Poincaré and Sobolev inequalities in the Monge-Ampère quasi-metric structure

by

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AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
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Abstract

In the first half of the 20th century, the initial study of so-called the Sobolev type inequalities was motivated by the question- Can one control the size of a function by the size of its gradient in the higher dimensional Euclidean space? Later in the second half, the Sobolev type inequalities found applications in proving some embedding theorems associated with the Sobolev space, and then to study the local behavior of solutions of certain elliptic partial differential equations, such as to prove the Harnack inequalities and the Holder's continuity. The Poincaré type inequalities were also studied together due to the similar phenomena and applications as of the Sobolev type inequalities.

At the earlier developmental stage, the Sobolev and Poincaré inequalities were established associated to the metric balls, more precisely to the Euclidean balls. The richness of applications of these inequalities in metric spaces motivated mathematicians to investigate such inequalities in various complicated geometrical structures. One of such geometrical structures is the space of homogeneous type, that is, the quasi-metric space equipped with the doubling measure.

The Poincaré and Sobolev type inequalities in the quasi-metric spaces are studied deeply throughout the first two decades of this 21st century. In 2008, G. Tian and X.J. Wang investigated Sobolev inequalities in a space of homogeneous type so-called the Monge-Ampère quasi-metric structure, for the first time, based on the geometry of the Monge-Ampère sections studied by Caffarelli and Gutièrrez in 1990s. Later in 2014, D. Maldonado developed Poincaré inequalities under the minimal assumptions in the Monge-Ampère quasi-metric structure.

In this dissertation we first focus on improving the known Poincaré inequalities in the Monge-Ampère quasi-metric structure by weakening the hypotheses, for instance with cheaper

assumptions on the Monge-Ampère measure, and then develop new such inequalities by imposing some stronger conditions on the Monge-Ampère measure. Finally, we present the application of these Poincaré inequalities in establishing the corresponding Sobolev inequalities. The proofs of both Poincaré and Sobolev inequalities developed earlier in the Monge-Ampère quasi-metric structure involve the Green's functions. We use a completely different approach to establish such inequalities, which is proudly a novelty of our work. Towards the end of this dissertation we study the geometry of the Monge-Ampère sections in the form of the Whitney decomposition.

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Acknowledgments

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Dedication

To my mother Durpati Ranabhat and to my late father Lal Bahadur Ranabhat

Chapter 1

Introduction to Poincaré and Sobolev inequalities

In this chapter we begin with the review of the classical Poincaré and Sobolev inequalities over the Euclidean balls and then briefly present the intuition of such inequalities in a space of homogeneous type. Let $B := B(x, r)$ be a ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$. Fix $1 \leq p < n$ and define $q := \frac{np}{n-p}$ $(> p)$. Then there exist constants $C_1, C_2 > 0$, depending only on n, r and p, such that

$$\forall f \in C^{\infty}(B), \quad \left(\int_B |f(x) - f_B|^q \, dx \right)^{\frac{1}{q}} \leq C_1 \left(\int_B |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}} \tag{1.1}$$

and

$$\forall f \in C_c^{\infty}(B), \quad \left(\int_B |f(x)|^q \, dx \right)^{\frac{1}{q}} \leq C_2 \left(\int_B |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}}. \tag{1.2}$$

Here above f_B is the average integral of f over the ball B, that is, $f_B := \frac{1}{|B|} \int_B f(x) \, dx$. We denote the space of infinitely differentiable functions defined in B by $C^{\infty}(B)$ and the space of infinitely differentiable functions with compact support in B by $C_c^{\infty}(B)$. We use the notation ∇f to represent the gradient of f in the classical sense. It is quite obvious to observe that the inequalities (1.1) and (1.2) hold true for all s with $1 \leq s \leq q$ due to the

Holder's inequality. The inequality (1.1) is known as the classical Poincaré inequality and the other inequality (1.2) as the classical Sobolev inequality. The proofs of these inequalities with different approaches can be found in many articles and books (for instance, see Chapter 1 in[1]).

Notice that the Sobolev inequalities are studied for the compactly supported continuously differentiable functions while the Poincaré inequalities don't require the functions to be compactly supported. These classical Poincaré and Sobolev inequalities involve the Euclidean balls, which are easily manageable in the sense that the surface area and volume can be computed precisely. On the other hand, the quasi-metric balls, in general, are not nice from this viewpoint.

For an open convex set $\Omega \subset \mathbb{R}^n$ and a strictly convex smooth function $\varphi : \Omega \to \mathbb{R}$, we define a space of homogeneous type so-called the Monge-Ampère quasi-metric structure, to be the triple $(\Omega, \delta_\varphi, \mu)$, whenever the measure μ is a doubling Borel measure on the Monge-Ampère sections of φ, and the map δ_φ is a quasi-distance. We will entirely stay inside the Monge-Ampère quasi-metric structure throughout chapter 2 to 7.

There have been significant developments on the Poincaré and Sobolev inequalities in the Monge-Ampère quasi-metric structure in recent years. The first authors to develop the groundbreaking $(q, 2)$-Sobolev inequality associated to the Monge-Ampère structure were G. Tian and X.J. Wang (see[2]). In 2013[3], D. Maldonado proved Sobolev inequalities associated to the Lebesgue measure and the Monge-Ampère measure analogous to the ones developed by G. Tian and X.J. Wang by using strictly weaker hypotheses and by allowing the exponent q to depend only on the dimension. A year later in 2014, D. Maldonado developed a $(1, 2)$-Poincaré inequality associated to the Monge-Ampère measure in[4] and an application is presented immediately in the same article to establish the following Harnack inequality for positive solutions of the linearized Monge-Ampère equation at a strictly convex smooth function φ.

Theorem 1.1 (Theorem 1.4 in[4])**.** *Suppose $\Omega \subset \mathbb{R}^n$ be open and bounded subset, and let μ_φ, the Monge-Ampère measure associated to a strictly convex smooth function φ, satisfies*

DC-doubling condition (that is $\mu_\varphi \in DC(\Omega, \delta_\varphi)$). Then there exist constants $C \geq 1$ and $\eta \in (0,1)$, depending only on the doubling constant and dimension n, such that for every Monge-Ampère section $S_\varphi(x_0, t) \subset\subset \Omega$, and every positive solution u of $L_\varphi u = 0$ in Ω, we have

$$\sup_{S_\varphi(x_0,\tau t)} u \leq C \inf_{S_\varphi(x_0,\tau t)} u, \tag{1.3}$$

where L_φ is the linearized Monge-Ampère operator at a strictly convex smooth function φ defined as

$$L_\varphi u(x) := \mathrm{trace}(A_\varphi(x)\, D^2\, u(x)) \quad x \in \Omega, \tag{1.4}$$

for twice differentiable function u, and here $A_\varphi(x)$ is the co-factor matrix of the Hessian matrix $D^2\varphi(x)$, that is,

$$A_\varphi(x) := \det D^2\varphi(x)\, D^2\varphi(x)^{-1}. \tag{1.5}$$

The operator L_φ is the linearization of the well known nonlinear Monge-Ampère operator

$$M\varphi(x) := \det D^2\varphi(x). \tag{1.6}$$

The other notations used in Theorem 1.1 will be studied in detail in chapter 2.

A strictly convex smooth function φ, associated to its Monge-Ampère sections and measure, models the geometric and measure theoretical approach in the analysis of regularity properties for the solutions to the linearized Monge-Ampère equation $L_\varphi(u) = 0$, as well as other singular/degenerate elliptic PDEs (see for instance[2;4–11]).

If a strictly convex function φ is three times differentiable at $x \in \Omega$ with $D^2\varphi(x) > 0$, we obtain

$$\begin{aligned} \mathrm{trace}(\mathcal{A}_\varphi(x) D^2 v(x)) &= \sum_{i,j} a_{ij}(x)\partial_{x_i}\partial_{x_j} v(x) \\ &= \sum_{i,j} \partial_{x_i}\left(a_{ij}(x)\partial_{x_j} v(x)\right) = \mathrm{div}(\mathcal{A}_\varphi \nabla v)(x), \end{aligned} \tag{1.7}$$

where $A_\varphi(x) = \{a_{ij}(x)\}$ is the co-factor matrix of $D^2\varphi(x)$. The second equality of (1.7) is due to the fact that the columns of $\mathcal{A}_\varphi$ are divergence free. That is,

$$\sum_i \partial_{x_i}\left(a_{ij}(x)\right) = 0, \quad \forall j.$$

Thus the linearized Monge-Ampère operator L_φ is a singular/degenerate elliptic operator that takes both the nondivergence and divergence forms. The divergence form of L_φ has naturally led to the study of various properties for the Monge-Ampère sections and measure that guarantee the existence of Sobolev, Poincaré, or other first-order inequalities related to $\mathcal{A}_\varphi$. In turn, such first-order inequalities have been crucial to the regularity theory for solutions to the linearized Monge-Ampère equation in[4;11] as well as to its applications to semi-geostrophic equations and optimal transport in[9;12] and capacitary estimates[13].

At this point knowing the history and rich applications of Poincaré and Sobolev inequalities in the Monge-Ampère quasi-metric structure, it is very appropriate to have some natural concerns, such as "Can we establish these inequalities by means of different approaches?" or "Can we improve these inequalities ?" This dissertation has been devoted to gathering answers in diverse circumstances to these questions. In the article[14], the authors established a series of improved versions of (q,p)-Poincaré and Sobolev inequalities associated to the Monge-Ampère measure and Lebesgue measure by increasing the exponent q on the left, by decreasing the exponent $p = 2$ on the right, and by replacing some of the expensive hypotheses by cheaper ones in the pre-existing corresponding inequalities. Also, a number of new Poincaré and Sobolev inequalities are developed in the same article considering a different set of hypotheses. Most importantly, the novelty in this work lies in the different approaches used in the development of these inequalities – approaches distinct from the techniques adopted in the early stages of their original developments.

In chapter 2, we study the Monge-Ampère cross sections of a given convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$, the Monge-Ampère measure associate to φ, μ_φ, the doubling conditions on the Monge-Ampère cross sections and some geometric properties of the Monge-Ampère cross sections. We also study the quasi-distance δ_φ, associated to a convex function φ, as well as other crucial techniques that will be employed to establish our main results in the subsequent chapters. The main results of this dissertation are dispersed throughout chapter 3 to 7, and the main results presented in chapter 2 to 6 were originally published in Maldonado-Ranabhat[14].

In chapter 3, we begin with the history of Poincaré inequality in the Monge-Ampère quasi-metric structure, and then develop the necessary ingredients, such as the reverse doubling property, the growth conditions and the self-improving properties for the Poincaré inequalities, to improve the known Poincaré inequality. The main results appearing in this chapter illustrate the improvements of previously known Poincaré inequality under the assumption that the Monge-Ampère measure satisfies the DC-doubling condition in the Monge-Ampère cross sections, and by weakening the assumption on the smoothness of the associated convex function φ.

In chapter 4, we develop new Poincaré inequalities when the Monge-Ampeère measure associated to the given convex function φ, μ_φ, satisfies the so-called Muckenhoupt's A_∞-weight condition. Both original Poincaré inequality and the ones improved in Chapter 3 carry the fixed exponent $p = 2$ on the right side of the inequalities. We observe that the Muckenhoupt's A_∞-weight condition introduced in this chapter is strictly stronger than the DC-doubling condition defined in Chapter 3. Under this new assumption, we sharpen the Poincaré inequalities by increasing the exponent 'q' on the left-hand side as well as by decreasing the size of the exponent $p = 2$ on the right-hand side.

In chapter 5, we present the Poincaré inequalities when the Monge-Ampeère measure satisfies the so-called Muckenhoupt's A_1-weight condition or the reverse Holder's RH_∞-condition. These conditions are even stronger than the A_∞-weight condition and the DC-doubling condition. The Muckenhoupt's A_1-weight condition corresponds to the Poincaré inequalities associated to the Monge-Ampère measure while the reverse Holder's RH_∞-condition corresponds to Lebesgue measure. With the aid of these stronger conditions, we develop new Poincaré inequalities with the exponent 'q' enlarged up to $\frac{2n}{n-2}$ on the left-sides of the inequalities whenever the dimension $n \geq 2$. In dimension 2, we will observe that we can feed any exponent $q > 1$ on the left-hand sides of the Poincaré inequalities, under the same set of of hypotheses, by allowing the constant to depend on 'q'.

Chapter 6 is primarily focused to explore applications of the Poincaré inequalities developed in Chapters 3, 4 and 5 in establishing Sobolev inequalities in the Monge-Ampère quasi-metric structure. The techniques employed to prove Sobolev inequalities in this chapter will be different form the ones adopted by G. Tian and X.J. Wang in[2] and by D. Maldonado in[4]. The Sobolev inequalities presented in this chapter will be either new or the improvement of pre-existing Sobolev inequalities. All the Sobolev inequalities under the assumption that the Monge-Ampère measure satisfies Muckenhoupt's A_∞-weight condition or A_1-weight condition or reverse Hölder RH_∞-condition are new, and only few of them will be presented for the purpose of illustration as well as to compare with existing literature.

Chapter 7 is the closing chapter of this dissertation. We begin this chapter with some review of the Whitney type decomposition in different geometrical structures. Then we present the Whitney decomposition of the Monge-Ampère sections. The main result presented in this chapter is also new and will be a part of future publication.

Chapter 2

Geometric properties of the Monge-Ampère cross sections

This chapter is devoted to setting some basic definitions and notations, and discuss preliminary results that will help to understand the flow of the materials in this dissertation. Section 2.1 provides brief reviews of convex functions and their properties. We introduce the Monge-Ampère measure and Monge-Ampère sections associated to the given convex function in Section 2.2. Section 2.3 starts with the discussion about the doubling measures in the Monge-Ampère sections. In Subsection 2.3.1 we illustrate the techniques of normalizing convex sets. In Subsection 2.3.3, we introduce the engulfing property, observe the connection of such property with the doubling measures in the Monge-Ampère sections. We conclude this chapter with an observation that the doubling condition implies the reverse doubling condition in Lemma 2.29 in Subsection 2.3.4.

2.1 Some properties of convex functions

Throughout this chapter, unless otherwise mentioned, we will assume that $\Omega \subset \mathbb{R}^n$ is open and convex.

Definition 2.1. *A function* $\varphi : \Omega \to \mathbb{R}$ *is convex if for all* $x, y \in \Omega$ *and* $\alpha \in [0,1]$*, we have*

$$\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y). \tag{2.1}$$

In other words, a function φ is convex if the secant line joining two points $(x, \varphi(x))$ and $(y, \varphi(y))$ lives above the graph of the function in between x and y.

One of the well known results about convex functions is that they are continuous. However, convex functions are not necessarily differentiable. For example, $\varphi(x) := |x|$ in $\mathbb{R}$ is convex, but not differentiable. As we will consider strictly convex functions in most of our work in this dissertation, let us see the definition and some associated results about such functions.

Definition 2.2. *A function* $\varphi : \Omega \to \mathbb{R}$ *is strictly convex if for all* x *and* y *with* $x \neq y$ *in* Ω *and* $\alpha \in (0,1)$*, we have*

$$\varphi(\alpha x + (1-\alpha)y) < \alpha\varphi(x) + (1-\alpha)\varphi(y). \tag{2.2}$$

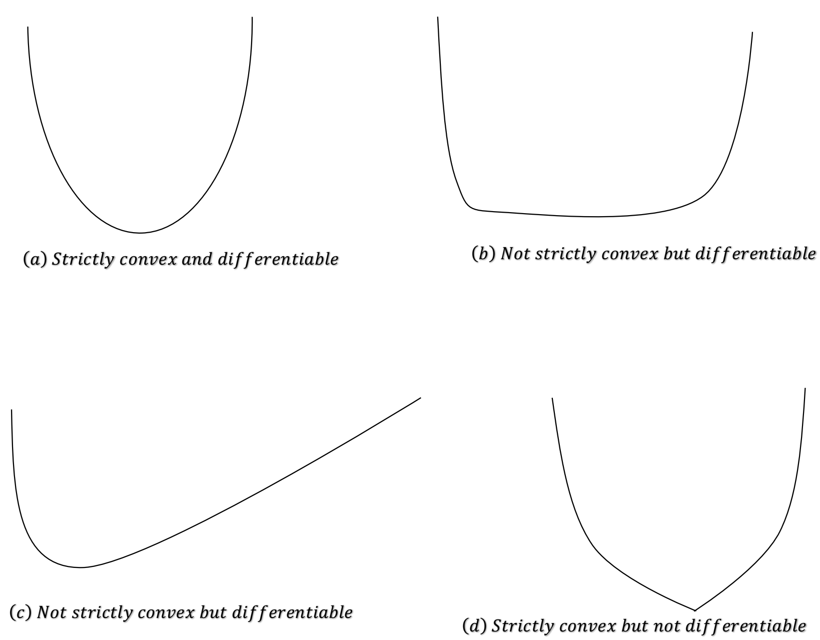

Figure 2.1: *Graphs related to strictly convexity*

Note the strict inequality in the definition of strictly convex function. A strictly convex function, in simpler words, means its graph doesn't contain line segments, for instance see Figure 2.1 above. Before looking at some examples, let us see two definitions associated to square matrices. The properties and details on such matrices will appear later.

Definition 2.3. *A square matrix $A \in \mathbb{R}^{n\times n}$ is positive semidefinite if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$, and positive definite if $\langle Ax, x\rangle > 0$ for all nonzero $x \in \mathbb{R}^n$. We simply write $A \geq 0$ to denote a positive semidefinite matrix and $A > 0$ for positive definite.*

All the affine transformations of the form

$$\varphi(x) := a^T Ax + b,\ (a \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, b \in \mathbb{R})$$

are non strictly convex functions. The quadratic functions of the form

$$\varphi(x) := x^T Ax + c^T x + d,\ (c \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, d \in \mathbb{R})$$

are convex if $A \geq 0$, and strictly convex if $A > 0$.

Another important class of convex functions is

$$\varphi_p(x) := \frac{|x|^p}{p}, x \in \mathbb{R}^n \text{ and } 1 \leq p < \infty.$$

These functions are strictly convex whenever $p > 1$. The influence of such strictly convex functions will be seen throughout this work. We will stress on other examples of convex functions whenever appropriate.

Unlike in dimension one, the test of convexity in general could be complex in higher dimension. The following straightforward theorem, stating the convexity implies the convexity through all the lines in the domain is sometimes useful to test the convexity in higher dimension.

Theorem 2.4. *A function $\varphi : \Omega \to \mathbb{R}$ is convex if and only if for all fixed $x, y \in \Omega$, the function $\psi(t) = \varphi(x + ty)$ is convex on its domain $\{t : x + ty \in \Omega\}$.*

As mentioned earlier, the convexity doesn't imply the differentiabilty in general. However, if a convex function is differentiable, we have very nice properties that allow us to test the convexity with the aid of derivatives. In fact, a function $\varphi \in C^2(\Omega)$ is convex if and only if $D^2\varphi(x) \geq 0$ for all $x \in \Omega$. Here $D^2\varphi$ is just the classical derivative φ'' in dimension $n = 1$, and it's the Hessian of φ defined as below for $n \geq 2$.

Definition 2.5. *Consider a function $\varphi : \Omega \to \mathbb{R}$ such that the second order partial derivatives exist. Then the Hessian of φ is defined by*

$$D^2\varphi(x) := \begin{bmatrix} \frac{\partial^2\varphi(x)}{\partial x_1^2} & \frac{\partial^2\varphi(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2\varphi(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2\varphi(x)}{\partial x_2 \partial x_1} & \frac{\partial^2\varphi(x)}{\partial x_2^2} & \cdots & \frac{\partial^2\varphi(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\varphi(x)}{\partial x_n \partial x_1} & \frac{\partial^2\varphi(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2\varphi(x)}{\partial x_n^2} \end{bmatrix}, \textit{for all } x \in \Omega.$$

Whenever the second order partial derivatives are continuous, the Hessian matrix is symmetric (that is, $D^2\varphi(x)^T = D^2\varphi(x)$).

The convexity of a function can also be characterized by using the first derivatives as well. To summarize this characterization by using first and second derivatives, we have the following theorem.

Theorem 2.6. *Consider that $\varphi : \Omega \to \mathbb{R}$ is twice differentiable. Then the following statements are equivalent:*

1. *φ is convex.*

2. *$\varphi(y) \geq \varphi(x) + \nabla\varphi(x)^T \cdot (y - x)$ for all $x, y \in \Omega$.*

3. *$D^2\varphi(x) \geq 0$ for all $x \in \Omega$.*

The proof of this theorem can be found in many books, for instance, see Chapter 3 in [15]. Observe that the strict inequalities in the second and third statements correspond to the strictly convexity of φ in Theorem 2.6.

If φ is a twice differentiable function, then the positive definiteness of the Hessian matrix $D^2\varphi$ (that is, $D^2\varphi > 0$) implies the strictly convexity of φ. To see this in dimension $n = 1$, assume that $\varphi'' > 0$ in Ω. Now by Taylor's theorem and the mean value theorem,

$$\varphi(y) = \varphi(x) + \varphi'(x)(y - x) + \frac{1}{2}\varphi''(\xi)(y - x)^2, \forall\, x < \xi < y.$$

This implies $\varphi(y) > \varphi(x) + \varphi'(x)(y - x)$ as $\varphi'' > 0$ in Ω. Hence, φ is strictly convex. Now in higher dimension $n \geq 2$, we use the Theorem 2.4. For this, consider $\psi(t) := \varphi(x + ty)$ such that $x + ty \in \Omega$. Then

$$\psi''(t) = y^T D^2\varphi(x + ty)y = \langle D^2\varphi(x + ty)y, y\rangle.$$

Since $D^2\varphi > 0$ in Ω, ψ is strictly convex. Hence φ is strictly convex.

The converse, however, is not true. That is, strictly convexity of a twice differetiable function does not imply $D^2\varphi > 0$. For example, we can see the function $\varphi(x) := x^4, x \in \mathbb{R}$.

Another ground-breaking result about convex functions is the following theorem by Aleksandrov.

Theorem 2.7 (see Page 242 in[16]). *A convex function $\varphi : \Omega \to \mathbb{R}$ is twice differentiable a.e. in Ω.*

We will discuss few other interesting properties associated to convex functions in Section 3.2 as they require to involve concepts on the convex conjugate.

2.2 The Monge-Ampère sections and the Monge-Ampère measure

Unless otherwise mentioned, we assume that $\varphi : \Omega \to \mathbb{R}$ is a convex and continuously differentiable function (that is, $\varphi \in C^1(\Omega)$) throughout this chapter hereafter.

2.2.1 The Monge-Ampère sections

Definition 2.8. *The Monge-Ampère section of* φ *centered at* $x \in \Omega$ *with height* $t > 0$ *is denoted by* $S_\varphi(x,t)$ *and is defined by*

$$S_\varphi(x,t) := \{y \in \Omega : \varphi(y) < \varphi(x) + \nabla\varphi(x) \cdot (y - x) + t\}. \tag{2.3}$$

In other words, the Monge-Ampère section $S_\varphi(x,t)$ is the set of all points $y \in \Omega$ such that $\varphi(y)$ is dominated by the tangent plane at x shifted by t units up (see Figure 2.2 below).

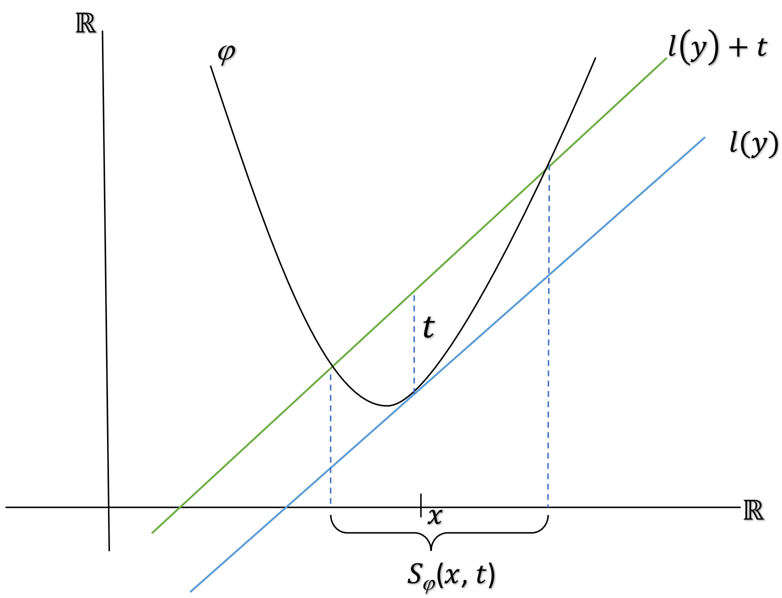

Figure 2.2: *The Monge-Ampère sections of* φ *at* x *where* $l(y) = \varphi(x) + \nabla\varphi(x) \cdot (y - x)$ *is the tangent line at* $(x, \varphi(x))$.

With the definition of

$$\delta_\varphi(x,y) := \varphi(y) - \varphi(x) - \nabla\varphi(x) \cdot (y - x), \tag{2.4}$$

the Monge-Ampère sections defined in (2.3) can be written as

$$S_\varphi(x,t) = \{y \in \Omega : \delta_\varphi(x,y) < t\}. \tag{2.5}$$

The Monge-Ampère section is non empty as its center always lies in the set. One can also observe that the Monge-Ampère sections are convex. In fact, if $y, z \in S_{\varphi}(x,t)$, then by definition

$$\varphi(y) < \varphi(x) + \nabla\varphi(x)\cdot(y-x) + t$$

and

$$\varphi(z) < \varphi(x) + \nabla\varphi(x)\cdot(z-x) + t.$$

As φ is convex, $\varphi(\alpha y + (1-\alpha)z) \leq \alpha\varphi(y) + (1-\alpha)\varphi(z)$. Now by substituting $\varphi(y)$ and $\varphi(z)$ from the above inequalities gives

$$\varphi(\alpha y + (1-\alpha)z) < \varphi(x) + \nabla\varphi(x)\cdot(\alpha y + (1-\alpha)z - x) + t.$$

So, the Monge-Ampère sections of a continuously differentiable convex function φ are non empty open convex sets. Moreover, if the function φ is strictly convex, then its Monge-Ampère sections will be bounded. Thus, We will consider functions to be continuously differentiable and strictly convex in order to have the Monge-Ampère sections open, bounded and convex. However, the assumption of differentiablity (and strictly convexity) is not required for the sake of definition of the Monge-Ampère section. If we simply assume that $\varphi : \Omega \to \mathbb{R}$ is convex, then the Monge-Ampère sections are defined as below:

Definition 2.9. *For given $t > 0$, and a supporting hyperplane $l(y) = \varphi(x) + p\cdot(y-x)$ of φ at $(x, \varphi(x))$ (that is, $\varphi(y) \geq l(y)$, $\forall y \in \Omega$), the Monge-Ampère section of φ centered at $\in \Omega$ with the height t is defined by*

$$S_{\varphi}(x,p,t) := \{y \in \Omega : \varphi(y) < l(y) + t\} = \{y \in \Omega : \varphi(y) < \varphi(x) + p\cdot(y-x) + t\}.$$

We may get more than one supporting hyperplane for φ at $(x, \varphi(x))$ if the function is not differentiable at x. However, if the function is differentiable at x, the supporting hyperplane of φ at $(x, \varphi(x))$ is unique and $p = \nabla\varphi(x)$. Consequently, the Monge-Ampère sections defined in (2.9) coincide with the ones defined earlier in (2.3).

We now see two examples of the Monge-Ampère sections, one for a non-differentiable convex function and the other for a differentiable convex function.

Example 2.10. *Consider a convex function $\varphi(x) := |x|$, $x \in \mathbb{R}^n$ whose graph is a cone. We get infinitely many supporting hyperplanes at the origin. If a supporting hyperplane is not parallel to any generator line of the cone, then the Monge-Ampère sections are ellipsoid. The supporting hyperplanes at the points other than at the origin coincide with a generator line of the cone, and hence the Monge-Ampère sections are the unbounded paraboloids in this case.*

Example 2.11. *Consider a convex function $\varphi_2(x) := \frac{|x|^2}{2}$, $x \in \mathbb{R}^n$, whose graph is a paraboloid. Then the Monge-Ampère sections of φ_2 reduce to the classical Euclidean balls.*

In fact,

$$\begin{aligned}
S_{\varphi_2}(x,t) &= \{y : \varphi_2(y) < \varphi_2(x) + \nabla\varphi_2(x)\cdot(y-x) + t\} \\
&= \{y : \frac{|y|^2}{2} < \frac{|x|^2}{2} + x\cdot(y-x) + t\} \\
&= \{y : \frac{|y|^2}{2} - \frac{|x|^2}{2} - x\cdot(y-x) < t\} \\
&= \{y : |y|^2 + |x|^2 - 2x\cdot y < 2t\} \\
&= \{y : |x-y|^2 < 2t\} \\
&= B(x, \sqrt{2t}).
\end{aligned}$$

2.2.2 The Monge-Ampère measure

Definition 2.12. *The Monge-Ampère measure associated to a convex function $\varphi \in C^1(\Omega)$ is denoted by μ_φ, and is defined by*

$$\mu_\varphi(E) := |\nabla\varphi(E)| \quad E \subset \Omega, E \text{ Borel set}, \tag{2.6}$$

where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^n$.

The Monge-Ampère measure defined as above is a Borel measure measure which is also locally finite. As in the definition for the Monge-Ampère sections, the differentiablity of φ is not required in the definition of the Monge-Ampère measure. We refer the readers to see chapter 3 in[7] as we are not going to consider such functions in this dissertation.

