

BOUNDEDNESS PROPERTIES OF BILINEAR
PSEUDODIFFERENTIAL OPERATORS

by

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B.S., Maranatha Baptist University, 2006

M.S., Kansas State University, 2010

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
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Abstract

Investigations of pseudodifferential operators are useful in a variety of applications. These include finding solutions or estimates of solutions to certain partial differential equations, studying boundedness properties of commutators and paraproducts, and obtaining fractional Leibniz rules.

A pseudodifferential operator is given through integration involving the Fourier transform of the arguments and a function called a symbol. Pseudodifferential operators were first studied in the linear case and results were obtained to advance both the theory and applicability of these operators. More recently, significant progress has been made in the study of bilinear, and more generally multilinear, pseudodifferential operators. Of special interest are boundedness properties of bilinear pseudodifferential operators which have been examined in a variety of function spaces. Since determining factors in the boundedness of these operators are connected to properties of the corresponding symbols, significant effort has been directed at categorizing the symbols according to size and decay conditions as well as at establishing the associated symbolic calculus. One such category, the bilinear Hörmander classes, plays a vital role in results concerning the boundedness of bilinear pseudodifferential operators in the setting of Lebesgue spaces in particular.

The new results in this work focus on the study of bilinear pseudodifferential operators with symbols in weighted Besov spaces of product type. Unlike the Hörmander classes, symbols in these Besov spaces are not required to possess infinitely many derivatives satisfying size or decay conditions. Even without this much smoothness, boundedness properties on Lebesgue spaces are obtained for bilinear operators with symbols in certain Besov spaces. Important tools in the proofs of these new results include the demonstration of appropriate estimates and the development of a symbolic calculus for some of the Besov spaces along

with duality arguments. In addition to the new boundedness results and as a byproduct of studying operators with symbols in Besov spaces, it is possible to quantify the smoothness of the symbols, in terms of the conditions that define the Hörmander classes, that is sufficient for boundedness of the operators in the context of Lebesgue spaces.

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Virginia Naibo

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Table of Contents

Table of Contents	viii
List of Figures	xi
Acknowledgements	xi
Dedication	xii
1 An Overview of Pseudodifferential Operators	1
2 Linear Pseudodifferential Operators	6
2.1 Introduction	6
2.2 The Linear Hörmander Classes	9
2.2.1 Definition and examples	9
2.2.2 Properties and symbolic calculus	11
2.3 Boundedness of Operators with Symbols in the Linear Hörmander Classes	14
2.3.1 L^2 boundedness of operators with symbols of order zero	15
2.3.2 L^p boundedness of operators with symbols of order m	19
2.4 Connections to Calderón-Zygmund Theory	20
2.5 Boundedness and the Smoothness of the Symbols	23
3 Bilinear Operators with Symbols in the Hörmander Classes	25
3.1 Introduction	25
3.2 The Bilinear Hörmander Classes	27

3.2.1	Symbolic calculus	29
3.3	Boundedness of Operators with Symbols in the Bilinear Hörmander Classes .	35
3.3.1	Symbols of order zero and Lebesgue spaces	36
3.3.2	Symbols of order m and Lebesgue spaces	42
3.3.3	Boundedness on Hardy spaces and BMO	54
3.4	Connections to Bilinear Calderón-Zygmund Theory	56
4	Bilinear Operators with Symbols in Besov Spaces	61
4.1	Introduction	61
4.2	Weighted Besov Spaces of Product Type and Related Classes	62
4.3	Boundedness Properties of Bilinear Operators with Symbols in $B_{\infty,q}^{s,m}(\mathbb{R}^{3n})$. .	66
4.4	An Upper Bound on the Number of Derivatives	68
4.5	Proof of Theorem 4.3.2	71
4.5.1	Symbols in $\mathcal{C}_m^0(\mathbb{R}^{3n})$ with compactly supported Fourier transforms . .	71
4.5.2	A crucial estimate	73
4.5.3	The completion of the proof	86
4.6	Proof of Theorem 4.3.3	88
4.6.1	A symbolic calculus for classes of x -independent symbols in Besov spaces of product type	89
4.6.2	The completion of the proof	97
4.7	Conclusions	98
	Bibliography	100
A	A Glossary of Selected Notation and Definitions	104
A.1	Frequently Used Notation	104
A.2	Function Spaces	105

A.3 Miscellaneous Definitions 107

List of Figures

3.1	Visualization of $m(p_1, p_2, \rho)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 \leq p_1, p_2 \leq \infty$	42
4.1	Visualization of $m(p_1, p_2, 0)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 \leq p_1, p_2, p \leq \infty$	66
A.1	Example of the system $\{\varphi_k\}_{k \geq 0}$ for $N = 1$	108

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Dedication

To my family for all their love and encouragement,
and especially to my father who was my first mentor in mathematics,
with love.

Chapter 1

An Overview of Pseudodifferential Operators

Over the last several decades, pseudodifferential operators have become an important field of study within analysis. Much of the initial interest in these operators was due to a concurrent enthusiasm for singular integral operators, for example, Calderón-Zygmund operators. Formally, an operator can be given as either a singular integral operator or a pseudodifferential operator; however, as they are both integral operators, the absolute convergence of the integral does not always hold in both settings. Even in situations where both are well defined operators, sometimes one form or the other is preferred, according to the goals of study. While more recent work has emphasized the study of these operators as important subjects in their own right, early development in the theories of pseudodifferential operators and singular integral operators was nurtured by the reciprocity between the two. To demonstrate this more fully, the connections between pseudodifferential operators and Calderón-Zygmund operators are presented at the end of Chapters 2 and 3 for the linear and bilinear cases, respectively.

One of the settings where pseudodifferential operators are of particular importance is in their applicability to partial differential equations. In fact, a linear differential operator is a

pseudodifferential operator though the class of pseudodifferential operators goes beyond the class of linear differential operators to include a variety of other operators. Pseudodifferential operators can sometimes be used to find solutions of certain differential equations; more frequently, they are used to find estimates on the size of solutions through the boundedness properties of the operator. The usefulness of pseudodifferential operators in the context of partial differential equations is explored in greater detail in Chapter 2 where we discuss in particular the topic of linear pseudodifferential operators.

A pseudodifferential operator is given through integration involving the Fourier transform of the arguments and a function called a symbol. The characteristics of the symbol determine to a large extent the behavior of the operator. In this work the boundedness properties of pseudodifferential operators will be examined in two cases: first when the symbols belong to what are known as the Hörmander classes, and secondly when the conditions that define the Hörmander classes are relaxed to allow for symbols with less smoothness. While a variety of options for rougher symbols exist, in this work, the symbols with less smoothness will for the most part belong to weighted Besov spaces of product type or related classes.

Symbols that belong to the Hörmander classes, either linear or bilinear, are smooth functions whose derivatives satisfy specific size and decay properties depending on the particular Hörmander class under consideration. This type of symbol arises naturally when considering pseudodifferential operators that can be realized as partial differential operators. In such a case, there is a connection between what is called the order of the specific Hörmander class to which the symbol of the operator belongs and the degree of the differential operator, as will be seen in Chapter 2. Furthermore, the Hörmander classes have a variety of properties that make them especially suited for the study of boundedness properties of the associated pseudodifferential operators. Possibly one of the most useful features of the Hörmander classes is what is called symbolic calculus. The symbolic calculus provides the tools to compute the symbols of the adjoints, the transposes, and the composition of operators with symbols in the Hörmander classes and also determines the Hörmander classes to which they

belong.

In the linear case, mathematicians such as Fefferman, Hörmander, Calderón, Coifman, Meyer, Stein, Vaillancourt, and others were able to establish boundedness properties of linear pseudodifferential operators on appropriate function spaces. Perhaps one of the most critical results, in the sense that it is used so frequently in the proofs of other results, is due to Hörmander and was later improved by Calderón and Vaillancourt. Their work establishes that operators with symbols in the Hörmander classes of order zero are bounded on L^2 . Using this result along with a variety of tools that include the symbolic calculus of the linear Hörmander classes, other mathematicians were able to characterize the boundedness properties of linear pseudodifferential operators with symbols in the Hörmander classes on the full scale of Lebesgue spaces and related function spaces.

In the process of proving boundedness, it can be demonstrated that the level of smoothness required in the definitions of the Hörmander classes is not actually needed. This led many to consider the question of just how much smoothness of the symbol is sufficient in order to prove continuity of the corresponding operator. In answering this question, many alternatives were considered including the examination of symbols in function spaces such as Sobolev and Besov spaces that naturally relate to the Hörmander classes.

Chapters 3 and 4 contain the heart of this work which explores the boundedness properties of bilinear pseudodifferential operators. In Chapter 3, we present known results from a variety of authors that provide background and motivation for the contributions of the author and Naibo presented in Chapter 4. Bilinear pseudodifferential operators are not simply an analog of the linear version but rather are operators of considerable interest in their own right. This is partially due to their usefulness not just in partial differential equations but also in the study of fractional Leibniz rules, paraproducts, and the boundedness properties of commutators where linear pseudodifferential operators are insufficient. More than that, as will be demonstrated both in Chapters 3 and 4, bilinear pseudodifferential operators also differ from their linear counterparts both in how they behave and in the techniques used

to prove the results that describe this behavior. In our discussion of the theory of bilinear pseudodifferential operators, some attention will be given to both the similarities and differences between the two cases.

The properties of the bilinear Hörmander classes and boundedness of the corresponding bilinear pseudodifferential operators have been studied by several authors including Bényi and Torres [5, 6], Bényi, Maldonado, Naibo, and Torres [4], Michalowski, Rule, and Staubach [28], Bényi, Bernicot, Maldonado, Naibo, and Torres [3], Miyachi and Tomita [30], Naibo [34, 33], Rodríguez-López and Staubach [35], and references therein. Recently, efforts in the setting of bilinear pseudodifferential operators with symbols in the Hörmander classes have been directed at expanding current boundedness results to a greater number of operators. This has been accomplished in two ways: first, by considering a broader array of function spaces than Lebesgue spaces, and secondly, by obtaining results that include as many of the Hörmander classes as possible.

However, as with the linear case, the proofs for boundedness of bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes do not require the level of smoothness inherent in the definition of these symbols. Therefore a natural question is whether boundedness for the operator can still be obtained even when the symbol is much rougher in the sense of requiring fewer derivatives. Thus in the final chapter of this work we consider recent results of the author and Naibo [21, 22] concerning boundedness of bilinear pseudodifferential operators with symbols in weighted Besov spaces of product type. These classes of symbols contain certain of the Hörmander classes as well as rougher symbols not contained in the Hörmander classes. In addition to obtaining boundedness properties for a larger class of operators, as a byproduct of considering symbols in Besov spaces, it is possible to quantify the level of smoothness of the symbols that is sufficient for boundedness of the associated operator, an endeavor that had not been previously undertaken.

The techniques used in the proofs of the new results presented in Chapter 4 differ according to the target space of the operators. In one instance, an important tool consists in

proving a statement on boundedness of operators with a symbol that satisfies a certain size condition and whose Fourier transform is compactly supported, but whose derivatives do not necessarily possess a decay restriction. In another instance, a symbolic calculus for certain Besov spaces is developed and used along with duality arguments and some additional tools.

Much of the notation used in this work is defined at the moment of implementation; however, some of the frequently used notation as well as some notation and definitions standard in the setting of analysis are given in Appendix [A](#).

Chapter 2

Linear Pseudodifferential Operators

2.1 Introduction

Though the study of bilinear pseudodifferential operators is the principal goal of this work, a brief review of the theory of linear pseudodifferential operators will provide historical context and occasional inspiration for the study of bilinear operators.

Definition 2.1.1. Let $\sigma(x, \xi)$ be a complex-valued, smooth function defined for $x, \xi \in \mathbb{R}^n$ called a symbol. The pseudodifferential operator associated to the symbol σ is defined by

$$T_\sigma f(x) := \int_{\mathbb{R}^n} \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n, \quad (2.1.1)$$

where \hat{f} is the Fourier transform of f defined as in (A.3.1).

In the last century, continuing into this one, significant study has been done in both the theory and applications of these operators. Part of the reason for this interest is due to the fact that linear pseudodifferential operators occur naturally in the study of partial differential equations. For example, consider the following linear partial differential operator in \mathbb{R}^n ,

$$P = I - \Delta, \quad (2.1.2)$$

where I is the identity operator and Δ is the Laplacian operator in \mathbb{R}^n . Using the properties of the Fourier transform, it follows that if $u \in \mathcal{S}(\mathbb{R}^n)$ then

$$\widehat{Pu}(\xi) = (1 + 4\pi^2|\xi|^2)\hat{u}(\xi), \quad \xi \in \mathbb{R}^n. \quad (2.1.3)$$

Taking the inverse Fourier transform we obtain that

$$Pu(x) = \int_{\mathbb{R}^n} (1 + 4\pi^2|\xi|^2)\hat{u}(\xi) e^{2\pi i x \cdot \xi} d\xi, \quad x \in \mathbb{R}^n.$$

That is, $P = T_{\sigma_P}$ with $\sigma_P(x, \xi) = 1 + 4\pi^2|\xi|^2$ for $x, \xi \in \mathbb{R}^n$. More generally, given a linear partial differential operator of degree $m \in \mathbb{N}$,

$$L = \sum_{|\gamma| \leq m} C_\gamma(x) \partial^\gamma, \quad x \in \mathbb{R}^n, \quad (2.1.4)$$

we have that

$$L = T_{\sigma_L} \text{ with } \sigma_L(x, \xi) = \sum_{|\gamma| \leq m} C_\gamma(x) (2\pi i \xi)^\gamma, \quad x, \xi \in \mathbb{R}^n. \quad (2.1.5)$$

In other words, linear partial differential operators are particular cases of pseudodifferential operators. Going back to the example (2.1.2), suppose we are given a function f defined in \mathbb{R}^n and asked to find u such that $Pu = f$. Though there are a variety of techniques for answering these kinds of questions, in this example, we could use (2.1.3) to get that

$$(1 + 4\pi^2|\xi|^2)\hat{u}(\xi) = \hat{f}(\xi), \quad \xi \in \mathbb{R}^n.$$

Then multiplying both sides by $(1 + 4\pi^2|\xi|^2)^{-1}$ and using the inverse Fourier transform formula yields

$$u(x) = \int_{\mathbb{R}^n} \frac{1}{(1 + 4\pi^2|\xi|^2)} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi, \quad x \in \mathbb{R}^n. \quad (2.1.6)$$

We then see that the solution u is given by $T_{\tilde{\sigma}_P}(f)$ where

$$\tilde{\sigma}_P(x, \xi) = \frac{1}{(1 + 4\pi^2|\xi|^2)} = \frac{1}{\sigma_P(x, \xi)}, \quad x, \xi \in \mathbb{R}^n.$$

Therefore, in this particular example both P and its inverse are pseudodifferential operators with symbols σ_P and $1/\sigma_P$, respectively. Note that the simplicity of obtaining the solution (2.1.6) via the Fourier transform was facilitated by the fact that the linear operator P has constant coefficients, which leads to the x -independent symbol σ_P . Such a simple process is not in general possible if the symbol σ_L in (2.1.5) does depend on x or if it is x -independent but vanishes for some ξ . However, while we may not be able to find the exact solution u as in (2.1.6) for a general linear differential operator L as in (2.1.4), for certain elliptic differential operators we can get close. Suppose L is an elliptic differential operator written as a pseudodifferential operator T_{σ_L} . It is then possible to find an “approximate inverse” $T_{\tilde{\sigma}_L}$ such that

$$T_{\tilde{\sigma}_L}T_{\sigma_L} + T_{e_1} = T_{\sigma_L}T_{\tilde{\sigma}_L} + T_{e_2} = I, \quad (2.1.7)$$

where I is the identity operator and T_{e_j} for $j = 1, 2$ is an error operator that is sufficiently easy to control. The fact that such an “inverse” operator is available is based on the existence of a symbolic calculus that will be explored in the next section.

Two other special examples of pseudodifferential operators include multipliers and pointwise multiplication by the symbol. A pseudodifferential operator is a multiplier when its symbol depends only on the variable ξ ; more precisely, if $\sigma(x, \xi) = \sigma(\xi)$ then $\widehat{T_\sigma(f)} = \sigma(\xi)\hat{f}(\xi)$. For instance, the operator P in (2.1.2) and its inverse are both multiplier operators. More generally, any linear differential operator with constant coefficients and its inverse (when it exists and under certain assumptions) are linear multipliers. Furthermore, when $\sigma(x, \xi) = \sigma(x)$, then it follows that $T_\sigma(f) = \sigma(x)f(x)$ and the operator is given by pointwise multiplication by the symbol.

In Section 2.2 we will examine in greater detail the symbols of the operators, in partic-

ular those that belong to the linear Hörmander classes. We first consider several specific examples of these symbols and demonstrate how they relate to the corresponding operator before discussing the properties of the symbol classes in general and especially the symbolic calculus. In Section 2.3 we divide the study of operators with symbols in the Hörmander classes into two cases: the first when the order of the symbol is zero, and secondly the more general case. This division allows us to clearly see the effect of the characteristics of the symbol on the boundedness properties of the operator. In Section 2.4 we explore the connections between linear pseudodifferential operators and Calderón-Zygmund singular integrals which, as was mentioned in Chapter 1, contributed to the evolution of the field. In the final section of this chapter, Section 2.5, we delve slightly into the idea of loosening the conditions that define the Hörmander classes and the corresponding effects on the operators. This section provides a glimpse of the linear results that prompted the work of Chapter 4.

2.2 The Linear Hörmander Classes

In (2.1.1), whether $T_\sigma f$ is well-defined for all $f \in \mathcal{S}(\mathbb{R}^n)$ depends on the properties of the symbol σ . Once sufficient conditions on σ are assumed so that the integral in (2.1.1) converges for all $f \in \mathcal{S}(\mathbb{R}^n)$, the primary goal is the study of boundedness properties of T_σ in the setting of diverse function spaces. In this context we now introduce important classes of symbols known as the Hörmander classes.

2.2.1 Definition and examples

Definition 2.2.1. Let $\sigma(x, \xi)$ be a complex-valued, smooth function defined for $x, \xi \in \mathbb{R}^n$, $m \in \mathbb{R}$, and $\rho, \delta \in [0, 1]$. The symbol σ is said to be in the linear Hörmander class $S_{\rho, \delta}^m$ if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$,

$$\sup_{x, \xi \in \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| (1 + |\xi|)^{-m - \delta|\alpha| + \rho|\beta|} < \infty. \quad (2.2.8)$$

The value m is called the *order* of the symbol.

The Hörmander classes were introduced by Hörmander in the 1960's with the purpose of studying solutions of certain partial differential operators; for details, we refer the reader to [24, 25] and references therein. As an example of symbols in these classes, consider the linear partial differential operator L introduced in Section 2.1 which has symbol

$$\sigma_L(x, \xi) = \sum_{|\gamma| \leq m} C_\gamma(x) (2\pi i \xi)^\gamma, \quad x, \xi \in \mathbb{R}^n,$$

where $m \in \mathbb{N}_0$. It easily follows that if the coefficients $C_\gamma(x)$ have bounded derivatives of all orders, then $\sigma_L \in S_{1,0}^m$. Another straight-forward example is the multiplier $(1 + |\xi|^2)^{\frac{m}{2}}$, defined for $\xi \in \mathbb{R}^n$ and $m \in \mathbb{R}$, which belongs to $S_{1,0}^m$.

As a final example of the Hörmander classes, suppose once again we are faced with the problem of finding approximate inverses to partial differential operators as in (2.1.7) but instead of considering just elliptic partial differential operators, we extend the problem to studying parabolic partial differential operators, for instance, the operator corresponding to the heat equation. In this context, we will work with symbols where the space and frequency variables are in \mathbb{R}^{n+1} ; $x = (t, x')$ with $t \in \mathbb{R}$ and $x' = (x_1, \dots, x_n) \in \mathbb{R}^n$ will play the role of the space variable, and $\xi = (\tau, \xi')$ with $\tau \in \mathbb{R}$ and $\xi' = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ will play the role of the frequency variable. Let H be the heat operator given by

$$H = \frac{\partial}{\partial t} - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.$$

Note that since H has constant coefficients, it can be thought of as a linear multiplier; indeed, its symbol is independent of the space variable and is given by $2\pi i \tau + 4\pi^2 |\xi'|^2$. However, this symbol vanishes at the origin. Then to solve the equation $Hu = f$ for some appropriate datum f we could try to find an approximate inverse \tilde{H} such that $H\tilde{H} = I + E$ where E is some appropriately small error term. It turns out that the symbols arising in

this process belong to Hörmander classes of order 0 or -1 and with $\rho = \frac{1}{2}$ or $\delta = \frac{1}{2}$. For instance, one could take $\tilde{H} = T_h$ where the symbol h is the multiplier in \mathbb{R}^{n+1} defined by

$$h(x, \xi) = h(\xi) = (2\pi i\tau + 4\pi^2|\xi'|^2)^{-1}\psi(\xi).$$

Here ψ is a smooth cut-off function that vanishes in a neighborhood of the origin and equals 1 for large (τ, ξ') . It can be shown that the symbol h belongs to the Hörmander class $S_{1/2,0}^{-1}$. By a smooth change of variables, we can transform the underlying space and achieve a new equation; this change of variables produces a corresponding symbol belonging to the class $S_{1/2,1/2}^{-1}$. Finally, we consider the symbols $h_{i,j}$ given by

$$h_{i,j}(x, \xi) = h_{i,j}(\xi) = \xi_i \xi_j h(\xi), \quad \text{for } 1 \leq i, j \leq n.$$

The symbols $h_{i,j}$ belong to $S_{1/2,0}^0$ while with the change of variables from before we can produce symbols that belong to the class $S_{1/2,1/2}^0$.

2.2.2 Properties and symbolic calculus

We start by stating some properties of the Hörmander classes which will prove useful.

- $S_{\rho_1, \delta_1}^{m_1} \subset S_{\rho_2, \delta_2}^{m_2}$ for $m_1 \leq m_2$, $\delta_1 \leq \delta_2$ and $\rho_1 \geq \rho_2$.
- If $\sigma \in S_{\rho, \delta}^m$ then $\partial_x^\alpha \partial_\xi^\beta \sigma \in S_{\rho, \delta}^{m + \delta|\alpha| - \rho|\beta|}$. In other words, taking a derivative with respect to any of the components in x , the space variable, increases the order of the symbol by δ , while taking a derivative with respect to any of the components in the frequency variable ξ , decreases the order of the symbol by ρ .
- If $\sigma_1 \in S_{\rho, \delta}^{m_1}$ and $\sigma_2 \in S_{\rho, \delta}^{m_2}$, the sum $\sigma_1 + \sigma_2$ lies in the Hörmander class $S_{\rho, \delta}^{\max(m_1, m_2)}$ while we can show via the Leibniz rule that the product $\sigma_1 \sigma_2$ is in the Hörmander class $S_{\rho, \delta}^{m_1 + m_2}$.

- Given $\sigma \in S_{\rho,\delta}^m$ and $s_1, s_2 \in \mathbb{N}_0$ we define

$$\|\sigma\|_{s_1, s_2} := \sup_{\substack{|\alpha| \leq s_1 \\ |\beta| \leq s_2}} \sup_{x, \xi \in \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| (1 + |\xi|)^{-m - \delta|\alpha| + \rho|\beta|}.$$

The topology induced by this family of norms turns $S_{\rho,\delta}^m$ into a Fréchet space.

- Continuity from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$: We now prove that for every $\sigma \in S_{\rho,\delta}^m$, T_σ is a well-defined operator that maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. If $f \in \mathcal{S}(\mathbb{R}^n)$, since $\sigma(x, \xi)$ is bounded by $(1 + |\xi|)^m$ uniformly in x and $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$, the integral defining $T_\sigma f(x)$ converges absolutely for every $x \in \mathbb{R}^n$. Using that $(1 - \Delta_\xi)^N e^{2\pi i x \cdot \xi} = (1 + 4\pi^2 |x|^2)^N e^{2\pi i x \cdot \xi}$ for $N \in \mathbb{N}_0$, we get

$$\begin{aligned} |T_\sigma f(x)| &= \left| \int_{\mathbb{R}^n} \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \right| \\ &= \left| \int_{\mathbb{R}^n} \sigma(x, \xi) \hat{f}(\xi) \frac{(1 - \Delta_\xi)^N e^{2\pi i x \cdot \xi}}{(1 + 4\pi^2 |x|^2)^N} d\xi \right| \\ &= \frac{1}{(1 + 4\pi^2 |x|^2)^N} \left| \int_{\mathbb{R}^n} (1 - \Delta_\xi)^N [\sigma(x, \xi) \hat{f}(\xi)] e^{2\pi i x \cdot \xi} d\xi \right| \\ &\lesssim \frac{1}{(1 + 4\pi^2 |x|^2)^N} \|\sigma\|_{0, 2N} \sum_{|\gamma|, |\beta| \leq M} \sup_{x \in \mathbb{R}^n} |x^\gamma \partial^\beta f(x)| \end{aligned}$$

for some $M \in \mathbb{N}_0$. Because $\sigma \in S_{\rho,\delta}^m$ and $f \in \mathcal{S}(\mathbb{R}^n)$, this last line is finite. We can proceed similarly to obtain an analogous estimate for $\partial^\alpha T_\sigma(f)$ for any $\alpha \in \mathbb{N}_0$ and therefore T_σ is continuous from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$.

- Schwartz kernel: If $\sigma \in S_{\rho,\delta}^m$, by the previous item and the Schwartz kernel theorem (see Hörmander [26, p. 129]) there exists a tempered distribution $K \in \mathcal{S}'(\mathbb{R}^{2n})$ such that

$$\int_{\mathbb{R}^n} T_\sigma(f)(x) g(x) dx = \langle K, f \otimes g \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^n),$$

where $(f \otimes g)(x, y) = f(x)g(y)$ for $x, y \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denotes the action of a tempered distribution on functions of the Schwartz class. Moreover, it can be proved that

$K(x, y) = k(x, x - y)$, where $k(x, y) = \mathcal{F}^{-1}(\sigma(x, \cdot))(y)$ and it holds that

$$T_\sigma f(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi = \langle \mathcal{F}^{-1}(\sigma(x, \cdot))(y), f(x - y) \rangle, \quad x \in \mathbb{R}^n, f \in \mathcal{S}(\mathbb{R}^n).$$

The distribution K is called the distributional kernel of T_σ .

Even though we have defined the Hörmander classes for $\rho, \delta \in [0, 1]$ we are mainly interested in the theory for the cases $0 \leq \delta \leq \rho \leq 1$. We refer the reader to Alvarez-Hounie [1] for the cases $\delta > \rho$.

The symbolic calculus.

Finally, we consider the symbolic calculus of the Hörmander classes, which is presented in the following theorem first proved by Hörmander in [24, Theorem 2.15, 2.7].

Theorem 2.2.2. *Let $0 \leq \delta \leq \rho \leq 1$ with $\delta < 1$ and $m \in \mathbb{R}$.*

(i) *If $\sigma \in S_{\rho, \delta}^m$ then the adjoint operator of T_σ is a pseudodifferential operator with symbol in $S_{\rho, \delta}^m$. More precisely, there exists $\sigma^* \in S_{\rho, \delta}^m$ such that*

$$\int_{\mathbb{R}^n} T_\sigma f(x) \overline{g(x)} dx = \int_{\mathbb{R}^n} f(x) \overline{T_{\sigma^*} g(x)} dx, \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

Moreover,

$$\sigma^*(x, \xi) = \sum_{|\alpha| < N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_x^\alpha \overline{\partial_\xi^\alpha \sigma(x, \xi)} + r_N(x, \xi), \quad N \in \mathbb{N}_0,$$

where $r_N \in S_{\rho, \delta}^{m+(\delta - \rho)N}$.

(ii) *If $\sigma_1 \in S_{\rho, \delta}^{m_1}$ and $\sigma_2 \in S_{\rho, \delta}^{m_2}$ then the composition $T_{\sigma_1} T_{\sigma_2}$ is a pseudodifferential operator*

with symbol σ in $S_{\rho,\delta}^{m_1+m_2}$. Moreover

$$\sigma(x, \xi) = \sum_{|\alpha| < N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_1(x, \xi) \partial_x^\alpha \sigma_2(x, \xi) + r_N(x, \xi), \quad N \in \mathbb{N}_0,$$

where $r_N \in S_{\rho,\delta}^{m_1+m_2+(\delta-\rho)N}$.

We will not give a proof of this theorem here as the proof of its bilinear counterpart will be presented in Chapter 3. The benefits of the symbolic calculus for pseudodifferential operators are considerable. With respect to the first part of Theorem 2.2.2, the usefulness often comes into play when attempting to prove boundedness of operators. For example, suppose T_σ is bounded from a Banach space X into another Banach space Y for every $\sigma \in S_{\rho,\delta}^m$, for some fixed m, ρ, δ with $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$ and $m \in \mathbb{R}$. By the first item in Theorem 2.2.2, the adjoint of T_σ is a pseudodifferential operator with symbol in $S_{\rho,\delta}^m$ and therefore the adjoint of T_σ is also bounded from X into Y . By duality we then obtain that T_σ is bounded from Y^* into X^* , where Y^* and X^* are the dual spaces of Y and X , respectively. The second item of Theorem 2.2.2 is equally valuable. The process described in Section 2.1 of finding an “approximate inverse” for an elliptic differential operator L relies on the smoothing properties of the asymptotic expansion of the symbol corresponding to the composition of two pseudodifferential operators. For more details on the construction of this inverse one can consult Stein [36, p.266].

2.3 Boundedness of Operators with Symbols in the Linear Hörmander Classes

The study of boundedness properties of linear pseudodifferential operators with symbols in the Hörmander classes is partly motivated by the need to obtain estimates of solutions to certain elliptic equations $Lu = f$, where L is as in (2.1.4) and f is a given datum. It

turns out that under appropriate assumptions, $L = T_{\sigma_L}$ where σ_L belongs to a Hörmander class, and (2.1.7) holds with a symbol $\tilde{\sigma}_L$ that also belongs to a Hörmander class. Roughly speaking, since the operator T_{e_1} is smooth, it follows that $u \sim T_{\tilde{\sigma}_L} f$. Then boundedness properties for $T_{\tilde{\sigma}_L}$ imply estimates for the solution u in terms of the datum f .

In this section we will present known results on boundedness of pseudodifferential operators with symbols in the Hörmander classes. Although continuity properties can be considered for many different function spaces, including Sobolev, Lipschitz, and Hardy spaces, we will concentrate the discussion on the boundedness of pseudodifferential operators in the setting of Lebesgue spaces. As observed in Section 2.2, $T_\sigma f$ is well-defined for $f \in \mathcal{S}(\mathbb{R}^n)$ and σ belonging to any of the Hörmander classes through the formula (2.1.1). Given two quasi-Banach spaces X and Y that contain $\mathcal{S}(\mathbb{R}^n)$ and such that $\mathcal{S}(\mathbb{R}^n)$ is dense in X , we will say that T_σ is bounded from X into Y if there exists a constant C_σ such that

$$\|T_\sigma f\|_Y \leq C_\sigma \|f\|_X \quad \forall f \in \mathcal{S}(\mathbb{R}^n). \quad (2.3.9)$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in X , there is a unique extension of the operator to X , which we still call T_σ , such that (2.3.9) holds for all $f \in X$.

2.3.1 L^2 boundedness of operators with symbols of order zero

One of the main results in the theory is the fact that symbols of order zero give rise to bounded operators on $L^2(\mathbb{R}^n)$. This is precisely stated in the following theorem.

Theorem 2.3.1. *Let $0 \leq \delta \leq \rho \leq 1$ with $\delta < 1$. If $\sigma \in S_{\rho,\delta}^0$ then T_σ is bounded from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.*

Theorem 2.3.1 was proved by Hörmander in [24] for $\delta < \rho$. The result for $\delta = \rho$, which implies the one by Hörmander, was proved by Calderón and Vaillancourt in [10] and [11]. We refer the reader to these articles for a proof of Theorem 2.3.1. However, just to illustrate

once more the usefulness of the symbolic calculus we will give a proof of Theorem 2.3.1 for the class $S_{1,0}^0$ following the work of Hörmander in [25].

