_3 + 2, \underbrace{2, 2, \cdots, 2}_{a_4 - 1 \text{ times}}, a_5 + 2, \cdots]^-.$$

For $R \geq 4$ even, the bound (1.13) is the best possible; for example, see Section 2.4.

Likewise, bounds of the form $\rho(\alpha) \geq C'(R)$ (giving (1.8) when $R = 2$ and an improvement when $R \geq 3$) were given in [13], where the a_i in (1.12) are the partial quotients in the nearest integer continued fraction expansion of α, and so $R \geq 2$. These bounds arise by constructing a γ' with $M(\alpha, \gamma') \geq C'(R)$. The values for small R are given in (1.18) below and the asymptotic behavior in (1.22). The goal here is to improve these $R \geq 3$ bounds when the a_i in (1.12) are the partial quotients in the negative continued fraction expansion of α rather than the nearest integer expansion. In addition, we improve (1.13) when R is odd. Of course, when $R \geq 2$ in the regular expansion or $R \geq 3$ in the negative expansion, the expansion eventually becomes the nearest integer expansion (in the remaining cases, $R = 1$ for the regular expansion and $R = 2$ for the negative expansion, the value of R can be much larger, even infinite, if one uses the nearest integer expansion). From this point on, the a_i in our definition of R will refer to the negative expansion, and we will assume that $R \geq 3$.

1.2 Main Results

This section presents the main results of this dissertation, which were initially published in [10, 11]. Chapters 3 and 4 will present the proof of these findings.

Consider a real number $\gamma^* \in (0, 1)$ which has the unique α-expansion

$$\gamma^* = \sum_{i=1}^{\infty} \frac{1}{2}(a_i - 2 + t_i) D_{i-1}, \tag{1.14}$$

where the sequence $\{t_i\}$ is given by

$$t_i = \begin{cases} 0, & \text{if } a_i \text{ is even,} \\ (-1)^{j+1}, & \text{if } a_i \text{ is the } j\text{th odd partial quotient.} \end{cases}$$

Notice that any two nonzero consecutive t_i have opposite signs. We define two numbers β and δ

$$\beta := [0; \overline{R_*}]^- = \frac{1}{2}\left(R_* - \sqrt{R_*^2 - 4}\right), \qquad \delta := [0; R_{**}, \overline{R_*}]^- = \frac{1}{R_{**} - \beta}, \tag{1.15}$$

where

$$R_*, \quad R_{**} := \begin{cases} R, \quad R+1, & \text{if } R \text{ is even,} \\ R+1, \quad R, & \text{if } R \text{ is odd.} \end{cases}$$

Setting

$$C(R) := \frac{(1-2\delta)(1-\beta)}{4(1-\delta\beta)}, \tag{1.16}$$

we observe that if R is even, then

$$C(R) = \frac{1}{4}\left(\frac{R-2}{\sqrt{R^2-4}+1}\right),$$

and if R is odd, then

$$C(R) = \frac{1}{4}\left(\frac{2R-2-\sqrt{(R+1)^2-4}}{\sqrt{(R+1)^2-4}-1}\right).$$

Henceforth, when referring to the function $C(R)$, it is implied that R relates to the negative expansion. Likewise, in the context of $C'(R)$, it should be understood that R corresponds to the nearest integer expansion.

The value of $M(\alpha, \gamma^*)$ gives us a lower bound for $\rho(\alpha)$.

Theorem 1.2.1. *Suppose that* (1.9) *gives the negative continued fraction expansion of* α

and $R = \liminf_{i\to\infty} a_i \geq 3$*. Then, with* γ^* *as in* (1.14) *and* $C(R)$ *as in* (1.16)*, we have*

$$\rho(\alpha) \geq M(\alpha, \gamma^*) \geq C(R). \tag{1.17}$$

In particular, when $R = 3$*,*

$$\rho(\alpha) \geq C(3) = \frac{1}{6\sqrt{3}+8} = \frac{1}{18.3923\ldots},$$

and when $R \geq 4$*,*

$$\rho(\alpha) \geq C(4) = \frac{1}{4\sqrt{3}+2} = \frac{1}{8.9282\ldots}.$$

For $R \geq 3$, the value of $C(R)$ improves the lower bound $C'(R)$ of Theorem 4 [13]:

R	$C'(R)^{-1}$	$C(R)^{-1}$
2	25.1592...	—
3	20.4874...	18.3923...
4	9.3372...	8.9282...
5	8.2500...	7.9497...
6	6.8120...	6.6568...
7	6.4643...	6.3431...
8	5.9109...	5.8306...

(1.18)

Of course, if all the $a_i \geq 3$ in the negative expansion, then the negative and nearest integer continued fraction expansions coincide. That is, the lower bound $C'(R)$, $R \geq 3$, actually applies to a much larger class of α than $C(R)$ (and so is not surprisingly smaller). Better bounds are also given in [13] when the nearest integer expansion coincides with the regular expansion. Notice that the bound $C(R)$ increases to $1/4$ as $R \to \infty$; in particular, from (1.13) and Theorem 1.2.1, when $R \geq 3$,

$$\rho(\alpha) = \frac{1}{4} \text{ if and only if } R = \infty. \tag{1.19}$$

Fukasawa [6] showed that (1.19) holds without the $R \geq 3$ condition when using the nearest integer continued fraction expansion (the restriction needed here since large partial quotients in the regular expansion will cause long strings of 2's in the negative expansion). We observe the asymptotic behavior of $C(R)$; when $R \geq 4$ is even,

$$C(R) = \frac{1}{4}\left(1 - \frac{3}{R} + \frac{5}{R^2} - \frac{E_1(R)}{R^3}\right), \quad 7.3268 < E_1(R) < 11, \tag{1.20}$$

and when $R \geq 3$ is odd,

$$C(R) = \frac{1}{4}\left(1 - \frac{3}{R} + \frac{4}{R^2} - \frac{E_2(R)}{R^3}\right), \quad 6.1279 < E_2(R) < 10. \tag{1.21}$$

We prove these identities in Chapter 3. For comparison, we note the bounds given in [13]:

$$C'(R) = \begin{cases} \frac{1}{4}\left(1 - \frac{3}{R} + \frac{4}{R^2} + O\left(R^{-3}\right)\right), & \text{if } R \text{ is even,} \\ \frac{1}{4}\left(1 - \frac{3}{R} + \frac{3}{R^2} + O\left(R^{-3}\right)\right), & \text{if } R \text{ is odd,} \end{cases} \tag{1.22}$$

with this lower bound asymptotically optimal (and hence $C'(R)$ inevitably smaller than $C(R)$ when R is even). When R is odd, it is not known whether the 3 in the $3/R^2$ term in (1.22) is optimal. Remember that R corresponds to the nearest integer expansion in the context of $C'(R)$.

Our lower bound $C(R)$ for $\rho(\alpha)$ is optimal when R is even.

Theorem 1.2.2. *With $R \geq 4$ even, if α has negative continued fraction expansion of period*

$$R+1, \underbrace{R, R, R, \cdots, R}_{l \;\; times}, \tag{1.23}$$

then $\rho(\alpha) \to C(R)$ as $l \to \infty$.

When $R \geq 3$ is odd, it will be clear from the proof of Theorem 1.2.1 that if α has negative continued fraction expansion of period $R+1, \underbrace{R, R, R, \cdots, R}_{l \text{ times}}$, then $M(\alpha, \gamma^*) \to C(R)$ as $l \to \infty$. So, the bound $M(\alpha, \gamma^*) \geq C(R)$ in Theorem 1.2.1 is still best possible. However, γ^* is no longer the best choice of γ; as we observe at the end of Chapter 3, for $R \geq 5$, these α have

$$\lim_{l\to\infty} \rho(\alpha) = \frac{\left(1 - 2\delta + \frac{2\delta\beta}{1+\beta}\right)(1-\beta)}{4(1-\delta\beta)} = \frac{1}{4}\left(1 - \frac{3}{R} + \frac{6}{R^2} + O\left(R^{-3}\right)\right). \tag{1.24}$$

We need a more complicated example to show the asymptotic sharpness of our lower bound when R is odd.

Theorem 1.2.3. *If R is odd and α has negative continued fraction expansion*

$$\alpha = [0; \overline{R, R, R+1, R, R+1, R+1, R, R+1, R+1, R, R+1}]^-,$$

then

$$\rho(\alpha) = \frac{1}{4}\left(1 - \frac{3}{R} + \frac{4}{R^2} + O\left(R^{-3}\right)\right).$$

For $R = 3, 5$ and 7, the period two examples

$$\rho\left([0; \overline{3,5}]^-\right) = \frac{13}{11\sqrt{165}} = \frac{1}{10.8690\ldots},$$
$$\rho\left([0; \overline{5,6}]^-\right) = \frac{589}{312\sqrt{195}} = \frac{1}{7.3970\ldots},$$
$$\rho\left([0; \overline{7,8}]^-\right) = \frac{3649}{1664\sqrt{182}} = \frac{1}{6.1519\ldots},$$

from [14] give upper bounds on the optimal $C(R)$.

When R is odd, upper bound (1.13) can be improved.

Theorem 1.2.4. *Suppose that (1.9) gives the negative continued fraction expansion of α and R is odd. Then*

$$\rho(\alpha) \leq \tilde{C}(R) := \frac{1}{4}\left(1 - \frac{1}{R}\right)\left(1 - \frac{1}{R^2}\right) = \frac{1}{4}\left(1 - \frac{1}{R} - \frac{1}{R^2} + \frac{1}{R^3}\right). \tag{1.25}$$

This upper bound is also best possible.

Theorem 1.2.5. *Suppose that* $\alpha = [0; \overline{R, NR}]^{-}$ *where* N *and* R *are positive integers with* R *odd. Define* γ_* *to have expansion* (1.11) *with* $t_{2i-1} = -1$ *and* $t_{2i} = N$*. Then*

$$\lim_{N\to\infty} M(\alpha, \gamma_*) = \tilde{C}(R).$$

Chapter 2

Preliminaries

In this chapter, we discuss a method to evaluate $M(\alpha, \gamma)$. Different algorithms have been used for computing $M(\alpha, \gamma)$, see [9]. In this dissertation, we will follow the approach of [12], which showed how $M(\alpha, \gamma)$ can be expressed in terms of the negative continued fraction expansion (1.9) of α and the corresponding α-expansion (1.11) of $\gamma \notin \mathbb{Z} + \alpha\mathbb{Z}$. So, we recall some notations and results from [12].

2.1 Negative Continued Fraction Expansion

Suppose that (1.9) gives the negative continued fraction expansion of α. Then, the corresponding convergents $\dfrac{p_n}{q_n} := [0; a_1, a_2, \ldots, a_n]^-$ are given by

$$p_{n+1} := a_{n+1}p_n - p_{n-1}, \;\; p_0 = 0, \;\; p_{-1} = -1,$$

$$q_{n+1} := a_{n+1}q_n - q_{n-1}, \quad q_0 = 1, \quad q_{-1} = 0.$$

Observe that the p_n and q_n are increasing sequences with

$$p_{n+1}q_n - p_nq_{n+1} = 1, \quad \frac{q_{n-1}}{q_n} = \bar{\alpha}_n, \quad \text{and} \quad q_n = \frac{q_n}{q_{n-1}} \cdots \frac{q_1}{q_0} = (\bar{\alpha}_1\bar{\alpha}_2 \cdots \bar{\alpha}_n)^{-1},$$

where $\bar{\alpha}_i$ is defined as in (1.10). For the negative expansion, the convergents p_n/q_n form an increasing sequence converging to α. Notice, increasing the size of any partial quotient decreases the size of α, that is, for any $c \geq 1$

$$[0; a_1, \ldots, a_{i-1}, a_i + c, \ldots]^- < [0; a_1, \ldots, a_{i-1}, a_i, \ldots]^-. \tag{2.1}$$

Defining $D_n := q_n\alpha - p_n$, we establish the following properties:

(i) $\alpha = \dfrac{p_{n+1} - p_n\alpha_{n+1}}{q_{n+1} - q_n\alpha_{n+1}}$.

(ii) $D_n = \alpha_0\alpha_1 \cdots \alpha_n = a_n D_{n-1} - D_{n-2} = \dfrac{1}{q_{n+1} - q_n\alpha_{n+1}}$.

(iii) $(a_1 - 1)D_0 + \sum_{i=2}^{\infty}(a_i - 2)D_{i-1} = 1$.

