Families and statistics of L-functions

by

Joshua Stucky

B. A., Union University, 2017

M. S., Kansas State University, 2019

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2022

Abstract

The first chapter of this dissertation provides a general introduction to the study of families of L-functions along with the necessary tools for understanding their behavior. In particular, we introduce the families studied in the second and third chapters of this dissertation and provide some prerequisite knowledge on these families.

The second chapter of this dissertation studies a family of L-functions attached to Hecke Grossencharacters and extends a geometric result of Ricci concerning the equidistribution of prime ideals of $\mathbb{Z}[i]$ in narrow sectors.

The third chapter of this dissertation studies a family of L-functions attached to automorphic forms on GL_2. Specifically, we investigate the sixth moment of the family of L-functions associated to holomorphic modular forms on GL_2 with respect to a congruence subgroup $\Gamma_1(q)$. We improve on previous work and obtain an unconditional upper bound of the correct order of magnitude.

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Major Professor
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Chapter 1

Introduction

1.1. Families of L-Functions

L-functions are among the central objects of study in modern analytic number theory, both because of the arithmetic problems they encode as well as for their own sake as interesting and mysterious mathematical objects. The prototypical example of an L-function is the Riemann zeta function, defined for $\mathrm{Re}\, s > 1$ by the Dirichlet series and Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is taken over all primes p. The function $\zeta(s)$ can be meromorphically continued to all of $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1. Moreover, the completed ζ function, defined by

$$\Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

where $\Gamma(s)$ is the standard gamma function, satisfies the functional equation

$$\Lambda(s) = \Lambda(1 - s).$$

In general, L-functions may be regarded as generalizations of ζ that possess many of the same properties, such as a Dirichlet series and Euler product representation as well as a functional equation.

Classically, the Riemann zeta function encodes the distribution of the prime numbers. If we denote by $\pi(x)$ the number of primes at most x, then the Prime Number Theorem states that

$$\pi(x) \sim \frac{x}{\log x}.$$

This is equivalent to the fact that ζ has no zeros ρ with $\operatorname{Re} \rho = 1$. Similarly, the existence of primes in short intervals $[x, x + x^\theta]$, $\theta < 1$, is closely connected with the distribution of the zeros of ζ through zero-density estimates. Through various analytic techniques, these can be connected with the *moments of* ζ,

$$I_{2k}(T) = \int_0^T \left|\zeta\left(\tfrac{1}{2} + it\right)\right|^{2k} dt.$$

Moments of L-functions play a crucial role in many number theoretic arguments, and thanks to the work of a number of authors, we have beautiful conjectures for the moments of many families of L-functions. These moments are closely related to *subconvexity estimates* for ζ, i.e. any estimate of the form

$$\left|\zeta\left(\tfrac{1}{2} + it\right)\right| \ll t^{\frac{1}{4} - \delta}$$

for some $\delta > 0$ (the exponent $\frac{1}{4}$ follows from the use of the Phragmen-Lindelöff convexity principle and the functional equation for ζ). It is conjectured that

$$I_{2k}(T) \sim C_k T (\log T)^{k^2}$$

for all integers $k \geq 1$. Note that this estimate implies the estimate

$$\left|\zeta\left(\tfrac{1}{2} + it\right)\right| \ll t^{\frac{1}{2k} + \varepsilon},$$

the implied constant depending only on k and ε.

The set

$$\mathcal{F} = \{\zeta(s+it) : 0 \leq t \leq T\}$$

is an example of a family of L-functions, and the moment I_{2k} can be regarded as the moment *at the central point* $s = \frac{1}{2}$. In general, moments of L-functions take the form

$$\sum_{f \in \mathcal{F}} L(\tfrac{1}{2}, f)^k,$$

where $L(s, f)$ is the L-function associated to f and $\mathcal{F}$ is some family of L-functions. In the case when the family $\mathcal{F}$ is continuous, the sum is an integral.

In Section 2, we discuss the family of L-functions we use in Chapter 2, giving some notation and definitions as well as briefly discussing the subconvexity estimate that is necessary for our arguments. In Section 3, we provide some background on the family of L-functions we study in Chapter 3.

1.2. Gaussian Primes and a Family of Grossencharacter L-Functions

We define the usual Gaussian integers as the ring of integers of the number field $\mathbb{Q}(i)$. That is, the Gaussian integers are the elements of

$$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}.$$

We are interested in the distribution of prime ideals in this ring. Since $\mathbb{Z}[i]$ is a principal ideal domain, the ideals are $(a + bi)$, where $a + bi$ is a Gaussian integer, and we may identify the ideals in $\mathbb{Z}[i]$ with the points $a+bi$ with, say, $a > 0$, $b \geq 0$. We say a Gaussian integer is prime if it generates a prime ideal in $\mathbb{Z}[i]$, and we call these *Gaussian primes*. When necessary, we may distinguish primes in $\mathbb{Z}$ by referring to them as *rational primes*. Classically, an ideal $\mathfrak{a}$ in $\mathbb{Z}[i]$ is prime if and only if one of the following hold:

1. $N\mathfrak{a} = p$ for some rational prime $p = 2$ or $p \equiv 1 \pmod 4$,

2. $\mathfrak{a} = (p)$ for $p \equiv 3 \pmod 4$.

We define the *norm* of an ideal to the field norm $N_{\mathbb{Q}(i)/\mathbb{Q}}$ of any one of its generators:

$$N\mathfrak{a} = N_{\mathbb{Q}(i)/\mathbb{Q}}(a + bi) = a^2 + b^2.$$

We define the *angular Grossencharacters*

$$\lambda^m(\mathfrak{a}) = e^{4im \arg(a+bi)}.$$

Here again, $a+bi$ is any one of the generators of $\mathfrak{a}$, and one may check that λ^m is independent of the choice of generator.

With the above notations, we define the Grossencharacter L-functions

$$L(s, \lambda^m) = \sum_{\mathfrak{a}} \frac{\lambda^m(\mathfrak{a})}{(N\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{\lambda^m(\mathfrak{p})}{(N\mathfrak{p})^s}\right)^{-1}, \qquad \text{Re } s > 1,$$

where the sum is over nonzero ideals and the product is over prime ideals.

The characters λ^m allow one to study the distribution of Gaussian primes in sectors, i.e. domains of $\mathbb{C}$ of the form

$$\{z \in \mathbb{C} : |4 \arg z - \theta| < \delta\},$$

where the absolute value is to be regarded mod 2π. The factor of 4 comes from the number of units in $\mathbb{Z}[i]$. As such, $\arg z$ may be regarded as mod $\frac{\pi}{2}$. Note that the harmonics λ^m appear as the Fourier coefficients of any function which is periodic mod $\frac{\pi}{2}$ (see Section 2.3 of Chapter 2).

Lastly, let us discuss the subconvexity estimate we need for our results in Chapter 2, namely Lemma 2.7.1, due originally to Ricci [19]. His result states that if $(4m^2 + t^2) \geq 4$, then

$$L\left(\tfrac{1}{2} + it, \lambda^m\right) \ll (m^2 + t^2)^{1/6} \log^3(m^2 + t^2).$$

This estimate should be compared to the Weyl subconvexity estimate for ζ, which is

$$\zeta\left(\tfrac{1}{2}+it\right) \ll t^{1/6+\varepsilon}.$$

In the case of ζ, the *conductor* of $\zeta\left(\frac{1}{2}+it\right)$ is of size $|t|$. Moreover, ζ is a degree 1 L-function, whereas $L(s,\lambda^m)$ is a degree 2 L-function, corresponding to the fact that $\mathbb{Q}(i)$ is a degree 2 extension of $\mathbb{Q}$. Since $L\left(\frac{1}{2}+it,\lambda^m\right)$ depends on the two parameters t and m, we see that $L\left(\frac{1}{2}+it,\lambda^m\right)$ has conductor $(|t|+|m|)^2 \asymp t^2+m^2$, and so we see that Ricci's estimate is the analogue of Weyl's subconvexity estimate for $L(s,\lambda^m)$.

1.3. The Family of GL_2 Automorphic L-Functions

In this section, we give some background on the family of L-functions studied in Chapter 3. We begin with the sets

$$\mathbb{H} = \left\{z = x+iy \in \mathbb{C} : y > 0\right\},$$

$$\Gamma(1) = SL_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc=1\right\}.$$

The group $\Gamma(1)$ acts on $\mathbb{H}$ on the left via the linear fractional transformations

$$\gamma z = \frac{az+b}{cz+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We are interested in functions f satisfying the following modularity condition:

$$f(\gamma z) = (cz+d)^k f(z) \tag{3.1}$$

for some integer $k \geq 0$ andγ belonging to certain subgroups of $\Gamma(1)$. If we define the weight-k slash operator by

$$f[\gamma]_k(z) = \frac{f(\gamma z)}{(cz+d)^k},$$

then the modularity condition (3.1) is equivalent to the invariance condition

$$f[\gamma]_k(z) = f(z).$$

1.3.1. **Congruence Subgroups**

To define these subgroups, we first define *principal congruence group of level* q by

$$\Gamma(q) = \left\{\gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{q}\right\}.$$

A *congruence subgroup* of $\Gamma(1)$ is then any subgroup which contains $\Gamma(q)$. Two important examples are

$$\Gamma_0(q) = \left\{\gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{q}\right\},$$

$$\Gamma_1(q) = \left\{\gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{q}\right\},$$

The parameter q above is called the *level* of the congruence subgroup. Note that congruence subgroups above contain the element

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and thus if f satisfies the transformation law (3.1), then $f(Tz) = f(z+1) = f(z)$. Thus f admits a Fourier series expansion of the form

$$f(z) = \sum_{n \in \mathbb{Z}} \lambda_f(n) n^{\frac{k-1}{2}} q^n, \tag{3.2}$$

where $q = e^{2\pi i z}$. As a caution to the reader, the use of q both as the level and as the variable $q = e^{2\pi i z}$ is unfortunate, but the usage is standard and there should be no ambiguity in context. Moreover, we shall not use the latter notation very much in the remainder of this dissertation.

1.3.2. **Holomorphic Modular Forms**

The functions we consider in Chapter 3 are examples of *modular forms* of weight k with respect to $\Gamma_1(q)$. We define holomorphic modular forms for generalizations of the spaces $\Gamma_0(q)$, into which $\Gamma_1(q)$ can be decomposed as a direct sum.

Definition. Let k, q be a positive integers and let χ be a Dirichlet character mod q. A function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k, level q and central character χ if f satisfies the following conditions:

1. f is holomorphic on $\mathbb{H}$,
2. $f[\gamma]_k(z) = \chi(d)f(z)$ for all $\gamma \in \Gamma_0(q)$,
3. f is holomorphic at every cusp α of $\Gamma_0(q)$.

To understand the last condition, we need to define the cusps of a congruence subgroup Γ. For a rational number m, the action of Γ on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ is given by

$$\gamma(m) = \frac{am + b}{cm + d}, \quad \gamma \in \Gamma.$$

Here $\gamma(\infty) = \frac{a}{c}$ and $\gamma(m) = \infty$ if $cm + d = 0$. If $c = 0$, then γ fixes ∞. The *cusps* of a congruence subgroup Γ are then the Γ-orbits of $\mathbb{P}^1(\mathbb{Q})$. It is nontrivial to show that this set is finite.

To define holomorphy at the cusps of Γ, we first define holomorphy at ∞. We say that f is holomorphic at ∞ if $\lambda_f(n) = 0$ for all $n < 0$. Alternatively, we may think of the function $q = q(z) = e^{2\pi i z}$ as a function from the upper half plane $\mathbb{H}$ to the punctured unit disk $\mathbb{D}^*$. Holomorphy at ∞ then means that the function $F(q) = F(q(z)) = f(z)$ extends to a function which is holomorphic at 0.

For a cusp α, let σ_α be the element of Γ such that $\sigma\infty = \alpha$. As before, the function $f[\sigma_\alpha]_k$ is periodic of period 1, and so can be defined as a function on a punctured unit disk, say $f[\sigma_\alpha]_k(z) = G(q)$. The modular form f is then holomorphic at the cusp α if G is extends to a function which is holomorphic at 0.

The set of all modular forms of weight k, level q, and central character χ form a complex vector space which we denote by $\mathcal{M}_k(\Gamma_0(q), \chi)$. An important subspace is the set of *cusp forms* of weight k, denoted by $\mathcal{S}_k(\Gamma_0(q), \chi)$. This set is defined by altering the third condition in the definition above: instead of requiring holomorphy at the cusps, we require the modular forms in $\mathcal{S}_k$ to vanish at the cusps (defined similarly). It follows that $\lambda_f(0) = 0$ for all cusp forms.

We note that if we take $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, then $(-1)^k f(z) = f[\gamma]_k(z) = \chi(-1)f(z)$, and so f is identically 0 unless $\chi(-1) = (-1)^k$, which we will now always assume.

One important feature of the spaces defined above is that we have the decompositions

$$\mathcal{M}_k(\Gamma_1(q)) = \bigoplus_{\substack{\chi(q) \\ \chi(-1)=(-1)^k}} \mathcal{M}_k(\Gamma_0(q), \chi),$$

$$\mathcal{S}_k(\Gamma_1(q)) = \bigoplus_{\substack{\chi(q) \\ \chi(-1)=(-1)^k}} \mathcal{S}_k(\Gamma_0(q), \chi).$$

Here $\mathcal{M}_k(\Gamma_1(q))$ and $\mathcal{S}_k(\Gamma_1(q))$ are defined analogously as the previous spaces, except that the transformation condition 2. is just $f[\gamma]_k(z) = f(z)$ for all $\gamma \in \Gamma_1(q)$.

1.3.3. **Hecke Eigenforms**

For a holomorphic modular form $f \in \mathcal{M}_k(\Gamma_0(q), \chi)$, we have

$$f(z) = \sum_{n \geq 0} \lambda_f(n) n^{\frac{k-1}{2}} e(nz).$$

The *Hecke L-function* associated to f is the Dirichlet series

$$L(f,s) = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s}.$$

Hecke showed that this L-function can be completed in an analogous way to the Riemann zeta function, and the resulting completed L-function satisfies a functional equation. However, we would also like our L-functions to have an Euler product representation. For these, we need the *Hecke operators*.

Let $k, q \geq 1$ and χ be a Dirichlet character with $\chi(-1) = (-1)^k$. The nth Hecke operator is defined by

$$T_n f(z) = \frac{1}{n} \sum_{ad=n} \chi(a) \sum_{0\leq b<d} f\left(\frac{az+b}{d}\right).$$

These operators satisfy a number of important properties, which we summarize in the following lemma.

Lemma 1.3.1.

1. *T_n takes modular forms to modular forms and cusp forms to cusp forms:*

$$T_n : \mathcal{M}_k(\Gamma_0(q),\chi) \to \mathcal{M}_k(\Gamma_0(q),\chi),$$

$$T_n : \mathcal{S}_k(\Gamma_0(q),\chi) \to \mathcal{S}_k(\Gamma_0(q),\chi).$$

2. *T_n acts on Fourier coefficients via*

$$T_n f(z) = \sum_{m\geq 0} \left(\sum_{d|(n,m)} \chi(d) \lambda_f\left(\frac{nm}{d^2}\right)\right) e(mz).$$

3. *The Hecke operators commute. More precisely*

$$T_m T_n = T_n T_m = \sum_{d|(n,m)} \chi(d) T_{\frac{mn}{d^2}},$$

and so in particular $T_m T_n = T_{mn}$ if $(m,n) = 1$.

4. *There is an orthonormal basis of the space $\mathcal{S}_k(\Gamma_0(q),\chi)$ of cusp forms consisting of eigenfunctions of all Hecke operators T_n with $(n,q)=1$.*

For a proof of this lemma, see Chapter 6 of [12] (Note that a different normalization is used in this reference, as the Fourier coefficients are defined by

$$f(z)=\sum_{n\in\mathbb{Z}}\lambda_f(n)q^n,$$

instead of (3.2)).

The last property is especially important, and the elements of this basis are called *Hecke eigenforms.* For a Hecke eigenform, let $\lambda(n)$ be such that

$$T_nf(z)=\lambda(n)f(z).$$

Comparing the first Fourier coefficients on both sides, we find that

$$\lambda(n)\lambda_f(1)=\lambda_f(n).$$

Normalizing so that the first Fourier coefficient is 1 (so long as $\lambda_f(1)\neq 0$. This is ensured, for instance, by assuming the level q is prime), we find that

$$\lambda_f(n)=\lambda(n).$$

That is, with this normalization, the nth Fourier coefficient of a Hecke eigenform *is* the Hecke eigenvalue of the nth Hecke operator. Moreover, the multiplicativity of the Hecke operators immediately implies that the Fourier coefficients of a Hecke eigenform are multiplicative, and so the associated L-function admits an Euler product. Specifically, if f is a Hecke eigenform and the level q is prime, then

$$L(f,s)=\sum_{n\geq 1}\frac{\lambda_f(n)}{n^s}=\prod_p\left(1-\frac{\lambda_f(p)}{p^s}+\frac{\chi(p)}{p^{2s}}\right).$$

Chapter 2

Gaussian Primes in Narrow Sectors

2.1. Introduction

A classical result of Huxley [9] states that for sufficiently large x and any $\theta > 7/12$, the interval $[x, x + x^\theta]$ contains a rational prime. In this chapter, we investigate an analogous problem about Gaussian primes. To be precise, let $\varphi \in \mathbb{R}$, $0 < \delta \leq \pi/2$, $0 < \theta \leq 1$, and x large. We are interested in the cardinality of the set

$$\left\{ a + bi \in \mathbb{Z}[i] : (a + bi) \text{ is prime}, \varphi < \arg(a + bi) \leq \varphi + \delta,\ x - x^\theta < a^2 + b^2 \leq x \right\}.$$

Here $(a + bi)$ denotes the ideal generated by $a + bi$. As is common in such problems, it is more convenient to count these ideals with a suitable weight. Denote by $\mathfrak{a}$ the ideal in $\mathbb{Z}[i]$ generated by $a + bi$ and by $N\mathfrak{a} = a^2 + b^2$ its norm. If we define

$$\Lambda(\mathfrak{a}) = \begin{cases} \log N\mathfrak{a} & \text{if } \mathfrak{a} = \mathfrak{p}^m \text{ with } \mathfrak{p} \text{ prime and } m \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

then our problem translates to obtaining an asymptotic estimate for

$$\psi(x, y; \varphi, \delta) = \sum_{\substack{x-y<N\mathfrak{a}\leq x \\ \varphi<\arg\mathfrak{a}\leq\varphi+\delta}} \Lambda(\mathfrak{a}).$$

Ricci [19] has shown that for all $\varepsilon > 0$ and $\delta \geq x^{-3/10+\varepsilon}$, one has

$$\psi(x, x; \varphi, \delta) \sim \frac{2\delta x}{\pi}.$$

We generalize this and prove the following

Theorem 2.1.1. *For any $\varepsilon > 0$, $\varphi \in \mathbb{R}$, x sufficiently large, $\theta > 7/10$, and $\delta x^\theta \geq x^{7/10+\varepsilon}$, we have*

$$\psi(x, x^\theta, \varphi, \delta) \sim \frac{2\delta x^\theta}{\pi}.$$

Geometrically, the parameters $x, \theta, \varphi, \delta$ describe a sector centered at the origin. The inner and outer radii of this sector are $\sqrt{x - x^\theta}$ and $\sqrt{x}$, and the sector is cut by rays emanating from the origin with angles φ and $\varphi + \delta$. Ricci's result gives the expected number of prime ideals in a sector so long as the inner radius $\sqrt{x - x^\theta}$ is essentially 0 and the angle δ between the rays is sufficiently wide. Theorem 2.1.1 claims the more general result that one obtains the expected number of prime ideals so long as the area of the sector is sufficiently large.