The Monge-Ampère measure for a twice continuously differentiable convex function can be expressed in the integral form. In fact from Page 5 in[7], if $\varphi \in C^2(\Omega)$ is a convex function, then the Monge-Ampère measure associated to φ satisfies

$$\mu_\varphi(E) = \int_E \det D^2\varphi(x)\, dx, \tag{2.7}$$

for every Borel sets $E \subset \Omega$.

We have this integral representation for the Monge-Ampère measure even with the weaker assumption on φ. Before we see this stronger result, let us recall the Sobolev space that will be of our interest.

Definition 2.13. *We say that a function $f : \Omega \to \mathbb{R}$ is locally integrable with respect to the Borel measure μ if*

$$\int_K |f|\, d\mu < \infty,$$

for all compact subset K of Ω.

Definition 2.14. *Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ be a multi-index and $|\alpha| := \alpha_1 + \cdots \alpha_n$. Then we say that a locally integrable function g is the α^{th}-weak derivative of $f \in L^1_{loc}(\Omega, d\mu)$ if*

$$\int_\Omega f\, D^\alpha \varphi\, d\mu = (-1)^{|\alpha|} \int_\Omega g\, \varphi\, d\mu, \tag{2.8}$$

for all $\varphi \in C_c^\infty(\Omega, d\mu)$, where $D^\alpha\varphi := \partial_{x_1^{\alpha_1}} \partial_{x_2^{\alpha_2}} \cdots \partial_{x_n^{\alpha_n}} \varphi$. The weak derivative of f is denoted by $D^\alpha f := g$.

Definition 2.15. *We say that a function $f \in W^{k,p}_{loc}(\Omega, d\mu)$ if for all multi-index α with $|\alpha| \leq k$,*

$$\|D^\alpha f\|_{L^p(K,d\mu)} < \infty, \textit{for all compact } K \subset \Omega. \tag{2.9}$$

That is, $D^\alpha f$ exists in the weak sense and $D^\alpha f \in L^p_{loc}(\Omega, d\mu)$.

The Sobolev norm of $f \in W^{k,p}(\Omega, d\mu)$ is usually defined by

$$\|f\|_{W^{k,p}(\Omega,\, d\mu)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^P(\Omega, d\mu)} \right)^{1/p}, \, 1 \leq p < \infty.$$

Most of our work in this dissertation will be focused with the Sobolev space $W^{2,n}_{loc}(\Omega, dx)$ (that is, $k = 2$ and $p = n$).

Recently, D. Maldonado proved (see Theorem 1 in[17]) that the Monge-Ampère measure can be expressed as in (2.7) by replacing the assumption $\varphi \in C^2(\Omega)$ with $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$. In fact, he proved that if $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$ is convex, then $\varphi \in C^1(\Omega)$ and $d\mu_\varphi(x) = \det D^2\varphi(x)\, dx$. This immediately implies that $\det D^2\varphi \in L^1_{loc}(\Omega, dx)$ and the integral representation (2.7).

2.3 Geometric properties

2.3.1 Doubling measures

We start this section with some basic definitions and notations.

Definition 2.16. *If U is bounded and measurable set, then the center of mass of U is the point x^* defined by*

$$x^* := \frac{1}{|U|} \int_U x\, dx, \tag{2.10}$$

where $|E|$ denotes the Lebesgue measure.

We note that the center of mass of a convex set lives inside it while this may not be the case for general bounded sets, such as annulli. The center of mass of some nice geometrical shapes matches with their center, for example, Euclidean balls and ellipsoids.

Definition 2.17. *For a given $0 < \lambda \leq 1$, the λ-contraction of the Monge-Ampère section*

$S_\varphi(x_0, t)$ *with respect to its center of mass is denoted by* $\lambda \odot S_\varphi(x_0, t)$ *and is defined by*

$$\lambda \odot S_\varphi(x_0, t) := \{x^* + \lambda(x - x^*) : x \in S_\varphi(x_0, t)\}. \tag{2.11}$$

We can see Figure 2.3 to visualize the concept of the contraction with respect to the center of mass.

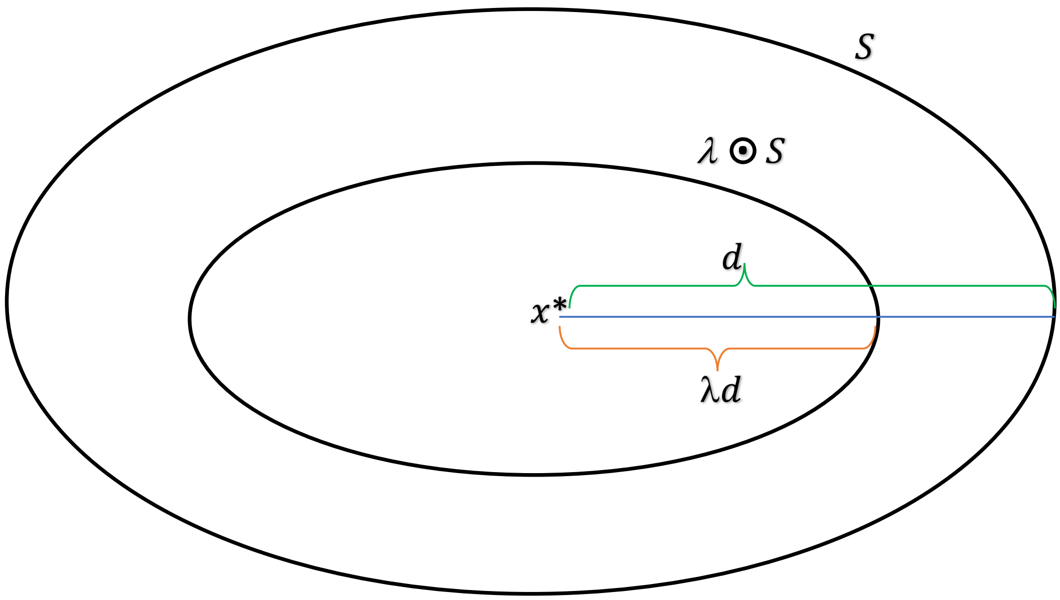

Figure 2.3: *λ-contraction of the Monge-Ampère section* $S := S_\varphi(x_0, t)$ *with respect to its center of mass* x^*.

For the case when $\lambda > 1$, the equation (2.11) is called the λ-dilation of $S_\varphi(x_0, t)$ as we enlarge the original set. The λ-contraction of $S_\varphi(x_0, t)$ is convex, but it is not necessarily the Monge-Ampère section.

For a given $0 < \lambda \leq 1$ and a section $S_\varphi(x_0, t)$, we denote $\lambda S_\varphi(x_0, t)$ to represent the λ-contraction (or λ-dilation when $\lambda > 1$) with respect to the parameter t of the section $S_\varphi(x_0, t)$. That is,

$$\lambda S_\varphi(x_0, t) := S_\varphi(x_0, \lambda t). \tag{2.12}$$

We note that the contraction $\lambda S_\varphi(x_0, t)$ defined in (2.12) has a different meaning than the contraction $\lambda \odot S_\varphi(x_0, t)$ defined in (2.11).

Definition 2.18. *A Borel measure μ on Ω is said to satisfy the DC-doubling condition, and write $\mu \in DC(\Omega, \delta_\varphi)$, if there exists a constant $C_D \geq 1$ such that for every section $S := S_\varphi(x,t) \subset\subset \Omega$, we have*

$$\mu(S) \leq C_D\, \mu\big(\tfrac{1}{2} \odot S\big). \tag{2.13}$$

The abbreviation DC here is to mean the doubling with respect to center of mass. In our work later, all the constant depending on the doubling constant from $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and dimension n will be called *geometric constants.* We now see another definition of the doubling measure associated to the parameter.

Definition 2.19. *A Borel measure μ on Ω is said to satisfy the DP-doubling condition, and write $\mu \in DP(\Omega, \delta_\varphi)$, if there exists a constant $C_P \geq 1$ such that for every section $S := S_\varphi(x,t) \subset\subset \Omega$, we have*

$$\mu(S) \leq C_P \mu\big(\tfrac{1}{2} S\big). \tag{2.14}$$

As an example of the Borel measure that satisfies the DP-doubling condition (2.14), we look at the following lemma by Caffarelli and Gutiérrez stating that the Lebesgue measure satisfies the DP-doubling condition on the Monge-Ampère sections.

Lemma 2.20 (Lemma 5.2 in[6])**.** *Consider a strictly convex function $\varphi \in C^1(\Omega)$. Then for every Monge-Ampère sections $S_\varphi(x,t) \subset\subset \Omega$, we have*

$$|S_\varphi(x,t)| \leq 2^n\, |S_\varphi(x, t/2)|. \tag{2.15}$$

Note that the Lebesgue measure satisfies the DP-doubling condition with precise size of constant $C = 2^n$. However, this constant may not be the optimal one.

The DC-doubling condition (2.13) implies the DP-doubling condition (2.14) (see Corollary 3.3.2 in[7]). The proof of this statement relies on the following geometric properties of the Monge-Ampère sections.

Lemma 2.21 (Lemma 3.3.1 in[7]). *Let* $0 < \lambda < 1$. *Then*

$$\lambda S_\varphi(x_0, t) \subset S_\varphi(x_0, \left(1 - (1-\lambda)\, n^{-3/2}\right) t).$$

The converse direction, however, is not true in general. For example, the Monge-Ampère measure associated to the strictly convex smooth function $\varphi(x) := e^x, x \in \mathbb{R}$ satisfies the DP-doubling condition (2.14) but not the DC-doubling condition (2.13) (see Remark 3.3.3 in[7]).

The connection between the measure theoretical phenomena and the geometrical structure of the Monge-Ampère sections is stressed in Subsection 2.3.3. For now, we see two examples of Monge-Ampère measure that satisfy the DC-doubling condition (2.13).

Example 2.22. *Let* $p(x)$ *be the polynomial in* $\mathbb{R}^n$. *Then the measure* $|p(x)|\, dx$ *satisfies the* DC*-doubling condition* (2.13) *on the Monge-Ampère sections of a strictly convex function* φ *with the doubling constant* C *depending only on the degree of the polynomial. In particular, the doubling constant doesn't depend on the coefficients of the polynomial. This result is proved in Remark 3.3.4 in*[7].

Example 2.23. *The Monge-Ampère measure,* μ_{φ_p}*, associated to the strictly convex function* $\varphi_p(x) := \dfrac{|x|^p}{p}$, $x \in \mathbb{R}^n$ *and* $1 < p < \infty$, *satisfies the* DC*-doubling condition* (2.13) *on the Monge-Ampère sections of* φ_p. *In fact,* μ_{φ_p} *satisfies even stronger condition than the* DC*-doubling condition with some restriction on* p *which we will see in Section 5.4.*

As the affine transformations play vital role on the study of geometric properties of Monge-Ampère sections, we now discuss the role of such transformations here.

For a real invertible matrix A, the map $T_A : \mathbb{R}^n \to \mathbb{R}^n$ given by $T_A(x) = Ax + b$, $b \in \mathbb{R}^n$ is an invertible affine transformation. Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable convex function and $\lambda > 0$. Now define

$$\psi_\lambda(y) := \tfrac{1}{\lambda}\, \varphi({T_A}^{-1} y). \tag{2.16}$$

Then ψ_λ is also a twice continuously differentiable convex function. In fact,

$$\nabla\psi_\lambda(y) = \tfrac{1}{\lambda}(A^{-1})^T\,\nabla\varphi({T_A}^{-1}y), \tag{2.17}$$

and

$$D^2\psi_\lambda(y) = \tfrac{1}{\lambda}(A^{-1})^T\,\nabla\varphi({T_A}^{-1}y)\,A^{-1}. \tag{2.18}$$

So for a Borel set $E \subset \mathbb{R}^n$,

$$\begin{aligned}
\mu_\psi(T_AE) &= \int_{T_AE} \det D^2\psi_\lambda(y)\,dy \\
&= \int_E \frac{1}{\lambda^n}\ \det(A^{-1})^T\ \det D^2\varphi(x)\ \det(A^{-1})\,|\det A|dx \\
&= \frac{1}{\lambda^n}\,|\det(A^{-1})|\,\mu_\varphi.
\end{aligned}$$

Thus the relationship between the Monge-Ampère measure associated to φ and ψ_λ is given by

$$\mu_\psi(T_AE) = \frac{1}{\lambda^n}\,|\det(A^{-1})|\,\mu_\varphi(E). \tag{2.19}$$

Now we see observe the comparison between the ma sections associated to φ and ϕ_λ. The Monge-Ampère section of ψ_λ centered at Tx_0 and height $\frac{t}{\lambda}$ is given by

$$\begin{aligned}
S_{\psi_\lambda}(T_Ax_0, \tfrac{t}{\lambda}) &= \{y \in \mathbb{R}^n : \psi_\lambda(y) < \psi_\lambda(T_Ax_0) + \nabla\psi_\lambda(T_Ax_0)\cdot(y - tx_0) + \tfrac{t}{\lambda}\} \\
&= \{y \in \mathbb{R}^n : \tfrac{1}{\lambda}\,\varphi({T_A}^{-1}y) < \tfrac{1}{\lambda}\varphi(x_0) + \tfrac{1}{\lambda}\nabla\varphi(x_0)\cdot({T_A}^{-1}y - x_0) + \tfrac{t}{\lambda}\} \\
&= \{T_Ax \in \mathbb{R}^n : \varphi(y) < \varphi(x) + \nabla\varphi(x)\cdot(y - x) + t\}, \text{with } y = T_Ax \\
&= T_A\,S_\varphi(x_0, t).
\end{aligned}$$

Thus the Monge-Ampère sections of φ and ψ_λ are associated by the equation

$$T_A\,(S_\varphi(x_0, t)) = S_{\psi_\lambda}(T_Ax_0, \tfrac{t}{\lambda}). \tag{2.20}$$

We now note that if the Monge-Ampère measure associated to φ, μ_φ, satisfies the DC-doubling condition (2.13), then so does the Monge-Ampère measure μ_{ψ_λ}. To see this, assume that μ_φ satisfies the DC-doubling condition (2.13) on the Monge-Ampère sections of φ and denote $S := S_\varphi(x_0, t)$. Then

$$\begin{aligned}\mu_{\psi_\lambda}(\tfrac{1}{2} \odot T_A S) &= |\nabla \psi_\lambda(\tfrac{1}{2} \odot T_A S)| = \tfrac{1}{\lambda^n} |\det A_{-1}|\, \mu_\varphi(\tfrac{1}{2} \odot S) \\ &\leq C\, \tfrac{1}{\lambda^n} |\det A_{-1}|\, \mu_\varphi(E) = C \mu_{\psi_\lambda}(T_A E).\end{aligned}$$

Observe that μ_{ψ_λ} satisfies the DC-doubling condition (2.13) with the same doubling constant produced for μ_φ. Similarly, if the Monge-Ampère measure μ_φ satisfies the DP-doubling condition (2.14), then so does the Monge-Ampère μ_{ψ_λ}. For this, we need the following immediate equation

$$T_A(\alpha(S_\varphi(x_0, t)) = \alpha T_A(S_\varphi(x_0, t)).$$

2.3.2 Normalization of convex sets

As normalization of convex sets is one of the crucial techniques in the study related to the geometry of the Monge-Ampère sections, we focus this section to build some background to deal with such techniques for later use. To do so, let us start with some quick reviews on ellipsoids.

One of the familiar form of the ellipsoid centered at $\tilde{x}$ and radii $a_1, \cdots, a_n$ is the set of points x such that

$$\frac{(x_1 - \tilde{x_1})^2}{a_1^2} + \cdots + \frac{(x_n - \tilde{x_n})^2}{a_n^2} \leq 1. \tag{2.21}$$

Now consider a diagonal matrix with diagonal entries $\lambda_i := \dfrac{1}{a_i^2}$:

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Then the matrix A is symmetric positive definite, and the ellipsoid given by (2.21) can be formulated as

$$E(A, \tilde{x}) = \{x : (x - \tilde{x})^T A (x - \tilde{x}) \le 1\}. \tag{2.22}$$

The volume of the ellipsoid is given by

$$|E(A, \tilde{x})| = \frac{\omega_n}{\sqrt{\det A}} = a_1 a_2 \cdots a_n \, \omega_n, \tag{2.23}$$

where ω_n represents the volume of the unit ball in $\mathbb{R}^n$.

A central result employed to normalize convex sets is the following theorem by F. John which guarantees the existence of an ellipsoid of minimum volume that inscribe bounded convex sets (see Figure 2.4).

Theorem 2.24 (Theorem 1.8.2 in[7])**.** *If $U \subset \mathbb{R}^n$ is a bounded convex set with nonempty interior and E is the ellipsoid of minimum volume containing U centered at the center of mass of U, then*

$$\alpha_n E \subset U \subset E, \tag{2.24}$$

where $\alpha_n = n^{-3/2}$ and αE denotes the α-contraction of E with respect to its center of mass.

Definition 2.25. *We say that a bounded convex set V is normalized if its center of mass is* 0 *and*

$$B(0, \alpha_n) \subset V \subset B(0, 1), \tag{2.25}$$

where $B(x, r)$ denotes the Euclidean Ball with center x and radius r.

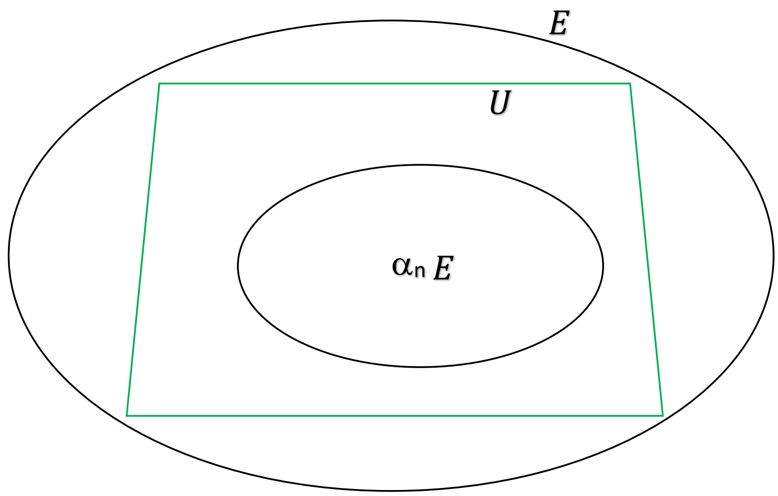

Figure 2.4: *Existence an of ellipsoid of minimum volume that inscribe bounded convex sets.*

Due to Theorem 2.24, every open bounded convex set can be normalized. In fact, for a given open bounded convex set Ω, there is an ellipsoid E of minimum volume such that (2.24) holds true. To see this, let $\tilde{T}$ be an affine transformation that maps the ellipsoid E to the unit Euclidean ball, that is, $\tilde{T}E = B(0,1)$. Then the center of mass of $\tilde{T}(\Omega)$ is 0 and

$$B(0, \alpha_n) \subset \tilde{T}(\Omega) \subset B(0,1), \tag{2.26}$$

as shown in figure 2.5

Thus the set $\tilde{T}(\Omega)$ is the normalization of Ω by the affine transformation $\tilde{T}$. In particular, since the Monge-Ampère sections $S_\varphi(x,t)$ are open convex and bounded (whenever φ is strictly convex), we can normalize them. That is, there is an affine trnasformation $\tilde{T}$ such that the center of mass of $\tilde{T}(S_\varphi(x,t))$ is 0, and

$$B(0, \alpha_n) \subset \tilde{T}(S_\varphi(x,t)) \subset B(0,1). \tag{2.27}$$

Now by defining $\psi(y) := \varphi(\tilde{T})^{-1}(y)$, we get $\tilde{T}S_\varphi(x,t) = S_\psi(\tilde{T}x,t)$ from (2.20). Consequently

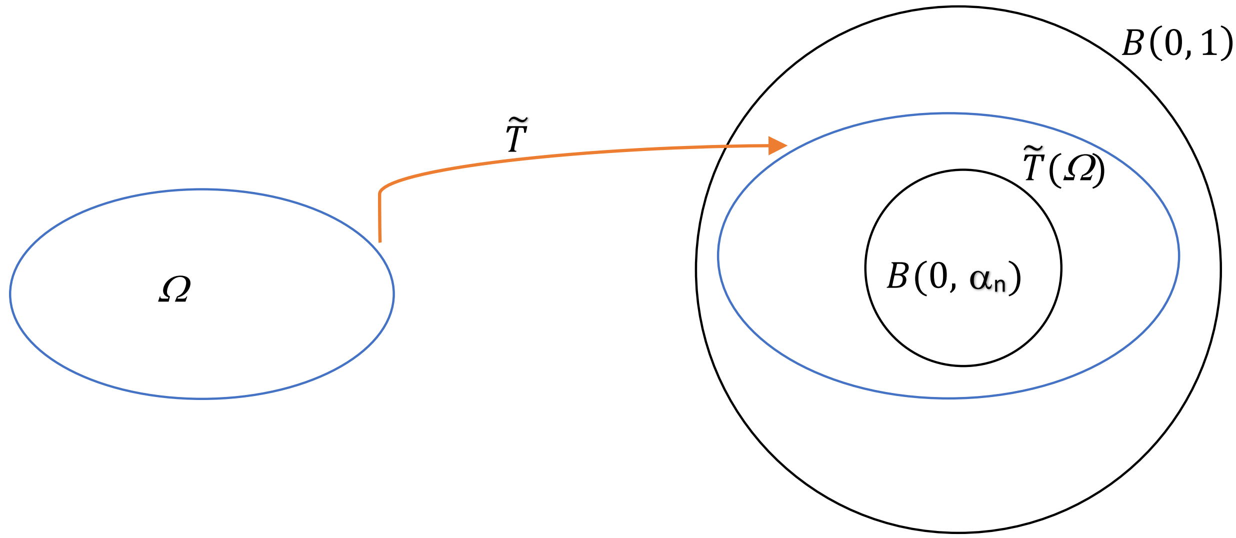

Figure 2.5: *Normalization of open bounded convex sets.*

from (2.27), we have

$$B(0, \alpha_n) \subset S_\psi(\tilde{T}x, t) \subset B(0, 1).$$

This implies

$$\alpha_n \omega_n \leq |S_\psi(\tilde{T}x, t)| \leq \omega_n.$$

These inequalities give the bound for the Lebesgue measure of the normalized Monge-Ampère section of φ. The techniques depicted here in this section will be used to prove the main results in the subsequent chapters.

2.3.3 The engulfing property

We recall that the Euclidean balls are the metric balls given by the metric induced by the Euclidean norm. Then for any point x in an Euclidean ball $B(x_0, r)$, we have

$$B(x_0, r) \subset B(x, 2r).$$

This inclusion is an immediate consequence of the triangle inequality of the associated metric. This property of Euclidean balls motivates us to study of similar phenomena associated to the Monge-Ampère sections, so-called the engulfing property.

Definition 2.26. *We say that the Monge-Ampère sections of a convex function* $\varphi \in C^1(\Omega)$ *have the engulfing property if there exists a geometric constant* $\Theta > 1$ *such that whenever* $x_0 \in \Omega$ *and* $\tau > 0$ *satisfy* $S_\varphi(x_0, \Theta^2\tau) \subset\subset \Omega$, *then for every* $x \in S_\varphi(x_0, \tau)$ *following inclusion holds true:*

$$S_\varphi(x_0, \tau) \subset S_\varphi(x, \Theta\tau). \tag{2.28}$$

This definition is depicted in Figure 2.6 here below.

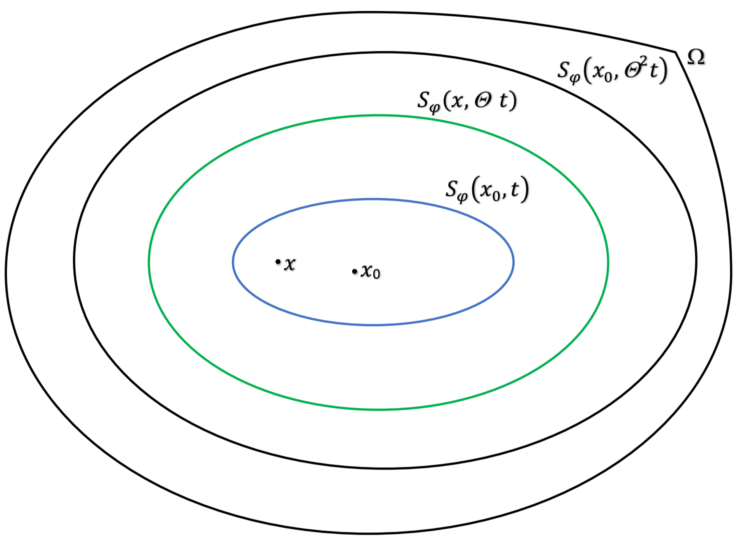

Figure 2.6: *The engulfing property of the Monge-Ampère sections.*

We will next observe that the Monge-Ampère sections of φ have this engulfing property whenever the map δ_φ defined in (2.4) is a quasi-distance in the following sense.

Definition 2.27. *A map* $\delta : \Omega \times \Omega \to [0, \infty)$ *is a quasi-distance if there exists* $K \geq 1$ *such that*

(i) $\delta(x, y) = 0$ *iff* $x = y$;

(ii) $\delta(x, y) \leq K\, \delta(y, x)$, *for all* $x, y \in \Omega$ *and*

(iii) $\delta(x, y) \leq K [\delta(x, z) + \delta(z, y)]$ *for all* $x, y, z \in \Omega$.

The space Ω associated with the quasi-distance δ, that is, (Ω, δ) is called the quasi-metric structure. The second condition in Definition 2.27 is named as a quasi-symmetry and the third one as a quasi-triangle inequality. We should pay attention on the quasi-symmetric condition (ii) in the above definition as we can find some mathematicians defining quasi-distance with symmetric condition instead of (ii) in our definition above. We also observe that the case $K = 1$ reduces this definition to a distance.

Now we see that if the map δ_φ defined in (2.4) is a quasi-distance, then the Monge-Ampère sections of φ have the engulfing property.

For $y \in S_\varphi(x_0, \tau)$, we have $\delta_\varphi(x_0, y) < \tau$. Then by the quasi-triangle inequality and the quasi-symmetry,

$$\delta_\varphi(x, y) \leq K [\delta_\varphi(x, x_0) + \delta_\varphi(x_0, y)] < K [K\delta_\varphi(x_0, x) + \tau] < 2K^2 \tau.$$

This verifies the inclusion of the engulfing property (2.28) with $\Theta = 2K^2$.

One of the most celebrated result in the study related to the Monge-Ampère sections is the following characterization of the DC-doubling condition by the engulfing property.

Theorem 2.28. *For* μ_φ*, the Monge-Ampère measure associated to a strictly convex function* $\varphi \in C^1(\Omega)$*, the following statements are equivalent:*

(i) μ_φ *satisfies the* DC*-doubling condition 2.13.*

(ii) The sections of φ *have the engulfing property 2.28.*

The direction (i) implies (ii) of Theorem 2.28 is proved by Gutiérrez (see Theorem 3.3.7 in[7]) and the result of other direction is proved by Forzani and Maldonado (see Theorem 8 in[18]). Note that the strictly convexity in the statement of this theorem can be relaxed by assuming the sections of φ are bounded.

We have already seen that if the map δ_φ defined in (2.4) is a quasi-distance, then the Monge-Ampère sections of φ have the engulfing property. On the other hand, the map δ_φ

restricted to a suitable region is a quasi-distance under the assumption of the engulfing property. In fact, the first condition of Definition 2.27 is immediate. Let us next briefly indicate how the inclusion (2.28) amounts to a quasi-symmetry and a quasi-triangle inequality for δ_φ in a restricted region.

For this, suppose $x, y \in \Omega$ such that

$$S_\varphi(x, \delta_\varphi(x, y)) \subset\subset \Omega \quad \text{and} \quad S_\varphi(y, \Theta\delta_\varphi(x, y)) \subset\subset \Omega, \tag{2.29}$$

and for $\varepsilon > 0$ sufficiently small we have $y \in S_\varphi(x, \delta_\varphi(x, y) + \varepsilon) \subset\subset \Omega$. Then the engulfing property implies

$$S_\varphi(x, \delta_\varphi(x, y) + \varepsilon) \subset S_\varphi(y, \Theta(\delta_\varphi(x, y) + \varepsilon)).$$

In particular, we have $\delta_\varphi(y, x) < \Theta(\delta_\varphi(x, y)+\varepsilon)$. Now by letting $\varepsilon \to 0$, we get the inequality

$$\delta_\varphi(y, x) \leq \Theta\delta_\varphi(x, y), \tag{2.30}$$

which represents the Θ-quasi symmetry of δ_φ. On the other hand, given $x, y, z \in \Omega$ such that

$$S_\varphi(z, \delta(z, y)), S_\varphi(z, \delta(y, z)), S_\varphi(x, \Theta\delta_\varphi(z, x)) \subset\subset \Omega, \tag{2.31}$$

assume first that $\delta_\varphi(z, x) \leq \delta_\varphi(z, y)$ to write, for $\varepsilon > 0$ small enough,

$$x \in S_\varphi(z, \delta_\varphi(z, x) + \varepsilon) \subset S_\varphi(z, \delta_\varphi(z, y) + \varepsilon) \subset\subset \Omega.$$

Now by applying the engulfing property to x and $S_\varphi(z, \delta_\varphi(z, y) + \varepsilon)$, we get

$$y \in S_\varphi(z, \delta_\varphi(z, y) + \varepsilon) \subset S_\varphi(x, \Theta(\delta_\varphi(z, x) + \varepsilon)).$$

In particular, $\delta_\varphi(x, y) < \Theta(\delta_\varphi(z, x) + \varepsilon)$ and by letting $\varepsilon \to \infty$, we get

$$\delta_\varphi(x, y) \leq \Theta\delta_\varphi(z, x). \tag{2.32}$$

Next, if $\delta_\varphi(z,x) > \delta_\varphi(z,y)$, we reverse the roles of x and y in the argument above, which requires the inclusions (2.31) with y replaced with x, to obtain

$$\delta_\varphi(y,x) \leq \Theta\delta_\varphi(z,y). \tag{2.33}$$

In addition, consider that

$$S_\varphi(y,\delta_\varphi(y,x)) \subset\subset \Omega \quad \text{and} \quad S_\varphi(x,\Theta\delta_\varphi(y,x)) \subset\subset \Omega. \tag{2.34}$$

Then the inequalities (2.30) (with x and y interchanged) and (2.33) give

$$\delta_\varphi(x,y) \leq \Theta^2\delta_\varphi(z,y). \tag{2.35}$$

Since (2.32) or (2.35) will hold true, it follows that

$$\delta_\varphi(x,y) \leq \Theta(\Theta\delta_\varphi(z,y) + \delta_\varphi(z,x)) \leq \Theta^2(\delta_\varphi(z,y) + \delta_\varphi(x,z)), \tag{2.36}$$

which effectively represents a Θ^2-quasi triangle inequality for δ_φ by considering the restrictions from the inclusions in 2.29, 2.31, and 2.34.