Proof. (i): Since $\alpha_{n+1} = [0; a_{n+2}, a_{n+3}, a_{n+4}, \cdots]^-$, we have

$$\alpha = [0; a_1, a_2, \cdots, a_{n+1}, 1/\alpha_{n+1}] = \frac{\frac{1}{\alpha_{n+1}}p_{n+1} - p_n}{\frac{1}{\alpha_{n+1}}q_{n+1} - q_n} = \frac{p_{n+1} - p_n\alpha_{n+1}}{q_{n+1} - q_n\alpha_{n+1}}.$$

(ii): We first show $D_n = \alpha_0\alpha_1 \cdots \alpha_n$. When $n = 0$, $D_0 = \alpha q_0 - p_0 = \alpha_0$. If $n = 1$, then

$$D_1 = \alpha q_1 - p_1 = \alpha\left(a_1 - \frac{1}{\alpha}\right) = \alpha(a_1 - a_1 + \alpha_1) = \alpha_0\alpha_1.$$

Now, suppose it is true for all $n \leq m$. Then,

$$\begin{aligned} D_{m+1} = \alpha q_{m+1} - p_{m+1} &= \alpha(a_{m+1}q_m - q_{m-1}) - (a_{m+1}p_m - p_{m-1}) \\ &= a_{m+1}(\alpha q_m - p_m) - (\alpha q_{m-1} - p_{m-1}) \\ &= a_{m+1}D_m - D_{m-1} \\ &= D_{m-1}\alpha_m\left(a_{m+1} - \frac{1}{\alpha_m}\right) \\ &= \alpha_0\alpha_1\alpha_2 \cdots a_{m-1}\alpha_m\alpha_{m+1}. \end{aligned}$$

To obtain $D_n = a_n D_{n-1} - D_{n-2}$, replace m by $n-1$. Finally,

$$D_n = q_n\alpha - p_n = q_n\left(\alpha - \frac{p_n}{q_n}\right) = q_n\left(\frac{p_{n+1} - p_n\alpha_{n+1}}{q_{n+1} - q_n\alpha_{n+1}} - \frac{p_n}{q_n}\right) = \frac{1}{q_{n+1} - q_n\alpha_{n+1}}.$$

(iii): Using $a_i D_{i-1} = D_i + D_{i-2}$ and $D_{-1} = \alpha q_{-1} - p_{-1} = 1$,

$$(a_1 - 1)D_0 + \sum_{i=2}^{\infty}(a_i - 2)D_{i-1} = D_1 + D_{-1} - D_0 + \sum_{i=2}^{\infty}(D_i + D_{i-2} - 2D_{i-1}) = 1. \qquad \square$$

2.2 Alpha-expansion of Gamma

In order to evaluate $M(\alpha, \gamma)$, we express $\gamma \in [0, 1)$ as a linear combination of the D_i. For this, we generate the integers b_i by the algorithm

$$\gamma_0 := \{\gamma\} = \gamma, \quad b_{i+1} := \left\lfloor \frac{\gamma_i}{\alpha_i} \right\rfloor, \quad \gamma_{i+1} := \left\{ \frac{\gamma_i}{\alpha_i} \right\} = \frac{\gamma_i}{\alpha_i} - b_{i+1}.$$

Lemma 2.2.1 (Page 4 [12]). *For any real $\gamma \in (0, 1)$, with the integers b_i as above, we have*

$$\gamma = \sum_{i=1}^{n} b_i D_{i-1} + \gamma_n D_{n-1} = \sum_{i=1}^{\infty} b_i D_{i-1}. \tag{2.2}$$

This gives the unique expansion of γ of the form $\sum_{i=1}^{\infty} b_i D_{i-1}$ with the following properties:

(i) $0 \leq b_i \leq a_i - 1$ *for all* i,

(ii) the sequence $\{b_i\}_i$ *does not contain a block of the form* $b_s = a_s - 1$ *for some* s *with* $b_j = a_j - 2$ *for all* $j > s$ *or with* $b_k = a_k - 1$ *for some* $k > s$ *and* $b_j = a_j - 2$ *for all* $k > j > s$.

Proof. First, we prove (2.2) by induction. When $n = 1$,

$$b_1 D_0 + \gamma_1 D_0 = D_0(b_1 + \gamma_1) = \alpha_0 \frac{\gamma_0}{\alpha_0} = \gamma.$$

Suppose it is true for all $n \le m$. Then,

$$\begin{aligned}\sum_{i=1}^{m+1} b_i D_{i-1} + \gamma_{m+1} D_m &= \sum_{i=1}^{m} b_i D_{i-1} + \gamma_m D_{m-1} - \gamma_m D_{m-1} + D_m(b_{m+1} + \gamma_{m+1}) \\ &= \gamma - \gamma_m D_{m-1} + D_m \frac{\gamma_m}{\alpha_m} \\ &= \gamma.\end{aligned}$$

The first property follows from the observation

$$b_i = \left\lfloor \frac{\gamma_{i-1}}{\alpha_{i-1}} \right\rfloor \le \left\lfloor \frac{1}{\alpha_{i-1}} \right\rfloor = \left\lceil \frac{1}{\alpha_{i-1}} \right\rceil - 1.$$

Suppose the sequence $\{b_i\}_i$ contain a block of the form $b_s = a_s - 1$ for some s and $b_j = a_j - 2$ for all $s < j \le m$. Then, since $a_i D_{i-1} = D_i + D_{i-2}$,

$$\begin{aligned}\sum_{i=s}^{m} b_i D_{i-1} = b_s D_{s-1} + \sum_{i=s+1}^{m} b_i D_{i-1} &= (a_s - 1) D_{s-1} + \sum_{i=s+1}^{m} (a_i - 2) D_{i-1} \\ &= D_s + D_{s-2} - D_{s-1} + \sum_{i=s+1}^{m} (D_i + D_{i-2} - 2D_{i-1}) \\ &= D_{s-2} - D_{m-1} + D_m.\end{aligned}$$

`Case 1.` If $k = m + 1$ and $b_{m+1} = a_{m+1} - 1$, then the expansion (2.2) gives

$$\begin{aligned}\gamma_{s-1} D_{s-2} &= \sum_{i=s}^{m} b_i D_{i-1} + (a_{m+1} - 1) D_m + \sum_{i=m+2}^{\infty} b_i D_{i-1} \\ &= D_{s-2} - D_{m-1} + D_m + D_{m+1} + D_{m-1} - D_m + \sum_{i=m+2}^{\infty} b_i D_{i-1} \\ &> D_{s-2},\end{aligned}$$

contradicting the fact that $\gamma_{s-1} < 1$.

`Case 2.` Suppose m is not finite. Observe that $D_{m-1} \to 0$ and $D_m \to 0$ as $m \to \infty$. Hence,

$$\gamma_{s-1} D_{s-2} = \sum_{i=s}^{\infty} b_i D_{i-1} = D_{s-2},$$

implying $\gamma_{s-1} = 1$, a contradiction.

Finally, we show the expansion (2.2) is unique. Suppose that there is another expansion of γ,

$$\gamma = \sum_{i=1}^{\infty} b'_i D_{i-1},$$

with the sequence $\{b'_i\}_i$ satisfying the given properties (i) and (ii). Then, there exists an index J such that $b_i = b'_i$ for all $i < J$ and $b_J \neq b'_J$. Defining

$$\gamma'_J := \frac{1}{D_{J-1}} \sum_{i=J+1}^{\infty} b'_i D_{i-1},$$

we get

$$\begin{aligned}
\gamma'_J D_{J-1} &= b'_{J+1} D_J + \sum_{i=J+2}^{\infty} b'_i D_{i-1} \\
&< (a_{J+1} - 1) D_J + \sum_{i=J+2}^{\infty} (a_i - 2) D_{i-1} \\
&= D_{J+1} + D_{J-1} - D_J + \sum_{i=J+2}^{\infty} (D_i + D_{i-2} - 2D_{i-1}) \\
&= D_{J-1}.
\end{aligned}$$

Also, we have $\gamma_J D_{J-1} < D_{J-1}$. Hence,

$$D_{J-1} \leq |b_J - b'_J| D_{J-1} = |\gamma_J - \gamma'_J| D_{J-1} < D_{J-1}. \qquad \square$$

Definition 1. *The expansion* (2.2) *is called the* α*-expansion of* γ*.*

We construct a sequence of integers t_k by

$$b_k = \frac{1}{2}(a_k - 2 + t_k),$$

which gives (1.11):

$$\gamma = \sum_{i=1}^{\infty} \frac{1}{2}(a_i - 2 + t_i) D_{i-1}. \tag{2.3}$$

Notice that t_k and a_k have the same parity, and by Lemma 2.2.1, we have the following properties:

(i) $-(a_k - 2) \leq t_k \leq a_k$,

(ii) the sequence $(t_k)_k$ does not contain a block of the form $t_s = a_s$ for some s with $t_k = a_k - 2$ for all $k > s$ or with $t_l = a_l$ for some $l > s$ and $t_k = a_k - 2$ for all $l > k > s$.

2.3 Sequence of Best Inhomogeneous Approximations

We now construct four different integer sequences so that running n through each sequence is sufficient to evaluate $M(\alpha, \gamma)$.

Define the integers $Q_k = Q_k(\alpha, \gamma)$ by

$$Q_k := \sum_{i=i}^{k} b_i q_{i-1}$$

and numbers $\xi_k := \frac{Q_k}{q_k}$. Observe that $0 \leq \xi_k \leq 1$. Also, we set

$$\lambda(n) = \lambda(n; \alpha, \gamma) := |n| \| n\alpha - \gamma \|.$$

Then, since $q_{i-1}\alpha - p_{i-1} = D_{i-1}$,

$$\begin{aligned}\lambda(Q_k) = Q_k\|Q_k\alpha - \gamma\| &= \xi_k q_k \left\| \sum_{i=1}^{k} b_i q_{i-1}\alpha - \sum_{i=1}^{k} b_i p_{i-1} - \sum_{i=1}^{\infty} b_i D_{i-1} \right\| \\ &= \xi_k q_k \left\| \sum_{i=k+1}^{\infty} b_i D_{i-1} \right\| \\ &= \xi_k \gamma_k q_k D_{k-1}\end{aligned}$$

for k large enough that

$$\sum_{i=k+1}^{\infty} b_i D_{i-1} < \frac{1}{2}.$$

Similarly,

$$\lambda(Q_k + q_{k-1}) = (\xi_k + \bar{\alpha}_k)(1 - \gamma_k) q_k D_{k-1},$$

$$\lambda(Q_k - (q_k - q_{k-1})) = |1 - \bar{\alpha}_k - \xi_k||1 - \alpha_k - \gamma_k| q_k D_{k-1},$$

$$\lambda(Q_k - q_k) = (1 - \xi_k)(\alpha_k + \gamma_k) q_k D_{k-1}.$$

Finally, we define

$$\begin{aligned} d_k^- &:= \sum_{j=1}^{k} t_j \left(\frac{q_{j-1}}{q_k}\right) = t_k\bar{\alpha}_k + t_{k-1}\bar{\alpha}_k\bar{\alpha}_{k-1} + t_{k-2}\bar{\alpha}_k\bar{\alpha}_{k-1}\bar{\alpha}_{k-2} + \cdots , \\ d_k^+ &:= \sum_{j=k+1}^{\infty} t_j \left(\frac{D_{j-1}}{D_{k-1}}\right) = t_{k+1}\alpha_k + t_{k+2}\alpha_k\alpha_{k+1} + t_{k+3}\alpha_k\alpha_{k+1}\alpha_{k+2} + \cdots . \end{aligned}$$

Then,

$$\begin{aligned}\xi_k = \frac{Q_k}{q_k} = \frac{1}{2q_k}\sum_{i=1}^{k}(a_i - 2 + t_i)q_{i-1} &= \frac{1}{2}\left(\frac{1}{q_k}\sum_{i=1}^{k}(q_i + q_{i-2} - 2q_{i-1}) + d_k^-\right) \\ &= \frac{1}{2}(1 - \bar{\alpha}_k + d_k^- - \frac{1}{q_k}).\end{aligned}$$

Similarly,

$$\gamma_k = \frac{1}{D_{k-1}} \sum_{i=1}^{k} b_i D_{i-1} = \frac{1}{2}(1 - \alpha_k + d_k^+),$$

and

$$q_k D_{k-1} = q_k \left(\frac{1}{q_n - q_{n-1}\alpha_n} \right) = \frac{1}{1 - \alpha_k \bar{\alpha}_k}.$$

Hence, as $k \to \infty$, we can replace $\lambda(Q_k), \lambda(Q_k + q_{k-1}), \lambda(Q_k - (q_k - q_{k-1}))$, and $\lambda(Q_k - q_k)$ by

$$\begin{aligned}
s_1(k) &:= \frac{1}{4}(1 - \bar{\alpha}_k + d_k^-)(1 - \alpha_k + d_k^+)/(1 - \alpha_k \bar{\alpha}_k), \\
s_2(k) &:= \frac{1}{4}(1 + \bar{\alpha}_k + d_k^-)(1 + \alpha_k - d_k^+)/(1 - \alpha_k \bar{\alpha}_k), \\
s_3(k) &:= \frac{1}{4}|1 - \bar{\alpha}_k - d_k^-||1 - \alpha_k - d_k^+|/(1 - \alpha_k \bar{\alpha}_k), \\
s_1(k) &:= \frac{1}{4}(1 + \bar{\alpha}_k - d_k^-)(1 + \alpha_k + d_k^+)/(1 - \alpha_k \bar{\alpha}_k).
\end{aligned}$$

It was observed in [12] that

$$-(1 - \bar{\alpha}_k) \leq d_k^- \leq (1 + \bar{\alpha}_k), \quad -(1 - \alpha_k) \leq d_k^+ \leq (1 + \alpha_k), \tag{2.4}$$

with $d_k^+ \geq 1 - \alpha_k$ (respectively $d_k^- \geq 1 - \bar{\alpha}_n$) if and only if the sequence $t_{k+1}, t_{k+2}, \ldots$ (respectively $t_k, t_{k-1}, \ldots$) has the form $t_j = a_j$ for some $j > k$ (respectively $j \leq k$) with $t_i = a_i - 2$ for any $k < i < j$ (respectively $j < i \leq k$). Note that $t_i = a_i$ if and only if $b_i = a_i - 1$. When only finitely many of the $b_i = a_i - 1$, it was shown in [12] that the sequence of best positive and negative inhomogeneous approximations lies amongst the integers

$$Q_k, \quad Q_k + q_{k-1}, \quad -(q_k - q_{k-1} - Q_k), \quad -(q_k - Q_k).$$

We will work with the value of $|n| \parallel n\alpha - \gamma \parallel$ for these four values of n expressed in the symmetrical form $s_1(k), \ldots, s_4(k)$ of Theorem 1 [12].