A note on the literature. It should be noted that Maknys [18] has claimed a result similar to Theorem 2.1.1, but with the exponent 11/16 in place of 7/10. However, Heath-Brown [8] has found an error in Maknys' proof of this result. He states that Maknys' proof, when corrected, yields the exponent $(221 + \sqrt{201})/320 = 0.7349...$. However, the result is potentially worse than 0.7349... because Maknys's proof depends on a zero density estimate (Theorem 2 of [17]), the proof of which also contains an error. In particular, there is an incorrect application of Theorem 1 of [16]. For a version of Theorem 1 of [16] which is applicable in the proof of Maknys' zero density result, see Theorem 6.2 and the end of Section 7 of [4].

Outline of the Proof To orient the reader, we provide an outline of the proof of Theorem 2.1.1. In Section 2.3, we begin by smoothing the angular and norm regions for $\psi(x, x^\theta, \varphi, \delta)$, and then express these regions via a sum of Hecke characters λ^m and an integral of $(N\mathfrak{a})^{it}$. The main term in Theorem 2.1.1 then arises from the contribution of the principal character. After applying an analogue of Heath-Brown's identity in $\mathbb{Z}[i]$ (see Lemma 2.2.6 below), we are left to bound a sum of $O((\log x)^{2J+2})$ expressions roughly of the form

$$\sum_{M \le m \le 2M} c_m \int_T^{2T} \tilde{V}\left(\tfrac{1}{2} + it\right) \sum_{\substack{\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_{2J} \\ N\mathfrak{a}_j \asymp N_j}} a_1(\mathfrak{a}_1) \cdots a_{2J}(\mathfrak{a}_{2J}) \frac{\lambda^m(\mathfrak{a})}{(N\mathfrak{a})^{1/2+it}} \, dt$$

for some parameters N_i. Here the c_m are Fourier coefficients and $\tilde{V}$ is a Mellin transform. Using estimates for c_m and $\tilde{V}$, this reduces to showing that

$$\sum_{M \le m \le 2M} \int_T^{2T} \left| F\left(\tfrac{1}{2} + it\right) \right| dt \ll \frac{x^{1/2}}{(\log x)^A},$$

where F is the Dirichlet series appearing in the penultimate display.

In Section 2.4, we reduce this to bounding the number R of pairs m, t for which a particular factor f of F attains a large value. Specifically, for such a pair m, t, we have

$$\left| \sum_{N\mathfrak{a} \asymp N} c(\mathfrak{a}) \lambda^m(\mathfrak{a}) (N\mathfrak{a})^{-it} \right| \gg W$$

for some divisor-bounded coefficients $c(\mathfrak{a})$ and $W > 0$. In Section 2.5, we use mean- and large-value estimates to bound R. Specifically, we use a hybrid large sieve estimate due to Coleman and an analogue of Huxley's large value result, also due to Coleman. Writing $G = \sum |c(\mathfrak{a})|^2$, these yield

$$R \ll NGW^{-2} + (M^2 + T^2) GW^{-2},$$

$$R \ll NGW^{-2} + (M^2 + T^2) NG^3 W^{-6},$$

respectively. We also use the "trivial" estimate

$$R \ll \min(M,T)NGW^{-2} + MTGW^{-2},$$

as well as a subconvexity result for the Hecke L-function, $L(s, \lambda^m)$, due to Ricci. There are a variety of ranges for N, M, T to consider when deciding which estimate to use. This requires a case analysis which is done in Sections 2.6 – 2.8. Here we also indicate the "worst cases" of N, M, T for which our estimates are sharp.

We note that with an optimal large sieve, one would have the estimate

$$R \ll NGW^{-2} + MTGW^{-2}. \tag{1.1}$$

Although such a large sieve is not available in the literature, this would not improve our results (it would, however, simplify the case analysis). This is because one of the worst cases in our analysis remains a worst case when using this estimate. See Section 2.9 for this discussion.

2.2. Notation and Preliminary Lemmas

We collect here some additional notation and lemmas we will need throughout the proof. The symbols $o, O, \ll, \gg, \asymp$ have their usual meanings. The letter ε denotes a sufficiently small positive real number, while A, B, C stand for an absolute positive constants, all of which may be different at each occurrence. For example, we may write

$$x^\varepsilon \log x \ll x^\varepsilon, \qquad (\log x)^B (\log x)^B \ll (\log x)^B$$

Any statement in which ε occurs holds for each positive ε, and any implied constant in such a statement is allowed to depend on ε. The implied constants in any statement involving the letters A, B, C are also allowed to depend on these variables. We also define the generalized j divisor function $\tau_j(\mathfrak{a})$ on ideals to be the number of ways to write $\mathfrak{a}$ as the product of j

nonzero ideals.

Similar to $\Lambda(\mathfrak{a})$, we define

$$\mu(\mathfrak{a}) = \begin{cases} (-1)^r & \text{if } \mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r \text{ with } \mathfrak{p}_i \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\arg \mathfrak{a}$ be the argument of any one of the generators of $\mathfrak{a}$ (which is unique mod $\pi/2$). For $m \in \mathbb{Z}$, we define the angular Hecke characters

$$\lambda^m(\mathfrak{a}) = e^{4im \arg \mathfrak{a}} = \left(\frac{\alpha}{|\alpha|}\right)^{4m},$$

which are primitive with conductor (1). Note that the character is well-defined since the particular generator α chosen for the definition above is immaterial. From these we get the Hecke L-functions, defined for $\operatorname{Re} s > 1$ by

$$L(s, \lambda^m) = \sum_{\mathfrak{a}} \frac{\lambda^m(\mathfrak{a})}{(N\mathfrak{a})^s}.$$

Here the sum is over all nonzero ideals of $\mathbb{Z}[i]$. These L-functions are absolutely convergent for $\operatorname{Re} s > 1$, and Hecke showed that, for $m \neq 0$, they have analytic continuation to all of $\mathbb{C}$ and satisfy a functional equation. We also have

$$\frac{1}{L(s, \lambda^m)} = \sum_{\mathfrak{a}} \frac{\lambda^m(\mathfrak{a})\mu(\mathfrak{a})}{(N\mathfrak{a})^s}, \qquad -\frac{L'(s, \lambda^m)}{L(s, \lambda^m)} = \sum_{\mathfrak{a}} \frac{\lambda^m(\mathfrak{a})\Lambda(\mathfrak{a})}{(N\mathfrak{a})^s},$$

which are also absolutely convergent for $\operatorname{Re} s > 1$. We summarize these facts in the following

Lemma 2.2.1. *The function $L(s, \lambda^m)$ satisfies the functional equation*

$$L(s, \lambda^m) = \gamma(s, \lambda^m) L(1 - s, \lambda^m), \tag{2.1}$$

where

$$\gamma(s,\lambda^m) = \pi^{2s-1}\frac{\Gamma(1-s+2\,|m|)}{\Gamma(s+2\,|m|)}.$$

If $m \neq 0$, then $L(s,\lambda^m)$ is entire, and otherwise it is meromorphic with a simple pole at $s = 1$ with residue $\frac{\pi}{4}$. We also have

$$L(s,\lambda^m) = L(s,\lambda^{-m}). \tag{2.2}$$

These results are standard. See [13], for instance. We will need several results on the behavior of these functions in the critical strip. These are given in the following pair of lemmas.

Lemma 2.2.2. *Let $V = (4m^2+t^2)^{1/2}$. Then there exist absolute constants $C,\delta > 0$ such that*

$$L(\sigma+it,\lambda^m) \ll V^{c(1-\sigma)^{3/2}}(\log V)^{2/3},$$

uniformly for $1-\delta < \sigma < 1$. It follows that there exists an absolute constant $C > 0$ such that $L(s,\lambda^m)$ has no zeros in the region

$$\sigma \geq 1 - C(\log V)^{-2/3}(\log\log V)^{-1/3}. \tag{2.3}$$

Lemma 2.2.3. *For σ in the region (2.3), we have*

$$\frac{L'(\sigma+it,\lambda^m)}{L(\sigma+it,\lambda^m)} \ll \log V, \qquad \frac{1}{L(\sigma+it,\lambda^m)} \ll \log V.$$

Lemma 2.2.2 follows from Theorems 1 and 2 of [3], and the proof of Lemma 2.2.3 follows closely the proof of Theorem 3.11 of [21]. Next, we need an estimate for the number of lattice points in a suitably regular sector.

Lemma 2.2.4. *Let $\varphi \in \mathbb{R}$, x and y be sufficiently large with $x^{1/2} \leq y \leq x$, and $x^{-1/2} \leq \delta \leq$*

$\pi/2$. If

$$\mathcal{N}(x,y,\varphi,\delta) = \#\left\{a+bi \in \mathbb{Z}[i] : \varphi \le \arg(a+bi) \le \varphi+\delta,\ x-y \le a^2+b^2 \le x\right\},$$

then $\mathcal{N}(x,y,\varphi,\delta) \ll \delta y$.

Lemma 2.2.5. *Let $\tau_j(\mathfrak{a})$ be the j-divisor function for $\mathbb{Z}[i]$. We have $\tau_j(\mathfrak{a}) \ll (N\mathfrak{a})^{\varepsilon}$, and for $y > x^{1/2}$ we also have*

$$\sum_{x-y<N\mathfrak{a}\le x} \tau_j(\mathfrak{a}) \ll y(\log x)^{j-1}.$$

For $\varphi \in \mathbb{R}$ and $x^{-1/2} < \delta \le \pi/2$, we also have

$$\sum_{\substack{x-y<N\mathfrak{a}\le x \\ \varphi\le\arg\mathfrak{a}\le\varphi+\delta}} \tau_j(\mathfrak{a}) \ll \delta y x^{\varepsilon}.$$

The implied constants above depend only on ε and j.

The proof of Lemma 2.2.4 is straightforward, and Lemma 2.2.5 follows from Shiu's work [20]. Our analysis makes use of an analogue of Heath-Brown's identity in $\mathbb{Z}[i]$ (see [7]). For technical reasons, it is more convenient to have a smoothed version of this identity. As such, let W be a smooth function supported on $[\frac{1}{2}, 2]$ such that

$$\sum_{n\ge 0} W(2^n t) = 1 \qquad \text{and} \qquad W^j(t) \ll t^{-j}$$

for all $0 < t \le 1$. Then we have the following

Lemma 2.2.6 (Heath-Brown's Identity)**.** *Let $X > 1$ and J be a positive integer, and let W*

be as above. Then for any $\mathfrak{a}$ with $N\mathfrak{a} \le X^J$, we have

$$\Lambda(\mathfrak{a}) = \sum_{j=1}^{J} \binom{J}{j} (-1)^{j-1} \sum_{\mathfrak{a}_1\cdots\mathfrak{a}_{2J}=\mathfrak{a}} \log(\mathfrak{a}_1)\mu(\mathfrak{a}_{J+1})\cdots\mu(\mathfrak{a}_{2J})$$
$$\times \sum_{\substack{n_1,\ldots,n_j\ge 0 \\ n_{J+1},\ldots,n_{J+j}\ge 0}} W\left(\frac{N\mathfrak{a}_1}{X^J/2^{n_1}}\right)\cdots W\left(\frac{N\mathfrak{a}_j}{X^J/2^{n_j}}\right) W\left(\frac{N\mathfrak{a}_{J+1}}{X/2^{n_{J+1}}}\right)\cdots W\left(\frac{N\mathfrak{a}_{J+j}}{X/2^{n_{J+j}}}\right)$$
$$\times W(N\mathfrak{a}_{j+1})\cdots W(N\mathfrak{a}_J) W(N\mathfrak{a}_{J+j+1})\cdots W(N\mathfrak{a}_{2J}).$$

Note that the terms on the last line simply force the ideals $\mathfrak{a}_{j+1},\ldots$ to have norm 1. The point of the lemma is that for $N\mathfrak{a} \le X^J$, the function $\Lambda(\mathfrak{a})$ can be decomposed into a linear combination of $O((\log X)^{2J})$ smooth sums of the form

$$\sum_{\mathfrak{a}_1\cdots\mathfrak{a}_{2J}=\mathfrak{a}} \log(\mathfrak{a}_1)\mu(\mathfrak{a}_{J+1})\cdots\mu(\mathfrak{a}_{2J}) W\left(\frac{N\mathfrak{a}_1}{N_1}\right)\cdots W\left(\frac{N\mathfrak{a}_{2J}}{N_{2J}}\right),$$

where $N_j = X^J/2^n$ or $X/2^n$ for some integer n, depending as $j \le J$.

2.3. Initial Decomposition

To estimate $\psi(x, x^\theta; \varphi, \delta)$, we begin by smoothing the angular region for $\mathfrak{a}$. For this, we need

Lemma 2.3.1. *Let $k \in \mathbb{Z}$ with $k \ge 0$ and let α, β, Δ, L be real numbers satisfying*

$$L > 0, \qquad 0 < \Delta < \frac{L}{2}, \qquad \Delta \le \beta - \alpha \le L - \Delta.$$

Then there exists an L-periodic function $P(x)$ with

$$P(t) = \frac{1}{L}(\beta - \alpha) + \sum_{m\ne 0} c_m e^{4imt}$$

which satisfies

$$P(t) = 1 \qquad \textit{if } t \in [\alpha, \beta],$$

$$P(t) = 0 \qquad \textit{if } t \in [\beta + \Delta, L + \alpha - \Delta],$$

$$P(t) \in [0,1] \quad \textit{for all } t,$$

and where the coefficients c_m *satisfy*

$$|c_m| \leq \begin{cases} \frac{1}{L}(\beta - \alpha), \\ \frac{L}{|m|}\left(\frac{kL}{\Delta\,|m|}\right)^k & \textit{if } m \neq 0, \end{cases} \tag{3.1}$$

where the factor involving k *is taken to equal 1 when* $k = 0$.

This result is classical. See, for example, Lemma A of Chapter 1, Section 2 of [15]. The special case $L = 1$ is proved there, but the lemma generalizes easily to arbitrary periods.

Let P be as in the lemma with $L = \frac{\pi}{2}$, $\alpha = \varphi$, $\beta = \varphi + \delta$, and $\Delta = \delta x^{-\varepsilon}$. Then

$$\psi(x, x^\theta; \varphi, \delta) = \sum_{x - x^\theta < N\mathfrak{a} \leq x} \Lambda(\mathfrak{a}) P(\arg \mathfrak{a}) + O\Bigg(\sum_{\substack{x - x^\theta < N\mathfrak{a} \leq x \\ \varphi - \Delta \leq \arg \mathfrak{a} \leq \varphi}} \Lambda(\mathfrak{a}) \Bigg) + O\Bigg(\sum_{\substack{x - x^\theta < N\mathfrak{a} \leq x \\ \varphi + \delta \leq \arg \mathfrak{a} \leq \varphi + \delta + \Delta}} \Lambda(\mathfrak{a}) \Bigg).$$

To estimate the error terms we note that the hypotheses of Theorem 2.1.1 imply that $x^\theta \gg x^{7/10+\varepsilon}$ and $\delta \gg x^{-3/10+\varepsilon}$. In particular, we have

$$x^\theta \geq x^{1/2} \qquad \text{and} \qquad \Delta \geq x^{-1/2}.$$

Since $\Lambda(\mathfrak{a}) \leq \log x$, we have by Lemma 2.2.4 that

$$\sum_{\substack{x - x^\theta < N\mathfrak{a} \leq x \\ \varphi - \Delta \leq \arg \mathfrak{a} \leq \varphi}} \Lambda(\mathfrak{a}) \ll (\log x) \mathcal{N}(x, x^\theta, \varphi - \Delta, \varphi) \ll (\log x) x^\theta \Delta = o(\delta x^\theta),$$

and similarly for the other error term. We expand $P(\arg\mathfrak{a})$ using its Fourier series and write

$$\sum_{x-x^\theta<N\mathfrak{a}\leq x}\Lambda(\mathfrak{a})P(\arg\mathfrak{a})=\sum_{x-x^\theta<N\mathfrak{a}\leq x}\Lambda(\mathfrak{a})\sum_m c_m\lambda^m(\mathfrak{a}).$$

We have

$$\sum_{x-x^\theta<N\mathfrak{a}\leq x}\Lambda(\mathfrak{a})=2\sum_{\substack{x-x^\theta<p\leq x\\ p\equiv 1 \pmod 4}}\log p+O(x^{1/2}\log x),$$

(see, for instance, display 7.4 in Chapter 2 of [19] for this computation). Since $x^\theta \gg x^{7/12+\varepsilon}$, the Siegel-Walfisz theorem in short intervals gives

$$2\sum_{\substack{x-x^\theta<p\leq x\\ p\equiv 1 \pmod 4}}\log p=x^\theta(1+o(1)),$$

and since $c_0=2\delta\pi^{-1}$, we obtain

$$\psi(x,x^\theta,\varphi,\delta)=\frac{2\delta x^\theta}{\pi}(1+o(1))+\sum_{x-x^\theta<N\mathfrak{a}\leq x}\Lambda(\mathfrak{a})\sum_{m\neq 0}c_m\lambda^m(\mathfrak{a}).$$

Using (3.1), we truncate the Fourier series at M_1 to obtain

$$\psi(x,x^\theta,\varphi,\delta)=\frac{2\delta x^\theta}{\pi}(1+o(1))+\sum_{x-x^\theta<N\mathfrak{a}\leq x}\Lambda(\mathfrak{a})\sum_{1\leq|m|\leq M_1}c_m\lambda^m(\mathfrak{a})+O\left(x^\theta\log x\left(\frac{\pi k x^\varepsilon}{2\delta M_1}\right)^k\right)$$

for any $k\geq 1$. Choosing $M_1=\delta^{-1}x^\varepsilon$, a sufficiently large choice of k depending only on ε makes the error term $o(\delta x^\theta)$, and so

$$\psi(x,x^\theta,\varphi,\delta)=\frac{2\delta x^\theta}{\pi}(1+o(1))+\sum_{1\leq|m|\leq M_1}c_m\sum_{x-x^\theta<N\mathfrak{a}\leq x}\Lambda(\mathfrak{a})\lambda^m(\mathfrak{a}).$$

Next, we smooth the norm-region for $\mathfrak{a}$. Let V be a smooth function satisfying

$$\begin{aligned} V(t) &= 1 && \text{if } t \in [x - x^\theta, x], \\ V(t) &= 0 && \text{if } t \in \mathbb{R} \smallsetminus [x - x^\theta - x^{\theta-\varepsilon}, x + x^{\theta-\varepsilon}], \\ V(t) &\in [0,1] && \text{for all } t. \end{aligned}$$

Then $\tilde{V}$ satisfies

$$\tilde{V}(s) \ll x^{\theta+\sigma-1} \qquad \text{and} \qquad \tilde{V}(s) \ll \frac{x^{\sigma+(A-1)(1-\theta+\varepsilon)}}{(1+|t|)^A} \tag{3.2}$$

for any real $A \geq 1$, where the implied constant depends only on A and σ. We obtain

$$\psi(x, x^\theta, \varphi, \delta) = \frac{2\delta x^\theta}{\pi}(1+o(1)) + \sum_{1\leq|m|\leq M_1} c_m \sum_{\mathfrak{a}} \Lambda(\mathfrak{a}) V(N\mathfrak{a}) \lambda^m(\mathfrak{a}) = \frac{2\delta x^\theta}{\pi}(1+o(1)) + \mathcal{S},$$

say, where the error in replacing the sharp cutoff with the smoothing function V has been absorbed into the error term $o(\delta x^\theta)$.