Thus if the Monge-Ampère sections of φ have the engulfing property or equivalently if the Monge-Ampère measure associated to φ, μ_φ, satisfies DC-doubling condition (due to Theorem 2.28), the map δ_φ defined in (2.4) is a quasi-distance (possibly with certain restrictions as mentioned above). Consequently, the space (Ω, δ_φ) takes the form of quasi-metric structure. In broader sense, the triple $(\Omega, \delta_\varphi, \mu_\varphi)$ is a space of of homogeneous type. Here after, the notation $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ is valid.

2.3.4 Doubling implies reverse doubling in (Ω, δ_φ)

In this section we recall reverse-doubling properties of doubling measures in the Monge-Ampère quasi-metric structure from[19]. From now on $\Theta > 1$ will always indicate the geometric constant from the engulfing property (2.28).

Lemma 2.29 (see[19], Section 2). *Fix $\varphi \in C^1(\Omega)$ with $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and let μ be a Borel measure on Ω which is DP-doubling with respect to the Monge-Ampère sections of φ. Then, for every $\alpha \in (0,1)$ there exists $\xi \in (0,1)$, depending only on α, the doubling constant of μ, and geometric constants, such that for every section $S_\varphi(x_0, t)$ with $S_\varphi(x_0, \Theta^2 t) \subset\subset \Omega$ we have*

$$\mu(S_\varphi(x_0, \alpha t)) \leq \xi \mu(S_\varphi(x_0, t)). \tag{2.37}$$

We note the constant ξ has to be smaller than 1. Otherwise the inequality (2.37) would be trivial because $S_\varphi(x_0, \alpha t) \subset S_\varphi(x_0, t)$.

Using Lemma 2.29, it was proved in[19] Section 2, that if $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ there exist geometric constants $C_D > 0$ and $\varepsilon \in (0,1)$ such that

$$\frac{\mu_\varphi(S_\varphi(x_0, t))}{\mu_\varphi(S_\varphi(x_0, t'))} \leq C_D \left(\frac{t}{t'}\right)^{n-\varepsilon} \tag{2.38}$$

for every section $S_\varphi(x_0, t)$ with $S_\varphi(x_0, t) \subset\subset \Omega$ and every $t' \in (0, t)$. Also, Lemma 2.29 will be useful in the proof of Theorem 3.19 and in the proof that every (q, p)-Poincaré inequality implies a corresponding Sobolev inequality in Chapter 6.

Chapter 3

Poincaré inequalities when $\mu_\varphi \in DC(\Omega, \delta_\varphi)$

Our objective in this chapter is to prove the $(q,2)$-Poincaré inequalities presented in Theorem 3.4 and Theorem 3.5. In Section 3.1, we review the history of Poincaré inequalities in the Monge-Ampère quasi-metric structure and then state our main results. Section 3.2 presents the overview of the convex conjugate. Section 3.3 is devoted to state and prove self-improving properties for the Poincaré inequalities in the Monge-Ampère quasi-metric structure. In Section 3.4, we prove $(1,2)$-Poincare inequality in Theorem 3.23 with respect to the Monge-Ampère measure by weakening the hypotheses of the first $(1,2)$-Poincare inequality developed in 2014. Finally, Section 3.5 provides the proof of the main results stated in Section 3.1.

3.1 Introduction and main results

The Poincaré and Sobolev inequalities in the Monge-Ampère quasi-metric structure will involve the Monge-Ampère gradient associated to the given convex function on the right sides of the inequalities. The Monge-Ampère gradient associated to a convex function φ, denoted by ∇^φ, for a function u differentiable at a point $x \in \Omega$ with $D^2\varphi(x) > 0$ is defined as

$$\nabla^{\varphi} u(x) := D^2\varphi(x)^{-\frac{1}{2}}\,\nabla u(x). \tag{3.1}$$

In general, the Monge-Ampère gradient is different from the classical gradient of a function. However, observe that the Monge-Ampère gradient associated to $\varphi_2(x) := \dfrac{|x|^2}{2}$ coincides with the classical gradient.

The first Poincaré inequality with respect to the Lebesgue measure in the Monge-Ampère quasi-metric structure was established by D. Maldonado in 2014, which reads as

Theorem 3.1 (Theorem 1.3 in[4])**.** *Given an open convex set $U \subset \mathbb{R}^n$ and $\varphi \in C^2(U)$ with $D^2\varphi > 0$ in U and $\mu_\varphi \in DC(U, \delta_\varphi)$ there exists a geometric constant $C_1^* > 0$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, t) \subset\subset U$ and every $h \in C^1(S)$ the following $(1, 2)$-Poincaré holds true in the Monge-Ampère quasi-metric structure with respect to the Lebesgue measure*

$$\fint_S |h(x) - h_S|\, dx \le C_1^*\, t^{\frac{1}{2}} \left(\fint_S |\nabla^{\varphi} h(x)|^2\, dx \right)^{\frac{1}{2}}, \tag{3.2}$$

where $h_S := \fint_S h(x)\, dx$.

With a combination of $(1, 2)$-Poincaré inequality with respect to Lebesgue measure in Theorem 3.1 and the change of variables from[3] Section 4 yields the following weak (1,2)-Poincaré inequality with respect to the Monge-Ampère measure, which has been essential to the Harnack inequalities in[11].

Theorem 3.2. *Fix $n \ge 2$ and $\varphi \in C^2(\Omega)$ with $D^2\varphi > 0$ in Ω and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Then, there exist constants $C_1, C_2 > 1$, depending only on the doubling constant from the condition $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and dimension n, such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, C_1 t) \subset\subset \Omega$ and every $u \in C^1(S_\varphi(x_0, C_1 t))$ the following Poincaré inequality holds true*

$$\fint_S |u(x) - u_S^{\mu_\varphi}| d\mu_\varphi(x) \le C t^{\frac{1}{2}} \left(\fint_{S_\varphi(x_0, C_1 t)} |\nabla^{\varphi} u(x)|^2 d\mu_\varphi(x) \right)^{\frac{1}{2}}, \tag{3.3}$$

where $u_S^{\mu_\varphi} := \fint_S u\, d\mu_\varphi$.

Note the difference between $(1, 2)$-Poincaré inequality and weak $(1, 2)$-Poincaré inequality is that the first notion involves the same set for the integration on both sides while the second one involves bigger set on the right side of the inequality.

Definition 3.3. *Consider a set $U \subset \mathbb{R}^n$. Then a function $f : U \to \mathbb{R}$ is called a Lipschitz continuous, and denoted by $f \in \mathrm{Lip}(U)$, if there exists a constant $C > 0$ such that for all $x, y \in U$*

$$|f(x) - f(y)| \leq C\,|x - y|.$$

We remark that the Lipschitz continuity can be generalized for functions mapping between metric spaces. But we will use the Lipschitz continuity defined on the Euclidean setting in our context. It is quite easy to observe that a continuously differentiable function defined on a closed interval $[a, b] \subset \mathbb{R}$ is Lipschitz continuous, but the reverse may not be true necessarily. For instance, the function $f(x) := |x|$, $x \in \mathbb{R}$ is Lipschitz continuous on $[-a, a]$ with $a > 0$ but not differentiable on that interval. Another important property of Lipschitz continuous functions is that they are differentiable a.e. and the derivatives are bounded a.e.

We are now in the position to list the following two theorems, one with respect to the Lebesgue measure and the other with respect to the Monge-Ampère measure, as our main results in this chapter.

Theorem 3.4. *Fix $n \geq 2$ and let $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$ be a strictly convex function with $D^2\varphi > 0$ a.e. in Ω and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Then, there exist geometric constants $K_1, K_2 > 1$ and $\epsilon_1 > 0$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_1 t) \subset\subset \Omega$ and every $u \in \mathrm{Lip}(K_1 S)$ we have*

$$\left(\fint_S |u(x) - u_S^{\mu_\varphi}|^{q_1} d\mu_\varphi(x) \right)^{\frac{1}{q_1}} \leq K_2\, t^{\frac{1}{2}} \left(\fint_{K_1 S} |\nabla^\varphi u(x)|^2 d\mu_\varphi(x) \right)^{\frac{1}{2}}, \tag{3.4}$$

where $q_1 := \frac{2n}{n-1} + \epsilon_1$ and $u_S^{\mu_\varphi} := \fint_S u(x) d\mu_\varphi(x)$.

As before we remark that the strict convexity of φ is not required in Theorem 3.4. However, we keep in mind the underlying hypothesis that all the Monge-Ampère sections involved are bounded.

The Poincaré inequality in Theorem 3.4 is an improvement of Theorem 3.2 from couple of viewpoints. The first thing to observe is that the hypotheses on Theorem 3.2 require $\varphi \in C^2(\Omega)$ with $D^2\varphi > 0$ in Ω which are replaced with the weaker hypotheses $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$ and $D^2\varphi > 0$ a.e. in Ω in Theorem 3.4. Second, Theorem 3.2 is a $(1,2)$-Poincaré inequality which is improved to $(q_1, 2)$-Poincaré inequality with $q_1 > 2$ in Theorem 3.4. We keep in mind that the average integral $\left(\fint_S |\cdot|^p\right)^{\frac{1}{p}}$ is increasing on p for $p \geq 1$. Finally, Theorem 3.4 establishes the weak $(q_1, 2)$-Poincaré inequality for the Lipschitz continuous functions while Theorem 3.2 is established for the continuously differentiable functions.

The second main result is the following weak $(q_1, 2)$-Poincaré inequality associated to the Lebesgue measure with the same size of parameter q_1 as in Theorem 3.4 and with an additional assumption that $\|(D^2\varphi)^{-1}\| \in L^n_{loc}(\Omega, d\mu_\varphi)$. Here onward, $\|\cdot\|$ deontes the L^2-norm.

Theorem 3.5. *Fix $n \geq 2$ and let $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$ be a strictly convex function such that $D^2\varphi > 0$ a.e. in Ω, $\|(D^2\varphi)^{-1}\| \in L^n_{loc}(\Omega, d\mu_\varphi)$, and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Then, there exist geometric constants $K_3, K_4 > 1$ and $\epsilon_1 > 0$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_3 t) \subset\subset \Omega$ and every $h \in \mathrm{Lip}(K_3 S)$ we have*

$$\left(\fint_S |h(x) - h_S|^{q_1}\, dx\right)^{\frac{1}{q_1}} \leq K_4\, t^{\frac{1}{2}} \left(\fint_{K_3 S} |\nabla^\varphi h(x)|^2\, dx\right)^{\frac{1}{2}}, \tag{3.5}$$

where $q_1 := \frac{2n}{n-1} + \epsilon_1$ and $h_S := \fint_S h(x)\, dx$.

We remark that the hypothesis $\|(D^2\varphi)^{-1}\| \in L^n_{loc}(\Omega, d\mu_\varphi)$ will only be used to prove local L^n-integrability of $D^2\psi$, where ψ is the convex conjugate of φ (see Section 3.2), but it will play no role in the behavior of the constants.

3.2 The convex conjugate

In this section, we discuss about the convex conjugate of functions and the relevant properties that play roles in the subsequent sections and chapters.

Definition 3.6. *The convex conjugate of a function $u : \Omega \to \mathbb{R}$ is the function $u^* : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$u^*(y) = \sup_{x\in\Omega} [x \cdot y - u(x)].$$

We observe that the convex conjugate u^* is a convex function no matter whether u is convex or not. In fact, for $t \in [0, 1]$,

$$\begin{aligned} u^*(ty + (1-t)z) &= \sup_{x\in\Omega} [x \cdot (ty + (1-t)z) - u(x)] \\ &= \sup_{x\in\Omega} [t(x \cdot y - u(x)) + (1-t)(x \cdot z - u(x))] \\ &\leq t \sup_{x\in\Omega} [x \cdot y - u(x)] + (1-t) \sup_{x\in\Omega} [x \cdot z - u(x)] \\ &= tu^*(y) + (1-t)u^*(z). \end{aligned}$$

Let us see an example to get the better picture of the convex conjugate.

Example 3.7. *Consider $\varphi(x) = \dfrac{|x|^p}{p}, 1 < p < \infty$ from $\mathbb{R} \to \mathbb{R}$. Then its convex conjugate is $\varphi^*(y) = \dfrac{|y|^q}{q}$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.*

To see this, let's recall the definition

$$\varphi^*(y) = \sup_{x\in\mathbb{R}} \left[x\, y - \frac{|x|^p}{p}\right], \text{ and let } h(x) = x\, y - \frac{|x|^p}{p}.$$

For the case $x \geq 0$, we have $h'(x) = y - x^{p-1}$. This implies that $x = y^{\frac{1}{p-1}}$ is the critical point.

Since $h''(x) = -(p-1)\, x^{p-2} < 0$, the critical point maximizes the function h. And hence,

$$h\left(y^{\frac{1}{p-1}}\right) = y^{\frac{1}{p-1}}\, y - y^{\frac{p}{p-1}} = \frac{y^q}{q}.$$

Hence $\varphi^*(y) = \dfrac{y^q}{q}$. We proceed in the similar way for the case $x < 0$.

Consider a strictly convex $\varphi \in C^1(\Omega)$. Then the gradient $\nabla\varphi$ is one-to-one. Now let

$\psi \in C^1(\nabla\varphi(\Omega))$ denote its convex conjugate. Then we have

$$\psi(\nabla\varphi(x)) = \langle \nabla\varphi(x), x\rangle - \varphi(x) \quad \forall x \in \Omega,$$

It is proved in[20] Section 26 that if φ is a strictly convex differentiable function, then its convex conjugate ψ is also strictly convex differentiable function such that $\nabla\psi = (\nabla\varphi)^{-1}$. That is,

$$\nabla\varphi(\nabla\psi(y)) = y \quad \forall y \in \nabla\varphi(\Omega), \tag{3.6}$$

$$\nabla\psi(\nabla\varphi(x)) = x \quad \forall x \in \Omega. \tag{3.7}$$

An another useful result associated to the convex conjugate, due to Forzani and Maldonado in 2004, is that if $\mu_\varphi \in DC(\Omega, \delta_\varphi)$, then $\mu_\psi \in DC(\nabla\varphi(\Omega), \delta_\psi)$ with a constant depending only on the constant from $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. More precisely,

Theorem 3.8 (Theorem 12 in[21])**.** *Consider a strictly convex function $\varphi \in C^1(\Omega)$ such that $\mu_\varphi \in DC(\Omega, \delta_\varphi)$, and let ψ be the convex conjugate of φ. Then $\mu_\psi \in DC(\nabla\varphi(\Omega), \delta_\psi)$ and there is a geometric constant $K^* > 0$ such that the section of φ and ψ have the following relation:*

$$S_\varphi(z, \tau/K^*) \subset \nabla\psi(S_\psi(\nabla\varphi(z), \tau)) \subset S_\varphi(z, K^*\tau), \tag{3.8}$$

for every section $S_\varphi(z, \tau)$ with $S_\varphi(z, K^*\tau) \subset\subset \Omega$.

We now add some details on the properties of convex functions that require to involve the convex conjugate. Recall from Section 2.1 that the strictly convexity of a smooth function doesn't guarantee the positive definiteness of its Hessian matrix everywhere. However, we have the following result that loses the positive definiteness on a set of measure zero.

Proposition 3.9. *If $\varphi \in C^2(\Omega)$ is strictly convex, then $D^2\varphi > 0$ a.e.in Ω.*

In order to sketch the proof of this proposition, we need the following theorem by Sard.

Theorem 3.10 (Theorem 1.1.15 in[7])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $g : \Omega \to \mathbb{R}^n$ is C^1 function in Ω. If $S_0 = \{x \in \Omega : \det Dg(x) = 0\}$, then $|g(S_0)| = 0$.*

Proof of Proposition 3.9: By the way of contradiction, we assume that $D^2\varphi \not> 0$ a.e. in Ω. Then there is a set $E \subset \Omega$ with positive measure such that $D^2\varphi(x) = 0$, $\forall x \in E$. Define $g := \nabla\varphi$. Note that $S_0 = \{x \in \Omega : \det Dg(x) = 0\} \supseteq E$. When Sard's Theorem is applied on $g = \nabla\varphi$, we get

$$0 = |g(S_0)| \geq |g(E)|.$$

That is, $|\nabla\varphi(E)| = 0$. Now let ψ be the convex conjugate of φ. Then ψ is C^2 and strictly convex as so is φ. Also, $\nabla\varphi$ is one-to-one as φ is strictly convex. So, we have $E = \nabla\psi(\nabla\varphi(E))$. Now since $\nabla\psi$ is also one-to-one (being the inverse of $\nabla\varphi$), it maps a set of measure zero to a set of measure zero. This implies that $|E| = 0$, a contradiction. □

We next remark that if the hypothesis $\varphi \in C^2(\Omega)$ of Proposition 3.9 is replaced by the weaker assumption $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$, then the Proposition 3.9 fails due to the following theorem.

Theorem 3.11 (Corollary 1 in[22]). *If $1 \leq p < n$, then there exists a strictly convex $W^{2,p}$-solution to the degenerate Monge-Ampère equation*

$$\det D^2\varphi = 0 \; a.e.$$

on the n-dimensional unit cube.

This is the reason why we assume $D^2\varphi > 0$ a.e. in Ω on the hypotheses of our statements along with the strictly convexity of $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$.

3.3 Self-improving properties for the Poincaré inequalities in the Monge-Ampère quasi-metric structure

Our objective in this section is to establish some results related to the self-improving properties. We will observe that Lemma 3.14 illustrates the improvement of weak L^p-norm to strong L^p-norm under suitable assumption on the associated Borel measure. Theorem 3.19

improves $(1,p)$-Poincaré inequality to (q,p)-Poincaré inequality with $q > 1$ when facilitated with an appropriate growth condition. This theorem will function as the heart of many results in this dissertation. Let us begin now with some definitions.

Definition 3.12. *Consider two Borel measures μ and ν on Ω. Then we say that μ is absolutely continuous with respect to ν, and write $\mu \ll \nu$, if for every Borel set $E \subset \Omega$, we have*

$$\nu(E) = 0 \Rightarrow \mu(E) = 0.$$

In our context, we are only interested with those Borel measures that are absolutely continuous with respect to the Lebesgue measure. A trivial example of such Borel measure is the Lebesgue measure itself. Another Borel measure that is absolutely continuous with respect to the Lebesgue measure is the Monge-Ampère measure. Indeed, let μ_φ be the Monge-Ampère measure of a convex function $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$. Then from Subsection 2.2.2,

$$\mu_\varphi(E) = \int_E \det D^2\varphi(x)\, dx,$$

for every Borel sets $E \subset \Omega$. Consequently, $|E| = 0$ implies $\mu_\varphi(E) = 0$.

Definition 3.13. *Let (X, Σ, μ) be a measurable space. Then a function $f : X \to \mathbb{C}$ is said to be in the weak L^p space, and write $f \in L^{p,w}(X, \mu)$, if there exists a constant $K > 0$ such that for all $\tau > 0$,*

$$\tau\, \mu\{x \in X : |f(x)| > \tau\}^{\frac{1}{p}} \leq K. \tag{3.9}$$

The weak L^p-norm of f is denoted by $\|f\|_{L^{p,w}}$ and is defined by

$$\|f\|_{L^{p,w}} := \sup_{\tau>0} \tau\, \mu\{x \in X : |f(x)| > \tau\}^{\frac{1}{p}},$$

which is same as the smallest possible constant K in the inequality (3.9). We note that the weak L^p-norm of a function f is smaller than the strong L^p-norm, that is, $\|f\|_{L^{p,w}} \leq \|f\|_{L^p}$.

Indeed,

$$\begin{aligned}
\|f\|_{L^p}^p &= \int_{\{x\in X:|f(x)|\le\tau\}} |f|^p\,d\mu + \int_{\{x\in X:|f(x)|>\tau\}} |f|^p\,d\mu \\
&\ge \int_{\{x\in X:|f(x)|\le\tau\}} \tau^p\,d\mu + \int_{\{x\in X:|f(x)|>\tau\}} |f|^p\,d\mu \\
&\ge \tau^p\,\mu\{x\in X:|f(x)|>\tau\}.
\end{aligned}$$

By taking the supremum on the right side, we get the desired inequality.

We recall that for a real valued function $u : X \to \mathbb{R}$, the positive and negative parts of u are defined by $u_+ := \max\{u, 0\}$ and $u_- := \max\{-u, 0\}$ respectively. Notice that $u_+ \ge, u_- \ge 0, u = u_+ - u_-$, and $|u| = u_+ + u_-$.

Another useful result that we will employ in the proof of Lemma 3.14 is the Cavalieri principle stating that if $u : X \to [0,\infty)$ is a μ-measurable function and $0 < q < \infty$, then

$$\int_X u^q\,d\mu = q\int_0^\infty \lambda^{q-1}\,\mu(\{x\in X : u(x)>\lambda\})\,d\lambda. \tag{3.10}$$

The proof of Cavalieri principle is quite straightforward. In fact, we can write $u(x)^q = q\int_0^{u(x)} \lambda^{q-1}\,d\lambda$. This implies

$$\begin{aligned}
\int_X u(x)^q &= q\int_X\left(\int_0^{u(x)}\lambda^{q-1}\,d\lambda\right)dx \\
&= q\int_X\left(\int_0^{\infty}\lambda^{q-1}\,\chi_{\{x:0<\lambda<u(x)\}}\,d\lambda\right)dx \\
&= q\int_0^{\infty}\lambda^{q-1}\left(\int_X \chi_{\{x:0<\lambda<u(x)\}}\,d\mu\right)d\lambda \\
&= q\int_0^{\infty}\lambda^{q-1}\,\mu(\{x\in X : u(x)>\lambda\})\,d\lambda.
\end{aligned}$$

We are now ready to state and prove the following lemma.

Lemma 3.14. *Let $\varphi \in C^1(\Omega)$ be a convex function with $D^2\varphi > 0$ a.e. in Ω and let μ be a Borel measure on Ω absolutely continuous with respect to the Lebesgue measure. Let S, S_0 be* *sections of* φ *with* $S \subset S_0 \subset\subset \Omega$ *and fix* $0 < p \le q < \infty$ *with* $q > 1$*. If, for some constant* $C_0 > 0$*, the inequality*

$$\tau^q \mu(\{x \in S : |u(x) - u_S^\mu| \ge \tau\}) \le C_0 \mu(S) \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{q}{p}} \tag{3.11}$$

holds true for every $\tau > 0$ *and* $u \in \mathrm{Lip}(S_0)$*, then*

$$\left(\fint_S |u - u_S^\mu|^q \, d\mu \right)^{1/q} \le C_1 \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \quad \forall u \in \mathrm{Lip}(S_0), \tag{3.12}$$

where $C_1 := 16 \left(1 + \left(\frac{q}{q-1}\right)^q\right)^{\frac{1}{q}} C_0^{\frac{1}{q}}$.

Notice that by taking the q^{th}-root and then supremum in the inequality (3.11) simplifies the quantity on the left to be the weak L^q-norm of $|u - u_S^\mu|$ which is eventually improved to the strong L^q-norm in the inequality (3.12).

Proof. Given $u \in \mathrm{Lip}(S_0)$, without loss of generality we may assume $u_S^\mu = 0$. Otherwise we can work with $v := u - u_S^\mu$. As $|u| = u_+ + u_-$, we will estimate the integrals $\fint_S u_+^p d\mu$ and $\fint_S u_-^p d\mu$.

Let $k_0 \in \mathbb{Z}$ such that

$$2^{k_0 - 1} \le \fint_S u_+ \, d\mu < 2^{k_0} \tag{3.13}$$

and for $k > k_0$ set

$$u_k := \begin{cases} 0, & u \le 2^k, \\ 2^k, & u \ge 2^{k+1}, \\ u - 2^k, & 2^k < u < 2^{k+1}. \end{cases} \tag{3.14}$$

Since u is differentiable a.e., $\nabla u_k = \nabla u \, \chi_{\{2^k < u < 2^{k+1}\}}$ (Lebesgue) a.e. in S_0 (see, for instance[23] Theorem 7.8) and consequently $\nabla^\varphi u_k = \nabla^\varphi u \, \chi_{\{2^k < u < 2^{k+1}\}}$ (Lebesgue) a.e. in S_0. In particular, $u_k \in \mathrm{Lip}(S_0)$ and, in view of (3.13),

$$(u_k)_S^\mu := \fint_S u_k \, d\mu \le \fint_S u_+ \, d\mu < 2^{k_0} \le 2^{k-1} \quad \forall k > k_0. \tag{3.15}$$

From the definition of u_k and the estimate (3.15) we get

$$\{x \in S : u(x) \geq 2^{k+1}\} \subset \{x \in S : u_k(x) = 2^k\}$$
$$= \{x \in S : |u_k(x) - (u_k)^\mu_S| = |2^k - (u_k)^\mu_S| \geq 2^{k-1}\}.$$

This implies

$$(2^{k-1})^q \,\mu(\{x \in S : u(x) \geq 2^{k+1}\}) \leq (2^{k-1})^q \,\mu(\{x \in S : |u_k(x) - (u_k)^\mu_S \geq 2^{k-1}\})$$
$$\leq C_0\mu(S) \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{q}{p}},$$

where the second inequality is due to (3.11) applied to u_k and $\tau = 2^{k-1}$. Define

$$I + II := \int_{\{x \in S : u(x) \geq 2^{k_0+2}\}} u_+^q \, d\mu + \int_{\{x \in S : u(x) < 2^{k_0+2}\}} u_+^q \, d\mu = \fint_S u_+^p d\mu.$$

Then,

$$I := \int_{\{x \in S : u(x) \geq 2^{k_0+2}\}} u_+^q \, d\mu = \sum_{k=k_0+1}^{\infty} \int_{\{x \in S : 2^{k+1} \leq u(x) < 2^{k+2}\}} u_+^q \, d\mu$$
$$\leq \sum_{k=k_0+1}^{\infty} 2^{(k+2)q} \mu(\{x \in S : 2^{k+1} \leq u(x) < 2^{k+2}\})$$
$$\leq 2^{3q} \sum_{k=k_0+1}^{\infty} 2^{(k-1)q} \mu(\{x \in S : |u_k(x) - (u_k)^\mu_S| \geq 2^{k-1}\})$$
$$\leq 2^{3q} C_0 \mu(S) \sum_{k=k_0+1}^{\infty} \left(\fint_{S_0} |\nabla^\varphi u_k|^p \, d\mu \right)^{\frac{q}{p}}$$
$$\leq 2^{3q} C_0 \mu(S) \left(\sum_{k=k_0+1}^{\infty} \fint_{S_0} |\nabla^\varphi u_k|^p \, d\mu \right)^{\frac{q}{p}}$$
$$= 2^{3q} C_0 \mu(S) \left(\sum_{k=k_0+1}^{\infty} \frac{1}{(\mu(S_0)} \int_{S_0 \cap \{x \in S : 2^{k+1} \leq u(x) < 2^{k+2}\}} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{q}{p}}$$
$$\leq 2^{3q} C_0 \mu(S) \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{q}{p}},$$

where the second last equality uses the facts that $\nabla^\varphi u_k = \nabla^\varphi u\, \chi_{\{2^k<u<2^{k+1}\}}$ (Lebesgue) a.e. in S_0 for every $k > k_0$ and that $q \geq p$. On the other hand, given $\zeta > 0$, due to Cavalieri principle, we get

$$\begin{aligned}
\fint_S u_+\, d\mu \leq \fint_S |u|\, d\mu &= \frac{1}{(\mu(S))} \int_S |u - u_S|\, d\mu \\
&= \frac{1}{(\mu(S))} \int_0^\zeta \mu(\{x \in S : |u(x) - u_S^\mu| \geq \tau\})\, d\tau + \frac{1}{\mu(S)} \int_\zeta^\infty \mu(\{x \in S : |u(x) - u_S^\mu| \geq \tau\})\, d\tau \\
&\leq \zeta + \frac{1}{\mu(S)} \int_\zeta^\infty \mu(\{x \in S : |u(x) - u_S^\mu| \geq \tau\})\, d\tau.
\end{aligned}$$

Then the inequality (3.11) applied to u gives

$$\begin{aligned}
\fint_S u_+\, d\mu &\leq \zeta + C_0 \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{q}{p}} \int_\zeta^\infty \frac{d\tau}{\tau^q} \\
&= \zeta \left(1 + C_0 \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{q}{p}} \zeta^{-q} \right).
\end{aligned}$$

Let us define $\zeta := C_0^{1/q} \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{1}{p}}$. We will see in Remark 3.15 that ζ defined here is finite under the assumption of integrability of certain function. So for this ζ, the above inequality simplifies to

$$\fint_S u_+\, d\mu \leq \frac{q C_0^{1/q}}{q-1} \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{1}{p}}. \tag{3.16}$$

We note that (3.16) holds true also in the case $\zeta = 0$. In fact, when $\zeta = 0$, the quantity on the right side of (3.12) is zero immediately. Now to see the quantity on the left is also zero, observe that $|\nabla^\varphi u|^p = 0$ a.e. in S_0. This implies $D^2\varphi(x)^{-1/2}\nabla u(x) = 0$ a.e. in S_0. Since $D^2\varphi > 0$ a.e. in S_0, we must have $\nabla u(x) = 0$ a.e. in S_0, and hence $u - u_S = 0$ a.e. in S.

