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Large-time behavior of the weak solution to 3D Navier-Stokes equations.

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Abstract

The weak solution to the Navier-Stokes equations in a bounded domain $D \subset \mathbb{R}^3$ with a smooth boundary is proved to be unique provided that it satisfies an additional requirement. This solution exists for all $t \geq 0$. In a bounded domain D the solution decays exponentially fast as $t \to \infty$ if the force term decays at a suitable rate.

Keywords: Navier-Stokes equations, weak solution, uniqueness theorem
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1. Introduction

Consider the problem

$$v' + (v, \nabla)v = -\nabla p + \nu \Delta v + f \text{ in } D, \quad \nabla \cdot v = 0, \tag{1}$$

$$v(x, 0) = v^0(x); \quad v|_S = 0. \tag{2}$$

Here $v = (v_m)_{m=1}^3$ is a vector function, $v' = \frac{dv}{dt}$, $D \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary S, $\nu = \text{const} > 0$ is the kinematic viscosity coefficient, v^0 and f are given functions, v and p are to be found. We assume throughout that $v^0(x) \in \mathring{H}^1(D)$, $\nabla \cdot v^0 = 0$, and $f \in L^2([0, T); H^1(D))$ for any $T < \infty$. We also assume that f decays fast as $t \to \infty$. Precise assumptions will be formulated in Section 2, in the proof of Lemma 2.1.

We use the standard notations: $\mathring{H}^1(D)$ is the closure of vector-functions $C_0^\infty(D)$ in the norm of the Sobolev space $H^1(D)$; V is the closure in $H^1(D)$ of the subset of $C_0^\infty(D)$ consisting of solenoidal vector fields, $\nabla \cdot v = 0$; (u, v) is the inner product in $H := L^2(D)$ of two vector functions in $\mathbb{R}^3$,

$$|u|^2 := (u, u), \quad ((u, v)) := (\nabla u, \nabla v), \quad ||u||^2 := ((u, u)).$$

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Definition 1. *A weak solution to (1)-(2) is a vector function* $v \in W := L^2([0,T];V)$ *satisfying the relation*

$$(v', \eta) + ((v \cdot \nabla)v, \eta) + \nu((v, \eta)) = (f, \eta) \quad \forall \eta \in W. \tag{3}$$

One proves that $(v', \eta) \in L^1([0,T))$ if $v \in V$. Indeed, $((v, \eta)) \in L^2([0,T))$ because $v \in L^2([0,T);V)$ and $\eta \in V$, so $\eta \in L^\infty([0,T);V)$. An integration by parts and Hölder's inequality yield

$$|((v \cdot \nabla)v, \eta)| = |-(vv, \nabla\eta)| \le ||v||^2_{L^4(D)}||\eta||.$$

Here $(vv, \nabla\eta) := (v_j v_m, \eta_{m,j})$, over the repeated indices summation is understood, v_m is the $m-$th Cartesian component of the vector function v, $\eta_{m,j} := \frac{\partial \eta_m}{\partial x_j}$. We use below the multiplicative inequality

$$||v||^2_{L^4(D)} \le c|v|^{1/2}||v||^{3/2}, \quad c = const > 0,$$

(see [5]), and the Young's inequality

$$ab \le \frac{\epsilon^p a^p}{p} + \frac{\epsilon^{-q} b^q}{q}, \ \forall \epsilon > 0; \quad \frac{1}{p} + \frac{1}{q} = 1, \ a, b > 0.$$

By c we denote throughout this paper *various* positive *time independent* constants. Using the Young's inequality with $\epsilon = 1$ and $p = 4$, one gets $|v|^{1/2}||v||^{3/2} \le \frac{|v|^2}{4} + \frac{3||v||^2}{4}$. Since $|v|^2 \in L^1([0,T))$ and $||v||^2 \in L^1([0,T))$, it follows from equation (3) that $(v', \eta) \in L^1([0,T))$ because all other terms in this equation are in $L^1([0,T))$. If (3) holds for all $\eta \in W$, then it holds for all $\eta \in V$, and vice versa, because the set of functions $\eta(x)\phi(t)$ for $\eta \in V$ and $\phi \in L^2([0,T))$ is dense in W. Thus, relation (3) is well defined for $v \in W$ and $\eta \in V$.