We now employ Lemma 2.2.6 with $X = (2x)^{1/J}$ for some integer $J \geq 1$ to be chosen. Then $\mathcal{S}$ is a linear combination of $O((\log x)^{2J})$ sums of the form

$$S = \sum_{1\leq|m|\leq M_1} c_m \sum_{\mathfrak{a}=\mathfrak{a}_1\cdots\mathfrak{a}_{2J}} a_1(\mathfrak{a}_1)\cdots a_{2J}(\mathfrak{a}_{2J}) W_1(N\mathfrak{a}_1)\cdots W_{2J}(N\mathfrak{a}_{2J}) \lambda^m(\mathfrak{a}) V(N\mathfrak{a}), \tag{3.3}$$

where

$$a_j(\mathfrak{a}) = \begin{cases} \log N\mathfrak{a} & \text{if } j = 1, \\ 1 & \text{if } 2 \leq j \leq J, \\ \mu(\mathfrak{a}) & \text{if } J+1 \leq j \leq 2J, \end{cases}$$

$W_j(k) = W(k/N_j)$, and $N_j = x/2^n$ or $X/2^n$ for some integer $n \geq 0$ depending as $j \leq J$.

It is natural to consider the Dirichlet series associated to the sums S. For each j and m, put

$$f_{j,m}(s) = \sum_{\mathfrak{a}} \frac{a_j(\mathfrak{a})\lambda^m(\mathfrak{a})W_j(N\mathfrak{a})}{(N\mathfrak{a})^s}$$

and also let

$$F_m(s) = \prod_{j=1}^{2J} f_{j,m}(s) = \sum_{\mathfrak{a}} \frac{a(\mathfrak{a})\lambda^m(\mathfrak{a})}{(N\mathfrak{a})^s},$$

where the coefficients satisfy

$$|a(\mathfrak{a})| \ll \tau_{2J}(\mathfrak{a}) \log x.$$

Then Mellin inversion gives

$$S = \frac{1}{2\pi i} \int_{(1/2)} \tilde{V}(s) \sum_{1\leq|m|\leq M_1} c_m F_m(s)\ ds.$$

For Re $(s) = \frac{1}{2}$, we have

$$F_m(s) \ll \log x \sum_{N\mathfrak{a}\leq 2x} \frac{\tau_{2J}(\mathfrak{a})}{(N\mathfrak{a})^{1/2}} \ll x^{1/2+\varepsilon}.$$

Also $|c_m| \ll \delta$. Truncating the integral at height T_1 and using (3.2) then gives

$$S = \frac{1}{2\pi i} \int_{1/2-iT_1}^{1/2+iT_1} \tilde{V}(s) \sum_{1\leq|m|\leq M_1} c_m F_m(s)\ ds + O\left(x^{1/2+\varepsilon} \frac{x^{1/2+(A-1)(1-\theta+\varepsilon)}}{T_1^{A-1}}\right)$$

for any $A \geq 1$. Choosing $T_1 = x^{1-\theta+\varepsilon}$ and taking A sufficiently large in terms of ε makes the error term negligible. We have $|c_m| \ll \delta$ and $|\tilde{V}\left(\frac{1}{2}+it\right)| \ll x^{\theta-1/2}$, so

$$S \ll \frac{\delta x^\theta}{x^{1/2}} \sum_{1\leq|m|\leq M_1} \int_{-T_1}^{T_1} \left|F_m\left(\tfrac{1}{2}+it\right)\right| dt \ll \frac{\delta x^\theta}{x^{1/2}} \sum_{1\leq m\leq M_1} \int_0^{T_1} \left|F_m\left(\tfrac{1}{2}+it\right)\right| dt,$$

the last inequality following from (2.2). We divide the ranges of m and t into dyadic intervals $[M, 2M]$ and $[T, 2T]$ for $M, T \geq 1$ along with the additional interval $[0, 1]$ for t. Theorem 2.1.1 now follows from

Lemma 2.3.2. *We have*

$$\sum_{M\le m\le 2M}\int_T^{2T}\left|F_m\left(\tfrac{1}{2}+it\right)\right|dt \ll \frac{x^{1/2}}{(\log x)^{2J+3}},$$

uniformly for $1\le M\le M_1$ *and* $1\le T\le T_1$. *The expression with an integral over* $[0,1]$ *also satisfies this bound.*

2.4. Reduction to Large Values

In this section, we reduce the proof of Lemma 2.3.2 to the estimation of the number of large values of a certain Dirichlet polynomials. We begin by letting Δ be a small parameter to be chosen and write $F_m(s)=G_m(s)H_m(s)$, where $H_m(s)$ is the product of those factors for which the lengths N_j satisfy $N_j\le x^{\Delta/J}$. Since

$$\left|f_{1,m}\left(\tfrac{1}{2}+it\right)\right|\ll N_1^{1/2}\log x;\qquad \left|f_{j,m}\left(\tfrac{1}{2}+it\right)\right|\ll N_j^{1/2},\ (j\ge 2),$$

we have

$$\left|H_m\left(\tfrac{1}{2}+it\right)\right|\ll Z^{1/2}\log x,$$

where Z is the product of those N_j with $N_j\le x^{\Delta/J}$. Then

$$\int_T^{2T}\sum_{M\le m\le 2M}\left|F_m\left(\tfrac{1}{2}+it\right)\right|dt \ll Z^{1/2}\log x\int_T^{2T}\sum_{M\le m\le 2M}\left|G_m\left(\tfrac{1}{2}+it\right)\right|dt. \tag{4.1}$$

We now bound the integral on the right (I, say) by a set of $O(T)$ well-spaced points t_n. We have

$$I\ll\sum_n\sum_{M\le m\le 2M}\left|G_m\left(\tfrac{1}{2}+it_n\right)\right|,$$

where $|t_l-t_n|\ge 1$ for $l\ne n$. For each triple j,m,n, let

$$\left|f_{j,m}\left(\tfrac{1}{2}+it_n\right)\right|=N_j^{\sigma(j,m,n)-1/2}(\log x)^4.$$

We need to show that $\sigma(j,m,n)$ cannot be too close to 1. We treat the case $j > J$, for which

$$f_{j,m}(s) = \sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})\lambda^m(\mathfrak{a})W_j(\mathfrak{a})}{(N\mathfrak{a})^s}.$$

The case $j \leq J$ would be very similar. By Mellin inversion

$$f_{j,m}\left(\tfrac{1}{2}+it\right) = \int_{(c)} L\left(\tfrac{1}{2}+it+s,\lambda^m\right)^{-1} N_j^s \tilde{W}_j(s)ds,$$

where $c = \frac{1}{2} + (\log x)^{-1}$. We have trivially that

$$\frac{1}{|L(1+(\log x)^{-1}+it,\lambda^m)|} \leq \zeta_K(1+(\log x)^{-1}) \ll \log x,$$

(here again ζ_K is the Dedekind zeta function for $\mathbb{Z}[i]$). Truncating the integral at height x^ε and using the rapid decay of $\tilde{W}$ gives

$$f_{j,m}\left(\tfrac{1}{2}+it\right) = \int_{c-ix^\varepsilon}^{c+ix^\varepsilon} L\left(\tfrac{1}{2}+it+s,\lambda^m\right)^{-1} N_j^s \tilde{W}_j(s)ds$$

with negligible error. We now use Lemmas 2.2.2 and 2.2.3 to move the line of integration to the left of $\mathrm{Re}\, s = \frac{1}{2}$. Then in the region

$$1-\eta \leq \mathrm{Re}\, w \leq \tfrac{1}{2}+c, \qquad |\mathrm{Im}\, w - t| \leq x,$$

where

$$\eta = C(\log x)^{-2/3}(\log\log x)^{-1/3},$$

we have

$$\frac{1}{L(w,\lambda^m)} \ll \log x$$

Moving the line of integration to $1/2-\eta$, we thus have

$$\left|f_{j,m}\left(\tfrac{1}{2}+it\right)\right| \ll (\log x) N_j^{1/2-\eta},$$

from which it follows that

$$\sigma(j, m, n) \le 1 - \eta$$

for x sufficiently large. We now split the available range for $\sigma(j, m, n)$ into $O(\log x)$ ranges $I_0 = (-\infty, \frac{1}{2})$ and

$$I_l = \left[\frac{1}{2} + \frac{l-1}{L}, \frac{1}{2} + \frac{l}{L}\right), \quad (1 \le l \le 1 + L/2,\ L = \lfloor \log x \rfloor),$$

For each j, l, let

$$C(j, l) = \left\{(m, t_n) : \max_{1 \le k \le 2J} \sigma(k, m, n) = \sigma(j, m, n) \text{ and } \sigma(j, m, n) \in I_l\right\}.$$

Since there are $O(\log x)$ classes $C(j, l)$, there must exist some class $\mathcal{C}$ for which

$$I \ll (\log x) \sum_{(m,t) \in \mathcal{C}} \left|G_m\left(\tfrac{1}{2} + it\right)\right|.$$

For $(m, t) \in \mathcal{C}$, we have

$$\left|G_m\left(\tfrac{1}{2} + it\right)\right| = \prod N_j^{\sigma(j,m,n)-1/2} \le \prod N_j^{l/L} = Y^{l/L},$$

where Y is the product of the N_j with $N_j > x^{\Delta/J}$. To simplify notation, let

$$\sigma = \frac{1}{2} + \frac{l-1}{L}, \quad f_m(s) = f_{j,m}(s), \quad N = N_j, \quad R = \#\mathcal{C}. \tag{4.2}$$

If $l = 0$, then $I \ll MT \log x$, so (4.1) gives

$$\int_T^{2T} \sum_{M \le m \le 2M} \left|F_m\left(\tfrac{1}{2} + it\right)\right| dt \ll Z^{1/2} M_1 T_1 (\log x)^2 \ll \delta^{-1} x^{1-\theta+\Delta+\varepsilon} \ll \frac{x^{1/2}}{(\log x)^A}$$

since we may assume $\Delta < \frac{1}{5}$ and $x^\theta \delta > x^{7/10}$. If $l \geq 1$, we have

$$I \ll (Y^{\sigma-1/2}) R \log x,$$

and so

$$\int_T^{2T} \sum_{M \leq m \leq 2M} \left| F_m\left(\tfrac{1}{2} + it\right)\right| dt \ll Z^{1/2} Y^{\sigma-1/2} R (\log x)^2.$$

Now since

$$Z^{1/2} Y^{\sigma-1/2} \ll Z^{1/2} (xZ^{-1})^{\sigma-1/2} \ll x^{1/2} (Zx^{-1})^{1-\sigma} \ll x^{1/2+(2\Delta-1)(1-\sigma)},$$

we find that

$$\int_T^{2T} \sum_{M \leq m \leq 2M} \left| F_m\left(\tfrac{1}{2} + it\right)\right| dt \ll x^{1/2} (\log x)^2 \left(\frac{R}{x^{(1-2\Delta)(1-\sigma)}} \right). \tag{4.3}$$

It remains to estimate R. For each $(t, m) \in \mathcal{C}$, we have

$$\left| f_m\left(\tfrac{1}{2} + it\right)\right| \gg N^{\sigma-1/2}.$$

Since $\sigma \leq 1 - \eta/2$ we see that Lemma 2.3.2 follows from the bound

$$R \ll x^{(1-3\Delta)(1-\sigma)} (\log x)^B \tag{4.4}$$

for any fixed $B > 0$, since then the expression on the right of (4.3) is bounded by taking $\sigma = 1 - \eta/2$, and the definition of η allows us to save arbitrary powers of $\log x$. To deduce the requisite bound for R, it is sufficient to show that

$$R \ll (MT)^{10(1-\sigma)/3} (\log x)^B \tag{4.5}$$

uniformly in M, T, σ, since $MT \leq M_1 T_1 = x^{1-\theta+\varepsilon} \delta^{-1} \leq x^{3/10-\varepsilon}$.

2.5. Mean and Large Value Results

To estimate R (see (4.2)), we will need several mean-value results of the form

$$\sum_{|m|\leq M} \int_{-T}^{T} \left| \sum_{N\mathfrak{a}\asymp N} c(\mathfrak{a})\lambda^m(\mathfrak{a})(N\mathfrak{a})^{-it} \right|^2 dt \ll \mathcal{D} \sum_{N\mathfrak{a}\asymp N} |c(\mathfrak{a})|^2 \tag{5.1}$$

for some $\mathcal{D} = \mathcal{D}(N, M, T)$, where $c(\mathfrak{a})$ are arbitrary complex coefficients defined on the ideals of $\mathbb{Z}[i]$. First, we have Coleman's hybrid large sieve (Theorem 6.2 of [4]).

Lemma 2.5.1 (Coleman)**.** *The estimate (5.1) holds with*

$$\mathcal{D} = M^2 + T^2 + N. \tag{5.2}$$

Additionally, we also have the following trivial estimate.

Lemma 2.5.2. *The estimate (5.1) holds with*

$$\mathcal{D} = MT + N \min(M, T). \tag{5.3}$$

Proof. For the case $T \leq M$, see [19], Theorem C. For the other case, the mean-value theorem for Dirichlet polynomials gives

$$\int_{-T}^{T} \left| \sum_{N\mathfrak{a}\asymp N} c(\mathfrak{a})\lambda^m(\mathfrak{a})(N\mathfrak{a})^{-it} \right|^2 dt = (T + O(N)) \sum_{N\mathfrak{a}\asymp N} |c(\mathfrak{a})|^2 .$$

Summing over m gives the other estimate.

□

Note that in each of the estimates above, the integral over t can be replaced by a sum over well-spaced points at the cost of a logarithmic factor, which will not affect our results.

For the problem at hand, the natural quantity to work with is MT, rather than the minimum or maximum of M and T. To this end, let

$$\mathcal{L} = \mathcal{L}(M,T) = \frac{|\log(M/T)|}{\log MT}$$

so that

$$\max(M^2,T^2) = (MT)^{1+\mathcal{L}} \qquad \text{and} \qquad \min(M^2,T^2) = (MT)^{1-\mathcal{L}}.$$

We will regard $\mathcal{L}$ as an arbitrary parameter assuming values in $[0,1]$. The estimates (5.2) and (5.3) become, respectively

$$(MT)^{1+\mathcal{L}} + N \qquad \text{and} \qquad MT + N(MT)^{(1-\mathcal{L})/2}.$$

We will apply these estimates to suitable powers of the polynomial $f_m\left(\frac{1}{2}+it\right)$. For any integer $g \geq 1$, we have

$$RN^{g(2\sigma-1)} \ll \sum_{(m,t)\in\mathcal{C}} \left| \sum_{\mathfrak{a}} \frac{a(\mathfrak{a})W(N\mathfrak{a})\lambda^m(\mathfrak{a})}{(N\mathfrak{a})^{1/2+it}} \right|^{2g} \ll \mathcal{D}(N^g,M,T) \sum_{N\mathfrak{a}\asymp N} \frac{|b(\mathfrak{a})|^2}{N\mathfrak{a}},$$

say where $|b(\mathfrak{a})| \leq \tau_g(\mathfrak{a})(\log x)^g$. Using Lemma 2.2.5 and partial summation, we find that the coefficient sum on the right is $O((\log x)^B)$ for some B which depends on g. Since g is bounded in terms of Δ, we find that B and the implied constant depend at most on our choice of Δ. Thus

$$RN^{g(2\sigma-1)} \ll \left(MT + N^g(MT)^{(1-\mathcal{L})/2}\right)(\log x)^B,$$

$$RN^{g(2\sigma-1)} \ll \left((MT)^{1+\mathcal{L}} + N^g\right)(\log x)^B.$$

We will also make use of the following large values result of Coleman (Theorem 7.3 of [4] with $\theta = 0$) which is proved using Huxley's subdivision method:

$$R \ll \left(N^{2g(1-\sigma)} + (M^2+T^2)N^{g(4-6\sigma)}\right)(\log x)^B \ll \left(N^{2g(1-\sigma)} + (MT)^{1+\mathcal{L}}N^{g(4-6\sigma)}\right)(\log x)^B.$$

For any integer $g \geq 1$, the estimates above give

$$R \ll \left((MT)^{1+\mathcal{L}} N^{g(1-2\sigma)} + N^{2g(1-\sigma)}\right) (\log x)^B, \tag{5.4}$$

$$R \ll \left(MTN^{g(1-2\sigma)} + (MT)^{(1-\mathcal{L})/2} N^{2g(1-\sigma)}\right) (\log x)^B, \tag{5.5}$$

$$R \ll \left((MT)^{1+\mathcal{L}} N^{g(4-6\sigma)} + N^{2g(1-\sigma)}\right) (\log x)^B. \tag{5.6}$$

The last estimate is useful only when $\sigma \geq 3/4$, and any time it is used, σ will be assumed to lie in this range. In each of the estimates above, the first summand decreases in g, and the second increases. Writing $N = (MT)^\beta$, one would like to choose

$$g = \frac{1+\mathcal{L}}{\beta}, \quad \frac{1+\mathcal{L}}{2\beta}, \quad \frac{1+\mathcal{L}}{2\beta(2\sigma-1)}, \tag{5.7}$$

respectively, so as to equalize the two summands in each estimate.