Now, the definition of $k_0 \in \mathbb{Z}$ from (3.13) and (3.16) imply

$$\begin{aligned} II := \int_{\{x\in S: u(x) < 2^{k_0+2}\}} u_+^q \, d\mu < 2^{(k_0+2)q}\mu(S) \le 2^{3q}\mu(S) \left(\fint_S u_+ \, d\mu \right)^q \\ \le 2^{3q} C_0 \left(\frac{q}{q-1} \right)^q \mu(S) \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{q}{p}}. \end{aligned}$$

Finally,

$$\int_S u_+^q d\mu = I + II \le C_1^q \mu(S) \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{q}{p}},$$

where $C_1^q := 2^{3q} C_0 \left(1 + \left(\frac{q}{q-1} \right)^q \right)$.

With similar reasoning, we get

$$\int_S u_-^q d\mu \le C_1^q \mu(S) \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{q}{p}},$$

Now combing these positive and negative parts

$$\begin{aligned} \left(\fint_S |u|^q \, d\mu \right)^{\frac{1}{q}} &= \left(\fint_S [u_+ + u_]^q \, d\mu \right)^{\frac{1}{q}} \\ &\le \left(\int_S u_+^q \, d\mu \right)^{\frac{1}{q}} + \left(\int_S u_-^q \, d\mu \right)^{\frac{1}{q}} \text{ (due to Minkowski inequality)} \\ &\le 16 \left(1 + \left(\frac{q}{q-1} \right)^q \right)^{\frac{1}{q}} C_0^{\frac{1}{q}} \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}}. \end{aligned}$$

□

Remark 3.15. *When $0 < p \le 2$ (which is the case we will be using), the condition $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\Omega, d\mu)$ guarantees that $\zeta := C_0^{1/q} \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}}$ in the proof of Lemma 3.14 is finite. Indeed,*

$$\begin{aligned} \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \le \left(\fint_{S_0} |\nabla^\varphi u|^2 \, d\mu \right)^{\frac{1}{2}} = \left(\fint_{S_0} \langle (D^2\varphi)^{-1} \nabla u, \nabla u \rangle \, d\mu \right)^{\frac{1}{2}} \\ \le \operatorname*{ess\,sup}_{S_0} |\nabla u| \left(\fint_{S_0} ||(D^2\varphi)^{-1}|| \, d\mu \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

Before presenting the main theorem of this section, let us briefly discuss about the Lebesgue point and the maximal operator associated to the Monge-Ampère sections. Recall that we assume $\Omega \subset \mathbb{R}^n$ is open and convex, and $\varphi \in C^1(\Omega)$ is strictly convex.

Definition 3.16. *Let $u : \Omega \to \mathbb{R}$ be a locally integrable function. Then a point $x \in \Omega$ is called the Lebesgue point of u if for sections $S_\varphi(x,t) \subset \Omega$,*

$$\lim_{t\to 0} \frac{1}{\mu(S_\varphi(x,t))} \int_{S_\varphi(x,t)} |u - u(x)|\, d\mu = 0. \tag{3.17}$$

The equation (3.17) implies that

$$u(x) = \lim_{t\to 0} \frac{1}{\mu(S_\varphi(x,t))} \int_{S_\varphi(x,t)} u\, d\mu.$$

We note that if u is continuous at $x \in \Omega$, then x is a Lebesgue point of u.

Consider two measurable spaces (x, Σ, μ) and (x, Σ', ν), and let T be an operator that maps $L^p(X)$ to a space of measurable functions defined on Y. Then for $1 \leq q < \infty$, we say that T is of weak type (p,q) if there exists a constant $C > 0$ such that for all $\alpha > 0$ and every $u \in L^P(X)$, we have

$$\nu(\{y \in Y : |Tu(y)| > \alpha\}) \leq \left(\frac{c\,\|u\|_{L^p}}{\alpha}\right)^q. \tag{3.18}$$

For the case $q = \infty$, we say that T is of weak type (p,∞) if for every $u \in L^P(X)$,

$$\|T\,u\|_{L^\infty} \leq C\ \|u\|_{L^p}\,.$$

And, we say that T is of strong type (p,q) if there exists a constant $C > 0$ such that for every $u \in L^P(X)$, we have

$$\|T\,u\|_{L^q} \leq C\,\|u\|_{L^p}\,. \tag{3.19}$$

We can easily verify that T is of strong type (p,q) implies T is of weak type (p,q). And due to Marcinkiewicz interpolation theorem (for instance, see Page 158 in[24]), if an operator T

is of weak $(1,1)$ type and strong (∞,∞) type, then T is of strong (p,p) type for every p with $1 < p \le \infty$.

Definition 3.17. *Suppose $U \subset \Omega$ and let $u \in L^1(U)$ with respect to a Borel measure μ. Then the noncentered maximal function restricted to U is defined by*

$$M_U\, u(x) := \sup_{t>0} \frac{1}{\mu(S_\varphi(x_0,t))} \int_{S_\varphi(x_0,t)} |u|\, d\mu. \tag{3.20}$$

where the supremum is taken over all sections $S_\varphi(x_0,t) \subset U$ containing x.

For the centered version of maximal function, we simply take supremum over all the sections $S_\varphi(x,t) \subset U$. That is, replacing $S_\varphi(x_0,t)$ by $S_\varphi(x,t)$ in (3.20) gives the definition for the centered version. Whenever the associated Borel measure μ is doubling with respect to the Monge-Ampère sections of φ (that is, either $\mu \in DC(\Omega, \delta_\varphi)$ or $\mu \in DP(\Omega, \delta_\varphi)$), then the centered and uncentered maximal functions are comparable.

Remark 3.18. *The maximal function M_U defined by (3.20) is finite a.e. in U, weak $(1,1)$ type (see[5] Section 5 or[25] Section 3.2), strong (∞,∞) type, and consequently strong (p,p) type for p with $1 < p \le \infty$ due to Marcinkiewicz interpolation theorem.*

We are now ready to state and prove the following main theorem.

Theorem 3.19. *Fix $\varphi \in C^1(\Omega)$ with $D^2\varphi > 0$ a.e. in Ω and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and let μ be a Borel doubling measure on Ω absolutely continuous with respect to Lebesgue measure satisfying the following conditions:*

(a) for some $C_P > 0$, $\lambda \ge 1$, and $p > 0$, the Poincaré inequality

$$\fint_S |u - u_S^\mu|\, d\mu \le C_P t^{\frac{1}{2}} \left(\fint_{\lambda S} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{1}{p}}, \tag{3.21}$$

with $u_S^\mu := \fint_S u\, d\mu$, holds true for every section $S := S_\varphi(x_0,t)$ with $\lambda S \subset\subset \Omega$ and every $u \in \mathrm{Lip}(\lambda S)$;

(b) for some $C_D > 0$ and $s > p/2$ it satisfies the growth condition

$$\mu(S_\varphi(z,r)) \le C_D \left(\frac{r}{r'}\right)^s \mu(S_\varphi(z,r')), \tag{3.22}$$

for all $0 < r' \le r$ and all sections $S_\varphi(z,r)$ with $S_\varphi(z,r) \subset\subset \Omega$.

Then,

$$\left(\fint_S |u - u_S^\mu|^{\frac{2sp}{2s-p}}\, d\mu\right)^{\frac{2s-p}{2sp}} \le C_{P,s} t^{\frac{1}{2}} \left(\fint_{\lambda\Theta^2 S} |\nabla^\varphi u|^p\, d\mu\right)^{\frac{1}{p}}, \tag{3.23}$$

for every section $S := S_\varphi(x_0,t)$ with $\lambda\Theta^2 S \subset\subset \Omega$ and every $u \in \mathrm{Lip}(\lambda\Theta^2 S)$, where $C_{P,s} > 0$ depends only on s, λ, Θ, C_P, and C_D.

Proof. Fix $S := S_\varphi(x_0,t)$ such that $S_\varphi(x_0,\lambda\Theta^2 t) \subset\subset \Omega$, $x \in S$, and $u \in \mathrm{Lip}(\lambda\Theta^2 S)$. For $j \in \mathbb{N}$ set $t_j := 2^{-j}t$ and $S_j := S_\varphi(x,t_j)$, for $j = 0$ set $S_0 := S_\varphi(x_0,\Theta^2 t)$ and $t_0 := \Theta^2 t$. Notice that these choices imply $S_{j+1} \subset S_j$ for every $j \in \mathbb{N}_0$ and, for $\lambda \ge 1$, $\lambda S_{j+1} \subset \lambda S_j$ for every $j \in \mathbb{N}_0$. To check this last inclusion when $j = 0$, we use that $x \in S = S_\varphi(x_0,t) \subset S_\varphi(x_0,\lambda t)$ and the second inclusion from (2.28) with "$\tau = \lambda t$" to obtain $\lambda S_1 \subset \lambda S_0$.

Due to the engulfing property (2.28), we get

$$x \in S_\varphi(x_0,t) \subset\subset S_\varphi(x_0,\Theta^2 t) \subset S_\varphi(x_0,\lambda\Theta^2 t) \subset\subset \Omega.$$

We can use (3.22) with $S = S_\varphi(x_0,t)$ and $S_j = S_\varphi(x,t_j)$ to obtain

$$t_j \le C_D^{1/s}\Theta t \left(\frac{\mu(S_\varphi(x,t_j))}{\mu(S_\varphi(x_0,t))}\right)^{1/s} \quad \forall j \in \mathbb{N}, \tag{3.24}$$

with $\mu(S_0) = \mu(S_\varphi(x_0,\Theta^2 t)) \le C_D\Theta^{2s}\mu(S_\varphi(x_0,t))$ from (3.22). Hence,

$$t_j \le C_D^{2/s}\Theta^3 t \left(\frac{\mu(S_j)}{\mu(S_0)}\right)^{1/s} \quad \forall j \in \mathbb{N}. \tag{3.25}$$

Notice that (3.25) is obviously true for $j = 0$ because $C_D > 1$ and $\Theta > 1$. In addition, $\mu(S_j) \le C_D\mu(S_{j+1})$ for every $j \in \mathbb{N}$ and, when $j = 0$, the doubling condition (3.22) and the fact that $S_\varphi(x_0,t) \subset S_\varphi(x,\Theta t)$ give

$$\mu(S_0) \le C_D\Theta^{2s}\mu(S_\varphi(x_0,t)) \le C_D\Theta^{2s}\mu(S_\varphi(x,\Theta t)) \\ \le 2^s C_D^2\Theta^{3s}\mu(S_\varphi(x,t/2)) = 2^s C_D^2\Theta^{3s}\mu(S_1). \tag{3.26}$$

Consequently, by using the fact that any $x \in \lambda\Theta^2 S$ is a Lebesgue point of u, the estimates above for $\mu(S_j)/\mu(S_{j+1})$ with $j \in \mathbb{N}_0$, the Poincaré inequality (3.21), and (3.25),

$$\begin{aligned} |u(x) - u^\mu_{S_0}| &= |\lim_{j\to\infty} u^\mu_{S_j} - u^\mu_{S_0}| = \lim_{j\to\infty} |u^\mu_{S_j} - u^\mu_{S_0}| \\ &\le \sum_{j=0}^\infty |u^\mu_{S_j} - u^\mu_{S_{j+1}}| \le \sum_{j=0}^\infty ⨍_{S_{j+1}} |u - u^\mu_{S_j}|\, d\mu \\ &\le 2^s C_D^2 \Theta^{3s} \sum_{j=0}^\infty ⨍_{S_j} |u - u^\mu_{S_j}|\, d\mu \\ &\le 2^s C_D^2 \Theta^{3s} C_P \sum_{j=0}^\infty t_j^{\frac12} \left(⨍_{\lambda S_j} |\nabla^\varphi u|^p\, d\mu \right)^{\frac1p} \\ &\le \frac{C_3 t^{\frac12}}{\mu(S_0)^{\frac{1}{2s}}} \sum_{j=0}^\infty \mu(S_j)^{\frac{1}{2s}} \left(⨍_{\lambda S_j} |\nabla^\varphi u|^p\, d\mu \right)^{\frac1p}, \end{aligned}$$

with $C_3 := 2^s C_D^{3/2} \Theta^{3s+3/2} C_P$. On the other hand, recalling that $t_0 := \Theta^2 t$,

$$\begin{aligned} |u^\mu_S - u^\mu_{S_0}| \le ⨍_S |u - u^\mu_{S_0}|\, d\mu &\le C_D\Theta^{2s} ⨍_{S_0} |u - u^\mu_{S_0}|\, d\mu \\ &\le C_D \Theta^{2s} C_P t^{\frac12} \left(⨍_{\lambda S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac1p}. \end{aligned}$$

Therefore, for every $x \in S$ we have

$$|u(x) - u^\mu_S| \le \frac{2C_3 t^{\frac12}}{\mu(S_0)^{\frac{1}{2s}}} \sum_{j=0}^\infty \mu(S_j)^{\frac{1}{2s}} \left(⨍_{\lambda S_j} |\nabla^\varphi u|^p\, d\mu \right)^{\frac1p} \tag{3.27}$$

Next, introduce

$$M(x) := \sup_{S'} ⨍_{S'} |\nabla^\varphi u|^p\, d\mu > 0$$

where the supremum is taken over all the sections $S' \subset \lambda\Theta^2 S_0$ with $x \in S'$. From Remark 3.18, it follows that $M(x)$ is finite for a.e. $x \in S$ and then $\frac{1}{M(x)} \int_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu > 0$ a.e. $x \in S$. Now, from Lemma 2.29, we obtain $\mu(\lambda S_j) \to 0$ as $j \to \infty$, and consequently, for a.e. $x \in S$, there is a smallest $j \in \mathbb{N}_0$ such that the inequality

$$\mu(\lambda S_j) \leq \frac{1}{M(x)} \int_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu \tag{3.28}$$

holds true. Let $j_0 \in \mathbb{N}_0$ denote such integer (which depends on x). Notice that if $j_0 = 0$ then equality occurs in (3.28) because

$$\frac{1}{\mu(\lambda S_j)} \int_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu \leq M(x).$$

In particular, we have

$$\mu(\lambda S_{j_0}) \leq \frac{1}{M(x)} \int_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu < \mu(\lambda S_{j_0 - 1}) \leq C_D \lambda^s \mu(S_{j_0-1}),$$

with $\mu(S_{j_0-1}) \leq C_D \mu(S_{j_0})$ if $j_0 > 1$ and, if $j_0 = 1$, we use (3.26) to obtain $\mu(S_{j_0-1}) = \mu(S_0) \leq C_D^2 \Theta^{3s} \mu(S_1)$. Hence,

$$\mu(\lambda S_{j_0}) \leq \frac{1}{M(x)} \int_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu \leq C_D^3 (\lambda\Theta^3)^s \mu(S_{j_0}). \tag{3.29}$$

Let us now split the sum from (3.27) into

$$\sum_{j=0}^{\infty} \mu(S_j)^{\frac{1}{2s}} \left(\fint_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} = \sum_{j=0}^{j_0-1} \cdots + \sum_{j=j_0}^{\infty} \cdots =: \Sigma' + \Sigma''.$$

Let us first consider Σ'. Notice that we can assume $j_0 \in \mathbb{N}$ (that is, $j_0 \geq 1$, because $\Sigma' = 0$ if $j_0 = 0$). Now, for $j_0 > 1$, Lemma 2.29 with $\alpha = 1/2$ implies

$$\mu(S_{j_0}) \leq \xi^{j_0 - j} \mu(S_j) \quad \forall j < j_0, \tag{3.30}$$

where $\xi \in (0,1)$ depends only on C_D and Θ. Now, if $j_0 = 1$ and $j = 0$, from the inclusion $S_1 = S_\varphi(x, t/2) \subset S_\varphi(x_0, \Theta^2 t) = S_0$, we get

$$\mu(S_1) \leq \mu(S_0) = \frac{1}{\xi}\xi^{j_0 - j}\mu(S_0). \tag{3.31}$$

Hence, for each $j_0 \in \mathbb{N}$, we have that $\mu(S_{j_0}) \leq \frac{1}{\xi}\xi^{j_0-j}\mu(S_j)$ for every $j < j_0$, which, in turn, implies (recall that $2s > p$)

$$\begin{aligned}
\Sigma' &:= \sum_{j=0}^{j_0-1} \frac{\mu(S_j)^{\frac{1}{2s}}}{\mu(\lambda S_j)^{\frac{1}{p}}} \left(\int_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \leq \sum_{j=0}^{j_0-1} \mu(S_j)^{\frac{1}{2s}-\frac{1}{p}} \left(\int_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \\
&\leq \xi^{\frac{1}{2s}-\frac{1}{p}} \mu(S_{j_0})^{\frac{1}{2s}-\frac{1}{p}} \sum_{j=0}^{j_0-1} \xi^{(\frac{1}{2s}-\frac{1}{p})(j-j_0)} \left(\int_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \\
&\leq C_4 \mu(S_{j_0})^{\frac{1}{2s}-\frac{1}{p}} \left(\int_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \leq C_5 M(x)^{\frac{1}{p}-\frac{1}{2s}} \left(\int_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{2s}},
\end{aligned}$$

where $C_4 := \xi^{\frac{1}{2s}-\frac{1}{p}} \sum_{k=1}^{\infty} \xi^{(\frac{1}{p}-\frac{1}{2s})k}$ and $C_5 := C_4[C_D^3(\lambda\Theta^3)^s]^{\frac{1}{p}-\frac{1}{2s}}$ and we used the second inequality from (3.29).

We now turn to Σ''. We first use Lemma 2.29 again with $\alpha = 1/2$ to write

$$\mu(S_j) \leq \xi^{j-j_0}\mu(S_{j_0}) \quad \forall j \geq j_0,$$

at least if $j_0 \in \mathbb{N}$. If $j_0 = 0$ the inclusion $S_1 \subset S_0$ and Lemma 2.29 give $\mu(S_j) \leq \xi^{j-1}\mu(S_1) \leq \frac{1}{\xi}\xi^{j-j_0}\mu(S_0) = \frac{1}{\xi}\xi^{j-j_0}\mu(S_{j_0})$. Hence,

$$\mu(S_j) \leq \frac{1}{\xi}\xi^{j-j_0}\mu(S_{j_0}) \quad \forall j \geq j_0 \geq 0. \tag{3.32}$$

Consequently, from (3.32) and (3.29),

$$\Sigma'' := \sum_{j=j_0}^{\infty} \mu(S_j)^{\frac{1}{2s}} \left(\fint_{\lambda S_j} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{1}{p}} \leq \xi^{-\frac{1}{2s}} \mu(S_{j_0})^{\frac{1}{2s}} M(x)^{\frac{1}{p}} \sum_{j=j_0}^{\infty} \xi^{\frac{j-j_0}{2s}}$$
$$\leq C_6 M(x)^{\frac{1}{p}-\frac{1}{2s}} \left(\int_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{1}{2s}},$$

with $C_6 := \xi^{-\frac{1}{2s}} \sum_{k=0}^{\infty} \xi^{\frac{k}{2s}}$. Going back to (3.27), we now have

$$|u(x) - u_S^{\mu}| \leq \frac{2C_3(C_5 + C_6)t^{\frac{1}{2}}}{\mu(S_0)^{\frac{1}{2s}}} M(x)^{\frac{1}{p}-\frac{1}{2s}} \left(\int_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{1}{2s}}$$
$$\leq C_7 t^{\frac{1}{2}} M(x)^{\frac{2s-p}{2sp}} \left(\fint_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{1}{2s}}, \tag{3.33}$$

with $C_7 := 2\lambda^{\frac{1}{2}} C_D^{\frac{1}{2s}} C_3(C_5 + C_6)$. Setting $q_s := \frac{2sp}{2s-p} > p$ and given $\tau > 0$, the inequality $|u(x) - u_S^{\mu}| \geq \tau$, the weak (1,1)-type of M with a constant $C_{1,1} > 0$ depending only on C_D and Θ (see Remark 3.18), and (3.33) then yield

$$\mu(\{x \in S : |u(x) - u_S^{\mu}| \geq \tau\})$$
$$\leq \mu\left(\left\{ x \in S : M(x) \geq C_7^{-q_s} \tau^{q_s} t^{-q_s/2} \left(\fint_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{-\frac{q_s}{2s}} \right\} \right)$$
$$\leq C_{1,1} C_7^{q_s} \tau^{-q_s} t^{q_s/2} \left(\fint_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{q_s}{2s}} \int_S |\nabla^{\varphi} u|^p \, d\mu$$
$$\leq C_0 \tau^{-q_s} \mu(S) \left(\fint_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{q_s}{2s}+1}$$
$$= C_0 \tau^{-q_s} \mu(S) \left(\fint_{\lambda \Theta^2 S} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{q_s}{p}},$$

with $C_0 := C_{1,1} C_7^{q_s} C_D \lambda^s t^{q_s/2}$. Hence, Lemma 3.14 applied with $q = q_s$ and $S_0 = \lambda\Theta^2 S$ imply (3.23). □

3.4 (1,2)-Poincaré inequality with a weaker hypothesis

Recall that Theorem 3.2 establishes $(1,2)$-Poincaré inequality under the assumption $\varphi \in C^2(U)$ with $D^2\varphi > 0$ everywhere in U. In this section we present $(1,2)$-Poincaré inequality with weaker hypotheses, that is, by replacing the hypotheses of Theorem 3.2 with $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ and $D^2\varphi > 0$ a.e., and the assumption on the doubling condition is carried as it is.

Let us first list some of the results that we will use to prove Theorem 3.23. The theorem below is useful to verify $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ implies the Monge-Ampère measure associated to the convolution function φ_ε satisfies the same doubling condition (that is, $\mu_{\varphi_\varepsilon} \in DC(\Omega, \delta_\varphi)$) in the proof of Theorem 3.23.

Theorem 3.20 (Theorem 4 in[21]). *Let $\varphi \in C^1(\Omega)$ be a strictly convex function. Then the following statements are equivalent.*

(i) The sections of φ have the engulfing property (2.28).

(ii) There exists a constant $K_1 > 1$ such that if $y \in S_\varphi(x,t)$ with $S_\varphi(x, K_1 t) \subset\subset \Omega$, then $x \in S_\varphi(y, K_1 t)$.

(iii) There exists a constant $K_2 > 1$ such that

$$\begin{aligned}\frac{K_2+1}{K_2}&[\varphi(y) - \varphi(x) - \nabla\varphi(x)\cdot(y-x)]\\ &\leq (\nabla\varphi(x) - \nabla\varphi(x))\cdot(y-x)\\ &\leq (K_2+1)[\varphi(y) - \varphi(x) - \nabla\varphi(x)\cdot(y-x)].\end{aligned}$$

We recall that whenever a matrix A is symmetric positive definite, its eigenvalues are positive and the largest eigenvalue gives its operator norm, that is $\|A\|_2 = \lambda_{\max}$. Moreover, a positive definite matrix is equivalent to a diagonal matrix. In particular, if $D^2\varphi(x) > 0$ and $\lambda_1(x), \cdots, \lambda_n(x)$ are its eigenvalues, then there exists an invertible matrix X such that $D^2\varphi(x) = X^{-1}\,D\,X$ with $D = \mathrm{diag}(\lambda_1(x), \cdots, \lambda_n(x))$. Consequently, $\det D^2\varphi(x) = \lambda_1(x)\lambda_2(x)\cdots\lambda_n(x)$, and $\Delta\varphi = \mathrm{trace}(D^2\varphi(x)) = \lambda_1(x) + \lambda_2(x) + \cdots + \lambda_n(x)$.

For the convenience of readers, let us also list the Fatou's lemma and the Lebesgue's dominated convergence theorem before heading to the proof of Theorem 3.23.

Lemma 3.21 (Fatou's lemma, see[26]). *Let (X, Σ, μ) be a measurable space. If $f_n : X \to [0, \infty]$ are measurable for all $n \in \mathbb{N}$, then*

$$\int_X \liminf_{\lim n\to\infty} f_n\, d\mu \leq \liminf_{\lim n\to\infty} \int_X f_n\, d\mu.$$

Theorem 3.22 (Lebesgue's dominated convergence theorem, see[26]). *Let $\{f_n\}$ be a sequence of measurable functions on a measurable space (X, Σ, μ) such that $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x \in E$. If there exists $g \in L^1(X)$ such that $|f_n(x) \leq g(x)|$ a.e. for all $n \in \mathbb{N}$, then f is integrable,*

$$\lim_{n\to\infty} \int_E |f_n - f|\, d\mu = 0, \quad \textit{and} \quad \lim_{n\to\infty} \int_E f_n\, d\mu = \int_E f\, d\mu.$$

Theorem 3.23. *Fix an open convex set $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ and $\varphi \in W^{2,n}_{loc}(\Omega)$ such that $D^2\varphi > 0$ a.e. in Ω and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Then, there exist geometric constants $C_3^* > 0$ and $K^* \geq 1$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, 2\,K^* t) \subset\subset \Omega$ and every $h \in C^1(S_\varphi(x_0, 2K^* t))$ the following Poincaré inequality holds true with respect to the Monge-Ampère measure μ_φ*

$$⨍_S |h(x) - h_S^{\mu_\varphi}| d\mu_\varphi(x) \leq C_3^*\, t^{\frac{1}{2}} \left(⨍_{S_\varphi(x_0, 2K^* t)} |\nabla^\varphi h(x)|^2 d\mu_\varphi(x) \right)^{\frac{1}{2}}. \tag{3.34}$$

Proof. Let $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ with $D^2\varphi > 0$ a.e. in Ω and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Given a section $S := S_\varphi(x_0, t) \subset\subset \Omega$ let $\Omega_S \subset \mathbb{R}^n$ be an open convex set such that $S \subset\subset \Omega_S \subset\subset \Omega$ set $\varepsilon_0 := \mathrm{dist}(\Omega_S, \partial\Omega)$ and for $0 < \varepsilon < \varepsilon_0$ and $x \in \Omega_S$ define

$$\varphi_\varepsilon(x) := \varphi * \eta_\varepsilon(x) = \int_{\mathbb{R}^n} \varphi(x - y)\eta_\varepsilon(y)\, dy \tag{3.35}$$

where $\eta \in C_c^\infty(\mathbb{R}^n)$, $\eta \geq 0$, $\text{supp}(\eta) \subset B(0,1)$ and $\|\eta\|_{L^1(\mathbb{R}^n)} = 1$ with $\eta_\varepsilon(y) := \varepsilon^{-n}\eta(\varepsilon^{-1}y)$. Then, for each $\varepsilon > 0$, we have that $\varphi_\varepsilon \in C^\infty(\Omega_S)$ with $D^2\varphi_\varepsilon > 0$ in Ω_S. Indeed, since $D^2\varphi_\varepsilon(x) = \int_{\mathbb{R}^n} D^2\varphi(x-y)\eta_\varepsilon(y)\,dy$, if we had $\langle D^2\varphi_\varepsilon(y_0)v, v\rangle = 0$ for some point $y_0 \in \Omega$ and non-zero vector $v \in \mathbb{R}^n \setminus \{0\}$, then it would follow that $\langle D^2\varphi(y_0 - y)v, v\rangle = 0$ for almost every $|y| < \varepsilon$, contradicting $D^2\varphi > 0$ a.e. in Ω. Also, φ_ε and $\nabla\varphi_\varepsilon$ converge to φ and $\nabla\varphi$, respectively, uniformly over compact subsets of Ω_S. Moreover, from the characterization of $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ in terms of the engulfing property in Theorem 3.20 we have that $\mu_{\varphi_\varepsilon} \in DC(\Omega, \delta_\varphi)$ for every $\varepsilon \in (0, \varepsilon_0)$ with constants depending only on the constant from $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ (and, in particular, independent of ε). In fact,

$$\begin{aligned}
\langle \nabla_{\varphi_\varepsilon}(x) - \nabla_{\varphi_\varepsilon}(y), x-y\rangle &= \Big\langle \int_{\mathbb{R}^n} \nabla\varphi(x-z)\,\eta_\varepsilon(z)\,dz - \int_{\mathbb{R}^n} \nabla\varphi(y-z)\,\eta_\varepsilon(z)\,dz,\ x-y\Big\rangle \\
&= \int_{\mathbb{R}^n} \langle \nabla\varphi(x-z) - \nabla\varphi(y-z),\ x-y\rangle\,\eta_\varepsilon(z)\,dz \\
&\approx \int_{\mathbb{R}^n} [\varphi(y-z) - \varphi(x-z) - \nabla\varphi(x-z)\cdot(y-x)]\,\eta_\varepsilon(z)\,dz. \\
&= \int_{\mathbb{R}^n} \varphi(y-z)\,\eta_\varepsilon(z)\,dz - \int_{\mathbb{R}^n} \varphi(x-z)\,\eta_\varepsilon(z)\,dz - \int_{\mathbb{R}^n} \nabla\varphi(x-z)\cdot(y-x)\eta_\varepsilon(z)\,dz. \\
&= \varphi_\varepsilon(y) - \varphi_\varepsilon(x) - \nabla_{\varphi_\varepsilon}(x)\cdot(y-x).
\end{aligned}$$