The questions of interest are: a) Is the weak solution unique? b) Does it exist globally, that is, for all $t \ge 0$? c) How does it behave as $t \to \infty$? d) Is it smooth if the data are smooth? e) Does the smooth solution to (1)-(2) exist globally? f) Does its smoothness improves if the smoothness of the data improves?

These questions were discussed in several books and many papers, see [1]-[7] and references therein. Existence of the weak solutions was proved in [5]-[7], but its uniqueness was not proved, and, for a long time, it has been an open problem to prove uniqueness of the weak solution. Local existence of the smooth solution and its uniqueness was proved in the cited books. The smoothness properties of the weak solution are improving locally if the smoothness of the data improves. Methods for proving this are developed in [3],[5]-[7], where theorems of this type can be found.

Let $W_1 \subset W$ denote a subset of W that consists of the elements v such that

$$||v(t)|| \le c. \tag{4}$$

By c here and below various positive constants, independent of t, are denoted.

The basic results of this paper include the proof of the uniqueness of the weak solution $v \in W_1$ and the decay estimates for the weak solutions as $t \to \infty$.

Theorem 1.1. *Problem (3) has at most one solution $v \in W_1$.*

Theorem 1.2. *A weak solution in W exists globally and decays exponentially fast as $t \to \infty$ provided that the force term decays sufficiently fast.*

The decay estimates for the solution of problem (3) are given in Lemma 2.1.

The known sufficient condition for the uniqueness of the weak solution is the Serrin's condition (see [7], p.276). If $v \in W_1$, or inequality (21) (see below) holds, then the Serrin's condition holds. Therefore, the result of Theorem 1.1 can be obtained as a consequence of the Serrin's uniqueness result (cf Theorem 1.5.1 on p.276 in [7]). Our proof is based on the estimates given in Lemma 2.1, it is short, and it uses minimal background.

The exponential decay of solutions to Navier-Stokes equations has been discussed in [7], p. 337, for the domains for which the Poincaré inequality holds. Our proof is different and shorter. Moreover, our estimates are valid, in contrast to the ones in [7], also in the case when the data do not decay exponentially fast as $t \to \infty$, see the last statement in Lemma 2.1. We derive estimates using a nonlinear differential inequality. The presentation in this paper is essentially self-contained.

In section 2 a proof of Theorem 1.1 is given and estimates of the solution as $t \to \infty$ are derived in Lemma 2.1. In Section 3 a proof of the existence part of Theorem 1.2 is given. In Section 4 the case of unbounded domain is discussed.

2. Proof of Theorem 1.1

2.1. Some inequalities.

If $D \subset \mathbb{R}^3$ is a bounded domain then $\mathring{H}^1(D) \subset L^q(D)$, $q < 6$, and

$$||v||^2_{L^4(D)} \leq c||v||^{1/2}_{L^2(D)}||\nabla v||^{3/2}_{L^2(D)} \leq \epsilon||v||^2 + \frac{c}{4\epsilon}|v|^2, \ \forall \epsilon > 0. \tag{5}$$

Similar inequalities hold also if $D = \mathbb{R}^3$. For example,

$$||v||^2_{L^4(\mathbb{R}^3)} = 2||v||^{1/2}_{L^2(\mathbb{R}^3)}||\nabla v||^{3/2}_{L^2(\mathbb{R}^3)} \leq \epsilon||\nabla v||^2_{L^2(\mathbb{R}^3)} + c(\epsilon)||v||^2_{L^2(\mathbb{R}^3)}, \tag{6}$$

where $\epsilon > 0$ can be arbitrarily small, and the Young's inequality was used.

Let $\eta = v$ in (3) and get

$$|v|^2 + 2\nu \int_0^t ||v||^2 ds = |v^0|^2 + 2\int_0^t (f, v)ds \leq c + \int_0^t |f||v|ds, \tag{7}$$

where $((v \cdot \nabla)v, v) = 0$ if $\nabla \cdot v = 0$ and $v|_S = 0$. If $\int_0^\infty |f(s)|ds < \infty$, then Lemma 2.1 (see formula (14) below) yields the estimate $\sup_{t\in[0,T)} |v(t)| \leq c$. This estimate and inequality (8) imply that $\sup_{t\in[0,T)} \int_0^t ||v(s)||^2 ds \leq c$.