Lemma 2.3.1 (Theorem 1 [12]). *If $\gamma \notin \mathbb{Z} + \alpha\mathbb{Z}$ and the α-expansion of γ has $b_i = a_i - 1$ (equivalently $t_i = a_i$) at most finitely many times, then*

$$M(\alpha, \gamma) = \liminf_{k\to\infty} \min\{\lambda(Q_k), \lambda(Q_k + q_{k-1}), \lambda(Q_k - (q_k - q_{k-1})), \lambda(Q_k - q_k)\}$$
$$= \liminf_{k\to\infty} \min\{s_1(k), s_2(k), s_3(k), s_4(k)\}.$$

Proof. First, we show

$$M_+(\alpha, \gamma) := \liminf_{n\to\infty} n||n\alpha - \gamma|| = \liminf_{k\to\infty} \min\{\lambda(Q_k), \lambda(Q_k + q_{k-1})\}.$$

Note that for any positive integer n, there exists an integer k such that $q_{k-1} \le n < q_k$. Take $z_k = \lfloor n/q_{k-1} \rfloor$, and repeat this process for $n - z_k q_{k-1}$ and so on. Then, we have

$$n = \sum_{i=1}^{k} z_i q_{i-1}, \quad z_k \ge 1,$$

with the sequence $\{z_i\}_i$ satisfying the properties given in Lemma 2.2.1. So,

$$\gamma' := \{n\alpha\} = \left\{ \sum_{i=1}^{k} z_i(\alpha q_{i-1} - p_{i-1}) \right\} = \sum_{i=1}^{k} z_i D_{i-1}$$

gives the α-expansion of $\{n\alpha\}$. We assume that $||n\alpha - \gamma|| = \pm(\{n\alpha\} - \gamma)$. Otherwise we have $||n\alpha - \gamma|| = 1 - (\{n\alpha\} - \gamma) > \gamma$ or $||n\alpha - \gamma|| = 1 + (\{n\alpha\} - \gamma) > 1 - \gamma$, and $n||n\alpha - \gamma|| \to \infty$ as $n \to \infty$.

Suppose $n = \sum_{i=1}^{k} z_i q_{i-1} \ne Q_k = \sum_{i=1}^{k} b_i q_{i-1}$. Then, there exists s with $1 \le s \le k$ such that $z_s \ne b_s$ with $z_i = b_i$ for all $i < s$.

`Case I:` If $z_s < b_s$, then $\{n\alpha\} < \gamma$ and

$$||n\alpha - \gamma|| = \gamma - \{n\alpha\} > \gamma_s D_{s-1} = ||Q_s\alpha - \gamma||.$$

This tells us that $Q_s \to \infty$ as $n \to \infty$ (since otherwise $\lambda(n) \to \infty$). If $s < k$, then

$$n \geq Q_s - (b_s - z_s)q_{s-1} + z_k q_{k-1} \geq Q_s - (a_s - 1)q_{s-1} + q_{k-1} > Q_s.$$

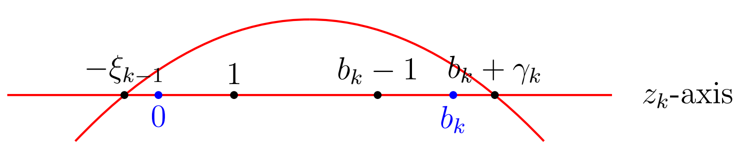
Hence, $\lambda(n) > \lambda(Q_s)$, and so such n can be safely ignored. Thus, it suffices to consider $s = k$. Then, $z_k < b_k$,

$$n = \sum_{i=1}^{k} z_i q_{i-1} = \sum_{i=1}^{k-1} b_i q_{i-1} + z_k q_{k-1} = Q_{k-1} + z_k q_{k-1},$$

and

$$\begin{aligned}
\lambda(n) &= (Q_{k-1} + z_k q_{k-1})\|\alpha Q_{k-1} - \gamma + \alpha z_k q_{k-1}\| \\
&= (Q_{k-1} + z_k q_{k-1})\left\|\sum_{i=1}^{k-1} b_i(\alpha q_{i-1} - p_{i-1}) - \sum_{i=1}^{k} b_i D_{i-1} - \gamma_k D_{k-1} + z_k(\alpha q_{k-1} - p_{k-1})\right\| \\
&= (Q_{k-1} + z_k q_{k-1})\| - b_k D_{k-1} - \gamma_k D_{k-1} + z_k D_{k-1}\| \\
&= (Q_{k-1} + z_k q_{k-1})(b_k - z_k + \gamma_k)D_{k-1},
\end{aligned}$$

which is quadratic in z_k, as shown in the following figure.



For $0 \leq z_k \leq b_k$, this quadratic is minimized for $z_k = 0$ or $z_k = b_k$. In this case, we have either $n = Q_k$ or $n = Q_{k-1}$. But, by our assumption, we have $0 < z_k < b_k$. Therefore, $\lambda(n) > \min\{\lambda(Q_k), \lambda(Q_{k-1})\}$.

`Case 2:` If $z_s > b_s$, then $\{n\alpha\} > \gamma$ and

$$||n\alpha - \gamma|| = \{n\alpha\} - \gamma = (z_s - b_s)D_{s-1} + \sum_{i=s+1}^{\infty} (z_i - b_i)D_{i-1}.$$

`Claim:` $s = k$ and $z_k = b_k + 1$.

Proof of the claim: Suppose we have either $s \neq k$ or $s = k$ and $z_k \geq b_k + 2$. Let $n' = n - q_{k-1}$. Then,

$$\|n'\alpha - \gamma\| = \|n\alpha - \gamma - \alpha q_{k-1} + p_{k-1}\| = \|\{n\alpha\} - \gamma - D_{k-1}\| = \|n\alpha - \gamma\| - D_{k-1},$$

and hence $\lambda(n') < \lambda(n)$. This completes the proof of the claim.

Since $s = k$ and $z_k = b_k + 1$, we get $n = Q_k + q_{k-1}$. Hence,

$$M_+(\alpha, \gamma) = \liminf_{k\to\infty} \min\{\lambda(Q_k), \lambda(Q_k + q_{k-1})\}.$$

Next, we show

$$M_-(\alpha, \gamma) := \liminf_{n\to-\infty} |n|||n\alpha - \gamma|| = \liminf_{k\to\infty} \min\{\lambda(Q_k - (q_k - q_{k-1})), \lambda(Q_k - q_k)\}.$$

Suppose there is an integer N such that $b_i \leq a_i - 2$ for all $i \geq N$. Consider the real number γ' whose α-expansion is

$$\gamma' = \sum_{i=N}^{\infty} (a_i - 2 - b_i) D_{i-1}.$$

Then, the corresponding integers Q'_k are given by

$$Q'_k = \sum_{i=N}^{k} (a_i - 2 - b_i) q_{i-1} = \sum_{i=N}^{k} (a_i - 2) q_{i-1} + \sum_{i=1}^{N-1} b_i q_{i-1} - Q_k,$$

and

$$\gamma + \sum_{i=1}^{N-1} (a_i - 2 - b_i) D_{i-1} + \gamma' = 1 - \alpha. \tag{2.5}$$

Using $D_{i-1} = \alpha q_{i-1} - p_{i-1}$ in (2.5), we get $\gamma = -\gamma' - m\alpha + l$, where

$$m = \sum_{i=1}^{N-1} (a_i - 2 - b_i) q_{i-1} + 1, \quad l = \sum_{i=1}^{N-1} (a_i - 2 - b_i) p_{i-1} + 1.$$

Now,

$$M_-(\alpha,\gamma) = M_-(\alpha, -\gamma' - m\alpha - l) = \liminf_{n\to-\infty} |n|||(n+m)\alpha + \gamma'||$$
$$= \liminf_{n\to-\infty} |n+m|||-(n+m)\alpha - \gamma'||$$
$$= M_+(\alpha,\gamma').$$

Therefore, by the first part of this proof, we have either $-(n+m) = Q'_k$ or $-(n+m) = Q'_k + q_{k-1}$, so that $n = Q_k - (q_k - q_{k-1})$ or $n = Q_k - q_k$. Hence,

$$M_-(\alpha,\gamma) = \liminf_{k\to\infty} \min\{\lambda(Q_k - (q_k - q_{k-1})), \lambda(Q_k - q_k)\}. \qquad \square$$

Throughout this dissertation, we use Lemma 2.3.1 to evaluate $M(\alpha,\gamma)$ for any α given by (1.9) and the corresponding α-expansion of γ of the form (1.11).

Corollary 2.3.1. *Suppose that $\gamma \notin \mathbb{Z} + \alpha\mathbb{Z}$. If $t_k = a_k$ at most finitely many times, then*

$$M(\alpha,\gamma) \le \frac{1}{4} \liminf_{k\to\infty} \frac{(1-\bar{\alpha}_k)(1-\alpha_k)}{1-\alpha_k\bar{\alpha}_k}.$$

Proof. Since $t_k = a_k$ at most finitely many times, we may assume that $d_k^- \le 1 - \bar{\alpha}_k$ and $d_k^+ \le 1 - \alpha_k$. Then,

$$\min\{s_1(k), s_3(k)\} \le \sqrt{s_1(k)s_3(k)} = \frac{\sqrt{((1-\bar{\alpha}_k)^2 - (d_k^-)^2)((1-\alpha_k)^2 - (d_k^+)^2)}}{4(1-\alpha_k\bar{\alpha}_k)}$$
$$\le \frac{(1-\bar{\alpha}_k)(1-\alpha_k)}{4(1-\alpha_k\bar{\alpha}_k)},$$

and hence

$$M(\alpha,\gamma) \le \liminf_{k\to\infty} \min\{s_1(k), s_3(k)\} \le \frac{1}{4} \liminf_{k\to\infty} \frac{(1-\bar{\alpha}_k)(1-\alpha_k)}{1-\alpha_k\bar{\alpha}_k}. \qquad \square$$

As we know, Lemma 2.3.1 is not valid when the α-expansion of γ has $t_k = a_k$ infinitely often. In such a case, we can use the following lemma to obtain an upper bound on $M(\alpha,\gamma)$.