Next, for each $0 < \varepsilon < \varepsilon_0$ let $\psi_\varepsilon : \nabla\varphi_\varepsilon(\Omega) \to \mathbb{R}$ denote the convex conjugate to φ_ε, which is smooth, strictly convex, and satisfies

$$\nabla\varphi_\varepsilon(\nabla\psi_\varepsilon(y)) = y \quad \forall y \in \nabla\varphi_\varepsilon(\Omega), \tag{3.36}$$

$$\nabla\psi_\varepsilon(\nabla\varphi_\varepsilon(x)) = x \quad \forall x \in \Omega. \tag{3.37}$$

Moreover, by Theorem 3.8 we have $\mu_{\psi_\varepsilon} \in DC(\delta_{\psi_\varepsilon}, \nabla\varphi_\varepsilon(\Omega))$ with a constant depending only on the constant from $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. In addition, there exists a constant $K^* > 1$, also depending only on the constant from $\mu_\varphi \in DC(\Omega, \delta_\varphi)$, such that

$$S_{\varphi_\varepsilon}(z, \tau/K^*) \subset \nabla\psi_\varepsilon(S_{\psi_\varepsilon}(\nabla\varphi_\varepsilon(z), \tau)) \subset S_{\varphi_\varepsilon}(z, K^*\tau), \tag{3.38}$$

for every section $S_{\varphi_\varepsilon}(z,\tau)$ with $S_{\varphi_\varepsilon}(z,K^*\tau) \subset\subset \Omega$. At this point, given a section $S_{\varphi_\varepsilon}(x_0,t)$ with $S_{\varphi_\varepsilon}(x_0,2K^*t) \subset\subset \Omega$, the second inclusion in (3.38) and (3.36) give

$$S_\varepsilon^* := S_{\psi_\varepsilon}(\nabla\varphi_\varepsilon(x_0),t) \subset \nabla\varphi_\varepsilon(S_{\varphi_\varepsilon}(x_0,K^*t)) \subset\subset \nabla\varphi_\varepsilon(\Omega). \tag{3.39}$$

Notice that from the fact that φ_ε and $\nabla\varphi_\varepsilon$ converge to φ and $\nabla\varphi$, respectively, uniformly over compact subsets of Ω we can assume that $\varepsilon > 0$ is small enough so that

$$\nabla\varphi_\varepsilon(S_{\varphi_\varepsilon}(x_0,K^*t)) \subset \nabla\varphi(S_\varphi(x_0,2K^*t)) \subset\subset \nabla\varphi(\Omega_S). \tag{3.40}$$

The next step is to apply (3.2) with ψ_ε in the section S_ε^*. Given a function $h \in C^1(S_\varphi(x_0,2K^*t))$ define $u \in C^1(\nabla\varphi(S_\varphi(x_0,2K^*t)))$ as $u(y) := h(\nabla\psi_\varepsilon(y))$. In particular, the inclusions (3.39) and (3.40) imply $u \in C^1(S_\varepsilon^*)$, so that the Poincaré inequality (3.2) applied with ψ_ε in the section S_ε^* to u reads as

$$\fint_{S_\varepsilon^*} |u(y) - u_{S_\varepsilon^*}|\,dy \le C_2^*\,t^{\frac{1}{2}} \left(\fint_{S_\varepsilon^*} |\nabla^{\psi_\varepsilon} u(y)|^2\,dy\right)^{\frac{1}{2}}. \tag{3.41}$$

By setting $y := \nabla\varphi_\varepsilon(x)$ for $x \in S_\varphi(x_0,2K^*t))$, and recalling (3.37), we get

$$\nabla h(x) = D^2\varphi_\varepsilon(x)\nabla u(\nabla\varphi_\varepsilon(x)) = D^2\psi_\varepsilon(y)^{-1}\nabla u(y)$$

and then

$$\begin{aligned} |\nabla^{\psi_\varepsilon} u(y)|^2 &= \langle D^2\psi_\varepsilon(y)^{-1}\nabla u(y), \nabla u(y)\rangle \\ &= \langle \nabla h(x), D^2\varphi_\varepsilon(x)^{-1}\nabla h(x)\rangle = |\nabla^{\varphi_\varepsilon} h(x)|^2. \end{aligned} \tag{3.42}$$

Hence, by changing variables $y = \nabla\varphi_\varepsilon(x)$ in (3.41) we get

$$\frac{1}{|S_\varepsilon^*|}\int_{\nabla\psi_\varepsilon(S_\varepsilon^*)} |h(x) - h_{S_\varepsilon}| \det D^2\varphi_\varepsilon(x)\, dx \\ \leq \frac{C_2^*\, t^{\frac{1}{2}}}{|S_\varepsilon^*|^{\frac{1}{2}}} \left(\int_{\nabla\psi_\varepsilon(S_\varepsilon^*)} |\nabla^{\varphi_\varepsilon} h(x)|^2 \det D^2\varphi_\varepsilon(x)\, dx\right)^{\frac{1}{2}} \tag{3.43}$$

where

$$h_{S_\varepsilon} := \frac{1}{|S_\varepsilon^*|}\int_{\nabla\psi_\varepsilon(S_\varepsilon^*)} h(x) \det D^2\varphi_\varepsilon(x)\, dx.$$

Notice that from the inclusions (3.38), (3.39), and (3.40) (with sufficiently small ε) it follows that

$$S_\varphi(x_0, t) \subset \nabla\psi_\varepsilon(S_\varepsilon^*) \subset S_\varphi(x_0, 2K^*t), \tag{3.44}$$

so that the integral on the left-hand side of (3.43) can be replaced with the integral over $S_\varphi(x_0, t)$ and the one on its right-hand side by the integral over $S_\varphi(x_0, 2K^*t)$. That is,

$$\frac{1}{|S_\varepsilon^*|}\int_{S_\varphi(x_0,t)} |h(x) - h_{S_\varepsilon}| \det D^2\varphi_\varepsilon(x)\, dx \\ \leq \frac{C_2^*\, t^{\frac{1}{2}}}{|S_\varepsilon^*|^{\frac{1}{2}}} \left(\int_{S_\varphi(x_0,2K^*t)} |\nabla^{\varphi_\varepsilon} h(x)|^2 \det D^2\varphi_\varepsilon(x)\, dx\right)^{\frac{1}{2}} \tag{3.45}$$

In addition, the inclusions (3.44), along with the fact that $\nabla\varphi_\varepsilon$ and $\nabla\psi_\varepsilon$ are the inverse of each other, imply

$$\mu_{\varphi_\varepsilon}(S_\varphi(x_0, t)) = |\nabla\varphi_\varepsilon(S_\varphi(x_0, t))| \leq |S_\varepsilon^*| \\ \leq |\nabla\varphi_\varepsilon(S_\varphi(x_0, 2K^*t))| = \mu_{\varphi_\varepsilon}(S_\varphi(x_0, 2K^*t)). \tag{3.46}$$

Moreover, since $\mu_{\varphi_\varepsilon}$ is doubling, we get

$$|S_\varepsilon^*| \leq \mu_{\varphi_\varepsilon}(S_\varphi(x_0, 2K^*t)) \leq \tilde{K}\mu_{\varphi_\varepsilon}(S_\varphi(x_0, t)) \leq \tilde{K}|S_\varepsilon^*|. \tag{3.47}$$

Thus the inequality (3.45) reduces to

$$\frac{1}{\tilde{K}\,\mu_{\varphi_\varepsilon}(S_\varphi(x_0,t))}\int_{S_\varphi(x_0,t)}|h(x)-h_{S_\varepsilon}|\det D^2\varphi_\varepsilon(x)\,dx \\ \leq \frac{C_2^*\,\tilde{K}^{\frac{1}{2}}\,t^{\frac{1}{2}}}{\mu_{\varphi_\varepsilon}(S_\varphi(x_0,2K^*t))^{\frac{1}{2}}}\left(\int_{S_\varphi(x_0,2K^*t)}|\nabla^{\varphi_\varepsilon}h(x)|^2\det D^2\varphi_\varepsilon(x)\,dx\right)^{\frac{1}{2}} \tag{3.48}$$

We are now in position to start taking limits as $\varepsilon\to 0$. From the definition of φ_ε in (3.35) we get that $D^2\varphi_\varepsilon(x)$ (or a subsequence) converges to $D^2\varphi(x)$ for a.e. $x\in\Omega$. In particular, $\det D^2\varphi_\varepsilon(x)$ converges to $\det D^2\varphi(x)$ for a.e. $x\in\Omega$. Let us first show that $\mu_{\varphi_\varepsilon}(F)$ converges to $\mu_\varphi(F)$ for every Borel set $F\subset S_\varphi(x_0,2K^*t)$. Indeed, since $S_\varphi(x_0,2K^*t)\subset\subset\Omega$ let S' denote a compact set such that $S_\varphi(x_0,2K^*t)\subset\subset S'\subset\subset\Omega$ and introduce $H(x):=\Delta\varphi(x)\,\chi_{S'}(x)$. Let us also assume that $\varepsilon<\varepsilon_1:=\operatorname{dist}(S_\varphi(x_0,2K^*t),\partial S')$ so that, for $x\in S_\varphi(x_0,2K^*t)$, we get $(\Delta\varphi*\eta_\varepsilon)(x)=(H*\eta_\varepsilon)(x)$. Then, for every $x\in S_\varphi(x_0,2K^*t)$, the arithmetic-geometric inequality implies

$$0<\det D^2\varphi_\varepsilon(x)\leq\Delta\varphi_\varepsilon(x)^n=(\Delta\varphi*\eta_\varepsilon)(x)^n=(H*\eta_\varepsilon)(x)^n\leq\mathcal{M}(H)(x)^n,$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function whose (n,n)-strong type (here is when we use $n\geq 2$) gives

$$\int_{S_\varphi(x_0,2K^*t)}\det D^2\varphi_\varepsilon(x)\,dx\leq\int_{S_\varphi(x_0,2K^*t)}\mathcal{M}(H)(x)^n\,dx\leq\|\mathcal{M}(H)\|^n_{L^n(\mathbb{R}^n,\,dx)} \\ \leq C_n\,\|H\|^n_{L^n(\mathbb{R}^n,\,dx)}=C_n\int_{S'}\Delta\varphi(x)^n\,dx<\infty,$$

where the hypothesis $\varphi\in W^{2,n}_{loc}(\Omega)$ guarantees the finiteness of the last integral. Notice from the explanation before Fatou's Lemma that for a convex function ϕ we always have $\frac{1}{n}\Delta\phi\leq\|D^2\phi\|\leq\Delta\phi$ almost everywhere.) Therefore, by using Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\mu_{\varphi_\varepsilon}(F)=\lim_{\varepsilon\to 0}\int_F \det D^2\varphi_\varepsilon(x)\,dx=\int_F\lim_{\varepsilon\to 0}\det D^2\varphi_\varepsilon(x)\,dx$$
$$=\int_F \det D^2\varphi(x)\,dx=\mu_\varphi(F),$$

for all $F\subset S_\varphi(x_0,2K^*t)$. That is, $\mu_{\varphi_\varepsilon}(F)$ converges to $\mu_\varphi(F)$ for every Borel set $F\subset S_\varphi(x_0,2K^*t)$ as claimed. Next, we will use Lebesgue's dominated convergence theorem on the integral

$$\int_{S_\varphi(x_0,2K^*t)}|\nabla^{\varphi_\varepsilon}h(x)|^2\det D^2\varphi_\varepsilon(x)\,dx.$$

Given $x\in S_\varphi(x_0,2K^*t)$ let $0<\lambda_{1,\varepsilon}(x)\le\cdots\le\lambda_{n,\varepsilon}(x)$ denote the eigenvalues of $D^2\varphi_\varepsilon(x)$ and using that $|\nabla^{\varphi_\varepsilon}h(x)|^2=\langle D^2\varphi_\varepsilon(x)^{-1}\nabla h(x),\nabla h(x)\rangle$ we get

$$|\nabla^{\varphi_\varepsilon}h(x)|^2\det D^2\varphi_\varepsilon(x)\le\Big(\sup_{S_\varphi(x_0,2K^*t)}|\nabla h|\Big)^2\|D^2\varphi_\varepsilon(x)^{-1}\|\det D^2\varphi_\varepsilon(x)$$

with

$$\|D^2\varphi_\varepsilon(x)^{-1}\|\det D^2\varphi_\varepsilon(x)=\frac{1}{\lambda_{1,\varepsilon}(x)}\prod_{j=1}^n\lambda_{j,\varepsilon}(x)\le\Big(\sum_{j=2}^n\lambda_{j,\varepsilon}(x)\Big)^{n-1}$$
$$<\Big(\sum_{j=1}^n\lambda_{j,\varepsilon}(x)\Big)^{n-1}=\Delta\varphi_\varepsilon(x)^{n-1}=\Delta\varphi*\eta_\varepsilon(x)^{n-1}\le\mathcal{M}(H)(x)^{n-1}.$$

Now for the case $n>2$ we obtain that $\mathcal{M}(H)^{n-1}\in L^1(S_\varphi(x_0,2K^*t),\,dx)$ by reasoning as above (that is, using the fact that the maximal function $\mathcal{M}$ is strong (p,p)-type for $p>1$). In fact, for the case $n>2$, we can use Hoilder's inequality to get the above result only with the assumption $\varphi\in W^{2,n-1}_{loc}(\Omega)$.

In the case $n=2$ we just use that

$$\|H*\eta_\varepsilon\|_{L^1(\mathbb{R}^2,\,dx)}\le\|H\|_{L^1(\mathbb{R}^2,\,dx)}\,\|\eta_\varepsilon\|_{L^1(\mathbb{R}^2,\,dx)}=\|H\|_{L^1(\mathbb{R}^2,\,dx)}<\infty.$$

Finally, by taking limits as $\varepsilon\to 0$ in (3.48) and by recalling the inequalities (3.46) and the doubling property of μ_φ, we obtain the Poincaré inequality (3.34). We keep in mind that we just use Fatou's lemma on its left-hand side as we don't whether the limit exist. But we have already seen that the limit of the integrand already exist on the right-side. This completes the proof. □

3.5 Proof of the main results

In this section we present the proof of the main results Theorem 3.4 and Theorem 3.5 stated on Section 3.1.

3.5.1 Proof of Theorem 3.4

The idea is to use Theorem 3.19 to improve the Poincaré inequality (3.34) from Theorem 3.23. Let us first observe that the condition $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$ with $D^2\varphi > 0$ a.e. in Ω implies that $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\Omega, d\mu_\varphi)$. In fact, even $\varphi \in W^{2,n-1}_{loc}(\Omega,\, dx)$ with $D^2\varphi > 0$ a.e. in Ω will do so. Let us see briefly how this statement is true. since $D^2\varphi(x) > 0$ for a.e. $x \in \Omega$, let $0 < \lambda_1(x) \le \cdots \le \lambda_n(x) < \infty$ denote the eigenvalues of $D^2\varphi(x)$. Then,

$$\begin{aligned} \|D^2\varphi(x)^{-1}\| \det D^2\varphi(x) &= \frac{1}{\lambda_1(x)} \prod_{j=1}^{n} \lambda_j(x) \le \left(\frac{1}{n-1} \sum_{j=2}^{n} \lambda_j(x) \right)^{n-1} \\ &\le \Delta\varphi(x)^{n-1} \in L^1_{loc}(\Omega,\, dx). \end{aligned}$$

Since $\mu_\varphi DC(\Omega, \delta_\varphi)$, we have the growth condition 2.38. That is,

$$\frac{\mu_\varphi(S_\varphi(x_0,t))}{\mu_\varphi(S_\varphi(x_0,t'))} \le C_D \left(\frac{t}{t'}\right)^{n-\varepsilon}.$$

Therefore, by using Theorem 3.19 with $p = 2$ and $s = n - \varepsilon$ the Poincaré inequality (3.34) self-improves to (3.4) since from our choices of p and s we get

$$\frac{2sp}{2s-p} = \frac{4(n-\varepsilon)}{2(n-\varepsilon)-2} = \frac{2(n-\varepsilon)}{(n-\varepsilon)-1} = \frac{2n}{n-1} + \varepsilon_1,$$

where $\varepsilon_1 := \frac{2(n-\varepsilon)}{(n-\varepsilon)-1} - \frac{2n}{n-1} > 0$ is a geometric constant. Also, notice that the requirement of $s > p/2$ in Theorem 3.19 is met since $s > p/2$ iff $n - \varepsilon > 1$ iff $n \geq 2$. $\square$

3.5.2 Proof of Theorem 3.5

The idea of the proof is to apply Theorem 3.4 to ψ, the convex conjugate of φ, and then do a change of variables. In order to see that $\psi \in W^{2,n}_{loc}(\nabla\varphi(\Omega), dy)$, we first notice from (3.7) that $\nabla\psi(\nabla\varphi(x)) = x$, $\forall x \in \Omega$. This implies that $D^2\varphi(x)\mathcal{D}^2\psi(\nabla\varphi(x)) = I$. Since $D^2\varphi > 0$ a.e. in Ω, we get $D^2\psi(\nabla\varphi(x)) = D^2\varphi(x)^{-1} > 0$ for a.e. $x \in \Omega$. Therefore, given a compact set $F \subset \Omega$ and changing variables $y := \nabla\varphi(x)$,

$$\int_{\nabla\varphi(F)} \|D^2\psi(y)\|^n \, dy = \int_F \|D^2\varphi(x)^{-1}\|^n \det D^2\varphi(x) \, dx < \infty,$$

where the finiteness of the last integral above follows from the hypothesis $\|(D^2\varphi)^{-1}\| \in L^n_{loc}(\Omega, d\mu_\varphi)$. Notice that $y = \nabla\varphi(x)$ is a valid change of variables because $\nabla\varphi$ is one-to-one and $\varphi \in W^{2,n}_{loc}(\Omega)$ (see[17] Section 3).

Now, given a section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_1K^*t) \subset\subset \Omega$ and $h \in C^1(S_\varphi(x_0, K_1K^*t))$ (where $K_1 > 1$ is the geometric constant from the Poincaré inequality (3.4) in Theorem 3.4 and $K^* > 1$ is the geometric constant from 3.8), by applying (3.4) to the section $S^* := S_\psi(\nabla\varphi(x_0), t)$ and the function $u(y) := h(\nabla\psi(y))$ we get

$$\left(\fint_{S^*} |u(y) - u^{\mu_\psi}_{S^*}|^q \, d\mu_\psi(y) \right)^{\frac{1}{q}} \leq K_2 \, t^{\frac{1}{2}} \left(\fint_{K_1 S^*} |\nabla^\psi u(y)|^2 \, d\mu_\psi(y) \right)^{\frac{1}{2}}, \tag{3.49}$$

where $q = \frac{2n}{n-1} + \epsilon_1$ and $u^{\mu_\psi}_{S^*} := \fint_{S^*} u(y) \, d\mu_\psi(y)$. Now, by changing variables $y = \nabla\varphi(x)$, using the second inclusion in (3.8), reasoning as in (3.42), and noticing that $\det D^2\psi(y) \det D^2\varphi(x) = 1$ for a.e. $x \in \Omega$, the integral on the right-hand side of (3.49) can be controlled by

$$\int_{K_1 S^*} |\nabla^\psi u(y)|^2 \, d\mu_\psi(y) \leq \int_{S_\varphi(x_0, K_1K^*t)} |\nabla^\varphi h(x)|^2 \, dx, \tag{3.50}$$

while, due to the first inclusion in (3.8), the integral on the left-hand side of (3.49) can be bound from below by the integral over $\nabla\varphi(S_\varphi(x_0, t/K^*))$.

On the other hand, the inclusions in (3.8) and the doubling property of the Lebesgue measure give

$$\mu_\psi(S^*) = |\nabla\psi(S^*)| \sim |S_\varphi(x_0, t)|, \tag{3.51}$$

where the implicit constants are geometric constants. Thus, the Poincaré inequality (3.5) follows, with $K_3 := K_1(K^*)^2 > 1$, from (3.49), (3.50), and (3.51). This completes the proof. □

Chapter 4

Poincaré inequalities when $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$

This chapter is devoted to establish new Poincaré inequalities under the assumption that the Monge-Ampère measure satisfies so-called the Muckenhoupt's A_∞-weight condition. Section 4.1 provides the definition of Muckenhoupt's A_∞-weight condition and its comparison with the DC-doubling condition introduced in Chapter 3, and then state the main results of this chapter in Theorem 4.3 and Theorem 4.4. Section establishes the $(1, 2-\varepsilon)$-Poincaré inequality with respect to Lebesgue measure in Theorem 4.8. We conclude this chapter with the proofs of the main results in Section 4.3.

4.1 Introduction and main results

Let us consider that $\varphi \in C^1(\Omega)$ be a strictly convex function defined on an open convex subset $\Omega \subset \mathbb{R}^n$ throughout this chapter.

For a given weight function w, that is, $w \geq 0$ and $w \in L^1_{loc}(\Omega)$, we define its associated Borel measure as

$$\mu_w(E) = w(E) := \int_E w(x)\, dx, \tag{4.1}$$

where $E \subset \Omega$ is a Borel set.

Definition 4.1. *We say that a weight w satisfies Muckenhoupt's A_∞-weight condition, and write $w \in A_\infty(\Omega, \delta_\varphi)$ or $\mu_w \in A_\infty(\Omega, \delta_\varphi)$, if there exist constants C_1, $C_2 > 1$ and $\theta > 0$ such that*

$$\frac{w(E)}{w(S)} \leq C_1 \left(\frac{|E|}{|S|}\right)^\theta, \tag{4.2}$$

for every section $S := S_\varphi(x,t)$ with $S_\varphi(x_0, C_2 t) \subset\subset \Omega$ and every measurable subset $E \subset S$. The definition for the Muckenhoupt's weight class for general $p \in [1, \infty]$ with some details will be presented in Section 5.1.

Recall from Subsection 2.2.2 that if $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ is convex, then the Monge-Ampère measure associated to φ satisfies $\mu_\varphi = \det D^2\varphi$ and condition (2.7):

$$\mu_\varphi(E) = \int_E \det D^2\varphi(x)\, dx,$$

for every Borel set $E \subset \Omega$. In this case, we will use the notations $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and $\det D^2\varphi \in DC(\Omega, \delta_\varphi)$ interchangeably. In the similar manner, $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$ and $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$ will have the same meaning. When $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$, all the constants depending only on the constants C_1, C_2 and θ from (4.2) and dimension n will be called *structural constants.*

Caffarelli and Gutiérrez defined a condition so-called μ_∞-condition in their article in 1997 ([6], Section 1). Let us recall this condition and compare with the Muckenhoupt's A_∞-weight condition that we have defined here above.

Definition 4.2. *Le φ be a convex function such that $\mu_\varphi = \det D^2\varphi$. Then μ_φ is said to satisfy μ_∞-condition if for given δ_1, $0 < \delta_1 < 1$, there exists δ_2, $0 < \delta_2 < 1$ and $C > 0$ such that*

$$\frac{|E|}{|S|} < \delta_2 \Rightarrow \frac{\mu_\varphi(E)}{\mu_\varphi(S)} < \delta_1, \tag{4.3}$$

for every section $S := S_\varphi(x,t)$ with $S_\varphi(x_0, Ct) \subset\subset \Omega$ and every measurable subset $E \subset S$.

We note that $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$ if and only if $\mu_\varphi \in \mu_\infty$. To look at the direction (4.2) implies (4.3), suppose $0 < \delta_1 < 1$, and define $\delta_2 := \left(\frac{\delta_1}{C_1}\right)^{\frac{1}{\theta}}$. Clearly, $0 < \delta_2 < 1$ as $C_1 > 1$

and $\delta_1 < 1$. Then for every measurable subset $E \subset S$, we have

$$\frac{|E|}{|S|} < \delta_2 \Rightarrow \frac{\mu_\varphi(E)}{\mu_\varphi(S)} < C_1 \left(\left(\frac{\delta_1}{C_1} \right)^{\frac{1}{\theta}} \right)^{\theta} = \delta_1.$$

The other direction (4.3) implies (4.2) is proved in[6] Theroem 6.

With some basic manipulation we can obtain the converse of (4.3), which will be considered to prove that $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$ implies μ_ψ, the Monge-Ampère measure associated to the convex conjugate of φ, satisfies the Muckenhoupt's A_∞-condition (that is, $\mu_\psi \in A_\infty(\nabla\varphi(\Omega), \delta_\psi))$ in Section 4.3. In fact, let $\delta_1, \delta_2 \in (0,1)$ such that (4.2) holds. For any given $F \subset S$, set $E = S - F$. Then $\frac{\mu_\varphi(E)}{\mu_\varphi(S)} > \delta_1 \Rightarrow \frac{|E|}{|S|} > \delta_2$. Now by substituting $E = S - F$, $\frac{\mu_\varphi(S-F)}{\mu_\varphi(S)} > \delta_1 \Rightarrow \frac{|S-F|}{|S|} > \delta_2$. Due to the definition of measure, $\frac{\mu_\varphi(S)-\mu_\varphi(F)}{\mu_\varphi(S)} > \delta_1 \Rightarrow \frac{|S|-|F|}{|S|} > \delta_2$. This implies $1 - \delta_1 > \frac{\mu_\varphi(F)}{\mu_\varphi(S)} \Rightarrow 1 - \delta_2 > \frac{|F|}{|S|}$. Thus, there exist $\alpha_1, \alpha_2 \in (0,1)$ such that for every $F \subset S$, we have

$$\frac{\mu_\varphi(F)}{\mu_\varphi(S)} < \alpha_2 \Rightarrow \frac{|F|}{|S|} < \alpha_1. \tag{4.4}$$

We next make an important observation from page 426 in[6] that $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$ implies $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. In fact, for given $\delta_1 \in (0,1)$, we can pick $\alpha \in (0,1)$ such that $\frac{|S-\alpha\odot S|}{|S|} = 1 - \alpha^n < \delta_2$. Indeed, for a given $\delta_2 \in (0,1)$ we can choose such α. Then from (4.3) with $E = S - \alpha \odot S$, we have $\frac{\mu_\varphi(S-\alpha\odot S)}{\mu_\varphi(S)} < \delta_1$. This implies

$$\mu_\varphi(S) = \mu_\varphi(S - \alpha \odot S) + \mu_\varphi(\alpha \odot S) < \delta_1 \mu_\varphi(S) + \mu_\varphi(\alpha \odot S).$$

$$\Rightarrow (1 - \delta_1)\, \mu_\varphi(S) < \mu_\varphi(\alpha \odot S).$$

$$\Rightarrow \mu_\varphi(S) < \frac{1}{1-\delta_1}\, \mu_\varphi(\alpha \odot S).$$

This establishes the inclusion $A_\infty(\Omega, \delta_\varphi) \subseteq DC(\Omega, \delta_\varphi)$. In fact, the inclusion is strict and we refer the readers to see Section 3 in[27] in order mark this gap. Now if the assumption $\det D^2\varphi \in DC(\Omega, \delta_\varphi)$ used in Chapter 3 is replaced with the strictly stronger condition $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$, then the exponent on the right-hand sides of the Poincaré inequalities (3.4) and (3.5) can be improved from 2 to $2 - \epsilon$ for some structural $0 < \epsilon < 1$. More precisely, we have the following two theorems as our main results in this chapter.

Theorem 4.3. *Fix $n \geq 2$ and let $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$ be a strictly convex function with $D^2\varphi > 0$ a.e. in Ω and $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$. Then, there exist structural constants $K_5, K_6 > 0$ and $\epsilon_0 > 0$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_5 t) \subset\subset \Omega$ and every $u \in \mathrm{Lip}(K_5 S)$ we have*

$$\left(\fint_S |u(x) - u_S^{\mu_\varphi}|^{q_0} d\mu_\varphi(x) \right)^{\frac{1}{q_0}} \leq K_6\, t^{\frac{1}{2}} \left(\fint_{K_5 S} |\nabla^\varphi u(x)|^{2-\epsilon_0} d\mu_\varphi(x)\, dx \right)^{\frac{1}{2-\epsilon_0}}, \tag{4.5}$$

with $q_0 := \frac{2(n-\epsilon_0)(2-\epsilon_0)}{2(n-\epsilon_0)-(2-\epsilon_0)} > 2$.

Theorem 4.4. *Fix $n \geq 2$ and let $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$ be a strictly convex function such that $D^2\varphi > 0$ a.e. in Ω, $\|(D^2\varphi)^{-1}\| \in L^n_{loc}(\Omega, d\mu_\varphi)$ and $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$. Then, there exist structural constants $K_7, K_8 \geq 1$ and $0 < \epsilon_0 < 1$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_7 t) \subset\subset \Omega$, and every $u \in \mathrm{Lip}(K_7 S)$ we have*

$$\left(\fint_S |u(x) - u_S|^{q_0}\, dx \right)^{\frac{1}{q_0}} \leq K_8\, t^{\frac{1}{2}} \left(\fint_{K_7 S} |\nabla^\varphi u(x)|^{2-\epsilon_0}\, dx \right)^{\frac{1}{2-\epsilon_0}}, \tag{4.6}$$

with $q_0 := \frac{2(n-\epsilon_0)(2-\epsilon_0)}{2(n-\epsilon_0)-(2-\epsilon_0)} > 2$.

Observe that (4.5) is the $(q, 2-\varepsilon)$ Poincaré inequality with respect to the Monge-Ampère measure while (4.6) is the $(q, 2-\varepsilon)$ Poincaré inequality with respect to Lebesgue measure with the additional assumption of integrability $\|(D^2\varphi)^{-1}\| \in L^n_{loc}(\Omega, d\mu_\varphi)$ on its hypothesis. The exponent on the left hand sides on both of these inequalities are same, but different from the ones in the Poincaré inequalities (3.4) and (3.5).

