PARAMETER ESTIMATION OF THE BLACK-SCHOLES-MERTON MODEL

by

Kubrom Hisho Teka

B.Sc., University of Asmara, 2002
M.S., Kansas State University, 2011
Ph.D., Kansas State University, 2012

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Major Professor
Prof. James Neill

Abstract

In financial mathematics, asset prices for European options are often modeled according to the Black-Scholes-Merton (BSM) model, a stochastic differential equation (SDE) depending on unknown parameters. A derivation of the solution to this SDE is reviewed, resulting in a stochastic process called geometric Brownian motion (GBM) which depends on two unknown real parameters referred to as the drift and volatility. For additional insight, the BSM equation is expressed as a heat equation, which is a partial differential equation (PDE) with well-known properties. For American options, it is established that asset value can be characterized as the solution to an obstacle problem, which is an example of a free boundary PDE problem. One approach for estimating the parameters in the GBM solution to the BSM model can be based on the method of maximum likelihood. This approach is discussed and applied to a dataset involving the weekly closing prices for the Dow Jones Industrial Average between January 2012 and December 2012.

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Chapter 1

Introduction

Oftentimes the mathematical modeling of numerous phenomena lead to solutions of differential equations which involve unknown parameters. Consider, for instance, the Black-Scholes model, or Black-Scholes-Merton model, which was first presented in[SB]. Merton and Scholes received the 1997 Noble prize in Economics for their work. Though ineligible for the prize due to his death in 1995, Black was mentioned as a contributor by the Swedish academy. It is a model for the asset prices given by the stochastic differential equation :

$$dS(t) = S(t)\{rdt + \sigma dW(t)\},\ t \geq 0, \tag{1.1}$$

where S is the asset value, $r \geq 0$ is the drift rate, σ is the volatility, $W(t)$ is Brownian motion, and t is time in years.

Equation (1.1) can also be written in integral form as:

$$S(t) = S(0) + r\int_0^t S(s)ds + \sigma\int_0^t S(s)dW(s), \quad t \in \mathbb{R}_+.$$

In chapter 2 we introduce **options.** Options are rights to buy or sell underlying assets for an exercise price (strike), which is fixed by the terms of the option contract. That is, the purchaser of the option is not obligated to buy or sell the asset. This decision will be based on the payoff, which is contingent on the underlying assets behavior. Both American and

European option contracts are discussed.

In chapter 3 the Black-Scholes model is derived, including a discussion of Brownian motion and Itó' s integral. The Black-Scholes equation used to price European call and put options based on an asset price dynamic model. This model is a stochastic dynamic systems model that may be written as a stochastic differential equation. See[FM]. The solution to this model takes the form of a stochastic process called geometric Brownian motion (GBM). The model contains two real parameters: drift (r) and volatility (σ). In chapter 4, we express the Black-Scholes model as a heat equation, a partial differential equation whose properties are well known.

We discuss free boundary problems in Chapter 5. We also establish that asset value can be characterized as the solution to an obstacle problem. In particular, the American option is expressed as an obstacle problem, which is an example of a free boundary problem.

In chapter 6, we discuss estimation procedures. Specifically we use the method of Maximum Likelihood Estimation (MLE) to estimate the parameters r and σ. We also construct a 95% confidence interval for the parameters of the GBM. In chapter 7, we do a simulation for sample mean variance of GBM. We conclude the report by giving a summary in chapter 8.

To summarize, in this report we establish relationships between Statistics, Finance and Partial Differential Equations.

Chapter 2

Option Theory

An **option** is the right (but not the obligation) to buy or sell a risky asset at a prespecified fixed price within a specified period. An option is a financial instrument that allows, among other things, one to make a bet on rising or falling values of an underlying asset. The underlying asset typically is a stock, or a parcel of shares of a company. An option is a contract between two parties about trading the asset at a certain future time. One party is the writer, often a bank, who fixes the terms of the option contract and sells the option. The other party is the holder, who purchases the option, paying the market price, which is called a premium[S].

The holder of the option must decide what to do with the rights that the option contract grants. The decision will depend on the market situation, and on the type of options. But first, a few definitions, as given in[S], are collected that are relevant to the discussion.

The **maturity date** T fixes the time horizon. At this date the rights of the holder expire, and for later times ($t > T$) the option is worthless. There are two basic types of options. The **call option** gives the holder the right to buy the underlying asset for an agreed price K, known as the **strike or exercise price** , by the date T. The **put option** gives the holder the right to sell the underlying asset for the price K by the date T. Thus, in summary, at time t the holder of the option can choose to:

1. sell the option at its current market price on some options exchange at $(t < T)$,
2. retain the option and do nothing,
3. exercise the option $(t \leq T)$, or
4. let the option expire worthless $(t \geq T)$.

If you are the writer of these options, you have received a premium, but may be forced to either buy or sell the underlying asset in the future, according to the terms of the contract.

Not every option can be exercised at any time $t \leq T$[S]. For **European options**, exercise is only permitted at expiration time T. **American options** can be exercised at any time up to and including the expiration date.

If we denote the current price of the underlying asset by S, then the **payoffs** at expiration, T, for a given strike price, K, of European calls and puts are, respectively,

$$V(S,T) = \max(S-K,0) = (S-K)^{+} \quad \text{and} \quad V(S,T) = \max(K-S,0) = (K-S)^{+}.$$

The value of $V(S,t)$ also depends on other factors. Dependence on the strike K and the maturity T is evident. Market parameters affecting the price are the interest rate r, the volatility σ of the price S_t, and dividends in case of a dividend-paying asset.

Chapter 3

The Black-Scholes Model

The Black-Scholes option pricing formula is the most famous continuous-time derivative pricing model which gives the price of a European put or call based on five quantities:

- the *intial price* of the underlying stock, which is known,
- the *strike price* of the option, which is known,
- the time to *expiration*, which is known,
- the *risk-free rate* during the lifetime of the option, which is assumed to be constant and can only be estimated,
- the *volatility* of the stock price, a constant that provides a measure of the fluctuation in the stock's price, and thus is a measure of the risk involved in the stock. This quantity can only be estimated as well.

In this chapter we will derive the Black-Scholes option pricing model. But first Brownian motion.

3.1 Brownian Motion

In 1827, just 35 years after the New York Stock Exchange was founded, an English botanist named Robert Brown studied the motion of small pollen grains immersed in a liquid medium[RO]. Brown wrote that pollen grains exhibited a continuous swarming motion when viewed under the microscope. Brownian motion is the most important stochastic process, being the archetype of Gaussian processes, of continuous time martingales, and of Markov processes. It is fundamental to the study of stochastic differential equations, financial mathematics, and filtering, for example. A formal mathematical construction of Brownian motion and its properties was first given by the mathematician Norbert Wiener beginning in 1918, based on Fourier series. Subsequently, martingale techniques have been employed to construct Brownian motion as well.

In this section we describe Brownian motion and construct the associated Itó stochastic integral. First, some relevant definitions are given. See[P]

3.1.1 Definition. A stochastic process is a family $(X_t)_{t \in T}$ of random variables $X_t : \Omega \to \mathbb{R}$ indexed by a set T.

Real-life examples of stochastic processes include:

- the time evolution of a risky asset. In this case X_t represents the price of the asset at time $t \in T$
- the time evolution of a physical parameter. For instance, X_t represents a temperature observed at time $t \in T$.

Brownian motion is a fundamental example of a stochastic process. Here we work on a probability space $(\Omega, \mathbb{F}, P)$, where $\Omega = C_0(\mathbb{R}_+)$ is the space of continuous real-valued functions on $\mathbb{R}_+$ started at 0.