Remark 2.3.2. We note that the class $S_{1,1}^0$ is not included in the statement of Theorem 2.3.1. It turns out that there exists $\sigma \in S_{1,1}^0$ such that T_σ is not bounded from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$. This can be seen by construction of such a symbols; see for instance [36, Proposition 2, p.272].

Before starting with the proof of Theorem 2.3.1 for the class $S_{1,0}^0$, we recall that (see Section 2.2.2)

$$T_\sigma f(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi = \langle \mathcal{F}^{-1}(\sigma(x, \cdot)), f(x - \cdot) \rangle, \quad (2.3.10)$$

where $\langle \cdot, \cdot \rangle$ denotes the action of a tempered distribution on functions in the Schwartz class. The following lemma establishes decay properties of the kernel of T_σ when $\sigma \in S_{1,0}^m$. We refer the reader to [36, p. 241] for its proof.

Lemma 2.3.3. *If $\sigma \in S_{1,0}^m$ then there exists $k(x, z) \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ such that $\mathcal{F}^{-1}(\sigma(x, \cdot))(z)$ coincides with k for $z \neq 0$ and*

$$|\partial_x^\beta \partial_z^\alpha k(x, z)| \leq C_{\alpha, \beta, N} |z|^{-n-m-|\alpha|-N}, \quad z \neq 0, x \in \mathbb{R}^n,$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0$ and all $N \geq 0$ such that $n + m + |\alpha| + N > 0$.

Proof of Theorem 2.3.1 for the class $S_{1,0}^0$. The proof will proceed in three steps. Boundedness from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ will first be proved for operators with symbols in $S_{1,0}^{-n-1}$ which is then implemented to show boundedness on $L^2(\mathbb{R}^n)$ for operators with symbols in $S_{1,0}^m$ for any negative m . The final step will use this last result to prove boundedness from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ for any operator with symbol in $S_{1,0}^0$.

Step 1. We will prove that if $\sigma \in S_{1,0}^{-n-1}$ then T_σ is bounded from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Let $\sigma \in S_{1,0}^{-n-1}$. Then $\sigma(x, \cdot) \in L^1(\mathbb{R}^n)$ and $k(x, z) := \mathcal{F}^{-1}(\sigma(x, \cdot))(z) = \int_{\mathbb{R}^n} \sigma(x, \xi) e^{2\pi i z \cdot \xi} d\xi$. By

Lemma 2.3.3, if we consider σ as a symbol of order zero (which we can do by the nesting property $S_{1,0}^{-n-1} \subset S_{1,0}^0$), then

$$|k(x, z)| \leq C_N |z|^{-n-N}, \quad x, z \in \mathbb{R}^n, z \neq 0, N \geq 0. \quad (2.3.11)$$

Moreover, since $\sigma \in S_{1,0}^{-n-1}$,

$$|k(x, z)| \leq \|\sigma\|_{0,0} \int_{\mathbb{R}^n} (1 + |\xi|)^{-n-1} d\xi < \infty, \quad x, z \in \mathbb{R}^n.$$

This and (2.3.11) with $N = n$ imply

$$|k(x, x - y)| \lesssim (1 + |x - y|^2)^{-n}, \quad x, y \in \mathbb{R}^n,$$

and therefore $K(x, y) := k(x, x - y)$ satisfies

$$\sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dx < \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dy < \infty.$$

Applying the Cauchy-Schwarz inequality with respect to the weight $|K(x, \cdot)|$,

$$\begin{aligned} |T_\sigma f(x)|^2 &= \left| \int_{\mathbb{R}^n} K(x, y) f(y) dy \right|^2 \\ &\leq \left(\int_{\mathbb{R}^n} |K(x, y)| dy \right) \left(\int_{\mathbb{R}^n} |K(x, y)| |f(y)|^2 dy \right) \\ &\leq \left(\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dy \right) \left(\int_{\mathbb{R}^n} |K(x, y)| |f(y)|^2 dy \right). \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}^n} |T_\sigma f(x)|^2 dx &\leq \left(\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dy \right) \int_{\mathbb{R}^{2n}} |K(x, y)| |f(y)|^2 dy dx \\ &\leq \left(\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dy \right) \left(\sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dx \right) \int_{\mathbb{R}^n} |f(y)|^2 dy, \end{aligned}$$

from which the desired result follows.

Step 2. We prove that for any $m < 0$ and $\sigma \in S_{1,0}^m$, T_σ is bounded from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. By the nested properties of the Hörmander classes, $\bigcup_{m < 0} S_{1,0}^m = \bigcup_{k \in \mathbb{Z}} S_{1,0}^{-1/2^k}$; it is then enough to prove this step for $S_{1,0}^{-1/2^k}$ with $k \in \mathbb{Z}$. By Step 1, the result is true for k sufficiently negative and therefore we can proceed by induction on k . Assuming that operators with symbols in $S_{1,0}^{-1/2^k}$ are bounded on $L^2(\mathbb{R}^n)$, we will prove boundedness on $L^2(\mathbb{R}^n)$ for operators with symbols in $S_{1,0}^{-1/2^{k+1}}$.

Let $\sigma \in S_{1,0}^{-1/2^{k+1}}$. By Theorem 2.2.2, the symbol of the composition $T_{\sigma^*}T_\sigma$ belongs to $S_{1,0}^{-1/2^k}$ and therefore

$$\|T_\sigma f\|_{L^2}^2 = \int_{\mathbb{R}^n} T_\sigma f(x) \overline{T_\sigma f(x)} dx = \int_{\mathbb{R}^n} f(x) \overline{T_{\sigma^*}T_\sigma f(x)} dx \leq \|f\|_{L^2} \|T_{\sigma^*}T_\sigma f\|_{L^2} \lesssim \|f\|_{L^2}^2,$$

where the induction hypothesis was used in the last inequality.

Step 3. We now prove boundedness on $L^2(\mathbb{R}^n)$ for any operator with symbol in $S_{1,0}^0$. Let $\sigma \in S_{1,0}^0$. We can easily check that for $F \in \mathcal{C}^\infty(\mathbb{C})$, $F(\sigma) \in S_{1,0}^0$. We consider $F \in \mathcal{C}^\infty(\mathbb{C})$ such that $F(z) = (1+z)^{1/2}$ for $z \in (0, \infty)$ and set $A := \sup_{x, \xi \in \mathbb{R}^n} |\sigma(x, \xi)|$. The symbol $A^2 - |\sigma(x, \xi)|^2$ is non-negative for all $x, \xi \in \mathbb{R}^n$ and belongs to $S_{1,0}^0$. Then $a(x, \xi) := F(A^2 - |\sigma(x, \xi)|^2)$ is in $S_{1,0}^0$ as well. From Theorem 2.2.2

$$\sigma^* = \bar{\sigma} + r_{\sigma^*} \quad \text{and} \quad a^* = \bar{a} + r_{a^*},$$

where r_{σ^*} and r_{a^*} are symbols in $S_{1,0}^{-1}$, and the symbols of $T_{\sigma^*}T_\sigma$ and $T_{a^*}T_a$ are given, respectively, by

$$\sigma\sigma^* + r_{\sigma\sigma^*} \quad \text{and} \quad aa^* + r_{aa^*},$$

where $r_{\sigma\sigma^*}$ and r_{aa^*} are symbols in $S_{1,0}^{-1}$. Putting everything together, we conclude that the

symbol of $T_{a^*}T_a + T_{\sigma^*}T_\sigma$ is given by

$$\sigma(\bar{\sigma} + r_{\sigma^*}) + r_{\sigma\sigma^*} + a(\bar{a} + r_{a^*}) + r_{aa^*} = |\sigma|^2 + |a|^2 + r = 1 + A^2 + r,$$

where r is a symbol in $S_{1,0}^{-1}$. Therefore

$$T_{a^*}T_a + T_{\sigma^*}T_\sigma = (1 + A^2)I + T_r,$$

where I is the identity operator and $r \in S_{1,0}^{-1}$. Then

$$\begin{aligned} \|T_\sigma f\|_{L^2}^2 &\leq \|T_\sigma f\|_{L^2}^2 + \|T_a f\|_{L^2}^2 = \int_{\mathbb{R}^n} f(x) \overline{T_{\sigma^*}T_\sigma f(x)} dx + \int_{\mathbb{R}^n} f(x) \overline{T_{a^*}T_a f(x)} dx \\ &\leq \|f\|_{L^2} \|(T_{\sigma^*}T_\sigma + T_{a^*}T_a)f\|_{L^2} = \|f\|_{L^2} \|(1 + A^2)f + T_r f\|_{L^2} \\ &\leq \|f\|_{L^2} ((1 + A^2)\|f\|_{L^2} + C\|f\|_{L^2}) \sim \|f\|_{L^2}^2, \end{aligned}$$

where in the last inequality we have used the result of Step 2 applied to r . The desired boundedness is then obtained. \square

2.3.2 L^p boundedness of operators with symbols of order m

In this section we briefly present the complete result in regards to L^p boundedness properties of operators with symbols in the Hörmander classes.

Theorem 2.3.4. *Let $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$ and $1 < p < \infty$. T_σ is bounded from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for every $\sigma \in S_{\rho,\delta}^m$ if and only if $m \leq n(\rho - 1) \left| \frac{1}{p} - \frac{1}{2} \right|$.*

The full statement of this theorem was the work of many authors. Boundedness on $L^p(\mathbb{R}^n)$ for operators with symbols in $S_{\rho,\delta}^m$ with $m < n(\rho - 1) \left| \frac{1}{p} - \frac{1}{2} \right|$ was proved by Hirschman [23] and Wainger [41] for constant-coefficient symbols and by Hörmander [24] for general symbols. The fact that operators with symbols in $S_{\rho,\delta}^m$ where $m > n(\rho - 1) \left| \frac{1}{p} - \frac{1}{2} \right|$ may fail to be bounded on $L^p(\mathbb{R}^n)$ is due to a counterexample of Hardy-Littlewood-Hirschman-Wainger.

Finally, boundedness for operators with symbols in $S_{\rho,\delta}^m$ where $m = n(\rho-1) \left| \frac{1}{p} - \frac{1}{2} \right|$ is proved by interpolation using the following key result corresponding to the endpoints $p = 1$ and $p = \infty$, proved for multipliers by Fefferman and Stein in [16] and for the general case by Fefferman in [15]. See also Section 2.3.1 for the case $m = 0$.

Theorem 2.3.5. *Let $0 \leq \delta \leq \rho \leq 1, \delta < 1$. If $\sigma \in S_{\rho,\delta}^{n(\rho-1)/2}$ then T_σ is bounded from $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$ and from the Hardy space $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*

We note that the index $\frac{n}{2}(\rho-1)$ in Theorem 2.3.5 is the value of $n(\rho-1) \left| \frac{1}{p} - \frac{1}{2} \right|$ in Theorem 2.3.4 when $p = 1$ or $p = \infty$. It can be proved that operators with symbols in $S_{\rho,\delta}^{n(\rho-1)/2}$ with ρ and δ as in Theorem 2.3.4 are not necessarily bounded from $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ or from $L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. However, Theorem 2.3.5 is a good substitute for the endpoint $p = 1$ and $p = \infty$ since $H^1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n)$. We refer the reader to the appendix for the definitions of $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$.

2.4 Connections to Calderón-Zygmund Theory

While the study of pseudodifferential operators is motivated partially because of its usefulness in the theory of partial differential equations, it is also important because of the connections to singular integral operators and, in particular, to the Calderón-Zygmund theory. When Calderón-Zygmund singular integrals were first studied, it became apparent that an alternative approach would be to examine these operators on the frequency side by means of the Fourier transform. With this idea in mind the implementation of pseudodifferential operators became useful and their study necessary. Along with growing interest in pseudodifferential operators, Calderón-Zygmund theory expanded to include a variety of other ideas. See for instance the books [14, 17, 36].

In this section we give the definition of standard kernels and Calderón-Zygmund operators and present the size and decay properties possessed by the kernels of pseudodifferential

operators with symbols in the Hörmander classes. Finally we note some connections between these two classes of operators.

Definition 2.4.1. A linear operator T is a Calderón-Zygmund operator if

- (i) T is bounded on $L^2(\mathbb{R}^n)$;
- (ii) there exists a standard kernel K such that for $f \in C^\infty(\mathbb{R}^n)$ with compact support

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y) dy, \quad x \notin \text{supp}(f), \quad (2.4.12)$$

where K is called a standard kernel if

$$|K(x, y)| \lesssim \frac{1}{|x - y|^n}, \quad \forall x, y \in \mathbb{R}^n, x \neq y,$$

and there exists $\delta > 0$ such that

$$|K(x, y) - K(x, z)| \lesssim \frac{|y - z|^\delta}{|x - y|^{n+\delta}} \quad \text{if } |x - y| > 2|y - z|,$$

and

$$|K(x, y) - K(w, y)| \lesssim \frac{|x - w|^\delta}{|x - y|^{n+\delta}} \quad \text{if } |x - y| > 2|x - w|.$$

Though the definition of a Calderón-Zygmund operator requires that T be bounded on $L^2(\mathbb{R}^n)$, an alternative but equivalent definition could instead require that the operator be bounded on $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Indeed, the following theorem holds:

Theorem 2.4.2. *If T is a Calderón-Zygmund operator, then T is bounded from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$, from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, and from $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.*

Estimates for the kernel of pseudodifferential operators with symbols in the Hörmander classes are given by the following result due to Alvarez and Hounie in [1].

Theorem 2.4.3. *Let $\sigma \in S_{\rho,\delta}^m$ with $0 < \rho \leq 1$, $0 \leq \delta < 1$, $m \in \mathbb{R}$, and let $K(x, y)$ denote the distributional kernel of the associated linear pseudodifferential operator T_σ defined by $K(x, y) := \mathcal{F}^{-1}(\sigma(x, \cdot))(x - y)$.*

(i) *(Pseudo-local property.) The distribution K is smooth outside the diagonal. Moreover, given $\alpha, \beta \in \mathbb{N}_0^n$ there exists $N_0 \in \mathbb{N}_0$ such that for each $N \geq N_0$,*

$$\sup_{x \neq y} |x - y|^N |\partial_x^\alpha \partial_y^\beta K(x, y)| < \infty.$$

(ii) *Suppose that σ has compact support in ξ uniformly with respect to x . Then K is smooth, and given $\alpha, \beta \in \mathbb{N}_0^n$, $N \in \mathbb{N}_0$,*

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \lesssim (1 + |x - y|)^{-N}, \quad \forall x, y \in \mathbb{R}^n.$$

(iii) *Suppose that $m + M + n < 0$ for some $M \in \mathbb{N}_0$. Then K is a bounded continuous function with bounded continuous derivatives of order less than or equal to M .*

(iv) *Suppose that $m + M + n = 0$ for some $M \in \mathbb{N}_0$. Then*

$$\sup_{|\alpha+\beta|=M} |\partial_x^\alpha \partial_y^\beta K(x, y)| \lesssim |\log |x - y||, \quad \forall x, y \in \mathbb{R}^n, x \neq y.$$

(v) *Suppose that $m + M + n > 0$ for some $M \in \mathbb{N}_0$. Then*

$$\sup_{|\alpha+\beta|=M} |\partial_x^\alpha \partial_y^\beta K(x, y)| \lesssim |x - y|^{-(m+M+n)/\rho}, \quad \forall x, y \in \mathbb{R}^n, x \neq y.$$

We note that using items (i) and (v), we can prove that if $\sigma \in S_{\rho,\delta}^m$ with $0 < \rho \leq 1$, $0 \leq \delta < 1$ and $m \leq (\rho - 1)(n + 1)$, then T_σ has a standard kernel. If in addition we assume that $m \leq n(\rho - 1) \left| \frac{1}{p} - \frac{1}{2} \right|$ where $0 \leq \delta \leq \rho \leq 1$ with $\delta < 1$ and $1 < p < \infty$, then by Theorem 2.3.4, the operator T_σ is bounded on $L^p(\mathbb{R}^n)$ and is therefore a Calderón-

Zygmund operator. In particular, operators with symbols in the class $S_{1,\delta}^0$ with $0 \leq \delta < 1$ are Calderón-Zygmund operators; by way of contrast, operators with symbols in $S_{1,1}^0$ do have standard kernels, but are not necessarily Calderón-Zygmund operators since they may fail to be bounded on Lebesgue spaces (see Remark 2.3.2).

We note that due to Theorem 2.3.4 and Theorem 2.4.2 many of the Hörmander classes give rise to operators that are not Calderón-Zygmund operators. In this sense Hörmander classes go beyond the Calderón-Zygmund theory and therefore the study of boundedness properties of T_σ with $\sigma \in S_{\rho,\delta}^m$ in the setting of Lebesgue spaces is legitimate.

2.5 Boundedness and the Smoothness of the Symbols

While the definition of symbols in the Hörmander classes requires that symbols be infinitely differentiable and satisfy (2.2.8) for all multi-indices α and β , the proofs for the boundedness of the corresponding operators do not actually require infinitely many derivatives be available. Often the proofs simply ask for N derivatives satisfying (2.2.8) for some sufficiently large N . The exact value of N may be irrelevant to the proofs; nonetheless, knowing the minimal regularity of a symbol sufficient for boundedness is an intriguing question in its own right and turns out to be useful in the applications. Work done by many authors including Coifman and Meyer [12], Cordes [13], Miyachi [29], Muramatu [32], Sugimoto [37, 38], Boulkhemair [8], and Tomita [39] have all contributed to this end.

For instance, in regards to the class $S_{0,0}^0$, if (2.2.8) is satisfied for $|\alpha|, |\beta| \leq [\frac{n}{2}] + 1$ (Cordes [13, Theorem B'_1]) or for $|\alpha| \leq [\frac{n}{2}] + 1$ and $\beta \in \{0, 1, 2\}^n$ (Coifman-Meyer [12, Corollary 3]), or for $\alpha, \beta \in \{0, 1\}^n$ (Cordes [13], Coifman-Meyer [12, Theorem 3]), then the corresponding pseudodifferential operator is bounded on $L^2(\mathbb{R}^n)$. The order $[\frac{n}{2}] + 1$ turns out to be critical as counterexamples show (see Coifman-Meyer [12, p. 12], Bourdaud-Meyer [9]). Miyachi [29], Muramatu [32], and Sugimoto [37] improved these results by considering fractional derivatives in terms of Besov spaces. Analogous results in connection

to the classes $S_{0,0}^m$, $m \leq -n|\frac{1}{p} - \frac{1}{2}|$, were obtained in relation to boundedness on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, by Miyachi [29], Sugimoto [38] and Tomita [39] with symbols belonging to certain Besov classes. As a consequence, operators with symbols in $S_{0,0}^m$ with $m \leq -n|\frac{1}{p} - \frac{1}{2}|$ are bounded on $L^p(\mathbb{R}^n)$ if the Hörmander condition (2.2.8) is satisfied for multi-indices α and β such that $|\alpha| \leq [\min(\frac{n}{2}, \frac{n}{p})] + 1$ and $|\beta| \leq [\max(\frac{n}{2}, \frac{n}{p})] + 1$, where $[s]$ denotes integer part of $s \in \mathbb{R}$.

Chapter 3

Bilinear Operators with Symbols in the Hörmander Classes

3.1 Introduction

In this chapter we begin examining bilinear pseudodifferential operators; specifically, those operators which have symbols that belong to the bilinear Hörmander classes. To begin, we have the following definition.

Definition 3.1.1. Let $\sigma(x, \xi, \eta)$ be a complex-valued, smooth function defined for $x, \xi, \eta \in \mathbb{R}^n$ called a symbol. The bilinear pseudodifferential operator associated to the symbol σ is defined by

$$T_\sigma(f, g)(x) := \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta, \quad f, g \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n. \quad (3.1.1)$$

Appropriate assumptions on σ will be assumed so that the integral in (3.1.1) converges absolutely for all $f, g \in \mathcal{S}(\mathbb{R}^n)$. Boundedness properties as well as applications of bilinear pseudodifferential operators will depend on the characteristics of their symbols. Before considering the properties of σ in greater detail, we present a few simple examples that

come from specific symbols.

- If σ is independent of ξ and η so that $\sigma(x, \xi, \eta) = \sigma(x)$, then

$$T_\sigma(f, g)(x) = \sigma(x)f(x)g(x), \quad x \in \mathbb{R}^n.$$

- If σ is x -independent so that $\sigma(x, \xi, \eta) = \sigma(\xi, \eta)$, then

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta, \quad x \in \mathbb{R}^n,$$

in which case the bilinear operator is called a bilinear multiplier in analogy to the linear case where the operator corresponds with multiplication by the symbol on the Fourier side.

- If $\sigma(x, \xi, \eta) = b(x)(2\pi i \xi)^\alpha (2\pi i \eta)^\beta$, where $\alpha, \beta \in \mathbb{N}_0^n$, then

$$T_\sigma(f, g)(x) = b(x) \partial^\alpha f(x) \partial^\beta g(x), \quad x \in \mathbb{R}^n.$$

In the next section, we define the bilinear Hörmander classes, present their properties, and establish the symbolic calculus. The symbolic calculus is useful in proving boundedness properties of the corresponding bilinear pseudodifferential operators which we consider in Section 3.3 first on Lebesgue spaces and then on Hardy spaces and *BMO*. We will examine several interesting cases that are unique to symbols of order zero and then complete the picture with symbols of any order m . In the final section of Chapter 3, we briefly explore the connections between bilinear pseudodifferential operators and the bilinear Calderón-Zygmund theory.

3.2 The Bilinear Hörmander Classes

As in the linear case, one possible way of classifying symbols is by their size and decay properties. In this section, we define the bilinear Hörmander classes and present several of their useful properties.

Definition 3.2.1. Let $\sigma(x, \xi, \eta)$ be a complex-valued, smooth function defined for $x, \xi, \eta \in \mathbb{R}^n$, $m \in \mathbb{R}$, and $\rho, \delta \in [0, 1]$. The symbol σ is said to be in the bilinear Hörmander class $BS_{\rho, \delta}^m$ if for all multi-indices $\alpha, \beta, \gamma \in \mathbb{N}_0^n$,

$$\sup_{x, \xi, \eta \in \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \langle \xi, \eta \rangle^{-m - \delta|\alpha| + \rho(|\beta| + |\gamma|)} < \infty \quad (3.2.2)$$

where $\langle \xi, \eta \rangle := 1 + |\xi| + |\eta|$. The value m is called the *order* of the symbol.

We examine now a few of the basic properties of the bilinear Hörmander symbols. Most of these are straightforward applications of the definition and are analogous to those of their linear counterparts.

- The bilinear Hörmander classes are nested in the following way:
 - If $m_1 \leq m_2$ then $BS_{\rho, \delta}^{m_1} \subseteq BS_{\rho, \delta}^{m_2}$.
 - If $\delta_1 \leq \delta_2$ then $BS_{\rho, \delta_1}^m \subseteq BS_{\rho, \delta_2}^m$.
 - If $\rho_1 \leq \rho_2$ then $BS_{\rho_1, \delta}^m \supseteq BS_{\rho_2, \delta}^m$.
- If $\sigma_1 \in BS_{\rho, \delta}^{m_1}$ and $\sigma_2 \in BS_{\rho, \delta}^{m_2}$ then $\sigma_1 + \sigma_2 \in BS_{\rho, \delta}^{\max(m_1, m_2)}$ and $\sigma_1 \sigma_2 \in BS_{\rho, \delta}^{m_1 + m_2}$.
- If $\sigma \in BS_{\rho, \delta}^m$ and $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, then $\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma \in BS_{\rho, \delta}^{m + \delta|\alpha| - \rho(|\beta| + |\gamma|)}$; that is, taking a derivative with respect to any of the components of x increases the order of the symbol by δ , while taking a derivative with respect to any component of the frequency variables ξ or η decreases the order of the symbol by ρ .
- It follows as in the linear case that if σ is in any of the Hörmander classes then T_σ is continuous from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. Moreover, a version of the Schwartz

kernel theorem gives that there exists $K \in \mathcal{S}'(\mathbb{R}^{3n})$ such that

$$\int_{\mathbb{R}^n} T_\sigma(f, g)(x) h(x) dx = \langle K, h \otimes f \otimes g \rangle, \quad f, g, h \in \mathcal{S}(\mathbb{R}^n),$$

where $(h \otimes f \otimes g)(x, y, z) = h(x)f(y)g(z)$ for $x, y, z \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denotes the action of a tempered distribution on functions of the Schwartz class. It can be proved that $K(x, y, z) = k(x, x - y, x - z)$, where $k(x, y, z) = \mathcal{F}^{-1}(\sigma(x, \cdot, \cdot))(y, z)$ and it holds that

$$\begin{aligned} T_\sigma(f, g)(x) &= \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \\ &= \langle \mathcal{F}^{-1}(\sigma(x, \cdot, \cdot))(y, z), f(x - y)g(x - z) \rangle, \end{aligned}$$

for $x \in \mathbb{R}^n$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. The distribution K is called the distributional kernel of the operator T_σ .

- Given $s_1, s_2 \in \mathbb{N}_0$ and $\sigma \in BS_{\rho, \delta}^m$ we define

$$\|\sigma\|_{s_1, s_2} := \sup_{\substack{|\alpha| \leq s_1 \\ |\beta|, |\gamma| \leq s_2}} \sup_{x, \xi, \eta \in \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \langle \xi, \eta \rangle^{-m - \delta|\alpha| + \rho(|\beta| + |\gamma|)}. \quad (3.2.3)$$

The family of norms $\{\|\sigma\|_{s_1, s_2}\}_{s_1, s_2 \in \mathbb{N}_0}$ makes $BS_{\rho, \delta}^m$ into a Fréchet space. Note that the notation for these norms does not explicitly show the indices m, ρ and δ ; the values of these parameters will be clear from the context.

Interestingly enough, while the definition of the Hörmander classes requires that (3.2.2) be satisfied for derivatives of all orders, this level of smoothness in the symbol is often unnecessary to prove boundedness properties of the corresponding operator. When this is the case, all that is required is that $\|\sigma\|_{s_1, s_2} < \infty$ for some $s_1, s_2 \in \mathbb{N}_0$ sufficiently large. The exact number of derivatives satisfying (3.2.2) that is sufficient for boundedness is part of the topic discussed in Chapter 4.

3.2.1 Symbolic calculus

In this section we present the symbolic calculus for the bilinear Hörmander classes. This symbolic calculus provides information about the symbols of the transposes of T_σ when σ belongs to a bilinear Hörmander class. Because the operator is bilinear, there are two transposes T_σ^{*1} and T_σ^{*2} of T_σ to be considered. These operators satisfy

$$\int_{\mathbb{R}^n} T_\sigma(f, g)(x) h(x) dx = \int_{\mathbb{R}^n} T_\sigma^{*1}(h, g)(x) f(x) dx = \int_{\mathbb{R}^n} T_\sigma^{*2}(f, h)(x) g(x) dx, \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

In the special case that the symbol σ does not depend on the space variable x , it easily follows that T_σ^{*1} and T_σ^{*2} have symbols $\sigma(-\xi - \eta, \eta)$ and $\sigma(\xi, -\xi - \eta)$, respectively.

Bényi, Maldonado, Naibo, and Torres proved in [4] that the bilinear Hörmander classes are closed under transpositions in general; they also developed the asymptotic expansions for the corresponding symbols of the transposes. More precisely,

Theorem 3.2.2. *Let $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, and $\sigma \in BS_{\rho, \delta}^m$.*

(i) *For $j = 1, 2$, $T_\sigma^{*j} = T_{\sigma^{*j}}$, where $\sigma^{*j} \in BS_{\rho, \delta}^m$.*

(ii) *For $N \in \mathbb{N}$ and $\delta < \rho$, σ^{*1} and σ^{*2} satisfy*

$$\sigma^{*1} - \sum_{|\alpha| < N} \frac{(2\pi i)^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha (\sigma(x, -\xi - \eta, \eta)) \in BS_{\rho, \delta}^{m+(\delta-\rho)N}$$

and

$$\sigma^{*2} - \sum_{|\alpha| < N} \frac{(2\pi i)^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_\eta^\alpha (\sigma(x, \xi, -\xi - \eta)) \in BS_{\rho, \delta}^{m+(\delta-\rho)N}.$$

By way of contrast with the results in Theorem 3.2.2, the classes $BS_{1,1}^0$ are not mentioned in this theorem because they are not closed under transposition. The proof of this statement will follow as a corollary to the boundedness properties discussed in Section 3.3 (See Remark 3.3.1). A proof for part (i) of Theorem 3.2.2 will be given following the work of Bényi et al.; for a proof for part (ii) see their result in [4, Theorem 2].

Proof of Theorem 3.2.2, part(i). We will prove the result for symbols with compact support. Since the estimates obtained will be independent of the support of the symbol, a limiting argument can be used to get the result for general symbols. We restrict our attention to the first transpose as the argument is completely analogous for the second transpose.

Fix $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$ with $\delta < 1$; let $\sigma \in BS_{\rho, \delta}^m$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. Using Fubini's theorem and an appropriate change of variables we obtain

$$\begin{aligned} \langle T_\sigma^{*1}(h, g), f \rangle &= \langle T_\sigma(f, g), h \rangle = \int_{\mathbb{R}^n} T_\sigma(f, g)(x) h(x) dx \\ &= \int_{\mathbb{R}^{3n}} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) h(x) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta dx \\ &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^{3n}} \sigma(y, -\xi - \eta, \eta) h(y) \hat{g}(\eta) e^{2\pi i \xi \cdot (x-y)} e^{2\pi i x \cdot \eta} dy d\xi d\eta \right] f(x) dx. \end{aligned}$$

Setting $\tau(y, \xi, \eta) := \sigma(y, -\xi - \eta, \eta)$ we then have

$$T_\sigma^{*1}(h, g)(x) = \int_{\mathbb{R}^{3n}} \tau(y, \xi, \eta) h(y) \hat{g}(\eta) e^{2\pi i \xi \cdot (x-y)} e^{2\pi i x \cdot \eta} dy d\xi d\eta,$$

and it easily follows that $\tau \in BS_{\rho, \delta}^m$ since $\sigma \in BS_{\rho, \delta}^m$, that is,

$$|\partial_y^\alpha \partial_\xi^\beta \partial_\eta^\gamma \tau(y, \xi, \eta)| \lesssim \langle \xi, \eta \rangle^{m + \delta |\alpha| - \rho(|\beta| + |\gamma|)}. \quad (3.2.4)$$

By an appropriate change of variables and Fubini's Theorem we can rewrite the operator T_σ^{*1} so that its symbol is given by

$$a(x, \xi, \eta) = \int_{\mathbb{R}^{2n}} \tau(x + y, z + \xi, \eta) e^{-2\pi i z \cdot y} dy dz. \quad (3.2.5)$$

We then must show that $a \in BS_{\rho, \delta}^m$. By (3.2.4) and since

$$\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma a(x, \xi, \eta) = \int_{\mathbb{R}^{2n}} \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \tau(x + y, z + \xi, \eta) e^{-2\pi i z \cdot y} dy dz,$$

it is enough to work with $\alpha = \beta = \gamma = 0$.

Fixing $\xi, \eta \in \mathbb{R}^n$ and setting $A := \langle \xi, \eta \rangle$, we will prove that

$$|a(x, \xi, \eta)| = \left| \int_{\mathbb{R}^{2n}} \tau(x + y, z + \xi, \eta) e^{-2\pi i z \cdot y} dy dz \right| \lesssim A^m$$

with a constant independent of the support of σ . To that end, we will divide the integral over $z \in \mathbb{R}^n$ into three regions depending on the size of $|z|$ by defining the sets

$$\Omega_1 = \{z : |z| \leq \frac{A^\delta}{2}\}, \quad \Omega_2 = \{z : \frac{A^\delta}{2} \leq |z| \leq \frac{A}{2}\}, \quad \Omega_3 = \{z : |z| \geq \frac{A}{2}\}.$$

Then

$$a(x, \xi, \eta) = \int_{\Omega_1} \int_y \cdots + \int_{\Omega_2} \int_y \cdots + \int_{\Omega_3} \int_y \cdots =: I_1 + I_2 + I_3$$

so that we need to prove $|I_j| \lesssim A^m$ for $j = 1, 2, 3$.