2.2. Large-time behavior of solutions.

Let us derive some estimates from (7). Denote

$$g(t) := |v|^2, \quad G(t) := ||v||^2, \quad b(t) := |f|.$$

Differentiate (7) with respect to t and get

$$g'(t) + 2\nu G(t) = 2(f, v) \leq 2b(t)g^{1/2}(t). \tag{8}$$

As was mentioned below formula (3), the derivative v' exists in the sense that for all $\eta \in W$ one has $(v', \eta) \in L^1([0,T))$ if $v \in V$. Let us assume that

$$\lim_{t\to\infty} b(t) = 0, \quad \lim_{t\to\infty} \frac{b'(t)}{b(t)} = 0. \tag{9}$$

If D is a finite domain then

$$||v||^2 \geq c_D|v|^2, \quad v \in \mathring{H}^1(D), \quad c_D = const > 0. \tag{10}$$

Thus, $G \geq c_D g$, and inequality (8) implies

$$g' + 2\gamma g \leq 2b(t)g^{1/2}, \quad \gamma := \nu c_D > 0, \quad t \geq 0. \tag{11}$$

Lemma 2.1. *Assume that $g \geq 0$ and inequality (11) holds. Then*

$$g^{1/2}(t) \leq e^{-\gamma t}g^{1/2}(0) + \frac{1}{2}\int_0^t e^{-\gamma(t-s)}b(s)ds, \tag{12}$$

and

$$g(t) \le 2e^{-2\gamma t} g(0) + \frac{1}{2}\left(\int_0^t e^{-\gamma(t-s)} b(s)ds\right)^2. \qquad (13)$$

Assume that $b(t) > 0$ and conditions (9) hold. Then

$$\lim_{t\to\infty} \frac{\int_0^t e^{-\gamma(t-s)} b(s)ds}{b(t)} = \frac{1}{\gamma}, \quad \gamma > 0. \qquad (14)$$

Proof of Lemma 2.1. Let $h(t) = g(t)e^{2\gamma t}$. Then $h(0) = g(0)$. Equation (11) implies $h' \le 2b(t)e^{\gamma t} h^{1/2}$. So,

$$h^{1/2}(t) \le h^{1/2}(0) + \int_0^t b(s)e^{\gamma s} ds,$$

and (12) follows. Inequality (13) follows from (12) since $(a+b)^2 \le 2(a^2+b^2)$. Relation (14) follows from the L'Hospital rule and conditions (9).

Lemma 2.1 is proved. □

Remark 1. If $b(t) = 0$ for $t > t_0$, then (13) yields

$$g(t) \le 2e^{-2\gamma t} g(0) + \frac{e^{-2\gamma t}}{2}\left(\frac{e^{\gamma t_0} - 1}{\gamma}\right)^2 = O(e^{-2\gamma t}).$$

If $b(t) = O(e^{-kt})$ and $k < \gamma$, then $g(t) \le O(e^{-2kt})$. If $k > \gamma$, then $g(t) \le O(e^{-2\gamma t})$. From (9) and (11)–(14) one gets $g'(t) \le O(e^{-\gamma t} + b(t))$. If $b(t) = O(e^{-\gamma t})$, then

$$g'(t) \le O(e^{-\gamma t}). \qquad (15)$$

Estimates in Lemma 2.1 and Remark 1 prove the part of Theorem 1.2. that deals with large-time behavior of the solution to (3). The last statement of Lemma 2.1 allows one to prove decay estimates when the decay of the data f, as $t \to \infty$ is much slower than an exponential. Remember that $b(t) = |f(t)|$ is defined by the data. Conditions (9) and the last statement of Lemma 2.1 allow one to estimate the rate of decay of the integral in formula (12). Conditions (9) hold, for example, if $c_1 t^{-a_1} \le b(t) \le ct^{-a}$ and $|b'(t)| \le ct^{-a-1}$, where $0 < a_1 \le a$, so the decay of the data is much slower than an exponential. This case is not covered by the results in [7].