3.1.2 Definition. Brownian motion is a stochastic process $(W_t)_{t \in R_+}$ such that

1. $W_0 = 0$ almost surely

2. the sample trajectories $t \to W_t$ are continuous with probability 1

3. for any finite sequence of times $t_0 < t_1 < ... < t_n$, the increments $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, ..., W_{t_n} - W_{t_{n-1}}$ are independent

4. for any times $0 \leq s < t$, $W_t - W_s$ is normally distributed with mean zero and variance $t - s$.

The existence of Brownian motion as a stochastic process (W_t) is given in[P].

Continuing the treatment as in[P], we regard Brownian motion as a random walk over infinitesimal time intervals of length Δt, with increments ΔW_t over the time interval $[t, t + \Delta t]$ given by

$$\Delta W_t = \pm\sqrt{\Delta t} \tag{3.1}$$

with equal probabilities.

By splitting the interval $[0, T]$ into N intervals

$$(\frac{k-1}{N}T, \frac{k}{N}T], \quad k = 1, ..., N,$$

of length $\Delta t = \frac{T}{N}$, with N large and letting

$$X_k = \pm\sqrt{T} = \pm\sqrt{N}\,\sqrt{\Delta t} = \sqrt{N}\,\Delta W_t$$

with $Var(X_k) = T$,

$$\Delta W_t = \frac{X_k}{\sqrt{N}} = \pm\sqrt{\Delta t}$$

is the increment of W_t over $((k-1)\Delta t, k\Delta t]$ and we get

$$W_T \simeq \sum_{0<t<T} \Delta W_t \simeq \frac{X_1 + ... + X_N}{\sqrt{N}}$$

Hence, by the central limit theorem we recover the fact that W_t has a centered Gaussian distribution with variance T, see point (4) of definition 3.1.2.

Next, to illustrate, we generate GBM sample paths with $S(0) = 1, r = 1$ and $\sigma = 0.2$. See figure 3.1.

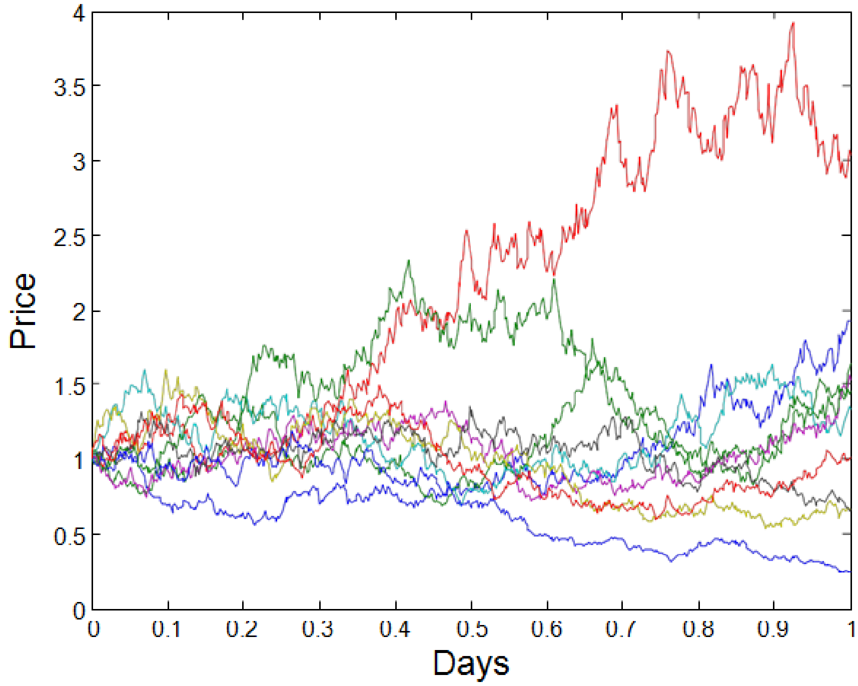

Figure 3.1: *Ten Geometric Brownian Motion Sample Path*

3.1.3 Definition. A stochastic process of the form $\{rt + W_t | t \geq 0\}$ where r is a constant and W_t is Brownian motion with volatility σ is called **Brownian motion** with **drift** r and **volatility** σ. If we let $S_t = e^{W_t}$, then the process $S_t, t \geq 0$ is said to be a **Geometric Brownian Motion.**

3.1.4 Definition. A Brownian motion process $\{Z_t | t \geq 0\}$ with drift $r = 0$ and volatility $\sigma = 1$ is called **Standard Brownian motion.** In this case Z_t has mean 0 and variance t.

If $\{W_t | t \geq 0\}$ is Brownian motion with drift r and variance σ^2, then we can write

$$W_t = rt + \sigma Z_t$$

where $\{Z_t | t \geq 0\}$ is Standard Brownian motion.

We next describe stochastic integrals, also known as Itó integrals, with respect to Brownian motion. See[K].

Consider integrals of a non-random simple process X_t, which is a function of t and does not depend on W_t. By definition a simple non-random process X_t is a process for which there exist times $0 \leq t_0 < t_1 < ... < t_n = T$ and constants $c_1, ..., c_n$, such that

$$X_t = \sum_{i=1}^{n} c_i \, \chi_{(t_{i-1}, t_i]}(t), \quad t \in \mathbb{R}_+.$$

3.1.5 Definition. The Itó integral $\int_0^\infty X_t \, dW_t$ is defined as the sum

$$\int_0^\infty X_t \, dW_t = \sum_{i=1}^{n} c_i \, (W_{t_i} - W_{t_{i-1}}). \tag{3.2}$$

As given in[P], the probability distribution of $\int_0^\infty X_t \, dW_t$ is independent of the representation of X_t. A relevant definition and results follow next.

3.1.6 Definition. A measurable function $f \in L^2(\mathbb{R}_+)$ if and only if $\int_0^\infty |f(x)|^2 \, dx < \infty$.

3.1.7 Proposition. *For $X_t \in L^2(\mathbb{R}_+)$, the integral $\int_0^\infty X_t \, dW_t$ has a centered Gaussian distribution*

$$\int_0^\infty X_t \, dW_t \simeq N(0, \int_0^\infty |X_t|^2 \, dt)$$

and we have the Itó isometry

$$E\{(\int_0^\infty X_t \, dW_t)^2\} = \int_0^\infty |X_t|^2 \, dt.$$

Proof. See Proposition 4.1 of[P]. ■

By a Taylor expansion,

$$df(x) = f'(x)dx + \frac{1}{2} f''(x)(dx)^2 + \frac{1}{3!} f'''(x)(dx)^3 + ...$$

and by applying Taylor's formula to Brownian motion

$$dW_t = W_{t+dt} - W_t,$$

and letting

$$df(W_t) = f(W_{t+dt}) - f(W_t),$$

we have

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2 + \frac{1}{3!}f'''(W_t)(dW_t)^3 + ...$$

By (3.1) and dropping higher order terms gives

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt, \quad \text{for a small } dt.$$

By integrating and applying the fundamental theorem of calculus gives the integral form of Itó's formula for Brownian motion

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds.$$

Next a general expression of Itó's formula is given which applies to an Itó process of the form

$$X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds, \quad t \in \mathbb{R}_+. \tag{3.3}$$

3.1.8 Theorem (Ito's formula for Itó process)**.** *For any Itó process $(X_t)_{t\in\mathbb{R}_+}$ of the form (3.3) and any $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R})$, we have*

$$f(t, X) = f(0, X_0) + \int_0^t V_s \frac{\partial f}{\partial x}(s, X_s)ds + \int_0^t U_s \frac{\partial f}{\partial x}(s, X_s)dW_s + \int_0^t \frac{\partial f}{\partial s}(s, X_s)ds$$
$$+ \frac{1}{2}\int_0^t |U_s|^2 \frac{\partial^2 f}{\partial x^2}(s, X_s)ds$$

Proof. See[P] ■

3.1.9 Theorem. *The solution to* (1.1) *is given by*

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}\ ,\ t \in \mathbb{R}_+. \tag{3.4}$$