Before proceeding to the estimates on the I_j 's we present some preliminary calculations. For $l_0 \in \mathbb{N}$ with $2l_0 > n$ and since $e^{-2\pi i z \cdot y} = (1 + 4\pi^2 A^{2\delta} |y|^2)^{-l_0} (1 + A^{2\delta} (-\Delta_z))^{l_0} e^{-2\pi i z \cdot y}$, integration by parts gives

$$a(x, \xi, \eta) = \int_{\mathbb{R}^{2n}} q(x, y, z, \xi, \eta) e^{-2\pi i z \cdot y} dy dz$$

where

$$q(x, y, z, \xi, \eta) = \frac{(1 + A^{2\delta} (-\Delta_z))^{l_0} \tau(x + y, z + \xi, \eta)}{(1 + 4\pi^2 A^{2\delta} |y|^2)^{l_0}}.$$

Next we proceed to estimate $(-\Delta_y)^l q$ for $l \in \mathbb{N}_0$. Note that

$$\begin{aligned} (-\Delta_y)^l q &= \sum_{\substack{|\alpha|=2l \\ \alpha_i \text{ even}}} C_\alpha \partial_y^\alpha q(x, y, z, \xi, \eta) \\ &= \sum_{\substack{|\alpha|=2l \\ \alpha_i \text{ even}}} \sum_{\beta \leq \alpha} C_{\alpha\beta} \partial_y^\beta ((1 + 4\pi^2 A^{2\delta} |y|^2)^{-l_0}) \partial_y^{\alpha-\beta} ((1 + A^{2\delta} (-\Delta_z))^{l_0} \tau(x + y, z + \xi, \eta)). \end{aligned} \tag{3.2.6}$$

We will get a bound by a power of A on the β derivatives of the first factor and the $\alpha - \beta$ derivatives of the second factor, in each term of the sum. For the β derivatives of the first factor, we have that

$$|\partial_y^\beta((1 + 4\pi^2 A^{2\delta}|y|^2)^{-l_0})| \leq C_{\beta l_0} A^{\delta|\beta|} (1 + A^{2\delta}|y|^2)^{-l_0}. \quad (3.2.7)$$

For the second factor we consider $P_{l_0} = \{\gamma = (\gamma_1, \dots, \gamma_n) : \gamma_i \text{ is even and } |\gamma| = 2j, j = 0, \dots, l_0\}$ and get

$$(1 + A^{2\delta}(-\Delta_z))^{l_0} \tau(x + y, z + \xi, \eta) = \sum_{\gamma \in P_{l_0}} C_\gamma A^{\delta|\gamma|} \partial_\xi^\gamma \tau(x + y, z + \xi, \eta).$$

Using that $\tau \in BS_{\rho, \delta}^m$, it follows that

$$\begin{aligned} & |\partial_y^{\alpha-\beta}((1 + A^{2\delta}(-\Delta_z))^{l_0} \tau(x + y, z + \xi, \eta))| \\ & \leq \sum_{\gamma \in P_{l_0}} C_{\gamma\alpha\beta} A^{\delta|\gamma|} (1 + |z + \xi| + |\eta|)^{m+\delta(|\alpha|-|\beta|)-\rho|\gamma|}. \end{aligned} \quad (3.2.8)$$

Putting (3.2.6), (3.2.7), and (3.2.8) together we get

$$\begin{aligned} & |(-\Delta_y)^l q| \lesssim \\ & (1 + A^{2\delta}|y|^2)^{-l_0} \sum_{\substack{|\alpha|=2l \\ \alpha_i \text{ even}}} \sum_{\beta \leq \alpha} C_{\alpha\beta l_0} A^{\delta|\beta|} \sum_{\gamma \in P_{l_0}} C_{\gamma\alpha\beta} A^{\delta|\gamma|} (1 + |z + \xi| + |\eta|)^{m+\delta(|\alpha|-|\beta|)-\rho|\gamma|}. \end{aligned} \quad (3.2.9)$$

As a final step before estimating each of the three integrals I_j we note that

$$\frac{1}{2}A \leq 1 + |z + \xi| + |\eta| \leq \frac{3}{2}A, \quad z \in \Omega_1 \cup \Omega_2, \quad (3.2.10)$$

and

$$1 + |z + \xi| + |\eta| \leq A + |z| \leq 3|z|, \quad z \in \Omega_3. \quad (3.2.11)$$

For I_1 we use the estimate (3.2.9) with $l = 0$. The inequalities from (3.2.10) and the fact that $\delta - \rho \leq 0$ give for $z \in \Omega_1$,

$$\begin{aligned} |q| &\leq (1 + A^{2\delta}|y|^2)^{-l_0} \sum_{\gamma \in P_{l_0}} C_\gamma A^{\delta|\gamma|} (1 + |z + \xi| + |\eta|)^{m-\rho|\gamma|} \\ &\leq (1 + A^{2\delta}|y|^2)^{-l_0} \sum_{\gamma \in P_{l_0}} C_\gamma A^{m+(\delta-\rho)|\gamma|} \lesssim (1 + A^{2\delta}|y|^2)^{-l_0} A^m. \end{aligned}$$

Therefore, since $2l_0 > n$,

$$|I_1| \lesssim A^m \int_{\Omega_1} \int_y \frac{1}{(1 + A^{2\delta}|y|^2)^{l_0}} dy dz \sim A^m.$$

For I_2 , integration by parts gives

$$\begin{aligned} \int_y q(x, y, z, \xi, \eta) e^{-2\pi iz \cdot y} dy &= \frac{1}{|z|^{2l_0}} \int_y q(x, y, z, \xi, \eta) (-\Delta_y)^{l_0} e^{-2\pi iz \cdot y} dy \\ &= \frac{1}{|z|^{2l_0}} \int_y (-\Delta_y)^{l_0} (q(x, y, z, \xi, \eta)) e^{-2\pi iz \cdot y} dy. \end{aligned}$$

Using (3.2.9) with $l = l_0$, (3.2.10), and $\delta - \rho \leq 0$ we get, for $z \in \Omega_2$,

$$\begin{aligned} |(-\Delta_y)^{l_0} q| &\leq (1 + A^{2\delta}|y|^2)^{-l_0} \sum_{\substack{|\alpha|=2l_0 \\ \alpha_i \text{ even}}} \sum_{\beta \leq \alpha} C_{\alpha\beta l_0} A^{\delta|\beta|} \sum_{\gamma \in P_{l_0}} C_{\gamma\alpha\beta} A^{\delta|\gamma|} A^{m+\delta(|\alpha|-|\beta|)-\rho|\gamma|} \\ &\leq (1 + A^{2\delta}|y|^2)^{-l_0} \sum_{\substack{|\alpha|=2l_0 \\ \alpha_i \text{ even}}} \sum_{\beta \leq \alpha} C_{\alpha\beta l_0} \sum_{\gamma \in P_{l_0}} C_{\gamma\alpha\beta} A^{m+\delta|\alpha|+(\delta-\rho)|\gamma|} \\ &\lesssim \frac{A^{m+2l_0\delta}}{(1 + A^{2\delta}|y|^2)^{l_0}}. \end{aligned}$$

Since $2l_0 > n$, it follows that

$$|I_2| \leq \int_{\Omega_2} \frac{1}{|z|^{2l_0}} \int_y \frac{A^{m+2l_0\delta}}{(1 + A^{2\delta}|y|^2)^{l_0}} dy dz \lesssim A^{m+2l_0\delta-\delta n} \int_{|z| \geq \frac{A^\delta}{2}} |z|^{-2l_0} dz \sim A^m.$$

For I_3 we will choose $l \in \mathbb{N}$ as necessary later. Again, integration by parts gives

$$\begin{aligned} \int_y q(x, y, z, \xi, \eta) e^{-2\pi i x \cdot y} dy &= \frac{1}{|z|^{2l}} \int_y q(x, y, z, \xi, \eta) (-\Delta_y)^l e^{-2\pi i z \cdot y} dy \\ &= \frac{1}{|z|^{2l}} \int_y (-\Delta_y)^l (q(x, y, z, \xi, \eta)) e^{-2\pi i z \cdot y} dy. \end{aligned}$$

Using (3.2.9) and (3.2.11), and defining $m_+ = \max(0, m)$, we get, for $z \in \Omega_3$,

$$\begin{aligned} &|(-\Delta_y)^l q| \\ &\lesssim (1 + A^{2\delta}|y|^2)^{-l_0} \sum_{\substack{|\alpha|=2l \\ \alpha_i \text{ even}}} \sum_{\beta \leq \alpha} C_{\alpha\beta l_0} A^{\delta|\beta|} \sum_{\gamma \in P_{l_0}} C_{\gamma\alpha\beta} A^{\delta|\gamma|} (1 + |z + \xi| + |\eta|)^{m + \delta(|\alpha| - |\beta|) - \rho|\gamma|} \\ &\lesssim (1 + A^{2\delta}|y|^2)^{-l_0} \sum_{\substack{|\alpha|=2l \\ \alpha_i \text{ even}}} \sum_{\beta \leq \alpha} C_{\alpha\beta l_0} \sum_{\gamma \in P_{l_0}} C_{\gamma\alpha\beta} |z|^{\delta(|\beta| + |\gamma|)} |z|^{m_+ + \delta(|\alpha| - |\beta|)} \\ &\lesssim (1 + A^{2\delta}|y|^2)^{-l_0} |z|^{m_+ + \delta(2l + 2l_0)}. \end{aligned}$$

Then

$$\begin{aligned} |I_3| &\lesssim \int_{\Omega_3} \frac{1}{|z|^{2l}} \int_y (1 + A^{2\delta}|y|^2)^{-l_0} |z|^{m_+ + \delta(2l + 2l_0)} dy dz \\ &\sim \int_{|z| \geq \frac{A}{2}} |z|^{m_+ + 2l_0\delta + 2l(\delta - 1)} dz \int_y (1 + A^{2\delta}|y|^2)^{-l_0} dy \\ &\sim A^{-\delta n} \int_{|z| \geq \frac{A}{2}} |z|^{m_+ + 2l_0\delta + 2l(\delta - 1)} dz. \end{aligned}$$

Since $0 \leq \delta < 1$ we can choose $l \in \mathbb{N}$ sufficiently large so that

$$m_+ + 2l_0\delta + 2l(\delta - 1) < -n \quad \text{and} \quad -\delta n + m_+ + 2l_0\delta + 2l(\delta - 1) + n < m.$$

Finally,

$$|I_3| \lesssim A^{-\delta n + m_+ + 2l_0\delta + 2l(\delta - 1) + n} \leq A^m.$$

□

Remark 3.2.3. It is important to recognize just how valuable this theorem is. We will see symbolic calculus referenced often in the next section; in such a situation, we mean an application of Theorem 3.2.2. As a concrete example of how the symbolic calculus is used in the bilinear setting, fix $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq \infty$ with $\delta < 1$, $1 \leq p_1, p_2, p \leq \infty$ and suppose that T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho, \delta}^m$. By Theorem 3.2.2 both $T_{\sigma^{*1}}$ and $T_{\sigma^{*2}}$ are also bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho, \delta}^m$. We can then use duality to infer that T_σ is bounded from $L^{p'}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^{p_1}(\mathbb{R}^n)$ and from $L^{p_1}(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho, \delta}^m$. Finally, we can interpolate to get boundedness of T_σ , for all $\sigma \in BS_{\rho, \delta}^m$, from $L^a(\mathbb{R}^n) \times L^b(\mathbb{R}^n)$ into $L^c(\mathbb{R}^n)$ for any a, b, c such that the point $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$ is in the convex hull of the points $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$, $(\frac{1}{p'}, \frac{1}{p_2}, \frac{1}{p_1})$ and $(\frac{1}{p_1}, \frac{1}{p'}, \frac{1}{p_2})$.

3.3 Boundedness of Operators with Symbols in the Bilinear Hörmander Classes

We next consider the important question of boundedness on Lebesgue spaces for bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes. Continuity properties of these operators in the context of other function spaces such as Besov and Triebel-Lizorkin spaces, Hardy spaces, and BMO can also be found in the literature; see for instance Bényi [2], Miyachi-Tomita [30], Naibo [33, 34] and references therein. However, we will mostly restrict our attention in this section to the results for Lebesgue spaces by first considering the case when the symbols are of order zero and then broadening our perspective to symbols of any order. For completeness, a brief account of boundedness results in the context of Hardy spaces and BMO is included at the end of this section.

As discussed in Section 3.2, T_σ is continuous from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$ for any

symbol σ in a Hörmander class. Given quasi-Banach spaces X, Y and Z that contain $\mathcal{S}(\mathbb{R}^n)$ and such that $\mathcal{S}(\mathbb{R}^n)$ is dense in X and Y , we will say that T_σ is bounded from $X \times Y$ into Z if there exists a constant C_σ such that

$$\|T_\sigma(f, g)\|_Z \leq C_\sigma \|f\|_X \|g\|_Y, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n). \quad (3.3.12)$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in X and Y , there is a unique extension of the operator to $X \times Y$, which we still call T_σ , such that (3.3.12) holds for all $f \in X$ and $g \in Y$.

3.3.1 Symbols of order zero and Lebesgue spaces

The discussion of boundedness on Lebesgue spaces of bilinear pseudodifferential operators with symbols in the Hörmander classes commences quite naturally with operators whose corresponding symbols are of order zero. In fact, we begin similarly to the way the theory itself developed. In this section we will see that the classes $BS_{1,1}^0$ and $BS_{\rho,\delta}^0$ for $0 \leq \delta \leq 1$ and $0 \leq \rho < 1$ contain symbols that produce unbounded operators in the setting of Lebesgue spaces. In constast, every operator with symbol in $BS_{1,\delta}^0$, $0 \leq \delta < 1$, is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for any $1 < p_1, p_2 < \infty$ and $\frac{1}{2} < p < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$; these classes are discussed in Section 3.3.2.

Bilinear Hörmander classes of order zero showcase both similarities and differences with their linear counterparts. On the one hand, the class $S_{1,1}^0$, like $BS_{1,1}^0$, contains some symbols whose associated operators are not bounded on Lebesgue spaces. Moreover both the linear classes $S_{1,\delta}^0$ and the bilinear classes $BS_{1,\delta}^0$ with $0 \leq \delta \leq 1$ are connected to the linear and bilinear Calderón-Zygmund theories, respectively; and for $0 \leq \delta < 1$ the corresponding operators are bounded on Lebesgue spaces. On the other hand, as was stated in Theorem 2.3.1, the operators with symbols in $S_{\rho,\delta}^0$, $0 \leq \delta \leq \rho < 1$, are bounded from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, while Theorem 3.3.2 below states that it is possible to have operators with symbols in $BS_{\rho,\delta}^0$ that are unbounded on Lebesgue spaces.

The class $BS_{1,1}^0$.

In [5], Bényi and Torres, mirroring work done in the linear case, proved that there are symbols in $BS_{1,1}^0$ for which the corresponding pseudodifferential operators are not bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for any $1 < p_1, p_2 < \infty$ and $\frac{1}{2} < p < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. The specific example below is given in the case when $n = 1$ but can be generalized to other dimensions.

Let $\psi \in \mathcal{S}(\mathbb{R})$ be such that $\text{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R} : 2^{-1/2} \leq |\xi| \leq 2^{1/2}\}$ and $\widehat{\psi}(\xi) \equiv 1$ for $2^{-1/4} \leq |\xi| \leq 2^{1/4}$ and define

$$\sigma(x, \xi, \eta) := \sum_{j=1}^{\infty} e^{-2^{j+1}\pi i x} \widehat{\psi}(2^{-j}(\xi^2 + \eta^2)^{1/2}). \quad (3.3.13)$$

Because the support of the function $\widehat{\psi}(2^{-j}(\xi^2 + \eta^2)^{1/2})$ is contained in $\{(\xi, \eta) \in \mathbb{R}^2 : 2^{j-1/2} \leq (\xi^2 + \eta^2)^{1/2} \leq 2^{j+1/2}\}$, at most one term in the sum (3.3.13) is nonzero for each (ξ, η) . This allows one to easily verify that $\sigma \in BS_{1,1}^0$.

We now consider $f \in \mathcal{S}(\mathbb{R})$ such that \hat{f} is real-valued and supported in $|\xi| \leq 1/2$ and set

$$f_N(x) := \sum_{j=10}^N \frac{1}{j} e^{2^{j+1}\pi i x} f(x).$$

Then, using that $\sigma(x, \xi, \eta) = e^{-2^{j+1}\pi i x}$ in the support of $\hat{f}(\xi - 2^j)\hat{f}(\eta)$, it follows that

$$\begin{aligned} T_\sigma(f_N, f)(x) &= \int_{\mathbb{R}^2} \sigma(x, \xi, \eta) e^{2\pi i x \cdot (\xi + \eta)} \sum_{j=10}^N \frac{1}{j} \hat{f}(\xi - 2^j) \hat{f}(\eta) d\xi d\eta \\ &= \sum_{j=10}^N \frac{1}{j} \int_{\mathbb{R}^2} e^{2\pi i x \cdot (\xi - 2^j)} e^{2\pi i x \cdot \eta} \hat{f}(\xi - 2^j) \hat{f}(\eta) d\xi d\eta \\ &= \left(\sum_{j=10}^N \frac{1}{j} \right) |f(x)|^2, \quad x \in \mathbb{R}, \end{aligned}$$

where the last equality is due to Plancherel's identity. By the orthogonality of the functions

$\hat{f}(\cdot - 2^j)$ for $j \geq 10$, we have that

$$\frac{1}{100} \|f\|_{L^2}^2 \leq \|f_N\|_{L^2}^2 = \left(\sum_{j=10}^N \frac{1}{j^2} \right) \|f\|_{L^2}^2 \lesssim \|f\|_{L^2}^2.$$

Therefore

$$\|T_\sigma(f_N, f)\|_{L^1} = \left(\sum_{j=10}^N \frac{1}{j} \right) \|f\|_{L^2}^2 \gtrsim \log N \|f_N\|_{L^2} \|f\|_{L^2},$$

and thus T_σ cannot be bounded from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ into $L^1(\mathbb{R})$.

This example together with the Calderón-Zygmund theory discussed in Section 3.4 proves that T_σ is not bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for any $1 < p_1, p_2 < \infty$ and $\frac{1}{2} < p < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

Remark 3.3.1. Though the class $BS_{1,1}^0$ produces some operators that are unbounded on Lebesgue spaces, Grafakos and Torres obtained in [20] conditions to guarantee boundedness. They proved that if $\sigma \in BS_{1,1}^0$ and T_σ^{*1} and T_σ^{*2} have symbols in $BS_{1,1}^0$ then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $1 < p_1, p_2 < \infty$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. As a consequence, the Hörmander class $BS_{1,1}^0$ is not closed under transposition as was mentioned in Section 3.2.1.

We add that boundedness properties for symbols in $BS_{1,1}^0$ in the context of Besov, Lipschitz, and Triebel-Lizorkin spaces have been obtained in Bényi [2] and Naibo [33] (see also references therein).

The classes $BS_{\rho,\delta}^0$ for $0 \leq \delta \leq \rho < 1$.

As mentioned, the class $BS_{1,1}^0$ is not the only class that produces unbounded operators on Lebesgue spaces. In this section we show that the Hörmander classes of order zero $BS_{\rho,\delta}^0$, with $0 \leq \delta \leq 1$ and $0 \leq \rho < 1$, also contain symbols that give rise to operators that are unbounded in this setting, as proven by Bényi, Bernicot, et al. in [3].

Theorem 3.3.2. For $0 \leq \delta \leq 1$, $0 \leq \rho < 1$, and $1 \leq p, p_1, p_2 < \infty$ where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, there exists $\sigma \in BS_{\rho, \delta}^0$, such that T_σ is not bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.

The structure of the proof of Theorem 3.3.2 is as follows: first, we prove that there are x -independent symbols in $BS_{0,0}^0$ that give rise to unbounded operators, implementing a result by Bényi and Torres in [6]. Then we prove by contradiction that $BS_{\rho, \delta}^0$ also produces unbounded operators for every ρ and δ under consideration using a scaling argument and Lemma 3.3.3 below as did Bényi, Bernicot, et al. in [3].

Lemma 3.3.3. Let $0 < p \leq \infty$, $1 \leq p_1, p_2 < \infty$, $1 \leq \delta, \rho \leq 1$ and suppose T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho, \delta}^m$. Then there exist $s_1, s_2 \in \mathbb{N}_0$ such that

$$\|T_\sigma\| \lesssim \|\sigma\|_{s_1, s_2}$$

for all $\sigma \in BS_{\rho, \delta}^m$.

Proof of Theorem 3.3.2. As mentioned, we first consider the class $BS_{0,0}^0$ and then $BS_{\rho, \delta}^0$ in the general case.

(a) **The class $BS_{0,0}^0$.** If $p_1 \neq 2$ we consider a symbol σ in $BS_{0,0}^0$ of the form $\sigma(x, \xi, \eta) = \sigma_1(\xi)$ such that σ_1 is not a multiplier in $L^{p_1}(\mathbb{R}^n)$. Then T_{σ_1} is not bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. The proof when $p_2 \neq 2$ is analogous.

Consider then $p_1 = p_2 = 2$. Suppose by contradiction that every x -independent symbol in $BS_{0,0}^0$ defines a bounded operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, and consider an x -independent symbol σ in $BS_{0,0}^0$ of the form $\sigma(\xi, \eta) = \tau(-\xi - \eta)$. By duality, then T_σ^{*1} maps $L^\infty(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. By the symbolic calculus (Theorem 3.2.2), the symbol of T_σ^{*1} belongs to $BS_{0,0}^0$ and is given by $\sigma^{*1}(\xi, \eta) = \sigma(-\xi - \eta, \eta) = \tau(\xi)$. It then follows that every operator with a symbol in $BS_{0,0}^0$ that depends only on ξ (think of τ), defines a bounded bilinear operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and from $L^\infty(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. By interpolation these operators would then be bounded from $L^{p_1}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^{2p_1/(2+p_1)}$ for any $2 < p_1 < \infty$, which contradicts our first case when $p_1 \neq 2$.

(b) **The class $BS_{\rho,\delta}^0$, general case.** Having established that some operators with symbols in $BS_{0,0}^0$ are unbounded on Lebesgue spaces we now consider more general symbols of order zero. Fix δ, ρ, p_1, p_2 and p as in the hypothesis. Suppose, by way of contradiction, that T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho,\delta}^0$. Consider an x -independent symbol $\sigma \in BS_{\rho,\delta}^0$ and, for multi-indices β and γ , set

$$C_{\beta,\gamma}(\sigma) := \sup_{\xi,\eta \in \mathbb{R}^n} |\partial_\xi^\beta \partial_\eta^\gamma \sigma(\xi, \eta)| \langle \xi, \eta \rangle^{\rho(|\beta|+|\gamma|)}.$$

For $\lambda > 0$ define $\sigma_\lambda(\xi, \eta) := \sigma(\lambda\xi, \lambda\eta)$, $\xi, \eta \in \mathbb{R}^n$. Then, for all multi-indices β, γ and $0 < \lambda < 1$, we have

$$\begin{aligned} |\partial_\xi^\beta \partial_\eta^\gamma \sigma_\lambda(\xi, \eta)| &= \lambda^{|\beta|+|\gamma|} |\partial_\xi^\beta \partial_\eta^\gamma \sigma(\lambda\xi, \lambda\eta)| \\ &\leq \lambda^{(1-\rho)(|\beta|+|\gamma|)} C_{\beta,\gamma}(\sigma) \langle \xi, \eta \rangle^{-\rho(|\beta|+|\gamma|)}, \end{aligned}$$

so that

$$C_{\beta,\gamma}(\sigma_\lambda) \leq \lambda^{(1-\rho)(|\beta|+|\gamma|)} C_{\beta,\gamma}(\sigma). \quad (3.3.14)$$

Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and define $f_{\lambda^{-1}}(x) := f(\frac{x}{\lambda})$ and $g_{\lambda^{-1}}(x) := g(\frac{x}{\lambda})$ for $x \in \mathbb{R}^n$. It easily follows that

$$T_\sigma(f, g)(x) = T_{\sigma_\lambda}(f_{\lambda^{-1}}, g_{\lambda^{-1}})(\lambda x).$$

By Lemma 3.3.3 there exist $s \in \mathbb{N}_0$ such that

$$\|T_{\sigma_\lambda}\| \lesssim \sup_{|\beta|, |\gamma| \leq s} C_{\beta,\gamma}(\sigma_\lambda),$$

with the implicit constant independent of σ and λ . The above along with the fact that

$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and (3.3.14) allow to obtain

$$\begin{aligned}
\|T_\sigma(f, g)\|_{L^p} &= \|T_{\sigma_\lambda}(f_{\lambda^{-1}}, g_{\lambda^{-1}})(\lambda \cdot)\|_{L^p} = \lambda^{-\frac{n}{p}} \|T_{\sigma_\lambda}(f_{\lambda^{-1}}, g_{\lambda^{-1}})\|_{L^p} \\
&\lesssim \lambda^{-\frac{n}{p}} \left(\sup_{|\beta|, |\gamma| \leq s} C_{\beta, \gamma}(\sigma_\lambda) \right) \|f_{\lambda^{-1}}\|_{L^{p_1}} \|g_{\lambda^{-1}}\|_{L^{p_2}} \\
&= \lambda^{-\frac{n}{p} + \frac{n}{p_1} + \frac{n}{p_2}} \left(\sup_{|\beta|, |\gamma| \leq s} C_{\beta, \gamma}(\sigma_\lambda) \right) \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \\
&\lesssim \left(\sup_{|\beta|, |\gamma| \leq s} \lambda^{(1-\rho)(|\beta|+|\gamma|)} C_{\beta, \gamma}(\sigma) \right) \|f\|_{L^{p_1}} \|g\|_{L^{p_2}},
\end{aligned}$$

and letting $\lambda \rightarrow 0$, it follows that

$$\|T_\sigma(f, g)\|_{L^p} \lesssim C_{0,0}(\sigma) \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}, \quad f, g \in \mathcal{S}(\mathbb{R}^n). \quad (3.3.15)$$

However, (3.3.15) cannot be true since it contradicts the fact that there are symbols in $BS_{0,0}^0$ that give rise to unbounded operators. Indeed, take an x -independent symbol σ in $BS_{0,0}^0$ such that T_σ is not bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ and let φ be an infinitely differentiable function in \mathbb{R}^{2n} supported in $|(\xi, \eta)| \leq 2$ and equal to one on $|(\xi, \eta)| \leq 1$. For each $\epsilon > 0$, set $\sigma^\epsilon(\xi, \eta) := \varphi(\epsilon\xi, \epsilon\eta)\sigma(\xi, \eta)$. Then $\sigma^\epsilon \in BS_{\rho, \delta}^0(\mathbb{R}^n)$ and $C_{0,0}(\sigma^\epsilon) \leq C_{0,0}(\sigma)$ for all $\epsilon > 0$. If (3.3.15) were true we would have

$$\|T_{\sigma^\epsilon}(f, g)\|_{L^p} \lesssim C_{0,0}(\sigma) \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and for all $\epsilon > 0$. But then as $\epsilon \rightarrow 0$, $T_{\sigma^\epsilon}(f, g) \rightarrow T_\sigma(f, g)$ pointwise. This fact, together with Fatou's Lemma implies

$$\|T_\sigma(f, g)\|_{L^p} \lesssim C_{0,0}(\sigma) \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, which is a contradiction. □

3.3.2 Symbols of order m and Lebesgue spaces

With the results that we have established for symbols of order zero, we next turn our attention to symbols of order m . We first observe that since the Hörmander classes of order zero are contained in the Hörmander classes of positive order, in view of the results of Section 3.3.1, the question of boundedness in Lebesgue spaces for operators with symbols in $BS_{\rho,\delta}^m$ is only interesting when $m < 0$ and $0 \leq \delta \leq \rho \leq 1$ or when $m \in \mathbb{R}$, $\rho = 1$ and $0 \leq \delta < 1$.

Given $1 \leq p_1, p_2 \leq \infty$ and $\frac{1}{2} \leq p \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, set

$$m(p_1, p_2, \rho) := n(\rho - 1) \left(\max \left\{ \frac{1}{2}, \frac{1}{p_1}, \frac{1}{p_2}, 1 - \frac{1}{p} \right\} + \frac{1}{2} \max \left\{ \frac{1}{p} - 1, 0 \right\} \right).$$

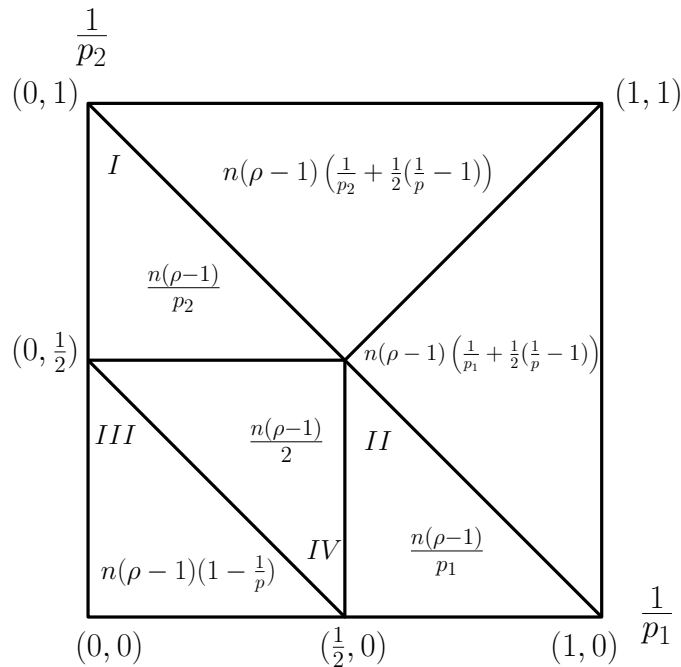


Figure 3.1: Visualization of $m(p_1, p_2, \rho)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 \leq p_1, p_2 \leq \infty$.

The value $m(p_1, p_2, \rho)$ plays an important role in the theory and the specific value depending on p_1 and p_2 is given in Figure 3.1. We will see that operators with symbols in $BS_{\rho,\delta}^m$ with $m < m(p_1, p_2, \rho)$ are bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. Furthermore, if

$1 \leq p, p_1, p_2 \leq \infty$ then operators with symbols of order $m > m(p_1, p_2, \rho)$ are not necessarily bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. Because of this, when $1 \leq p_1, p_2, p \leq \infty$, we call $m(p_1, p_2, \rho)$ a *critical order* for boundedness from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. The statements of these results are made precise in Theorems 3.3.4 and 3.3.5.

The following theorem states sufficient conditions for boundedness in terms of the order of the Hörmander classes.

Theorem 3.3.4. *Let $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, $1 \leq p_1, p_2 \leq \infty$, and $\frac{1}{2} \leq p \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. If $m < m(p_1, p_2, \rho)$ there exist $s_1, s_2 \in \mathbb{N}_0$ such that*

$$\|T_\sigma(f, g)\|_{L^p} \lesssim \|\sigma\|_{s_1, s_2} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and all $\sigma \in BS_{\rho, \delta}^m$.

Theorem 3.3.4 as it now stands is the product of several authors. A partial result in the spirit of Theorem 3.3.4 was proved by Michalowski, Rule, and Staubach in [28], which includes the boundedness from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ for operators with symbols in the classes of order $m < m(2, 2, \rho)$. The statement for boundedness in the case when $p \geq 1$ comes from the work of Bényi, Bernicot, et al. in [3]; while the case for $p < 1$ relies on the work of Rodríguez-López and Staubach in [35].

The following result by Miyachi and Tomita [30] complements Theorem 3.3.4 in that it includes necessary conditions in terms of the order m for the operators with symbols in $BS_{\rho, \delta}^m$ to be bounded when $1 \leq p, p_1, p_2 \leq \infty$. Moreover, Theorem 3.3.5 also indicates that for $\rho = \delta = 0$, $m = m(p_1, p_2, 0)$ is also sufficient for boundedness from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.