Lemma 2.3.2 (Lemma 1 [12])**.** *Suppose that* $\gamma \notin \mathbb{Z} + \alpha\mathbb{Z}$*. If* $t_k = a_k$ *infinitely often, then*

$$M(\alpha, \gamma) \leq \frac{1}{4} \liminf_{\substack{k\to\infty \\ t_k = a_k}} \frac{\bar{\alpha}_k}{1 - \alpha_k \bar{\alpha}_k}.$$

Proof. Recall that it is not possible to have $d_k^- \geq 1 - \bar{\alpha}_k$ and $d_k^+ \geq 1 - \alpha_k$ at the same time. When $t_k = a_k$, we must have $d_k^- \geq 1 - \bar{\alpha}_k$, and so $d_k^+ \leq 1 - \alpha_k$. Thus,

$$\min\{s_3(k), s_4(k)\} \leq \sqrt{s_3(k)s_4(k)} = \frac{\sqrt{(\bar{\alpha}_k^2 - (1 - d_k^-)^2)(1 - (\alpha_k + d_k^+)^2)}}{4(1 - \alpha\bar{\alpha}_k)} \leq \frac{\bar{\alpha}_k}{4(1 - \alpha\bar{\alpha}_k)}.$$

∎

If $R \geq 3$, then $\bar{\alpha}_k, \alpha_k \leq [0; \overline{3}]^- = \frac{3-\sqrt{5}}{2}$ as $k \to \infty$. Observe that $\bar{\alpha}_k \leq (1 - \bar{\alpha}_k)(1 - \alpha_k)$. Then, it follows from Corollary 2.3.1 and Lemma 2.3.2 that for any γ, we have

$$M(\alpha, \gamma) \leq \frac{1}{4} \liminf_{k\to\infty} \frac{(1 - \bar{\alpha}_k)(1 - \alpha_k)}{1 - \alpha_k \bar{\alpha}_k}. \tag{2.6}$$

Note that $1 - \alpha_k \leq 1 - \alpha_k \bar{\alpha}_k$. Also, we have $a_k = R$ infinitely often, and hence

$$\bar{\alpha}_k = \frac{1}{(R - \bar{\alpha}_{k-1})} \geq \frac{1}{R}.$$

Thus, if $R \geq 3$, then for any γ, we get

$$M(\alpha, \gamma) \leq \frac{1}{4} \liminf_{k\to\infty} (1 - \bar{\alpha}_k) \leq \frac{1}{4}\left(1 - \frac{1}{R}\right),$$

giving (1.13):

$$\rho(\alpha) \leq \frac{1}{4}\left(1 - \frac{1}{R}\right).$$

2.4 Upper Bound (1.13) is Optimal when R is Even

We present an example, as suggested in Theorem 2 [12], to demonstrate the optimality of the upper bound (1.13) when R is even. If the partial quotients a_k in the expansion (1.9) are even for all $k \geq J$, then we can take $t_k = -(a_k - 2)$ for $k < J$ and $t_k = 0$ for all $k \geq J$ in (1.11). Hence, $d_k^- \to 0$, $d_k^+ \to 0$ as $k \to \infty$, and Lemma 2.3.1 gives

$$M(\alpha, \gamma) = \frac{1}{4} \liminf_{k\to\infty} \frac{(1-\bar{\alpha}_k)(1-\alpha_k)}{1-\bar{\alpha}_k\alpha_k}, \tag{2.7}$$

where

$$\gamma = \frac{1}{2}\sum_{i=1}^{\infty}(a_i - 2 + t_i)D_{i-1} = \frac{1}{2}\sum_{i=J}^{\infty}(a_i - 2)D_{i-1} = \frac{1}{2}(D_{J-2} - D_{J-1}).$$

In addition, if $a_k \geq 4$ for all $k \geq L$, then it follows from (2.6) and (2.7) that

$$\rho(\alpha) = \frac{1}{4} \liminf_{k\to\infty} \frac{(1-\bar{\alpha}_k)(1-\alpha_k)}{1-\bar{\alpha}_k\alpha_k}. \tag{2.8}$$

Now, suppose $R \geq 4$ is even, and $\alpha = [0; \overline{R, 2N}]^-$. Then, as $N \to \infty$,

$$\alpha_{2k}, \bar{\alpha}_{2k-1} = \frac{1}{R - \frac{1}{2N - O(1)}} \to \frac{1}{R},$$
$$\alpha_{2k-1}, \bar{\alpha}_{2k} = \frac{1}{2N - O(1)} \to 0.$$

From (2.8), we get

$$\lim_{N\to\infty} \rho(\alpha) = \frac{1}{4}\left(1 - \frac{1}{R}\right).$$

Chapter 3

Lower Bounds on the Largest Inhomogeneous Approximation Constant

In this chapter, we prove the main results related to lower bounds on $\rho(\alpha)$ when $R \geq 3$: Theorem 1.2.1, Theorem 1.2.2, and Theorem 1.2.3. In addition, we show (1.20), (1.21), and (1.24).

3.1 Key Lemma

We shall make frequent use of the following simple observation.

Lemma 3.1.1. *If $\lambda > \mu > 0$, then $f(z) = \dfrac{1-\lambda z}{1-\mu z}$ is decreasing for $0 \leq \lambda z < 1$.*

In particular, if $\lambda_1, \lambda_2 \geq 1$ and $0 \leq x \leq \alpha$, $0 \leq y \leq \beta$, with $\lambda_1\alpha, \lambda_2\beta < 1$, then

$$\frac{(1-\lambda_1 x)(1-\lambda_2 y)}{1-xy} \geq \left(\frac{1-\lambda_1\alpha}{1-\alpha y}\right)(1-\lambda_2 y) \geq \frac{(1-\lambda_1\alpha)(1-\lambda_2\beta)}{1-\alpha\beta}.$$

Proof. Plainly $f'(z) = -(\lambda-\mu)/(1-z\mu)^2 < 0$ for $0 \leq z < \mu^{-1}$. □

3.2 Proof of Theorem 1.2.1

Since any two nonzero consecutive t_i in (1.14) have opposite signs, we have

$$|d_k^-| \leq \bar{\alpha}_k, \quad |d_k^+| \leq \alpha_k, \quad d_k^- d_k^+ \leq 0. \tag{3.1}$$

From (3.1) and (1.6), we have $s_2(k), s_4(k) \geq \frac{1}{4} \geq M(\alpha, \gamma^*)$. Hence, by Lemma 2.3.1,

$$M(\alpha, \gamma^*) = \liminf_{k \to \infty} \min\{s_1(k), s_3(k)\}. \tag{3.2}$$

Since we are evaluating lim inf on k, from now on, whenever we see the index k, it will be understood that we are letting $k \to \infty$. Also, we may assume that $a_i \geq R$ for all i.

Observe that changing the signs of t_i only interchanges $s_1(i)$ with $s_3(i)$. Hence, as long as we check both signs on the t_i, it will be enough to show that

$$s_3(k) \geq C(R).$$

We also observe that interchanging the pairs (a_{k-i}, t_{k-i}) with (a_{k+1+i}, t_{k+1+i}) for all $i \geq 0$ only interchanges $\bar{\alpha}_k$ with α_k and d_k^- with d_k^+.

3.2.1 R is Even

The proof when R is even is straightforward.

Proof of Theorem 1.2.1 when R is even. In this case we have $R_* = R$, $R_{**} = R + 1$, and (1.15) becomes

$$\beta = [0; \overline{R}]^- = \frac{1}{R - \beta} \text{ and } \delta = \frac{1}{R + 1 - \beta} = [0; R + 1, \overline{R}]^-,$$

with

$$\beta - \delta = \frac{1}{(R - \beta)(R + 1 - \beta)} = \delta\beta,$$

and hence $\beta = \delta + \delta\beta > \delta$.

If a_k is odd and $t_k = 1$, then $a_k \geq R+1$, $d_k^- \leq \bar{\alpha}_k$, $d_k^+ \leq 0$, and

$$s_3(k) \geq \frac{(1-2\bar{\alpha}_k)(1-\alpha_k)}{4(1-\bar{\alpha}_k\alpha_k)} \geq \frac{(1-2\delta)(1-\beta)}{4(1-\delta\beta)},$$

where the last inequality follows from Lemma 3.1.1, since $\bar{\alpha}_k \leq \delta, \alpha_k \leq \beta$ by property (2.1). As observed above this also covers the case a_{k+1} odd with $t_{k+1} = 1$.

If a_k is odd and $t_k = -1$ with a_{k+1} even (likewise a_{k+1} odd, $t_{k+1} = -1$ with a_k even), then we have $t_{k+1} = 0$, $d_k^- \leq 0$, $d_k^+ \leq \alpha_k\alpha_{k+1} \leq \alpha_k\beta$, and Lemma 3.1.1 with $\alpha_k, \bar{\alpha}_k \leq \beta$ gives

$$s_3(k) \geq \frac{(1-\bar{\alpha}_k)(1-(1+\beta)\alpha_k)}{4(1-\bar{\alpha}_k\alpha_k)} \geq \frac{(1-\beta)(1-\beta(1+\beta)}{4(1-\beta^2)} \geq \frac{(1-\beta)(1-2\delta)}{4(1-\delta\beta)}, \tag{3.3}$$

since $\delta < \beta$ and $\beta(1+\beta) < 2\delta$ by observing

$$2\delta - \beta(1+\beta) = \frac{2}{R+1-\beta} - \frac{1+\beta}{R-\beta} = \delta\beta(R - \beta(R-\beta) - 1 - 2\beta) = \delta\beta(R-2-2\beta) > 0.$$

This just leaves the case that a_k and a_{k+1} both are even. If $d_k^- \leq 0$ and $d_k^+ \geq 0$ (likewise $d_k^- \geq 0$ and $d_k^+ \leq 0$), then $d_k^+ \leq \alpha_k\alpha_{k+1} \leq \alpha_k\beta$, and again we have (3.3). □

3.2.2 *R* is Odd

In this case, we have $R_* = R+1$, $R_{**} = R$, and (1.15) becomes

$$\beta = [0; \overline{R+1}]^- = \frac{1}{R+1-\beta} \quad \text{and} \quad \delta = \frac{1}{R-\beta} = [0; R, \overline{R+1}]^-,$$

with $\delta = \beta + \delta\beta > \beta$.

We first establish some lemmas. Assume in both lemmas that $R \geq 3$ is odd and $\gamma = \gamma^*$.

Lemma 3.2.1. *Suppose that $\theta < 1$. If a_{k+1} is odd and $t_{k+1} = 1$, then*

$$\frac{1-\alpha_k-d_k^+}{1-\theta\alpha_k} \geq \frac{1-2\delta}{1-\theta\delta}. \tag{3.4}$$

Likewise, if a_k is odd and $t_k = 1$, then

$$\frac{1-\bar{\alpha}_k - d_k^-}{1-\theta\bar{\alpha}_k} \geq \frac{1-2\delta}{1-\theta\delta}. \tag{3.5}$$

Proof. Notice that it suffices to show the inequality (3.4) when $k = 0$. That is

$$A := \frac{1-\alpha-d_0^+}{1-\theta\alpha} \geq \frac{1-2\delta}{1-\theta\delta},$$

where $\alpha_0 = \alpha = [0; a_1, a_2, \cdots]^-$, and $d_0^+ = t_1\alpha + t_2\alpha\alpha_1 + \cdots$.

If $\alpha \leq \delta$ (for example the case when the a_i, $i \geq 2$, are all even), then

$$A \geq \frac{1-2\alpha}{1-\theta\alpha} \geq \frac{1-2\delta}{1-\theta\delta}, \tag{3.6}$$

from (3.1) and Lemma 3.1.1. So, suppose that $\alpha > \delta$, and let a_{n+1}, $n \geq 1$, be the odd partial quotient such that a_i is even for all $1 < i < n+1$. Notice we must have $a_1 = a_{n+1} = R$ and $a_i = R+1$ for $1 < i < n+1$, else $\alpha < \delta$. Since $t_1 = 1$, $t_2 = t_3 = \cdots = t_n = 0$, $t_{n+1} = -1$, and trivially $\alpha_{n+1} \leq [0; \bar{3}]^- \leq 1/2$, we have

$$d_0^+ \leq \alpha - \alpha\alpha_1\cdots\alpha_n + \alpha\alpha_1\cdots\alpha_n\alpha_{n+1} \leq \alpha - \frac{1}{2}\alpha\alpha_1\cdots\alpha_n,$$

and

$$A \geq \frac{1-2\alpha+\frac{1}{2}\alpha\alpha_1\cdots\alpha_n}{1-\theta\alpha} > \frac{1-2\alpha+\frac{1}{2}\alpha\alpha_1\cdots\alpha_n}{1-\theta\delta}. \tag{3.7}$$

Setting $\nu := [0; a_1, a_2, \ldots, a_n, a_{n+1}+2, a_{n+2}, \ldots]^-$, we have $\nu < \delta$, and we just need to show that

$$\alpha - \nu \leq \frac{1}{4}\alpha\alpha_1\cdots\alpha_n, \tag{3.8}$$

to obtain $1 - 2\alpha + \frac{1}{2}\alpha\alpha_1\cdots\alpha_n \geq 1 - 2\nu \geq 1 - 2\delta$ and (3.4).