Proof. First rewrite (1.1) in integral form as

$$S(t) = S(0) + r\int_0^t S(s)ds + \sigma\int_0^t S(s)\,dW(s), \quad t \in \mathbb{R}_+.$$

By applying Itó's formula to $f(S(t)) = \log S(t)$ with $f(x) = \log x$, we have

$$\begin{aligned} d\text{log}S(t) &= rS(t)\,f'(S(t))dt + \sigma S(t)f'(S(t))dW(t) + \frac{1}{2}\sigma^2 S^2(t)\,f''(S(t))dt \\ &= rdt + \sigma dW(t) - \frac{1}{2}\sigma^2 dt \\ &= \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t). \end{aligned}$$

Hence,

$$\begin{aligned} \text{log}S_t - \text{log}S(0) &= \int_0^t d\log S(x) \\ &= \int_0^t (r - \frac{1}{2}\sigma^2)\,dx + \int_0^t \sigma\,dW(x) \\ &= (r - \frac{1}{2}\sigma^2)t + \sigma\,W(t), \quad t \in \mathbb{R}_+, \end{aligned}$$

and

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)}\ ,\ t \in \mathbb{R}_+.$$

■

Asset prices can be modeled by the following stochastic differential equation (SDE)[R] :

$$dS(t) = S(t)\{rdt + \sigma dW(t)\},\ t \geq 0,$$

where S is the asset value, $r \geq 0$ is the drift rate, σ is the volatility, $W(t)$ is Brownian motion (as described in the preceding), and t is time in years. Because of the dW term, S

itself is also a random variable.

3.1.10 Lemma. *If S satisfies the above SDE, then the value function $V(S,t)$ satisfies the following*

$$dV = \sigma S \frac{\partial V}{\partial S} dW + \left(rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt$$

Proof. See[Hu] ∎

3.2 The Black- Scholes Equation

For the value function $V(S,t)$, the Black-Scholes equation is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = 0 \tag{3.5}$$

where $\delta = 0$ for the European option, as no dividend is paid.

The assumptions that led to (3.5) are[S]:

1. There are no arbitrage opportunities. (Arbitrage means the existence of a portfolio which requires no investment initially, and which with guarantee makes no loss but very likely a gain at maturity.
2. The market is frictionless. This means that there are no transaction costs (fees or taxes), the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, and all securities and credits are available at any time and in any size.
3. The asset price follows a geometric Brownian motion.
4. r and σ are constant for $0 \leq t \leq T$. No dividends are paid in that time period. The option is European.

Under the above assumptions, there exists a closed form solution to some types of option. For some of the more advanced option contracts, the closed form solution may not be possible to determine, and thus a numerical solution is usually obtained.

A closed form solution for the European put option given by[Hu] is

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}s^2}\,ds,\ d_1 = \frac{\log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},\text{ and } d_2 = \frac{\log\frac{S}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Chapter 4

Basic Partial Differential Equations

When considering a partial differential equation, there are few things one needs to ask. The first question is: Do we have a **well-posed problem?** In other words, does a solution to the problem exist and is it unique. And the other question: Is the solution **well-behaved?** In other words, does the solution depend continuously on the initial and boundary conditions, so that a small perturbation in the conditions does not bring a big change in the solution?

One also wants to know the regularity of the solution, that is, which conditions give the best regularity etc. In this chapter we describe the techniques of deriving the solution of the Black-Scholes by converting the Black-Scholes equation to a heat equation, a partial differential equation whose properties are well known.

4.1 The Heat Equation

The **heat equation**, also know as the **diffusion equation**, in $\mathbb{R}$ is given by:

$$\begin{cases} u_\tau = u_{xx} & \text{for } x \in \mathbb{R},\ \tau > 0 \\ u(x,0) = g(x). \end{cases} \tag{4.1}$$

By using the Fourier transform, it is known that the solution of equation (4.1) is:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{\mathbb{R}} g(y) e^{-\frac{(x-y)^2}{4\tau}}\, dy \quad \text{for } -\infty < x < \infty, \quad \tau > 0. \tag{4.2}$$

4.2 Black-Scholes as a Heat Equation

The main purpose of this section is to express the Black-Scholes equation as a heat equation. As described in the above section, the solution of a heat equation is known, and consequently this will allow us to find the solution of the Black-Scholes for a European call.

The Black-Scholes equation and boundary conditions for a European call with values $V(S,t)$, as described in (3.5) , is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

with

$$V(0,t) = 0, \qquad V(S,t) \sim S \quad \text{as } S \to \infty,$$

and

$$V(S,t) = (S-K)^+.$$

To write the Black-Scholes equation as a heat equation requires a series of change variables[DHW]. The first change of variable is to let

$$S = Ke^x, \qquad t = T - 2\tau/\sigma^2, \qquad V = Kv(x,\tau).$$

This results in

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (b-1)\frac{\partial v}{\partial x} - bv \quad \text{and} \quad v(x,0) = (e^x - 1)^+$$

where $b = 2r/\sigma^2$.

If we do a change of variable one more time by setting $v = e^{\alpha x+\beta\tau}u(x,\tau)$, for some constants α and β to be specified later, we get

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 v}{\partial x^2} + (b-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - bu.$$

Now set $\alpha = -1/2(b-1)$ and $\beta = -1/4(b+1)^2$, to obtain

$$v = e^{-1/2(b-1)x - 1/4(b+1)^2\tau}u(x,\tau)$$

where

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } -\infty < x < \infty, \quad \tau > 0,$$

with

$$u(x,0) = g(x) := \left(e^{1/2(b+1)x} - e^{1/2(b-1)x}\right)^+.$$

Now that we have transformed the Black-Scholes equation into a heat equation, the solution of the Black-Scholes equation will take the form of (4.2) .

In the next chapter, the American option will be expressed as a free boundary, specifically as an obstacle problem.

Chapter 5

Free boundary Problems

Free boundary problems deal with solving partial differential equations (PDEs) in a domain, a part of whose boundary is unknown in advance. That portion of the boundary is called a free boundary. In addition to the standard boundary conditions that are needed in order to solve the PDEs, an additional condition must be imposed at the free boundary. One then seeks to determine both the free boundary and the solution of the differential equations.

Free boundary problems have applications in Finance, and we will express the American option as an obstacle problem, which is an example of a free boundary problem. A brief description of the obstacle problem is first given.

5.1 The Obstacle Problem

What happens when we pull an elastic membrane down over an obstacle?

To formulate what is happening mathematically:

assume the membrane is given by the graph

$$u : B_1 \subset \mathbb{R}^n \longrightarrow \mathbb{R},\ u \equiv 0 \quad \text{on} \quad \partial B_1,\ \text{and}\ \varphi : B_1 \subset \mathbb{R}^n \longrightarrow \mathbb{R},\ \varphi < 0 \quad \text{on} \quad \partial B_1.$$