4.2 $(1, 2-\varepsilon)$ Poincaré inequality

We recall that Theorem 3.1 provides the $(1,2)$-Poincaré inquality with respect to Lebesgue measure under the assumption that $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. In this section, we put our efforts to reduce the size of exponent 2 on the right-side to $2-\varepsilon$ for some $\varepsilon \in (0,1)$, by considering strictly stronger hypothesis $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$, which is presented in Theorem 4.8. In order to do so, let us begin by stating some of the ingredients to be used in this theorem.

Lemma 4.5 (Lemma 3.2.1 in[7]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex open set and $\varphi \in C^1(\Omega)$ be a convex function such that $\varphi \leq 0$ on $\partial\Omega$. Then for every $x \in \Omega$,*

$$|\nabla\varphi(x)| \leq \frac{-\varphi(x)}{\operatorname{dist}(x, \partial\Omega)}.$$

More generally, if $\overline{E} \subset \Omega$, then

$$\nabla\varphi(E) \subset B\left(0, \frac{\max_E(-\varphi)}{\operatorname{dist}(E, \partial\Omega)}\right).$$

This lemma above helps us to estimate the size of the gradient of given convex function. We next state a lemma that describe the behaviour of the norm of the Hessian matrix $D^2\varphi$ whenever $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$.

Lemma 4.6 (Lemma 3.1 in[4]). *Let $\varphi \in C^1(\Omega)$ be a strictly convex function such that $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$. Then $\|D^2\phi(x)\| \in A_\infty(\Omega, \delta_\varphi)$ with constants depending only on the $A_\infty(\Omega, \delta_\varphi)$-constants for μ_φ and dimension n. That is, there exist constants $C_0 > 1$ and $\varepsilon_0 > 0$, depending only on the $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$-constants for μ_φ and dimension n, such that*

$$\left(\fint_S \left\|D^2\varphi(x)\right\|^{1+\varepsilon_0}(x)\, dx\right)^{\frac{1}{1+\varepsilon_0}} \leq C_0 \fint_S \|D^2\varphi(x)\|\, dx, \tag{4.7}$$

for every section $S := S_\varphi(x,t) \subset\subset \Omega$.

Let us state one more result which helps us to compare the product of the Monge-Ampère measure and Lebesgue measure of the Monge-Ampère sections with their heights.

Lemma 4.7 (Corollary 4 in[18]). *Let $\varphi \in C^1(\Omega)$ be a strictly convex function such that $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Then there exists constants $C_1, C_2 > 0$, depending only on the doubling* *constant and dimension, such that for every section $S := S_\varphi(x,t) \subset\subset \Omega$, it holds that*

$$C_1\, t^n \le \mu_\varphi(S)\, |S| \le C_2\, t^n.$$

Theorem 4.8. *Fix an open convex set $U \subset \mathbb{R}^n$ and $\varphi \in C^2(U)$ with $D^2\varphi > 0$ in U and $\mu_\varphi \in A_\infty(U, \delta_\varphi)$. Then, there exist constants $N_1, \epsilon > 0$, depending only on the constants from $\mu_\varphi \in A_\infty(U, \delta_\varphi)$ and dimension n, such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0,t) \subset\subset U$ and every $h \in C^1(S)$ the following $(1, 2-\epsilon)$-Poincaré holds true in the Monge-Ampère quasi-metric structure with respect to the Lebesgue measure*

$$\fint_S |h(x) - h_S|\, dx \le N_1\, t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi h(x)|^{2-\epsilon}\, dx \right)^{\frac{1}{2-\epsilon}}, \tag{4.8}$$

where $h_S := \fint_S h(x)\, dx$.

Proof. The proof of Theorem 4.8 goes along the lines of the proof of[4] Theorem 1.3. We will follow the notation in[6] Section 1 regarding the normalization technique of a given section $S := S_\varphi(x_0, t)$. Thus, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation such that $B(0, n^{-3/2}) \subset T(S) \subset B(0, 1)$. In particular, $\alpha_n \le |S||\det T| \le \beta_n$ for some positive dimensional constants α_n, β_n. Always as in[6] Section 1, let $\lambda > 0$ and φ^* be defined by

$$\lambda^n := \frac{\mu_\varphi(S)}{|\det T|} \quad \text{and} \quad \varphi^*(y) := \frac{1}{\lambda}\varphi(T^{-1}y) - \bar{l}(y) - \frac{t}{\lambda},$$

where $\bar{l}$ is a linear function.

For $\psi_\lambda(y) := \frac{1}{\lambda}\varphi(T^{-1}y)$ and $\bar{l}(y) := \psi_\lambda(y) + \nabla\psi_\lambda(Tx_0) \cdot (y - Tx_0)$, we get

$$T(S) = T(S_\varphi(x_0,t)) = S_{\psi_\lambda}\left(Tx_0, \frac{t}{\lambda}\right) = \{y : \varphi^*(y) < 0\} \quad \text{and} \quad \varphi^* = 0 \quad \text{on} \quad \partial(T(S)).$$

We note that

$$D^2\varphi^*(y) = \frac{1}{\lambda}(T^{-1})^t D^2\varphi(T^{-1}y) T^{-1} = D^2\psi_\lambda(y). \tag{4.9}$$

This implies

$$\det D^2\varphi^*(y) = \frac{1}{\lambda^n}\frac{1}{|\det T|^2}\det D^2\varphi(T^{-1}y), \quad \text{as} \quad \det(\lambda A) = \lambda^n \det(A).$$

Denote $\bar{\mu} := \mu^*_\varphi$. Then by simple change of variable and using the definition of λ, we obtain

$$\bar{\mu}(T(S)) = \int_{T(S)} \det D^2\varphi^*(y)\, dy = 1.$$

From Lemma 4.7 we have

$$C_3 t \le \lambda \le C_4 t, \tag{4.10}$$

where $C_3, C_4 > 0$ depend on the doubling constant from $\mu_\varphi \in DC(U, \delta_\varphi)$ and the dimension n. Now from the first few lines of the proof of Theorem 2 in[6] or Lemma 4.5 or Lemma 3.2 in[28], there exists a constant $C_5 > 0$, also depending only on the doubling constants from $\mu_\varphi \in DC(U, \delta_\varphi)$ and dimension n, such that

$$⨍_{T(S)} \Delta\varphi^*(y)\, dy \le C_5. \tag{4.11}$$

Next note that $\mu_\varphi \in A_\infty(U, \delta_\varphi)$ implies $\mu_{\varphi^*} \in A_\infty(T(U), \delta_{\varphi^*})$ with the same set of constants. In fact, suppose μ_φ satisfies (4.2), Then

$$\begin{aligned}\frac{\mu_{\varphi^*}(T(E))}{\mu_{\varphi^*}(T(S))} &= \frac{\frac{1}{\lambda^n |\det T|}\mu_\varphi(E)}{\frac{1}{\lambda^n |\det T|}\mu_\varphi(S)} \le C_1\left(\frac{|E|}{|S|}\right)^\theta \\ &= C_1\left(\frac{|\det T|\,|E|}{|\det T|\,|S|}\right)^\theta = C_1\left(\frac{|T(E)|}{|T(S)|}\right)^\theta.\end{aligned}$$

Now, by Lemma 4.6 when applied to φ^* (recall that $\|D^2\varphi^*\| \le \Delta\varphi^*$ as φ is convex), there exist constants $C_6 > 1$ and $0 < \epsilon_0 < 1$, depending only on the constants from $\mu_\varphi \in A_\infty(U, \delta_\varphi)$ and dimension n, such that

$$\left(⨍_{T(S)} \Delta\varphi^*(y)^{1+\epsilon_0}\, dy\right)^{\frac{1}{1+\epsilon_0}} \le C_6 ⨍_{T(S)} \Delta\varphi^*(y)\, dy. \tag{4.12}$$

Then, given $h \in C^1(S)$ let $\bar{u} \in C^1(T(S))$ be defined as $\bar{u}(y) = h(T^{-1}y)$. Thus, the usual $(1,1)$-Poincaré inequality applied to $\bar{u}$ on the convex set $T(S)$ (recall that $B(0, n^{-3/2}) \subset T(S) \subset B(0,1)$) yields

$$\fint_{T(S)} |\bar{u}(y) - \bar{u}_{T(S)}|\, dy \leq C_n \fint_{T(S)} |\nabla \bar{u}(y)|\, dy, \tag{4.13}$$

where $C_n > 0$ is a dimensional constant, and by changing variables $y = Tx$ in (4.13) we obtain

$$\fint_S |h(x) - h_S|\, dx \leq C_n \fint_S |(T^{-1})^t \nabla h(x)|\, dx. \tag{4.14}$$

Next, notice that from the identity (4.9) and the fact that $\|D^2\varphi\| \leq \Delta\varphi$ we get

$$\|(T^{-1})^t D^2\varphi(x) T^{-1}\| \leq \lambda \Delta\varphi^*(Tx),$$

which followed by the simple matrix identity

$$\|(T^{-1})^t D^2\varphi(x)^{\frac{1}{2}}\|^2 = \|(T^{-1})^t D^2\varphi(x) T^{-1}\|,$$

gives $\|(T^{-1})^t D^2\varphi(x)^{\frac{1}{2}}\|^2 \leq \lambda\Delta\varphi^*(Tx)$. Consequently,

$$\begin{aligned}\left(\fint_S \|(T^{-1})^t D^2\varphi(x)^{\frac{1}{2}}\|^{2(1+\epsilon_0)}\, dx\right)^{\frac{1}{1+\epsilon_0}} &\leq \lambda\left(\fint_{T(S)} \Delta\varphi^*(y)^{1+\epsilon_0}\, dy\right)^{\frac{1}{1+\epsilon_0}} \\ &\leq C_5 C_6 \lambda,\end{aligned}$$

where the last inequality follows from (4.12) and (4.11). Finally, by setting $p := 2(1+\epsilon_0)$

and recalling that $\nabla^{\varphi} h = D^2\varphi^{-\frac{1}{2}}\nabla h$,

$$\begin{aligned}
&\fint_S |(T^{-1})^t \nabla h(x)|\,dx = \fint_S |(T^{-1})^t D^2\varphi(x)^{\frac{1}{2}} D^2\phi(x)^{-\frac{1}{2}}\nabla h(x)|\,dx\\
&\leq \fint_S \|(T^{-1})^t D^2\varphi(x)^{\frac{1}{2}}\| |D^2\varphi(x)^{-\frac{1}{2}}\nabla h(x)|\,dx\\
&\leq \left(\fint_S \|(T^{-1})^t D^2\varphi(x)^{\frac{1}{2}}\|^p\,dx\right)^{\frac{1}{p}} \left(\fint_S |\nabla^{\varphi} h(x)|^{p'}\,dx\right)^{\frac{1}{p'}}\\
&\leq (C_5C_6\lambda)^{\frac{1}{2}} \left(\fint_S |\nabla^{\varphi} h(x)|^{p'}\,dx\right)^{\frac{1}{p'}} \leq (C_4C_5C_6t)^{\frac{1}{2}} \left(\fint_S |\nabla^{\varphi} h(x)|^{p'}\,dx\right)^{\frac{1}{p'}},
\end{aligned}$$

where $p' = 2-\epsilon$ with $\epsilon := 2\epsilon_0/(1+2\epsilon_0) \in (0,1)$, which combined with (4.14) proves (4.8). $\square$

4.3 Proof of the main results

Before proceeding to the proofs of the main results, let us first outline the proof of the fact, to be used in the proof of Theorem 4.3, that $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$ implies that $\mu_\psi \in A_\infty(\nabla\varphi(\Omega), \delta_\psi)$, where ψ is the convex conjugate of φ. That is, the A_∞-property is preserved, quantitatively, under conjugation. Fix $\varphi \in C^1(\Omega)$ such that $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$. In particular, $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and the sections of φ have the engulfing property. Now, since a section $S_\varphi(x,t)$ coincides with the set $\{y \in \Omega : \delta_\varphi(x,y) < t\}$, the quasi-symmetry and quasi-triangle inequality for δ_φ, allows to think of the interior sections (meaning sections with $S_\varphi(x,t) \subset\subset \Omega$) as balls in a space of homogeneous type. Consequently, the usual characterizations of the Muckenhoupt's class A_∞ hold true, see for instance see Section 4.1 above and Section 5 in [6]. Thus, the fact that $\mu_\psi \in A_\infty(\nabla\varphi(\Omega), \delta_\psi)$ will be a consequence, for instance, of the existence of structural constants $\alpha_0, \beta_0 \in (0,1)$ and $M_0 \geq 1$ such that for every section $S_\psi(y,t)$ with $S_\psi(y, M_0t) \subset\subset \nabla\varphi(\Omega)$ and every measurable set $F \subset S_\psi(y,t)$ the implication

$$\mu_\psi(F) \leq \alpha_0 \mu_\psi(S_\psi(y,t)) \Longrightarrow |F| \leq \beta_0 |S_\psi(y,t)| \tag{4.15}$$

holds true. To see this, let us assume that μ_φ satisfies (4.2) with constants $C_1, C_2 \geq 1$ and $\theta \in (0,1)$ and fix a section $S_\psi := S_\psi(y,t)$ and a measurable set $F \subset S_\psi$. From the second inclusion in (3.8), setting $x := \psi(y)$ we get

$$\nabla\psi(F) \subset \nabla\psi(S_\psi) \subset S_\varphi(x, K^*t). \tag{4.16}$$

Now, setting $E := \nabla\psi(F)$, the fact that $\nabla\varphi$ and $\nabla\psi$ are inverses to each other gives

$$\mu_\varphi(E) = |\nabla\varphi(E)| = |F|$$

and the first inclusion in (3.8) and the doubling property (2.38) for μ_φ imply

$$\begin{aligned}\mu_\varphi(S_\varphi(x, K^*t)) &\le C_D(K^*)^{2(n-\varepsilon)}\mu_\varphi(S_\varphi(x, t/K^*))\\ &\le C_D(K^*)^{2(n-\varepsilon)}\mu_\varphi(\nabla\psi(S_\psi)) = C_D(K^*)^{2(n-\varepsilon)}|S_\psi|.\end{aligned}$$

On the other hand, $|E| = |\nabla\psi(F)| = \mu_\psi(F)$ and, from (4.16), $\mu_\psi(S_\psi) = |\nabla\psi(S_\psi)| \le |S_\varphi(x, K^*t)|$. Hence, by using (4.2) with $E := \nabla\psi(F)$ and $S_\varphi(x, K^*t)$ (and this requires $S_\varphi(x, K^*C_2t) \subset\subset \Omega$) it follows that

$$\begin{aligned}\frac{|F|}{C_D(K^*)^{2(n-\varepsilon)}|S_\psi|} &\le \frac{\mu_\varphi(E)}{\mu_\varphi(S_\varphi(x, K^*t))}\\ &\le C_1\left(\frac{|E|}{|S_\varphi(x, K^*t)|}\right)^\theta \le C_1\left(\frac{\mu_\psi(F)}{\mu_\psi(S_\psi)}\right)^\theta.\end{aligned}$$

Consequently, by taking $\alpha_0 \in (0, 1)$ so that $\beta_0 := C_D(K^*)^{2(n-\varepsilon)}C_1\alpha_0^\theta \in (0, 1)$, the implication (4.15) holds true with structural constants $\alpha_0, \beta_0 \in (0, 1)$.

Now we are ready to compile the techniques presented in this chapter (as well as in previous chapters) to complete the proofs of Theorem 4.3 and Theorem 4.4.

Fortunately, the proof of Theorem 4.3 follows along the lines of the proof of Theorem 3.4. First, Theorem 4.8 (used in lieu of Theorem 3.1) implies a version of Theorem 3.23 where the exponent 2 on the right-hand side of (3.34) can be replaced by $2 - \epsilon$. Note that we just outlined the proof of the fact that the A_∞ property is qualitatively preserved under convex conjugation. It is also quantitatively preserved by the approximations φ_ε due to the fact that φ_ε and $\nabla\varphi_\varepsilon$ converge uniformly on compact sets. Set $\epsilon_0 := \min\{\varepsilon, \epsilon\}$ with $\varepsilon > 0$ the geometric constant from (2.38). Then, just as in Section 3.5.1, Theorem 3.19 applied with μ as the Monge-Ampère measure, $p = 2 - \epsilon_0$, and $s = n - \epsilon_0$, yields with

$$q_0 := \frac{2(n-\epsilon_0)(2-\epsilon_0)}{2(n-\epsilon_0)-(2-\epsilon_0)},$$

and (4.5) follows. □

The proof of Theorem 4.4 goes just like the one of Theorem 3.5, but now instead of using Theorem 3.4 we use Theorem 4.3 with ψ, the convex conjugate of φ, and then change variables $y = \nabla\varphi(x)$. □

Chapter 5

Poincaré inequalities when $\mu_\varphi \in A_1(\Omega, \delta_\varphi)$ or $\mu_\varphi \in RH_\infty(\Omega, \delta_\varphi)$

Our objective in this chapter is to present more new Poincaré inequalities by considering even more stronger assumptions on their hypotheses. Section 5.1 first introduces the Muckenhoupt's $A_1(\Omega, \delta_\varphi)$-condition and reverse Hölder's $RH_\infty(\Omega, \delta_\varphi)$-condition and then illustrates the comparison with the weight conditions introduced in Chapter 3 and Chapter 4. We state and prove the sharp Poincaré inequalities associated to the Monge-Ampère measure when $\mu_\varphi \in A_1(\Omega, \delta_\varphi)$ in Section 5.2 and associated to Lebesgue measure in Section 5.3. We conclude this chapter by recording a list of convex functions whose determinants of Hessian matrices satisfy the weight-conditions $A_1(\Omega, \delta_\varphi)$ or $RH_\infty(\Omega, \delta_\varphi)$ in Section 5.4.

5.1 Introduction to Muckenhoupt's $A_1(\Omega, \delta_\varphi)$-condition and reverse Hölder's $RH_\infty(\Omega, \delta_\varphi)$-condition

As always, assume that $\Omega \subset \mathbb{R}^n$ is open and convex and $\varphi \in C^1(\Omega)$ is a strictly convex function. Let us first introduce the definition of Muckenhoupt's weight class and reverse Hölder's inequality which will be the central assumptions in this chapter.

Definition 5.1. *Let $1 \leq p < \infty$. A weight function w defined in Ω is said to satisfy*

Muckenhoupt's weight class A_p*, and write* $w \in A_p(\Omega, \delta_\varphi)$*, if there exists* $\Theta_1, H_1 \geq 1$ *such that for every section* $S_\varphi(x_0, t)$ *with* $S_\varphi(x_0, \Theta_1 t) \subset\subset \Omega$*,*

$$\left(\fint_{S_\varphi(x_0,t)} w(x)\, dx \right) \left(\fint_{S_\varphi(x_0,t)} w(x)^{-1/(p-1)}\, dx \right)^{p-1} \leq H_1, \quad \textit{when} \quad 1 < p < \infty, \tag{5.1}$$

and

$$\fint_{S_\varphi(x_0,t)} w(x)\, dx \leq H_1 \operatorname*{ess\,inf}_{S_\varphi(x_0,t)} w(x) \quad \textit{when} \quad p = 1. \tag{5.2}$$

Definition 5.2. *Let* $1 < r \leq \infty$*. A weight function* w *defined in* Ω *is said to satisfy reverse Hölder's class, and write* $w \in RH_r(\Omega, \delta_\varphi)$*, if there exists* $\Theta_\infty, H_\infty \geq 1$ *such that for every section* $S_\varphi(x_0, t)$ *with* $S_\varphi(x_0, \Theta_\infty t) \subset\subset \Omega$*,*

$$\left(\fint_{S_\varphi(x_0,t)} w(x)^r\, dx \right)^{\frac{1}{r}} \leq H_\infty \fint_{S_\varphi(x_0,t)} w(x)\, dx, \quad \textit{when} \quad 1 < r < \infty, \tag{5.3}$$

and

$$\operatorname*{ess\,sup}_{S_\varphi(x_0,t)} w(x) \leq H_\infty \fint_{S_\varphi(x_0,t)} w(x)\, dx, \quad \textit{when} \quad r = \infty. \tag{5.4}$$

The reason to use constants with suffix 1 in the definition of Muckenhoupt's weight class A_p and ∞ in the definition of reverse Hölder's class RH_r is that our main results in this chapter will be based only on Muckenhoupt's weight class A_p with $p = 1$ or the reverse Hölder's class RH_r with $r = \infty$.

Let us record the facts that if $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$ or $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$, then $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$, quantitatively (see Section 5 in [6]). In fact, to prove the fact $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$ implies $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$, we can assume that $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$ and verify (4.3). By definition,

$$\det D^2\varphi(x) \leq H_\infty \fint_S w(x)\, dx = H_\infty \frac{\mu_\varphi(S)}{|S|} \quad \text{a.e. in} \quad S := S_\varphi(x_0, t).$$

Integrating over $E \subset S$ gives

$$\int_E \det D^2\varphi(x)\,dx \le H_\infty \frac{\mu_\varphi(S)}{|S|}|E|.$$
$$\Rightarrow \frac{\mu_\varphi(E)}{\mu_\varphi(S)} \le H_\infty \frac{|E|}{|S|}.$$

The choice of $\delta_2 = \frac{\delta_1}{H_\infty}$ verifies (4.3) as required.

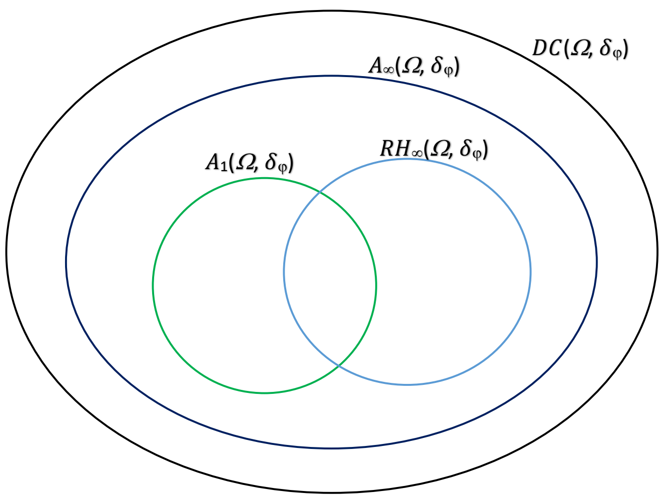

Figure 5.1: *Comparison of weight conditions.*

As illustrated in Figure 5.1 above, the hierarchy of the classes for the weight functions on which we rely in this dissertation can be expressed with the following inclusions.

$$A_1(\Omega, \delta_\varphi) \cup RH_\infty(\Omega, \delta_\varphi) \subsetneq A_\infty(\Omega, \delta_\varphi) \subsetneq DC(\Omega, \delta_\varphi). \tag{5.5}$$

Notice that the strictness in the second inclusion of (5.5) is discussed in Section 4.1. The strictness on the first inclusion will be immediate from the examples of weights satisfying $A_1(\Omega, \delta_\varphi)$-condition and $RH_\infty(\Omega, \delta_\varphi)$-condition in Section 5.4. Let us see a lemma from[29] that sums up more general properties of Muckenhoupt's weight classes and the reverse Hölder classes.

Lemma 5.3. *The following properties hold true:*

(i) $w \in A_p$, $1 < p < \infty$ *if and only if* $w^{1-p'} \in A_{p'}$ *where* $\frac{1}{p} + \frac{1}{p'} = 1$.

(ii) $A_1 \subset A_p \subset A_q$ *for* $1 \le p \le q \le \infty$.

(iii) $RH_\infty \subset RH_s \subset RH_r$ *for* $1 < r \le s \le \infty$.

(iv) $A_\infty = \bigcup_{1 \le p < \infty} A_p = \bigcup_{1 < r \le \infty} RH_r$.

As before, we remark that if $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$ or $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$, the constants depending only the corresponding pairs (Θ_1, H_1) or $(\Theta_\infty, H_\infty)$ and dimension n will also be called structural constants.

5.2 Poincaré inequalities when $\mu_\varphi \in A_1(\Omega, \delta_\varphi)$

In this section, we state two theorems under the assumption that $\mu_\varphi \in A_1(\Omega, \delta_\varphi)$, and the proofs will be given in Subsection 5.2.1.

Theorem 5.4. *Fix $n \ge 3$ and let $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ be a strictly convex function with $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$ and $\det D^2\varphi > 0$ a.e. in Ω. Then, there exist structural constants $K_9, K_{10} \ge 1$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_9 t) \subset\subset \Omega$ and every $u \in \mathrm{Lip}(K_9 S)$ we have*

$$\left(\fint_S |u(x) - u_S^{\mu_\varphi}|^{\frac{2n}{n-2}} d\mu_\varphi(x)\right)^{\frac{n-2}{2n}} \le K_{10}\, t^{\frac{1}{2}} \left(\fint_{K_9 S} |\nabla^\varphi u(x)|^2 d\mu_\varphi(x)\right)^{\frac{1}{2}}. \tag{5.6}$$

In addition, there exists a structural constant $\epsilon_0 > 0$ such that for every $0 < \epsilon \le \epsilon_0$ there is a constant $K_\epsilon > 0$, depending only on ϵ and structural constants, such that

$$\left(\fint_S |u(x) - u_S^{\mu_\varphi}|^{q_\epsilon} d\mu_\varphi(x)\right)^{\frac{1}{q_\epsilon}} \le K_\epsilon\, t^{\frac{1}{2}} \left(\fint_{K_9 S} |\nabla^\varphi u(x)|^{2-\epsilon} d\mu_\varphi(x)\right)^{\frac{1}{2-\epsilon}}, \tag{5.7}$$

with $q_\epsilon := \frac{n(2-\epsilon)}{n-(2-\epsilon)} > 2$.

Under the assumption that $\mu_\varphi \in A_1(\Omega, \delta_\varphi)$ and $n \ge 3$, the exponent q in the Poincaré inequality (5.6) is $\frac{2n}{n-2}$ which is larger than the exponents on the left-hand sides in the

Poincaré inequalities presented in previous chapters under the assumptions $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$ or $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. In dimension 2, we can have a larger exponent on the left hand size, depending on some structural constant $\epsilon_0 > 0$, but in this case the constant appearing in the corresponding Poincaré inequality will depend on ϵ_0 as well. More precisely,

Theorem 5.5. *Assume $n = 2$ and let $\varphi \in W^{2,2}_{loc}(\Omega)$ be a strictly convex function with $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$. Then, there exist structural constants $K_9 \geq 1$ and $0 < \epsilon_0 < 1$, such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_9 t) \subset\subset \Omega$, every $u \in \mathrm{Lip}(K_9 S)$, and every $0 < \epsilon \leq \epsilon_0$ we have*

$$\left(\fint_S |u(x) - u_S^{\mu_\varphi}|^{q_\epsilon} d\mu_\varphi(x)\right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon \, t^{\frac{1}{2}} \left(\fint_{K_9 S} |\nabla^\varphi u(x)|^{2-\epsilon} d\mu_\varphi(x)\right)^{\frac{1}{2-\epsilon}}, \tag{5.8}$$

with $q_\epsilon := 2(2-\epsilon)/\epsilon$ and $K_\epsilon > 0$ depends only on ϵ and structural constants.

5.2.1 Proofs of Theorems 5.4 and 5.5

Since $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ is a strictly convex function with $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$, from[19] Section 4 we have that there exists a structural constant $M_1 > 0$ such that the Monge-Ampère measure satisfies the growth condition

$$\frac{\mu_\varphi(S_\varphi(x_0, t))}{\mu_\varphi(S_\varphi(x_0, t'))} \leq M_1 \left(\frac{t}{t'}\right)^{\frac{n}{2}}, \tag{5.9}$$

for every section $S_\varphi(x_0, t)$ with $S_\varphi(x_0, \Theta_1 t) \subset \Omega$ and every $0 < t' < t$. Therefore, Theorem 3.19 applied with $\mu = \mu_\varphi$, $p = 2$ (the right-hand side exponent from the Poincaré inequality from Theorem 3.23), and $s = n/2$ (the growth exponent from (5.9)), yields

$$q = \frac{2sp}{2s - p} = \frac{2n}{n-2},$$

which is finite in the case $n \geq 3$, and (5.6) follows.

On the other hand, let $\epsilon_0 > 0$ be the structural constant from Theorem 4.3 so that for every $0 < \epsilon \leq \epsilon_0$, the inequality (4.5) implies (since $q > 2$)

$$\fint_S |u(x) - u_S^{\mu_\varphi}| d\mu_\varphi(x) \leq K_6 \, t^{\frac{1}{2}} \left(\fint_{K_5 S} |\nabla^\varphi u(x)|^{2-\epsilon} d\mu_\varphi(x) \, dx \right)^{\frac{1}{2-\epsilon}}. \tag{5.10}$$

Now we use Theorem 3.19 applied with $\mu = \mu_\varphi$, $p = 2 - \epsilon$ (the right-hand side exponent from (5.10)), and $s = n/2$ (the growth exponent from (5.9)) to obtain the inequality (5.7) with

$$q = \frac{2sp}{2s - p} = \frac{n(2-\epsilon)}{n - (2-\epsilon)}. \tag{5.11}$$

Notice that in the case $n = 2$, the expression for q in (5.11) reduces to $q = 2(2-\epsilon)/\epsilon$ and (5.8) follows. □

5.3 Poincaré inequalities when $\mu_\varphi \in RH_\infty(\Omega, \delta_\varphi)$

In this section, we state two theorems under the assumption that $\mu_\varphi \in RH_\infty(\Omega, \delta_\varphi)$, and their proofs will be given in Subsection 5.3.1.