Recall that

$$\alpha = [0; a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots]^- = \frac{p_n - p_{n-1}\alpha_n}{q_n - q_{n-1}\alpha_n} = \frac{p_{n+1} - p_n\alpha_{n+1}}{q_{n+1} - q_n\alpha_{n+1}}. \tag{3.9}$$

Similarly, we obtain

$$\nu = \frac{p_n - p_{n-1}\frac{1}{a_{n+1}+2-\alpha_{n+1}}}{q_n - q_{n-1}\frac{1}{a_{n+1}+2-\alpha_{n+1}}} = \frac{p_{n+1} - p_n\alpha_{n+1} + 2p_n}{q_{n+1} - q_n\alpha_{n+1} + 2q_n},$$

and $p_{n+1}q_n - p_nq_{n+1} = 1$ gives

$$\alpha - \nu = \frac{2}{(q_{n+1} - q_n\alpha_{n+1})(q_{n+1} - q_n\alpha_{n+1} + 2q_n)} = \frac{2\alpha\alpha_1\cdots\alpha_n}{(q_{n+1} + (2-\alpha_{n+1})q_n)}.$$

So, (3.8) just needs $q_{n+1} + (2-\alpha_{n+1})q_n \geq 8$. Plainly $q_{n+1} \geq q_2 \geq 3\cdot 3 - 1 = 8$. ☐

Lemma 3.2.2. *Suppose that $\theta < 1$. If a_{k+1} is even and $d_k^+ \leq 0$, then*

$$\frac{1-\alpha_k - d_k^+}{1-\theta\alpha_k} \geq \frac{1-\beta}{1-\theta\beta}.$$

Proof. We proceed as in the proof of Lemma 3.2.1. Suppose $k = 0$. Then, we show

$$A := \frac{1-\alpha-d_0^+}{1-\theta\alpha} \geq \frac{1-\beta}{1-\theta\beta}.$$

If $\alpha \leq \beta$, then

$$A \geq \frac{1-\alpha}{1-\theta\alpha} \geq \frac{1-\beta}{1-\theta\beta}. \tag{3.10}$$

Assume $\alpha > \beta$, and let $a_{n+1}, n \geq 1$, be the odd partial quotient such that a_i is even for all $1 \leq i \leq n$. We must have $a_1 = a_2 = \cdots = a_n = R+1$ and $a_{n+1} = R$, else $\alpha < \beta$. Since $t_1 = t_2 = \ldots = t_n = 0$, $t_{n+1} = -1$, and $\alpha_{n+1} \leq 1/2$,

$$d_0^+ \leq -\alpha\alpha_1\cdots\alpha_n + \alpha\alpha_1\cdots\alpha_n\alpha_{n+1} \leq -\frac{1}{2}\alpha\alpha_1\cdots\alpha_n,$$

and

$$A \geq \frac{1-\alpha+\frac{1}{2}\alpha\alpha_1\cdots\alpha_n}{1-\theta\beta}.$$

Set $\nu := [0; a_1, a_2, \ldots, a_n, a_{n+1}+2, a_{n+2}, \ldots]^- < \beta$. This time we just need to show $\alpha - \nu \leq \frac{1}{2}\alpha\alpha_1\cdots\alpha_n$, which reduces to $q_{n+1}+(2-\alpha_{n+1})q_n \geq 4$. Plainly, $q_{n+1} \geq q_2 \geq 3\cdot 4 - 1 = 11$. ☐

Proof of Theorem 1.2.1 when R is odd. We set $\sigma := [0;\overline{R}]^-$. We need to show that $s_3(k) \geq C(R)$. If a_k and a_{k+1} both are odd, then without loss of generality, we can assume $t_k = -1$ and $t_{k+1} = 1$. Plainly $d_k^- \leq -\bar{\alpha}_k + \bar{\alpha}_k\bar{\alpha}_{k-1} \leq -\bar{\alpha}_k + \bar{\alpha}_k\sigma$ (using $\bar{\alpha}_k \leq \sigma$), and by Lemma 3.2.1, Lemma 3.1.1 (using $\bar{\alpha}_k \leq \sigma$), and $\sigma > \beta$, we have

$$s_3(k) \geq \frac{(1-\bar{\alpha}_k\sigma)(1-\alpha_k-d_k^+)}{1-\bar{\alpha}_k\alpha_k} \geq \frac{(1-\bar{\alpha}_k\sigma)(1-2\delta)}{4(1-\bar{\alpha}_k\delta)} \geq \frac{(1-2\delta)(1-\sigma^2)}{4(1-\sigma\delta)} > \frac{(1-2\delta)(1-\sigma^2)}{4(1-\beta\delta)} > C(R),$$

since $\sigma^2 < \frac{1}{2}\sigma < \beta$.

Now it suffices to consider the following three cases: (i), a_k and a_{k+1} both are even, (ii), $(a_k, t_k) = (odd, -1)$ and a_{k+1} is even, (iii), $(a_k, t_k) = (odd, 1)$ and a_{k+1} is even. For (i), we have either $d_k^- \leq 0$ and $d_k^+ \leq \alpha_k\alpha_{k-1} \leq \sigma^2$ or $d_k^+ \leq 0$ and $d_k^- \leq \bar{\alpha}_k\bar{\alpha}_{k-1} \leq \sigma^2$, and for (ii), we have $d_k^- \leq 0$, $d_k^+ \leq \sigma^2$. In each case, it can be readily seen that

$$s_3(k) \geq \frac{1}{4}(1-\sigma)(1-\sigma-\sigma^2) = \frac{1}{4}(1-2\sigma+\sigma^3) \geq \frac{1}{4}(1-2\delta) \geq C(R),$$

using that

$$\sigma - \delta = \frac{1}{R-\sigma} - \frac{1}{R-\beta} = (\sigma-\beta)\sigma\delta = (1-\beta+\sigma)\sigma^2\delta\beta < \frac{3}{2}\beta\sigma^3 < \frac{1}{2}\sigma^3.$$

For (iii), we apply Lemma 3.2.1 and Lemma 3.2.2 to get

$$s_3(k) = \frac{(1-\bar{\alpha}_k-d_k^-)(1-\alpha_k-d_k^+)}{4(1-\bar{\alpha}_k\alpha_k)} \geq \frac{(1-2\delta)(1-\alpha_k-d_k^+)}{4(1-\delta\alpha_k)} \geq \frac{(1-2\delta)(1-\beta)}{4(1-\delta\beta)}. \qquad \square$$

3.3 Proof of the Asymptotic Behaviors (1.20) and (1.21)

Suppose $R \geq 4$ is even. Then, using $\delta = \frac{\beta}{1+\beta}$ in $C(R)$, we obtain

$$E_1(R) = R^3\left(1 - \frac{3}{R} + \frac{5}{R^2} - \frac{1-2\beta+\beta^2}{1+\beta-\beta^2}\right). \tag{3.11}$$

One can use $R\beta = 1+\beta^2$ in (3.11) to write

$$E_1(R) = \frac{11-2\beta(6-3\beta+\beta^2)}{1+(1-2\beta)/R} = 11 + O\left(\frac{1}{R}\right).$$

Notice that $E_1(R) \nearrow 11$. Since $\beta = 2-\sqrt{3}$ when $R=4$, we have $E_1(4) = (524-256\sqrt{3})/11 = 7.3268\ldots$. This completes the proof of (1.20).

Now, suppose $R \geq 3$ odd. In this case, one can use $\delta = \frac{\beta}{1-\beta}$ and $R\beta = 1-\beta+\beta^2$ to write

$$E_2(R) = \frac{10-2\beta(11-7\beta+2\beta^2)}{1-2\beta/R} = 10 + O\left(\frac{1}{R}\right),$$

with $E_2(3) = (348-162\sqrt{3})/11 = 6.1279\ldots$, $E_2(R) \nearrow 10$, and (1.21) is clear.

3.4 Proof of Theorem 1.2.2

We assume that α has expansion (1.9) of period (1.23) with $R \geq 4$ even. In this case, we have $\bar{\alpha}_k, \alpha_k \leq \beta$ for any k.

Suppose first that γ has an expansion (1.11) with $t_i = 0$ when $a_i = R$ and $t_i = \pm 1$ when $a_i = R+1$, for all sufficiently large i. If $a_k = R+1$ and $t_k = 1$, then

$$\bar{\alpha}_k = [0; R+1, \underbrace{R, R, \ldots, R}_{l\ \ times}, \ldots],\ \ \alpha_k = [0; \underbrace{R, R, \ldots, R}_{l\ \ times}, \ldots],$$

and hence

$$\bar{\alpha}_k \to \delta,\ \alpha_k \to \beta, \quad d_k^- \to \delta,\ d_k^+ \to 0, \quad s_3(k) \to C(R), \qquad \text{as } k, l \to \infty.$$

Likewise, if $a_k = R+1$ and $t_k = -1$, then $s_1(k) \to C(R)$ as $k, l \to \infty$. Hence, these γ cannot contribute a value $M(\alpha,\gamma)$ strictly greater than $C(R)$ to $\rho(\alpha)$ as $l \to \infty$. By Theorem 1.2.1, we have $M(\alpha, \gamma^*) \geq C(R)$, and hence

$$\lim_{l \to \infty} M(\alpha, \gamma^*) = C(R). \tag{3.12}$$

It remains to show that $M(\alpha, \gamma) \leq C(R)$ as $l \to \infty$ for the remaining $\gamma \notin \mathbb{Z} + \alpha\mathbb{Z}$; that is, those γ having an expansion (1.11) with $|t_i| \geq 2$ infinitely often.

Observe that changing the signs of t_i only interchanges $s_1(i)$ with $s_3(i)$ and $s_2(i)$ with $s_4(i)$. Thus, if we eliminate any block of t_i from consideration, then the same will be true for its negative. Also, interchanging $\bar{\alpha}_k$ with α_k and d_k^- with d_k^+ does not change $s_1(k)$ and $s_3(k)$ (and interchanges $s_2(k), s_4(k)$). Hence, if we eliminate a block of t_i from consideration, then the same will be true for the reversed block of t_i (on a reversed block of a_i).

If $t_k = a_k$ infinitely often, then by Lemma 2.3.2,

$$M(\alpha, \gamma) \leq \liminf_{\substack{k \to \infty \\ t_k = a_k}} \frac{\bar{\alpha}_k}{4(1 - \bar{\alpha}_k \alpha_k)}, \tag{3.13}$$

and hence

$$M(\alpha, \gamma) \leq \frac{\beta}{4(1 - \beta^2)} \leq \frac{(\beta + \beta^2 - \delta\beta)}{4(1 - \beta^2 + \beta^2 - \delta\beta)} < C(R),$$

on observing that $(1 - 2\delta)(1 - \beta) = 1 - 3\beta + 4\delta\beta$ and $4\beta + \beta^2 - 5\delta\beta < 1$ (equivalently $R > 4 - \delta(3 - 2\beta)$). Thus, we may assume that $t_i = a_i$ at most finitely many times. In this case, we have $|d_k^-| \leq 1 - \bar{\alpha}_k$ and $|d_k^+| \leq 1 - \alpha_k$ for suitably large k. Hence, $0 \leq (1 - d_k^-)^2 - \bar{\alpha}_k^2 < (1 - d_k^-)^2$ and $0 \leq 1 - (\alpha_k + d_k^+)^2 < 1$, and we notice that

$$\sqrt{s_3(k) s_4(k)} = \frac{\sqrt{\left((1 - d_k^-)^2 - \bar{\alpha}_k^2\right)\left(1 - (\alpha_k + d_k^+)^2\right)}}{4(1 - \bar{\alpha}_k \alpha_k)},$$

giving

$$\min\{s_3(k), s_4(k)\} \leq \frac{1 - d_k^-}{4(1 - \bar{\alpha}_k \alpha_k)}. \tag{3.14}$$

Similarly,

$$\min\{s_1(k), s_2(k)\} \leq \frac{1+d_k^-}{4(1-\bar{\alpha}_k\alpha_k)}.$$

We establish the following lemmas. In each case, we are assuming that $t_i = a_i$ at most finitely many times.

Lemma 3.4.1. *If the sequence $\{t_i\}_i$ in the expansion (1.11) of γ has infinitely many blocks of the form $t_k, t_{k+j} > 0$, for some $j > 0$, with at least one of a_k, a_{k+j} even, and $t_i = 0$ for any $k < i < k+j$, then $M(\alpha,\gamma) < C(R)$ as $l \to \infty$.*

Proof. Without loss of generality suppose that a_k is even. Then $a_k = R$, $t_k \geq 2$, and with $\theta = [0; \overline{R+1}]^-$, we have

$$\bar{\alpha}_k \geq [0; R, \overline{R+1}]^- = \frac{1}{R-\theta} = \frac{1}{(R+1-\theta)-1} = \frac{1}{1/\theta - 1} = \frac{\theta}{1-\theta}, \quad \alpha_k \geq \delta \quad \text{as} \quad l \to \infty.$$

Also,

$$\delta - \theta = \frac{1}{R+1-\beta} - \frac{1}{R+1-\theta} = \delta\theta(\beta-\theta).$$

Plainly, $d_k^+ \geq 0$, and by (2.4), we get

$$d_k^- = t_k\bar{\alpha}_k + \bar{\alpha}_k(t_{k-1}\bar{\alpha}_{k-1} + t_{k-2}\bar{\alpha}_{k-1}\bar{\alpha}_{k-2} + \cdots) \geq 2\bar{\alpha}_k + \bar{\alpha}_k d_{k-1}^- > \bar{\alpha}_k.$$

Hence,

$$s_3(k) \leq \frac{(1-2\bar{\alpha}_k)(1-\alpha_k)}{4(1-\bar{\alpha}_k\alpha_k)} \leq \frac{\left(1 - \dfrac{2\theta}{1-\theta}\right)(1-\delta)}{4(1-\delta\beta)} \leq \frac{1-3\theta}{4(1-\delta\beta)} < C(R),$$

where the second inequality follows from Lemma 3.1.1 and $\theta/(1-\theta) < \beta$, the third from $\delta > \theta$, and the last since $1 - 3\theta = 1 - 3\delta + 3\delta\theta(\beta-\theta)$ while $(1-2\delta)(1-\beta) = 1 - 3\delta + \delta\beta$. The result follows from Lemma 2.3.1. ☐

We can now assume that the sequence $\{t_i\}_i$ in the expansion (1.11) eventually does not contain any block (or its negative) of the type excluded by Lemma 3.4.1.