Theorem 3.3.5. *Assume $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ with the ranges of p_1, p_2 and p as indicated below.*

- (i) *Let $1 \leq p_1, p_2, p \leq \infty$ and $0 \leq \rho < 1$. If every operator T_σ with $\sigma \in BS_{\rho, \rho}^m$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, then $m \leq m(p_1, p_2, \rho)$.*

(ii) Let $1 < p_1, p_2 \leq \infty$ and $1 \leq p < \infty$. Every operator T_σ with $\sigma \in BS_{0,0}^m$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ if and only if $m \leq m(p_1, p_2, 0)$.

Remark 3.3.6. The result of Miyachi and Tomita in [30] in regards to item (ii) of Theorem 3.3.5 is broader in the sense that it also includes indices p_1, p_2 and p below. However, their result when any of the indices p_1, p_2 and p are less than or equal to one is a statement about the boundedness of bilinear pseudodifferential operators on Hardy spaces. We postpone to Section 3.3.3 precise statements of these facts along with other results corresponding to the situation $p_1 = p_2 = p = \infty$ due to Bényi, Bernicot, et al. in [3] and Naibo in [34].

The rest of this section is devoted to the proof of Theorem 3.3.4 for the case $p \geq 1$ though we note that the proof for boundedness for $p < 1$ is similar in spirit to the proof of Michalowski et al. [28] that we present for $p_1 = p_2 = 2$. We refer the reader to [30] for a proof of Theorem 3.3.5.

The proof of Theorem 3.3.4 for $p \geq 1$.

Before proceeding to the proof of Theorem 3.3.4, we first state two intermediary results and a definition. We start with a lemma which will prove useful in obtaining boundedness from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and is from the work of Bényi, Bernicot, et al. in [3].

Lemma 3.3.7. *Let $m \in \mathbb{R}$, $0 \leq \delta, \rho \leq 1$, $\sigma \in BS_{\rho,\delta}^m$ and $s \in \mathbb{N}_0$ with s even and $s > 2n$.*

(i) *If $0 < R \leq 1$ and $\text{supp}(\sigma) \subset \{(x, \xi, \eta) : |\xi| + |\eta| \leq R\}$ then*

$$\|T_\sigma(f, g)\|_{L^\infty} \lesssim R^{2n} \|\sigma\|_{0,s} \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad f, g \in L^\infty(\mathbb{R}^n).$$

(ii) *If $R \geq 1$ and $\text{supp}(\sigma) \subset \{(x, \xi, \eta) : R \leq |\xi| + |\eta| \leq 4R\}$ then*

$$\|T_\sigma(f, g)\|_{L^\infty} \lesssim R^{m+n(1-\rho)} \|\sigma\|_{0,s} \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad f, g \in L^\infty(\mathbb{R}^n).$$

Sometimes, as is the case in the proof of Theorem 3.3.4, we fix one of the functions f or g and treat the bilinear operator as a linear one. In such a situation, Lemma 3.3.9 from [28] will be useful. Here we consider a variation of the linear Hörmander classes $S_{\rho,0}^m$ which we will denote $L^p S_{\rho}^m$.

Definition 3.3.8. Let $1 \leq p \leq \infty$, $m \in \mathbb{R}$, and $0 \leq \rho \leq 1$. A symbol $\sigma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ belongs to the class $L^p S_{\rho}^m$ if for every $\alpha \in \mathbb{N}_0^n$,

$$\sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-m+\rho|\alpha|} \left\| \partial_{\xi}^{\alpha} \sigma(\cdot, \xi) \right\|_{L^p} < \infty.$$

With this definition we can now state the following:

Lemma 3.3.9. *Let $0 \leq \rho \leq 1$, $1 \leq p < \infty$, $1 < p_1 \leq \infty$ and $2 \leq p_2 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, and $m < \frac{n(\rho-1)}{p_2'}$ where $\frac{1}{p_2} + \frac{1}{p_2'} = 1$. Then there exists $l \in \mathbb{N}_0$ such that*

$$\|T_{\sigma}(f)\|_{L^p} \lesssim \sup_{|\alpha| \leq l} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-m+\rho|\alpha|} \left\| \partial_{\xi}^{\alpha} \sigma(\cdot, \xi) \right\|_{L^{p_2}} \|f\|_{L^{p_1}},$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all $\sigma \in L^{p_2} S_{\rho}^m$.

With these lemmas in mind, we are ready to prove the case $p \geq 1$ of Theorem 3.3.4. For $p_1 = p_2 = 2$ we follow the work of Michalowski et al. [28], while the remainder of the details come from Bényi, Bernicot, et al. [3].

Proof of Theorem 3.3.4. First, we begin with the case $p = p_1 = p_2 = \infty$ which corresponds to the point $(0, 0)$ in Figure 3.1. Next we treat the boundedness from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ corresponding to the point $(1/2, 1/2)$. Once these two cases are established, we will then use symbolic calculus, duality and complex interpolation to complete the proof of the theorem for the range $1 \leq p \leq \infty$.

For $p = p_1 = p_2 = \infty$ we have that $m(p_1, p_2, \rho) = n(\rho - 1)$. Let $m < n(\rho - 1)$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$ and $\{\psi_j\}_{j \in \mathbb{N}_0}$ be a Littlewood-Paley partition of unity as in (A.3.3) with

$N = 2n$ and $\xi, \eta \in \mathbb{R}^n$. We decompose the symbol $\sigma(x, \xi, \eta)$ as

$$\sigma(x, \xi, \eta) = \sum_{j=0}^{\infty} \sigma_j(x, \xi, \eta), \quad (3.3.16)$$

where $\sigma_j(x, \xi, \eta) := \sigma(x, \xi, \eta)\psi_j(\xi, \eta)$.

From Lemma 3.3.7 with $R = 2^j$ and $s \in \mathbb{N}_0$ with $s > 2n$ we get that

$$\begin{aligned} \|T_{\sigma_j}(f, g)\|_{L^\infty} &\lesssim 2^{j(m+n(1-\rho))} \|\sigma\|_{0,s} \|f\|_{L^\infty} \|g\|_{L^\infty} \\ &\lesssim 2^{j(m+n(1-\rho))} \|\sigma\|_{0,s} \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad f, g \in \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

Therefore

$$\begin{aligned} \|T_\sigma(f, g)\|_{L^\infty} &\leq \sum_{j=0}^{\infty} \|T_{\sigma_j}(f, g)\|_{L^\infty} \\ &\lesssim \|\sigma\|_{0,s} \sum_{j=0}^{\infty} 2^{j(m+n(1-\rho))} \|f\|_{L^\infty} \|g\|_{L^\infty} \\ &\lesssim \|\sigma\|_{0,s} \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad f, g \in \mathcal{S}(\mathbb{R}^n), \end{aligned}$$

where we have used that $m < n(\rho - 1)$ proving this case.

We next prove boundedness when $p_1 = p_2 = 2$. Since $m(2, 2, \rho) = \frac{n}{2}(\rho - 1)$, we then consider symbols $\sigma \in BS_{\rho, \delta}^m$ with $m < \frac{n}{2}(\rho - 1)$. Once again we use a partition of unity $\{\varphi_k\}_{k \geq 0}$ as in (A.3.3), but with $N = n$, and decompose the symbol as

$$\sigma(x, \xi, \eta) = \sum_{j,k=0}^{\infty} \sigma_{j,k}(x, \xi, \eta), \quad (3.3.17)$$

with $\sigma_{j,k}(x, \xi, \eta) := \sigma(x, \xi, \eta)\varphi_j(\xi)\varphi_k(\eta)$. Considering the bilinear operator with symbol $\sigma_{j,k}$

we have, for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} T_{\sigma_{j,k}}(f, g)(x) &= \int_{\mathbb{R}^{2n}} \sigma_{j,k}(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \sigma_{j,k}(x, \xi, \eta) \hat{g}(\eta) e^{2\pi i x \cdot \eta} d\eta \right) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi. \end{aligned} \quad (3.3.18)$$

We set $S_{j,k}(g; x, \xi) := \int_{\mathbb{R}^n} \sigma_{j,k}(x, \xi, \eta) \hat{g}(\eta) e^{2\pi i x \cdot \eta} d\eta$, which is a linear pseudodifferential operator for each fixed ξ with symbol $\sigma_{j,k}(x, \xi, \eta)$ as a function of x and η . We also consider, for each fixed $g \in \mathcal{S}(\mathbb{R}^n)$, the linear pseudodifferential operator with symbol $S_{j,k}(g; x, \xi)$, this is,

$$T_{S_{j,k}(g; \cdot, \cdot)}(f)(x) := \int_{\mathbb{R}^n} S_{j,k}(g; x, \xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

We now analyze the symbols $\sigma_{j,k}$ according to the relation between j and k . First, we consider the case when $j \leq k$ and let (ξ, η) be in the support of $\sigma_{j,k}$. If $j = k = 0$ then both $|\xi| \lesssim 1$ and $|\eta| \lesssim 1$, and if $j = 0$ but $k > 0$ then $|\xi| \lesssim 1$ while $|\eta| \sim 2^k$; for the remaining values of the indices j and k we have that $2^j \sim |\xi| \lesssim |\eta| \sim 2^k$. Fixing $\epsilon > 0$ sufficiently small, we conclude that

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_{j,k}(x, \xi, \eta)| &\leq C_{\alpha, \beta, \gamma} (2^j + 2^k)^{m + \delta|\alpha| - \rho(|\beta| + |\gamma|)} \\ &\leq C_{\alpha, \beta, \gamma} 2^{j(m_1 - \epsilon - \rho|\beta|)} 2^{k(m_2 - \epsilon + \delta|\alpha| - \rho|\gamma|)}, \quad x, \xi, \eta \in \mathbb{R}^n, \end{aligned}$$

for $m_1, m_2 \leq 0$ such that $m_1 + m_2 = m + 2\epsilon$. Note that the above inequality tells that $\partial_\xi^\beta \sigma_{j,k}(x, \xi, \eta)$ belongs to the Hörmander class $S_{\rho, \delta}^{m_2}$ as a function of x and η and for each ξ fixed. Since $m_2 \leq 0$, linear operators with symbols in the class $S_{\rho, \delta}^{m_2}$ are bounded on L^2 (see Theorem 2.3.1) and therefore,

$$\left\| \partial_\xi^\beta S_{j,k}(g; \cdot, \xi) \right\|_{L^2} \lesssim 2^{j(m_1 - \rho|\beta| - \epsilon)} 2^{-k\epsilon} \|g\|_{L^2}$$

so that $S_{j,k}(g; \cdot, \cdot) \in L^2 S_\rho^{m_1}$. Therefore an application of Lemma 3.3.9 with the assumption

$m_1 < \frac{n}{2}(\rho - 1)$ yields

$$\|T_{\sigma_{j,k}}(f, g)\|_{L^1} \lesssim 2^{-j\epsilon} 2^{-k\epsilon} \|f\|_{L^2} \|g\|_{L^2}. \quad (3.3.19)$$

For the terms such that $k < j$, we can repeat the same argument reversing the roles of ξ and η . Putting these two arguments together and summing in j and k we obtain boundedness from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ provided $m < \frac{n}{2}(\rho - 1)$.

We next use the boundedness just proved corresponding to the points $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ in Figure 3.1 along with symbolic calculus, duality, and interpolation to complete the proof of the theorem in the case $1 \leq p \leq \infty$.

The mapping property from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ for every operator with symbol in $BS_{\rho,\delta}^m$ where $m < n(\rho - 1)$ and part of the reasoning from Remark 3.2.3, based on duality and the symbolic calculus of the bilinear Hörmander classes, give:

- (a) T_σ is bounded from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho,\delta}^m$ with $m < n(\rho - 1)$,
- (b) T_σ is bounded from $L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho,\delta}^m$ with $m < n(\rho - 1)$,
- (c) T_σ is bounded from $L^\infty(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho,\delta}^m$ with $m < n(\rho - 1)$.

Similarly, the boundedness from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ for every operator with symbol in $BS_{\rho,\delta}^m$ where $m < \frac{n(\rho-1)}{2}$ implies:

- (d) T_σ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho,\delta}^m$ with $m < \frac{n(\rho-1)}{2}$,
- (e) T_σ is bounded from $L^\infty(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho,\delta}^m$ with $m < \frac{n(\rho-1)}{2}$,
- (f) T_σ is bounded from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho,\delta}^m$ with $m < \frac{n(\rho-1)}{2}$.

Summarizing, we have so far proved the result for $m < m(p_1, p_2, \rho)$ with p_1 and p_2 corresponding to the points $(0, 0)$, $(1, 0)$, $(0, 1)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{2})$ in Figure 3.1.

The boundedness from $L^\infty(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$ for all $\sigma \in BS_{\rho,\delta}^m$ with $m < m(\infty, p_2, \rho)$ and $2 < p_2 < \infty$, which corresponds to the segment from $(0, 0)$ to $(0, \frac{1}{2})$ in Figure 3.1, is achieved by looking at the operator $T_\sigma(f, g)$ as a trilinear operator of σ , f and

g and using trilinear complex interpolation. Indeed, this follows from [7, Theorem 4.4.1] and the facts that if $\theta \in (0, 1)$, $m = (1 - \theta)m_1 + \theta m_2$ and $\frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}$, where $m_1, m_2 \in \mathbb{R}$ and $1 \leq r, r_1, r_2 \leq \infty$, the complex interpolation method gives

$$(L^{r_1}(\mathbb{R}^n), L^{r_2}(\mathbb{R}^n))_{[\theta]} = L^r(\mathbb{R}^n) \quad \text{and} \quad (BS_{\rho, \delta}^{m_1}, BS_{\rho, \delta}^{m_2})_{[\theta]} = BS_{\rho, \delta}^m.$$

The first fact is well known (see for instance [7, Theorem 5.1.1]) while the second fact, for which an appropriate norm is assumed, corresponds to [3, Lemma 2.7]. An analogous reasoning applies to the segments from $(0, \frac{1}{2})$ to $(0, 1)$, from $(0, 1)$ to $(\frac{1}{2}, \frac{1}{2})$, from $(\frac{1}{2}, \frac{1}{2})$ to $(1, 0)$, from $(1, 0)$ to $(\frac{1}{2}, 0)$ and from $(\frac{1}{2}, 0)$ to $(0, 0)$.

Finally, we use bilinear complex interpolation to get the result for p_1 and p_2 such that $(\frac{1}{p_1}, \frac{1}{p_2})$ is in the interior of one of the four smaller triangular regions in Figure 3.1, taking into account that $m(p_1, p_2, \rho)$ is constant along horizontal segments in region I , $m(p_1, p_2, \rho)$ is constant along vertical segments in region II , $m(p_1, p_2, \rho)$ is constant along diagonal segments in region III , and $m(p_1, p_2, \rho)$ is constant in region IV .

□

To finish the proof of Theorem 3.3.4 we now include the proofs of Lemma 3.3.7 and Lemma 3.3.9.

Proof of Lemma 3.3.7. We make use of the distributional kernel $K(x, y, z)$ with $K(x, y, z) = k(x, x - y, x - z)$ and $k(x, y, z) = \mathcal{F}^{-1}(\sigma(x, \cdot, \cdot))(y, z)$. Then

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} K(x, y, z) f(y) g(z) dy dz, \quad x \in \mathbb{R}^n.$$

For part (i), it is enough to show that for $s \in \mathbb{N}$ with s even and $s > 2n$

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^{2n}} |k(x, y, z)| dy dz \lesssim R^{2n} \|\sigma\|_{0, s}. \quad (3.3.20)$$

Let $s = 2t$ with $t \in \mathbb{N}_0$ and $t > n$. By assumption, σ is smooth with compact support in ξ and η so that

$$\begin{aligned} (1 + 4\pi^2|(y, z)|^2)^t k(x, y, z) &= \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) (1 - \Delta_\xi - \Delta_\eta)^t (e^{2\pi i \xi \cdot y} e^{2\pi i \eta \cdot z}) d\xi d\eta \\ &= \int_{\mathbb{R}^{2n}} (1 - \Delta_\xi - \Delta_\eta)^t (\sigma(x, \xi, \eta)) e^{2\pi i \xi \cdot y} e^{2\pi i \eta \cdot z} d\xi d\eta \end{aligned} \quad (3.3.21)$$

Since $\sigma \in BS_{\rho, \delta}^m$, $|(1 - \Delta_\xi - \Delta_\eta)^t \sigma(x, \xi, \eta)| \leq \|\sigma\|_{0, s} \langle \xi, \eta \rangle^{m-s\rho}$ for $x, \xi, \eta \in \mathbb{R}^n$. Because we are integrating over the domain $|\xi| + |\eta| \leq R$ and by hypothesis $R \leq 1$ we use that $\langle \xi, \eta \rangle = (1 + |\xi| + |\eta|) \sim 1$ to get

$$(1 + 4\pi^2|(y, z)|^2)^t |k(x, y, z)| \leq \|\sigma\|_{0, s} \int_{|\xi|+|\eta|\leq R} d\xi d\eta \sim \|\sigma\|_{0, s} R^{2n}, \quad x, y, z \in \mathbb{R}^n.$$

From this calculation we conclude that

$$|k(x, y, z)| \lesssim \frac{R^{2n} \|\sigma\|_{0, s}}{(1 + 4\pi^2|(y, z)|^2)^t}, \quad x, y, z \in \mathbb{R}^n,$$

so that (3.3.20) follows since $t > n$, proving part (i).

For part (ii) we again use the distributional kernel form of the operator T_σ and note that it is enough to show that if $s = 2t$ with $t > n$ and $t \in \mathbb{N}_0$ then

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^{2n}} |k(x, y, z)| dy dz \lesssim R^{m+n(1-\rho)} \|\sigma\|_{0, s}. \quad (3.3.22)$$

We first split the integral as follows:

$$\int_{\mathbb{R}^{2n}} |k(x, y, z)| dy dz = \int_{|y|+|z|\leq R^{-\rho}} |k(x, y, z)| dy dz + \int_{|y|+|z|\geq R^{-\rho}} |k(x, y, z)| dy dz =: I_1 + I_2.$$

In order to estimate I_1 , we use the Cauchy-Schwarz inequality, Plancherel's identity and the

fact that $R \geq 1$ to get

$$\begin{aligned}
I_1^2 &\lesssim R^{-2\rho n} \int_{|y|+|z|\leq R^{-\rho}} |k(x, y, z)|^2 dy dz \\
&\lesssim R^{-2\rho n} \int_{|\xi|+|\eta|\sim R} |\sigma(x, \xi, \eta)|^2 d\xi d\eta \\
&\lesssim \|\sigma\|_{0,0}^2 R^{-2\rho n} \int_{|\xi|+|\eta|\sim R} (1 + |\xi| + |\eta|)^{2m} d\xi d\eta \\
&\lesssim \|\sigma\|_{0,0}^2 R^{-2\rho n} R^{2m+2n} = \|\sigma\|_{0,0}^2 R^{2(m+n(1-\rho))}.
\end{aligned}$$

For the second integral I_2 , we first multiply and divide by $(2\pi|(y, z)|)^{2t}$ and then use the Cauchy-Schwarz inequality, that $t > n$, the equality

$$(2\pi|(y, z)|)^{2t} k(x, y, z) = \mathcal{F}_{2n}^{-1}((-\Delta_\xi - \Delta_\eta)^t(\sigma(x, \cdot, \cdot)))(y, z),$$

Plancherel's identity, and that $R \geq 1$ to get the estimate:

$$\begin{aligned}
I_2^2 &\lesssim \left(\int_{|y|+|z|\geq R^{-\rho}} \frac{1}{(2\pi|(y, z)|)^{4t}} dy dz \right) \left(\int_{|y|+|z|\geq R^{-\rho}} |(2\pi|(y, z)|)^{2t} k(x, y, z)|^2 dy dz \right) \\
&\lesssim R^{\rho(4t-2n)} \int_{|\xi|+|\eta|\sim R} |(-\Delta_\xi - \Delta_\eta)^t \sigma(x, \xi, \eta)|^2 d\xi d\eta \\
&\lesssim \|\sigma\|_{0,s}^2 R^{\rho 2(s-n)} \int_{|\xi|+|\eta|\sim R} (1 + |\xi| + |\eta|)^{2(m-\rho s)} d\xi d\eta \\
&\lesssim \|\sigma\|_{0,s}^2 R^{\rho 2(s-n)} R^{2(m-\rho s+n)} = \|\sigma\|_{0,s}^2 R^{2(m+n(1-\rho))}.
\end{aligned}$$

Finally the estimates for I_1 and I_2 yield (3.3.22) completing the proof of part (ii). \square

For the proof of Lemma 3.3.9 we use the following well-known result (see for instance [17, Theorem 2.1.10]).

Proposition 3.3.10. *Suppose that $\phi : \mathbb{R}^n \rightarrow [0, \infty)$ is integrable, non-increasing, and radial and, for $f \in L^1_{loc}(\mathbb{R}^n)$, consider the Hardy-Littlewood maximal function*

$$M(f)(x) = \sup_{x \in B} \left(\frac{1}{|B|} \int_B |f(y)| dy \right), \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all Euclidean balls B in \mathbb{R}^n containing x . Then

$$\int_{\mathbb{R}^n} \phi(y) |f(x-y)| dy \leq \|\phi\|_{L^1} M(f)(x), \quad x \in \mathbb{R}^n.$$

Proof of Lemma 3.3.9. Given $l \in \mathbb{N}$, let

$$\tau_j(y) := \begin{cases} 2^{-j\rho n/p'_2}, & |y| \leq 2^{-j\rho}; \\ 2^{-j\rho(n/p'_2-l)} |y|^l, & |y| > 2^{-j\rho}. \end{cases}$$

Then for l sufficiently large we have that

$$\left(\int_{\mathbb{R}^n} |\tau_j(y)|^{-p'_2} dy \right)^{\frac{1}{p'_2}} \lesssim 1. \quad (3.3.23)$$

For a Littlewood-Paley partition of unity in \mathbb{R}^n , $\{\varphi_j\}_{j \in \mathbb{N}}$, we let $\sigma_j(x, \xi) := \sigma(x, \xi) \varphi_j(\xi)$ and study the operators T_{σ_j} . Setting $K_j(x, y) := \mathcal{F}^{-1}(\sigma_j(x, \cdot))(y)$ we have

$$\begin{aligned} |T_{\sigma_j} f(x)| &= \left| \int_{\mathbb{R}^n} K_j(x, y) f(x-y) dy \right| \\ &\leq \left(\int_{\mathbb{R}^n} |K_j(x, y) \tau_j(y)|^{p_2} dy \right)^{\frac{1}{p_2}} \left(\int_{\mathbb{R}^n} \left| \frac{f(x-y)}{\tau_j(y)} \right|^{p'_2} dy \right)^{\frac{1}{p'_2}} \\ &\lesssim \sum_{|\alpha| \leq l} 2^{-j(n\rho/p'_2 - |\alpha|\rho)} \left(\int_{\mathbb{R}^n} |\partial_\xi^\alpha \sigma_j(x, \xi)|^{p'_2} d\xi \right)^{\frac{1}{p'_2}} \left(\int_{\mathbb{R}^n} \left| \frac{f(x-y)}{\tau_j(y)} \right|^{p'_2} dy \right)^{\frac{1}{p'_2}} \quad (3.3.24) \\ &\lesssim \sum_{|\alpha| \leq l} 2^{-j(n\rho/p'_2 - |\alpha|\rho)} \left(\int_{\mathbb{R}^n} |\partial_\xi^\alpha \sigma_j(x, \xi)|^{p'_2} d\xi \right)^{\frac{1}{p'_2}} \left(M(f^{p'_2})(x) \right)^{\frac{1}{p'_2}}, \end{aligned}$$

where we have used Hölder's inequality, the Hausdorff-Young inequality, Proposition 3.3.10 and (3.3.23). If $p > 1$, the use of Hölder's inequality, the boundedness of the Hardy-Littlewood maximal function, Minkowski's inequality and the fact that $\sigma \in L^{p_2} S_\rho^m$ yield

$$\begin{aligned}
\|T_{\sigma_j}(f)\|_{L^p} &\lesssim \sum_{|\alpha| \leq l} 2^{-j\rho(n\rho/p'_2 - |\alpha|\rho)} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\partial_\xi^\alpha \sigma_j(x, \xi)|^{p'_2} d\xi \right)^{\frac{p}{p'_2}} \left(M(f^{p'_2})(x) \right)^{\frac{p}{p'_2}} dx \right)^{\frac{1}{p}} \\
&\lesssim \sum_{|\alpha| \leq l} 2^{-j(n\rho/p'_2 - |\alpha|\rho)} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\partial_\xi^\alpha \sigma_j(x, \xi)|^{p'_2} d\xi \right)^{\frac{p_2}{p'_2}} dx \right)^{\frac{1}{p_2}} \|f\|_{L^{p_1}} \quad (3.3.25) \\
&\lesssim \sum_{|\alpha| \leq l} 2^{-j(n\rho/p'_2 - |\alpha|\rho)} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\partial_\xi^\alpha \sigma_j(x, \xi)|^{p_2} dx \right)^{\frac{p'_2}{p_2}} d\xi \right)^{\frac{1}{p_2}} \|f\|_{L^{p_1}} \\
&\lesssim 2^{j(m-n(\rho-1)/p'_2)} \sup_{|\alpha| \leq l} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-m+\rho|\alpha|} \|\partial_\xi^\alpha \sigma(\cdot, \xi)\|_{L^{p_2}} \|f\|_{L^{p_1}}
\end{aligned}$$

Summing in j and using that $m < \frac{n(\rho-1)}{p'_2}$ we obtain

$$\|T_\sigma(f)\|_{L^p} \leq \sum_{j=0}^{\infty} \|T_{\sigma_j}(f)\|_{L^p} \lesssim \sup_{|\alpha| \leq l} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-m+\rho|\alpha|} \|\partial_\xi^\alpha \sigma(\cdot, \xi)\|_{L^{p_2}} \|f\|_{L^{p_1}}.$$

If $p = 1$, then $p'_2 = p_1$ and therefore we cannot use the boundedness of the Hardy-Littlewood maximal function in (3.3.25). Instead, we take the $L^1(\mathbb{R}^n)$ norm of the inequality in (3.3.24) and apply Hölder's inequality to get

$$\begin{aligned}
\int_{\mathbb{R}^n} |T_{\sigma_j} f(x)| dx &\lesssim \sum_{|\alpha| \leq l} 2^{-j(n\rho/p'_2 - |\alpha|\rho)} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\partial_\xi^\alpha \sigma_j(x, \xi)|^{p'_2} d\xi \right)^{\frac{p_2}{p'_2}} dx \right)^{\frac{1}{p_2}} \\
&\quad \times \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \frac{f(x-y)}{\tau_j(y)} \right|^{p_1} dy dx \right)^{\frac{1}{p_1}} \\
&\lesssim \sum_{|\alpha| \leq l} 2^{-j(n\rho/p'_2 - |\alpha|\rho)} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\partial_\xi^\alpha \sigma_j(x, \xi)|^{p'_2} d\xi \right)^{\frac{p_2}{p'_2}} dx \right)^{\frac{1}{p_2}} \|f\|_{L^{p_1}},
\end{aligned}$$

where in the last line we have used (3.3.23). We then proceed as in the case $p > 1$. \square

3.3.3 Boundedness on Hardy spaces and BMO

In this section we briefly describe results on boundedness of operators in the case when $p_1 = p_2 = p = \infty$ and the symbols are in the class $BS_{\rho,\delta}^m$ with critical order $m = m(\infty, \infty, \rho) = n(\rho - 1)$, and when $p_1 \leq 1$ or $p_2 \leq 1$ and the symbols belong to various Hörmander classes.

The case $p_1 = p_2 = p = \infty$ for the critical order $m(\infty, \infty, \rho) = n(\rho - 1)$.

The theorems in Section 3.3.2 state, in particular, that operators with symbols in $BS_{\rho,\delta}^m$ are bounded from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ if $m < n(\rho - 1)$, but may fail to be bounded from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ if $m > n(\rho - 1)$. When $m = n(\rho - 1)$ it is expected that boundedness from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ fails for some symbols in $BS_{\delta,\rho}^{n(\rho-1)}$ while it is conjectured that boundedness from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$ holds for every operator with symbol in this critical class. We recall that the space $BMO(\mathbb{R}^n)$, defined in the appendix, contains the space $L^\infty(\mathbb{R}^n)$. The conjecture was proved for $0 < \rho < \frac{1}{2}$ and $\delta = 0$ by Bényi, Bernicot, et al. in [3], for $\delta = \rho = 0$ by Miyachi and Tomita in [30] and for $0 < \delta \leq \rho < \frac{1}{2}$ by Naibo in [34]. More precisely,

Theorem 3.3.11. *If $0 \leq \delta \leq \rho < \frac{1}{2}$ then there exist $s_1, s_2 \in \mathbb{N}_0$ such that*

$$\|T_\sigma(f, g)\|_{BMO} \lesssim \|\sigma\|_{s_1, s_2} \|f\|_{L^\infty} \|g\|_{L^\infty},$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and all $\sigma \in BS_{\rho,\delta}^{n(\rho-1)}$.

We note that Theorem 3.3.11 is a bilinear counterpart of the result by C. Fefferman [15] stated in Theorem 2.3.5.

The case of indices p_1 and p_2 below 1.

To this point, we have only considered boundedness results on Lebesgue spaces for indices $1 \leq p_1, p_2 \leq \infty$. Miyachi and Tomita explored in [30] the cases when p_1 or p_2 are smaller

than 1, which we proceed to present. Define the index $\tilde{m}(p_1, p_2, \rho)$ as

$$\tilde{m}(p_1, p_2, \rho) := n(\rho - 1) \left(\max \left\{ \frac{1}{2}, \frac{1}{p_1}, \frac{1}{p_2}, 1 - \frac{1}{p}, \frac{1}{p} - \frac{1}{2} \right\} \right),$$

where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. It can be easily seen that $\tilde{m}(p_1, p_2, \rho) = m(p_1, p_2, \rho)$ when $p \geq 1$; however when $p < 1$ it holds that $\tilde{m}(p_1, p_2, \rho) \geq m(p_1, p_2, \rho)$ with equality only when $p_1 = p_2$. In order to get this improvement in the index, Miyachi and Tomita considered the Hardy spaces $h^q(\mathbb{R}^n)$ and $H^q(\mathbb{R}^n)$ for $0 < q \leq 1$ rather than Lebesgue spaces. Hardy spaces can be defined for any $0 < q < \infty$ and we refer the reader to the appendix for their definitions. The following relations are well-known (see for instance the books [18, 36]):

$$h^q(\mathbb{R}^n) = H^q(\mathbb{R}^n) = L^q(\mathbb{R}^n), \quad \text{for } 1 < q \leq \infty, \quad (3.3.26)$$

$$H^q(\mathbb{R}^n) \subset h^q(\mathbb{R}^n), \quad \text{for } 0 < q \leq \infty,$$

$$H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n),$$

with continuity in norms. We also consider the Banach space $bmo(\mathbb{R}^n)$ (see appendix for a definition) which satisfies $L^\infty(\mathbb{R}^n) \subset bmo(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ with continuous inclusions. Let

$$X^q(\mathbb{R}^n) := \begin{cases} h^q(\mathbb{R}^n) & \text{if } 0 < q \leq 1, \\ L^q(\mathbb{R}^n) & \text{if } 1 < q < \infty, \\ bmo(\mathbb{R}^n) & \text{if } q = \infty. \end{cases}$$

We are now ready to state the continuity properties of bilinear pseudodifferential operators in this setting, due to Miyachi and Tomita [30]:

Theorem 3.3.12. *Let $m \in \mathbb{R}$ and $0 < p_1, p_2, p \leq \infty$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.*

- (i) *Every operator T_σ with $\sigma \in BS_{0,0}^m$ is bounded from $X^{p_1}(\mathbb{R}^n) \times X^{p_2}(\mathbb{R}^n)$ into $X^p(\mathbb{R}^n)$ if and only if $m \leq \tilde{m}(p_1, p_2, 0)$.*

(ii) For $0 \leq \rho \leq 1$, if every operator T_σ with $\sigma \in BS_{\rho,\rho}^m$ is bounded from $H^{p_1}(\mathbb{R}^n) \times H^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ (with $L^p(\mathbb{R}^n)$ replaced by $BMO(\mathbb{R}^n)$ if $p_1 = p_2 = p = \infty$) then $m \leq \tilde{m}(p_1, p_2, \rho)$.