Unfortunately, g must be chosen to be an integer, and this adds a fair amount of complication to our analysis. The optimal choices for g in (5.4) – (5.6) are obtained by taking the floor of the values in (5.7), or 1 plus the floor. Thus, unconditionally, we have $R \ll (MT)^{\min(\alpha_1,\ldots,\alpha_6)}$, where

$$\begin{aligned}
\alpha_1(\mathcal{L},\beta,\sigma) &= 1+\mathcal{L}+\beta\left\lfloor\frac{1+\mathcal{L}}{\beta}\right\rfloor(1-2\sigma),\\
\alpha_2(\mathcal{L},\beta,\sigma) &= 2\beta\left\lfloor\frac{1+\mathcal{L}}{\beta}+1\right\rfloor(1-\sigma),\\
\alpha_3(\mathcal{L},\beta,\sigma) &= 1+\beta\left\lfloor\frac{1+\mathcal{L}}{2\beta}\right\rfloor(1-2\sigma),\\
\alpha_4(\mathcal{L},\beta,\sigma) &= \frac{1-\mathcal{L}}{2}+2\beta\left\lfloor\frac{1+\mathcal{L}}{2\beta}+1\right\rfloor(1-\sigma),\\
\alpha_5(\mathcal{L},\beta,\sigma) &= 1+\mathcal{L}+\beta\left\lfloor\frac{1+\mathcal{L}}{2\beta(2\sigma-1)}\right\rfloor(4-6\sigma),\\
\alpha_6(\mathcal{L},\beta,\sigma) &= 2\beta\left\lfloor\frac{1+\mathcal{L}}{2\beta(2\sigma-1)}+1\right\rfloor(1-\sigma),
\end{aligned} \tag{5.8}$$

where $\alpha_1, \alpha_3, \alpha_5$ apply only when the expression in the floor brackets is at least 1. We also define $\mathcal{A}_0$ to be the minimum of these six estimates and $\mathcal{A}_i = \min(\alpha_{2i-1}, \alpha_{2i})$ for $i = 1, 2, 3$.

Our analysis now proceeds by fixing β and σ and understanding the behavior of $\mathcal{A}_0$ as $\mathcal{L}$ ranges between 0 and 1. For this, we will need the following propositions which describe the behavior of $\mathcal{A}_i$ for $i = 1, 2, 3$. The proofs of these propositions are very similar, so we only prove Proposition 2.5.3. For notational brevity, we also suppress the dependence of α_i and $\mathcal{A}_i$ on β and σ.

Proposition 2.5.3. *Fix $\beta \in (0, \frac{5}{6})$ and $\sigma \in (\frac{7}{10}, \frac{3}{4})$. For $n \in \mathbb{Z}$, define*

$$\mathcal{L}_n^{(1,d)} = \beta n - 1 \qquad \text{and} \qquad \mathcal{L}_n^{(1,e)} = \mathcal{L}_n^{(1,d)} + 2\beta(1-\sigma).$$

Then on $[0,1] \cap \left[\mathcal{L}_n^{(1,d)}, \mathcal{L}_{n+1}^{(1,d)}\right)$, we have

$$\mathcal{A}_1(\mathcal{L}) = \begin{cases} 1 + \mathcal{L} + \beta n(1-2\sigma) & \text{if } \mathcal{L} \leq \mathcal{L}_n^{(1,e)}, \\ 2\beta(n+1)(1-\sigma) & \text{if } \mathcal{L} \geq \mathcal{L}_n^{(1,e)}. \end{cases}$$

In particular, $\mathcal{A}_1(\mathcal{L})$ is a continuous non-decreasing function of $\mathcal{L}$ on $[0,1]$.

Proposition 2.5.4. *Fix $\beta \in (0, \frac{5}{3})$ and $\sigma \in (\frac{7}{10}, 1)$. For $n \in \mathbb{Z}$, define*

$$\mathcal{L}_n^{(2,d)} = 2\beta n - 1 \qquad \text{and} \qquad \mathcal{L}_n^{(2,e)} = \mathcal{L}_n^{(2,d)} + 4\beta(1-\sigma).$$

Then on $[0,1] \cap \left[\mathcal{L}_n^{(2,d)}, \mathcal{L}_{n+1}^{(2,d)}\right)$, we have

$$\mathcal{A}_2(\mathcal{L}) = \begin{cases} 1 + \beta n(1-2\sigma) & \text{if } \mathcal{L} \leq \mathcal{L}_n^{(2,e)}, \\ \frac{1-\mathcal{L}}{2} + 2\beta(n+1)(1-\sigma) & \text{if } \mathcal{L} \geq \mathcal{L}_n^{(2,e)}. \end{cases}$$

In particular, $\mathcal{A}_2(\mathcal{L})$ is a continuous non-increasing function of $\mathcal{L}$ on $[0,1]$.

Proposition 2.5.5. *Fix $\beta \in (0, \frac{5}{6})$ and $\sigma \in (\frac{3}{4}, 1)$. For $n \in \mathbb{Z}$, define*

$$\mathcal{L}_n^{(3,d)} = 2\beta(2\sigma-1)n - 1 \qquad \text{and} \qquad \mathcal{L}_n^{(3,e)} = \mathcal{L}_n^{(3,d)} + 2\beta(1-\sigma).$$

Then on $[0,1] \cap \left[\mathcal{L}_n^{(3,d)}, \mathcal{L}_{n+1}^{(3,d)}\right)$*, we have*

$$\mathcal{A}_3(\mathcal{L}) = \begin{cases} 1 + \mathcal{L} + \beta n(4 - 6\sigma) & \textit{if } \mathcal{L} \leq \mathcal{L}_n^{(3,e)}, \\ 2\beta(n+1)(1-\sigma) & \textit{if } \mathcal{L} \geq \mathcal{L}_n^{(3,e)}. \end{cases}$$

In particular, $\mathcal{A}_3(\mathcal{L})$ *is a continuous non-decreasing function of* $\mathcal{L}$ *on* $[0,1]$.

As a remark on notation, the superscripts d and e appearing above are no parameters, but rather indicate that the variables using these superscripts are points of discontinuity or equality, respectively.

Proof of Proposition 2.5.3. A short computation shows that the solutions to $\alpha_1(\mathcal{L}) = \alpha_2(\mathcal{L})$ are given by

$$\mathcal{L}_m^{(1,e)} = m\beta - 1 + 2\beta(1-\sigma), \qquad \text{for } \frac{1}{\beta} - 2(1-\sigma) \leq m \leq \frac{2}{\beta} - 2(1-\sigma),$$

and that the points of discontinuity of $\mathcal{A}_1(\mathcal{L})$ are given by

$$\mathcal{L}_n^{(1,d)} = n\beta - 1, \qquad \frac{1}{\beta} \leq n \leq \frac{2}{\beta}.$$

Since $\mathcal{L}_{m+1}^{(1,e)} - \mathcal{L}_m^{(1,e)} = \mathcal{L}_{n+1}^{(1,d)} - \mathcal{L}_n^{(1,d)} = \beta$, and since $\sigma \neq 1$, there is a unique point of intersection, say $\mathcal{L}_{m_n}^{(1,e)}$, between each pair $\mathcal{L}_n^{(1,d)}$, $\mathcal{L}_{n+1}^{(1,d)}$ of points of discontinuity, and it is easy to check that $m_n = n$. Moreover, for a fixed value of $\left\lfloor \frac{1+\mathcal{L}}{\beta} \right\rfloor$, i.e. on the interval between two points of discontinuity, it is clear that α_1 increases in $\mathcal{L}$, and α_2 is constant. Thus $\mathcal{A}_1$ is non-decreasing on each interval $\left[\mathcal{L}_n^{(1,d)}, \mathcal{L}_{n+1}^{(1,d)}\right)$. Finally, we note that $\alpha_2\left(\mathcal{L}_{n-1}^{(1,d)}\right) = \alpha_2\left(\mathcal{L}_n^{(1,d)} - \varepsilon\right) = \alpha_1\left(\mathcal{L}_n^{(1,d)}\right)$ for all $\varepsilon > 0$ sufficiently small. Thus $\mathcal{A}_1$ is continuous, proving the last statement of the proposition.

□

It is worth noting that the results of these propositions extend to some slightly wider ranges of β and σ. For clarity of exposition, we have included only the ranges we need for our analysis. From these propositions, we can also deduce the following upper bounds, which have the benefit of being linear in $\mathcal{L}$.

Corollary 2.5.6. *For all* $\mathcal{L} \in [0,1]$, $\beta \in (0, \frac{2}{3})$, *and* $\sigma \in \left(\frac{7}{10}, 1\right)$, *we have* $\mathcal{A}_i \leq \mathcal{B}_i$, *where*

$$\mathcal{B}_1(\mathcal{L},\beta,\sigma) = 2(1+\mathcal{L}+\beta)(1-\sigma) - 4\beta(1-\sigma)^2,$$

$$\mathcal{B}_2(\mathcal{L},\beta,\sigma) = \left(\tfrac{1}{2}-\sigma\right)\left(1+\mathcal{L}-2\beta(1-\sigma)\right) + 2\beta(1-\sigma),$$

$$\mathcal{B}_3(\mathcal{L},\beta,\sigma) = \left(\frac{1-\sigma}{2\sigma-1}\right)\left(1+\mathcal{L}-2\beta(1-\sigma)\right) + 2\beta(1-\sigma).$$

Proof. The functions $\mathcal{B}_i$ are the linear interpolations of the points $\left(\mathcal{L}_m^{(i,e)}, \mathcal{A}_i\left(\mathcal{L}_m^{(i,e)}\right)\right)$. □

2.6. Short Polynomials

We are now ready to apply the estimates in Section 2.5 to estimate the quantity R. We will need a subconvexity estimate for Hecke L-functions (Lemma 2.7.1 below) to eliminate certain ranges of $\mathcal{L}, \beta, \sigma$. This will require the coefficients $a(\mathfrak{a})$ to be smooth, which is ensured by $N > X$. As such, the present section is devoted to the case $N \leq X$, where we do not require subconvexity. We divide into several cases.

Case 1.1: $MT \leq X$

Choose g so that

$$X^2 \leq N^g \leq X^3.$$

Then $(MT)^{1+\mathcal{L}} \leq (MT)^2 \leq X^2$, so by (5.4), we have

$$R \ll \left(X^{2+2(1-2\sigma)} + X^{6(1-\sigma)}\right)(\log x)^B \ll x^{6(1-\sigma)/J}(\log x)^B.$$

This gives (4.4) so long as $J > 6$ and Δ is sufficiently small.

Case 1.2: $MT > X$, $\beta > \frac{2}{3}$

In this case, we have $\beta < 1$. If $\beta > \frac{2}{3}$, then $(MT)^{1+\mathcal{L}} \leq X^3$. Similar to Case 1, we choose g so that

$$X^3 \leq N^g \leq X^4$$

and apply (5.4) to obtain

$$R \ll x^{8(1-\sigma)/J}(\log x)^B.$$

We obtain (4.4) so long as $J > 8$ and Δ is sufficiently small.

Case 1.3: $MT > X$, $\beta \leq \frac{2}{3}$, $\sigma \leq \frac{3}{4}$

Here it suffices to use the estimates $\mathcal{B}_1$ and $\mathcal{B}_2$. A short computations shows that $\mathcal{B}_1(\mathcal{L}) \leq \frac{10}{3}(1-\sigma)$ so long as

$$\mathcal{L} \leq \left(\frac{2}{3} - \beta(2\sigma - 1)\right)(1-\sigma) = \mathcal{L}^*,$$

say. Since $\mathcal{B}_2$ decreases in $\mathcal{L}$, it suffices to check that $\mathcal{B}_2(\mathcal{L}^*) \leq \frac{10}{3}(1-\sigma)$. Another computation shows that this inequality holds so long as

$$\beta \leq \frac{10\sigma - 9}{12\sigma^2 - 24\sigma + 9}.$$

The expression on the right decreases in σ, and substituting $\sigma = \frac{3}{4}$, we see that $\mathcal{B}_2(\mathcal{L}^*) \leq \frac{10}{3}(1-\sigma)$ so long as $\beta \leq \frac{2}{3}$.

Case 1.4: $MT > X$, $\beta \leq \frac{2}{3}$, $\sigma > \frac{3}{4}$

The proof of this case is very similar to Case 3, except that we use $\mathcal{A}_3$ in place of $\mathcal{A}_1$. Note that Cases 3 and 4 do not use any information about the size of MT compared to X. As such, Cases 3 and 4 actually cover the entire range $\mathcal{L} \in [0, 1]$, $\beta \leq 2/3$, $\sigma \in (\frac{7}{10}, 1)$.

2.7. Long Polynomials: Subconvexity and Simplifications

We now suppose that $N > X$, in which case we may apply the following subconvexity estimate for Hecke L-functions.

Lemma 2.7.1 (Ricci). *If* $(4m^2 + t^2) \geq 4$, *then*

$$L\left(\tfrac{1}{2} + it, \lambda^m\right) \ll (m^2 + t^2)^{1/6} \log^3(m^2 + t^2).$$

For a proof, see [19], Chapter 2, Theorem 4. Since $N > X$, the coefficients of $f_m(1/2+it)$ are smooth and we may write (in the case $j > 1$)

$$f_m\left(\tfrac{1}{2} + it\right) = \frac{1}{2\pi i} \int_{(0)} L\left(\tfrac{1}{2} + it + s, \lambda^m\right) \tilde{W}(s) N^s ds. \tag{7.1}$$

We have $m \geq 1$ always, so Lemma 2.7.1 yields

$$\begin{aligned} f_m\left(\tfrac{1}{2} + it\right) &\ll \int_{-\infty}^{\infty} \frac{(m^2 + t^2 + y^2)^{1/6} \log^3(m^2 + t^2 + y^2)}{(1 + |y|)^A} \, dy \\ &\ll (M^2 + T^2)^{1/6} \log^3(M^2 + T^2) \\ &\ll (MT)^{(1+\mathcal{L})/6} \log^3(M^2 + T^2). \end{aligned}$$

If $j = 1$, we write

$$f_m\left(\tfrac{1}{2} + it\right) = \log N \sum_{\mathfrak{a}} W\left(\frac{N\mathfrak{a}}{N}\right) \frac{\lambda^m(\mathfrak{a})}{(N\mathfrak{a})^{1/2+it}} + \sum_{\mathfrak{a}} W\left(\frac{N\mathfrak{a}}{N}\right) \log\left(\frac{N\mathfrak{a}}{N}\right) \frac{\lambda^m(\mathfrak{a})}{(N\mathfrak{a})^{1/2+it}}.$$

The first sum is handled in the same way as before. If $W(y)$ is replaced by $W^*(y) = W(y)\log y$, then $\tilde{W}^*$ decays rapidly on vertical lines just as $\tilde{W}$, and so in this case we obtain

$$f_m\left(\tfrac{1}{2} + it\right) \ll (MT)^{(1+\mathcal{L})/6} \log^3(M^2 + T^2) \log N.$$

Since $\left|f_m\left(\frac{1}{2}+it\right)\right| \gg (MT)^{\beta(\sigma-1/2)}(\log x)^4$, we deduce that

$$\sigma \leq \frac{1}{2} + \frac{1+\mathcal{L}}{6\beta}. \tag{7.2}$$

We can also make a few simplifying assumptions. We may assume

$$\sigma > \frac{7}{10},$$

for otherwise

$$R \leq MT \leq (MT)^{10(1-\sigma)/3}.$$

In particular, we have $\sigma \leq \frac{7}{10}$ if $\beta \geq \frac{5}{6}(1+\mathcal{L})$. From the remarks in Case 4, we may also assume $\beta > \frac{2}{3}$. Thus we may limit our analysis to the situation in which

$$\frac{2}{3} < \beta < \frac{5}{6}(1+\mathcal{L}) \leq \frac{5}{3}. \tag{7.3}$$

2.8. Long Polynomials: Case Checking

Case 2.1: $\beta \leq \frac{5}{6}$, $\sigma \leq \frac{3}{4}$

Fix $\sigma \in (\frac{7}{10}, \frac{3}{4}]$ and $\beta \in (\frac{2}{3}, \frac{5}{6}]$. We determine the largest value $\mathcal{L}^*$ of $\mathcal{L}$ for which $\mathcal{A}_1$ is sufficient. Since $\mathcal{A}_1$ is continuous and non-decreasing, we can compute $\mathcal{L}^*$ as follows. We have

$$2\beta(n+1)(1-\sigma) \leq \frac{10}{3}(1-\sigma)$$

so long as $n \leq \frac{5}{3\beta} - 1$. If $\beta \neq \frac{5}{6}$, then since $\left\lfloor \frac{5}{3\beta} \right\rfloor = 2$ in the present case, it follows that $\mathcal{L}^*$ lies in the interval $\left[\mathcal{L}_2^{(1,d)}, \mathcal{L}_2^{(1,e)}\right)$. The value $\mathcal{L}^*$ is then given by the solution to

$$1 + \mathcal{L}^* - 2\beta(2\sigma - 1) = \frac{10}{3}(1-\sigma).$$

If $\beta = \frac{5}{6}$, then $\mathcal{A}_1(\mathcal{L}) = \frac{10}{3}(1-\sigma)$ for all $\mathcal{L} \in \left[\mathcal{L}_1^{(1,e)}, \mathcal{L}_2^{(1,d)}\right]$, so we may take $\mathcal{L}^* = \mathcal{L}_2^{(1,d)} = \frac{2}{3}$. Thus in this case also, $\mathcal{L}^*$ is given by the solution to the equation above.