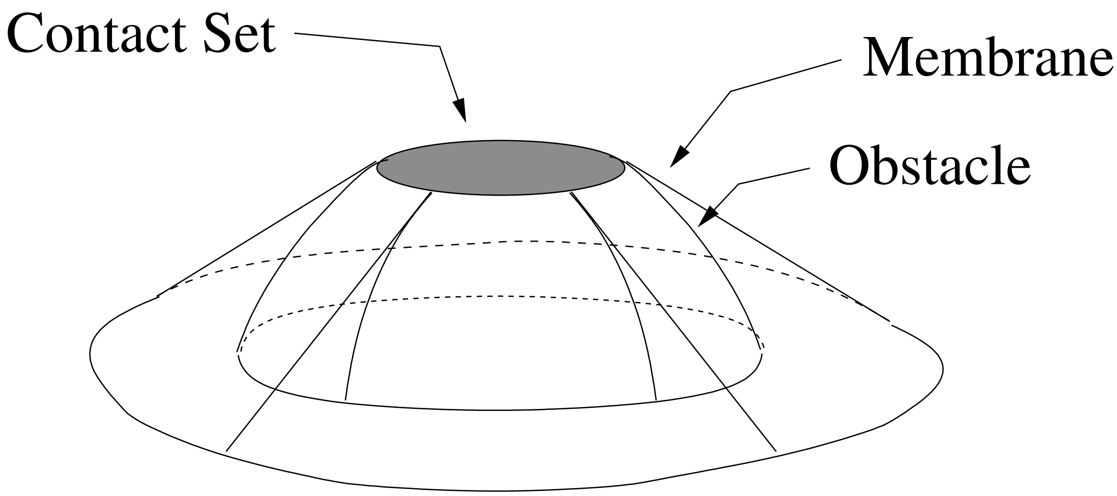

Figure 5.1: *The Obstacle problem*

We want to find a function u (the "membrane") which minimizes the Area integral:

$$I_1(u) := \int_{B_1} \sqrt{1 + |\nabla u|^2} \qquad \text{among } \ u \ \text{ satisfying :}$$

- $u = 0$ on ∂B_1 (i.e. the membrane is "pinned down") and
- $u \geq \varphi$ in B_1 (i.e. the membrane is above the obstacle).

From the calculus of variations, it follows that functions which minimize I_1 in a neighborhood among functions with fixed boundary data satisfy the minimal surface equation. Observe that for a small deflection of the membrane, $|\nabla u|^2$ is the first important term in the Taylor expansion of $\sqrt{1 + |\nabla u|^2}$. (i.e. $\sqrt{1+x} \approx 1 + \frac{1}{2}x$, for x small.) Thus, we want to find a function u which minimizes

$$I_2(u) := \int_{B_1} |\nabla u|^2 \qquad \text{i.e. Energy - The Dirichlet Integral}$$

among u satisfying:

- $u = 0$ on ∂B_1 (i.e. the membrane is "pinned down") and
- $u \geq \varphi$ in B_1 (i.e. the membrane is above the obstacle).

It is known that functions which locally minimize I_2 satisfy Laplace's equation. Linearizing the Area integral is standard in the study of the obstacle problem, mainly because it adds technical simplification in that it changes the operator from nonlinear to linear, without altering the real difficulties of the problem.

Therefore, the obstacle problem involves finding a function u which solves the problem[C]:

$$\text{minimize} \quad \int_{B_1} |\nabla u|^2 \, dx \;\; \text{among all functions } u \in K_\varphi,$$

where we define K_φ to be the closed convex set:

$$K_\varphi := \{u \in W_0^{1,2}(B_1),\ u \geq \varphi\}$$

We also have

$$\Delta u = 0, \text{ for } u > \varphi \quad \text{and} \quad \Delta u \leq 0, \text{ everywhere.} \tag{5.1}$$

If we define the "height function" $w := u - \varphi$, and we let $f := -\Delta\varphi$, then w satisfies:

$$\Delta w = \chi_{\{w>0\}} f \ .$$

Since the problem above is variational, existence and uniqueness of solutions follows. Regularity of the solution has been studied by many authors, and in the case where φ is smooth, Frehse showed in 1972 that the solutions belong to $C^{1,1}$. Finally, in Caffarelli's famous Acta paper in 1977, the regularity of the free boundary was addressed in the case where f was Hölder continuous and positive.

Since the obstacle problem was formulated, but especially in the last 15 years, there has been interest in extending some of these results to related problems. Ki-Ahm Lee studied the case where the Laplacian is replaced with a fully nonlinear (but smooth) operator. Blank studied the case where the function f was not assumed to be Hölder continuous. Blank and

Teka[BT] studied the the case where the Laplacian is replaced with a general second order elliptic operator in nondivergence form. Many people (e.g. Blanchet, Caffarelli, Dolbeault, Monneau, Petrosyan, Shahgholian, Weiss) have recently studied the case where the Laplacian is replaced with the Heat Operator.

5.2 American Options as an Obstacle Problem

The explicit formula discussed in chapter 4 is valid for European options where early exercise is not allowed. It does not necessarily give the values for American options which has the additional feature that exercise is permitted at any time during the life of the option.

A European option can have a value that is smaller than the payoff. This cannot happen with American options[S]. If, for instance, an American put would have a value $V_p^{Am} < (K-S)^+$, one would simultaneously purchase the asset and the put, and exercise immediately. Thus, there is an obvious arbitrage opportunity, even though this opportunity would not last long before the value of the option is pushed up by the demand of the arbitragers. The same is true for an American call if $V_c^{Am} < (S-K)^+$. Therefore, the following constraints are imposed:

$$V_p^{Am} \geq (K-S)^+ \text{ and } V_c^{Am} \geq (S-K)^+ \text{ for all } \ (S,t) \tag{5.2}$$

Thus, there must be a value of S for which it is optimal from the holder's point of view to exercise the American option. The valuation of the American options is therefore more complicated, since at each time we have to determine not only the option value, but also, for each value of S, whether or not it should be exercised. This is what is known as a **free boundary problem**[DHW].

If we focus on an American put, then for $0 \leq t \leq T$ and for a **contact point** $S_f(t) \in (0, K)$

we have

$$V_p^{Am}(S,t) > (K-S)^+ \quad \text{for } S > S_f(t)$$

and

$$V_p^{Am}(S,t) = (K-S) \quad \text{for } S \leq S_f(t).$$

The curve S_f is the boundary separating the area with $V >$ payoff and the area with $V =$ payoff. The curve S_f of a put is illustrated in the diagram below.

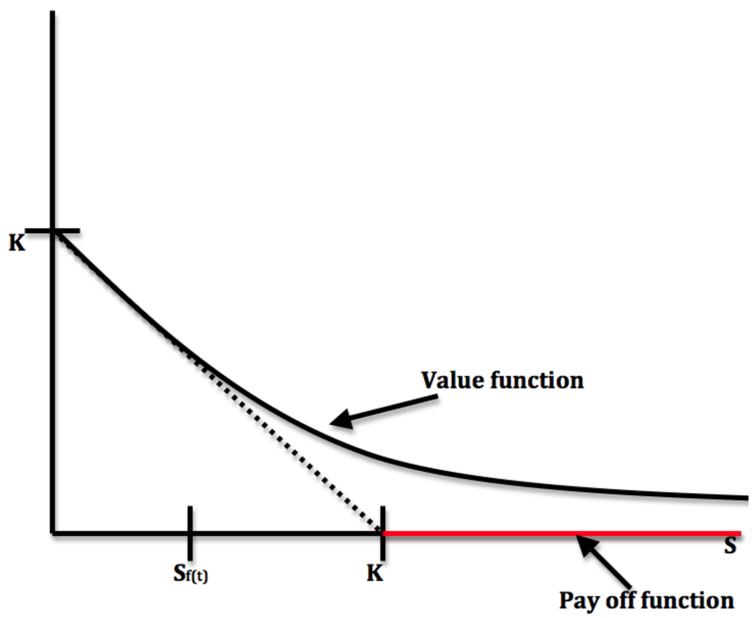

Figure 5.2: *American put - pay off function*

A priori, just as in the obstacle problem, the location of the boundary S_f is unknown, the curve is free. This explains why the problem of calculating $V_p^{Am}(S,t)$ for $S > S_f(t)$ is called **free boundary problem,** and specifically the obstacle problem.

Let us introduce an operator $\tilde{L}$ defined by

$$\tilde{L}V := \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV.$$

Thus, (3.5) can be written as

$$\frac{\partial V}{\partial t} + \tilde{L}V = 0.$$

For the case of a put, $S \leq S_f$ and we have

$$V = K - S,\ \frac{\partial V}{\partial t} = 0,\ \frac{\partial V}{\partial S} = -1,\ \frac{\partial^2 V}{\partial S^2} = 0.$$

Thus,

$$\frac{\partial V}{\partial t} + \tilde{L}V = -(r-\delta)S - r(K-S) = \delta S - rK < 0.$$

But since $\delta S < rK$, we have

$$\frac{\partial V}{\partial t} + \tilde{L}V < 0.$$

The same holds for the call.