Note that, in view of (3.3.26), the above theorem covers the results of Theorem 3.3.5.

3.4 Connections to Bilinear Calderón-Zygmund Theory

In this section we mention the connection between bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes and the bilinear Calderón-Zygmund theory.

In order to describe a bilinear Calderón-Zygmund operator, it is first necessary to consider bilinear Calderón-Zygmund (CZ) kernels. Denote by Δ the diagonal of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, that is, $\Delta := \{(x, x, x) : x \in \mathbb{R}^n\}$.

Definition 3.4.1. A bilinear Calderón-Zygmund kernel is a locally integrable function $K(x, y, z)$ defined on $(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta$ that satisfies the size estimate

$$|K(x, y, z)| \lesssim \frac{1}{(|x - y| + |x - z| + |y - z|)^{2n}} \quad (3.4.27)$$

for all $(x, y, z) \in \mathbb{R}^{3n} \setminus \Delta$ and the following regularity conditions for some $\epsilon > 0$:

$$|K(x, y, z) - K(\tilde{x}, y, z)| \lesssim \frac{|x - \tilde{x}|^\epsilon}{(|x - y| + |x - z| + |y - z|)^{2n+\epsilon}} \quad (3.4.28)$$

whenever $|x - \tilde{x}| \leq \frac{1}{2} \max(|x - z|, |x - y|)$,

$$|K(x, y, z) - K(x, \tilde{y}, z)| \lesssim \frac{|y - \tilde{y}|^\epsilon}{(|x - y| + |x - z| + |y - z|)^{2n+\epsilon}} \quad (3.4.29)$$

whenever $|y - \tilde{y}| \leq \frac{1}{2} \max(|x - y|, |y - z|)$, and finally

$$|K(x, y, z) - K(x, y, \tilde{z})| \lesssim \frac{|z - \tilde{z}|^\epsilon}{(|x - y| + |x - z| + |y - z|)^{2n+\epsilon}} \quad (3.4.30)$$

whenever $|z - \tilde{z}| \leq \frac{1}{2} \max(|x - z|, |y - z|)$.

The typical example of a bilinear CZ kernel is a kernel K that in addition to the size estimate (3.4.27) satisfies

$$|\nabla K(x, y, z)| \lesssim \frac{1}{(|x - y| + |x - z| + |y - z|)^{2n+1}},$$

where ∇ denotes the gradient in \mathbb{R}^{3n} .

We are now ready to define a Calderón-Zygmund operator.

Definition 3.4.2. An operator T is a bilinear Calderón-Zygmund operator if

- (i) there exists a bilinear Calderón-Zygmund kernel K such that

$$T(f, g)(x) = \int_{\mathbb{R}^{2n}} K(x, y, z) f(y) g(z) dy dz$$

for $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$ with compact support and $x \notin \text{supp}(f) \cap \text{supp}(g)$;

- (ii) T is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

Condition (ii) in the definition specifically requires that the operator T be bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$; however, $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ could be replaced with $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$, respectively, for some $1 < p_1, p_2, p < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. This is a consequence of Theorem 3.4.3 which states that Calderón-Zygmund operators are bounded on a variety of spaces.

Theorem 3.4.3. *Let T be a bilinear Calderón-Zygmund operator. If $1 \leq p_1, p_2 \leq \infty$ and $\frac{1}{2} \leq p < \infty$, with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, then T satisfies the following statements.*

- (i) If $1 < p_1, p_2$ then T can be extended to a bounded operator from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ where $L^{p_1}(\mathbb{R}^n)$ or $L^{p_2}(\mathbb{R}^n)$ should be replaced by $L_c^\infty(\mathbb{R}^n)$ if $p_1 = \infty$ or $p_2 = \infty$, respectively.
- (ii) If $p_1 = 1$ or $p_2 = 1$, then T can be extended to a bounded operator from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$ where $L^{p_1}(\mathbb{R}^n)$ or $L^{p_2}(\mathbb{R}^n)$ should be replaced by $L_c^\infty(\mathbb{R}^n)$ if $p_1 = \infty$ or $p_2 = \infty$, respectively.
- (iii) T can be extended to a bounded operator from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into BMO .

Not all of the proven results concerning boundedness of Calderón-Zygmund operators are listed here. A more thorough exposition is given by Grafakos and Torres [20].

As was mentioned in Section 3.2, bilinear pseudodifferential operators have an associated distributional kernel on the space domain. The following size and decay estimates for the kernels of pseudodifferential operators with symbols in the bilinear Hörmander classes is due to Bényi, et al. in [4].

Theorem 3.4.4. *Let $\sigma \in BS_{\rho,\delta}^m$, $0 < \rho \leq 1$, $0 \leq \delta < 1$, $m \in \mathbb{R}$. Denote by $K(x, y, z)$ the distributional kernel of the associated bilinear pseudodifferential operator T_σ , this is $K(x, y, z) = \mathcal{F}^{-1}(\sigma(x, \cdot, \cdot))(x - y, x - z)$ for $x, y, z \in \mathbb{R}^n$, and set*

$$S(x, y, z) := |x - y| + |x - z| + |y - z|, \quad x, y, z \in \mathbb{R}^n.$$

- (i) Given $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, there exists $N_0 \in \mathbb{N}_0$ such that for each $N \geq N_0$,

$$\sup_{(x,y,z):S(x,y,z)>0} S(x, y, z)^N |\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x, y, z)| < \infty.$$

- (ii) Suppose that σ has compact support in (ξ, η) uniformly in x . Then K is smooth, and given $\alpha, \beta, \gamma \in \mathbb{N}^n$ and $N_0 \in \mathbb{N}$, there exists $C > 0$ such that for all $x, y, z \in \mathbb{R}^n$ with $S(x, y, z) > 0$

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x, y, z)| \leq C(1 + S(x, y, z))^{-N_0}.$$

(iii) Suppose that $m + M + 2n < 0$ for some $M \in \mathbb{N}_0$. Then K is a bounded continuous function with bounded continuous derivatives of order $\leq M$.

(iv) Suppose that $m + M + 2n = 0$ for some $M \in \mathbb{N}_0$. Then there exists a constant $C > 0$ such that for all $x, y, z, \in \mathbb{R}^n$ with $S(x, y, z) > 0$,

$$\sup_{|\alpha+\beta+\gamma|=M} |\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x, y, z)| \leq C |\log |S(x, y, z)||.$$

(v) Suppose that $m + M + 2n > 0$ for some $M \in \mathbb{N}_0$. Then, given $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, there exists a positive constant C such that for all $x, y, z, \in \mathbb{R}^n$ with $S(x, y, z) > 0$,

$$\sup_{|\alpha+\beta+\gamma|=M} |\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x, y, z)| \leq C S(x, y, z)^{-(m+M+2n)/\rho}.$$

(vi) Suppose that $m + \epsilon + 2n > 0$ for some $\epsilon \in (0, 1)$. Then there exists a positive constant C such that for all $x, y, z, u \in \mathbb{R}^n$ with $S(x, y, z) > 0$ and $|u| \leq S(x, y, z)$,

$$\begin{aligned} & |K(x, y, z) - K(x + u, y, z)| + |K(x, y, z) - K(x, y + u, z)| \\ & + |K(x, y, z) - K(x, y, z + u)| \leq C |u|^\epsilon S(x, y, z)^{-(m+\epsilon+2n)/\rho}. \end{aligned}$$

All constants in the above inequalities depend linearly on $\|\sigma\|_{s_1, s_2}$ for some $s_1, s_2 \in \mathbb{N}_0$.

While not all bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes are Calderón-Zygmund operators, we can now prove that some are by applying Theorem 3.4.4 and Theorem 3.3.4. The content of Theorem 3.4.5 and its proof come from Bényi, Bernicot, et al. in [3].

Theorem 3.4.5. *Let $0 \leq \delta \leq \rho \leq 1$ with $\delta < 1$ and $\rho > 0$, and set $m_{cz} := 2n(\rho - 1)$. If $\sigma \in BS_{\rho, \delta}^m$ and $m < m_{cz}$ then T_σ is a bilinear Calderón-Zygmund operator.*

Proof of Theorem 3.4.5. To prove that the operators T_σ described in the hypothesis are Calderón-Zygmund operators we need to prove that the associated distributional kernel, say $K(x, y, z)$, satisfies both the size and regularity conditions required of a CZ kernel and that the operator is bounded on at least one triple of Lebesgue spaces. Let $\sigma \in BS_{\rho, \delta}^m$ with $m < m_{cz}$. Then $BS_{\rho, \delta}^m \subset BS_{\rho, \delta}^{m_{cz}}$ and from Theorem 3.4.4 part (v) applied to $BS_{\rho, \delta}^{m_{cz}}$, the kernel $K(x, y, z)$ of T_σ satisfies

$$|K(x, y, z)| \lesssim \frac{1}{(|x - y| + |x - z| + |y - z|)^{2n}}.$$

It is enough to prove the regularity conditions for m such that $2n(\rho - 1) - t < m < 2n(\rho - 1) = m_{cz}$ for some sufficiently small positive t . Using this range of m we can find $\epsilon \in (0, 1)$ such that

$$m + 2n + \epsilon > 0 \text{ and } \frac{m + 2n + \epsilon}{\rho} = 2n + \epsilon.$$

Part (vi) of Theorem 3.4.4 yields

$$\begin{aligned} & |K(x, y, z) - K(x + u, y, z)| + |K(x, y, z) - K(x, y + u, z)| \\ & + |K(x, y, z) - K(x, y, z + u)| \lesssim \frac{|u|^\epsilon}{(|x - y| + |x - z| + |y - z|)^{2n + \epsilon}}, \end{aligned}$$

where $|u| \leq |x - y| + |x - z| + |y - z|$.

Finally, since $m < m_{cz} < n(\rho - 1)/2$, Theorem 3.3.4 with $p = 1$ and $p_1 = p_2 = 2$ states that T_σ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

Therefore, T_σ is a bilinear Calderón-Zygmund operator. □

Chapter 4

Bilinear Operators with Symbols in Besov Spaces

4.1 Introduction

When examining the proofs of the boundedness properties for bilinear pseudodifferential operators with symbols in the Hörmander classes, it becomes apparent that the symbols possess more smoothness than is actually required by the proof. In this chapter we examine the effect of relaxing condition (3.2.2) in the definition of the Hörmander classes. We will prove boundedness properties in the setting of Lebesgue spaces for bilinear operators associated to symbols in various Besov spaces of product type and quantify the smoothness of the symbols that is sufficient for boundedness. Furthermore, since the Besov spaces to be studied strictly contain the bilinear Hörmander classes $BS_{0,0}^m$ for $m \in \mathbb{R}$, a connection will be drawn between the new theorems presented in this chapter and the results of Chapter 3. The techniques employed in our treatment of operators with symbols in the Besov spaces will be different from those used for the Hörmander classes. For instance, taking derivatives of the symbols in order to perform integration by parts is no longer allowed for symbols in the Besov classes since the symbols are rough. Important tools in the proofs of these

new results include the demonstration of appropriate estimates and the development of a symbolic calculus for some of the Besov classes along with duality arguments. The results of this chapter appear in Herbert-Naibo [21, 22].

In Section 4.2 we define weighted Besov spaces of product type and explore the connections between these spaces and the Hörmander classes $BS_{0,0}^m$. In Section 4.3 we present new results concerning boundedness properties of bilinear pseudodifferential operators with symbols in the Besov classes. As a byproduct, we are able to quantify the smoothness of the symbols that is sufficient for boundedness in terms of the norms that define the Hörmander classes; these ideas are discussed in Section 4.4. Sections 4.5 and 4.6 contain the details of the proofs of the new results stated in Section 4.3, and in the final section of Chapter 4 we present a summary of the results in this chapter.

4.2 Weighted Besov Spaces of Product Type and Related Classes

In this section we define the classes of symbols of interest to us, we mention some of their properties, and we establish their connection with the Hörmander classes.

Let w and w_0 be functions defined in \mathbb{R}^N which satisfy the following conditions:

$$\begin{aligned} w_0 &\in \mathcal{S}(\mathbb{R}^N), \quad \text{supp}(w_0) \subset \{\xi \in \mathbb{R}^N : |\xi| \leq 2\}, \\ w &\in \mathcal{S}(\mathbb{R}^N), \quad \text{supp}(w) \subset \{\xi \in \mathbb{R}^N : \tfrac{1}{2} \leq |\xi| \leq 2\}, \\ w_k(\xi) &:= w(2^{-k}\xi), \quad k \in \mathbb{N}, \quad \sum_{k=0}^{\infty} w_k(\xi) = 1, \quad \xi \in \mathbb{R}^N. \end{aligned} \tag{4.2.1}$$

This is, $\{w_k\}_{k \in \mathbb{N}_0}$ is a Littlewood-Paley partition of unity in \mathbb{R}^N . Given $\xi, \eta \in \mathbb{R}^n$ we set $\langle \xi, \eta \rangle := 1 + |\xi| + |\eta|$ as in Chapter 3. In the following definitions, the Fourier transform and inverse Fourier transform as well as the L^r norm are taken in \mathbb{R}^{3n} . For $m \in \mathbb{R}$, $0 < r, q \leq \infty$,

and $s \in \mathbb{R} \cup \mathbb{R}^3 \cup \mathbb{R}^{3n}$, we define the Besov spaces $B_{r,q}^{s,m}(\mathbb{R}^{3n})$ as follows:

- Given $s \in \mathbb{R}$ and functions w_0 and w satisfying (4.2.1) with $N = 3n$, $B_{r,q}^{s,m}(\mathbb{R}^{3n})$ denotes the space of complex-valued functions $\sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, such that

$$\|\sigma\|_{B_{r,q}^{s,m}} := \left(\sum_{k \in \mathbb{N}_0} (2^{s \cdot k} \|\langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \hat{\sigma})\|_{L^r})^q \right)^{\frac{1}{q}} < \infty,$$

with the corresponding modification for $q = \infty$. Note that when $m = 0$, this agrees with the usual definition of Besov spaces in \mathbb{R}^{3n} .

- Given $s = (s_1, s_2, s_3) \in \mathbb{R}^3$ and functions w_0 and w satisfying (4.2.1) with $N = n$, $B_{r,q}^{s,m}(\mathbb{R}^{3n})$ denotes the space of complex-valued functions $\sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, such that

$$\|\sigma\|_{B_{r,q}^{s,m}} := \left(\sum_{k \in \mathbb{N}_0^3} (2^{s \cdot k} \|\langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \hat{\sigma})\|_{L^r})^q \right)^{\frac{1}{q}} < \infty,$$

where for $k = (k_1, k_2, k_3)$, $w_k(x, \xi, \eta) := w_{k_1}(x)w_{k_2}(\xi)w_{k_3}(\eta)$, and with the corresponding modification for $q = \infty$.

- Given $s = (s_1, \dots, s_{3n}) \in \mathbb{R}^{3n}$ and functions w_0 and w satisfying (4.2.1) with $N = 1$, $B_{r,q}^{s,m}(\mathbb{R}^{3n})$ denotes the space of complex-valued functions $\sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, such that

$$\|\sigma\|_{B_{r,q}^{s,m}} := \left(\sum_{k \in \mathbb{N}_0^{3n}} (2^{s \cdot k} \|\langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \hat{\sigma})\|_{L^r})^q \right)^{\frac{1}{q}} < \infty,$$

where for $k = (k_1, \dots, k_{3n})$, $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$, and $\eta = (\eta_1, \dots, \eta_n)$, $w_k(x, \xi, \eta) := w_{k_1}(x_1) \cdots w_{k_n}(x_n)w_{k_{n+1}}(\xi_1) \cdots w_{k_{2n}}(\xi_n)w_{k_{2n+1}}(\eta_1) \cdots w_{k_{3n}}(\eta_n)$, with the corresponding modification for $q = \infty$.

It can be proved that, for all s, m, r, q as in the definitions above, the space $B_{r,q}^{s,m}(\mathbb{R}^{3n})$ is independent of the choice of w_0 and w satisfying (4.2.1) and is contained in $\mathcal{S}'(\mathbb{R}^{3n})$, that it

is a quasi-Banach space (Banach space if $1 \leq r, q \leq \infty$), that it contains $\mathcal{S}(\mathbb{R}^{3n})$, and that $\mathcal{S}(\mathbb{R}^{3n})$ is dense if $0 < r, q < \infty$. We refer the reader to Sugimoto [37], where a variety of Besov spaces of product type are defined and many of their properties are presented.

We next define classes of symbols that are closely connected with both $B_{r,q}^{s,m}(\mathbb{R}^{3n})$ and the Hörmander classes $BS_{0,0}^m$. Given $s \in \mathbb{N}_0^k$ where $k = 3$ or $k = 3n$ and $m \in \mathbb{R}$, a complex-valued functions $\sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, belongs to $\mathcal{C}_m^s(\mathbb{R}^{3n})$ if it satisfies one of the following conditions.

- If $s = (s_1, s_2, s_3) \in \mathbb{N}_0^3$:

$$\begin{aligned} & \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma \in \mathcal{C}(\mathbb{R}^{3n}) \text{ for } \alpha, \beta, \gamma \in \mathbb{N}_0^n, |\alpha| \leq s_1, |\beta| \leq s_2, |\gamma| \leq s_3, \text{ and} \\ & \|\sigma\|_{\mathcal{C}_m^s} := \sup_{\substack{|\alpha| \leq s_1 \\ |\beta| \leq s_2, |\gamma| \leq s_3}} \sup_{x, \xi, \eta \in \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \langle \xi, \eta \rangle^{-m} < \infty. \end{aligned} \quad (4.2.2)$$

- If $s = (s_1, \dots, s_{3n}) \in \mathbb{N}_0^{3n}$:

$$\begin{aligned} & \partial_x^{(\alpha_1, \dots, \alpha_n)} \partial_\xi^{(\alpha_{n+1}, \dots, \alpha_{2n})} \partial_\eta^{(\alpha_{2n+1}, \dots, \alpha_{3n})} \sigma \in \mathcal{C}(\mathbb{R}^{3n}) \text{ for } \alpha_j \in \mathbb{N}_0, \alpha_j \leq s_j, j = 1, \dots, 3n, \text{ and} \\ & \|\sigma\|_{\mathcal{C}_m^s} := \sup_{\substack{\alpha_j \leq s_j \\ j=1, \dots, 3n}} \sup_{x, \xi, \eta \in \mathbb{R}^n} |\partial_x^{(\alpha_1, \dots, \alpha_n)} \partial_\xi^{(\alpha_{n+1}, \dots, \alpha_{2n})} \partial_\eta^{(\alpha_{2n+1}, \dots, \alpha_{3n})} \sigma(x, \xi, \eta)| \langle \xi, \eta \rangle^{-m} < \infty. \end{aligned} \quad (4.2.3)$$

We note that $\|\sigma\|_{s_1, s_2} = \|\sigma\|_{\mathcal{C}_m^{(s_1, s_2, s_2)}}$ for any $s_1, s_2 \in \mathbb{N}_0$ and where $\|\sigma\|_{s_1, s_2}$ is as in (3.2.3) with $\rho = \delta = 0$.

The following chain of continuous proper inclusions shows the connection between the classes introduced in this section and the bilinear Hörmander classes:

$$BS_{0,0}^m \subsetneq \mathcal{C}_m^{[s]+1}(\mathbb{R}^{3n}) \subsetneq B_{\infty,1}^{s,m}(\mathbb{R}^{3n}) \subsetneq \mathcal{C}_m^{[s]}(\mathbb{R}^{3n}), \quad (4.2.4)$$

where s has positive components, $[s]$ denotes the vector of the same dimension as s and

components given by the integer parts of the components of s , and adding 1 to a vector means adding 1 to each component of the vector.

The first inclusion in (4.2.4) is straightforward and the rest of the inclusions are a consequence of the following proposition, which will be useful in the proof of some of our results (see [37, Theorems 1.3.2, 1.3.5, and 1.3.9 and Corollary 1.3.1] for a proof). Given two vectors s and \tilde{s} of the same dimension, the notation $s > \tilde{s}$ ($s \geq \tilde{s}$, etc) used below is meant component-wise.

Proposition 4.2.1. (a) *Let $0 < r \leq \infty$, s and \tilde{s} be vectors of real numbers of the same dimension (dimension 1, 3 or $3n$), and $m, \tilde{m} \in \mathbb{R}$. Then following continuous inclusions hold:*

$$(i) B_{r,q}^{s,m}(\mathbb{R}^{3n}) \subset B_{r,\tilde{q}}^{s,\tilde{m}}(\mathbb{R}^{3n}), \text{ if } 0 < q \leq \tilde{q} \leq \infty \text{ and } m \leq \tilde{m};$$

$$(ii) B_{r,q}^{s,m}(\mathbb{R}^{3n}) \subset B_{r,\tilde{q}}^{\tilde{s},m}(\mathbb{R}^{3n}), \text{ if } 0 < q, \tilde{q} \leq \infty \text{ and } \tilde{s} < s \text{ component-wise};$$

$$(iii) \mathcal{C}_m^s(\mathbb{R}^{3n}) \subsetneq B_{\infty,q}^{\tilde{s},m}(\mathbb{R}^{3n}), \text{ if } 0 < q \leq \infty, 0 < \tilde{s} < s \text{ component-wise and } s \text{ has components in } \mathbb{N};$$

$$(iv) B_{\infty,1}^{s,m}(\mathbb{R}^{3n}) \subset \mathcal{C}_m^s(\mathbb{R}^{3n}), \text{ if } s \text{ has components in } \mathbb{N}_0.$$

(b) *If $1 \leq r, \tilde{r} \leq \infty$, $0 < q, \tilde{q} \leq \infty$, s, \tilde{s} are vectors of the same dimension (dimension 1, 3, or $3n$) with positive components, and $m \in \mathbb{R}$, then $B_{r,q}^{s,m}(\mathbb{R}^{3n}) = B_{\tilde{r},\tilde{q}}^{\tilde{s},m}(\mathbb{R}^{3n})$ if and only if $r = \tilde{r}$, $q = \tilde{q}$ and $s = \tilde{s}$.*

(c) *Let $1 \leq r \leq \infty$, $0 < q \leq \infty$, $m \in \mathbb{R}$, $s = (s_1, \dots, s_{3n}) \in \mathbb{R}^{3n}$, $s_k > 0$, $k = 1, \dots, 3n$, $\tilde{s} = (s_1 + \dots + s_n, s_{n+1} + \dots + s_{2n}, s_{2n+1} + \dots + s_{3n})$ and $\tilde{\tilde{s}} = s_1 + \dots + s_{3n}$. Then the following continuous inclusions hold:*

$$B_{r,q}^{\tilde{\tilde{s}},m}(\mathbb{R}^{3n}) \subsetneq B_{r,q}^{\tilde{s},m}(\mathbb{R}^{3n}) \subsetneq B_{r,q}^{s,m}(\mathbb{R}^{3n}).$$

4.3 Boundedness Properties of Bilinear Operators with Symbols in $B_{\infty,q}^{s,m}(\mathbb{R}^{3n})$

In this section we present two of the main results of Chapter 4. We start by recalling the index $m(p_1, p_2, \rho)$ introduced in Section 3.3.2 for the particular case $\rho = 0$, $1 \leq p, p_1, p_2 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$:

$$m(p_1, p_2, 0) = -n \max \left\{ \frac{1}{2}, \frac{1}{p_1}, \frac{1}{p_2}, 1 - \frac{1}{p} \right\}.$$

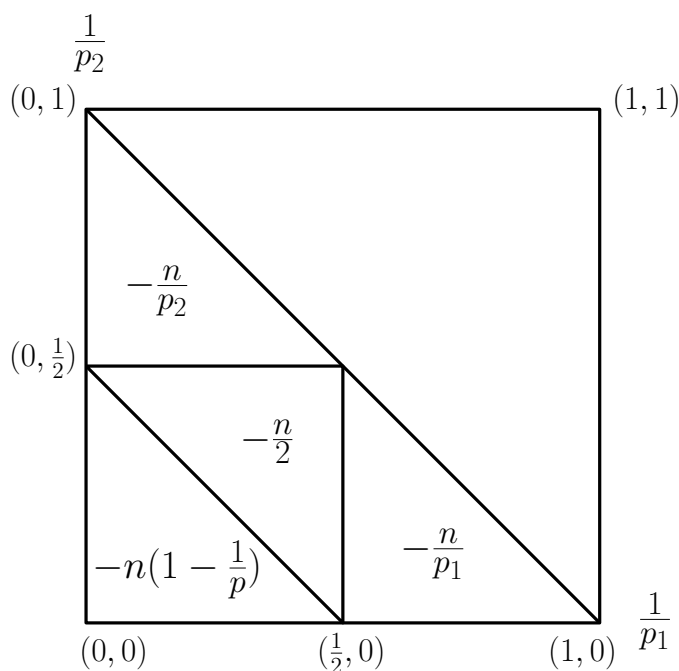


Figure 4.1: Visualization of $m(p_1, p_2, 0)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 \leq p_1, p_2, p \leq \infty$.

Definition 4.3.1. Given $1 \leq p \leq \infty$, $s(p)$ will denote the number $\min(\frac{n}{2}, \frac{n}{p}) + 2 \max(\frac{n}{2}, \frac{n}{p})$, or the 3-dimensional vector $(\min(\frac{n}{2}, \frac{n}{p}), \max(\frac{n}{2}, \frac{n}{p}), \max(\frac{n}{2}, \frac{n}{p}))$, or the $3n$ -dimensional vector (s_1, \dots, s_{3n}) where $s_1 = \dots = s_n = \min(\frac{1}{2}, \frac{1}{p})$ and $s_{n+1} = \dots = s_{3n} = \max(\frac{1}{2}, \frac{1}{p})$. It will be clear from the context which of these definitions of $s(p)$ is being used in each case.

The first theorem we state addresses boundedness for operators with symbols in the classes $B_{\infty,q}^{s,m}(\mathbb{R}^{3n})$ in the setting of Lebesgue spaces with indices larger than or equal to 2.

This range of indices corresponds to the triangle with vertices $(0,0)$, $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ in Figure 4.1.

Theorem 4.3.2 (Herbert-Naibo [21]). *Let $2 \leq p, p_1, p_2 \leq \infty$ be related by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $m < m(p_1, p_2, 0)$, $s(p)$ be as in Definition 4.3.1 and s be a vector of the same dimension as $s(p)$. The following statements hold true:*

(a) *If $0 < q \leq 1$ and $s \geq s(p)$, then*

$$\|T_\sigma(f, g)\|_{L^p} \lesssim \|\sigma\|_{B_{\infty, q}^{s, m}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}},$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and all $\sigma \in B_{\infty, q}^{s, m}(\mathbb{R}^{3n})$.

(b) *If $1 < q \leq \infty$ and $s > s(p)$, then*

$$\|T_\sigma(f, g)\|_{L^p} \lesssim \|\sigma\|_{B_{\infty, q}^{s, m}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}},$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and all $\sigma \in B_{\infty, q}^{s, m}(\mathbb{R}^{3n})$.

One of the main tools for the proof of Theorem 4.3.2 is a new result concerning boundedness of operators with symbols in the classes $\mathcal{C}_m^0(\mathbb{R}^{3n})$ whose Fourier transform is compactly supported. The statement of this result, its proof, and the proof of Theorem 4.3.2 are presented in Section 4.5.

The next result presented in this chapter refers to boundedness properties from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ of bilinear operators with x -independent symbols in the Besov classes.

Theorem 4.3.3 (Herbert-Naibo [22]). *Let $1 \leq p_1, p_2 \leq \infty$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$, $m < m(p_1, p_2, 0)$, $s(1)$ be as in Definition 4.3.1 and s be a vector of the same dimension as $s(1)$. The following statements hold true:*

(a) If $0 < q \leq 1$ and $s \geq s(1)$, then

$$\|T_\sigma(f, g)\|_{L^1} \lesssim \|\sigma\|_{B_{\infty,q}^{s,m}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}},$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and all x -independent symbols σ in $B_{\infty,q}^{s,m}(\mathbb{R}^{3n})$.

(b) If $1 < q \leq \infty$ and $s > s(1)$, then

$$\|T_\sigma(f, g)\|_{L^1} \lesssim \|\sigma\|_{B_{\infty,q}^{s,m}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}},$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and all x -independent symbols σ in $B_{\infty,q}^{s,m}(\mathbb{R}^{3n})$.

The proof of Theorem 4.3.3 is based on a new result regarding a symbolic calculus for the Besov classes of x -independent symbols and on the use of Theorem 4.3.2. The statement of this new result, its proof, and the proof of Theorem 4.3.3 are presented in Section 4.6. Additional results concerning the minimal smoothness conditions for bilinear multipliers in terms of Sobolev regularity were proven by Grafakos, Miyachi, and Tomita [19], Miyachi and Tomita [31], and references therein.

4.4 An Upper Bound on the Number of Derivatives

As a consequence of Theorems 4.3.2 and 4.3.3 we obtain two corollaries for the bilinear Hörmander classes $BS_{0,0}^m$, where m is below the critical order. These corollaries give estimates for the number of derivatives of the symbols needed to exist and satisfy

$$\sup_{x,\xi,\eta \in \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \langle \xi, \eta \rangle^{-m} < \infty, \quad (4.4.5)$$

in order for the corresponding pseudodifferential operator to be bounded on Lebesgue spaces.

We first note that by (4.2.4), we have, in particular, that

$$BS_{0,0}^m \subsetneq \mathcal{C}_m^{[s(p)]+1}(\mathbb{R}^{3n}) \subsetneq B_{\infty,1}^{s(p),m}(\mathbb{R}^{3n}), \quad (4.4.6)$$

where $s(p)$ is as in Definition 4.3.1. The inclusions in (4.4.6) and part (a) of Theorem 4.3.2 for $q = 1$ imply:

Corollary 4.4.1 (Herbert-Naibo [21]). *Let $2 \leq p, p_1, p_2 \leq \infty$ be related by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $m < m(p_1, p_2, 0)$ and $s(p)$ be as in Definition 4.3.1. Then*

$$\|T_\sigma(f, g)\|_{L^p} \lesssim \|\sigma\|_{\mathcal{C}_m^{[s(p)]+1}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and all $\sigma \in \mathcal{C}_m^{[s(p)]+1}(\mathbb{R}^{3n})$.

In addition, (4.4.6) along with part (a) of Theorem 4.3.3 for $q = 1$ give:

Corollary 4.4.2 (Herbert-Naibo [22]). *Let $1 \leq p_1, p_2 \leq \infty$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$, $m < m(p_1, p_2, 0)$ and $s(1)$ be as in Definition 4.3.1. Then*

$$\|T_\sigma(f, g)\|_{L^1} \lesssim \|\sigma\|_{\mathcal{C}_m^{[s(1)]+1}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and all x -independent symbols σ in $\mathcal{C}_m^{[s(1)]+1}(\mathbb{R}^{3n})$.