Theorem 5.6. *Fix $n \geq 3$ and let $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ be a strictly convex function with $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$ and $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\Omega, dx)$. Then, there exist structural constants $K_{11}, K_{12} \geq 1$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_{11}t) \subset\subset \Omega$ and every $u \in \mathrm{Lip}(K_{11}S)$ we have*

$$\left(\fint_S |u(x) - u_S|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{2n}} \leq K_{12} \, t^{\frac{1}{2}} \left(\fint_{K_{11} S} |\nabla^\varphi u(x)|^2 \, dx \right)^{\frac{1}{2}}. \tag{5.12}$$

In addition, there exists a structural constant $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ there is a constant $K_\epsilon > 0$, depending only on ϵ and structural constants, such that

$$\left(\fint_S |u(x) - u_S|^{q_\epsilon} \, dx \right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon \, t^{\frac{1}{2}} \left(\fint_{K_9 S} |\nabla^\varphi u(x)|^{2-\epsilon} \, dx \right)^{\frac{1}{2-\epsilon}}, \tag{5.13}$$

with $q_\epsilon := \frac{n(2-\epsilon)}{n-(2-\epsilon)} > 2$.

We note that the Poincaré inequalities under the assumption $\mu_\varphi \in A_1(\Omega, \delta_\varphi)$ in Theorem 5.7 and Theorem 5.8 differ from the theorems in this section just by the measure associated to integrate the quantities. This difference is occurred from the growth conditions as we obtain a growth condition with respect to the Monge-Ampère measure when $\mu_\varphi \in A_1(\Omega, \delta_\varphi)$ and with respect to Lebesgue measure when $\mu_\varphi \in RH_\infty(\Omega, \delta_\varphi)$. As in Section 5.2, we have the following theorem in dimension 2.

Theorem 5.7. *Assume $n = 2$ and let $\varphi \in W^{2,2}_{loc}(\Omega)$ be a strictly convex function with $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$ and $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\Omega, dx)$. Then, there exist structural constants $K_{11} \geq 1$ and $0 < \epsilon_0 < 1$, such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_{11}t) \subset\subset \Omega$, every $u \in \mathrm{Lip}(K_{11}S)$, and every $0 < \epsilon \leq \epsilon_0$ we have*

$$\left(\fint_S |u(x) - u_S|^{q_\epsilon} \, dx \right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon \, t^{\frac{1}{2}} \left(\fint_{K_{11}S} |\nabla^\varphi u(x)|^{2-\epsilon} \, dx \right)^{\frac{1}{2-\epsilon}}, \tag{5.14}$$

with $q_\epsilon := 2(2-\epsilon)/\epsilon$ and $K_\epsilon > 0$ depends only on ϵ and structural constants.

5.3.1 Proofs of Theorems 5.6 and 5.7

Since $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ is a strictly convex function with $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$, from[19] Section 3 we now have that there exists a structural constant $M_\infty > 0$ such that the Lebesgue measure satisfies the growth condition

$$\frac{|S_\varphi(x_0, t)|}{|S_\varphi(x_0, t')|} \leq M_\infty \left(\frac{t}{t'} \right)^{\frac{n}{2}}, \tag{5.15}$$

for every section $S_\varphi(x_0, t)$ with $S_\varphi(x_0, \Theta_\infty t) \subset \Omega$ and every $0 < t' < t$.

Hence, the proofs of Theorems 5.6 and 5.7 follow as the ones of Theorems 5.4 and 5.5. Indeed, the same reasoning from Section 5.2.1 but now using Theorem 3.5 instead of Theorem 3.23, and Theorem 4.4 instead of Theorem 4.3, as well as using Theorem 3.19 with the Lebesgue measure instead of μ_φ (but always with $s = n/2$ as in (5.15)) yields (5.12), (5.13), and (5.14). $\square$

5.4 Examples of weight functions satisfying $A_1(\Omega, \delta_\varphi)$ or $RH_\infty(\Omega, \delta_\varphi)$ conditions

This section is intended to record ample examples of convex functions φ with $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$ or $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$, which we borrow from[19]. Let us begin with a convention of a notation associated to a convex function for which the Monge-Ampère measure and Lebesgue measure are comparable.

Definition 5.8. *We say that* $\det D^2\varphi \sim 1$ *in* Ω *if there exists constants* $0 < \Lambda_1 \leq \Lambda_2$ *such that*

$$\Lambda_1 \leq \det D^2\varphi(x) \leq \Lambda_2, \quad \forall x \in \Omega. \tag{5.16}$$

A trivial but very important convex function that satisfies (5.16) is the quadratic function $\varphi_2(x) := \frac{1}{2}|x|^2$, $x \in \mathbb{R}^n$. In fact, this quadratic convex function plays a crucial role to bridge in between the study related to the Monge-Ampère structure and the Euclidean setting. G. Tina and X.J. Wang illustrated the following two non-trivial examples of strictly convex non-smooth functions that satisfy (5.16) in[2] Section 1.

Example 5.9. *The strictly convex function defined by*

$$\varphi(x) := \frac{x_1^2}{\log|\log|x|^2|} + x_2^2 \log|\log|x|^2|, \quad x \in \mathcal{B}(0,r) \subset \mathbb{R}^2, \tag{5.17}$$

satisfies (5.16).

Example 5.10. *The strictly convex function defined by*

$$\varphi(x) := \begin{cases} x_1^4 + \frac{3x_2^2}{2x_1^2} & \text{if} \quad |x_2| < |x_1|^3, \\ \frac{1}{2}x_1^2 x_2^{2/3} + 2x_2^{4/3} & \text{if} \quad |x_2| \geq |x_1|^3, \end{cases} \tag{5.18}$$

satisfies (5.16).

We now list the examples of weights satisfying $A_1(\Omega, \delta_\varphi)$-condition taken from[19].

(i) The case $\det D^2\varphi \sim 1$ in Ω in the sense of (5.16). Here $\Theta_1 = 1$ and $H_1 = \Lambda_2/\Lambda_1$. Thus, for instance, the function $\varphi_2(x) := \frac{1}{2}|x|^2$ and the ones defined on (5.17) and (5.18) all satisfy the $A_1(\Omega, \delta_\varphi)$-condition.

(ii) The case $\det D^2\varphi \sim |q|^{-a}$ in Ω with $q = q(x)$ polynomial and $0 < a < 1/\deg(q)$. Here $\Theta_1 = 1$ and the constant $H_1 \geq 1$ depends only on a, dimension n, and deg(q), the degree of q but not on the coefficients of the polynomial q.

(iii) The case $\varphi_p(x) := \frac{1}{p}|x|^p$, $x \in \mathbb{R}^n$ and $2 - 1/n < p \leq 2$. Here $\Theta_1 = 1$ and the constant $H_1 \geq 1$ depends only on p and n. The author in[19] used above example (ii) with the fact that

$$\det D^2\varphi_p(x) = (p-1)\,|x|^{n(p-2)},\ \forall x \in \mathbb{R}^n \setminus \{0\}$$

in order to verify $A_1(\Omega, \delta_\varphi)$-condition in this case.

(iv) The case $\varphi_P(x) := \sum\limits_{j=1}^{n} \frac{1}{p_j(p_j-1)}|x_j|^{p_j}$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $P := (p_1, \ldots, p_n) \in (1, 2]^n$. Here $\Theta_1 = 1$ and $H_1 \geq 1$ depends only on $p_1, \ldots, p_n$, and n.

Next we list the examples of weight functions that satisfy $RH_\infty(\Omega, \delta_\varphi)$-condition taken from[19].

(a) The case $\det D^2\varphi \sim 1$ in Ω. As before, here $\Theta_\infty = 1$ and $H_\infty = \Lambda_2/\Lambda_1$.

(b) The case $\det D^2\varphi \sim |q|^a$ with $q = q(x)$ polynomial and $a > 0$. Here $\Theta_\infty = 1$ and $H_\infty \geq 1$ depends only on a, n, and the degree of q (and not on its coefficients).

(c) The case when φ is a convex polynomial in $\mathbb{R}^n$. Here $\Theta_\infty = 1$ and $H_\infty \geq 1$ depends only on n and the degree of φ (and not on its coefficients.)

(d) The case $\varphi_p(x) := \frac{1}{p}|x|^p$ with $2 \leq p < \infty$. Here $\Theta_\infty = 1$ and $H_\infty \geq 1$ depends only on p and n.

(e) The case $\varphi_P(x) := \sum\limits_{j=1}^{n} \frac{1}{p_j(p_j-1)}|x_j|^{p_j}$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $P := (p_1, \ldots, p_n) \in [2, \infty)^n$. Here $\Theta_\infty = 1$ and $H_1 \geq 1$ depends only on $p_1, \ldots, p_n$, and n.

From the list of these examples, we observe that some weight functions satisfy both $A_1(\Omega, \delta_\varphi)$ and $RH_\infty(\Omega, \delta_\varphi)$-conditions (that is, $A_1(\Omega, \delta_\varphi) \cap RH_\infty(\Omega, \delta_\varphi) \neq \emptyset$). Also, we can conclude that the inclusions $A_1(\Omega, \delta_\varphi) \subset A_\infty(\Omega, \delta_\varphi)$ and $RH_\infty(\Omega, \delta_\varphi) \subset A_\infty(\Omega, \delta_\varphi)$ are strict.

Chapter 6

Sobolev inequalities in the Monge-Ampère quasi-metric structure

This chapter is devoted to present Sobolev inequalities in the Monge-Ampère quasi-metric structure corresponding to the Poincaré inequalities studied in previous chapters. We start with a brief of history of Sobolev inequalities in the Monge-Ampère quasi-metric structure in Section 6.1. Then, Section 6.2 provides discussion on how to obtain Sobolev inequalities from Poincaré inequalities and records some new and improved Sobolev inequalities. In Section 6.3, we discuss the applications of both Poincare and Sobolev inequalities and also provide the comparison and connection of new such inequalities with the existing literature.

6.1 History of Sobolev inequalities in the Monge-Ampère setting

As indicated in Chapter 1, the first order inequalities such as Sobolev and Poincaré inequalities are widely applicable to investigate the local behaviour of solutions to the elliptic partial differential equations, such as to study the local estimates Harnack inequality and Hölder's continuity. In contrast, G. Tian and X.J Wang were able to develop the following Sobolev inequality in the Monge-Ampère quasi-metric structure, relying on the Hanrnak inequality for the solution of certain elliptic partial differential equation.

Theorem 6.1 (Theorem 3.1 in[2]). *Fix $n > 2$ and $\varphi \in C^2(\Omega)$ with $D^2\varphi > 0$ in Ω such that*

(a) $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$ and

(b) there exist $\theta \geq 0, \sigma > 0$, $C_1, C_2 > 0$ such that

$$C_1|S|^{1+\theta} \leq \mu_\varphi(S) \leq C_2|S|^{\frac{1}{n-1}+\sigma} \tag{6.1}$$

for every section $S := S_\varphi(x,t) \subset\subset \Omega$.

Then the following Sobolev inequality holds true for every $u \in C_c^\infty(\Omega)$

$$\left(\int_\Omega |u(x)|^p d\mu_\varphi(x)\right)^{\frac{1}{p}} \leq C \left(\int_\Omega |\nabla^\varphi u(x)|^2 d\mu_\varphi(x)\right)^{\frac{1}{2}} \tag{6.2}$$

where $d\mu_\varphi(x) = \det D^2\varphi(x)\, dx$, $p := \frac{2n(1+\theta)}{(n-1)(1+\theta)-1} > 2$, and the constant $C > 0$ depends on the constants from the $A_\infty(\Omega, \delta_\varphi)$-condition and (6.1) as well as on the diameter of Ω.

Theorem 6.1 is the first Sobolev inequality developed in the Monge-Ampère quasi-metric structure. Notice that one of the hypotheses is the Monge-Ampère measure μ_φ should satisfy Muckenhoupt's A_∞-weight condition. In particular, if we replace the A_∞-condition with the strictly stronger condition $\det D^2\varphi \sim 1$, in the sense of (5.16), it follows from Example 3 in[2] that the Sobolev inequality (6.2) holds true with $p = 2n/(n-2)$. Consequently, whenever $n > 2$, we recover the classical Sobolev inequality in the Euclidean setting from the choice $\varphi(x) = \frac{1}{2}|x|^2$.

The proof of Theorem 6.1 basically relies on the Harnack inequality by Cafarelli and Gutiérrez[6] and a crucial lemma by G. Tian and X.J. Wang[2], which read as

Theorem 6.2 (Harnack inequality). *Let φ be a strictly convex smooth function in $\mathbb{R}^n$ such that $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$. Then there exist constants $\beta > 1 > \tau > 0$, depending only on the $A_\infty(\Omega, \delta_\varphi)$-constants and dimension n, such that if u is any nonnegative solution of $L_\varphi u = 0$*

in $S_\varphi(x,t)$*, we have*

$$\sup_{S_\varphi(x,\tau t)} u \leq \beta \inf_{S_\varphi(x,\tau t)} u.$$

Lemma 6.3 (Crucial lemma)**.** *Suppose there is an integrable, almost everywhere positive function* μ *such that for any* $t > 0$*,*

$$\mu(\{x \in \Omega : G(x,y) > t\}) \leq K\, t^{-p/2},$$

where $p > 2$ *and* $K > 0$ *are constants. Suppose also that* μ *satisfies the doubling condition, namely there exists a constant* $b > 0$ *such that*

$$\mu(B(x,2r)) \leq b\, \mu(B(x,2r))$$

for every ball $B(x,r) \subset\subset \Omega$*. Then for smooth function* $u \in C_c^\infty(\Omega)$*, we have the inequality*

$$\left(\int_\Omega |u|^p \, d\mu\right)^{\frac{1}{p}} \leq C \left(\int_\Omega \sum a_{ij} u_{x_i} u_{x_j} \, dx\right)^{\frac{1}{2}},$$

where the constant C *depends only on* $n, p, b,$ *and* K*.*

Here above $G(x,y)$ is the Green's function of the linear elliptic operator $L = \sum \partial_{x_i}(a_{ij}(x)\partial_{x_j})$ in a bounded domain Ω, namely $G(\cdot,y)$ is a positive solution of

$$-L[G(\cdot,y)] = \delta_y \quad \text{in} \quad \Omega,$$
$$G(\cdot,y) = 0 \quad \text{on} \quad \partial\Omega,$$

where δ_y is a Dirac measure at $y \in \Omega$. This crucial lemma, in other words, estimates a rate of decay for the distribution function of the Green's function associated to the linearized Monge-Ampère operator (1.4) in Ω.

Few years later, D. Maldonado established an improved version of Theorem 6.1, namely,

Theorem 6.4 (Theorem 1 in[3])**.** *Fix* $n > 1$ *and let* $\varphi \in C^2(\Omega)$ *with* $D^2\varphi > 0$ *in* Ω *and* $\mu_\varphi \in$

$DC(\Omega, \delta_\varphi)$. Then the following Sobolev inequality holds true for every section $S := S_\varphi(x_0, t)$ with $S \subset\subset \Omega$ and every $u \in C^1_c(S)$

$$\left(\fint_S |u(x)|^{\frac{2n}{n-1}} d\mu_\varphi(x) \right)^{\frac{n-1}{2n}} \leq C t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^2 d\mu_\varphi(x) \right)^{\frac{1}{2}}, \tag{6.3}$$

where the constant $C > 0$ depends only on the doubling constant from the condition $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and dimension n.

We observe that Theorem 6.4 is established by completely dropping the condition (6.1) and by replacing the another assumption $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$ with $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ from the hypotheses of Theorem 6.1. Moreover, the exponent q in the Sobolev inequality (6.3) depends only on the dimension while such parameter rely on couple of other constants in the Sobolev inequality (6.2). Most importantly, the constant C on (6.3) depends only on the DC-doubling constant and dimension n, but such constant in (6.2) relies on several constants including the set Ω itself.

The Sobolev inequality (6.3) has played a key role in the implementation of Moser's iterations in[11] towards Harnack's inequality for nonnegative solutions of certain singular/degenerate elliptic PDEs.

Most recently, when proving Hölder regularity of solutions to the 2D dual semigeostrophic equation by means of the linearized Monge-Ampère equation under the assumption $\det D^2\varphi \sim 1$ in the sense of (5.16) (which, in particular, renders $d\mu_\varphi \sim dx$), N. Q. Le proved

Theorem 6.5 (Proposition 2.6 in[9])**.** *Fix $n = 2$ and $\varphi \in C^2(\Omega)$ with $\det D^2\varphi \sim 1$ in the sense of (5.16). Then, given $q \in (0, \infty)$ there exists a constant $C > 0$, depending only on q and Λ_1, Λ_2 from (5.16), such that the following Sobolev inequality holds true for every section $S := S_\varphi(x_0, t)$ with $S \subset\subset \Omega$ and every $u \in C^1_c(S)$*

$$\left(\fint_S |u(x)|^q \, dx \right)^{\frac{1}{q}} \leq C t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^2 \, dx \right)^{\frac{1}{2}}. \tag{6.4}$$

The proofs of both Theorems 6.4 and 6.5 rely variations of the aforementioned crucial lemma 6.3.

6.2 Improved and new Sobolev inequalities

In this section we point out that from each one of the Poincaré inequalities presented in previous chapters a corresponding Sobolev inequality can be obtained. This is possible due to a well-known fact that weak (q,p)-Poincaré inequalities with respect to a reverse-doubling measure imply (q,p)-Sobolev ones (see for instance Theorem 5.51 in[25]). For the sake of completeness, we briefly sketch the proof in our context. Given a section $S := S_\varphi(x,t) \subset\subset \Omega$, $u \in \mathrm{Lip}_c(S)$, that is, $u \in \mathrm{Lip}(S)$ with compact support within S, and $q \geq 1$, we have

$$\begin{aligned} |u^\mu_{2S}| \leq \fint_{2S} |u|\chi_S \, d\mu &\leq \left(\fint_{2S} |u|^q \, d\mu\right)^{\frac{1}{q}} \left(\frac{\mu(S)}{\mu(2S)}\right)^{1-1/q} \\ &\leq \left(\fint_{2S} |u|^q \, d\mu\right)^{\frac{1}{q}} \xi^{1-1/q}, \end{aligned}$$

where $\xi \in (0,1)$ is the constant from the reverse-doubling property in Lemma 2.29 corresponding to $\alpha = 1/2$. On the other hand, since

$$\left(\fint_{2S} |u|^q \, d\mu\right)^{\frac{1}{q}} \leq \left(\fint_{2S} |u - u^\mu_{2S}|^q \, d\mu\right)^{\frac{1}{q}} + |u^\mu_{2S}|,$$

it then follows that

$$\left(\fint_{2S} |u|^q \, d\mu\right)^{\frac{1}{q}} \leq \frac{1}{1-\xi^{1-1/q}} \left(\fint_{2S} |u - u^\mu_{2S}|^q \, d\mu\right)^{\frac{1}{q}},$$

which combined with an arbitrary weak (q,p)-Poincaré inequality

$$\left(\fint_{2S} |u - u^\mu_{2S}|^q \, d\mu\right)^{\frac{1}{q}} \leq C_P t^{\frac{1}{2}} \left(\fint_{2\lambda S} |\nabla^\varphi u|^p \, d\mu\right)^{\frac{1}{p}},$$

for some $\lambda \geq 1$, and recalling that u is supported in S, yields the Sobolev inequality

$$\left(\fint_S |u|^q \, d\mu \right)^{\frac{1}{q}} \leq \frac{C_P t^{\frac{1}{2}}}{1 - \xi^{1-1/q}} \left(\fint_S |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}}. \tag{6.5}$$

As an illustration and for future reference, we state the Sobolev inequalities that follow from the Poincaré inequalities in Theorems 5.4 and 5.5.

Theorem 6.6. *Fix $n \geq 3$ and let $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ be a strictly convex function with $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$. Then, there exist structural constants $K_9, K_{10} \geq 1$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_9 t) \subset\subset \Omega$ and every $u \in \mathrm{Lip}_c(S)$ we have*

$$\left(\fint_S |u(x)|^{\frac{2n}{n-2}} d\mu_\varphi(x) \right)^{\frac{n-2}{2n}} \leq K_{10} \, t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^2 d\mu_\varphi(x) \right)^{\frac{1}{2}}. \tag{6.6}$$

In addition, there exists a structural constant $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ there is a constant $K_\epsilon > 0$, depending only on ϵ and structural constants, such that

$$\left(\fint_S |u(x)|^{q_\epsilon} d\mu_\varphi(x) \right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon \, t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^{2-\epsilon} d\mu_\varphi(x) \right)^{\frac{1}{2-\epsilon}}, \tag{6.7}$$

with $q_\epsilon := \frac{n(2-\epsilon)}{n-(2-\epsilon)} > 2$.

Theorem 6.7. *Assume $n = 2$ and let $\varphi \in W^{2,2}_{loc}(\Omega)$ be a strictly convex function with $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$. Then, there exist structural constants $K_9 \geq 1$ and $0 < \epsilon_0 < 1$, such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_9 t) \subset\subset \Omega$, every $u \in \mathrm{Lip}_c(S)$, and every $0 < \epsilon \leq \epsilon_0$ we have*

$$\left(\fint_S |u(x)|^{q_\epsilon} d\mu_\varphi(x) \right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon \, t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^{2-\epsilon} d\mu_\varphi(x) \right)^{\frac{1}{2-\epsilon}}, \tag{6.8}$$

with $q_\epsilon := 2(2-\epsilon)/\epsilon$ and $K_\epsilon > 0$ depends only on ϵ and structural constants.

Remark 6.8. *Notice that Theorem 6.7 extends Proposition 2.6 in[9], that is, Theorem 6.5 from Section 6.1, by weakening the assumption $\det D^2\varphi \sim 1$, in the sense of (5.16), to $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$.*

Remark 6.9. *Poincaré and Sobolev inequalities such as the ones in Theorems 5.4, 5.5, and Theorems 6.6, 6.7, respectively, play a central role in the implementation of Moser's iterations for solutions to the linearized Monge-Ampère equation, as described in*[11] *Section 2.4.*

6.3 Examples and applications

We conclude this chapter by discussing the further applications and connections of Theorems presented in previous chapters as our main results as well as their corresponding Sobolev inequalities with related inequalities in the existing literature. We also visualize such inequalities associated to certain convex functions from the list of examples illustrated in Section 5.4.

As mentioned in Remark 6.9, the Poincaré inequalities presented as our main results in Chapter 3, Chapter 4 and Chapter 5 will find applications in the implementation of Moser's iterations for certain degenerate/singular PDEs. Also, the comments below Theorem 3.4 and Remark 6.8 point out how they improve upon a few previously known results. In addition, in view of the examples in Section 5.4, all our main results presented in previous chapters give rise to a large variety of new or improved Poincaré and Sobolev inequalities some of which complement or extend inequalities from the existing literature. As an illustration, in this section we take a look at just a couple of such inequalities. We start by mentioning the following Sobolev inequality by Tian and Wang in[2] when φ is a strictly convex polynomial in $\mathbb{R}^n$.

Theorem 6.10 (Theorem 1.1 in[2])**.** *Let φ be a strictly convex polynomial in $\mathbb{R}^n$, $n \geq 3$. Then, for any bounded domain $\Omega \subset B_R(0)$ and any function $u \in C_0^\infty(\Omega)$,*

$$\left(\int_\Omega |u(x)|^p d\mu_\varphi(x)\right)^{\frac{1}{p}} \leq C \left(\int_\Omega |\nabla^\varphi u(x)|^2 d\mu_\varphi(x)\right)^{\frac{1}{2}}, \tag{6.9}$$

where $p > 2$ depends on n and φ and C also depends on R.

By using Theorem 5.6 and Example (c 5.4) of weights satisfying $RH_\infty(\Omega, \delta_\varphi)$-condition in Section 5.4 provide a Poincaré inequality with respect to the Lebesgue measure which, in turn, yields a related Sobolev inequality (as described in Section 6.2) that complements Theorem 6.10 where the Monge-Ampère measure is replaced with Lebesgue measure and with a finer tuning on the constants. More precisely,

Theorem 6.11. *Fix $n \geq 3$ and let φ be a strictly convex polynomial with $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\mathbb{R}^n, dx)$. Then, there exist constants $K_{11}, K_{12} \geq 1$, depending only on the degree of φ and dimension n, such that for every section $S := S_\varphi(x_0, t)$ we have*

$$\left(\fint_S |u(x) - u_S|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{2n}} \leq K_{12}\, t^{\frac{1}{2}} \left(\fint_{K_{11}S} |\nabla^\varphi u(x)|^2\, dx\right)^{\frac{1}{2}}$$

for every $u \in \mathrm{Lip}(K_{11}S)$, as well as

$$\left(\fint_S |u(x)|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{2n}} \leq K_{12}\, t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^2\, dx\right)^{\frac{1}{2}}$$

for every $u \in \mathrm{Lip}_c(S)$.

We next see Poincaré and Sobolev inequalities associated to convex function φ_P defined in Section 5.4 example (iv 5.4). For this let us first recall the following Sobolev inequality proved by Cabré and Ros-Oton.

Theorem 6.12 (Theorem 1.3(a) in [30]). *Suppose $A = (a_1, \ldots, a_n) \in \mathbb{R}^n$, with $a_j \geq 0$ for every $j = 1, \ldots, n$ and let $1 \leq p < D := n + a_1 + \cdots + a_n$. Then there exists $C_p > 0$ such that for every $u \in C^1_c(\mathbb{R}^n)$*

$$\left(\int_{\mathbb{R}^n_*} |u(x)|^{p_*} x^A\, dx\right)^{\frac{1}{p_*}} \leq C_p \left(\int_{\mathbb{R}^n_*} |\nabla u(x)|^p x^A\, dx\right)^{\frac{1}{p}}, \tag{6.10}$$

where $p_* := \frac{pD}{D-p}$, $x^A := \prod_{j=1}^{n} |x_j|^{a_j}$, and

$$\mathbb{R}^n_* := \{(x_1, \ldots, x_n) : \text{ with } x_j > 0 \text{ whenever } a_j > 0\}.$$

Now, by means of the Poincaré inequalities from Section 5.2 and Example (iv 5.4) we will next obtain Poincaré and Sobolev inequalities related to the weight x^A as in (6.10) but now in the case $-1/n < a_j \leq 0$ for every $j = 1, \dots, n$. Indeed, for $-1/n < a_j \leq 0$ set $p_j := 2 + a_j \in (1, 2]$ and

$$\varphi_P(x) := \sum_{j=1}^{n} \tfrac{1}{p_j(p_j-1)} |x_j|^{p_j}, x = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{6.11}$$

as in Example (iv 5.4). Then

$$D^2\varphi_P(x) = \begin{bmatrix} |x_1|^{a_1} & 0 & \cdots & 0 \\ 0 & |x_2|^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & |x_n|^{a_n} \end{bmatrix},$$

so that $\det D^2\varphi_P(x) = \prod_{j=1}^{n} |x_j|^{a_j} = x^A$. Notice that the condition $-1/n < a_j \leq 0$ for every $j = 1, \dots, n$ guarantees that $\varphi_P \in W^{2,n}_{loc}(\mathbb{R}^n, dx)$. Also, for a.e. $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $u \in C^1(\mathbb{R}^n)$,

$$\begin{aligned} \nabla^{\varphi_P} u(x) &= D^2\varphi_P(x)^{-\frac{1}{2}} \nabla u(x) \\ &= (|x_1|^{-\frac{a_1}{2}} u_1(x), \cdots, |x_n|^{-\frac{a_n}{2}} u_n(x)) \end{aligned}$$

and consequently

$$|\nabla^{\varphi_P} u(x)| = \left(\sum_{j=1}^{n} |x_j|^{-a_j} |u_j(x)|^2 \right)^{\frac{1}{2}}.$$

Moreover, by[27] Lemma 6 the Monge-Ampère sections of φ_P are related to the ones of $\varphi_{p_j}(x) := \frac{1}{p_j}|x|^{p_j}$, $x \in \mathbb{R}$, by means of the inclusions

$$S_{\varphi_P}(y, t) \subset S_{\varphi_{p_1}}(y_1, t) \times \cdots \times S_{\varphi_{p_n}}(y_n, t) \subset S_{\varphi_P}(y, n\,t),$$

for every $y = (y_1, \ldots, y_n) \in \mathbb{R}^n, y_j \in \mathbb{R}, j = 1, \ldots, n$, and $t > 0$.

Finally, by using Theorem 5.4 with the convex function φ_P, we obtain the following Poincaré and Sobolev inequalities.

Theorem 6.13. *Fix $n \geq 3$ and let φ_P be the strictly convex function defined in (6.11). Then, there exist constants $K_9, K_{10} \geq 1$, depending only on $a_1, \ldots, a_n \in (-1/n, 0]$ and dimension n, such that for every section $S := S_{\varphi_P}(x_0, t)$ we have*

$$\left(\fint_S |u(x) - u_S^{\mu_{\varphi_P}}|^{\frac{2n}{n-2}} \, d\mu_{\varphi_P}(x) \right)^{\frac{n-2}{2n}} \leq K_{10} \, t^{\frac{1}{2}} \left(\fint_{K_9 S} |\nabla^{\varphi_P} u(x)|^2 \, d\mu_{\varphi_P}(x) \right)^{\frac{1}{2}}$$

for every $u \in \mathrm{Lip}(K_9 S)$, as well as

$$\left(\fint_S |u(x)|^{\frac{2n}{n-2}} \, d\mu_{\varphi_P}(x) \right)^{\frac{n-2}{2n}} \leq K_{10} \, t^{\frac{1}{2}} \left(\fint_S |\nabla^{\varphi_P} u(x)|^2 \, d\mu_{\varphi_P}(x) \right)^{\frac{1}{2}}$$

for every $u \in \mathrm{Lip}_c(S)$, where $d\mu_{\varphi_P}(x) = x^A \, dx$.