We have $\mathcal{A}_1(\mathcal{L}) \leq \frac{10}{3}(1-\sigma)$ so long as $\mathcal{L} \leq \mathcal{L}^*$. Since $\mathcal{A}_2$ is continuous and non-increasing, to estimate the remaining range of $\mathcal{L}$, it suffices to check that $\mathcal{A}_2(\mathcal{L}^*) \leq \frac{10}{3}(1-\sigma)$. To evaluate $\mathcal{A}_2(\mathcal{L}^*)$, we need to determine n^* such that the interval $\left[\mathcal{L}_{n^*}^{(2,d)}, \mathcal{L}_{n^*+1}^{(2,d)}\right)$ contains $\mathcal{L}^*$. A short computation shows that in the present case, we have

$$n^* = \left\lfloor \frac{5}{3\beta}(1-\sigma) + (2\sigma - 1) \right\rfloor = 1.$$

If $\mathcal{L}^* \leq \mathcal{L}_1^{(2,e)}$, then

$$\mathcal{A}_2(\mathcal{L}^*) = 1 + \beta(1-2\sigma) = 1 - \beta + 2\beta(1-\sigma) \leq \frac{10}{3}(1-\sigma),$$

where the last inequality follows from $1 - \beta \leq \frac{1}{6} < \frac{5}{12} \leq \frac{5}{3}(1-\sigma)$. Otherwise if $\mathcal{L}^* > \mathcal{L}_1^{(2,e)}$, then again we have

$$\mathcal{A}_2(\mathcal{L}^*) = \frac{1-\mathcal{L}^*}{2} + 4\beta(1-\sigma) \leq \frac{1-\mathcal{L}_1^{(2,e)}}{2} + 4\beta(1-\sigma) = 1 - \beta + 2\beta(1-\sigma) \leq \frac{10}{3}(1-\sigma).$$

Case 2.2: $\beta \leq \frac{5}{6}$, $\sigma > \frac{3}{4}$

Fix $\sigma \in (\frac{3}{4}, 1)$ and $\beta \in (\frac{2}{3}, \frac{5}{6}]$. The arguments for this case and the next are very similar to Case 2.1, so we will be fairly brief. As in Case 2.1, we determine the largest value $\mathcal{L}^*$ of $\mathcal{L}$ for which $\mathcal{A}_3$ is sufficient. Arguing as in that case, we find that $\mathcal{L}^*$ is given by the solution to

$$1 + \mathcal{L}^* - 2\beta(6\sigma - 4) = \frac{10}{3}(1-\sigma).$$

We now check that $\mathcal{A}_2(\mathcal{L}^*) \leq \frac{10}{3}(1-\sigma)$. As before, we have $\mathcal{L}^* \in \left[\mathcal{L}_1^{(2,d)}, \mathcal{L}_2^{(2,d)}\right)$. If $\mathcal{L}^* \leq \mathcal{L}_1^{(2,e)}$, then

$$\mathcal{A}_2(\mathcal{L}^*) = 1 + \beta(1-2\sigma) = 1 - \beta + 2\beta(1-\sigma) \leq \frac{10}{3}(1-\sigma)$$

so long as $\sigma \le \frac{9}{10}$, where the last inequality follows from $1 - \beta \le \frac{1}{6} \le \frac{5}{3}(1-\sigma)$. If $\sigma > \frac{9}{10}$, then $\mathcal{L}^* > 1$, so $\mathcal{A}_3$ suffices for all $\mathcal{L} \in [0,1]$. If $\mathcal{L}^* > \mathcal{L}_1^{(2,e)}$, then just as in Case 2.1 we have

$$\mathcal{A}_2(\mathcal{L}^*) \le \frac{10}{3}(1-\sigma).$$

Case 2.3: $\beta > \frac{5}{6}$

Fix $\sigma \in (\frac{7}{10}, 1)$ and $\beta \in (\frac{5}{6}, \frac{5}{3})$. By the subconvexity restriction (7.2), we may assume $\mathcal{L} > 3\beta(2\sigma - 1) - 1 = \mathcal{L}^*$, say, and since $\mathcal{A}_2$ is non-increasing in $\mathcal{L}$, it suffices to check that $\mathcal{A}_2(\mathcal{L}^*) \le \frac{10}{3}(1-\sigma)$. The inequalities $\mathcal{L}_0^{(2,e)} \le \mathcal{L}^* \le \mathcal{L}_1^{(2,d)}$ are easy to verify (the interval $\left[\mathcal{L}_0^{(2,e)}, \mathcal{L}_1^{(2,d)}\right]$ may intersect only part of $[0,1]$, but this is immaterial). It follows that

$$\begin{aligned}\mathcal{A}_2(\mathcal{L}^*) &= \frac{1 - (3\beta(2\sigma-1) - 1)}{2} + 2\beta(1-\sigma) = 1 + 3\beta(\tfrac{1}{2} - \sigma) + 2\beta(1-\sigma) \\ &= 1 - \frac{3\beta}{2} + 5\beta(1-\sigma) \le \frac{10}{3}(1-\sigma)\left(1 - \frac{3\beta}{2}\right) + 5\beta(1-\sigma) = \frac{10}{3}(1-\sigma).\end{aligned}$$

2.9. Optimality of $\frac{10}{3}$

There are two sets of values of $\mathcal{L}, \beta, \sigma$ which show that the constant $\frac{10}{3}$ is optimal in our analysis. These are

$$\mathcal{L} = \frac{3}{5},\ \beta = \frac{4}{3},\ \sigma = \frac{7}{10} \qquad \text{and} \qquad \mathcal{L} = 1,\ \beta = \frac{5}{6},\ \sigma = \frac{9}{10}.$$

In the cases above where these values occur, one may check that the inequalities used are sharp, and so $\frac{10}{3}$ cannot be improved. The optimal large sieve (1.1) would eliminate the need for the variable $\mathcal{L}$, but the particular case $\beta = \frac{5}{6}, \sigma = \frac{9}{10}$ remains a worst case when using this estimate.

Chapter 3

The Sixth Moment of Automorphic L-Functions

3.1. Introduction

Moments of L-functions are among the central objects of study in modern analytic number theory, and there is a vast literature on the subject. In this chapter, we shall be concerned with a family of L-functions attached to automorphic forms on GL_2. Specifically, we consider the sixth moment of L-functions associated to the family of holomorphic modular forms with respect to the congruence subgroup $\Gamma_1(q)$ (see [11] for definitions). Our work is motivated by the work of Djanković [5] and Chandee and Li [2] on this family. For a detailed introduction to this family of L-functions, see the introductions of the above two papers.

Let $S_k(\Gamma_1(q))$ denote the space of holomorphic cusp forms on $\Gamma_1(q)$. We assume $k \geq 3$ is an odd integer and q is prime (these assumptions are made mostly to eliminate oldforms). Then $S_k(\Gamma_1(q))$ is a Hilbert space with the Petersson's inner product

$$\langle f, g \rangle = \int_{\Gamma_1(q)\backslash \mathbb{H}} f(z)\overline{g}(z)y^{k-2}dx\ dy,$$

and

$$S_k(\Gamma_1(q)) = \bigoplus_{\chi \pmod q} S_k(\Gamma_0(q), \chi).$$

Let $\mathcal{H}_\chi$ be an orthogonal basis for $S_k(\Gamma_0(q), \chi)$ consisting of Hecke cusp forms, normalized so that the first Fourier coefficient is 1. For each $f \in \mathcal{H}_\chi$, we let $L(f,s)$ be the L-function associated to f, defined for Re $s > 1$ as

$$L(f,s) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1},$$

where $\{\lambda_f(n)\}$ are the Hecke eigenvalues of f. In general, these satisfy the Hecke relation

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \chi(d)\lambda_f\left(\frac{mn}{d^2}\right). \tag{1.1}$$

We define the completed L-function as

$$\Lambda\left(f, \tfrac{1}{2}+s\right) = \left(\frac{q}{4\pi^2}\right)^{\frac{s}{2}} \Gamma\left(s + \frac{k}{2}\right) L\left(f, \tfrac{1}{2}+s\right), \tag{1.2}$$

which satisfies the functional equation

$$\Lambda\left(f, \tfrac{1}{2}+s\right) = i^k \overline{\eta_f} \Lambda\left(\overline{f}, \tfrac{1}{2}-s\right)$$

where $|\eta_f| = 1$ when f is a newform. We define the harmonic average over $\mathcal{H}_\chi$ by

$$\sideset{}{^h}\sum_{f \in \mathcal{H}_\chi} \alpha_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{H}_\chi} \frac{\alpha_f}{||f||^2}.$$

We are interested in the sixth moment

$$\mathcal{M}(q) = \frac{2}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_\chi} \left|L\left(f, \tfrac{1}{2}\right)\right|^6.$$

The conjectured asymptotic (see [2]) is

$$\mathcal{M}(q) = \frac{2}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_\chi} \left| L\left(f, \tfrac{1}{2}\right)\right|^6 \sim 42\mathscr{C}_3 \left(1 - \frac{1}{q}\right)^4 \left(1 + \frac{4}{q} + \frac{1}{q^2}\right) C_q^{-1} \left(\frac{(\log q)^9}{9!}\right),$$

where $\mathscr{C}_3$, C_q^{-1} are certain explicit constants. Using the asymptotic large sieve for the Fourier coefficients of cusp forms developed by Iwaniec and Xiaoqing Li [14], Djanković [5] has shown

$$\mathcal{M}(q) \ll q^\varepsilon$$

for any $\varepsilon > 0$, whereas Chandee and Xiannan Li [2] have obtained the following asymptotic formula for the smoothed sixth moment:

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_\chi} \int_{-\infty}^{\infty} \left|\Lambda\left(f, \tfrac{1}{2} + it\right)\right|^6 dt$$
$$\sim 42\mathscr{C}_3 \left(1 - \frac{1}{q}\right)^4 \left(1 + \frac{4}{q} + \frac{1}{q^2}\right) C_q^{-1} \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left|\Gamma(\tfrac{k}{2} + it)\right|^6 dt.$$

Note that that the integral in t is quite short due to the presence of the gamma function. Building on these results, we prove

Theorem 3.1.1. *Let q be prime and $k \geq 3$. Then, as $q \to \infty$, we have*

$$\mathcal{M}(q) \ll (\log q)^9.$$

Our proof of Theorem 3.1.1 adheres closely to the work of Chandee and Li [2] and may be seen as an application of their ideas that avoids many of the technical details of their proof. Although we sacrifice an asymptotic, our result has the benefit of having no integral in t and being of the correct order of magnitude.

3.1.1. **Notation**

We retain the notational conventions of Chapter 2, except for the use of the symbol λ, which we shall use either for Fourier coefficients or as a summation variable, and also put $e(x) = e^{2\pi i x}$. We use a bold letter such as $\boldsymbol{a}$ to denote the pair of variables a_1, a_2 and write $f(\boldsymbol{a})$ to indicate that f is a function depending on these variables. However, we use $\boldsymbol{n}$ and $\boldsymbol{N}$ to indicate the pairs (n, m) and (N, M), respectively. We write $n \asymp N$ to denote the condition $c_1 N \leq n \leq c_2 N$ for some suitable constants $0 < c_1 < c_2$. The use of the notation $\sum^*$ in a sum such as $\sum^*_{x(c)}$ indicates that the is sum over residue classes x which are coprime to the modulus of the sum, in this case c. In such a sum, we denote by $\overline{x}$ the inverse of x mod c. All other notation should be clear from context.

3.2. Outline of the Proof

To help orient the reader, we provide an outline for the proof. First, after applying the approximate functional equation for $L\left(f, \frac{1}{2}\right)^3$, the main object we need to understand is roughly of the form

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi(-1) = (-1)^k}} \sum_{f \in \mathcal{H}_\chi}^h \sum_{m,n \asymp q^{\frac{3}{2}}} \frac{\tau_3(m)\tau_3(n)\lambda_f(m)\lambda_f(n)}{\sqrt{mn}}.$$

We apply the functional equation for $L\left(f, \frac{1}{2}\right)^3$ rather than for $L\left(f, \frac{1}{2}\right)^6$ to avoid unbalanced sums in m and n (i.e. $m, n \asymp q^{3/2}$ rather than the weaker condition $mn \leq q^3$). We note that the t integral used [2] is included for precisely the same reason. It is also worth noting that the application of Cauchy's inequality in (3.2) immediately precludes any hope of obtaining an asymptotic formula by our method, as we completely ignore the arithmetic of the root numbers η_f. Applying Peterson's formula to the average over $f \in \mathcal{H}_\chi$ leads to diagonal terms $m = n$ and off-diagonal terms. The diagonal terms are evaluated fairly easily in Section 3.4.

The off-diagonal terms involve sums of the form

$$\sum_{m,n \asymp q^{\frac{3}{2}}} \frac{\tau_3(m)\tau_3(n)}{\sqrt{mn}} \frac{2}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \sum_c S_\chi(m,n;cq) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq}\right),$$

where $S_\chi(m,n;cq)$ is the Kloosterman sum defined in (4.1) and J_{k-1} is the usual Bessel function of order $k-1$.

The most important range for c is in the transition region for the Bessel function, i.e. $c \asymp q^{\frac{1}{2}}$. To focus on this region, we truncate the sum in c using the Weil estimate for Kloosterman sums. The details of this truncation are given in Section 3.5. The conductor of the Kloosterman sum is then essentially of size $cq \asymp q^{\frac{3}{2}}$. To understand the correlations between the Kloosterman sums and the Bessel functions, we apply harmonic analysis in the form of the Voronoi formula of Ivić [10]. Before doing so, we reduce the conductor in the Kloosterman sums by taking advantage of the average over χ. The conductor lowering trick of [2] (see Lemma 3.6.1) produces new Kloosterman sums of the form

$$e\left(\frac{m+n}{cq}\right) \sideset{}{^*}\sum_{x(c)} e\left(\frac{\overline{q}(x-1)m + \overline{q}(\overline{x}-1)n}{c}\right),$$

where the conductor is now reduced to $c \asymp q^{\frac{1}{2}}$ and the exponential in front may be treated as a smooth function with small derivatives. Applying the Voronoi formula then produces a single main term and eight error terms. The details of these transformations are given in Section 3.6. The main term is estimated in Section 3.7, and it is here that we require a more delicate analysis of the Laurent series coefficients D_{-i} of the third-order Estermann zeta function $E_3\left(s, \frac{\lambda}{\eta}\right)$ (see (6.8), (7.1), and (7.2)). This is accomplished via Lemma 3.7.4, in which we improve the trivial estimate $D_{-i}(\eta) \ll \eta^{-1+\varepsilon}$ to

$$D_{-i}(\eta) \ll \frac{\tau_2(\eta)(\log \eta)^{3-i}}{\eta}.$$

In order to apply this estimate without losing too much from the triangle inequality, we need to suitably transform the sum. This is accomplished by an application of Poisson summation (in the form of Lemma 3.7.2), along with some identities involving integrals of Bessel functions. The details of these transformations are given in Subsection 3.7.2.

It turns out that the Laurent coefficients D_{-i} are not multiplicative for $i = 1, 2$. In order to extract a residue of $\zeta(s)$ at $s = 1$ after the above transformations, we require a somewhat delicate analysis of certain complex-valued arithmetic functions (see Proposition 3.7.5), in contrast to the simple contour shifting argument used in [2]. We then estimate the remaining sum by elementary means to obtain a final estimate of $(\log q)^9$ for the main term. These computations are given in Subsection 3.7.3.

Finally, the eight error terms (i.e. the dual sums arising from Voronoi summation) are estimated in Section 3.8. The main aspect of the calculations in this section is that the dual sums are short. This is precisely the reason for reducing the conductor in the Kloosterman sums using Lemma 3.6.1. The details of these calculations are standard but technical and follow closely the arguments of [2].

3.3. Approximate Functional Equation

As is standard in such problems, we begin with an approximation functional equation for $L(f, 1/2)^3$. The derivation of this is standard. For our purposes, it suffices to cite equation (2.5) of [5], which is

$$\begin{aligned} L(f,1/2)^3 &= \sum_{a\geq 1}\sum_{b\geq 1}\sum_{n\geq 1} \frac{\mu(a)\chi(b)\tau_3(b)\lambda_f(an)\tau_3(n)}{(a^3b^2n)^{\frac{1}{2}}} U\left(\frac{a^3b^2n}{q^{\frac{3}{2}}}\right) \\ &\quad + (i^k\overline{\eta_f})^3 \sum_{a\geq 1}\sum_{b\geq 1}\sum_{n\geq 1} \frac{\mu(a)\overline{\chi}(b)\tau_3(b)\overline{\lambda_f}(an)\tau_3(n)}{(a^3b^2n)^{\frac{1}{2}}} U\left(\frac{a^3b^2n}{q^{\frac{3}{2}}}\right), \end{aligned}$$

where

$$U(y) = \frac{1}{2\pi i}\int_{(2)} y^{-s}\gamma^3(s)\left(e^{s^2}\right)^3\frac{ds}{s}, \qquad \gamma(s) = (2\pi)^{-s}\frac{\Gamma(\frac{k}{2}+s)}{\Gamma(\frac{k}{2})}.$$

Here we have chosen the specific function e^{s^2} to ensure that $U(y)$ is real when y is real. This is mainly for notational simplicity. The function U satisfies

$$\begin{aligned} &U(y) \ll (1+y)^{-A}, \\ &U(y) = 1 + O(y^A) \quad \text{as } y \to 0 \end{aligned} \tag{3.1}$$

for any $A > 1$. Applying Cauchy's inequality, we have

$$\mathcal{M}(q) \ll \frac{2}{\varphi(q)} \sum_{\substack{\chi(q) \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_\chi} \left| \sum_{a,b,n \geq 1} \frac{\mu(a)\tau_3(b)\tau_3(n)\chi(ab)\lambda_f(an)}{(a^3b^2n)^{\frac{1}{2}}} U\left(\frac{a^3b^2n}{q^{\frac{3}{2}}}\right) \right|^2. \tag{3.2}$$

Expanding the square and rearranging, we obtain

$$\begin{aligned} \mathcal{M}(q) \ll &\sum_{\substack{a_1,b_1,n\geq 1 \\ a_2,b_2,m\geq 1}} \cdots \sum \frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(n)\tau_3(b_2)\tau_3(m)}{(a_1^3b_1^2n)^{\frac{1}{2}}(a_2^3b_2^2m)^{\frac{1}{2}}} U\left(\frac{a_1^3b_1^2n}{q^{\frac{3}{2}}}\right) U\left(\frac{a_2^3b_2^2m}{q^{\frac{3}{2}}}\right) \\ &\times \left(\frac{2}{\varphi(q)} \sum_{\substack{\chi(q) \\ \chi(-1)=(-1)^k}} \chi(a_1b_1)\overline{\chi}(a_2b_2)\right)\left(\sideset{}{^h}\sum_{f\in\mathcal{H}_\chi} \lambda_f(a_1n)\overline{\lambda_f}(a_2m)\right). \end{aligned}$$

Note that by (3.1), the terms with $a_1^3b_1^2n, a_2^3b_2^2m \gg q^{\frac{3}{2}+\varepsilon}$ give a contribution of q^{-2022}.