Remark 4.4.3. Taking into account the value of $[s(p)] + 1$ as a vector in \mathbb{R}, \mathbb{R}^3 , and \mathbb{R}^{3n} , respectively, we remark that $\sigma \in \mathcal{C}_m^{[s(p)]+1}(\mathbb{R}^{3n})$ means that in Corollary 4.4.1, since $p \geq 2$,

$$\sup_{|\alpha+\beta+\gamma| \leq [n(1+\frac{1}{p})]+1} \sup_{x, \xi, \eta \in \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \langle \xi, \eta \rangle^{-m} < \infty \quad (4.4.7)$$

or

$$\sup_{\substack{|\alpha| \leq [\frac{n}{p}]+1 \\ |\beta|, |\gamma| \leq [\frac{n}{2}]+1}} \sup_{x, \xi, \eta \in \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \langle \xi, \eta \rangle^{-m} < \infty \quad (4.4.8)$$

or

$$\sup_{\alpha, \beta, \gamma \in \{0, 1\}^n} \sup_{x, \xi, \eta \in \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \langle \xi, \eta \rangle^{-m} < \infty, \quad (4.4.9)$$

while in Corollary 4.4.2, since the symbols do not depend on x and $p = 1$, we have

$$\sup_{|\beta + \gamma| \leq [\frac{5}{2}n] + 1} \sup_{\xi, \eta \in \mathbb{R}^n} |\partial_\xi^\beta \partial_\eta^\gamma \sigma(\xi, \eta)| \langle \xi, \eta \rangle^{-m} < \infty \quad (4.4.10)$$

or

$$\sup_{|\beta|, |\gamma| \leq n + 1} \sup_{\xi, \eta \in \mathbb{R}^n} |\partial_\xi^\beta \partial_\eta^\gamma \sigma(\xi, \eta)| \langle \xi, \eta \rangle^{-m} < \infty \quad (4.4.11)$$

or

$$\sup_{\beta, \gamma \in \{0, 2\}^n} \sup_{\xi, \eta \in \mathbb{R}^n} |\partial_\xi^\beta \partial_\eta^\gamma \sigma(\xi, \eta)| \langle \xi, \eta \rangle^{-m} < \infty. \quad (4.4.12)$$

Remark 4.4.4. Consider the symbol

$$\sigma(x, \xi, \eta) = (1 + |x|^2)^{-\frac{n}{2p}} e^{-2\pi i x \cdot \xi} e^{-2\pi i x \cdot \eta} e^{-|\xi|^2} e^{-|\eta|^2}, \quad x, \xi, \eta \in \mathbb{R}^n, \quad 0 < p < \infty.$$

Elementary computations show that $\sigma \in \mathcal{C}_m^{(s_1, [\frac{n}{p}])}(\mathbb{R}^{3n})$ for all $s_1 \in \mathbb{N}$ and for all $m \in \mathbb{R}$, where $\mathcal{C}_m^{(s_1, [\frac{n}{p}])}(\mathbb{R}^{3n})$ is defined as $\mathcal{C}_m^{(s_1, [\frac{n}{p}], [\frac{n}{p}])}(\mathbb{R}^{3n})$ but requiring $|\beta + \gamma| \leq [\frac{n}{p}]$ instead of $|\beta|, |\gamma| \leq [\frac{n}{p}]$ (see (4.2.2)). Since $\|T_\sigma(f, g)\|_p = \infty$ for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, then T_σ is not bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, regardless of the values of p_1 and p_2 . At least in the case $p = 2$, this raises the question as to whether the condition (4.4.8) can be changed so that $|\beta + \gamma| \leq [\frac{n}{2}] + 1$ rather than $|\beta|, |\gamma| \leq [\frac{n}{2}] + 1$ is required. This example also hints to the fact that at least $[\frac{n}{p}] + 1$ derivatives with respect to the frequency variables are needed for boundedness in the case $0 < p < 2$ and $m \leq m(p_1, p_2, 0)$.

4.5 Proof of Theorem 4.3.2

As mentioned in Section 4.3, a useful tool in the proof of Theorem 4.3.2 is the boundedness properties for operators with symbols in the subclass of $\mathcal{C}_m^0(\mathbb{R}^{3n})$ whose Fourier transforms have compact support. We present the statement of this result as Theorem 4.5.1 in Section 4.5.1. In Section 4.5.2 a crucial estimate to obtain Theorem 4.5.1 is proved. Finally all the pieces are put together in Section 4.5.3, where the proofs of Theorems 4.5.1 and 4.3.2, respectively, are presented.

4.5.1 Symbols in $\mathcal{C}_m^0(\mathbb{R}^{3n})$ with compactly supported Fourier transforms

We consider the class of symbols $\mathcal{C}_m^0(\mathbb{R}^{3n})$ whose Fourier transforms have compact support. More precisely, let $\sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, be a complex-valued function satisfying

$$\text{supp}(\hat{\sigma}) \subset \prod_{j=1}^{3n} [-r_j, r_j] \quad (4.5.13)$$

for some $1 \leq r_j < \infty$, $j = 1, \dots, 3n$, and

$$\|\sigma\|_{\mathcal{C}_m^0} = \sup_{x, \xi, \eta \in \mathbb{R}^n} |\sigma(x, \xi, \eta)| \langle \xi, \eta \rangle^{-m} < \infty. \quad (4.5.14)$$

Theorem 4.5.1 (Herbert-Naibo [21]). *Let $2 \leq p, p_1, p_2 \leq \infty$ be related by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and let $\sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, be a complex-valued function satisfying (4.5.13) and (4.5.14) for some $1 \leq r_j < \infty$, $j = 1, \dots, 3n$, and some $m < m(p_1, p_2, 0)$, respectively. Then*

$$\|T_\sigma(f, g)\|_{L^p} \lesssim \|\sigma\|_{\mathcal{C}_m^0} (r_1 \cdots r_n)^{\frac{1}{p}} (r_{n+1} \cdots r_{3n})^{\frac{1}{2}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \quad (4.5.15)$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, and with the implicit constant independent of σ and r_j for $j = 1, \dots, 2n$.

A version of Theorem 4.5.1 in the linear case was first proved by Boulkhemair [8] for L^2 boundedness and extended by Tomita [39] for L^p boundedness.

Remark 4.5.2. Let σ satisfy (4.5.13) and (4.5.14) for some $1 \leq r_j < \infty$, $j = 1, \dots, 3n$, and some $m \in \mathbb{R}$. The computations below show that $\sigma \in B_{\infty,1}^{s,m}(\mathbb{R}^{3n})$ for any $s = (s_1, \dots, s_{3n}) \in \mathbb{R}^{3n}$ with positive components and that

$$\|\sigma\|_{B_{\infty,1}^{s,m}} \lesssim r_1^{s_1} \dots r_{3n}^{s_{3n}} \|\sigma\|_{C_m^0}.$$

Indeed, if σ satisfies (4.5.13) and (4.5.14), using the notation in the definition of $B_{\infty,1}^{s,m}(\mathbb{R}^{3n})$ for $s \in \mathbb{R}^{3n}$ given in Section 4.2 we have

$$\|\sigma\|_{B_{\infty,1}^{s,m}} = \sum_{k \in \mathbb{N}_0^{3n}} 2^{s \cdot k} \|\langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \hat{\sigma})\|_{L^\infty} = \sum_{\substack{k \in \mathbb{N}_0^{3n} \\ k_j \leq [\log_2(r_j)]+1}} 2^{s \cdot k} \|\langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \hat{\sigma})\|_{L^\infty},$$

in view of the supports of w_k and $\hat{\sigma}$. Now,

$$\begin{aligned} |\langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \hat{\sigma})(x, \xi, \eta)| &= |\langle \xi, \eta \rangle^{-m} (\check{w}_k * \sigma)(x, \xi, \eta)| \\ &= \left| \langle \xi, \eta \rangle^{-m} \int_{\mathbb{R}^{3n}} \check{w}_k(y, a, b) \sigma(x - y, \xi - a, \eta - b) dy da db \right| \\ &\lesssim \|\sigma\|_{C_m^0} \langle \xi, \eta \rangle^{-m} \int_{\mathbb{R}^{3n}} |\check{w}_k(y, a, b)| \langle \xi - a, \eta - b \rangle^m dy da db \\ &\lesssim \|\sigma\|_{C_m^0} \int_{\mathbb{R}^{3n}} |\check{w}_k(y, a, b)| \langle a, b \rangle^{|m|} dy da db, \end{aligned}$$

where we have used that $\langle \xi, \eta \rangle \lesssim \langle \xi - a, \eta - b \rangle \langle a, b \rangle$ for $m < 0$ and that $\langle \xi, \eta \rangle^{-1} \lesssim \langle \xi - a, \eta - b \rangle^{-1} \langle a, b \rangle$ for $m \geq 0$. Finally we note that the last integral is bounded by a constant independent of $k \in \mathbb{N}_0^{3n}$ and conclude that

$$\|\sigma\|_{B_{\infty,1}^{s,m}} \lesssim \sum_{\substack{k \in \mathbb{N}_0^{3n} \\ k_j \leq [\log_2(r_j)]+1}} 2^{s \cdot k} \|\sigma\|_{C_m^0} \sim r_1^{s_1} \dots r_{3n}^{s_{3n}} \|\sigma\|_{C_m^0},$$

where we have used that $s_j > 0$ for $j = 1, \dots, 3n$. In particular, if σ is as in the statement of Theorem 4.5.1 then $\sigma \in B_{\infty,1}^{s(p),m}(\mathbb{R}^{3n})$, where $s(p)$ is the $3n$ -dimensional vector in Definition 4.3.1, and

$$\|\sigma\|_{B_{\infty,1}^{s(p),m}} \lesssim (r_1 \cdots r_n)^{\frac{1}{p}} (r_{n+1} \cdots r_{3n})^{\frac{1}{2}} \|\sigma\|_{C_m^0}.$$

4.5.2 A crucial estimate

We start with some definitions followed by the statement and proof of Theorem 4.5.3 which constitutes an essential ingredient in our proof of Theorem 4.5.1.

For $f, g, h, \varphi, \psi, \theta$ in the Schwarz class $\mathcal{S}(\mathbb{R}^n)$ define:

$$\begin{aligned} V(f, g, h)(y, a, b) &:= \int_{\mathbb{R}^{3n}} e^{2\pi i(y \cdot x + a \cdot \xi + b \cdot \eta)} \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \bar{h}(x) dx d\xi d\eta \\ &= \int_{\mathbb{R}^n} e^{2\pi i y \cdot x} \bar{h}(x) f(x+a) g(x+b) dx, \end{aligned}$$

$$W(f, g, h, \varphi, \psi, \theta)(x, \xi, \eta) := \int_{\mathbb{R}^{3n}} e^{-2\pi i(x \cdot y + \xi \cdot a + \eta \cdot b)} \varphi(y) \psi(a) \theta(b) V(f, g, h)(y, a, b) dy da db.$$

We note that $V(f, g, h)$ and $W(f, g, h, \varphi, \psi, \theta)$ belong to $\mathcal{S}(\mathbb{R}^{3n})$ for any $f, g, h, \varphi, \psi, \theta \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 4.5.3 (Herbert-Naibo [21]). *Let $2 \leq p, p_1, p_2 \leq \infty$ be related by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $m < m(p_1, p_2, 0)$. Then*

$$\begin{aligned} &\int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W(f, g, h, \varphi, \psi, \theta)(x, \xi, \eta)| dx d\xi d\eta \\ &\lesssim \sum_{\alpha, \beta, \gamma \in \{0,1,2,3\}^n} \|\widehat{\partial^\alpha \psi}\|_{L^2} \|\widehat{\partial^\beta \theta}\|_{L^2} \|\widehat{\partial^\gamma \varphi}\|_{L^{p'}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p'}}, \end{aligned} \tag{4.5.16}$$

for all functions $f, g, h, \psi, \theta \in \mathcal{S}(\mathbb{R}^n)$ and φ of the form $\varphi(x) = \prod_{j=1}^n \varphi_j(x_j)$, where $x = (x_1, \dots, x_n)$ and $\varphi_j \in \mathcal{S}(\mathbb{R})$ for $j = 1, \dots, n$.

The proof of Theorem 4.5.1 through the use of Theorem 4.5.3 is inspired by ideas in Hwang-Lee [27], also used in Tomita [39], in the linear case. Due to the bilinear setting, the proof of Theorem 4.5.3 requires new ideas.

The following lemma will be useful; see Bergh-Löfström [7, p. 17].

Lemma 4.5.4. *Let $2 \leq r < \infty$ and $r' \leq q \leq r$. There exists a positive constant $C_{r,q}$ such that*

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^q \langle \xi \rangle^{-n(1-\frac{q}{r})} d\xi \right)^{\frac{1}{q}} \leq C_{r,q} \|f\|_{L^{r'}}.$$

Proof of Theorem 4.5.3. Fix $f, g, h, \varphi, \psi, \theta \in \mathcal{S}(\mathbb{R}^n)$; we will write $W(x, \xi, \eta)$ instead of $W(f, g, h, \varphi, \psi, \theta)(x, \xi, \eta)$. We note that all changes in the order of integration in the following steps are justified in view of the smoothness and decay of the integrands. Using that $h(t) = \int_{\mathbb{R}^n} e^{2\pi i t \cdot \tau} \hat{h}(\tau) d\tau$ and making the change of variables $a + t \rightarrow a$ and $b + t \rightarrow b$, we get

$$\begin{aligned} W(x, \xi, \eta) &= \int_{\mathbb{R}^{3n}} e^{-2\pi i(x \cdot y + \xi \cdot a + \eta \cdot b)} \varphi(y) \psi(a) \theta(b) \left(\int_{\mathbb{R}^n} e^{2\pi i t \cdot y} \bar{h}(t) f(t+a) g(t+b) dt \right) dy da db \\ &= \int_{\mathbb{R}^{5n}} e^{-2\pi i(x \cdot y + \xi \cdot (a-t) + \eta \cdot (b-t))} \varphi(y) \psi(a-t) \theta(b-t) e^{2\pi i t \cdot (y-\tau)} \bar{h}(\tau) f(a) g(b) d\tau dt dy da db. \end{aligned}$$

Denoting $A_{a,b}(t) := \psi(a+t)\theta(b+t)$, the integral in t can be written as

$$\int_{\mathbb{R}^n} e^{2\pi i t \cdot (\xi + \eta + y - \tau)} \psi(a-t) \theta(b-t) dt = \widehat{A_{a,b}}(\xi + \eta + y - \tau).$$

Incorporating this into the formula for W and making the change of variable $\xi + \eta + y - \tau \rightarrow y$, it follows that

$$\begin{aligned} W(x, \xi, \eta) &= \int_{\mathbb{R}^{4n}} e^{-i2\pi(x \cdot y + \xi \cdot a + \eta \cdot b)} \varphi(y) \widehat{A_{a,b}}(\xi + \eta + y - \tau) \bar{h}(\tau) f(a) g(b) d\tau dy da db \\ &= \int_{\mathbb{R}^{3n}} e^{-2\pi i(x \cdot (-\xi - \eta + \tau) + \xi \cdot a + \eta \cdot b)} \bar{h}(\tau) f(a) g(b) \left(\int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} \varphi(y + \tau - \xi - \eta) \widehat{A_{a,b}}(y) dy \right) d\tau da db. \end{aligned}$$

We have $\varphi_j(y_j + \tau_j - \xi_j - \eta_j) = \varphi_j(\tau_j - \xi_j - \eta_j) + y_j \int_0^1 \varphi_j^{(1)}(s_j y_j + \tau_j - \xi_j - \eta_j) ds_j$ where

$\varphi_j^{(1)}$ denotes the first derivative of φ_j . Therefore, defining

$$\mathcal{J}_k := \{\vec{j} = (j_1, \dots, j_n) \in \{1, \dots, n\}^n : j_l \neq j_{\tilde{l}} \text{ if } l \neq \tilde{l}, j_1 < \dots < j_k, j_{k+1} < \dots < j_n\}$$

for $k = 0, \dots, n$, it follows that

$$\begin{aligned} \varphi(y + \tau - \xi - \eta) &= \prod_{j=1}^n \left(\varphi_j(\tau_j - \xi_j - \eta_j) + y_j \int_0^1 \varphi_j^{(1)}(s_j y_j + \tau_j - \xi_j - \eta_j) ds_j \right) \\ &= \sum_{\substack{k=0, \dots, n \\ (j_1, \dots, j_n) \in \mathcal{J}_k}} \left(\prod_{l=1}^k \varphi_{j_l}(\tau_{j_l} - \xi_{j_l} - \eta_{j_l}) \right) \left(\prod_{l=k+1}^n y_{j_l} \int_0^1 \varphi_{j_l}^{(1)}(s_{j_l} y_{j_l} + \tau_{j_l} - \xi_{j_l} - \eta_{j_l}) ds_{j_l} \right), \end{aligned}$$

with the products $\prod_{l=1}^0$ and $\prod_{l=n+1}^n$ being interpreted as 1. We then obtain that

$$W(x, \xi, \eta) = \sum_{\substack{k=0, \dots, n \\ \vec{j} \in \mathcal{J}_k}} W_{k, \vec{j}}(x, \xi, \eta),$$

where for $k = 0, \dots, n$ and $\vec{j} = (j_1, \dots, j_n) \in \mathcal{J}_k$,

$$\begin{aligned} W_{k, \vec{j}}(x, \xi, \eta) &:= \int_{\mathbb{R}^{3n}} e^{-2\pi i(x \cdot (-\xi - \eta + \tau) + \xi \cdot a + \eta \cdot b)} \tilde{h}(\tau) f(a) g(b) S_{k, \vec{j}}(x, \tau, \xi, \eta, a, b) d\tau da db, \\ S_{k, \vec{j}}(x, \tau, \xi, \eta, a, b) &:= \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} \varphi_{k, \vec{j}}(\tau - \xi - \eta) \Phi_{k, \vec{j}}(y, \tau, \xi, \eta) \widehat{A}_{a, b}(y) dy, \\ \varphi_{k, \vec{j}}(\tau - \xi - \eta) &:= \prod_{l=1}^k \varphi_{j_l}(\tau_{j_l} - \xi_{j_l} - \eta_{j_l}), \quad (\varphi_{0, \vec{j}}(\tau - \xi - \eta) := 1), \\ \Phi_{k, \vec{j}}(y, \tau, \xi, \eta) &:= \prod_{l=k+1}^n y_{j_l} \int_0^1 \varphi_{j_l}^{(1)}(s_{j_l} y_{j_l} + \tau_{j_l} - \xi_{j_l} - \eta_{j_l}) ds_{j_l} \\ &= y_{j_{k+1}} \cdots y_{j_n} \int_{[0,1]^{n-k}} \prod_{l=k+1}^n \varphi_{j_l}^{(1)}(s_{j_l} y_{j_l} + \tau_{j_l} - \xi_{j_l} - \eta_{j_l}) ds_{j_{k+1}} \cdots ds_{j_n}, \end{aligned}$$

with $\Phi_{n, \vec{j}} := 1$.

It is then enough to prove the inequality (4.5.16) for each $W_{k, \vec{j}}$. We will distinguish

between the cases $k = n$ and $k \in \{0, \dots, n-1\}$.

Case $k = n$. Here $\vec{j} = (1, \dots, n)$ and therefore

$$\begin{aligned} W_{n,\vec{j}}(x, \xi, \eta) &= \int_{\mathbb{R}^{3n}} e^{-2\pi i(x \cdot (-\xi - \eta + \tau) + \xi \cdot a + \eta \cdot b)} \bar{h}(\tau) f(a) g(b) S_{n,\vec{j}}(x, \tau, \xi, \eta, a, b) d\tau da db \\ &= \int_{\mathbb{R}^{3n}} e^{-2\pi i(x \cdot (-\xi - \eta + \tau) + \xi \cdot a + \eta \cdot b)} \bar{h}(\tau) f(a) g(b) \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} \varphi(\tau - \xi - \eta) \widehat{A_{a,b}}(y) dy d\tau da db \\ &= \int_{\mathbb{R}^{3n}} e^{-2\pi i(x \cdot (-\xi - \eta + \tau) + \xi \cdot a + \eta \cdot b)} \bar{h}(\tau) f(a) g(b) \varphi(\tau - \xi - \eta) A_{a,b}(-x) d\tau da db. \end{aligned}$$

Using that $A_{a,b}(-x) := \psi(a-x)\theta(b-x)$ and defining $F_x(a) := f(a)\psi(a-x)$, $G_x(b) := g(b)\theta(b-x)$, and $H_x(\tau) := \bar{h}(\tau)\hat{\varphi}(x-\tau)$,

$$\begin{aligned} W_{n,\vec{j}}(x, \xi, \eta) &= e^{2\pi i x \cdot (\xi + \eta)} \left(\int_{\mathbb{R}^n} e^{-2\pi i x \cdot \tau} \bar{h}(\tau) \varphi(\tau - \xi - \eta) d\tau \right) \\ &\quad \times \left(\int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot a} f(a) \psi(a-x) da \right) \left(\int_{\mathbb{R}^n} e^{-2\pi i \eta \cdot b} g(b) \theta(b-x) db \right) \\ &= e^{2\pi i x \cdot (\xi + \eta)} \left(\int_{\mathbb{R}^n} e^{-2\pi i x \cdot \tau} \bar{h}(\tau) \varphi(\tau - \xi - \eta) d\tau \right) \widehat{F_x}(\xi) \widehat{G_x}(\eta) \\ &= e^{2\pi i x \cdot (\xi + \eta)} \left(\int_{\mathbb{R}^n} \bar{h}(\tau) \mathcal{F}^{-1}(e^{-2\pi i x \cdot \cdot} \varphi(\cdot - \xi - \eta))(\tau) d\tau \right) \widehat{F_x}(\xi) \widehat{G_x}(\eta) \\ &= \left(\int_{\mathbb{R}^n} \bar{h}(\tau) e^{2\pi i \tau \cdot (\xi + \eta)} \hat{\varphi}(x - \tau) d\tau \right) \widehat{F_x}(\xi) \widehat{G_x}(\eta) = \check{H}_x(\xi + \eta) \widehat{F_x}(\xi) \widehat{G_x}(\eta). \end{aligned}$$

Applying Hölder's inequality with respect to (ξ, η) and Plancherel's identity, we obtain

$$\begin{aligned} \int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W_{n,\vec{j}}(x, \xi, \eta)| dx d\xi d\eta &= \int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |\check{H}_x(\xi + \eta) \widehat{F_x}(\xi) \widehat{G_x}(\eta)| dx d\xi d\eta \quad (4.5.17) \\ &\leq \int_{\mathbb{R}^n} \|F_x\|_2 \|G_x\|_2 \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} dx. \end{aligned}$$

Using inequality (4.5.17) we now consider the cases $p < \infty$ and $p = \infty$ separately. First, for

$p < \infty$, Hölder's inequality in x then gives

$$\begin{aligned}
& \int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W_{n,\vec{j}}(x, \xi, \eta)| dx d\xi d\eta \tag{4.5.18} \\
& \leq \left(\int_{\mathbb{R}^n} \|F_x\|_{L^2}^p \|G_x\|_{L^2}^p dx \right)^{\frac{1}{p}} \left\| \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \right\|_{L^{p'}} \\
& \leq \left(\int_{\mathbb{R}^n} \|F_x\|_{L^2}^{p_1} dx \right)^{\frac{1}{p_1}} \left(\int_{\mathbb{R}^n} \|G_x\|_{L^2}^{p_2} dx \right)^{\frac{1}{p_2}} \left\| \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \right\|_{L^{p'}}.
\end{aligned}$$

(With obvious changes if $p_1 = \infty$ or $p_2 = \infty$). Recalling that $F_x(a) = f(a)\psi(a-x)$ and that $p_1 \geq 2$, it follows that

$$\left(\int_{\mathbb{R}^n} \|F_x\|_{L^2}^{p_1} dx \right)^{\frac{1}{p_1}} = \| |f|^2 * |\psi(\cdot)|^2 \|_{L^{\frac{p_1}{2}}}^{\frac{1}{2}} \leq \| |f|^2 \|_{L^{\frac{p_1}{2}}}^{\frac{1}{2}} \| |\psi|^2 \|_{L^1}^{\frac{1}{2}} = \|f\|_{L^{p_1}} \|\psi\|_2. \tag{4.5.19}$$

Similarly,

$$\left(\int_{\mathbb{R}^n} \|G_x\|_{L^2}^{p_2} dx \right)^{\frac{1}{p_2}} \leq \|g\|_{L^{p_2}} \|\theta\|_{L^2}. \tag{4.5.20}$$

(Again, with obvious changes if $p_1 = \infty$ or $p_2 = \infty$). We now look at the factor

$$\left\| \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \right\|_{L^{p'}}$$

in (4.5.18). Recall that $m < m(p_1, p_2, 0) = -n(1 - \frac{1}{p})$, then $m = -n(1 - \frac{1}{p}) - \varepsilon$, for some $\varepsilon > 0$. Set $m_1 := -\frac{n}{2} - \varepsilon$ and $m_2 := -\frac{n}{2}(1 - \frac{2}{p})$, then $m_1 < -\frac{n}{2}$, $m_2 \leq 0$ (since $p \geq 2$), and $m_1 + m_2 = m$. The change of variable $\eta \rightarrow \eta - \xi$ and the fact that $\langle \xi, \eta - \xi \rangle^{2m} \leq \langle \xi \rangle^{2m_1} \langle \eta \rangle^{2m_2}$ imply

$$\begin{aligned}
\left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} &= \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta - \xi \rangle^{2m} |\check{H}_x(\eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}^n} \langle \xi \rangle^{2m_1} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \langle \eta \rangle^{2m_2} |\check{H}_x(\eta)|^2 d\eta \right)^{\frac{1}{2}}.
\end{aligned}$$

Note that the integral in ξ is finite; moreover, by Lemma 4.5.4

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \langle \eta \rangle^{2m_2} |\check{H}_x(\eta)|^2 d\eta \right)^{\frac{1}{2}} &= \left(\int_{\mathbb{R}^n} \langle \eta \rangle^{-n(1-\frac{2}{p})} |\check{H}_x(\eta)|^2 d\eta \right)^{\frac{1}{2}} \\ &\lesssim \|H_x\|_{L^{p'}} = \|\bar{h}(\cdot)\hat{\varphi}(x-\cdot)\|_{L^{p'}}, \end{aligned}$$

which implies

$$\left\| \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \right\|_{L^{p'}} \lesssim \|h\|_{L^{p'}} \|\hat{\varphi}\|_{L^{p'}}. \quad (4.5.21)$$

By (4.5.18), (4.5.19), (4.5.20), and (4.5.21), we obtain

$$\int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W_{n,\vec{j}}(x, \xi, \eta)| dx d\xi d\eta \lesssim \|\hat{\psi}\|_{L^2} \|\hat{\theta}\|_{L^2} \|\hat{\varphi}\|_{L^{p'}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p'}}. \quad (4.5.22)$$

Next, we consider the case when $p = p_1 = p_2 = \infty$ for which we will also prove (4.5.22).

Again (4.5.17) gives

$$\begin{aligned} \int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W_{n,\vec{j}}(x, \xi, \eta)| dx d\xi d\eta & \quad (4.5.23) \\ &\leq \sup_{x \in \mathbb{R}^n} (\|F_x\|_{L^2} \|G_x\|_{L^2}) \left\| \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \right\|_{L^1} \\ &\leq \|f\|_{L^\infty} \|g\|_{L^\infty} \|\hat{\psi}\|_{L^2} \|\hat{\theta}\|_{L^2} \left\| \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \right\|_{L^1}. \end{aligned}$$

The treatment of $\left\| \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \right\|_{L^1}$ in (4.5.23) is slightly different regarding the selection of m_1 and m_2 below when compared to the case $2 \leq p < \infty$. Since $m(\infty, \infty, 0) = -n$ and $m < m(\infty, \infty, 0)$, we have $m = -n - \epsilon$ for some $\epsilon > 0$. Set $m_1 = m_2 := -\frac{n}{2} - \frac{\epsilon}{2}$. The change of variable $\eta \rightarrow \eta - \xi$ and the fact that $\langle \xi, \eta - \xi \rangle^{2m} \leq \langle \xi \rangle^{2m_1} \langle \eta \rangle^{2m_2}$

imply

$$\left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}^n} \langle \xi \rangle^{2m_1} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \langle \eta \rangle^{2m_2} |\check{H}_x(\eta)|^2 d\eta \right)^{\frac{1}{2}}.$$

The integral in ξ is finite and

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \langle \eta \rangle^{2m_2} |\check{H}_x(\eta)|^2 d\eta \right)^{\frac{1}{2}} &= \left(\int_{\mathbb{R}^n} \langle \eta \rangle^{-n-\epsilon} |\check{H}_x(\eta)|^2 d\eta \right)^{\frac{1}{2}} \\ &\lesssim \|\check{H}_x\|_{L^\infty} \lesssim \|H_x\|_{L^1} = \|\bar{h}(\cdot)\hat{\varphi}(x - \cdot)\|_{L^1}, \end{aligned}$$

which implies

$$\left\| \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\check{H}_x(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \right\|_{L^1} \lesssim \|h\|_{L^1} \|\hat{\varphi}\|_{L^1}. \quad (4.5.24)$$

We then obtain

$$\int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W_{n\vec{j}}(x, \xi, \eta)| dx d\xi d\eta \lesssim \|\hat{\psi}\|_{L^2} \|\hat{\theta}\|_{L^2} \|\hat{\varphi}\|_{L^1} \|f\|_{L^\infty} \|g\|_{L^\infty} \|h\|_{L^1} \quad (4.5.25)$$

proving inequality (4.5.22) for $p = p_1 = p_2 = \infty$ and thus completing the case $k = n$.