Lemma 3.4.2. *If γ has infinitely many $t_k \geq 3$, then $M(\alpha, \gamma) < C(R)$ as $l \to \infty$.*

Proof. If $a_k = R$ and $t_k \geq 4$, then $d_k^- \geq 4\bar{\alpha}_k + \bar{\alpha}_k d_{k-1}^- \geq 3\bar{\alpha}_k$, and by (3.14), we have

$$M(\alpha, \gamma) \leq \frac{1 - 3\bar{\alpha}_k}{4(1 - \bar{\alpha}_k \beta)} \leq \frac{1 - 3\delta}{4(1 - \delta\beta)} = \frac{1 - 2\delta - \beta + \delta\beta}{4(1 - \delta\beta)} < C(R),$$

where the second inequality follows from Lemma 3.1.1 and $\bar{\alpha}_k > \delta$. So, we can assume that $|t_i| \leq 2$ if $a_i = R$. Suppose $a_k = R + 1$ and $t_k \geq 3$. Then, $d_k^- \geq 3\bar{\alpha}_k - 2\bar{\alpha}_k\bar{\alpha}_{k-1}$, $d_k^+ \geq -2\alpha_k$, and as $l \to \infty$

$$\begin{aligned} s_3(k) &\leq \frac{(1 - 4\bar{\alpha}_k + 2\bar{\alpha}_k\bar{\alpha}_{k-1})(1 + \alpha_k)}{4(1 - \bar{\alpha}_k \alpha_k)} \to \frac{(1 - 4\delta + 2\delta\beta)(1 + \beta)}{4(1 - \delta\beta)} \\ &= \frac{(1 - 4\delta + \beta - 2\delta\beta + 2\delta\beta^2)}{4(1 - \delta\beta)} = \frac{(1 - 2\delta - \beta + 2\delta\beta^2)}{4(1 - \delta\beta)} < C(R). \end{aligned}$$ □

So, $M(\alpha, \gamma) < C(R)$ from Lemma 2.3.1 if this happens for infinitely many k.

We now also assume that $|t_i| \leq 2$ for all sufficiently large i.

Lemma 3.4.3. *If γ has infinitely many of the following blocks, then $M(\alpha, \gamma) < C(R)$ as $l \to \infty$.*

(i) $t_k = 1$ and $t_{k+1} = -2$, or

(ii) $t_k = 0$ and $t_{k+1} = -2$.

Proof. (i): Note that $a_k = R + 1$ and $a_{k+1} = R$. Plainly, $d_k^- \leq \bar{\alpha}_k$, $d_k^+ \leq -2\alpha_k + 2\alpha_k\alpha_{k+1}$, and as $l \to \infty$

$$s_1(k) \leq \frac{(1 - 3\alpha_k + 2\alpha_k\alpha_{k+1})}{4(1 - \bar{\alpha}_k \alpha_k)} \to \frac{(1 - 3\beta + 2\beta^2)}{4(1 - \delta\beta)} = \frac{(1 - 2\delta - \beta + 2\delta\beta^2)}{4(1 - \delta\beta)} < C(R).$$

(ii): In this case, $a_k = a_{k+1} = R$. With $\theta = [0; \overline{R+1}]^-$, we have $\alpha_k \geq \dfrac{1}{R-\theta} =: \lambda$. Since $\bar{\alpha}_i, \alpha_i \leq \beta$ for any i, we have $d_k^- \leq 2\bar{\alpha}_k\bar{\alpha}_{k-1} \leq 2\beta\bar{\alpha}_k$, $d_k^+ \leq -2\alpha_k + 2\alpha_k\alpha_{k+1} \leq -2\alpha_k + 2\beta^2$, and hence

$$\begin{aligned} s_1(k) &\leq \frac{(1-(1-2\beta)\bar{\alpha}_k)(1-3\alpha_k+2\beta^2)}{4(1-\bar{\alpha}_k\beta)} \\ &\leq \frac{(1-\delta+2\beta\delta)(1-3\lambda+2\beta^2)}{4(1-\delta\beta)} \\ &\leq \frac{(1-3\lambda+2\beta^2)}{4(1-\delta\beta)} < C(R), \end{aligned}$$

where the second inequality follows from Lemma 3.1.1 and $\bar{\alpha}_k > \delta$. For the last inequality observe that, $\beta - \lambda = \beta\lambda(\beta - \theta) < \lambda\beta^2$ so that $1 - 3\lambda + 2\beta^2 < 1 - 3\beta + \beta^2(2 + 3\lambda)$ while $(1-2\delta)(1-\beta) = 1 - 3\beta + \beta^2(4 - 4\delta)$. Infinitely many such k would give $M(\alpha, \gamma) < C(R)$ by Lemma 2.3.1. □

Proof of Theorem 1.2.2. From Lemma 3.4.2 and Lemma 3.4.3, we see that a γ with infinitely many $|t_i| \geq 2$ has $M(\alpha, \gamma) \leq C_0(R) < C(R)$ as $l \to \infty$ (where $C_0(R)$ is made explicit in the proof). Hence, $\lim_{l\to\infty} \rho(\alpha) = \lim_{l\to\infty} M(\alpha, \gamma^*) = C(R)$. □

3.5 Proof of Theorem 1.2.3

Suppose that $\alpha = [0; \overline{R, R, R+1, R, R+1, R+1, R, R+1, R+1, R, R+1}]^-$ with R odd. By Theorem 1.2.1 and (1.21), we have

$$\rho(\alpha) \geq M(\alpha, \gamma^*) \geq C(R) = \frac{1}{4}\left(1 - \frac{3}{R} + \frac{4}{R^2} + O(R^{-3})\right),$$

so we just need to show that all γ have

$$M(\alpha, \gamma) \leq \frac{1}{4}\left(1 - \frac{3}{R} + \frac{4}{R^2} + O(R^{-3})\right). \tag{3.15}$$

Since $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$, we have

$$[0; R, R, \cdots]^{-} = \frac{1}{R - \frac{1}{R - \cdots}} = \frac{1}{R}\left(1 - \frac{1}{R(R-\cdots)}\right)^{-1} = \frac{1}{R} + O(R^{-3}).$$

Similarly, we observe the following

$$\begin{aligned} [0; R, R+1, \cdots]^{-} &= \frac{1}{R} + O(R^{-3}), \\ [0; R+1, R \text{ or } R+1, \cdots]^{-} &= \frac{1}{R} - \frac{1}{R^2} + O(R^{-3}), \end{aligned}$$

and so for our α, we have

$$\frac{1}{1 - \bar{\alpha}_k \alpha_k} = 1 + \bar{\alpha}_k \alpha_k + (\bar{\alpha}_k \alpha_k)^2 + \cdots = 1 + \frac{1}{R^2} + O(R^{-3}). \tag{3.16}$$

Now if γ has $t_k = a_k$ infinitely often, then from (3.13)

$$M(\alpha, \gamma) \leq \frac{1}{4}\left(\frac{1}{R} + O(R^{-3})\right)\left(1 + \frac{1}{R^2} + O(R^{-3})\right) = \frac{1}{4}\left(\frac{1}{R} + O(R^{-3})\right),$$

so we can assume that γ has only finitely many $t_i = a_i$. In view of (3.16), we write

$$\tilde{s}_j(k) = 4(1 - \bar{\alpha}_k \alpha_k) s_j(k), \quad j = 1, \ldots, 4,$$

and (3.15) amounts to showing that there are infinitely many k with an

$$\tilde{s}_j(k) \leq 1 - \frac{3}{R} + \frac{3}{R^2} + O(R^{-3}). \tag{3.17}$$

We proceed as in the proof of Theorem 1.2.2 successively eliminating blocks of t_i, recalling that when we eliminate a block, we also eliminate its negative or reverse (by interchanging $s_j(k)$).

By (3.14), we will get (3.17) if γ has infinitely many k with

$$d_k^- \geq \frac{3}{R} - \frac{3}{R^2} + O(R^{-3}). \tag{3.18}$$

We use this to rule out large $|t_i|$. If $t_k \geq 5$, then we have

$$d_k^- \geq 5\bar{\alpha}_k + \bar{\alpha}_k d_{k-1}^- > 4\bar{\alpha}_k \geq \frac{4}{R} + O(R^{-2}) \geq \frac{3}{R} - \frac{3}{R^2} + O(R^{-3}).$$

If we have $t_k = 4$ with $|t_{k-1}| \leq 4$, then $d_{k-1}^- = O(R^{-1})$ and again

$$d_k^- = \frac{4}{R} + O(R^{-2}).$$

So, we can assume that $|t_i| \leq 3$ for all but finitely many i. If we have infinitely many blocks $a_k, a_{k-1} = R, R+1$ (or their reverse) with $t_k = 3$, then

$$d_k^- \geq 3\bar{\alpha}_k - 2\bar{\alpha}_k\bar{\alpha}_{k-1} + O(R^{-3}) = \frac{3}{R} - \frac{2}{R^2} + O(R^{-3}).$$

Hence, we can assume that (all but finitely many) $t_i = \pm 1$ if $a_i = R$ and $t_i = 0, \pm 2$ if $a_i = R + 1$.

First we rule out infinitely many consecutive positive or consecutive negative t_i. If $t_k, t_{k+1} > 0$, then $d_k^- \geq \bar{\alpha}_k + O(R^{-2}), d_k^+ \geq \alpha_k + O(R^{-2})$ and

$$\begin{aligned}\tilde{s}_3(k) &\leq (1 - 2\bar{\alpha}_k + O(R^{-2}))((1 - 2\alpha_k + O(R^{-2})) \\ &= \left(1 - \frac{2}{R} + O(R^{-2})\right)\left(1 - \frac{2}{R} + O(R^{-2})\right) \\ &= 1 - \frac{4}{R} + O(R^{-2}).\end{aligned}$$

Next we rule out infinitely many blocks $t_{k-1}, t_k, t_{k+1}, t_{k+2} = 0, 1, 0, 0$ (or their reverse

$0,0,1,0$ or their negatives) since $d_k^- = \bar\alpha_k + O(R^{-3})$, $d_k^+ = O(R^{-3})$ and

$$\begin{aligned}\tilde s_3(k) &= \Big(1 - 2\bar\alpha_k + O(R^{-3})\Big)\Big(1 - \alpha_k + O(R^{-3})\Big)\\ &= \left(1 - \frac{2}{R} + O(R^{-3})\right)\left(1 - \frac{1}{R} + \frac{1}{R^2} + O(R^{-3})\right) = 1 - \frac{3}{R} + \frac{3}{R^2} + O(R^{-3}).\end{aligned}$$

If $t_k, t_{k+1} = 2, 0$, then $d_k^- = 2\bar\alpha_k + O(R^{-2})$, $d_k^+ = O(R^{-2})$ and

$$\begin{aligned}\tilde s_3(k) &= \Big(1 - 3\bar\alpha_k + O(R^{-2})\Big)\Big(1 - \alpha_k + O(R^{-2})\Big)\\ &= \left(1 - \frac{3}{R} + O(R^{-2})\right)\left(1 - \frac{1}{R} + O(R^{-2})\right) = 1 - \frac{4}{R} + O(R^{-2}).\end{aligned}$$

Hence blocks $R+1, R+1$ must eventually have $t_k, t_{k+1} = 0,0$ or $2,-2$ or $-2,2$.