Thus, American options satisfy

$$\frac{\partial V}{\partial t} + \tilde{L}V \leq 0.$$

Now let us consider the obstacle problem in $\mathbb{R}^1$. Graphically this can be described as in fig. 5.2.

We express equation (5.1) as

$$u'' = 0,\ \text{for } u > \varphi \quad \text{and} \quad u'' \leq 0, \quad u > \varphi.$$

It turns out we can express American options as

1. If $V >$ payoff, then Black-Scholes equation $\frac{\partial V}{\partial t} + \tilde{L}V = 0$

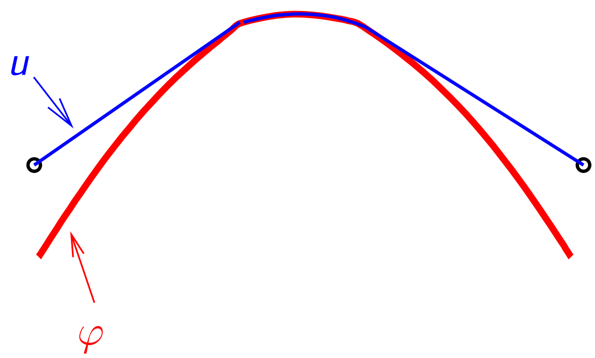

Figure 5.3: *The Obstacle problem in* $\mathbb{R}^1$

2. If V = payoff, then Black-Scholes inequality $\frac{\partial V}{\partial t} + \tilde{L}V < 0$

Expressing the Black-Scholes model in terms of an obstacle problem serves to aid better understanding of the model as the theory of the obstacle problem is well developed. The purpose of this section was to establish the relationship between the Black-Scholes model and the obstacle problem for future investigation.

Chapter 6

Overview of estimation procedure

From (3.4) , the solution to the Black-Scholes model is given by

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t} \ , \ t \in \mathbb{R}_+.$$

Thus, for $S(t) > 0$,

$$\ln(S(t)) = \ln(S(0)) + (r - \frac{1}{2}\sigma^2)t + \sigma W_t \ , \ t \in \mathbb{R}_+,$$

and for $k = 1, 2, ...,$

$$\ln(S(t_k)) - \ln(S(t_{k-1})) = (r - \frac{1}{2}\sigma^2)(t_k - t_{k-1}) + \sigma(W_{t_k} - W_{t_{k-1}}) \ , \ t_k \in \mathbb{R}_+ \tag{6.1}$$

where $W_{t_k} - W_{t_{k-1}}$ is normally distributed with mean zero and variance $t_k - t_{k-1}$ so that

$$\ln(S(t_k)) - \ln(S(t_{k-1})) \sim N((r - \frac{1}{2}\sigma^2)(t_k - t_{k-1}), \sigma^2(t_k - t_{k-1}))$$

for discrete time points t_k.

Further, as given in[B], the mean and variance of $S(t)$ are, respectively,

$$E[S(t)] = S(0)\, e^{rt} \quad \text{and} \quad \text{Var}[S(t)] = e^{2rt}S^2(0)\left(e^{\sigma^2 t} - 1\right).$$

Also, as noted by[B], to simulate this process, the continuous equation between discrete instants $t_0 < t_1 < ... < t_n$ needs to be solved as follows:

$$S(t_k) = S(t_{k-1})e^{[(r-\frac{1}{2}\sigma^2)(t_k-t_{k-1})+\sigma\sqrt{t_k-t_{k-1}}\,Z_k]} \tag{6.2}$$

where $Z_1, Z_2, ...Z_n$ are independent random draws from the standard normal distribution.

The parameters to be estimated are $\theta = (r, \sigma)$ in the GBM process. Since GBM is a continuous time process, an approximate MLE based on discrete observations will be obtained (see[AP] and[HN]).

6.1 Maximum Likelihood Estimation

The maximum likelihood estimation method is illustrated on data comprised of the weekly closing prices for the Dow Jones Industrial Average for the year 2012. This data is given in the Appendix.

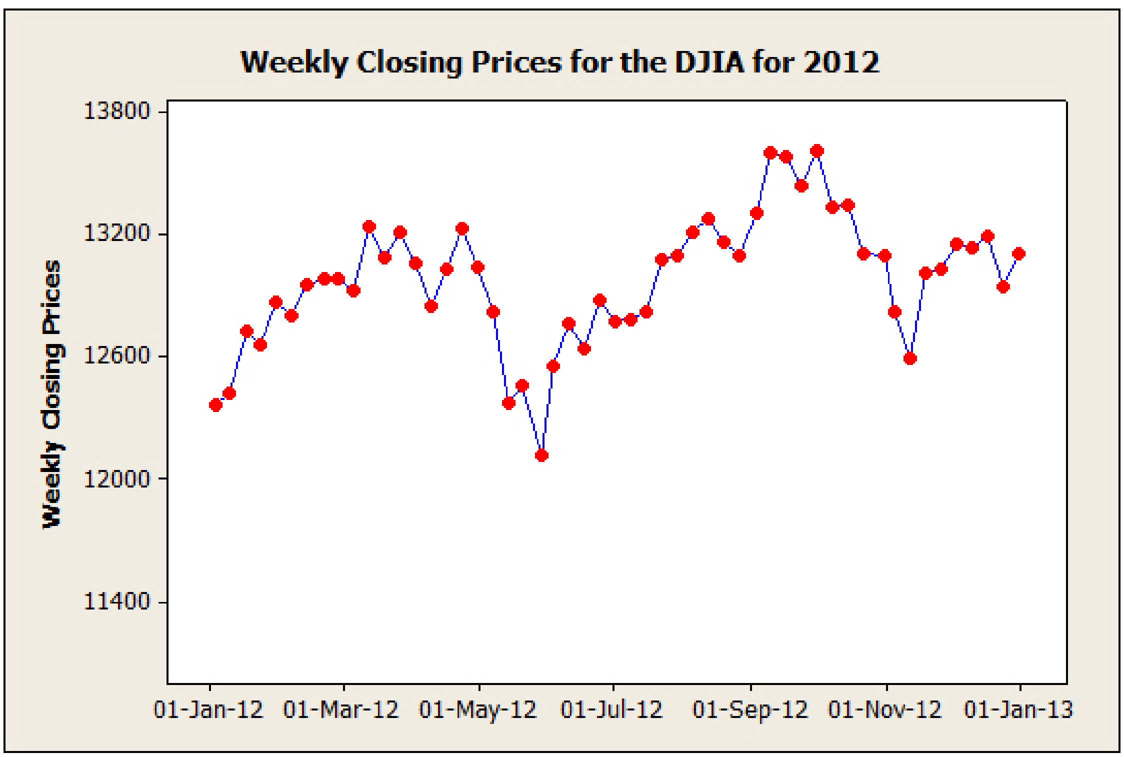

Figure 6.1: *Weekly Closing Prices for the DJIA between Jan 2012 - Dec 2012*

First set $x(t_k) := \ln(S(t_k)) - \ln(S(t_{k-1}))$. As indicated in figures 6.2 and 6.3, the autocorrelation and partial autocorrelation graphs for $x(t_k)$ do not exhibit any significant lags. Thus, an assumption of independence for the $x(t_k)$ is tenable. Moreover, $x(t_k)$, satisfies the assumption of normality. See figure 6.4.