3.4. Orthogonality and the Diagonal Contribution

We now apply the orthogonality relations for χ and λ_f given by the following lemma.

Lemma 3.4.1. *The orthogonality relation for Dirichlet characters is*

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi(q) \\ \chi(-1)=(-1)^k}} \chi(m)\overline{\chi}(n) = \begin{cases} 1 & \text{if } m \equiv n(q),\ (mn,q)=1, \\ (-1)^k & \text{if } m \equiv -n(q),\ (mn,q)=1, \\ 0 & \text{otherwise.} \end{cases}$$

Petersson's formula gives

$$\sum_{f\in\mathcal{H}_\chi}^{h} \lambda_f(n)\overline{\lambda}_f(m) = \delta_{m=n} + \sigma_\chi(m,n),$$

where

$$\sigma_\chi(m,n) = 2\pi i^{-k}\sum_{c=1}^{\infty}(cq)^{-1}S_\chi(m,n;cq)J_{k-1}\left(\frac{4\pi}{cq}\sqrt{mn}\right)$$

and S_χ is the Kloosterman sum defined by

$$S_\chi(m,n,cq) = \sum_{a(cq)}^{*}\chi(a)e\left(\frac{am+\overline{a}n}{cq}\right). \tag{4.1}$$

Here $\sum^*$ denotes a sum over residues a with $(a,cq)=1$ and $\overline{a}$ satisfies $a\overline{a}\equiv 1(cq)$.

Lemma 3.4.1 gives

$$\mathcal{M}(q) \ll \mathcal{D} + \mathcal{OD},$$

where $\mathcal{OD}$ is given by (5.1) and $\mathcal{D} = \mathcal{D}_+ + \mathcal{D}_-$ with

$$\mathcal{D}_+ = \sum_{\substack{a_1,b_1,n,a_2,b_2,m\geq 1\\ a_1b_1\equiv a_2b_2(q)\\ a_1n=a_2m\\ (a_1b_1a_2b_2,q)=1}}\cdots\sum \frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(n)\tau_3(b_2)\tau_3(m)}{(a_1^3b_1^2n)^{\frac{1}{2}}(a_2^3b_2^2m)^{\frac{1}{2}}}U\left(\frac{a_1^3b_1^2n}{q^{\frac{3}{2}}}\right)U\left(\frac{a_2^3b_2^2m}{q^{\frac{3}{2}}}\right)$$

and $\mathcal{D}_-$ is the same sum but multiplied by $(-1)^k$ with the condition $a_1b_1 \equiv a_2b_2 \pmod q$ replaced by $a_1b_1 \equiv -a_2b_2 \pmod q$. The only relevant case is when $a_1b_1 = a_2b_2$ in $\mathcal{D}_+$, since in the other cases we have $a_1b_1 \geq q/4$ or $a_2b_2 \geq q/4$, which means that $a_1^3b_1^2n \gg q^2$ or $a_2^3b_2^2m \gg q^2$. Thus

$$\mathcal{D} = \sum_{\substack{a_1,b_1,n,a_2,b_2,m\geq 1\\ a_1b_1=a_2b_2\\ a_1n=a_2m\\ (a_1b_1a_2b_2,q)=1}}\cdots\sum \frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(n)\tau_3(b_2)\tau_3(m)}{(a_1^3b_1^2n)^{\frac{1}{2}}(a_2^3b_2^2m)^{\frac{1}{2}}}U\left(\frac{a_1^3b_1^2n}{q^{\frac{3}{2}}}\right)U\left(\frac{a_2^3b_2^2m}{q^{\frac{3}{2}}}\right) + O(q^{-2022}). \tag{4.2}$$

Neglecting the error term, we open the factors of U and write

$$\mathcal{D} = \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} q^{\frac{3}{2}(s_1+s_2)} \mathscr{D}(1+s_1+s_2)\gamma^3(s_1)\gamma^3(s_2)e^{3(s_1^2+s_2^2)} \frac{ds_1}{s_1} \frac{ds_2}{s_2}. \tag{4.3}$$

where

$$\mathscr{D}(s) = \sum_{\substack{a_1,b_1,n\geq 1\\(a_1b_1,q)=1}} \frac{\mu(a_1)\tau_3(b_1)\tau_3(n)}{(a_1^3b_1^2n)^s} \sum_{\substack{a_2,b_2,m\\a_2b_2=a_1b_1\\a_2m=a_1n\\(a_2b_2,q)=1}} \mu(a_2)\tau_3(b_2)\tau_3(m).$$

We write $\mathscr{D}(s)$ as the Euler product

$$\mathscr{D}(s) = \prod_p \mathscr{D}_p(s),$$

where

$$\mathscr{D}_p(s) = \sum_{\substack{a_1,b_1,n,a_2,b_2,m\geq 0\\a_1+b_1=a_2+b_2\\a_1+n=a_2+m}} \cdots \sum \frac{\mu(p^{a_1})\mu(p^{a_2})\tau_3(p^{b_1})\tau_3(p^n)\tau_3(p^{b_2})\tau_3(p^m)}{p^{s(3a_1+2b_1+n)}} = 1 + \frac{9}{p^s} + \cdots$$

for $p \neq q$ and

$$\mathscr{D}_q(s) = \sum_{n\geq 0} \frac{\tau_3(q^n)^2}{q^{sn}} = 1 + \frac{9}{q^s} + \cdots .$$

Thus

$$\mathscr{D}(s) = \zeta^9(s)H(s)$$

for some $H(s)$ that is analytic for Re $s > 1/2$. After the change of variables $u = s_1 + s_2$, $s = s_2$, we have

$$\mathcal{D} = \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(4)} q^{\frac{3}{2}u} \zeta^9(1+u)H(1+u)\gamma^3(u-s)\gamma^3(s)e^{3(u^2-2us)} \frac{du}{u-s} \frac{ds}{s}.$$

The rapid decay of $\gamma(s)$ and e^{3u^2} on vertical lines allows us to move the line of integration in s to Re $s = -1$ and the integration in u to Re $u = -\frac{1}{2} + \varepsilon$. In doing so, we pass a simple pole at $s = 0$ and poles of orders 9 and 10 at $u = 0$. Thus

$$\mathcal{D} = R_1 + R_2 + E_1 + E_2,$$

where

$$\begin{aligned}
R_1 &= \operatorname*{Res}_{u=0} \left[q^{\frac{3}{2}u} \zeta^9(1+u) H(1+u) \gamma^3(u) e^{3u^2} u^{-1} \right], \\
R_2 &= \frac{1}{2\pi i} \int\limits_{(-1)} \gamma^3(s) \operatorname*{Res}_{u=0} \left[q^{\frac{3}{2}u} \zeta^9(1+u) H(1+u) \gamma^3(u-s) e^{3u^2 - 2us} \frac{1}{u-s} \right] \frac{ds}{s}, \\
E_1 &= \frac{1}{2\pi i} \int\limits_{(-\frac{1}{2}+\varepsilon)} q^{\frac{3}{2}u} \zeta^9(1+u) H(1+u) \gamma^3(u) e^{3u^2} \frac{du}{u}, \\
E_2 &= \frac{1}{(2\pi i)^2} \int\limits_{(-1)} \int\limits_{(-\frac{1}{2}+\varepsilon)} q^{\frac{3}{2}u} \zeta^9(1+u) H(1+u) \gamma^3(u-s) \gamma^3(s) e^{3(u^2 - 2us)} \frac{du}{u-s} \frac{ds}{s}.
\end{aligned}$$

Using Stirling's formula and the rapid decay of e^{3u^2}, we see that

$$E_1, E_2 \ll q^{-3/4+\varepsilon}.$$

A straightforward calculation shows that

$$R_1 \asymp (\log q)^9.$$

In R_2, the leading order term of the residue (in terms of q) is of the form

$$\frac{C(\log q)^8}{2\pi i} \int\limits_{(-1)} \gamma^3(s) \gamma^3(-s) \frac{ds}{s^2} \asymp (\log q)^8.$$

We deduce that $R_2 \asymp (\log q)^8$, from which it follows that

$$\mathcal{D} \asymp (\log q)^9.$$

We proceed now to our treatment of the off-diagonal terms, which constitutes the remainder of the proof

3.5. Truncation of the The Off-Diagonal Terms

The off-diagonal contribution is

$$\begin{aligned}\mathcal{OD} = &\sum_{\substack{a_1,b_1,n\geq 1\\ a_2,b_2,m\geq 1}}\cdots\sum \frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(n)\tau_3(b_2)\tau_3(m)}{(a_1^3b_1^2n)^{\frac{1}{2}}(a_2^3b_2^2m)^{\frac{1}{2}}} U\left(\frac{a_1^3b_1^2n}{q^{\frac{3}{2}}}\right)U\left(\frac{a_2^3b_2^2m}{q^{\frac{3}{2}}}\right)\\ &\times\left(\frac{2}{\varphi(q)}\sum_{\substack{\chi(q)\\ \chi(-1)=(-1)^k}}\chi(a_1b_1)\overline{\chi}(a_2b_2)\right)\left(2\pi i^{-k}\sum_{c\geq 1}(cq)^{-1}\sum_{a\overline{a}\equiv 1(cq)}\chi(a)e\left(\frac{aa_2m+\overline{a}a_1n}{cq}\right)\right)\\ =&\frac{1}{q}\sum_{\substack{a_1,b_1,n\geq 1\\ a_2,b_2,m\geq 1}}\cdots\sum \frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(n)\tau_3(b_2)\tau_3(m)}{(a_1^3b_1^2n)^{\frac{1}{2}}(a_2^3b_2^2m)^{\frac{1}{2}}} U\left(\frac{a_1^3b_1^2n}{q^{\frac{3}{2}}}\right)U\left(\frac{a_2^3b_2^2m}{q^{\frac{3}{2}}}\right)\\ &\times 2\pi i^{-k}\sum_{c\geq 1}\frac{1}{c}\sum_{a\overline{a}\equiv 1(cq)}e\left(\frac{aa_2m+\overline{a}a_1n}{cq}\right)\left(\frac{2}{\varphi(q)}\sum_{\substack{\chi(q)\\ \chi(-1)=(-1)^k}}\chi(aa_1b_1)\overline{\chi}(a_2b_2)\right).\end{aligned}$$

As in [2], we introduce the operator $\mathcal{K}g = i^{-k}g + i^k\overline{g}$ for notational convenience. Let f be a smooth function supported on $[\frac{1}{2}, 3]$ such that

$$\sum_{j\in\mathbb{Z}} f\left(\frac{t}{2^j}\right) = 1$$

for all $t \geq 0$. Inserting two of these dyadic partitions of unity into the sums over n, m and using the orthogonality relations for χ, we find that the off-diagonal contribution is

$$\begin{aligned}&\frac{2\pi}{q}\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\ (a_1a_2b_1b_2,q)=1}}\frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(b_2)}{(a_1^3b_1^2a_2^3b_2^2)^{\frac{1}{2}}}\sum_N^{\mathrm{D}}\sum_M^{\mathrm{D}}\sum\sum_{n,m\geq 1}\frac{\tau_3(n)\tau_3(m)}{(nm)^{\frac{1}{2}}}\\ &\qquad\times\sum_{c\geq 1}\frac{1}{c}\mathcal{G}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{n},\boldsymbol{N},c)\mathcal{K}\sideset{}{^*}\sum_{\substack{a(cq)\\ a\equiv\overline{a_1b_1}a_2b_2(q)}} e\left(\frac{aa_2m+\overline{a}a_1n}{c}\right),\end{aligned} \tag{5.1}$$

where

$$\mathcal{G}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{n},\boldsymbol{N},c)=U\left(\frac{a_1^3b_1^2n}{q^{\frac{3}{2}}}\right)U\left(\frac{a_2^3b_2^2m}{q^{\frac{3}{2}}}\right)f\left(\frac{n}{N}\right)f\left(\frac{m}{M}\right)J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1na_2m}\right).$$

Here $\sum^{\mathrm{D}}_{N}$ denotes a dyadic sum over $N=2^j$.

We now truncate the sum in c. Letting $C=q^{-\frac{2}{3}}\sqrt{a_1a_2NM}$, we write

$$\mathcal{OD}=\mathscr{M}+\mathscr{K}_1+\mathscr{K}_2,$$

where

$$\mathscr{K}_i=\frac{2\pi}{q}\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\(a_1a_2b_1b_2,q)=1}}\frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(b_2)}{(a_1^3b_1^2a_2^3b_2^2)^{\frac{1}{2}}}\sum^{\mathrm{D}}_{N,M}\mathscr{S}_i(\boldsymbol{a},\boldsymbol{b},\boldsymbol{N}) \tag{5.2}$$

with

$$\mathscr{S}_1(\boldsymbol{a},\boldsymbol{b},\boldsymbol{N})=\sum_{\substack{c\geq 1\\ q|c}}\frac{1}{c}\sum_{n,m\geq 1}\frac{\tau_3(n)\tau_3(m)}{(nm)^{\frac{1}{2}}}\mathcal{F}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{n},\boldsymbol{N},c),$$

$$\mathscr{S}_2(\boldsymbol{a},\boldsymbol{b},\boldsymbol{N})=\sum_{\substack{c>C\\ (c,q)=1}}\frac{1}{c}\sum_{n,m\geq 1}\frac{\tau_3(n)\tau_3(m)}{(nm)^{\frac{1}{2}}}\mathcal{F}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{n},\boldsymbol{N},c),$$

and

$$\mathcal{F}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{n},\boldsymbol{N},c)=\mathcal{G}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{n},\boldsymbol{N},c)\mathcal{K}\sum^{*}_{\substack{a(cq)\\ a\equiv\overline{a_1b_1}a_2b_2(q)}}e\left(\frac{aa_2m+\overline{a}a_1n}{cq}\right).$$

The quantity $\mathscr{M}$ is defined like $\mathscr{K}_2$, except with the condition $c\leq C$ replaced by $c>C$. We now prove the following proposition.

Proposition 3.5.1. *For $C=q^{-\frac{2}{3}}\sqrt{a_1a_2NM}$, we have*

$$\mathscr{K}_1+\mathscr{K}_2\ll q^{-\frac{5}{12}+\varepsilon}.$$

For the proof of this proposition and for our arguments in Section 3.8, we will need several properties of the J-Bessel functions. These are summarized in the following lemma. These results are standard and can all be found in [22].

Lemma 3.5.2. *We have*

$$J_{k-1}(2\pi x) = \frac{1}{\pi\sqrt{x}}\left(W(2\pi x)e\left(x-\frac{k}{4}+\frac{1}{8}\right)+\overline{W}(2\pi x)e\left(-x+\frac{k}{4}-\frac{1}{8}\right)\right), \tag{5.3}$$

where $W^{(j)}(x) \ll_{j,k} x^{-j}$. *Moreover,*

$$J_{k-1}(2\pi x) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{x^{2\ell+k+1}}{\ell!(\ell+k-1)!}, \tag{5.4}$$

and

$$J_{k-1}(x) \ll \min(x^{-\frac{1}{2}}, x^{k-1}). \tag{5.5}$$

Proof of Proposition 3.5.1. To treat $\mathscr{K}_1$, we begin by writing

$$\mathscr{S}_1(\boldsymbol{a},\boldsymbol{b},\boldsymbol{N}) = \sum_{r=1}^{\infty}\frac{1}{q^r}\sum_{\substack{c\geq 1\\(c,q)=1}}\frac{1}{c}\sum_{n,m\geq 1}\frac{\tau_3(n)\tau_3(m)}{(nm)^{\frac{1}{2}}}\mathcal{F}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{n},\boldsymbol{N},cq^r).$$

For a fixed r, we use the Chinese Remainder Theorem and the Weil bound to see that the modulus of the Kloosterman sum in $\mathcal{F}$ is

$$\left|\sideset{}{^*}\sum_{\substack{a(cq^{r+1})\\ a\equiv\overline{a_1b_1}a_2b_2(q)}} e\left(\frac{aa_2m+\overline{a}a_1n}{cq^{r+1}}\right)\right| = \left|\sideset{}{^*}\sum_{x \pmod{q^r}} e\left(\frac{xa_2m+\overline{x}a_1n}{q^r}\right)\right|\left|\sideset{}{^*}\sum_{y \pmod{c}} e\left(\frac{ya_2m+\overline{y}a_1n}{c}\right)\right|$$
$$\ll (cq^r)^{\frac{1}{2}+\varepsilon}\sqrt{(a_1n,a_2m,c)(n,m,q^r)}.$$

From (5.5), we have

$$J_{k-1}\left(\frac{4\pi}{cq^{r+1}}\sqrt{a_1na_2m}\right) \ll \left(\frac{\sqrt{a_1a_2NM}}{cq^{r+1}}\right)^2,$$

and so

$$\begin{aligned}\mathscr{S}_1(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{N}) &\ll a_1 a_2 (NM)^{\frac{1}{2}} \sum_{r=1}^{\infty} \frac{1}{q^{\frac{5r}{2}+2-\varepsilon}} \sum_{\substack{c\geq 1\\(c,q)=1}} \frac{1}{c^{\frac{5}{2}-\varepsilon}} \sum_{\substack{n\asymp N\\m\asymp M}} \tau_3(n)\tau_3(m)\sqrt{(a_1 n, a_2 m, c)(n, m, q^r)}\\
&\ll a_1 a_2 (NM)^{\frac{1}{2}} \sum_{r=2}^{\infty} \frac{1}{q^{2r-\varepsilon}} \sum_{c\geq 1} \frac{1}{c^{2-\varepsilon}} \sum_{\substack{n\asymp N\\m\asymp M}} \tau_3(n)\tau_3(m)\\
&\ll \frac{a_1 a_2 (NM)^{\frac{3}{2}}}{q^{4-\varepsilon}}.\end{aligned}$$