If $t_{k-1}, t_k, t_{k+1} = -2, 1, -2$, then $d_k^- = \bar\alpha_k - 2\bar\alpha_k\bar\alpha_{k-1} + O(R^{-3})$, $d_k^+ \le -2\alpha_k + 2\alpha_k\alpha_{k+1} + O(R^{-3})$ and

$$\begin{aligned}\tilde s_1(k) &\le \Big(1 - 2\bar\alpha_k\bar\alpha_{k-1} + O(R^{-3})\Big)\Big(1 - 3\alpha_k + 2\alpha_k\alpha_{k+1} + O(R^{-3})\Big)\\ &= \left(1 - \frac{2}{R^2} + O(R^{-3})\right)\left(1 - \frac{3}{R} + \frac{5}{R^2} + O(R^{-2})\right) = 1 - \frac{3}{R} + \frac{3}{R^2} + O(R^{-3}).\end{aligned}$$

Hence if we have a block $a_{k-2}, a_{k-1}, a_k, a_{k+1}, a_{k+2} = R+1, R+1, R, R+1, R+1$ with $t_k = \pm 1$ then we must have $t_{k+1}, t_{k+2} = 0, 0$ and $t_{k-1}, t_{k-2} = \mp 2, \pm 2$ (or vice versa in which case we use the reverse). Consider then the block

$$a_{k+1}, \ldots, a_{k+6} = R+1, R+1, R, R+1, R, R, \quad t_{k+1}, t_{k+2} = 0, 0.$$

Assuming that $t_{k+3} = 1$ (or use the negative), then having ruled out $0,0,1,0$, we must have $t_{k+4}, t_{k+5}, t_{k+6} = -2, 1, -1$, and finally

$$\begin{aligned}d_{k+4}^- &= -2\bar\alpha_{k+4} + \bar\alpha_{k+4}\bar\alpha_{k+3} + O(R^{-3}),\\ d_{k+4}^+ &= \alpha_{k+4} - \alpha_{k+4}\alpha_{k+5} + O(R^{-3}),\end{aligned}$$

and hence

$$\begin{aligned}\tilde{s}_1(k+4) &= \Big(1 - 3\bar{\alpha}_{k+4} + \bar{\alpha}_{k+4}\bar{\alpha}_{k+3} + O(R^{-3}\Big)\Big(1 - \alpha_{k+4}\alpha_{k+5} + O(R^{-3})\Big) \\ &= \left(1 - \frac{3}{R} + \frac{4}{R^2} + O(R^{-3})\right)\left(1 - \frac{1}{R^2} + O(R^{-3}\right) = 1 - \frac{3}{R} + \frac{3}{R^2} + O(R^{-3}). \quad \square\end{aligned}$$

3.6 Proof of (1.24)

If in the previous proof we had taken α to have period $R, \underbrace{R+1, \ldots, R+1}_{l \text{ times}}$, with $R \geq 5$ odd, then (3.15) would still hold, except for those γ whose t_i eventually consist of zeros one side of the ± 1 and blocks of $\mp 2, \pm 2$ the other. For these γ, if $t_k = 1$ inside a block $\ldots, 0, 0, 1, -2, 2, \ldots$, then $d_k^- \to \delta$, $d_k^+ \to -2\beta/(1+\beta)$ and $d_{k-1}^- \to 0$, $d_{k-1}^+ \to \delta - 2\delta\beta/(1+\beta)$ as $l \to \infty$, and

$$\begin{aligned} s_1(k) &\to \frac{1 - 3\beta + \frac{2\beta^2}{1+\beta}}{4(1-\delta\beta)} =: D(R), \\ s_3(k-1) &\to \frac{\left(1 - 2\delta + \frac{2\delta\beta}{1+\beta}\right)(1-\beta)}{4(1-\delta\beta)} =: C_1(R), \end{aligned}$$

with $s_1(k-1) > s_3(k-1)$ and $s_3(k) > (1-2\delta)/4(1-\alpha\beta) > D(R)$. Likewise for the negatives and reverses. At all places, we have

$$s_2(k), s_4(k) > (1 - \bar{\alpha}_k)(1 - \alpha_k)/4(1 - \beta\alpha_k) > (1-\delta)^2/4(1-\delta\beta) > C_1(R).$$

Moreover, using $\delta = \beta/(1-\beta)$ in $C_1(R)$, it can be readily seen that $C_1(R) = D(R)$.

If $t_k, t_{k+1} = 0, 0$, and $d_k^- \to 0$, $\bar{\alpha}_k \to \beta$ (likewise if $d_k^+ \to 0$, $\alpha_k \to \beta$), then $d_k^+ \leq \alpha_k\alpha_{k+1} \leq \alpha_k\delta$ and hence

$$s_1(k), s_3(k) \geq \frac{(1-\beta)(1-\alpha_k(1+\delta))}{4(1-\beta\alpha_k)} > \frac{(1-\beta)(1-\delta-\delta^2)}{4(1-\delta\beta)} > C_1(R).$$

If $t_k, t_{k+1} = -2, 2$, then $d_{k-1}^- > \beta$ (because either $t_{k-1} = 2$ or $t_{k-1} = 1$ with $d_{k-1}^- \to \delta$),

$d_k^- = -2\bar{\alpha}_k + \bar{\alpha}_k d_{k-1}^- \geq -2\bar{\alpha}_k + \bar{\alpha}_k \beta$, $d_k^+ \geq 2\alpha_k - 2\alpha_k\alpha_{k+1} \geq \beta(2-2\delta)$, and

$$s_1(k) \geq \frac{(1-\bar{\alpha}_k(3-\beta))\,(1+\beta(1-2\delta))}{4(1-\bar{\alpha}_k\beta)} \geq \frac{(1-3\delta+\delta\beta)(1+\beta-2\beta\delta)}{4(1-\delta\beta)} > D(R),$$

for $R \geq 9$ using

$$(1-3\delta+\delta\beta)(1+\beta-2\beta\delta) = (1-3\beta+2\beta^2) + \beta(1-9\beta+4\delta^2\beta),$$

replacing $\bar{\alpha}_k$ by $1/(R+1-\delta)$ instead of δ in the second inequality and checking numerically for $R = 5$ and 7. Likewise for $s_3(k)$ and for $2, -2$. Hence, these γ have $M(\alpha, \gamma) \to C_1(R)$ as $l \to \infty$.

The proof of Theorem 1.2.3 immediately gives (1.24) for suitably large R. To see that it is true for all $R \geq 5$, we show that $M(\alpha, \gamma) < C_1(R)$ for the other γ. Notice, if we let $l \to \infty$, then $(1-\bar{\alpha}_k\alpha_k)^{-1} \leq (1-\delta\beta)^{-1}$; so it will be enough to show that the remaining γ have infinitely many

$$\tilde{s}_j(k) \leq 1 - 3\beta + \frac{2\beta^2}{1+\beta} = \left(1 - 2\delta + \frac{2\delta\beta}{1+\beta}\right)(1-\beta) =: \tilde{C}_1(R).$$

We repeat the steps of the proof of Theorem 1.2.3; successively ruling out certain blocks of t_i (or their negatives and reverses) occurring infinitely often. We can rule out $t_k = a_k$ since by (3.13),

$$\tilde{s_j} \leq \delta \leq 1 - 3\beta \leq \tilde{C_1}(R).$$

To eliminate large $|t_k|$, it suffices to replace (3.18) by

$$d_k^- \geq 3\beta - \frac{2\beta^2}{1+\beta}.$$

When $t_k \geq 4$, we have $d_k^- = 4\bar{\alpha}_k + \bar{\alpha}_k d_{k-1}^- > 3\beta$. Then, for $t_k = 3$, we get

$$d_k^- \geq 3\delta - \frac{2\delta\beta}{1-\beta} > 3\beta.$$

Now, we can suppose that $t_k = \pm 1$ if $a_k = R$ and $t_k = 0, \pm 2$ if $a_k = R+1$. If $t_k = 2$ and $t_{k+j} > 0$ for some $j \geq 1$, with $t_i = 0$ for any $k < i < k+j$, then $d_k^- = 2\bar{\alpha}_k + \bar{\alpha}_k d_{k-1}^- \geq \bar{\alpha}_k + \bar{\alpha}_k \bar{\alpha}_{k-1}, d_k^+ \geq 0$, and

$$\tilde{s}_3(k) \leq (1 - 2\beta - \beta^2)(1-\beta) = (1 - 2\delta + 2\delta\beta - \beta^2)(1-\beta) < \tilde{C}_1(R).$$

If $t_k, t_{k+1} = 2, 0$, then

$$\tilde{s}_3(k) \leq (1 - 3\beta + 2\beta\delta)\,(1 - \beta + 2\beta\delta) = 1 - 3\beta + 2\beta^2 - 2\beta^3 - \lambda < \tilde{C}_1(R),$$

with $\lambda = \beta(1 - 5\beta + 2\beta^2(1 - 2\delta^2)) > 0$. If $t_{k-1}, t_k, t_{k+1} = 0, 1, 0$, then as $l \to \infty$

$$\tilde{s}_3(k) \to (1 - 2\delta)(1 - \beta) < \tilde{C}_1(R).$$

Finally, if $t_{k-1}, t_k, t_{k+1} = -2, 1, -2$, then

$$\tilde{s}_1(k) \leq \left(1 - \frac{2\delta\beta}{1+\beta}\right)\left(1 - 3\beta + \frac{2\beta^2}{1+\beta}\right) < \tilde{C}_1(R).$$

For the asymptotic formula, we recall that

$$\delta = \frac{1}{R} + O(R^{-3}), \quad \beta = \frac{1}{R} - \frac{1}{R^2} + O(R^{-3}), \quad \frac{1}{1 - \delta\beta} = \frac{1}{R^2} + O(R^{-3}).$$

Hence,

$$C_1(R) = D(R) = \frac{1 - 3\beta + 2\beta^2 + O(R^{-3})}{4(1 - \delta\beta)} = \frac{1}{4}\left(1 - \frac{3}{R} + \frac{6}{R^2} + O(R^{-3})\right). \qquad \square$$

Chapter 4

Upper Bounds on the Largest Inhomogeneous Approximation Constant

In this chapter, we prove the main results associated with upper bounds on $\rho(\alpha)$ when R is odd: Theorem 1.2.4 and Theorem 1.2.5.

4.1 Proof of Theorem 1.2.4

Proof of Theorem 1.2.4. Suppose that α has $R \geq 3$ odd. Setting $\beta := [0; \overline{R}]^-$, we can assume that $\alpha_k \leq \beta$ and $\bar{\alpha}_k \leq \beta$ as $k \to \infty$. When $R = 3$ we, have $\beta = \frac{1}{2}(3 - \sqrt{5})$. We need to show that $M(\alpha, \gamma) \leq \tilde{C}(R)$ for any $\gamma \notin \mathbb{Z} + \alpha\mathbb{Z}$.

From (3.13), we can assume that (1.11) does not have $t_k = a_k$ infinitely often, for otherwise, as $k \to \infty$, we have

$$\frac{\bar{\alpha}_k}{4(1 - \bar{\alpha}_k \alpha_k)} = \frac{1}{4(a_k - \bar{\alpha}_{k-1} - \alpha_k)} \leq \frac{1}{4(R - 2\beta)} \leq \frac{1}{4\sqrt{5}} < \tilde{C}(3) \leq \tilde{C}(R). \tag{4.1}$$

Hence, by Lemma 2.3.1, we just need to show that there are infinitely many k with

$$\min\{s_1(k), s_2(k), s_3(k), s_4(k)\} \leq \tilde{C}(R).$$

If $a_k = R$ and $t_k \geq 3$, then $d_k^- = 3\bar{\alpha}_k + \bar{\alpha}_k d_{k-1}^- > 2\bar{\alpha}_k$ and $\alpha_k \leq \beta$, and by (3.14), we get

$$\min\{s_3(k), s_4(k)\} \leq \frac{1 - 2\bar{\alpha}_k}{4(1 - \bar{\alpha}_k\beta)}.$$

Since $f(x) = \frac{(1-2x)}{(1-\beta x)}$ has $f'(x) = -\frac{(2-\beta)}{(1-\beta x)^2} < 0$ and $\bar{\alpha}_k > 1/R$, we have

$$\min\{s_3(k), s_4(k)\} \leq \frac{1 - \frac{2}{R}}{4\left(1 - \frac{\beta}{R}\right)} < \tilde{C}(R),$$

the latter inequality since

$$\begin{aligned}\left(1 - \frac{1}{R}\right)\left(1 - \frac{1}{R^2}\right)\left(1 - \frac{\beta}{R}\right) - \left(1 - \frac{2}{R}\right)& \\ = \frac{1}{R}\left(1 - \frac{1}{R}\left(1 - \frac{1}{R}\right) - \beta\left(1 - \frac{1}{R}\right)\left(1 - \frac{1}{R^2}\right)\right) &> 0.\end{aligned}$$

Likewise, if $t_k \leq -3$, we get $d_k^- < -2\bar{\alpha}_k$ and

$$\min\{s_1(k), s_2(k)\} \leq \frac{1 + d_k^-}{4(1 - \bar{\alpha}_k\alpha_k)} \leq \tilde{C}(R).$$

So, we can assume that the $a_k = R$ have $t_k = \pm 1$. Suppose now that $a_k = R$ with $t_k = -1$ so that

$$d_k^- = -\bar{\alpha}_k + d_{k-1}^-\bar{\alpha}_k.$$

If $t_k = +1$ then we simply switch the roles of $s_1(k)$ and $s_3(k)$ and replace d_{k-1}^-, d_k^+ by $-d_{k-1}^-, -d_k^+$, respectively, in what follows.