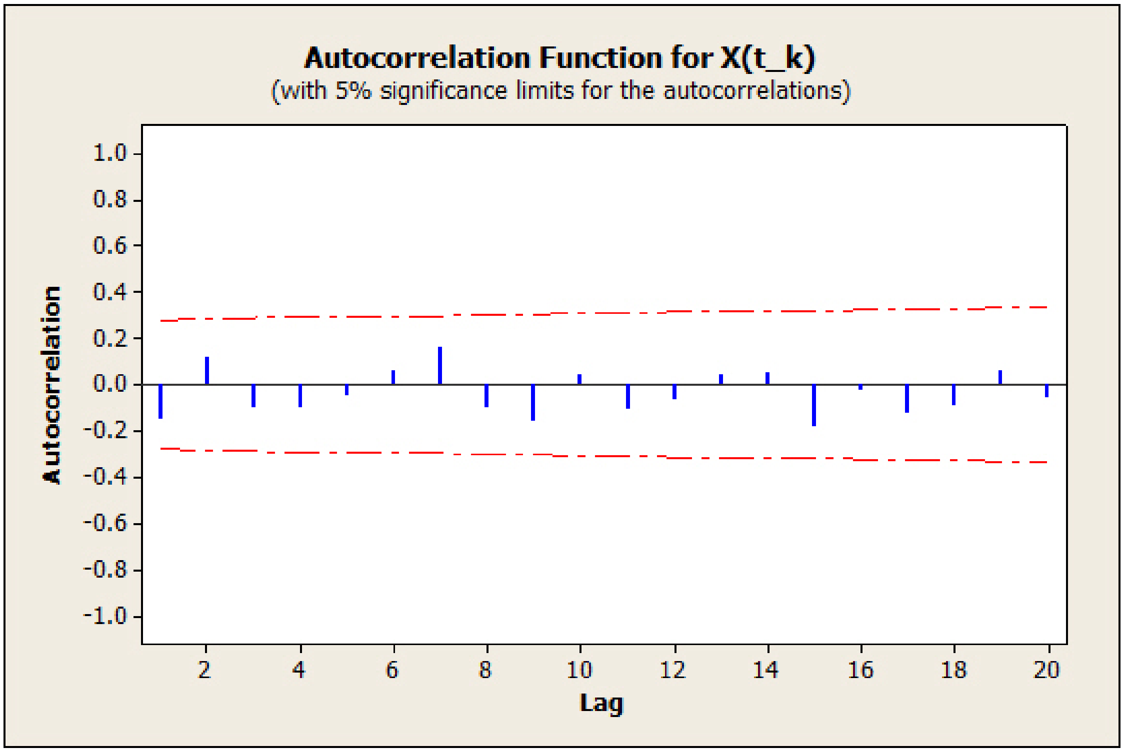

Figure 6.2: *Autocorrelation Function for* $x(t_k)$

With independence and normality tenable, the log likelihood function is given by

$$L(\theta) = \sum_{k=1}^{n} \ln\left(f_\theta(x(t_k))\right) \tag{6.3}$$

where the probability density function f_θ is given by

$$f_\theta(x(t_k)) = \frac{1}{S(t_k)\,\sigma\,\sqrt{2\pi(t_k - t_{k-1})}} \exp\left(-\frac{\left[x(t_k) - (r - \frac{1}{2}\sigma^2)(t_k - t_{k-1})\right]^2}{2\sigma^2(t_k - t_{k-1})}\right).$$

To determine $\hat{\theta}$, let $\Delta t := t_k - t_{k-1}$ in (6.1) . The mean and variance parameters are

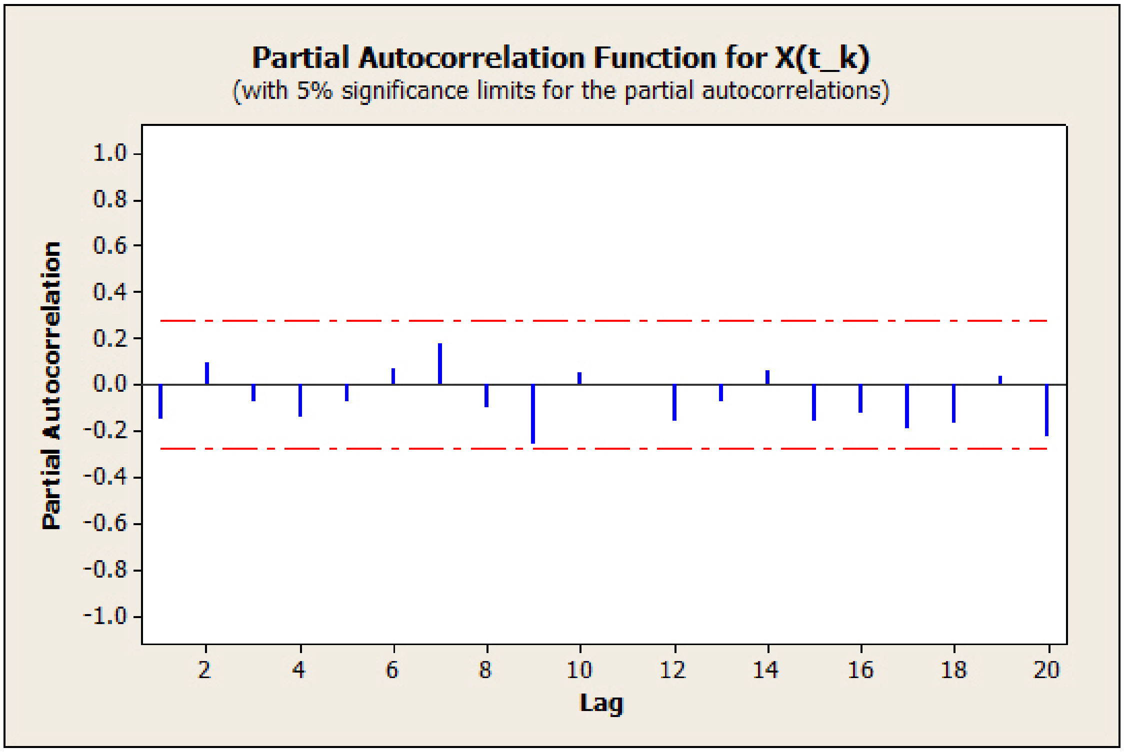

Figure 6.3: *Partial Autocorrelation Function for* $x(t_k)$

identified and computed as

$$\hat{m} := \left(\hat{r} - \frac{1}{2}\hat{\sigma^2}\right)\Delta t \text{ and } \hat{v} := \hat{\sigma^2}\Delta t. \tag{6.4}$$

The estimates for the GBM parameters are then deduced from the estimates of m and v[B]. To get a closed form expression for the parameters m and v, the derivative of the above density function is taken with respect to these parameters and setting the resulting derivatives equal to zero, yielding

$$\hat{m} = \sum_{k=1}^{n} \frac{x(t_k)}{n} \quad \text{and} \quad \hat{v} = \sum_{k=1}^{n} \frac{(x(t_k) - \hat{m})^2}{n}. \tag{6.5}$$