Here we have bounded the gcds by c and q^r, respectively. Returning to (5.2), we conclude that

$$\mathscr{K}_1 \ll \frac{1}{q^{5-\varepsilon}} \sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\a_1^3 b_1^2 N, a_2^3 b_2^2 M \ll q^{\frac{3}{2}+\epsilon}}} \sum\sum\sum \sum_{N,M}^{\mathrm{D}} \frac{\tau_3(b_1)\tau_3(b_2)}{a_1 b_1 a_2 b_2} (NM)^{\frac{3}{2}} \ll q^{-\frac{1}{2}+\varepsilon}.$$

We now turn to $\mathscr{K}_2$. Again by the Chinese Remainder Theorem and the Weil bound, the modulus of the Kloosterman sum in $\mathcal{F}$ is

$$\left| \sum_{\substack{a(cq)\\a\equiv\overline{a_1 b_1} a_2 b_2 (q)}}^{*} e\left(\frac{a a_2 m + \overline{a} a_1 n}{cq}\right) \right| = \left| \sum_{y \pmod c}^{*} e\left(\frac{y a_2 m + \overline{y} a_1 n}{c}\right) \right| \ll c^{\frac{1}{2}+\varepsilon}\sqrt{(a_1 n, a_2 m, c)}.$$

Using (5.5) once more, we see that

$$
\begin{aligned}
\mathscr{S}_2(\boldsymbol{a},\boldsymbol{b},\boldsymbol{N}) &\ll q^{\varepsilon}(NM)^{\frac{k}{2}-1}\sum_{c>C} c^{\varepsilon-\frac{1}{2}}\left(\frac{\sqrt{a_1a_2}}{cq}\right)^{k-1}\sum_{\substack{n\asymp N\\ m\asymp M}}\sqrt{(a_1n,a_2m,c)}\\
&\ll q^{\varepsilon}(NM)^{\frac{k}{2}-1}\sum_{c>C} c^{\varepsilon-\frac{1}{2}}\left(\frac{\sqrt{a_1a_2}}{cq}\right)^{k-1}\sum_{d|c}\sqrt{d}\sum_{\substack{n\asymp N\\ m\asymp M\\ (a_1n,a_2m,c)=d}}1\\
&\ll q^{\varepsilon}(NM)^{\frac{k}{2}-1}\sum_{c>C} c^{\varepsilon-\frac{1}{2}}\left(\frac{\sqrt{a_1a_2}}{cq}\right)^{k-1}\sum_{d|c}\sqrt{d}\sum_{\substack{n\asymp N\\ d|a_1n}}\sum_{\substack{m\asymp M\\ d|a_2m}}1\\
&\ll q^{\varepsilon}(NM)^{\frac{k}{2}}\sum_{c>C} c^{\varepsilon-\frac{1}{2}}\left(\frac{\sqrt{a_1a_2}}{cq}\right)^{k-1}\sum_{d|c}\frac{(d,a_1)(d,a_2)}{d^{\frac{3}{2}}}\\
&\ll q^{\varepsilon}\frac{(a_1a_2NM)^{\frac{k}{2}}}{q^{k-1}}C^{\frac{3}{2}-k+\varepsilon}\ll q^{\varepsilon}\frac{(a_1a_2NM)^{\frac{3}{4}}}{q^{\frac{5}{3}}},
\end{aligned}
$$

so long as $k\geq 5$. On the fifth line above we have used the estimate $(d,a_i)\leq\sqrt{da_i}$. Once again returning to (5.2), we conclude that

$$
\mathscr{K}_2 \ll \frac{1}{q^{\frac{8}{3}-\varepsilon}}\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\ a_1^3b_1^2N,\,a_2^3b_2^2M\ll q^{\frac{3}{2}+\epsilon}}}\sideset{}{^{\mathrm{D}}}\sum_{N,M}\frac{\tau_3(b_1)\tau_3(b_2)}{(a_1a_2)^{\frac{3}{4}}b_1b_2}(NM)^{\frac{3}{4}}\ll q^{-\frac{5}{12}+\varepsilon}.
$$

□

3.6. Voronoi Summation

It remains to estimate

$$
\begin{aligned}
&\frac{2\pi}{q}\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\ (a_1a_2b_1b_2,q)=1}}\frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(b_2)}{(a_1^3b_1^2a_2^3b_2^2)^{\frac{1}{2}}}\sideset{}{^{\mathrm{D}}}\sum_{N,M}\sum_{n,m\geq 1}\frac{\tau_3(n)\tau_3(m)}{(nm)^{\frac{1}{2}}}U_1\left(a_1^3b_1^2n,a_2^3b_2^2m\right)\\
&\times f\left(\frac{n}{N}\right)f\left(\frac{m}{M}\right)\sum_{c\leq C}\frac{1}{c}J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1na_2m}\right)\mathcal{K}\sideset{}{^*}\sum_{\substack{a(cq)\\ a\equiv\overline{a_1b_1}a_2b_2(q)}}e\left(\frac{aa_2m+\overline{a}a_1n}{cq}\right).
\end{aligned}
\tag{6.1}
$$

Before applying Voronoi summation, we reduce the conductor in the Kloosterman sum using Lemma 5.3 of [2], which we cite in the following form.

Lemma 3.6.1. *Let c, q, m, n be positive integers with $(cmn, q) = 1$, and let*

$$Y(u,v) = \sum_{\substack{a(cq) \\ a \equiv \overline{m}n(q)}}^{*} e\left(\frac{au + \overline{a}v}{cq}\right).$$

Then

$$Y(u,v) = e\left(\frac{n^2u + m^2v}{cqmn}\right) \sum_{x(c)}^{*} e\left(\frac{\overline{q}(mx - n)u}{mc}\right) e\left(\frac{\overline{q}(n\overline{x} - m)v}{nc}\right).$$

Applying Lemma 3.6.1 with $m = a_1a_2$, $n = a_2b_2$, the expression in (6.1) is

$$\begin{aligned}
&\frac{2\pi}{q} \sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2 \geq 1 \\ (a_1a_2b_1b_2,q)=1}} \frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(b_2)}{(a_1^3b_1^2a_2^3b_2^2)^{\frac{1}{2}}} \sum_{N,M}^{\mathrm{D}} \sum_{n,m \geq 1} \frac{\tau_3(n)\tau_3(m)}{(nm)^{\frac{1}{2}}} U_1\left(a_1^3b_1^2n, a_2^3b_2^2m\right) \\
&\qquad \times f\left(\frac{n}{N}\right) f\left(\frac{m}{M}\right) \sum_{\substack{c \leq C \\ (c,q)=1}} \frac{1}{c} J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1na_2m}\right) \mathcal{K} e\left(\frac{(a_1b_1)^2a_1n + (a_2b_2)^2a_2m}{cqa_1b_1a_2b_2}\right) \\
&\qquad \times \sum_{x(c)}^{*} e\left(\frac{\overline{q}(a_2b_2\overline{x} - a_1b_1)a_1n}{a_2b_2c}\right) e\left(\frac{\overline{q}(a_1b_1x - a_2b_2)a_2m}{a_1b_1c}\right) \\
&= \frac{2\pi}{q} \sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2 \geq 1 \\ (a_1a_2b_1b_2,q)=1}} \frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(b_2)}{(a_1^3b_1^2a_2^3b_2^2)^{\frac{1}{2}}} \sum_{N,M}^{\mathrm{D}} \sum_{c \leq C} \frac{1}{c} \sum_{x(c)}^{*} \mathcal{K}\, \mathcal{S}(c,x),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}(c,x) &= \mathcal{S}(c,x;\boldsymbol{a},\boldsymbol{b},\boldsymbol{N}) \\
&= \sum_{n \geq 1} \tau_3(n) e\left(\frac{\overline{q}(a_2b_2\overline{x} - a_1b_1)a_1n}{a_2b_2c}\right) \sum_{m \geq 1} \tau_3(m) e\left(\frac{\overline{q}(a_1b_1x - a_2b_2)a_2m}{a_1b_1c}\right) \\
&\qquad \times F_1(n)F_2(m)J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1na_2m}\right),
\end{aligned}$$

and

$$F_1(y) = y^{-\frac{1}{2}} f\left(\frac{y}{N}\right) e\left(\frac{a_1^2 b_1 y}{cqa_2b_2}\right) U\left(\frac{a_1^3 b_1^2 y}{q^{\frac{3}{2}}}\right),$$
$$F_2(y) = y^{-\frac{1}{2}} f\left(\frac{y}{M}\right) e\left(\frac{a_2^2 b_2 y}{cqa_1b_1}\right) U\left(\frac{a_2^3 b_2^2 y}{q^{\frac{3}{2}}}\right).$$

We write

$$\begin{aligned}\frac{\lambda_1}{\eta_1} &= \frac{\overline{q}(a_2b_2\overline{x} - a_1b_1)a_1}{a_2b_2c},\\ \frac{\lambda_2}{\eta_2} &= \frac{\overline{q}(a_1b_1x - a_2b_2)a_2}{a_1b_1c},\end{aligned} \tag{6.2}$$

where $(\lambda_1, \eta_1) = (\lambda_2, \eta_2) = 1$, and define

$$\begin{aligned}\mathcal{U}(c; \boldsymbol{y}) &= F_1(y_1)F_2(y_2)\\ &= \frac{i^{-k}}{(y_1y_2)^{\frac{1}{2}}} f\left(\frac{y_1}{N}\right) f\left(\frac{y_2}{M}\right) e\left(\frac{a_1^2b_1y_1}{cqa_2b_2} + \frac{a_2^2b_2y_2}{cqa_1b_1}\right) U\left(\frac{a_1^3b_1^2y_1}{q^{\frac{3}{2}}}\right) U\left(\frac{a_2^3b_2^2y_2}{q^{\frac{3}{2}}}\right).\end{aligned}$$

Thus

$$\mathcal{S}(c,x) = \sum_{n\geq 1} F_1(n)\tau_3(n) e\left(\frac{\lambda_1 n}{\eta_1}\right) \sum_{m\geq 1} F_2(m)\tau_3(m) e\left(\frac{\lambda_2 m}{\eta_2}\right) J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1na_2m}\right).$$

We now apply Theorem 2 of [10] with the same notations used there, except with A_3^- in place of Ivic's B_3, first to the sum over m, and then to the sum over n, to obtain

$$\mathcal{S}(c,x) = \sum_{j=1}^{9} \mathcal{T}_j(c,x),$$

where

$$\begin{aligned}\mathcal{T}_1(c,x) = \operatorname*{Res}_{s_1=1} \operatorname*{Res}_{s_2=1} \Bigg[& E_3\left(s_1, \frac{\lambda_1}{\eta_1}\right) E_3\left(s_2, \frac{\lambda_2}{\eta_2}\right)\\ & \times \int_0^\infty \int_0^\infty \mathcal{U}(c; y_1, y_2) J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1a_2y_1y_2}\right) y_1^{s_1-1} y_2^{s_2-1} dy_1\, dy_2 \Bigg],\end{aligned} \tag{6.3}$$

$$\mathcal{T}_2(c,x) = \frac{\pi^{\frac{3}{2}}}{\eta_2^3} \operatorname*{Res}_{s_1=1} E_3\left(s_1, \frac{\lambda_1}{\eta_1}\right) \int_0^\infty F_1(y_1) y_1^{s_1-1} \sum_{m\geq 1} A_3^+\left(m, \frac{\lambda_2}{\eta_2}\right) \int_0^\infty F_2(y_2) U_3\left(\frac{\pi^3 m y_2}{\eta_2^3}\right) \times J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1 a_2 y_1 y_2}\right) dy_1\, dy_2, \tag{6.4}$$

$$\mathcal{T}_3(c,x) = \frac{i\pi^{\frac{3}{2}}}{\eta_2^3} \operatorname*{Res}_{s_1=1} E_3\left(s_1, \frac{\lambda_1}{\eta_1}\right) \int_0^\infty F_1(y_1) y^{s_1-1} \sum_{m\geq 1} A_3^-\left(m, \frac{\lambda_2}{\eta_2}\right) \int_0^\infty F_2(y_2) V_3\left(\frac{\pi^3 m y_2}{\eta_2^3}\right) \times J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1 a_2 y_1 y_2}\right) dy_1\, dy_2, \tag{6.5}$$

$\mathcal{T}_4$ and $\mathcal{T}_5$ are defined as $\mathcal{T}_2$ and $\mathcal{T}_3$, but one swaps all subscripts of 1 and 2,

$$\mathcal{T}_6(c,x) = \frac{\pi^3}{\eta_1^3\eta_2^3} \sum_{n\geq 1}\sum_{m\geq 1} A_3^+\left(n, \frac{\lambda_1}{\eta_1}\right) A_3^+\left(m, \frac{\lambda_2}{\eta_2}\right) \int_0^\infty \int_0^\infty F_1(y_1) F_2(y_2) \times U_3\left(\frac{\pi^3 n y_1}{\eta_1^3}\right) U_3\left(\frac{\pi^3 m y_2}{\eta_2^3}\right) J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1 y_1 a_2 y_2}\right) dy_1\, dy_2, \tag{6.6}$$

$\mathcal{T}_7$ is defined similar to $\mathcal{T}_6$ except we replace the leading coefficient by its negative, A^+ with A^-, and U_3 with V_3,

$$\mathcal{T}_8(c,x) = \frac{i\pi^3}{\eta_1^3\eta_2^3} \sum_{n\geq 1}\sum_{m\geq 1} A_3^+\left(n, \frac{\lambda_1}{\eta_1}\right) A_3^-\left(m, \frac{\lambda_2}{\eta_2}\right) \int_0^\infty \int_0^\infty F_1(y_1) F_2(y_2) \times U_3\left(\frac{\pi^3 n y_1}{\eta_1^3}\right) V_3\left(\frac{\pi^3 m y_2}{\eta_2^3}\right) J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1 y_1 a_2 y_2}\right) dy_1\, dy_2, \tag{6.7}$$

and $\mathcal{T}_9$ is defined as $\mathcal{T}_8$, but one swaps all the subscripts of 1 and 2. Here and throughout, E_3 denotes the third-order Estermann zeta function:

$$E\left(s, \frac{\lambda}{\eta}\right) = \sum_{n=1}^{\infty} \frac{\tau_3(n) e\left(\frac{n\lambda}{\eta}\right)}{n^s}. \tag{6.8}$$

Theorem 3.1.1 now follows from the following two propositions which we prove in Sections 3.7 and 3.8, respectively.

Proposition 3.6.2. *Let*

$$\mathcal{R}(q) = \frac{2\pi}{q} \sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\(a_1a_2b_1b_2,q)=1}} \frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(b_2)}{(a_1^3b_1^2a_2^3b_2^2)^{\frac{1}{2}}} \sum_{N,M}^{\mathrm{D}} \sum_{c\leq C} \frac{1}{c} \sum_{x(c)}^{*} \mathcal{K}\mathcal{T}_1(c,x).$$

Then

$$\mathcal{R}(q) \ll (\log q)^9.$$

Proposition 3.6.3. *For* $j = 2, \ldots, 9$, *let*

$$\mathcal{E}_j(q) = \frac{2\pi}{q} \sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\(a_1a_2b_1b_2,q)=1}} \frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(b_2)}{(a_1^3b_1^2a_2^3b_2^2)^{\frac{1}{2}}} \sum_{N,M}^{\mathrm{D}} \sum_{c\leq C} \frac{1}{c} \sum_{x(c)}^{*} \mathcal{K}\mathcal{T}_j(c,x).$$

Then

$$\mathcal{E}_j(q) \ll q^{-\frac{1}{8}+\varepsilon}$$

3.7. Proof of Proposition 3.6.2

Recall that

$$\mathcal{T}_1(c,x) = \operatorname*{Res}_{s_1=1}\operatorname*{Res}_{s_2=1}\Bigg[E_3\left(s_1,\frac{\lambda_1}{\eta_1}\right)E_3\left(s_2,\frac{\lambda_2}{\eta_2}\right) \\ \times \int_0^\infty\int_0^\infty \mathcal{U}(c;y_1,y_2)J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1a_2y_1y_2}\right)y_1^{s_1-1}y_2^{s_2-1}dy_1\ dy_2\Bigg].$$

To estimate the contribution from $\mathcal{T}_1$, we first compute the residues via the Laurent series for E_3. For $j_1, j_2 \geq 0$, let

$$\mathcal{I}_1(c;j_1,j_2) = \frac{1}{j_1!j_2!}\int_0^\infty\int_0^\infty \mathcal{U}(c;y_1,y_2)(\log y_1)^{j_1}(\log y_2)^{j_2}J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1a_2y_1y_2}\right)dy_1\ dy_2$$

so that

$$\int_0^\infty \int_0^\infty \mathcal{U}(c; y_1, y_2) J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1 a_2 y_1 y_2}\right) y_1^{s_1-1} y_2^{s_2-1} dy_1 \, dy_2$$
$$= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} \mathcal{I}_1(c; j_1, j_2)(s_1-1)^{j_1}(s_2-1)^{j_2}.$$

For $(\lambda, \eta) = 1$, we have (see (2.13) of [5])

$$E_3\left(s, \frac{\lambda}{\eta}\right) = \frac{D_{-3}(\eta)}{(s-1)^3} + \frac{D_{-2}(\eta)}{(s-1)^2} + \frac{D_{-1}(\eta)}{s-1} + D_0 + \cdots, \tag{7.1}$$

where

$$\begin{aligned} D_{-3}(\eta) &= \frac{1}{\eta^2}\sum_{\alpha_1=1}^{\eta}\sum_{\alpha_2=1}^{\eta} \mathbf{1}(\eta|\alpha_1\alpha_2), \\ D_{-2}(\eta) &= \frac{1}{\eta^2}\sum_{\alpha_1=1}^{\eta}\sum_{\alpha_2=1}^{\eta} \mathbf{1}(\eta|\alpha_1\alpha_2)\left(3\gamma_0\left(\frac{\alpha_1}{\eta}\right) - 3\log\eta\right), \\ D_{-1}(\eta) &= \frac{1}{\eta^2}\sum_{\alpha_1=1}^{\eta}\sum_{\alpha_2=1}^{\eta} \mathbf{1}(\eta|\alpha_1\alpha_2)\Bigg(\frac{9}{2}(\log\eta)^2 - 9\gamma_0\left(\frac{\alpha_1}{\eta}\right)\log\eta \\ &\qquad + 3\gamma_0\left(\frac{\alpha_1}{\eta}\right)\gamma_0\left(\frac{\alpha_2}{\eta}\right) + 3\gamma_1\left(\frac{\alpha_1}{\eta}\right)\Bigg). \end{aligned} \tag{7.2}$$

Here $\mathbf{1}(\eta|\alpha_1\alpha_2)$ is 1 if η divides $\alpha_1\alpha_2$ and 0 otherwise, and γ_0, γ_1 are generalized Stieltjes constants defined by

$$\zeta(s, r) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \gamma_n(r)(s-1)^n, \tag{7.3}$$

where $\zeta(s, r)$ is the Hurwitz zeta function. Thus

$$\mathcal{T}_1(x, c) = \sum_{\substack{1\le l_1, l_2\le 3 \\ 0\le j_1, j_2\le 2 \\ j_i - l_i = -1}} D_{-l_1}(\eta_1) D_{-l_2}(\eta_2) \mathcal{I}_1(c; j_1, j_2),$$

and so

$$\mathcal{R}(q) = \frac{2\pi}{q} \sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\ (a_1a_2b_1b_2,q)=1}} \frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(b_2)}{(a_1^3b_1^2a_2^3b_2^2)^{\frac{1}{2}}} \times \mathcal{K} \sum_{c\leq C} \frac{1}{c} \sideset{}{^*}\sum_{x(c)} \sum_{\substack{1\leq l_1,l_2\leq 3\\ 0\leq j_1,j_2\leq 2\\ j_i-l_i=-1}} D_{-l_1}(\eta_1)D_{-l_2}(\eta_2) \sideset{}{^{\mathrm{D}}}\sum_{N,M} \mathcal{I}_1(c;j_1,j_2). \tag{7.4}$$

As discussed in Section 3.2, we would like to estimate the factors of D_{-l_i} using Lemma 3.7.4 below. However, we will lose too much in our upper bound if we ignore the oscillations present in the integrals $\mathcal{I}_1(c;j_1,j_2)$. To take advantage of these oscillations, we apply several Fourier-analytic manipulations to suitably transform the sum over c in (7.4). This requires several technical lemmas which we collect in the following subsection. The manipulations themselves are then performed in Subsection 3.7.2. Finally, we conclude the proof of Proposition 3.6.2 in Subsection 3.7.3 by applying Lemma 3.7.4 and estimating the remaining Dirichlet series via elementary means.