`Case 1.` Suppose that d_{k-1}^-, $d_k^+ \le \frac{1}{R}$. Then,

$$s_1(k) = \frac{\left(1 - 2\bar{\alpha}_k + d_{k-1}^-\bar{\alpha}_k\right)\left(1 - \alpha_k + d_k^+\right)}{4(1 - \bar{\alpha}_k\alpha_k)} \le \frac{\left(1 - \bar{\alpha}_k(2 - \frac{1}{R})\right)\left(1 - \alpha_k + \frac{1}{R}\right)}{4(1 - \bar{\alpha}_k\alpha_k)}.$$

Since $\bar{\alpha}_k(1 + \frac{1}{R}) < 1$ and $\bar{\alpha}_k > \frac{1}{R}$, we get

$$s_1(k) < \frac{\left(1 - \bar{\alpha}_k\left(2 - \frac{1}{R}\right)\right)\left(1 - \alpha_k + \frac{1}{R}\right)}{4\left(1 - \alpha_k/\left(1 + \frac{1}{R}\right)\right)} = \frac{1}{4}\left(1 - \bar{\alpha}_k\left(2 - \frac{1}{R}\right)\right)\left(1 + \frac{1}{R}\right)$$
$$< \frac{1}{4}\left(1 - \frac{1}{R}\left(2 - \frac{1}{R}\right)\right)\left(1 + \frac{1}{R}\right) = \tilde{C}(R).$$

`Case 2.` Suppose that d_{k-1}^-, $d_k^+ \ge \frac{1}{R}$. Then

$$s_3(k) = \frac{\left(1 - d_{k-1}^-\bar{\alpha}_k\right)\left(1 - \alpha_k - d_k^+\right)}{4(1 - \bar{\alpha}_k\alpha_k)} \le \frac{\left(1 - \frac{\bar{\alpha}_k}{R}\right)\left(1 - \alpha_k - \frac{1}{R}\right)}{4(1 - \bar{\alpha}_k\alpha_k)},$$

This time, using $\bar{\alpha}_k(1 - \frac{1}{R}) < 1$ and $\bar{\alpha}_k > \frac{1}{R}$, we get

$$s_3(k) < \frac{\left(1 - \frac{\bar{\alpha}_k}{R}\right)\left(1 - \alpha_k - \frac{1}{R}\right)}{4\left(1 - \alpha_k/(1 - \frac{1}{R})\right)} = \frac{1}{4}\left(1 - \frac{\bar{\alpha}_k}{R}\right)\left(1 - \frac{1}{R}\right) < \tilde{C}(R).$$

`Case 3.` Suppose that $d_{k-1}^- \le \frac{1}{R}$ and $d_k^+ \ge \frac{1}{R}$.

We observe that

$$d_k^- = -\bar{\alpha}_k + d_{k-1}^-\bar{\alpha}_k \le -\left(1 - \frac{1}{R}\right)\bar{\alpha}_k,$$

and

$$\min\{s_1(k), s_3(k)\} \le \sqrt{s_1(k)s_3(k)} = \frac{1}{4}\sqrt{S},$$

with

$$S = \frac{\left((1-\bar{\alpha}_k)^2 - (d_k^-)^2\right)\left((1-\alpha_k)^2 - (d_k^+)^2\right)}{(1-\bar{\alpha}_k\alpha_k)^2}$$
$$\leq \frac{\left((1-\bar{\alpha}_k)^2 - \left(1-\frac{1}{R}\right)^2\bar{\alpha}_k^2\right)\left((1-\alpha_k)^2 - \frac{1}{R^2}\right)}{(1-\bar{\alpha}_k\alpha_k)^2}.$$

Hence, we have

$$S < \left((1-\bar{\alpha}_k)^2 - \left(1-\frac{1}{R}\right)^2\bar{\alpha}_k^2\right)\left(1-\frac{1}{R^2}\right)$$
$$< \left(\left(1-\frac{1}{R}\right)^2 - \left(1-\frac{1}{R}\right)^2\frac{1}{R^2}\right)\left(1-\frac{1}{R^2}\right) = \tilde{C}(R)^2, \tag{4.2}$$

the first inequality in (4.2) holding since

$$(1-\bar{\alpha}_k\alpha_k)^2\left(1-\frac{1}{R^2}\right) - \left((1-\alpha_k)^2 - \frac{1}{R^2}\right)$$
$$= \alpha_k\left(2 - (2-\bar{\alpha}_k\alpha_k)\bar{\alpha}_k\left(1-\frac{1}{R^2}\right) - \alpha_k\right) > 0,$$

and the second since $\bar{\alpha}_k > 1/R$ and

$$f(x) = (1-x)^2 - \left(1-\frac{1}{R}\right)^2 x^2 = 1 - 2x + \left(\frac{2}{R} - \frac{1}{R^2}\right)x^2$$

is plainly decreasing on $0 \leq x \leq \frac{1}{2}$.

`Case 4.` Suppose that $d_{k-1}^- \geq \frac{1}{R}$ and $d_k^+ \leq \frac{1}{R}$.

This is almost the same as Case 3, except we observe that

$$d_{k-1}^+ = (t_k + d_k^+)\alpha_{k-1} \leq -\left(1-\frac{1}{R}\right)\alpha_{k-1},$$

and use

$$\min\{s_1(k-1), s_3(k-1)\} \leq \sqrt{s_1(k-1)s_3(k-1)} = \frac{1}{4}\sqrt{S},$$

with

$$S = \frac{\left((1-\bar{\alpha}_{k-1})^2 - (d^-_{k-1})^2\right)\left((1-\alpha_{k-1})^2 - (d^+_{k-1})^2\right)}{(1-\bar{\alpha}_{k-1}\alpha_{k-1})^2}$$
$$\leq \frac{\left((1-\bar{\alpha}_{k-1})^2 - \frac{1}{R^2}\right)\left((1-\alpha_{k-1})^2 - (1-\frac{1}{R})^2\alpha^2_{k-1}\right)}{(1-\bar{\alpha}_{k-1}\alpha_{k-1})^2}.$$

The proof follows, using $\bar{\alpha}_{k-1}$ and α_{k-1} in place of α_k and $\bar{\alpha}_k$, respectively. □

4.2 Proof of Theorem 1.2.5

Proof of Theorem 1.2.5. Suppose that α and γ_* have $a_{2k-1} = R$, $t_{2k-1} = -1$, $a_{2k} = NR$, and $t_{2k} = N$ for all k. Plainly, as $N \to \infty$,

$$\bar{\alpha}_{2k-1}, \alpha_{2k} = [0; \overline{R, NR}]^- = \frac{1}{R - \frac{1}{NR - O(1)}} \to \frac{1}{R},$$
$$\alpha_{2k-1}, \bar{\alpha}_{2k} = [0; \overline{NR, R}]^- = \frac{1}{NR - O(1)} \to 0.$$

Clearly, we have $d^-_k = O(1)$ and $d^+_k = O(1)$ for any k. Thus, as $k \to \infty$,

$$d^-_{2k-1} = t_{2k-1}\bar{\alpha}_{2k-1} + (t_{2k-2} + d^-_{2k-3})\bar{\alpha}_{2k-2}\bar{\alpha}_{2k-1} = -\bar{\alpha}_{2k-1} + (N + d^-_{2k-3})\bar{\alpha}_{2k-2}\bar{\alpha}_{2k-1}$$
$$= \bar{\alpha}_{2k-1}\left(-1 + \frac{N + O(1)}{NR - O(1)}\right) \to \frac{1}{R}\left(-1 + \frac{1}{R}\right) = -\frac{1}{R} + \frac{1}{R^2},$$

and

$$d^+_{2k-1} = (N + d^+_{2k})\alpha_{2k-1} = \frac{N + O(1)}{NR - O(1)} \to \frac{1}{R}.$$

Likewise, $d^+_{2k} \to -\frac{1}{R} + \frac{1}{R^2}$ and $d^-_{2k} \to \frac{1}{R}$. So, as $k \to \infty$,

$$s_1(2k-1), s_1(2k) \to \frac{1}{4}\left(1 - \frac{2}{R} + \frac{1}{R^2}\right)\left(1 + \frac{1}{R}\right) = \tilde{C}(R),$$
$$s_3(2k-1), s_3(2k) \to \frac{1}{4}\left(1 - \frac{1}{R^2}\right)\left(1 - \frac{1}{R}\right) = \tilde{C}(R),$$

while $s_4(2k-1) > s_2(2k-1)$ and $s_2(2k) > s_4(2k)$ with

$$s_2(2k-1), s_4(2k) \to \frac{1}{4}\left(1+\frac{1}{R^2}\right)\left(1-\frac{1}{R}\right) > \tilde{C}(R).$$

Hence, as $N \to \infty$,

$$M(\alpha, \gamma_*) = \liminf_{k\to\infty} \min\{s_1(k), s_2(k), s_3(k), s_4(k)\} \to \tilde{C}(R). \qquad \square$$

Bibliography

[1] E. S. Barnes and H. P. F Swinnerton-Dyer. The inhomogeneous minima of binary quadratic forms. I. *Acta Math.*, 87:259–323, 1952.

[2] E. S. Barnes and H. P. F Swinnerton-Dyer. The inhomogeneous minima of binary quadratic forms. II. *Acta Math.*, 88:279–316, 1952.

[3] J. W. S. Cassels. *An Introduction to Diophantine Approximation.* Cambridge Univ. Press, London/New York, 1957.

[4] H. Davenport. Indefinite binary quadratic forms, and Euclid's algorithm in real quadratic fields. *Proc. London Math. Soc.*, 53(2):65–82, 1951.

[5] V. Ennola. On the first inhomogeneous minimum of indefinite binary quadratic forms and Euclid's algorithm in real quadratic fields. *Ann. Univ. Turkuensis Ser AI*, 28:9–58, 1958.

[6] S. Fukasawa. Über die grossenordnung des absoluten betrages von einer linearen inhomogenen form, i, ii & iv. *Japan J. Math.*, 3:1-26, 91-106, 1926, & 4:147–167, 1927.

[7] H. J. Godwin. On a conjecture of Barnes and Swinnerton-Dyer. *Proc. Camb. Phil. Soc.*, 59(3):519–522, 1963.

[8] J. H. Grace. Note on a Diophantine approximation. *Proc. London Math. Soc.*, 17: 316–319, 1918.

[9] T. Komatsu. On inhomogeneous continued fraction expansions and inhomogeneous Diophantine approximation. *J. Number Theory*, 62(1):192–212, 1997.

[10] B. Paudel and C. Pinner. Lower bounds on the largest inhomogeneous approximation constant. *Unif. Distrib. Theory*, 18(2):77–96, 2023.

[11] B. Paudel and C. Pinner. An upper bound on the inhomogeneous approximation constants. *arXiv:2301.12270 [math.NT]*, to appear in Integers.

[12] C. G. Pinner. More on inhomogeneous Diophantine approximation. *J. Théor. Nombres Bordeaux*, 13(2):539–557, 2001.

[13] C. G. Pinner. Lower bounds on the two-sided inhomogeneous approximation constant. *arXiv:1603.06178[math.NT]*, 2016.

[14] C. G. Pinner. On the inhomogeneous spectrum of period two quadratics. *arXiv:1603.06179[math.NT]*, 2016.

[15] J. Pitman. Davenport's constant for indefinite binary quadratic forms. *Acta Arith.*, 6: 37–46, 1960.

[16] A. Rockett and P. Szüsz. *Continued Fractions*. World Scientific, Singapore, 1992.

Appendix A

Proof of Minkowski's Corollary 1.1.3

Define a pair of linear forms

$$L_1(x, y) := \lambda_1 x + \mu_1 y = \alpha x - y,$$

$$L_2(x, y) := \lambda_2 x + \mu_2 y = x.$$

Clearly, $\frac{\mu_1}{\lambda_1} = \frac{-1}{\alpha}$ is irrational and

$$|\Delta| = |\lambda_1 \mu_2 - \lambda_2 \mu_1| = 1.$$

Choose $\rho_1 = -\gamma$ and $\rho_2 = 0$. Then, by Minkowski's Theorem 1.1.2, for any $\epsilon > 0$, there exist integers n and m with

$$|L_1(n, m) + \rho_1||L_2(n, m) + \rho_2| \leq \frac{1}{4}|\Delta| \quad \text{and} \quad |L_1(n, m) + \rho_1| < \epsilon.$$

That is,

$$|n||n\alpha - m - \gamma| \leq \frac{1}{4} \quad \text{and} \quad |n\alpha - m - \gamma| < \epsilon.$$

The first inequality implies that

$$|n|||n\alpha - \gamma|| \leq \frac{1}{4}.$$

Since $\gamma \neq n\alpha - m$, we have $0 < |n\alpha - m - \gamma| < \epsilon$. Therefore, we can get infinitely many n by taking sufficiently small different values for $\epsilon > 0$. $\square$