Thus, for the dataset under consideration,

$$\hat{m} = \sum_{k=1}^{52} \frac{x(t_k)}{52} = 0.00112$$

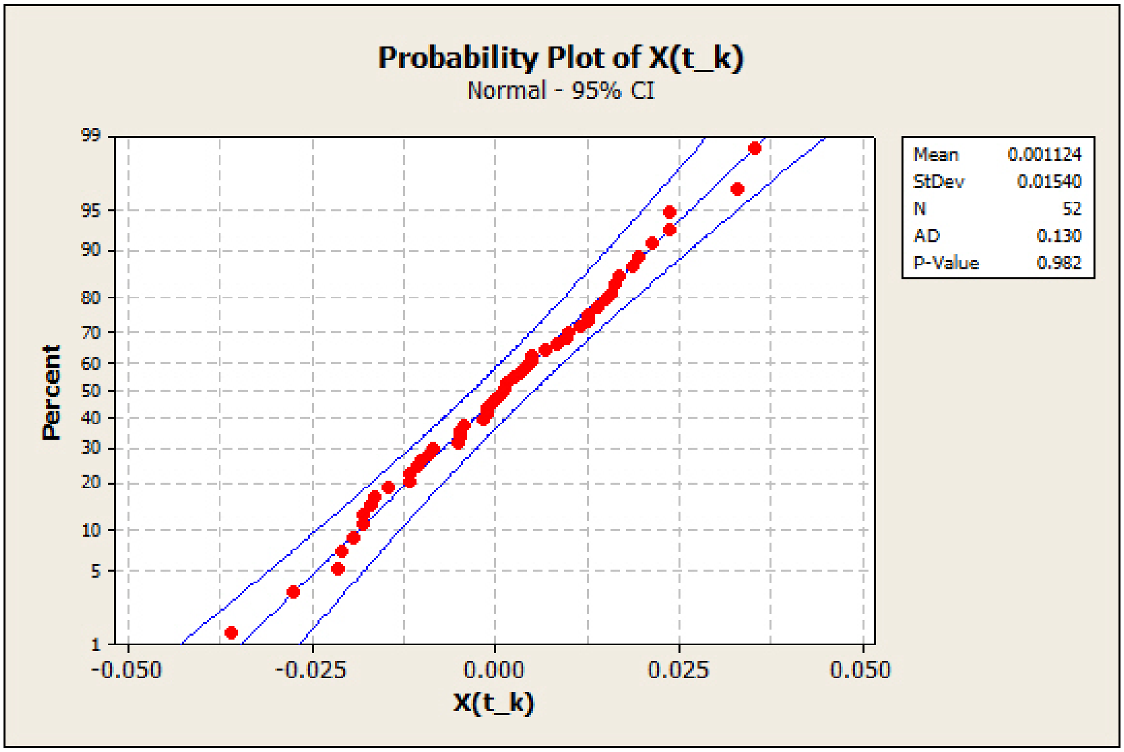

Figure 6.4: *Probability plot for* $x(t_k)$

and

$$\hat{v} = \sum_{k=1}^{n} \frac{(y(x(t_k)) - \hat{m})^2}{n} = \sum_{k=1}^{52} \frac{(x(t_k) - 0.00112)^2}{52} = 0.00023.$$

6.2 Confidence Level

The 95% confidence intervals for the parameters m and v, respectively, are given by [B]

$$\hat{m} - 1.96\frac{\sqrt{\hat{v}}}{\sqrt{n}} \leq m \leq \hat{m} + 1.96\frac{\sqrt{\hat{v}}}{\sqrt{n}} \tag{6.6}$$

and

$$\frac{n}{\chi^2_{n,0.025}}\,\hat{v} \leq v \leq \frac{n}{\chi^2_{n,0.975}}\,\hat{v} \tag{6.7}$$

where $\chi^2_{n,0.025}$ and $\chi^2_{n,0.975}$ are the quantiles of the chi-square distribution with n degrees of freedom corresponding to a 95% confidence level.

The computed 95% confidence intervals for the mean and variance as given in equations (6.6) and (6.7) , respectively are

$$0.00112 - 1.96\frac{\sqrt{0.00023}}{\sqrt{52}} \leq m \leq 0.00112 + 1.96\frac{\sqrt{0.00023}}{\sqrt{52}}$$
$$-0.003 \leq m \leq 0.005$$

and

$$\frac{52}{594.62423519}\,0.00023 \leq v \leq \frac{52}{467.16334532}\,0.00023$$
$$0.00002 \leq v \leq 0.00003.$$

For $\Delta t = \frac{1}{52}$, and from (6.4) ,

$$\hat{\sigma^2} = \frac{\hat{v}}{\Delta t} = (0.00023)(52) = 0.01196 \quad \text{and hence} \quad \hat{\sigma} = 0.1094$$

and

$$\hat{r} = \frac{1}{2} * 0.01196 + (52)(0.00112) = 0.06422.$$

Thus,

$$\hat{r} = 0.06422 \quad \text{and} \quad \hat{\sigma} = 0.1094. \tag{6.8}$$

In the next chapter, simulated parameter distributions are considered based on the so-called first parameters being taken as the estimates of $\hat{r}$ and $\hat{\sigma}$. Such can be considered in the absence of explicit relationships as discussed in[B].

Chapter 7

Simulations for Sample Mean and Variance of GBM

We generated 15,000 data sets each containing 52 normal random values with mean $\hat{m} =$ 0.00112 and standard deviation of $\hat{v} = 0.01525$. This can be done in Matlab by using:

$$normrnd(0.00112, 0.01525, 15000, 52)$$

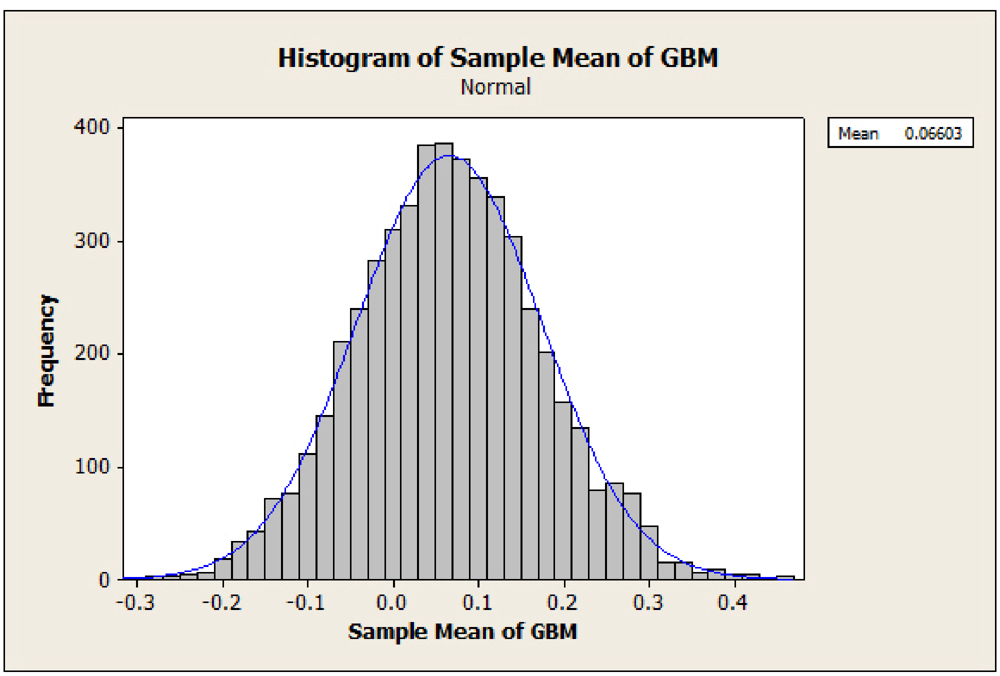

Figure 7.1: *Simulation of Sample Mean of GBM*

The mean and variance are computed for of each of the 15,000 data sets. These values are used in (6.4) , to compute the sample estimates for the GBM parameters corresponding to each of the 15,000 data sets. The Δt in this case is $\frac{1}{52}$ as we are considering the weekly closing prices for the DJIA.