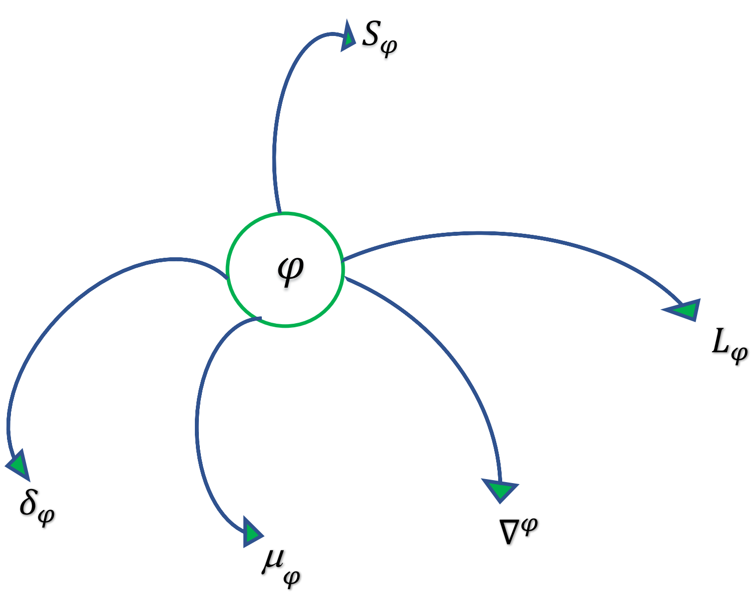

Figure 6.1: *Notions related to the given convex function in the Monge-Ampère quasi-metric structure.*

Throughout the discussion from Chapter 1 to Chapter 6, the readers shall observe that for each continuously differentiable strictly convex function φ defined in an open convex set Ω satisfying so-called the DC-doubling condition, we have a family of notions related to φ as shown in Figure 6.1. That is, we have the Monge-Ampère sections associated to φ which can be viewed as quasi-metric balls, a measure μ_φ, a quasi-distance δ_φ which in turn produces a space of homogeneous type: $(\Omega, \delta_\varphi, \mu_\varphi)$, a linearized Monge-Ampère operator L_φ, and the Monge-Ampère gradient ∇^φ. We then can study Poincaré and Sobolev inequalities associated to these notions in the Monge-Ampère quasi-metric structure. Thus for each "nice" convex function, we obtain a pair of Poincaré and Sobolev inequalities associated to it. In general, when a family of convex functions satisfying certain conditions is given, we can produce a zoo of Poincaré and Sobolev inequalities associated to the given collection of convex functions, which can be celebrated as a beauty of the study in the Monge-Ampère quasi-metric structure. In addition, the study associated to some particular choices of convex functions in the Monge-Ampère setting reduces to the study in the celebrated geometrical structure known as the Euclidean space, and consequently the first order inequalities and their applications generated in the Monge-Ampère quasi-metric structure translate to the ones in the Euclidean settings.

Chapter 7

The Whitney decomposition of the Monge-Ampère sections

7.1 Introduction and main result

A class of essential components in the study related to Analysis, Differentiation and Geometry consists of covering lemmas. The Whitney type covering lemma, named after the American mathematician Hassler Whitney, is one of such covering lemmas. The covering lemma, originally introduced by Whitney in 1934, provides the partition of an open set with non empty boundary into a disjoint sequence of half-open cubes such that the diameters of the cubes are comparable to the distance from the corresponding cubes to the boundary of the given set, namely,

Theorem 7.1 (Theorem 2.1 in[31]). *Let G be any open subset of $\mathbb{R}^n$ with non empty boundary ∂G. Then there is a countable collection of disjoint half-open cubes $\{Q_j\}_{j\geq 1}$ such that*

$$G = \bigcup_{j\geq 1} Q_j \quad \textit{and} \quad 1 \leq \frac{\operatorname{dist}(Q_j, \partial G)}{\operatorname{diam}(Q_j)} < 3.$$

When the requirement of disjointness is relaxed in Theorem 7.1, then a collection of open cubes $\{Q_j\}$ can be found such that $G = \bigcup_{j\geq 1} Q_j$ with $\operatorname{dist}(Q_j, \partial G) = 3\operatorname{diam}(Q_j)$ and for each

$x \in \mathbb{R}^n$,

$$\sum_{j=1}^{\infty} \chi_{Q_j}(x) \leq \alpha_n,$$

α_n being a constant that depends only on dimension n.

The Whitney type decompositions, later in the second half of 20th century, were investigated widely in order to obtain similar partitions involving geometric shapes other than cubes, for instance metric balls, generally by imposing some extra conditions. We now state a Whitey decomposition in a doubling metric space from[31] as below.

Theorem 7.2 (Theorem 2.3 in[31])**.** *Consider (X, δ) be a metric space and let μ be a doubling measure defined on X (that is, there exists $C > 0$ such that for every metric balls $B(x, r) \subset X$, $\mu(B(x, 2r)) \leq C\mu(B(x, r))$). Then for every open set $G \subset X$ with non empty boundary, there exists a countable collection of balls $\{B(x_j, r_j)\}_{j\geq 1}$ and $K > 0$, depending only on C, such that*

(a) $G = \bigcup_{j\geq 1} B(x_j, r_j)$;

(b) $r_j \leq \operatorname{dist}(B(x_j, r_j), \partial G) \leq 4r_j$, *for all* $j \in \mathbb{N}$ *and*

(c) $\sum_{j\geq 1} \chi_{B(x_j, r_j)}(x) \leq K\chi_G(x)$, *for all* $x \in X$.

The Whitney decomposition in the doubling quasi-metric space, such as Theorem 7.2 as well as the ones involving the cubes, find applications in establishing the well-known lemma of Calderón and Zygmund. The readers can find more examples and applications of the Whitney type decomposition mentioned here above in the lecture notes by Coifman and Weiss in[32].

Whenever a Whitney type decomposition is intended for a set in the Euclidean space involving Euclidean balls or cubes, it is doable as we discussed above. In fact, these are the nice scenarios for such decomposition as cubes and Euclidean balls are the geometric shapes that behave very well. On the other hand, the scenario will be slightly complicated for general metric spaces. However, the good news with metric spaces is that the metric balls have nice properties due to the symmetric condition and triangle inequality of the associated metric. But some complexity arise in order to find Whitney type decompositions whenever the given space is a doubling quasi-metric space with the quasi-distance that satisfy quasi-symmetric condition and quasi-triangle inequality as in the Definition 2.27. In other words, the extra factor $K > 1$ in the quasi-triangle inequality and quasi-symmetric condition create many troubles in the construction of partitions of given sets.

Our objective in this chapter is to provide the Whitney decomposition for the Monge-Ampère sections in the Monge-Ampère quasi-metric structure. We keep in mind that the Monge-Ampère quasi-metric structure is associated with the quasi-distance in the sense of Definition 2.27. Let us state the main result of this chapter here below.

Theorem 7.3. *Let $\varphi \in C^1(\Omega)$ be a strictly convex function and the sections of φ have the engulfing property with the engulfing constant $\Theta \geq 1$. Also, let $S_0 := S_\varphi(x_0, t_0)$ be a Monge-Ampère section of φ with $S_\varphi(x_0, \Theta^2 t_0) \subset\subset \Omega$ and fix ε such that $0 < \varepsilon < \frac{1}{10\Theta^6}$. Then there exists a countable collection of Monge-Ampère sections $\{S_j := S_\varphi(x_j, t_j)\}_{j\geq 1}$ in S_0 and a constant $C > 0$, depending only on Θ and dimension n, such that*

(i) S_j *are mutually disjoint for* $j \geq 1$;

(ii) $\bigcup_{j\geq 1} \Theta^2 S_j = S_0$, *where* $\Theta^2 S_j = S_\varphi(x_j, \Theta^2 t_j), j \geq 1$;

(iii) $t_j = \varepsilon\, \delta_\varphi(S_j, \partial S_0)$ *for* $j \geq 1$;

(iv) $\sum_{j\geq 1} \chi_{\Theta^2 S_j}(x) \leq C\, \chi_{S_0}(x), \quad \forall x \in \Omega$;

(v) If $\Theta^2 S_i \cap \Theta^2 S_j \neq \emptyset$, *then* $\frac{1}{2\Theta^2} < \frac{t_i}{t_j} < 2\Theta^2$.

The proof of Theorem 7.3 is inspired from[33]. We know that every quasi-distance is comparable to a power of a distance. That is, if (X, δ) is a quasi-metric space (with quasi-triangle constant K, say), then there exists a distance d and constants $C, \alpha > 0$ (depending on K) such that

$$\frac{1}{C}\, d(x,y)^\alpha < \delta(x,y) < C\, d(x,y)^\alpha, \quad \text{for every } x, y \in X.$$

This is known as the metrization of quasi-metric spaces (a proof is due to Macias and Segovia[34]). Also, the readers can see[35] and[36] for other details and different approaches related to the the metrization techniques of quasi-metric spaces. Therefore, the Whitney decomposition for quasi-distances follows from the decomposition for distances by just re-scaling the balls to quasi-balls. Despite the metrization result above, we give a proof of Theorem 7.3 that is more direct.

Let us discuss about some ingredients that we will require to prove Theorem 7.3. Recall from Lemma 2.20 that the Lebesgue measure is doubling in the Monge-Ampère sections. That is, $|S_\varphi(x,t)| \leq 2^n\,|S_\varphi(x,t/2)|$, for every Monge-Ampère sections $S_\varphi(x,t) \subset\subset \Omega$. With this, a simple manipulation provides that for every $\lambda \geq 1$, we have

$$|S_\varphi(x,\lambda\, t)| \leq (2\lambda)^n\,|S_\varphi(x,t)|, \tag{7.1}$$

for every Monge-Ampère sections $S_\varphi(x,\lambda\, t) \subset\subset \Omega$. In fact, for a given $\lambda \geq 1$, there exists $k \in \mathbb{N}$ such that $2^{k-1} \leq \lambda \leq 2^k$. Then

$$\begin{aligned}|S_\varphi(x,\lambda\, t)| &\leq |S_\varphi(x,2^k\, t)| \leq 2^n|S_\varphi(x,2^{k-1}\, t)|\\ &\leq (2^n)^2|S_\varphi(x,2^{k-2}\, t)|\\ &\;\vdots\\ &\leq (2^n)^k|S_\varphi(x,t)| \leq (2\lambda)^n|S_\varphi(x,t)|.\end{aligned}$$

Next we just state a lemma in the spirit of Subsection 2.3.3.

Lemma 7.4. *Assume that the Monge-Ampère sections of φ have the engulfing property in Ω with the engulfing constant $\Theta > 1$. (That is, there exists a geometric constant $\Theta > 1$ such that whenever $x_0 \in \Omega$ and $\tau > 0$ satisfy $S_\varphi(x_0,\Theta^2\tau) \subset\subset \Omega$, then for every $x \in S_\varphi(x_0,\tau)$ we have $S_\varphi(x_0,\tau) \subset S_\varphi(x,\Theta\tau)$.) Then δ_φ defined in (2.4) satisfies*

(a) $\delta_\varphi(x,y) \leq \Theta\,\delta_\varphi(y,x)$, $\forall x,y \in S_\varphi(x_0,\tau)$;

(b) $\delta_\varphi(x,y) \leq \Theta\,[\delta_\varphi(z,x) + \delta_\varphi(z,y)]$, $\forall x,y,z \in S_\varphi(x_0,\tau)$ *and*

(c) $\delta_\varphi(x,y)) \leq \Theta\,[\Theta\,\delta_\varphi(x,z)+\delta_\varphi(z,y)] \leq \Theta^2\,[\delta_\varphi(x,z)+\delta_\varphi(z,y)],\ \forall x,y,z \in S_\varphi(x_0,\tau)$.

The size of the constants in the inequalities in part (b) and (c) in Lemma 7.4 are based on the the direction of the distance measured. In particular, we observe that the distance $\delta_\varphi(x,y)$ is bounded by the quantity smaller than the one used for the quasi-triangle inequality in Subsection 2.3.3, which is same as the right-side of (c) in Lemma 7.4. We will consider the smallest possible quantities to bound certain distances in order to achieve smallest constant in the proof of Theorem 7.3.

Let us also see a definition that plays a role in the proof of Theorem 7.3.

Definition 7.5 (Geodesic property)**.** *Let (Ω,δ) be a quasi-metric structure. Then a $\delta-$ball $B_0 \subset \Omega$ (or equivalently the quasi-distance δ) is said to possess the geodesic property if for every $\delta-$ball B contained in B_0 with center x_B, we have for each $x \in B$ there is a continuous one-to-one curve $\gamma = \gamma_{x_B x}(t), 0 \leq t \leq 1$, in B with $\gamma(0)=x_B, \gamma(1)=x$ and $\delta(x_B,z)=\delta(x_B,y)+\delta(y,z),\ \forall\, y,z \in \gamma$ with $y=\gamma(s), z=\gamma(t)$ where $0 \leq s \leq t \leq 1$.*

We can easily observe that the Euclidean balls in $\mathbb{R}^n$ possess the geodesic property. We know that metric balls (or metric spaces) behave very well compared to the quasi-metric balls. However, in general, metric balls may not possess the geodesic property. For example, the $\delta-$balls in the discrete metric space $(\mathbb{Z},\delta)$ where

$$\delta(x,y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y. \end{cases}$$

Another example of quasi-metric balls that have the geodesic property consists of the Monge-Ampère sections due to the following theorem by D. Maldonado.

Theorem 7.6 (Theorem 16 in[10])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open convex set and $\varphi \in C^1(\Omega)$ be a strictly convex function. Then the Monge-Ampère quasi-distance δ_φ defined in 2.4 possesses the following geodesic property:*

Given $x,z \in \Omega$ with $S_\varphi(x,\delta_\varphi(x,z)) \subset\subset \Omega$ and $0<r<\delta_\varphi(x,z)$, there exists $y \in S_\varphi(x,\delta_\varphi(x,z))$ such that $\delta_\varphi(x,y)=r$ and $\delta_\varphi(x,z)=\delta_\varphi(x,y)+\delta_\varphi(y,z)$.

7.2 Proof of Theorem 7.3

Let us fix $S_0 := S_\varphi(x_0, t_0)$ such that $S_\varphi(x_0, \Theta^2 t_0) \subset\subset \Omega$ and ε with $0 < \varepsilon < \dfrac{1}{10\theta^6}$. We first prove (i) and (iii). For this we begin by claiming that given $x \in S_0$, there is a Monge-Ampère section $S_\varphi(x, t)$ such that

$$t = \varepsilon\, \delta_\varphi(S_\varphi(x, t), \partial S_0). \tag{7.2}$$

To see this, define

$$f(t) := \delta_\varphi(S_\varphi(x, t), \partial S_0), \quad 0 \le t \le \delta_\varphi(x, \partial S_0).$$

Clearly, $f(0) = \delta_\varphi(x, \partial S_0)$ and $f(\delta_\varphi(x, \partial S_0)) = 0$. Since δ_φ is continuous, so is f. This implies

$$g(t) := \varepsilon\, \delta_\varphi(S_\varphi(x, t), \partial S_0), \quad 0 \le t \le \delta_\varphi(x, \partial S_0)$$

is also continuous on $[0, \delta_\varphi(x, \partial S_0)]$, which maps onto $[0, \varepsilon\delta_\varphi(x, \partial S_0)] \subset [0, \delta_\varphi(x, \partial S_0)]$. Then by the fixed point theorem, there exists $t' \in [0, \delta_\varphi(x, \partial S_0)]$ such that $g(t') = t'$, and hence the claim (7.2) follows.

Now define $t_1 := \sup\{t > 0 : \exists\, x \in S_0 \text{ satisfying } t = \varepsilon\, \delta_\varphi(S_\varphi(x, t), \partial S_0)\}$. Then we can pick $z_k \in S_0$ and $u_k > 0$ such that $u_k = \varepsilon\delta_\varphi(S_\varphi(z_k, u_k), \partial S_0)$ and u_k is an increasing sequence converging to t_1. Since $\bar{S}_0$ is compact, z_k has a convergent subsequence, say $z_k \to x_1 \in S_0$. We next claim that

$$t_1 = \varepsilon\, \delta_\varphi(S_\varphi(x_1, t_1), \partial S_0). \tag{7.3}$$

In order to prove this claim, let's pick $\tilde{z}_k \in \partial S_\varphi(z_k, u_k)$ such that

$$u_k = \varepsilon\, \delta_\varphi(S_\varphi(z_k, u_k), \partial S_0) = \varepsilon\, \delta_\varphi(\tilde{z}_k, \partial S_0).$$

By the same reason as above, we may assume that $\tilde{z}_k$ has a convergent subsequence, say $\tilde{z}_k \to z_0 \in S_0$. Since $\tilde{z}_k \in \partial S_\varphi(z_k, u_k)$, $u_k \to t_1$ and δ_φ is continuous, we get

$$\delta_\varphi(x_1, z_0) = \lim_{k\to\infty} \delta_\varphi(z_k, \tilde{z}_k) = \lim_{k\to\infty} u_k = t_1.$$

That is, $z_0 \in \partial S_\varphi(x_1, t_1)$ and

$$t_1 = \lim_{k\to\infty} u_k = \varepsilon \lim_{k\to\infty} \delta_\varphi(\tilde{z}_k, \partial S_0) = \varepsilon\delta_\varphi(z_0, \partial S_0) \geq \varepsilon\, \delta_\varphi(S_\varphi(x_1, t_1), \partial S_0).$$

For the other inequality, we pick $y \in \partial S_\varphi(x_1, t_1)$ such that

$$\delta_\varphi(S_\varphi(x_1, t_1), \partial S_0) = \delta_\varphi(y, \partial S_0).$$

Now, if we can pick $y_k \in \overline{S(z_k, u_k)}$ such that $y_k \to y$, then

$$t_1 = \lim_{k\to\infty} u_k = \varepsilon \lim_{k\to\infty} \delta_\varphi(S_\varphi(z_k, u_k), \partial S_0) \leq \varepsilon \lim_{k\to\infty} \delta_\varphi(y_k, \partial S_0) = \varepsilon\delta_\varphi(y_0, \partial S_0).$$

In order to pick such y_k, for a given k, assume that $y \notin S_\varphi(z_k, u_k)$ (otherwise we can choose $y_k = y$). Then by the geodesic property, we can pick y_k on the line segment joining z_k and y such that $\delta_\varphi(z_k, y_k) = u_k$ (that is, $y_k \in \partial S_\varphi(z_k, u_k)$) and $\delta_\varphi(z_k, y) = \delta_\varphi(z_k, y_k) + \delta_\varphi(y_k, y)$. So,

$$\lim_{k\to\infty} \delta_\varphi(y_k, y) = \lim_{k\to\infty} [\delta_\varphi(z_k, y) - u_k] = \delta_\varphi(x_1, y) - t_1 = 0.$$

This proves our claim (7.3) and hence name the Monge-Ampère section $S_1 := S_\varphi(x_1, t_1)$. Clearly S_1 is the largest Monge-Ampère section in S_0 satisfying (iii). Next define

$$t_2 := \sup\{t > 0 : \exists\, x \in S_0 \setminus S_1 \;\text{ with } t = \varepsilon\, \delta_\varphi(S(x, t), \partial S_0) \text{ and } S_\varphi(x, t) \cap S_1 = \emptyset\}.$$

Then by proceeding similarly as above, we get another Monge-Ampère section $S_2 = S_\varphi(x_2, t_2)$ which is the second largest Monge-Ampère section in S_0 satisfying (iii) and disjoint to S_1. Continuing this process proves (i) and (iii).

Now we proceed to prove (ii). The inclusion $\subseteq$ is obvious. Indeed, for all j,

$$\Theta^2 t_j = \Theta^2 \varepsilon\, \delta_\varphi(S_j, \partial S_0) \leq \Theta^2 \varepsilon\, \delta_\varphi(x_j, \partial S_0) \leq \delta_\varphi(x_j, \partial S_0).$$

So, $S_\varphi(x_j, \Theta^2 t_j) \subset S_0$ for all j. This implies

$$\bigcup_{j\geq 1} \Theta^2 S_j \subset S_0.$$

For the other inclusion $\supseteq$, let's pick $x \in S_0$ and show that $x \in \Theta^2 S_j$ for some j. We know that for $x \in S_0$, there exists a Monge-Ampère section $S_\varphi(x,t)$ such that $t = \varepsilon\, \delta_\varphi(S_\varphi(x,t), \partial S_0)$. Clearly, $t \leq t_1$. If $S_\varphi(x,t) \cap S_1 \neq \emptyset$, then by the engulfing property $S_\varphi(x,t) \subset \theta^2 S_1$. If $S_\varphi(x,t) \cap S_1 = \emptyset$, we still have $t \leq t_2$ and continue the process. We now claim that this process will terminate. That is, there exists j_0 such that $S_\varphi(x,t) \cap S_{j_0} \neq \emptyset$. If possible, let's assume that there is no such j. This would mean that $t \leq t_j$ for infinitely many $j = 1, 2, \cdots$ By the engulfing property, $x_j \in S_\varphi(x_0, t_0)$ implies $S_\varphi(x_0, t_0) \subset S_\varphi(x_j, \Theta t_0)$. So,

$$\begin{aligned} |S_\varphi(x_0, t_0)| \leq |S_\varphi(x_j, \Theta t_0)| &= \left|S_\varphi\Big(x_j, \frac{\Theta t_0}{t_j} t_j\Big)\right| \\ &\leq \left|S_\varphi\Big(x_j, \frac{\Theta t_0}{t} t_j\Big)\right| \quad \text{since} \quad t \leq t_j, \forall j \\ &\leq \left(\frac{2\Theta t_0}{t}\right)^n |S_\varphi(x_j, t_j)| \quad \text{due to (7.1)}. \end{aligned}$$

Consequently, $|S_0| \geq \sum_{j\geq 1} |S_j| \geq \left(\frac{t}{2\Theta t_0}\right)^n |S_0| \sum_{j \geq 1} 1 = \infty$, a contradiction. This completes the proof of (ii).

Next, we proceed to prove (iv). We already know $\Theta^2 S_j \subset S_0$, for all j. Let $y \in \partial(\Theta^2 S_j)$ and $z \in \partial S_0$. Then by Lemma 7.4, $\delta_\varphi(x_j, z) \leq \Theta[\Theta \delta_\varphi(x_j, y) + \delta_\varphi(y, z)] \leq \Theta[\Theta^3 t_j + \delta_\varphi(y,z)]$. This implies

$$\delta_\varphi(y,z) \geq \frac{1}{\Theta}\delta_\varphi(x_j, z) - \Theta^3 t_j \geq \frac{1}{\Theta}\delta_\varphi(S_j, \partial S_0) - \Theta^3 t_j \geq \frac{1-\Theta^4\varepsilon}{\Theta\varepsilon} t_j.$$

That is,

$$t_j \leq \frac{\Theta\varepsilon}{1-\Theta^4\varepsilon}\delta_\varphi(\Theta^2 S_j, \partial S_0), \ \forall j. \tag{7.4}$$

Let $x \in S_0$ and denote $\alpha := \delta_\varphi(x, \partial S_0)$. If $x \in \Theta^2 S_j$, then $\delta_\varphi(\Theta^2 S_j, \partial S_0) \leq \alpha$ and hence due to (7.4), $t_j \leq \dfrac{\Theta\,\varepsilon}{1-\Theta^4\varepsilon}\alpha$. Since $\dfrac{1}{1-\Theta^4\varepsilon} < \dfrac{10}{9}$ from the assumption, we have

$$t_j \leq \frac{10}{9}\Theta\varepsilon\alpha.$$

Let's set $\varepsilon_1 := \dfrac{10}{9}\Theta^4\varepsilon < \dfrac{1}{9}$. Then for those $\Theta^2\, S_j$ containing x, we have

$$\Theta^3 t_j < \varepsilon_1\alpha. \tag{7.5}$$

By the engulfing property and using the inequality (7.5), $x \in \Theta^2 S_j$ implies $\Theta^2 S_j \subset S_\varphi(x, \Theta^3 t_j) \subset S_\varphi(x, \varepsilon_1\alpha)$. That is, all the $\Theta^2 S_j$ containing x lie in $S_\varphi(x, \varepsilon_1\alpha)$. Since $\alpha = \delta_\varphi(x, \partial S_0)$, there exist $\bar{z} \in \partial S_0$ such that $\alpha = \delta_\varphi(x, \bar{z})$. For any $y \in S_\varphi(x, \varepsilon_1\alpha)$, we have $\delta_\varphi(x, y) < \varepsilon_1\alpha$. So,

$$\delta_\varphi(y, \partial S_0) \leq \delta_\varphi(y, \bar{z}) \leq \Theta[\delta_\varphi(x, y) + \delta_\varphi(x, \bar{z})] < \Theta(1+\varepsilon_1)\alpha.$$

Since $1+\varepsilon_1 < \dfrac{10}{9}$,

$$\delta_\varphi(y, \partial S_0) < \frac{10}{9}\Theta\alpha, \quad \text{whenever} \quad x \in \Theta^2 S_j. \tag{7.6}$$

Inequality (7.6) gives the upper bound for $\delta_\varphi(y, \partial S_0)$. We now estimate lower bound for this quantity. For any $z \in \partial S_0$,

$$\alpha = \delta_\varphi(x, \bar{z}) \leq \delta_\varphi(x, z) \leq \Theta[\Theta\delta_\varphi(x, y) + \delta_\varphi(y, z)].$$

This implies

$$\delta_\varphi(y, z) \geq \frac{1-\Theta^2\varepsilon_1}{\Theta}\alpha.$$

Since $1-\Theta^2\varepsilon_1 > \dfrac{8}{9}$,

$$\delta_\varphi(y, \partial S_0) > \frac{8}{9\Theta}\alpha, \quad \text{whenever} \quad x \in \Theta^2 S_j. \tag{7.7}$$

So, combining inequalities from (7.6) and (7.7), we see that the distance from any $y \in$

$S_\varphi(x, \varepsilon_1\alpha)$ to ∂S_0 is in between $\dfrac{8}{9\Theta}\alpha$ and $\dfrac{10}{9}\Theta\alpha$. Since $\Theta^2 S_j \subset S_\varphi(x, \varepsilon_1\alpha)$, in particular,

$$\frac{8}{9\Theta}\alpha < \delta_\varphi(S_j, \partial S_0) < \frac{10}{9}\Theta\alpha.$$

That is,

$$\frac{8}{9\Theta}\alpha < \frac{t_j}{\varepsilon} < \frac{10}{9}\Theta\alpha, \quad \text{whenever} \quad x \in \Theta^2 S_j. \tag{7.8}$$

We know that $\Theta^2 S_j \subset S_\varphi(x, \varepsilon_1\alpha)$ whenever $x \in \Theta^2 S_j$. In particular, $x_j \in S_\varphi(x, \varepsilon_1\alpha)$. Then by the engulfing property $S_\varphi(x, \varepsilon_1\alpha) \subset S(x_j, \Theta\varepsilon_1\alpha)$. So,

$$|S_\varphi(x, \varepsilon_1\alpha)| \leq |S_\varphi(x_j, \Theta\varepsilon_1\alpha)| = \left|S_\varphi\Big(x_j, \frac{\Theta\varepsilon_1\alpha}{t_j}t_j\Big)\right|.$$

From inequalities (7.5) and (7.8), we obtain $\dfrac{\Theta\varepsilon_1\alpha}{t_j} \leq \dfrac{5}{4}\Theta^6$. Now by using inequality (7.1),

$$|S_\varphi(x, \varepsilon_1\alpha)| \leq \left|S_\varphi\Big(x_j, \frac{5\Theta^6}{4}t_j\Big)\right| \leq \left(\frac{5\Theta^6}{2}\right)^n |S_\varphi(x_j, t_j)|.$$

That is,

$$x \in \Theta^2 S_j \Rightarrow |S_\varphi(x, \varepsilon_1\alpha)| \leq C\,|S_j|, \tag{7.9}$$

where $C := \left(\dfrac{5\Theta^6}{2}\right)^n$, depending only on n and Θ. Clearly, there are finitely many of S_j satisfying the inequality (7.9). Otherwise, $|S_\varphi(x, \varepsilon_1\alpha)| = 0$ because $|S_\varphi(x_j, t_j)| \to 0$ as $j \to \infty$, which is impossible.

If N is the number of the Monge-Ampère sections S_j such that $x \in \Theta^2 S_j$, then $N \leq C$. In fact, if S_{j_0} is the Monge-Ampère section of smallest Lebesgue measure among all the S_j for which $x \in \Theta^2 S_j$, then from (7.9)

$$C\,|S_{j_0}| \geq |S_\varphi(x, \varepsilon_1\alpha)| \geq \sum_{j : x \in \Theta^2 S_j} |S_j| \geq N|S_{j_0}|.$$

Since N is also the number of the Monge-Ampère sections $\Theta^2 S_j$ containing x,

$$N = \sum_{j \geq 1} \chi_{\Theta^2 S_j}(x) \leq C.$$

This proves (iv). Finally to prove (v), let $x \in \Theta^2 S_i \cap \Theta^2 S_j$. Then from inequality (7.8),

$$\frac{8}{9\Theta}\alpha < \frac{t_i}{\varepsilon} < \frac{10}{9}\Theta\alpha \quad \text{and} \quad \frac{8}{9\Theta}\alpha < \frac{t_j}{\varepsilon} < \frac{10}{9}\Theta\alpha.$$

So, we have

$$\frac{t_i}{t_j} \leq \frac{10}{8}\Theta^2 < 2\Theta^2 \quad \text{and} \quad \frac{8}{10\Theta^2} < \frac{t_i}{t_j}$$

Hence,

$$\frac{1}{2\Theta^2} < \frac{t_i}{t_j} < 2\Theta^2.$$

This completes the proof. □

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