Case $k \in \{0, \dots, n-1\}$. Fix $k \in \{0, \dots, n-1\}$ and $\vec{j} = (j_1, \dots, j_n) \in \mathcal{J}_k$. Without loss of generality we can assume that $j_l = l$ for $l = 1, \dots, n$. Note first that since $(2\pi i)^{n-k} y_{k+1} y_{k+2} \cdots y_n \widehat{A_{a,b}}(y) = \mathcal{F}(\partial_{t_{k+1}} \cdots \partial_{t_n} A_{a,b}(t))(y)$ then

$$\begin{aligned} S_{k,\vec{j}}(x, \tau, \xi, \eta, a, b) &:= (2\pi i)^{k-n} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} \varphi_{k,\vec{j}}(\tau - \xi - \eta) \mathcal{F}(\partial_{t_{k+1}} \cdots \partial_{t_n} A_{a,b}(t))(y) \\ &\quad \times \int_{[0,1]^{n-k}} \prod_{l=k+1}^n \varphi_l^{(1)}(s_l y_l + \tau_l - \xi_l - \eta_l) ds_{k+1} \cdots ds_n dy. \end{aligned}$$

Now, defining $\vec{e}_{k,\vec{j}} \in \mathbb{R}^n$ as the vector with components equal to 1 at positions l , $l =$

$k + 1, \dots, n$, and 0 otherwise, we have,

$$\begin{aligned}\mathcal{F}(\partial_{t_{k+1}} \cdots \partial_{t_n} A_{a,b})(y) &= \int_{\mathbb{R}^n} e^{-2\pi i y \cdot t} \partial_{t_{k+1}} \cdots \partial_{t_n} (\psi(a+t)\theta(b+t)) dt \\ &= \int_{\mathbb{R}^n} e^{-2\pi i y \cdot t} \sum_{\alpha_1 + \alpha_2 = \vec{e}_{k,\bar{j}}} C_{\alpha_1, \alpha_2} \partial_t^{\alpha_1} \psi(a+t) \partial_t^{\alpha_2} \theta(b+t) dt.\end{aligned}$$

Using the fact that

$$(1 - \partial_{t_1}^2) \cdots (1 - \partial_{t_n}^2) e^{-2\pi i y \cdot t} = e^{-2\pi i y \cdot t} \prod_{j=1}^n (1 + 4\pi^2 y_j^2)$$

and integration by parts, we obtain

$$\begin{aligned}\mathcal{F}(\partial_{t_{k+1}} \cdots \partial_{t_n} A_{a,b})(y) \\ = \sum_{\alpha_1 + \alpha_2 = \vec{e}_{k,\bar{j}}} \frac{C_{\alpha_1, \alpha_2}}{\prod_{j=1}^n (1 + 4\pi^2 y_j^2)} \int_{\mathbb{R}^n} e^{-2\pi i y \cdot t} (1 - \partial_{t_1}^2) \cdots (1 - \partial_{t_n}^2) (\partial_t^{\alpha_1} \psi(a+t) \partial_t^{\alpha_2} \theta(b+t)) dt.\end{aligned}$$

We now note that

$$(1 - \partial_{t_1}^2) \cdots (1 - \partial_{t_n}^2) = \sum_{d=0}^n (-1)^d \sum_{\gamma \in \mathcal{H}_d} \partial_t^\gamma$$

where $\mathcal{H}_d := \{\gamma \in \mathbb{R}^n : \gamma \text{ has } d \text{ entries equal to } 2 \text{ and all others equal to } 0\}$, and therefore

$$\begin{aligned}(1 - \partial_{t_1}^2) \cdots (1 - \partial_{t_n}^2) (\partial_t^{\alpha_1} \psi(a+t) \partial_t^{\alpha_2} \theta(b+t)) \\ = \sum_{d=0}^n (-1)^d \sum_{\gamma \in \mathcal{H}_d} \partial_t^\gamma (\partial_t^{\alpha_1} \psi(a+t) \partial_t^{\alpha_2} \theta(b+t)) \\ = \sum_{d=0}^n (-1)^d \sum_{\gamma \in \mathcal{H}_d} \sum_{\gamma_1 + \gamma_2 = \gamma} C_{\gamma_1, \gamma_2} (\partial^{\alpha_1 + \gamma_1} \psi)(a+t) (\partial^{\alpha_2 + \gamma_2} \theta)(b+t).\end{aligned}$$

This leads to

$$\begin{aligned} & \mathcal{F}(\partial_{t_{k+1}} \cdots \partial_{t_n} A_{a,b})(y) \\ &= \sum_{\alpha_1 + \alpha_2 = \vec{e}_{k,\vec{j}}} \sum_{d=0}^n \sum_{\gamma \in \mathcal{H}_d} \sum_{\gamma_1 + \gamma_2 = \gamma} \frac{C_{\alpha_1, \alpha_2, \gamma_1, \gamma_2}}{\prod_{j=1}^n (1 + 4\pi^2 y_j^2)} \int_{\mathbb{R}^n} e^{-2\pi i y \cdot t} (\partial^{\alpha_1 + \gamma_1} \psi)(a+t) (\partial^{\alpha_2 + \gamma_2} \theta)(b+t) dt. \end{aligned}$$

Then $S_{k,\vec{j}}(x, \tau, \xi, \eta, a, b)$ is a finite linear combination of terms of the form

$$\begin{aligned} S_{k,\vec{j},1}(x, \tau, \xi, \eta, a, b) &:= \varphi_{k,\vec{j}}(\tau - \xi - \eta) \int_{\mathbb{R}^n} \frac{e^{-2\pi i x \cdot y}}{\prod_{j=1}^n (1 + 4\pi^2 y_j^2)} \\ &\quad \times \int_{\mathbb{R}^n} e^{-2\pi i y \cdot t} (\partial^{\alpha_1 + \gamma_1} \psi)(a+t) (\partial^{\alpha_2 + \gamma_2} \theta)(b+t) dt \\ &\quad \times \int_{[0,1]^{n-k}} \prod_{l=k+1}^n \varphi_l^{(1)}(s_l y_l + \tau_l - \xi_l - \eta_l) ds_{k+1} \cdots ds_n dy \end{aligned}$$

and it is enough to analyze

$$W_{k,\vec{j},1}(x, \xi, \eta) := \int_{\mathbb{R}^{3n}} e^{-2\pi i(x \cdot (-\xi - \eta + \tau) + \xi \cdot a + \eta \cdot b)} \bar{h}(\tau) f(a) g(b) S_{k,\vec{j},1}(x, \tau, \xi, \eta, a, b) d\tau da db.$$

Now,

$$S_{k,\vec{j},1}(x, \tau, \xi, \eta, a, b) = \varphi_{k,\vec{j}}(\tau - \xi - \eta) \int_{\mathbb{R}^n} (\partial^{\alpha_1 + \gamma_1} \psi)(a+t) (\partial^{\alpha_2 + \gamma_2} \theta)(b+t) Q_1(x, t, \tau, \xi, \eta) dt$$

where

$$Q_1(x, t, \tau, \xi, \eta) := \int_{\mathbb{R}^n} \int_{[0,1]^{n-k}} \frac{e^{-2\pi i(x+t) \cdot y}}{\prod_{j=1}^n (1 + 4\pi^2 y_j^2)} \prod_{l=k+1}^n \varphi_l^{(1)}(s_l y_l + \tau_l - \xi_l - \eta_l) ds_{k+1} \cdots ds_n dy.$$

Using that

$$(1 - \partial_{y_1})^2 \cdots (1 - \partial_{y_n})^2 e^{-2\pi i(x+t) \cdot y} = e^{-2\pi i(x+t) \cdot y} \prod_{j=1}^n (1 + 2\pi i(t_j + x_j))^2$$

and integration by parts give

$$\begin{aligned}
Q_1(x, t, \tau, \xi, \eta) &= \frac{1}{\prod_{j=1}^n (1 + 2\pi i(t_j + x_j))^2} \int_{\mathbb{R}^n} \int_{[0,1]^{n-k}} e^{-2\pi i(x+t) \cdot y} \\
&\quad \times \left(\prod_{l=1}^k (1 + \partial_{y_l})^2 \left(\frac{1}{1 + 4\pi^2 y_l^2} \right) \right) \left(\prod_{l=k+1}^n (1 + \partial_{y_l})^2 \left(\frac{\varphi_l^{(1)}(s_l y_l + \tau_l - \xi_l - \eta_l)}{1 + 4\pi^2 y_l^2} \right) \right) \\
&\quad \times ds_{k+1} \dots ds_n dy \\
&= \frac{1}{\prod_{j=1}^n (1 + 2\pi i(t_j + x_j))^2} \int_{\mathbb{R}^n} \int_{[0,1]^{n-k}} e^{-2\pi i(x+t) \cdot y} \left(\prod_{l=1}^k H_l(y_l) \right) \\
&\quad \times \left(\prod_{l=k+1}^n \sum_{j=1}^3 \varphi_l^{(j)}(s_l y_l + \tau_l - \xi_l - \eta_l) s_l^{j-1} H_l(y_l) \right) ds_{k+1} \dots ds_n dy,
\end{aligned}$$

where $H_l \in L^1(\mathbb{R})$ for $l = 1, \dots, n$ and $\varphi_l^{(j)}$ denotes the j th derivative of φ_l . Then we have that $S_{k, \vec{j}, 1}(x, \tau, \xi, \eta, a, b)$ is a finite linear combination of terms of the form

$$S_{k, \vec{j}, 2}(x, \tau, \xi, \eta, a, b) := \varphi_{k, \vec{j}}(\tau - \xi - \eta) \int_{\mathbb{R}^n} (\partial^{\alpha_1 + \gamma_1} \psi)(a + t) (\partial^{\alpha_2 + \gamma_2} \theta)(b + t) Q_2(x, t, \tau, \xi, \eta) dt,$$

where

$$\begin{aligned}
Q_2(x, t, \tau, \xi, \eta) &:= \frac{1}{\prod_{j=1}^n (1 + 2\pi i(t_j + x_j))^2} \int_{\mathbb{R}^n} \int_{[0,1]^{n-k}} e^{-2\pi i(x+t) \cdot y} H(y) \\
&\quad \times \left(\prod_{l=k+1}^n \varphi_l^{(j_l)}(s_l y_l + \tau_l - \xi_l - \eta_l) s_l^{j_l-1} \right) ds_{k+1} \dots ds_n dy,
\end{aligned}$$

with $H(y) := H_1(y_1) \dots H_n(y_n) \in L^1(\mathbb{R}^n)$ and j_l equal to 1, 2, or 3. It is then enough to analyze

$$W_{k, \vec{j}, 2}(x, \xi, \eta) := \int_{\mathbb{R}^{3n}} e^{-2\pi i(x \cdot (-\xi - \eta + \tau) + \xi \cdot a + \eta \cdot b)} \bar{h}(\tau) f(a) g(b) S_{k, \vec{j}, 2}(x, \tau, \xi, \eta, a, b) d\tau da db.$$

Setting $F_{t, \alpha_1, \gamma_1}(a) := f(a) (\partial^{\alpha_1 + \gamma_1} \psi)(a + t)$ and $G_{t, \alpha_2, \gamma_2}(b) := g(b) (\partial^{\alpha_2 + \gamma_2} \theta)(b + t)$, the integrals

in a and b in $W_{k,\vec{j},2}$ are given by

$$\int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot a} f(a) (\partial^{\alpha_1 + \gamma_1} \psi)(a+t) da \int_{\mathbb{R}^n} e^{-2\pi i \eta \cdot b} g(b) (\partial^{\alpha_2 + \gamma_2} \theta)(b+t) db = \widehat{F_{t,\alpha_1,\gamma_1}}(\xi) \widehat{G_{t,\alpha_2,\gamma_2}}(\eta).$$

For the integral in τ we have

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \tau} \bar{h}(\tau) \varphi_{k,\vec{j}}(\tau - \xi - \eta) \left(\prod_{l=k+1}^n \varphi_l^{(j_l)}(s_l y_l + \tau_l - \xi_l - \eta_l) s_l^{j_l - 1} \right) d\tau \\ &= \int_{\mathbb{R}^n} \overline{\widehat{h(\cdot + x)}}(\tau) \varphi_{k,\vec{j}}(\tau - \xi - \eta) \left(\prod_{l=k+1}^n \varphi_l^{(j_l)}(s_l y_l + \tau_l - \xi_l - \eta_l) s_l^{j_l - 1} \right) d\tau \\ &= \int_{\mathbb{R}^n} \bar{h}(\tau + x) \mathcal{F}^{-1} \left(\varphi_{k,\vec{j}}(\cdot - \xi - \eta) \left(\prod_{l=k+1}^n \varphi_l^{(j_l)}(s_l y_l + \cdot_l - \xi_l - \eta_l) s_l^{j_l - 1} \right) \right) (\tau) d\tau \\ &= \int_{\mathbb{R}^n} \bar{h}(\tau + x) \mathcal{F}^{-1} \left(\varphi_{k,\vec{j}}(\cdot) \left(\prod_{l=k+1}^n \varphi_l^{(j_l)}(\cdot_l) s_l^{j_l - 1} \right) \right) (\tau) e^{-2\pi i \sum_{l=k+1}^n s_l y_l \tau_l} e^{2\pi i (\xi + \eta) \cdot \tau} d\tau \\ &= \mathcal{F}^{-1} (H_{x,\bar{s}_k,\bar{y}_k}) (\xi + \eta), \end{aligned}$$

where $\bar{s}_k := (s_{k+1}, \dots, s_n)$, $\bar{y}_k := (y_{k+1}, \dots, y_n)$, and

$$H_{x,\bar{s}_k,\bar{y}_k}(\tau) := \bar{h}(\tau + x) \mathcal{F}^{-1} \left(\varphi_{k,\vec{j}}(\cdot) \left(\prod_{l=k+1}^n \varphi_l^{(j_l)}(\cdot_l) s_l^{j_l - 1} \right) \right) (\tau) e^{-2\pi i \sum_{l=k+1}^n s_l y_l \tau_l}.$$

It then follows that

$$\begin{aligned} W_{k,\vec{j},2}(x, \xi, \eta) &= \int_{\mathbb{R}^{2n}} \int_{[0,1]^{n-k}} \frac{e^{2\pi i x \cdot (\xi + \eta)} e^{-2\pi i (x+t) \cdot y} H(y)}{\prod_{j=1}^n (1 + 2\pi i (t_j + x_j))^2} \\ &\quad \times \widehat{F_{t,\alpha_1,\gamma_1}}(\xi) \widehat{G_{t,\alpha_2,\gamma_2}}(\eta) \mathcal{F}^{-1} (H_{x,\bar{s}_k,\bar{y}_k}) (\xi + \eta) ds_{k+1} \dots ds_n dy dt. \end{aligned}$$

Multiplying by $\langle \xi, \eta \rangle^m$ on both sides of the last equality, integrating with respect to x , ξ , η after taking modulus, and applying the Cauchy-Schwarz inequality in ξ and η and

Plancherel's identity, we have

$$\int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W_{k,\vec{j},2}(x, \xi, \eta)| d\xi d\eta dx \leq \int_{\mathbb{R}^{3n}} |H(y)| \int_{[0,1]^{n-k}} \frac{\|F_{t,\alpha_1,\gamma_1}\|_2 \|G_{t,\alpha_2,\gamma_2}\|_{L^2}}{\prod_{j=1}^n (1 + |t_j + x_j|^2)} \quad (4.5.26)$$

$$\times \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\mathcal{F}^{-1}(H_{x,\bar{s}_k,\bar{y}_k})(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} ds_{k+1} \dots ds_n dy dt dx. \quad (4.5.27)$$

At this point we divide the case $k \in \{0, \dots, n-1\}$ into the two possibilities $2 \leq p < \infty$ and $p = \infty$. For $2 \leq p < \infty$, we now apply Hölder's inequality with respect to x and t to get

$$\begin{aligned} & \int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W_{k,\vec{j},2}(x, \xi, \eta)| d\xi d\eta dx \\ & \leq \int_{\mathbb{R}^n} |H(y)| \int_{[0,1]^{n-k}} \left(\int_{\mathbb{R}^{2n}} \left(\frac{\|F_{t,\alpha_1,\gamma_1}\|_{L^2} \|G_{t,\alpha_2,\gamma_2}\|_{L^2}}{\prod_{j=1}^n (1 + |t_j + x_j|^2)^{\frac{1}{2}}} \right)^p dt dx \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_{\mathbb{R}^{2n}} \left(\frac{(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\mathcal{F}^{-1}(H_{x,\bar{s}_k,\bar{y}_k})(\xi + \eta)|^2 d\xi d\eta)^{\frac{1}{2}}}{\prod_{j=1}^n (1 + |t_j + x_j|^2)^{\frac{1}{2}}} \right)^{p'} dx dt \right)^{\frac{1}{p'}} ds_{k+1} \dots ds_n dy \\ & \lesssim \int_{\mathbb{R}^n} |H(y)| \int_{[0,1]^{n-k}} \left(\int_{\mathbb{R}^n} \|F_{t,\alpha_1,\gamma_1}\|_{L^2}^p \|G_{t,\alpha_2,\gamma_2}\|_{L^2}^p dt \right)^{\frac{1}{p}} \quad (4.5.28) \\ & \quad \times \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\mathcal{F}^{-1}(H_{x,\bar{s}_k,\bar{y}_k})(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{p'}{2}} dx \right)^{\frac{1}{p'}} ds_{k+1} \dots ds_n dy, \end{aligned}$$

where we have used that p and p' are both larger than 1. The factor in (4.5.28) given by $(\int_{\mathbb{R}^n} \|F_{t,\alpha_1,\gamma_1}\|_{L^2}^p \|G_{t,\alpha_2,\gamma_2}\|_{L^2}^p dt)^{\frac{1}{p}}$ is handled in the same way as $(\int_{\mathbb{R}^n} \|F_x\|_{L^2}^p \|G_x\|_{L^2}^p dx)^{\frac{1}{p}}$ in the case $k = n$ and satisfies

$$\left(\int_{\mathbb{R}^n} \|F_{t,\alpha_1,\gamma_1}\|_{L^2}^p \|G_{t,\alpha_2,\gamma_2}\|_{L^2}^p dt \right)^{\frac{1}{p}} \lesssim \|\widehat{\partial^{\alpha_1+\gamma_1}\psi}\|_{L^2} \|\widehat{\partial^{\alpha_2+\gamma_2}\theta}\|_{L^2} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}. \quad (4.5.29)$$

For the other factor, we proceed as in the case $k = n$ using Lemma 4.5.4 to get

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\mathcal{F}^{-1}(H_{x, \bar{s}_k, \bar{y}_k})(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{p'}{2}} dx \right)^{\frac{1}{p'}} \\ & \lesssim \left(\int_{\mathbb{R}^{2n}} |H_{x, \bar{s}_k, \bar{y}_k}(\tau)|^{p'} d\tau dx \right)^{\frac{1}{p'}} \lesssim \|h\|_{L^{p'}} \left\| \widehat{\varphi_{k, \vec{j}}} \prod_{j=k+1}^n \widehat{\varphi_l^{(j_i)}} \right\|_{L^{p'}}. \end{aligned} \quad (4.5.30)$$

Putting (4.5.28), (4.5.29), and (4.5.30) together and using that $s_j \in [0, 1]$ and $H \in L^1(\mathbb{R}^n)$, we get

$$\begin{aligned} & \int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W_{k, \vec{j}, 2}(x, \xi, \eta)| d\xi d\eta dx \\ & \lesssim \|\widehat{\partial^{\alpha_1 + \gamma_1} \psi}\|_{L^2} \|\widehat{\partial^{\alpha_2 + \gamma_2} \theta}\|_{L^2} \left\| \widehat{\varphi_{k, \vec{j}}} \prod_{j=k+1}^n \widehat{\varphi_l^{(j_i)}} \right\|_{L^{p'}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p'}}. \end{aligned} \quad (4.5.31)$$

Finally, we consider the case $p = p_1 = p_2 = \infty$ for $k \in \{0, \dots, n-1\}$. From (4.5.26) we get

$$\begin{aligned} & \int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W_{k, \vec{j}, 2}(x, \xi, \eta)| d\xi d\eta dx \lesssim \sup_t (\|F_{t, \alpha_1, \gamma_1}\|_{L^2} \|G_{t, \alpha_2, \gamma_2}\|_{L^2}) \int_{\mathbb{R}^n} |H(y)| \\ & \quad \times \int_{[0, 1]^{n-k}} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{2n}} \langle \xi, \eta \rangle^{2m} |\mathcal{F}^{-1}(H_{x, \bar{s}_k, \bar{y}_k})(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} dx \right) ds_{k+1} \dots ds_n dy. \end{aligned} \quad (4.5.32)$$

For the supremum in t we have

$$\sup_t (\|F_{t, \alpha_1, \gamma_1}\|_{L^2} \|G_{t, \alpha_2, \gamma_2}\|_{L^2}) \lesssim \|\widehat{\partial^{\alpha_1 + \gamma_1} \psi}\|_{L^2} \|\widehat{\partial^{\alpha_2 + \gamma_2} \theta}\|_{L^2} \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad (4.5.33)$$

while for the other factor, we proceed as in the case $k = n$ corresponding to $p = p_1 = p_2 = \infty$

and get

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \langle \xi, \eta \rangle^{2m} |\mathcal{F}^{-1}(H_{x, \bar{s}, \bar{y}_k})(\xi + \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} dx \\ & \lesssim \int_{\mathbb{R}^{2n}} |H_{x, \bar{s}_k, \bar{y}_k}(\tau)| d\tau dx \lesssim \|h\|_{L^1} \left\| \widehat{\varphi_{k, \vec{j}}} \prod_{j=k+1}^n \widehat{\varphi_l^{(j_i)}} \right\|_{L^1}. \end{aligned} \quad (4.5.34)$$

Putting (4.5.32), (4.5.33), and (4.5.34) together and using that $s_j \in [0, 1]$ and $H \in L^1(\mathbb{R}^n)$, we get

$$\begin{aligned} & \int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W_{k, \vec{j}, 2}(x, xi, \eta)| d\xi d\eta dx \\ & \lesssim \left\| \widehat{\partial^{\alpha_1 + \gamma_1} \psi} \right\|_{L^2} \left\| \widehat{\partial^{\alpha_2 + \gamma_2} \theta} \right\|_{L^2} \left\| \widehat{\varphi_{k, \vec{j}}} \prod_{j=k+1}^n \widehat{\varphi_l^{(j_i)}} \right\|_{L^1} \|f\|_{L^\infty} \|g\|_{L^\infty} \|h\|_{L^1}. \end{aligned} \quad (4.5.35)$$

Recalling the ranges of the number of derivatives used, (4.5.22) along with (4.5.31) and (4.5.25) along with (4.5.35) lead to (4.5.16) for $2 \leq p < \infty$ and $p = \infty$, respectively. \square

4.5.3 The completion of the proof

We first note that if the components of s are positive, the action of $\sigma \in B_{\infty, q}^{s, m}$ as a tempered distribution is given by

$$\langle \sigma, F \rangle := \int_{\mathbb{R}^{3n}} \sigma(x, \xi, \eta) F(x, \xi, \eta) dx d\xi d\eta, \quad \text{for all } F \in \mathcal{S}(\mathbb{R}^{3n}).$$

This implies that

$$\int_{\mathbb{R}^{3n}} \sigma(x, \xi, \eta) \hat{F}(x, \xi, \eta) dx d\xi d\eta = \langle \hat{\sigma}, F \rangle, \quad \text{for all } F \in \mathcal{S}(\mathbb{R}^{3n}),$$

with an analogous formula when the Fourier transform is replaced by the inverse Fourier transform.

Proof of Theorem 4.5.1. Let $\sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, satisfy (4.5.13) and (4.5.14). Consider $\varphi(x) = \prod_{j=1}^n \Phi\left(\frac{x_j}{r_j}\right)$, $\psi(x) = \prod_{j=1}^n \Psi\left(\frac{x_j}{r_{j+n}}\right)$ and $\theta(x) = \prod_{j=1}^n \Theta\left(\frac{x_j}{r_{j+2n}}\right)$, $x = (x_1, \dots, x_n)$, where Φ , Ψ and Θ are functions in $\mathcal{S}(\mathbb{R})$ supported in $[-2, 2]$ and identically equal to 1 in $[-1, 1]$. Then, for $f, g, h \in \mathcal{S}(\mathbb{R}^n)$, in view of the definitions of V and W and the support of $\hat{\sigma}$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} T_\sigma(f, g)(x) \bar{h}(x) dx &= \int_{\mathbb{R}^{3n}} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \bar{h}(x) dx d\xi d\eta \\ &= \langle \hat{\sigma}, V(f, g, h) \rangle \\ &= \langle \hat{\sigma}, (\varphi \otimes \psi \otimes \theta) V(f, g, h) \rangle \\ &= \int_{\mathbb{R}^{3n}} \sigma(x, \xi, \eta) W(f, g, h, \varphi, \psi, \theta)(x, \xi, \eta) dx d\xi d\eta, \end{aligned}$$

where $(\varphi \otimes \psi \otimes \theta)(y, a, b) := \varphi(y)\psi(a)\theta(b)$. Theorem 4.5.3 then implies

$$\begin{aligned} \left| \int_{\mathbb{R}^n} T_\sigma(f, g)(x) \bar{h}(x) dx \right| &\lesssim \|\sigma\|_{C_m^0} \int_{\mathbb{R}^{3n}} \langle \xi, \eta \rangle^m |W(f, g, h, \varphi, \psi, \theta)(x, \xi, \eta)| dx d\xi d\eta \\ &\lesssim \|\sigma\|_{C_m^0} \sum_{\alpha, \beta, \gamma \in \{0, 1, 2, 3\}^n} \|\widehat{\partial^\alpha \psi}\|_{L^2} \|\widehat{\partial^\beta \theta}\|_{L^2} \|\widehat{\partial^\gamma \varphi}\|_{L^{p'}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p'}}. \end{aligned}$$

A simple computation shows that

$$\|\widehat{\partial^\gamma \varphi}\|_{L^{p'}} \lesssim (r_1 \cdots r_n)^{\frac{1}{p}}, \quad \|\widehat{\partial^\alpha \psi}\|_{L^2} \lesssim (r_{n+1} \cdots r_{2n})^{\frac{1}{2}}, \quad \|\widehat{\partial^\beta \theta}\|_{L^2} \lesssim (r_{2n+1} \cdots r_{3n})^{\frac{1}{2}},$$

where we have used that $r_j \geq 1$, $j = 1, \dots, 3n$, and therefore (4.5.15) follows. \square

Next we use Theorem 4.5.1 to prove Theorem 4.3.2.

Proof of Theorem 4.3.2. In view of Proposition 4.2.1, it is enough to prove part (a) for $s(p) = (s_1, \dots, s_{3n})$ where $s_1 = \dots = s_n = \frac{1}{p}$ and $s_{n+1} = \dots = s_{3n} = \frac{1}{2}$. Consider $\{w_j\}_{j \in \mathbb{N}_0}$ as in (4.2.1) with $N = 1$ and for $k = (k_1, \dots, k_{3n}) \in \mathbb{R}^{3n}$ set $w_k(x, \xi, \eta) :=$

$w_{k_1}(x_1) \cdots w_{k_n}(x_n) w_{k_{n+1}}(\xi_1) \cdots w_{k_{2n}}(\xi_n) w_{k_{2n+1}}(\eta_1) \cdots w_{k_{3n}}(\eta_n)$. Then for $f, g, h \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\int_{\mathbb{R}^{3n}} T_\sigma(f, g)(x) \bar{h}(x) dx &= \int_{\mathbb{R}^{3n}} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \bar{h}(x) dx d\xi d\eta \\
&= \langle \hat{\sigma}, V(f, g, h) \rangle \\
&= \left\langle \sum_{k \in \mathbb{N}_0^{3n}} w_k \hat{\sigma}, V(f, g, h) \right\rangle \\
&= \sum_{k \in \mathbb{N}_0^{3n}} \langle w_k \hat{\sigma}, V(f, g, h) \rangle \\
&= \sum_{k \in \mathbb{N}_0^{3n}} \int_{\mathbb{R}^{3n}} \mathcal{F}^{-1}(w_k \hat{\sigma})(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \bar{h}(x) dx d\xi d\eta \\
&= \sum_{k \in \mathbb{N}_0^{3n}} \int_{\mathbb{R}^n} T_{\sigma_k}(f, g)(x) \bar{h}(x) dx d\xi d\eta,
\end{aligned}$$

where $\sigma_k(x, \xi, \eta) := \mathcal{F}^{-1}(w_k \hat{\sigma})(x, \xi, \eta)$. It then follows that

$$\|T_\sigma(f, g)\|_{L^p} \lesssim \sum_{k \in \mathbb{N}_0^{3n}} \|T_{\sigma_k}(f, g)\|_{L^p}.$$

We note that σ_k satisfies (4.5.13), since $\text{supp}(\hat{\sigma}_k) \subset \prod_{l=1}^{3n} [-2^{k_l+1}, 2^{k_l+1}]$, and (4.5.14) since $\sigma \in B_{\infty,1}^{s(p),m}(\mathbb{R}^{3n})$. Theorem 4.5.1 implies

$$\|T_\sigma(f, g)\|_{L^p} \lesssim \sum_{k \in \mathbb{N}_0^{3n}} 2^{s(p) \cdot k} \|\sigma_k\|_{C_m^0} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} = \|\sigma\|_{B_{\infty,1}^{s(p),m}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$

□

4.6 Proof of Theorem 4.3.3

In Section 4.6.1 we state and prove a new result in regards to a symbolic calculus for classes of x -independent symbols belonging to Besov spaces of product type, which we then use in

Section 4.6.2 to proof Theorem 4.3.3.

4.6.1 A symbolic calculus for classes of x -independent symbols in Besov spaces of product type

We start with a remark about the norm of x -independent symbols belonging to the Besov classes introduced in Section 4.2. Let w and w_0 be as in (4.2.1) with $N = n$. If σ is an x -independent symbol in $B_{r,q}^{s,m}(\mathbb{R}^{3n})$ and $s = (s_1, s_2, s_3) \in \mathbb{R}^3$, it easily follows that

$$\|\sigma\|_{B_{\infty,1}^{s,m}} \sim \left(\sum_{k \in \mathbb{N}_0^2} (2^{\bar{s} \cdot k} \|\langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \hat{\sigma})\|_{L^r})^q \right)^{\frac{1}{q}}, \quad (4.6.36)$$

where $\bar{s} = (s_2, s_3)$, $w_k(\xi, \eta) = w_{k_2}(\xi)w_{k_3}(\eta)$ for $k = (k_2, k_3) \in \mathbb{N}_0^2$, \mathcal{F}^{-1} and $\hat{\cdot}$ denote inverse Fourier transform and Fourier transform in \mathbb{R}^{2n} , respectively, and the L^r norm is taken in \mathbb{R}^{2n} . An analogous remark corresponds to the case when $s \in \mathbb{R}^{3n}$.

We now state our result regarding a symbolic calculus for such symbols.

Theorem 4.6.1 (Herbert-Naibo [22]). *Let $m \in \mathbb{R}$, $2 \leq r \leq \infty$, and suppose that σ is an x -independent symbol. Then*

$$\sigma \in B_{\infty,1}^{s(1),m}(\mathbb{R}^{3n}) \Rightarrow \sigma^{*1}, \sigma^{*2} \in B_{\infty,1}^{s(r),m}(\mathbb{R}^{3n}),$$

where $s(1)$ and $s(r)$ have the same dimension (3 or $3n$) and are given as in Definition 4.3.1.

Moreover

$$\|\sigma^{*j}\|_{B_{\infty,1}^{s(r),m}} \lesssim \|\sigma\|_{B_{\infty,1}^{s(1),m}}, \quad j = 1, 2, \quad (4.6.37)$$

with the implicit constant independent of σ .

The following lemma will be useful in the proof of Theorem 4.6.1. We state it in \mathbb{R}^{2n} because it is convenient for our setting, but more general versions also hold (compare with

Sugimoto [37] or Triebel [40, p. 25-28]). Let $d = (d_1, \dots, d_{2n}) \in \mathbb{R}^{2n}$. Given $\xi, \eta \in \mathbb{R}^n$ set $\langle \xi, \eta \rangle_{d^{-1}} := \langle (d_1^{-1}\xi_1, \dots, d_n^{-1}\xi_n), (d_{n+1}^{-1}\eta_1, \dots, d_{2n}^{-1}\eta_n) \rangle$. If h is a function defined in \mathbb{R}^{2n} denote $S_d(h)(y_1, \dots, y_{2n}) := h(d_1 y_1, \dots, d_{2n} y_{2n})$ and $S_{d^{-1}}(h)(y_1, \dots, y_{2n}) := h(d_1^{-1} y_1, \dots, d_{2n}^{-1} y_{2n})$.

Lemma 4.6.2. *Let $1 \leq r \leq \infty$ and $t \in \mathbb{R}$. Then for every continuous function $g(\xi, \eta)$ defined for $\xi, \eta \in \mathbb{R}^n$ such that $\|\langle \xi, \eta \rangle^t g\|_{L^r} < \infty$ and every $M \in \mathcal{S}(\mathbb{R}^{2n})$,*

$$\|\langle \xi, \eta \rangle^t \mathcal{F}^{-1} M \hat{g}\|_{L^r} \leq \|\langle \xi, \eta \rangle^{|t|} \mathcal{F}^{-1} M\|_{L^1} \|\langle \xi, \eta \rangle^t g\|_{L^r}, \quad (4.6.38)$$

and, more generally,

$$\|\langle \xi, \eta \rangle_{d^{-1}}^t \mathcal{F}^{-1} M \hat{g}\|_{L^r} \leq \|\langle \xi, \eta \rangle_{d^{-1}}^{|t|} \mathcal{F}^{-1} M\|_{L^1} \|\langle \xi, \eta \rangle_{d^{-1}}^t g\|_{L^r} \quad (4.6.39)$$

for any $d \in \mathbb{R}^{2n}$. In particular, if $d = (d_1, \dots, d_{2n})$ and $d_i \geq 1$ for $i = 1, \dots, 2n$,

$$\|\langle \xi, \eta \rangle^t \mathcal{F}^{-1} M \hat{g}\|_{L^r} \leq \|\langle \xi, \eta \rangle^{|t|} \mathcal{F}^{-1} S_d(M)\|_{L^1} \|\langle \xi, \eta \rangle^t g\|_{L^r}. \quad (4.6.40)$$

Proof of Lemma 4.6.2. We have $\langle u + y, v + z \rangle^t \lesssim \langle u, v \rangle^{|t|} \langle y, z \rangle^t$ for all $u, v, y, z \in \mathbb{R}^n$. Then

$$|\langle \xi, \eta \rangle_{d^{-1}}^t \mathcal{F}^{-1}(M \hat{g})(\xi, \eta)| \lesssim \int_{\mathbb{R}^{2n}} \langle a, b \rangle_{d^{-1}}^{|t|} |\check{M}(a, b)| \langle \xi - a, \eta - b \rangle_{d^{-1}}^t |g(\xi - a, \eta - b)| da db,$$

from where (4.6.39) follows by Minkowski's integral inequality. For (4.6.40), apply (4.6.39) with M replaced by $S_d(M)$ and g replaced by $S_{d^{-1}}(g)$ and note that $\langle \xi, \eta \rangle_{d^{-1}}^{|t|} \leq \langle \xi, \eta \rangle^{|t|}$ since $d_i \geq 1$ for $i = 1, \dots, 2n$. □

Proof of Theorem 4.6.1. We will prove the result for σ^{*1} , with the result for σ^{*2} following in an analogous way. Fix $m \in \mathbb{R}$ and let σ be an x -independent symbols in $B_{\infty,1}^{s(1),m}(\mathbb{R}^{3n})$. It easily follows that $\sigma^{*1}(\xi, \eta) = \sigma(-\xi - \eta, \eta)$.