□

2.3. Proof of the uniqueness of the solution to (3) in the space W_1.

Suppose that $v, w \in W$ solve (3). Let $u = v - w$. Subtract (3) with w in place of v from (3) and get

$$(u', \eta) + \nu((u, \eta)) + ((u \cdot \nabla)v, \eta) + ((w \cdot \nabla)u, \eta) = 0, \ \ \forall \eta \in W. \qquad (16)$$

Take $\eta = u$ and use the relation $((w \cdot \nabla)u, u) = 0$ which holds for $u, w \in W$. Denote $h := |u|^2$, $H := ||u||^2$. Then relation (16) and Hölder's inequality yield

$$h'(t) + 2\nu H(t) \leq ||v(t)|| ||u||^2_{L^4(D)} \leq c||u||^2_{L^4(D)}, \tag{17}$$

where the assumption $||v(t)|| \leq c$ was used. From (17) one gets

$$h(t) + 2\nu \int_0^t H(s) ds \leq c \int_0^t ||u||^2_{L^4(D)} ds. \tag{18}$$

Using inequality (5) one gets

$$||u||^2_{L^4(D)} \leq c|u|^{1/2}||u||^{3/2} \leq \nu H + c(\nu)h, \tag{19}$$

where the Young's inequality was used.

Since $H \geq 0$, inequalities (17) and (19) yield $h' \leq ch$, and $h(0) = 0$ by the assumption. Therefore

$$h(t) \leq c \int_0^t h(s) ds, \quad h(0) = 0. \tag{20}$$

This implies that $h = 0 \ \forall t \geq 0$. The assumption (4) was crucial for the proof. Theorem 1.1 is proved. □

Remark 2. A slight variation of the above argument shows that the additional assumption (4) can be replaced by the assumption

$$\int_0^t ||v(s)||^4 ds \leq c. \tag{21}$$

Recall that $c > 0$ is independent of t.

3. Global existence of the weak solution

In this Section the existence part of Theorem 1.2 is proved. The exponential decay of the solution follows from the estimates proved in lemma 2.1 provided that $b(t) = |f|$ decays exponentially fast. If the weak solution exists globally and is unique, then a smooth solution, if it exists globally, has to be equal to the weak solution due to the uniqueness of the solution. Therefore, the weak solution has to be smooth if a smooth solution exists.

The global existence of the weak solution was proved, for example, in [5]-[7]. We give a slightly different proof. Let $D \subset \mathbb{R}^3$ be a bounded domain. Denote by $\{\phi_j\}_{j=1}^{\infty}$ the eigenvectors of the Stokes operator $-P\Delta$ in $H = L^2(D)$, where P is the Helmholtz-Leray projector (see [1], [5], [6] or [7]).

These eigenvectors are orthonormal in H, and form a basis of V. They solve the problem:

$$-P\Delta\phi_j = \lambda_j\phi_j,\ \phi_j \in V; \quad 0 < \lambda_1 \le \lambda_2 \dots, \lim_{j\to\infty} \lambda_j = \infty; \quad ((\phi_j, \phi_i)) = \lambda_j \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. Let us look for a solution to (3) of the form $v_m = \sum_{j=1}^{m} c_{jm}(t)\phi_j(x)$, where $c_{jm}(t)$ are unknown functions. If v_m is substituted in equation (3) with $\eta = \phi_j$, then one gets:

$$c'_{jm}(t) + \nu\lambda_j c_{jm}(t) + ((v_m \cdot \nabla)v_m, \phi_j) = (f, \phi_j) := f_j(t),\ c_{jm}(0) = (v^0, \phi_j). \tag{22}$$

Multiplying this equation by c_{jm}, summing up over j from $j = 1$ to $j = m$, taking into account that

$$((v_m \cdot \nabla)v_m, v_m) = 0, \quad \sum_{j=1}^{m} \lambda_j c_{jm}^2 = ((v_m, v_m)) := G_m,$$

and denoting

$$g_m := g_m(t) := \sum_{j=1}^{m} c_{jm}^2(t),$$

one gets

$$g'_m + 2\nu\lambda_1 g_m \le g'_m + 2\nu G_m \le 2|P_m f| g_m^{0.5}, \tag{23}$$

where the inequality $\lambda_1 g_m \le G_m$ was used, λ_1 depends on D, and

$$P_m f := \sum_{j=1}^{m} f_j\phi_j, \quad \lim_{m\to\infty} |P_m f - f| = 0.$$

Inequality (23) and Lemma 2.1 imply that

$$g_m(t) \le ce^{-2\gamma t}, \tag{24}$$

where $\gamma := \nu\lambda_1$, the constant $c > 0$ does not depend on m, and it is assumed that $|f(t)| \le O(e^{-2\gamma t})$. The system (22) of ordinary differential equations with the quadratic nonlinearity