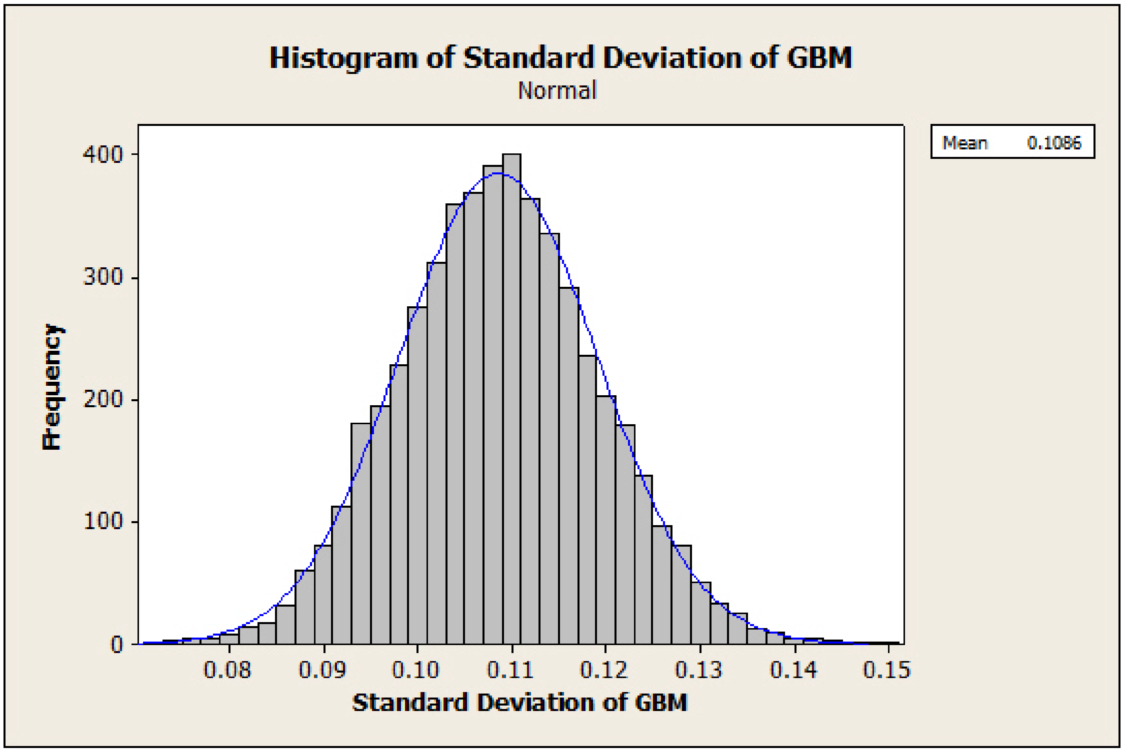

Figure 7.2: *Simulation of Sample Standard Deviations of GBM*

The frequency histograms for the sample estimates are given in figure 7.1 and 7.2, respectively. The sample mean, 0.06603, and sample standard deviation, 0.1086, are observed to be very close to the closed form theoretical estimates found in (6.8) . These computations were done in Matlab. The Matlab code is given below.

Code: For Simulation of Sample mean and Standard deviations of GBM

$Z = normrnd(0.00112, 0.01525, 15000, 52);$

$R = Z';$

$m = sum(R)/52;$

$p = m';$

$for\ k = 1 : 15000;$

$for\ j = 1 : 52;$

$D(k, j) = (Z(k, j) - p(k, 1)).^2;$

end;

end;

$V = sum(D')/52;$

$V = V';$

$t = 1/52;$

$S = sqrt(V/t)$

$A = 0.5 * S.^2 + (1/t) * p$

Chapter 8

Summary

Stochastic differential equations are utilized in financial applications, as well as science and engineering applications involving chemical engineering and neurobiology, for example. Such equations generally involve unknown parameters, requiring statistical estimation techniques. The Black-Scholes-Merton model used in financial mathematics for European options has been considered in this report, and the method of maximum likelihood for parameter estimation used for estimating the unknown drift and volatility parameters. This approach is discussed and applied to a dataset involving the weekly closing prices for the Dow Jones Industrial Average between January 2012 and December 2012. In addition, it was established that asset values for American options can be characterized as the solution to an obstacle problem, which is an example of a free boundary partial differential equation problem. For future work, comparison of maximum likelihood estimation techniques with recent methods involving data smoothing along with a generalization of profiled estimation would be of interest[CHR].

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Appendix A

Data

Weekly closing price for the Dow Jones Industrial Average between Jan, 2012 and Dec, 2012.

DJIA				
Date	t_k	S(t_k)	ln(S(t_k))	x(t_k)=ln(S(t_k)) - ln(S(t_k-1))
3-Jan-12	0	12,359.92	9.422214258	
9-Jan-12	1	12,422.06	9.427229203	0.005014945
17-Jan-12	2	12,720.48	9.450968572	0.023739369
23-Jan-12	3	12,660.46	9.44623903	-0.004729542
30-Jan-12	4	12,862.23	9.462050389	0.015811359
6-Feb-12	5	12,801.23	9.457296539	-0.00475385
13-Feb-12	6	12,949.87	9.468841028	0.011544489
21-Feb-12	7	12,982.95	9.471392237	0.002551209
27-Feb-12	8	12,977.57	9.470977762	-0.000414476
5-Mar-12	9	12,922.02	9.466688112	-0.00428965
12-Mar-12	10	13,232.62	9.490440272	0.02375216
19-Mar-12	11	13,080.73	9.478895434	-0.011544838
26-Mar-12	12	13,212.04	9.488883814	0.00998838
2-Apr-12	13	13,060.14	9.477320123	-0.011563692
9-Apr-12	14	12,849.59	9.461067183	-0.016252939
16-Apr-12	15	13,029.26	9.474952876	0.013885693
23-Apr-12	16	13,228.31	9.490114509	0.015161632
30-Apr-12	17	13,038.27	9.475644158	-0.014470351
7-May-12	18	12,820.60	9.458808531	-0.016835627
14-May-12	19	12,369.38	9.422979343	-0.035829188
21-May-12	20	12,454.83	9.429863778	0.006884436
29-May-12	21	12,118.57	9.402494266	-0.027369513
4-Jun-12	22	12,554.20	9.43781055	0.035316284
11-Jun-12	23	12,767.17	9.454632311	0.016821761
18-Jun-12	24	12,640.78	9.444683375	-0.009948937
25-Jun-12	25	12,880.09	9.463437987	0.018754613
2-Jul-12	26	12,772.47	9.455047352	-0.008390635
9-Jul-12	27	12,777.09	9.455409002	0.00036165
16-Jul-12	28	12,822.57	9.458962178	0.003553176
23-Jul-12	29	13,075.66	9.478507766	0.019545587
30-Jul-12	30	13,096.17	9.4800751	0.001567334
6-Aug-12	31	13,207.95	9.4885742	0.0084991
13-Aug-12	32	13,275.20	9.493652912	0.005078712
20-Aug-12	33	13,157.97	9.484782938	-0.008869974
27-Aug-12	34	13,090.84	9.479668028	-0.00511491
4-Sep-12	35	13,306.64	9.496018438	0.01635041
10-Sep-12	36	13,593.37	9.517337453	0.021319015
17-Sep-12	37	13,579.47	9.516314372	-0.00102308
24-Sep-12	38	13,437.13	9.50577705	-0.010537323

1

1-Oct-12	39	13,610.15	9.518571117	0.012794067
8-Oct-12	40	13,328.85	9.497686138	-0.020884979
15-Oct-12	41	13,343.51	9.498785403	0.001099265
22-Oct-12	42	13,107.21	9.480917739	-0.017867664
31-Oct-12	43	13,093.16	9.479845235	-0.001072504
5-Nov-12	44	12,815.39	9.458402071	-0.021443164
12-Nov-12	45	12,588.31	9.440523885	-0.017878187
19-Nov-12	46	13,009.53	9.473437445	0.03291356
26-Nov-12	47	13,025.58	9.474670395	0.001232951
3-Dec-12	48	13,155.13	9.484567076	0.00989668
10-Dec-12	49	13,135.01	9.483036463	-0.001530612
17-Dec-12	50	13,190.84	9.487277928	0.004241465
24-Dec-12	51	12,938.11	9.467932499	-0.01934543
31-Dec-12	52	13,104.14	9.48068349	0.012750991