Consider first the case when $s(1), s(r) \in \mathbb{R}^3$ and note that $s(1) = (\frac{n}{2}, n, n)$ and $s(r) =$

$(\frac{n}{r}, \frac{n}{2}, \frac{n}{2})$. Let w and w_0 be radial functions that satisfy (4.2.1) for $N = n$. In view of (4.6.36) we have to prove that

$$\sum_{k \in \mathbb{N}_0^2} 2^{(\frac{n}{2}, \frac{n}{2}) \cdot k} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \lesssim \sum_{k \in \mathbb{N}_0^2} 2^{(n, n) \cdot k} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \hat{\sigma}) \right\|_{L^\infty} \quad (4.6.41)$$

where $w_k(\xi, \eta) = w_{k_1}(\xi)w_{k_2}(\eta)$ for $k = (k_1, k_2) \in \mathbb{N}_0^2$, \mathcal{F}^{-1} and $\widehat{\cdot}$ denote inverse Fourier transform and Fourier transform in \mathbb{R}^{2n} , respectively, and the L^∞ norm is taken in \mathbb{R}^{2n} .

Given $k = (k_1, k_2) \in \mathbb{N}_0^2$ and noting that $\widehat{\sigma^{*1}}(a, b) = \hat{\sigma}(-a, b - a)$, a change of variables gives

$$\mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}})(\xi, \eta) = \mathcal{F}^{-1}(w_{k_1}(a)w_{k_2}(b - a)\hat{\sigma}(a, b))(-\eta - \xi, \eta).$$

Since $\langle \xi, \eta \rangle \sim \langle \xi + \eta, \eta \rangle$, it then follows that

$$\left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \sim \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_{k_1}(a)w_{k_2}(b - a)\hat{\sigma}(a, b)) \right\|_{L^\infty}. \quad (4.6.42)$$

We will divide the summation in $(k_1, k_2) \in \mathbb{N}_0^2$ according to the following regions:

$$\begin{aligned} R &= \mathbb{N}^2, \quad R_1 = \{(k_1, 0) : k_1 \geq 3\}, \quad R_2 = \{(0, k_2) : k_2 \geq 3\}, \\ R_3 &= \{(0, 0), (0, 1), (0, 2), (1, 0), (2, 0)\}. \end{aligned}$$

Define $\tilde{w}(a) := \sum_{l=-2}^2 w(2^{-l}a)$ for $a \in \mathbb{R}^n$ and observe that $\tilde{w}_j(a) := \tilde{w}(2^{-j}a) = \sum_{l=j-2}^{j+2} w_l(a) \equiv 1$ for $2^{j-2} \leq |a| \leq 2^{j+2}$ and $j \geq 2$. Define $\tilde{w}_j(a) := \sum_{l=0}^{j+2} w_l(a)$ for $j = 0, 1$ and $a \in \mathbb{R}^n$; then $\tilde{w}_j(a) \equiv 1$ for $|a| \leq 2^{j+2}$ and $j = 0, 1$. For $(k_1, k_2) \in \mathbb{N}_0^2$ and $a, b \in \mathbb{R}^n$ set $h_{(k_1, k_2)}(a, b) := w_{k_1}(a)\tilde{w}_{k_2}(b)$.

Summation in region R : Consider the following subregions

$$R_A = \{(k_1, k_2) \in \mathbb{N}^2 : k_1 - k_2 > 2\}, \quad R_B = \{(k_1, k_2) \in \mathbb{N}^2 : k_1 - k_2 < -2\},$$

$$R_C = \{(k_1, k_2) \in \mathbb{N}^2 : -2 \leq k_1 - k_2 \leq 2\}.$$

We first estimate the summation in region R_A . If $(k_1, k_2) \in R_A$,

$$\text{supp}(w_{k_1}(a)w_{k_2}(b-a)) \subset \{(a, b) : 2^{k_1-1} \leq |a| \leq 2^{k_1+1} \text{ and } \frac{1}{2}2^{k_1-1} \leq |b| \leq \frac{9}{8}2^{k_1+1}\}$$

and therefore $w_{k_1}(a)w_{k_2}(b-a) = \tilde{w}_{k_1}(a)w_{k_2}(b-a)w_{k_1}(a)\tilde{w}_{k_1}(b)$. Then (4.6.42) implies

$$\left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \sim \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}[\tilde{w}_{k_1}(a)w_{k_2}(b-a)\mathcal{F}(\mathcal{F}^{-1}(h_{(k_1, k_1)}\hat{\sigma}))](a, b) \right\|_{L^\infty}.$$

By (4.6.40) in Lemma 4.6.2 it follows that,

$$\begin{aligned} & \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}[\tilde{w}_{k_1}(a)w_{k_2}(b-a)\mathcal{F}(\mathcal{F}^{-1}(h_{(k_1, k_1)}\hat{\sigma}))](a, b) \right\|_{L^\infty} \\ & \lesssim \left\| \langle \xi, \eta \rangle^{|m|} \mathcal{F}^{-1}[\tilde{w}_{k_1}(2^{k_2}a)w_{k_2}(2^{k_2}(b-a))] \right\|_{L^1} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(h_{(k_1, k_1)}\hat{\sigma}) \right\|_{L^\infty}. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{(k_1, k_2) \in R_A} 2^{\frac{n}{2}(k_1+k_2)} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \\ & \lesssim \sum_{k_1=4}^{\infty} \sum_{k_2=1}^{k_1-3} 2^{\frac{n}{2}(k_1+k_2)} \left\| \langle \xi, \eta \rangle^{|m|} \mathcal{F}^{-1}[\tilde{w}(2^{k_2-k_1}a)w(b-a)] \right\|_{L^1} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(h_{(k_1, k_1)}\hat{\sigma}) \right\|_{L^\infty}. \end{aligned}$$

An elementary computation shows that for $1 \leq k_2 \leq k_1 - 3$,

$$\left\| \langle \xi, \eta \rangle^{|m|} \mathcal{F}^{-1}[\tilde{w}(2^{k_2-k_1}a)w(b-a)] \right\|_{L^1} \lesssim 1,$$

which implies that

$$\sum_{k_2=1}^{k_1-3} 2^{\frac{n}{2}(k_1+k_2)} \left\| \langle \xi, \eta \rangle^{|m|} \mathcal{F}^{-1}[\tilde{w}(2^{k_2-k_1}a)w(b-a)] \right\|_{L^1} \lesssim 2^{nk_1} \leq 2^{n(k_1+k_1)}.$$

We have therefore obtain

$$\sum_{k \in R_A} 2^{(\frac{n}{2}, \frac{n}{2}) \cdot k} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \lesssim \sum_{k_1=4}^{\infty} 2^{n(k_1+k_1)} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(h_{(k_1, k_1)} \hat{\sigma}) \right\|_{L^\infty}. \quad (4.6.43)$$

We now look at the summation in the region R_B . Note that if $(k_1, k_2) \in R_B$, then

$$\text{supp}(w_{k_1}(a)w_{k_2}(b-a)) \subset \{(a, b) : 2^{k_1-1} \leq |a| \leq 2^{k_1+1} \text{ and } \frac{1}{2} 2^{k_2-1} \leq |b| \leq \frac{9}{8} 2^{k_2+1}\}.$$

For $1 \leq k_1 \leq k_2 - 3$ we have $w_{k_1}(a)w_{k_2}(b-a) = w_{k_2}(b-a)\tilde{w}_{k_2}(b)w_{k_1}(a)\tilde{w}_{k_2}(b)$, and (4.6.42) implies

$$\left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \sim \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}[w_{k_2}(b-a)\tilde{w}_{k_2}(b)\mathcal{F}(\mathcal{F}^{-1}(h_{(k_1, k_2)} \hat{\sigma}))(a, b)] \right\|_{L^\infty}.$$

By (4.6.40) in Lemma 4.6.2 it is enough to prove that

$$\left\| \langle \xi, \eta \rangle^{|m|} \mathcal{F}^{-1}(w_{k_2}(2^{k_2}(b-a))\tilde{w}_{k_2}(2^{k_2}b)) \right\|_{L^1} \lesssim 2^{\frac{n}{2}(k_1+k_2)}$$

for $1 \leq k_1 \leq k_2 - 3$, $k_2 \geq 4$, which follows immediately since the L^1 norm appearing above is independent of k_1 and k_2 . As a consequence we obtain

$$\sum_{k \in R_B} 2^{(\frac{n}{2}, \frac{n}{2}) \cdot k} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \lesssim \sum_{k_2=4}^{\infty} \sum_{k_1=1}^{k_2-3} 2^{n(k_1+k_2)} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(h_{(k_1, k_2)} \hat{\sigma}) \right\|_{L^\infty}. \quad (4.6.44)$$

For $(k_1, k_2) \in R_C$ it follows that

$$\text{supp}(w_{k_1}(a)w_{k_2}(b-a)) \subset \{(a, b) : 2^{k_1-1} \leq |a| \leq 2^{k_1+1} \text{ and } |b| \leq 10 \cdot 2^{k_1}\}.$$

Set $\chi_{k_1} := \sum_{j=0}^{k_1+4} w_j$ for $k_1 \in \mathbb{N}$; since $\chi_{k_1}(b) = 1$ for all b in the set $\{b : |b| \leq 10 \cdot 2^{k_1}\}$ then

$$w_{k_1}(a)w_{k_2}(b-a) = \sum_{j=0}^{k_1+4} w_{k_2}(b-a)\chi_{k_1}(b)w_{k_1}(a)w_j(b).$$

From this and (4.6.42) it follows that for $k_1 \in \mathbb{N}$,

$$\begin{aligned} & \sum_{k_2=\max(0, k_1-2)}^{k_1+2} 2^{\frac{n}{2}(k_1+k_2)} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_{(k_1, k_2)} \widehat{\sigma^{*1}}) \right\|_{L^\infty} \\ & \lesssim \sum_{j=0}^{k_1+4} \sum_{k_2=\max(0, k_1-2)}^{k_1+2} 2^{\frac{n}{2}(k_1+k_2)} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}[w_{k_2}(b-a)\chi_{k_1}(b)\mathcal{F}(\mathcal{F}^{-1}(w_{(k_1, j)}\hat{\sigma}))](a, b) \right\|_{L^\infty}. \end{aligned}$$

By (4.6.40) in Lemma 4.6.2 the desired inequality will be implied by

$$\sum_{k_2=\max(0, k_1-2)}^{k_1+2} 2^{\frac{n}{2}(k_1+k_2)} \left\| \langle \xi, \eta \rangle^{|m|} \mathcal{F}^{-1}(w_{k_2}(2^{k_1}(b-a))\chi_{k_1}(2^{k_1}b)) \right\|_{L^1} \lesssim 2^{n(k_1+j)}$$

for $0 \leq j \leq k_1 + 4$. The L^1 norms are bounded by a constant independent of k_1 and k_2 and therefore the above inequality follows from the fact that $2^{nk_1} \leq 2^{n(k_1+j)}$ for $j \geq 0$. We have therefore obtained that

$$\sum_{k \in R_C} 2^{\left(\frac{n}{2}, \frac{n}{2}\right) \cdot k} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \lesssim \sum_{k_1=1}^{\infty} \sum_{j=0}^{k_1+4} 2^{n(k_1+j)} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_{(k_1, j)}\hat{\sigma}) \right\|_{L^\infty}. \quad (4.6.45)$$

Summation in region R_1 : In view of (4.6.42), we have to estimate

$$\sum_{k_1=3}^{\infty} 2^{\frac{n}{2}k_1} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_{k_1}(a)w_0(b-a)\hat{\sigma}(a, b)) \right\|_{L^\infty}. \quad (4.6.46)$$

For $k_1 \geq 3$ it holds that

$$\text{supp}(w_{k_1}(a)w_0(b-a)) \subset \{(a, b) : 2^{k_1-1} \leq |a| \leq 2^{k_1+1} \text{ and } 2^{k_1-2} \leq |b| \leq 2^{k_1+2}\}$$

and therefore

$$w_{k_1}(a)w_0(b-a) = w_0(b-a)\tilde{w}_{k_1}(b)w_{k_1}(a)\tilde{w}_{k_1}(b).$$

It easily follows that

$$2^{\frac{n}{2}k_1} \|\langle \xi, \eta \rangle^{|m|} \mathcal{F}^{-1}[w_0(b-a)\tilde{w}_{k_1}(b)]\|_{L^1} \lesssim 2^{2nk_1},$$

and reasoning as above we obtain

$$\sum_{k \in R_1} 2^{(\frac{n}{2}, \frac{n}{2}) \cdot k} \|\langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}})\|_{L^\infty} \lesssim \sum_{k_1=3}^{\infty} 2^{n(k_1+k_1)} \|\langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(h_{(k_1, k_1)} \hat{\sigma})\|_{L^\infty}. \quad (4.6.47)$$

Summation in region R_2 : In this case we have to estimate, again by (4.6.42),

$$\sum_{k_2=3}^{\infty} 2^{\frac{n}{2}k_2} \|\langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_0(a)w_{k_2}(b-a)\hat{\sigma}(a, b))\|_{L^\infty}.$$

For $k_2 \geq 3$ it holds that

$$\text{supp}(w_0(a)w_{k_2}(b-a)) \subset \{(a, b) : |a| \leq 2 \text{ and } 2^{k_2-2} \leq |b| \leq 2^{k_2+2}\}$$

and therefore

$$w_0(a)w_{k_2}(b-a) = w_{k_2}(b-a)\tilde{w}_{k_2}(b)w_0(a)\tilde{w}_{k_2}(b).$$

The estimate

$$2^{\frac{n}{2}k_2} \|\langle \xi, \eta \rangle^{|m|} \mathcal{F}^{-1}[w_{k_2}(2^{k_2}(b-a))\tilde{w}_{k_2}(2^{k_2}b)]\|_{L^1} \lesssim 2^{nk_2},$$

follows from the fact that the L^1 norm is independent of k_2 . As a consequence, it follows that

$$\sum_{k \in R_2} 2^{(\frac{n}{2}, \frac{n}{2}) \cdot k} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \lesssim \sum_{k_2=3}^{\infty} 2^{nk_2} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(h_{(0, k_2)} \hat{\sigma}) \right\|_{L^\infty}. \quad (4.6.48)$$

Summation in region R_3 : We first observe that for $(k_1, k_2) \in R_3$

$$\text{supp } (w_{k_1}(a)w_{k_2}(b-a)) \subset \{(a, b) : |a| \leq 8 \text{ and } |b| \leq 16\}.$$

Therefore, for (k_1, k_2) in region R_3 it follows that

$$\sum_{k \in R_3} 2^{(\frac{n}{2}, \frac{n}{2}) \cdot k} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \lesssim \sum_{k_1=0}^2 \sum_{k_2=0}^4 2^{n(k_1+k_2)} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_{(k_1, k_2)} \hat{\sigma}) \right\|_{L^\infty}. \quad (4.6.49)$$

Inequalities (4.6.43), (4.6.44), (4.6.45), (4.6.47), (4.6.48) and (4.6.49) then lead to the desired estimate (4.6.41).

We now briefly describe the proof when $s(1), s(r) \in \mathbb{R}^{3n}$. Note that $s(1)$ has its first n components equal to $\frac{1}{2}$ and the rest of them equal to 1, while $s(r)$ has its first n components equal to $\frac{1}{r}$ and the rest of them equal to $\frac{1}{2}$. Let w and w_0 be radial functions that satisfy (4.2.1) for $N = 1$. Reasoning as in the previous case, it is enough to prove that

$$\begin{aligned} \sum_{(k_1, \dots, k_{2n}) \in \mathbb{N}_0^{2n}} 2^{\frac{1}{2}(k_1 + \dots + k_{2n})} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \\ \lesssim \sum_{(k_1, \dots, k_{2n}) \in \mathbb{N}_0^{2n}} 2^{k_1 + \dots + k_{2n}} \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \hat{\sigma}) \right\|_{L^\infty}, \end{aligned}$$

where for $k = (k_1, \dots, k_{2n})$ we have $w_k(\xi, \eta) = w_{k_1}(\xi_1) \cdots w_{k_n}(\xi_n) w_{k_{n+1}}(\eta_1) \cdots w_{k_{2n}}(\eta_n)$, \mathcal{F}^{-1} and $\hat{\cdot}$ denote inverse Fourier transform and Fourier transform in \mathbb{R}^{2n} , respectively, and the

L^∞ norm is taken in \mathbb{R}^{2n} . For such k , we have

$$\left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_k \widehat{\sigma^{*1}}) \right\|_{L^\infty} \sim \left\| \langle \xi, \eta \rangle^{-m} \mathcal{F}^{-1}(w_{K_1}(a) w_{K_2}(b-a) \hat{\sigma}(a, b)) \right\|_{L^\infty}, \quad (4.6.50)$$

where $w_{K_1}(a) = w_{k_1}(a_1) \cdots w_{k_n}(a_n)$ and $w_{K_2}(x) = w_{k_{n+1}}(b_1) \cdots w_{k_{2n}}(b_n)$. The process is now similar to the case previously treated but much heavier in notation and the result follows by dividing the summation in $(k_1, \dots, k_{2n}) \in \mathbb{N}_0^{2n}$ based on the regions R, R_1, R_2 and R_3 for each pair $(k_j, k_{n+j}), j = 1, \dots, n$. \square

4.6.2 The completion of the proof

We now have all the tools needed to complete the proof of Theorem 4.3.3.

Proof of Theorem 4.3.3. By the inclusion properties indicated in Proposition 4.2.1, it is enough to work with the classes $B_{\infty,1}^{s(1),m}(\mathbb{R}^{3n})$ where $m < m(p_1, p_2, 0)$ and $s(1)$ is in \mathbb{R}^{3n} .

Let $\sigma \in B_{\infty,1}^{s(1),m}(\mathbb{R}^{3n})$. If $1 \leq p_1 \leq 2$ then $(\frac{1}{p_1}, \frac{1}{p_2})$ is in the line segment joining $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ in Figure 4.1 and $m(p_1, p_2, 0) = -\frac{n}{p_1}$. Since $2 \leq p_2 \leq \infty$, Theorem 4.6.1 implies that $\sigma^{*1} \in B_{\infty,1}^{s(p_2),m}(\mathbb{R}^{3n})$. Note that $(\frac{1}{\infty}, \frac{1}{p_2})$ is in the segment joining $(0, 0)$ with $(0, \frac{1}{2})$ of Figure 4.1 and $m(\infty, p_2, 0) = -\frac{n}{p_2} = -\frac{n}{p_1} = m(p_1, p_2, 0)$. By Theorem 4.3.2, it follows that $T_{\sigma^{*1}}$ is bounded from $L^\infty(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$ and

$$\|T_{\sigma^{*1}}(f, g)\|_{L^{p_2}} \lesssim \|\sigma^{*1}\|_{B_{\infty,1}^{s(p_2),m}} \|f\|_{L^\infty} \|g\|_{L^{p_2}}.$$

Duality then gives that T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and, in view of (4.6.37),

$$\|T_\sigma(f, g)\|_{L^1} \lesssim \|\sigma^{*1}\|_{B_{\infty,1}^{s(p_2),m}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \lesssim \|\sigma\|_{B_{\infty,1}^{s(1),m}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}},$$

as desired.

If $2 \leq p_1 \leq \infty$ then $(\frac{1}{p_1}, \frac{1}{p_2})$ is in the line segment joining $(0, 1)$ and $(\frac{1}{2}, \frac{1}{2})$ and $m(p_1, p_2, 0) = -\frac{n}{p_2}$. We can then proceed in a similar way as above, with σ^{*2} instead of σ^{*1} , to conclude that T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ with the operator norm controlled by $\|\sigma\|_{B_{\infty,1}^{s(1),m}}$. \square

4.7 Conclusions

In this section, we present a brief summary of the conclusions of Chapter 4 and consider possible directions and methods for extending these results.

Let $1 \leq p, p_1, p_2 \leq \infty$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and assume $m < m(p_1, p_2, 0)$. For $0 < q \leq 1$ we consider $s \geq s(p)$ while for $1 < q \leq \infty$ we set $s > s(p)$, where $s(p)$ is as in Definition 4.3.1. With these indices, we obtained boundedness from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for bilinear pseudodifferential operators with symbols in various classes. In particular,

(a) For $2 \leq p, p_1, p_2 \leq \infty$, T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $\sigma \in B_{\infty,q}^{s,m}(\mathbb{R}^{3n})$,

(b) For $2 \leq p, p_1, p_2 \leq \infty$, T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $\sigma \in \mathcal{C}_m^0(\mathbb{R}^{3n})$ such that $\text{supp}(\hat{\sigma})$ is compact,

and

(c) For $p = 1$, T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ for all x -independent $\sigma \in B_{\infty,q}^{s,m}(\mathbb{R}^{3n})$.

We note that in the above mentioned cases the index $m(p_1, p_2, 0)$ is optimal for the classes $B_{\infty,q}^{s,m}(\mathbb{R}^{3n})$ in the sense that if $m > m(p_1, p_2, 0)$ there are symbols in $B_{\infty,q}^{s,m}(\mathbb{R}^{3n})$ for which the corresponding operators are not bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. This follows from the fact that $BS_{0,0}^m \subset B_{\infty,q}^{s,m}(\mathbb{R}^{3n})$ along with the results of Chapter 3 corresponding to $BS_{0,0}^m$.

As a byproduct of the results stated in items (a), (b), and (c), we obtained an upper bound (in terms of $s(p)$) on the number of derivatives of the symbol satisfying (4.4.5) that is sufficient for boundedness of the corresponding operator in the setting of Lebesgue spaces, improving in this sense results related to $BS_{0,0}^m$. An open question along these lines is whether $s(p)$ is sharp; that is, if $t < s(p)$ and $m < m(p_1, p_2, 0)$, are there symbols in $B_{\infty,q}^{t,m}$ for which the corresponding operators are not bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$? As was mentioned in Remark 4.4.4, it seems likely that an improvement could be made at least in certain cases by considering a slight modification in the definition of the classes of symbols involved.

Finally, in order to prove the result corresponding to item (c), we developed a symbolic calculus for Besov classes of x -independent symbols. In view of items (a) and (c), boundedness corresponding to $1 < p < 2$ remains open for the case of x -independent symbols. It is possible that an argument based on complex interpolation of the Besov classes of x -independent symbols along with the use of trilinear complex interpolation of operators could be used to prove boundedness for the associated operators in the range $1 < p < 2$. In addition, an open question is whether a similar symbolic calculus holds for general Besov classes, which could potentially lead to extending the results of item (a) to the range $1 \leq p < 2$. These and many other questions about the boundedness of bilinear pseudodifferential operators for various types of symbols and various functions spaces provide intriguing possibilities for the future.

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Appendix A

A Glossary of Selected Notation and Definitions

A.1 Frequently Used Notation

- \mathbb{N}_0 is the set of non-negative integers.
- The symbol \lesssim is used in place of $\leq C$ where C is a positive constant that may depend on some parameters but not on the functions or symbols involved in the inequality. Similarly, $x \sim y$ means $c_1 y \leq x \leq c_2 y$ for some positive constants c_1 and c_2 that are uniform in x and y .
- \hat{f} and $\mathcal{F}(f)$ are used interchangeably to denote the Fourier transform of a tempered distribution f . Analogously, $\check{\cdot}$ and \mathcal{F}^{-1} denote the inverse Fourier transform operator. For the definition of the Fourier transform and its inverse, see (A.3.1) and (A.3.2) in Section A.3.
- If X is a Banach (quasi-Banach) space then $\|\cdot\|_X$ denotes the norm (quasi-norm) in X .

- For $x \in \mathbb{R}^n$, $|x|$ stands for the usual Euclidean norm; while for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| := \alpha_1 + \dots + \alpha_n$.
- For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n}$ and $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$.
- Given $\xi, \eta \in \mathbb{R}^n$, $\langle \xi, \eta \rangle := 1 + |\xi| + |\eta|$. The notation $\langle \cdot, \cdot \rangle$ for arguments other than vectors in a Euclidean space always refers to the action of a tempered distribution on functions in the Schwartz class.
- $S_{\rho, \delta}^m$ and $BS_{\rho, \delta}^m$ are the linear and bilinear Hörmander classes, respectively (see Definitions 2.2.1 and 3.2.1).
- $B_{r, q}^{s, m}$ is a weighted Besov space of product type (see Section 4.2).
- For Banach or quasi-Banach spaces X and Y , $X \subset Y$ means X is a subset of Y ; $X \subsetneq Y$ means X is a proper subset of Y . Such inclusions are said to be continuous if $\|x\|_Y \lesssim \|x\|_X$ for all $x \in X$.

A.2 Function Spaces

Continuous and differentiable functions. The space $\mathcal{C}(\mathbb{R}^n)$ denotes the set of continuous complex-valued functions on \mathbb{R}^n . For $k \in \mathbb{N}$ or $k = \infty$ we define $\mathcal{C}^k(\mathbb{R}^n) := \{f : \mathbb{R}^n \rightarrow \mathbb{C} : \partial^\alpha f \in \mathcal{C}(\mathbb{R}^n) \text{ for } |\alpha| \leq k\}$. The notation $\mathcal{S}(\mathbb{R}^n)$ is used for the Schwartz class in \mathbb{R}^n , this is the space of infinitely differentiable complex-valued functions on \mathbb{R}^n which decrease rapidly at infinity. The space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Lebesgue spaces. For $0 < p < \infty$ we denote by $L^p(\mathbb{R}^n)$ the space of measurable functions f for which

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, $L^\infty(\mathbb{R}^n)$ is the space of all essentially bounded measurable functions, that is, those f satisfying

$$\|f\|_{L^\infty} := \inf \{C \geq 0 : |f(x)| \leq C \text{ for almost every } x\} < \infty.$$

As usual two functions in $L^p(\mathbb{R}^n)$ are equal if they coincide everywhere except possibly in a set of zero Lebesgue measure. For $1 \leq p \leq \infty$, the space $L^p(\mathbb{R}^n)$ is a Banach space (a Hilbert space when $p = 2$) with the norm $\|\cdot\|_{L^p}$; while for $0 < p < 1$, $L^p(\mathbb{R}^n)$ is a quasi-Banach space with the quasi norm $\|\cdot\|_{L^p}$.

The conjugate exponent of p is denoted by p' , where $\frac{1}{p} + \frac{1}{p'} = 1$. For $1 \leq p < \infty$, $L^{p'}(\mathbb{R}^n)$ is the dual space for $L^p(\mathbb{R}^n)$.

The notation $L_c^\infty(\mathbb{R}^n)$ is used for the subspace of functions in $L^\infty(\mathbb{R}^n)$ which have compact support. The space $L_{\text{loc}}^1(\mathbb{R}^n)$ is composed of all locally integrable functions in \mathbb{R}^n .

Hardy spaces. Let $0 < p \leq \infty$ and consider $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x) dx \neq 0$. The Hardy space $H^p(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^p} := \left\| \sup_{t>0} |\phi_t * f| \right\|_{L^p} < \infty,$$

where $\phi_t(x) := t^{-n} \phi(x/t)$. The local Hardy spaces $h^p(\mathbb{R}^n)$ consist of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{h^p} := \left\| \sup_{0<t<1} |\phi_t * f| \right\|_{L^p} < \infty.$$

Neither $H^p(\mathbb{R}^n)$ nor $h^p(\mathbb{R}^n)$ depend on the choice of the test function ϕ . It is clear that $H^p(\mathbb{R}^n) \hookrightarrow h^p(\mathbb{R}^n)$ for $0 < p < \infty$ and it can be proved that $H^p(\mathbb{R}^n) = h^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$ and that $H^1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$.

The space of functions with bounded mean oscillation. For $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ and a cube $Q \subset \mathbb{R}^n$, set $f_Q := \frac{1}{|Q|} \int_Q f(y) dy$. The space $BMO(\mathbb{R}^n)$ is defined as the class of functions

$f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy < \infty.$$

The space $bmo(\mathbb{R}^n)$ consists of all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{bmo} := \sup_{|Q| \leq 1} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy + \sup_{|Q|=1} \frac{1}{|Q|} \int_Q |f(y)| dy < \infty.$$

It follows that $L^\infty(\mathbb{R}^n) \subset bmo(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. Moreover, the dual spaces of $H^1(\mathbb{R}^n)$ and $h^1(\mathbb{R}^n)$ are, respectively, $BMO(\mathbb{R}^n)$ and $bmo(\mathbb{R}^n)$.

A.3 Miscellaneous Definitions

The Fourier transform and the inverse Fourier transform. If $f \in L^1(\mathbb{R}^n)$ the Fourier transform of f and the inverse Fourier transform of f are given, respectively, by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \tag{A.3.1}$$

and

$$\check{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi} dx. \tag{A.3.2}$$

If $F \in \mathcal{S}'(\mathbb{R}^n)$ then \hat{F} is the tempered distribution defined by $\langle \hat{F}, f \rangle := \langle F, \hat{f} \rangle$ for $f \in \mathcal{S}(\mathbb{R}^n)$, where, as was mentioned in Section A.1, $\langle \cdot, \cdot \rangle$ is used to denote the action of a tempered distribution on the functions of the Schwartz class. Similarly, $\langle \check{F}, f \rangle := \langle F, \check{f} \rangle$ for $f \in \mathcal{S}(\mathbb{R}^n)$. The notations \mathcal{F} and \mathcal{F}^{-1} are used interchangeably with $\hat{\cdot}$ and $\check{\cdot}$, respectively.

Boundedness of linear and bilinear operators. Let X, Y and Z be quasi-Banach

spaces. We say that a linear operator T is bounded from X into Y if

$$\|Tf\|_Y \lesssim \|f\|_X \quad \text{for all } f \in X.$$

A bilinear operator T is bounded from $X \times Y$ into Z if

$$\|T(f, g)\|_Z \lesssim \|f\|_X \|g\|_Y \quad \text{for all } f \in X, g \in Y.$$

Littlewood-Paley partition of unity. Let φ and φ_0 be functions defined in \mathbb{R}^N which satisfy the following conditions:

$$\begin{aligned} \varphi_0 &\in \mathcal{S}(\mathbb{R}^N), \quad \text{supp}(\varphi_0) \subset \{\xi \in \mathbb{R}^N : |\xi| \leq 2\}, \\ \varphi &\in \mathcal{S}(\mathbb{R}^N), \quad \text{supp}(\varphi) \subset \{\xi \in \mathbb{R}^N : \frac{1}{2} \leq |\xi| \leq 2\}, \\ \varphi_k(\xi) &:= \varphi(2^{-k}\xi), \quad k \in \mathbb{N}, \quad \sum_{k=0}^{\infty} \varphi_k(\xi) = 1, \quad \xi \in \mathbb{R}^N. \end{aligned} \tag{A.3.3}$$

The system $\{\varphi_k\}_{k \in \mathbb{N}_0}$ is called a Littlewood-Paley partition of unity in \mathbb{R}^N . An example of such system is produced if φ_0 is as above, $\varphi_0 \equiv 1$ in the set $\{\xi \in \mathbb{R}^N : |\xi| \leq 1\}$, and $\varphi(\xi) := \varphi_0(\xi) - \varphi_0(2\xi)$ for $\xi \in \mathbb{R}^n$.

A visual example of functions φ_0 and φ_k for dimension $N = 1$ is given in the following figure:

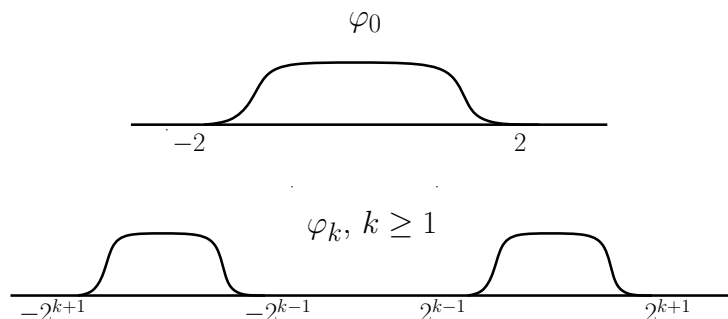


Figure A.1: Example of the system $\{\varphi_k\}_{k \geq 0}$ for $N = 1$.