$$((v_m \cdot \nabla)v_m, \phi_j) = \sum_{p,q=1}^{m} c_p(t)c_q(t)((\phi_p \cdot \nabla)\phi_q, \phi_j)$$

has a local solution by the standard result. Estimate (24) shows that the local solution is bounded uniformly with respect to t, and, consequently, the functions $c_{jm}(t)$, $1 \le j \le m$, exist globally, that is, for all $t \ge 0$. Furthermore, there exists a subsequence, as $m \to \infty$, denoted c_{jm} again, that converges weakly in $L^2([0,T))$ to a sequence $\{c_j\}_{j=1}^{\infty}$, $c_j = c_j(t)$. From the estimate (24) one concludes that

$$g(t) := \sum_{j=1}^{\infty} c_j^2(t) \le ce^{-2\gamma t}. \tag{25}$$

Therefore, $c_j(t) = O(e^{-\gamma t})$ as $t \to \infty$. Moreover, $\int_0^t G_m(s)ds$ is bounded uniformly with respect to m and $t \ge 0$. To prove this one uses an inequality similar to (7):

$$g_m + 2\nu \int_0^t G_m(s)ds \le g_m(0) + 2\int_0^t b(s) g_m^{1/2}(s)ds,$$

and an estimate of $g_m^{1/2}$ similar to (12). It follows from (25) that $\sum_{j=1}^{\infty} c_j^2(t) \in L^{\infty}([0,T))$. If the subsequence c_{jm} converges weakly to c_j, then v_m converges weakly to a function v in W. Let us check that the limiting function $v = \sum_{j=1}^{\infty} c_j(t)\phi_j$ solves (3). Integrating equation (3) with respect to t one obtains

$$(v,\eta) + \nu \int_0^t ((v,\eta))ds + \int_0^t ((v\cdot\nabla)v,\eta)ds = (v^0,\eta) + \int_0^t (f,\eta)ds, \quad \forall \eta \in V. \tag{26}$$

Let us compare (26) with the relation

$$(v_m,\eta) + \nu \int_0^t ((v_m,\eta))ds + \int_0^t ((v_m\cdot\nabla)v_m,\eta)ds = (v_m^0,\eta) + \int_0^t (P_m f,\eta)ds. \tag{27}$$

Passing to the limit $m \to \infty$ in (27) yields (26). The passage is straightforward in all the terms, except for the term $\int_0^t ((v_m\cdot\nabla)v_m,\eta)ds$. This term can be rewritten as $-\int_0^t (v_m v_m, \nabla\eta)ds$. The embedding operator from V to H is compact. Therefore the weak convergence of v_m in $L^2([0,T);V)$ implies the convergence of the term $-\int_0^t (v_m v_m,\nabla\eta)ds$ to the integral $-\int_0^t (vv,\nabla\eta)ds = \int_0^t ((v\cdot\nabla)v,\eta)ds$. Thus, one can pass to the limit in (27) and get (26). If equation (26) holds, then one can differentiate (26) with respect to t and obtain relation (3) for all $\eta \in V$. The set of the products $\eta h_j(t)$, where $\eta \in V$ and the set $\{h_j(t)\}$ forms a basis of $L^2([0,T))$, is dense in the set W in the norm of $L^2([0,T);V)$. Therefore, if relation (3) holds for all $\eta \in V$ it holds also for all $\eta \in W$. Consequently, the limiting function v satisfies (3). The existence part of Theorem 1.2 is proved. $\square$

4. Unbounded domain

Assume in this section that $D = \mathbb{R}^3$. Then inequality (10) does not hold. We want to outline the proof of the uniqueness result similar to Theorem 1.1 for unbounded domain $\mathbb{R}^3$. Using inequality (6) one gets an analog of inequality (18)

$$||u||^2_{L^4(\mathbb{R}^3)} \leq \nu H(t) + ch(t). \tag{28}$$

This inequality and an inequality similar to (17) yield an analog of inequality (20), and the uniqueness theorem follows as in the case of a bounded domain D. This yields

Theorem 4.1. *If $D = \mathbb{R}^3$ then problem (3) has at most one solution in W_1.*

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