3.7.1. **Preliminary Lemmas**

The first three of these are Lemmas 7.1–7.3 of [2].

Lemma 3.7.1. *Let* $(a,\ell)=1$. *We have*

$$\sideset{}{^*}\sum_{\substack{x(c\ell)\\ x\equiv a(\ell)}} 1 = c\prod_{\substack{p|c\\ p\nmid \ell}}\left(1-\frac{1}{p}\right),$$

where the sum is over x *coprime to* $c\ell$.

Lemma 3.7.2. *Let* α,β,y_1,y_2 *be nonnegative real numbers satisfying* $\alpha y_1, \beta y_2 \ll q^2$ *and define*

$$T(y_1,y_2,\alpha,\beta) = \sum_{\delta=1}^{\infty} \frac{1}{\delta} J_{k-1}\left(\frac{4\pi}{\delta}\sqrt{\alpha\beta y_1 y_2}\right)\mathcal{K}e\left(\frac{\alpha y_1}{\delta}+\frac{\beta y_2}{\delta}\right).$$

Further, let $L=q^{100}$ *and* w *be a smooth function on* $\mathbb{R}_{\geq 0}$ *with* $w(x)=1$ *if* $0\leq x\leq 1$ *and*

$w(x) = 0$ if $x > 2$. Then for any $A > 0$, we have

$$T = 2\pi \sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right) J_{k-1}\left(4\pi\sqrt{\alpha y_1 \ell}\right) J_{k-1}\left(4\pi\sqrt{\beta y_2 \ell}\right)$$
$$- 2\pi \int_0^{\infty} w\left(\frac{\ell}{L}\right) J_{k-1}\left(4\pi\sqrt{\alpha y_1 \ell}\right) J_{k-1}\left(4\pi\sqrt{\beta y_2 \ell}\right) d\ell + O(q^{-2022}).$$

Lemma 3.7.3. *Let w and L be as in Lemma 3.7.2 and let u be a complex number with $|\mathrm{Re}\, u| \ll \frac{1}{\log q}$. Then*

$$\sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right) \frac{1}{\ell^{1+u}} - \int_0^{\infty} w\left(\frac{\ell}{L}\right) \frac{1}{\ell^{1+u}}\, d\ell = \zeta(1+u) + O(q^{-20}).$$

It can be seen from the proof of this last lemma that the error term is holomorphic in u, and thus we can differentiate in u to see that

$$\sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right) \frac{(\log \ell)^j}{\ell^{1+u}} - \int_0^{\infty} w\left(\frac{\ell}{L}\right) \frac{(\log \ell)^j}{\ell^{1+u}}\, d\ell = (-1)^j \zeta^{(j)}(1+u) + O(q^{-20}). \tag{7.5}$$

The last lemma we need is an improvement on the bound $D_{-i}(\eta) \ll \eta^{-1+\varepsilon}$ used by Djanković.

Lemma 3.7.4. *For $i = 1, 2, 3$, we have*

$$D_{-i}(\eta) \ll \frac{\tau_2(\eta)(\log \eta)^{3-i}}{\eta}.$$

Proof. Note that $D_{-3}(\eta)$ is multiplicative, so

$$D_{-3}(\eta) = \prod_{p^r || \eta} D_{-3}(p^r).$$

At a prime power, we have

$$D_{-3}(p^r) = \frac{1}{p^{2r}} \sum_{j=0}^{r} \sum_{\substack{\alpha_1=1 \\ p^j || \alpha_1}}^{p^r} \sum_{\alpha_2=1}^{p^r} \mathbf{1}(p^r \mid \alpha_1\alpha_2) = \frac{1}{p^{2r}} \sum_{j=0}^{r} \sum_{\substack{\alpha_1=1 \\ p \nmid \alpha_1}}^{p^{r-j}} \sum_{\alpha_2=1}^{p^r} \mathbf{1}(p^{r-j} \mid \alpha_2)$$
$$= \frac{1}{p^{2r}} \sum_{j=0}^{r} \varphi(p^{r-j}) p^j = \frac{1}{p^r}\left(1 + r\left(1 - \frac{1}{p}\right)\right) = \frac{r+1}{p^r}\left(1 - \frac{r}{p(r+1)}\right),$$

and so

$$D_{-3}(\eta) = \frac{\tau_2(\eta)}{\eta} \prod_{p^r || \eta}\left(1 - \frac{r}{p(r+1)}\right) \ll \frac{\tau(\eta)}{\eta}.$$

To estimate D_{-2} and D_{-1}, we need the following result of Berndt (see [1], Theorem 2) for the generalized Stieltjes constants (7.3): for $x \in (0,1]$ and $n \geq 1$, we have

$$\gamma_n(x) \ll_n \frac{(\log x)^n}{x}.$$

Combining this with our estimate for D_{-3}, we see that

$$D_{-2}(\eta) \ll \frac{1}{\eta} \sum_{\alpha_1=1}^{\eta} \sum_{\alpha_2=1}^{\eta} \frac{\mathbf{1}(\eta | \alpha_1\alpha_2)}{\alpha_1} + \frac{\tau(\eta)\log\eta}{\eta}.$$

We then have

$$\sum_{\alpha_1=1}^{\eta} \sum_{\alpha_2=1}^{\eta} \frac{\mathbf{1}(\eta|\alpha_1\alpha_2)}{\alpha_1} = \sum_{d|\eta} \sum_{\substack{\alpha_1=1 \\ (\alpha_1,\eta)=\frac{\eta}{d}}}^{\eta} \sum_{\alpha_2=1}^{\eta} \frac{\mathbf{1}(\eta|\alpha_1\alpha_2)}{\alpha_1}$$
$$= \sum_{d|\eta} \frac{d}{\eta} \sum_{\substack{\alpha_1=1 \\ (\alpha_1,d)=1}}^{d} \frac{1}{\alpha_1} \sum_{\alpha_2=1}^{\eta} \mathbf{1}(d|\alpha_2) = \sum_{d|\eta} \sum_{\substack{\alpha_1=1 \\ (\alpha_1,d)=1}}^{d} \frac{1}{\alpha_1} \ll \tau(\eta)\log\eta,$$

which gives the claimed estimate for D_{-2}. Finally, for D_{-1}, our arguments above give

$$\begin{aligned} D_{-1}(\eta) &\ll \sum_{\alpha_1=1}^{\eta}\sum_{\alpha_2=1}^{\eta} \mathbf{1}(\eta|\alpha_1\alpha_2)\left((\log\eta)^2 + \frac{\eta\log\eta}{\alpha_1} + \frac{\eta^2}{\alpha_1\alpha_1} + \frac{\eta}{\alpha_1}\log\left(\frac{\eta}{\alpha_1}\right)\right) \\ &\ll \frac{\tau(\eta)(\log\eta)^2}{\eta} + \sum_{\alpha_1=1}^{\eta}\sum_{\alpha_2=1}^{\eta}\frac{\mathbf{1}(\eta|\alpha_1\alpha_2)}{\alpha_1\alpha_2}. \end{aligned}$$

Following our treatment of D_{-2}, we have

$$\begin{aligned} \sum_{\alpha_1=1}^{\eta}\sum_{\alpha_2=1}^{\eta}\frac{\mathbf{1}(\eta|\alpha_1\alpha_2)}{\alpha_1\alpha_2} &= \sum_{d|\eta}\sum_{\substack{\alpha_1=1\\(\alpha_1,\eta)=\frac{\eta}{d}}}^{\eta}\sum_{\alpha_2=1}^{\eta}\frac{\mathbf{1}(\eta|\alpha_1\alpha_2)}{\alpha_1\alpha_2} = \sum_{d|\eta}\frac{d}{\eta}\sum_{\substack{\alpha_1=1\\(\alpha_1,d)=1}}^{d}\frac{1}{\alpha_1}\sum_{\alpha_2=1}^{\eta}\frac{\mathbf{1}(d|\alpha_2)}{\alpha_2} \\ &= \frac{1}{\eta}\sum_{d|\eta}\sum_{\substack{\alpha_1=1\\(\alpha_1,d)=1}}^{d}\frac{1}{\alpha_1}\sum_{a=1}^{\eta/d}\frac{1}{a} \ll \frac{\tau(\eta)(\log\eta)^2}{\eta}, \end{aligned}$$

which gives the claimed estimate for D_{-3}.

□

3.7.2. Fourier-Analytic Manipulations

Returning to (7.4), we first transform the sums over N, M, c, x. This will allow us to determine the main term (in terms of $\log q$) of the double residue $\mathcal{R}(q)$, which we then estimate by purely arithmetic means in Subsection 3.7.3.

By the decay of the Bessel function, we may extend the sum over $c \leq C$ to all $c \geq 1$ in a similar way as our truncation in Section 3.5. Note that after extending the sum, the only parts of $\mathcal{R}(q)$ that depend on N, M are the factors of f. Because of the absolute convergence of the integrals $\mathcal{I}_1(c; j_1, j_2)$, we may execute the dyadic sums over N, M to see that

$$\begin{aligned} \sum_{N,M}^{\mathrm{D}}\mathcal{I}_1(c;j_1,j_2) = \int_0^\infty\int_0^\infty &\frac{(\log y_1)^{j_1}(\log y_2)^{j_2}}{(y_1y_2)^{\frac{1}{2}}}U_1\left(a_1^3b_1^2y_1, a_2^3b_2^2y_2\right) \\ &\times e\left(\frac{a_1^2b_1y_1}{cqa_2b_2} + \frac{a_2^2b_2y_2}{cqa_1b_1}\right)J_{k-1}\left(\frac{4\pi}{cq}\sqrt{a_1a_2y_1y_2}\right)dy_1\,dy_2. \end{aligned}$$

Thus $\mathcal{R}(q)$ is

$$\frac{2\pi}{q}\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\(a_1a_2b_1b_2,q)=1\\a_i^3b_i^2\ll q^{3/2+\varepsilon}}}\frac{\mu(a_1)\mu(a_2)\tau_3(b_1)\tau_3(b_2)}{(a_1^3b_1^2a_2^3b_2^2)^{\frac{1}{2}}}\sum_{\substack{1\leq l_1,l_2\leq 3\\0\leq j_1,j_2\leq 2\\j_i-l_i=-1}}\mathcal{I}_2(l_1,l_2,j_1,j_2) \tag{7.6}$$

where

$$\mathcal{I}_2(l_1,l_2,j_1,j_2)=\int_0^\infty\int_0^\infty\frac{(\log y_1)^{j_1}(\log y_2)^{j_2}}{(y_1y_2)^{\frac{1}{2}}}U_1\left(a_1^3b_1^2y_1,a_2^3b_2^2y_2\right)\mathcal{C}(l_1,l_2;y_1,y_2)dy_1\,dy_2,$$

and

$$\begin{aligned}\mathcal{C}(l_1,l_2;y_1,y_2)&=\mathcal{K}\sum_{c\geq 1}\frac{1}{c}J_{k-1}\left(\frac{4\pi\sqrt{a_1a_2y_1y_2}}{cq}\right)e\left(\frac{a_1^2b_1y_1}{cqa_2b_2}+\frac{a_2^2b_2y_2}{cqa_1b_1}\right)\sum_{x(c)}^{*}D_{-l_1}(\eta_1)D_{-l_2}(\eta_2)\\&=\mathcal{K}\sum_{c\geq 1}\frac{\mathcal{F}(c)}{c}\sum_{x(c)}^{*}D_{-l_1}(\eta_1)D_{-l_2}(\eta_2),\end{aligned}$$

say, where η_1,η_2 are as in (6.2). We write

$$(a_1b_1,a_2b_2)=\lambda,\qquad a_1b_1=u_1\lambda,\qquad a_2b_2=u_2\lambda,\qquad (u_1\overline{x}-u_2,c)=(u_2x-u_1,c)=\delta, \tag{7.7}$$

where $(u_1,u_2)=1$, and so

$$\eta_1=\frac{u_2c/\delta}{(a_1,u_2c/\delta)},\qquad \eta_2=\frac{u_1c/\delta}{(a_2,u_1c/\delta)}. \tag{7.8}$$

We now focus on estimating the integrals $\mathcal{I}_2$ in (7.6). To do so, we first transform the cums $\mathcal{C}$. We proceed by fixing the value of δ in the sum over x and writing

$$\begin{aligned}\mathcal{C}(l_1,l_2;y_1,y_2)&=\mathcal{K}\sum_{c\geq 1}\sum_{\delta|c}\frac{\mathcal{F}(c)}{c}\sum_{\substack{x(c)\\(u_2x-u_1,c)=\delta}}^{*}D_{-l_1}\left(\frac{u_2c/\delta}{(a_1,u_2c/\delta)}\right)D_{-l_2}\left(\frac{u_1c/\delta}{(a_2,u_1c/\delta)}\right)\\&=\mathcal{K}\sum_{\delta\geq 1}\frac{1}{\delta}\sum_{c\geq 1}\frac{\mathcal{F}(c\delta)}{c}D_{-l_1}\left(\frac{u_2c}{(a_1,u_2c)}\right)D_{-l_2}\left(\frac{u_1c}{(a_2,u_1c)}\right)\sum_{\substack{x(c\delta)\\(\frac{u_2x-u_1}{\delta},c)=1}}^{*}1.\end{aligned} \tag{7.9}$$

Möbius inversion in the sum over x gives

$$\sum_{\substack{x(c\delta) \\ (\frac{u_2x-u_1}{\delta},c)=1}}^{*} 1 = \sum_{b|c} \mu(b) \sum_{\substack{x(c\delta) \\ u_2x\equiv u_1(b\delta)}}^{*} 1.$$

Note that since $(x, b\delta) = (u_1, u_2) = 1$, the congruence $u_2 x \equiv u_1(b\delta)$ has a solution in x if and only if $(u_1u_2, b\delta) = 1$. Applying Lemma 3.7.1, the last line of (7.9) can be written

$$\sum_{c\geq 1} c \sum_{\substack{b\geq 1 \\ (u_1u_2,b)=1}} \frac{\mu(b)}{b} D_{-l_1}\left(\frac{u_2cb}{(a_1, u_2cb)}\right) D_{-l_2}\left(\frac{u_1cb}{(a_2, u_1cb)}\right) \mathcal{K} \sum_{\substack{\delta\geq 1 \\ (u_1u_2,\delta)=1}} \frac{\mathcal{F}(cb\delta)}{\delta} \prod_{\substack{p|c \\ p\nmid b\delta}} \left(1-\frac{1}{p}\right).$$

Several more applications of Möbius inversion give

$$\sum_{\substack{\delta\geq 1 \\ (u_1u_2,\delta)=1}} \frac{\mathcal{F}(cb\delta)}{\delta} \prod_{p|(c,b\delta h)} \left(1-\frac{1}{p}\right) = \sum_{h|u_1u_2} \frac{\mu(h)}{h} \sum_{\gamma|c} \frac{1}{\gamma} \sum_{g|\frac{c}{\gamma}} \frac{\mu(g)}{g} \prod_{\substack{p|c \\ p\nmid b\gamma h}} \left(1-\frac{1}{p}\right) \sum_{\delta\geq 1} \frac{\mathcal{F}(cbhg\gamma\delta)}{\delta},$$

and thus

$$\begin{aligned} \mathcal{C}(l_1, l_2; y_1, y_2) = &\sum_{h|u_1u_2} \frac{\mu(h)}{h} \sum_{\substack{b\geq 1 \\ (u_1u_2,b)=1}} \frac{\mu(b)}{b} \sum_{c\geq 1} D_{-l_1}\left(\frac{u_2cb}{(a_1, u_2cb)}\right) D_{-l_2}\left(\frac{u_1cb}{(a_2, u_1cb)}\right) \\ &\times \sum_{\gamma|c} \frac{1}{\gamma} \sum_{g|\frac{c}{\gamma}} \frac{\mu(g)}{g} \prod_{\substack{p|c \\ p\nmid b\gamma h}} \left(1-\frac{1}{p}\right) \mathcal{K} \sum_{\delta\geq 1} \frac{\mathcal{F}(cbhg\gamma\delta)}{\delta}. \end{aligned}$$

We now apply Lemma 3.7.2 with

$$\alpha = \frac{a_1^2b_1}{a_2b_2cbhg\gamma\delta q}, \qquad \beta = \frac{a_2^2b_2}{a_1b_1cbhg\gamma\delta q}.$$

Note that $\alpha y_1, \beta y_2 \ll q^{\frac{3}{2}+\varepsilon}$ by the decay of U. Thus with negligible error, we have

$$\mathcal{K}\sum_{\delta\geq 1}\frac{\mathcal{F}(cbhg\gamma\delta)}{\delta}$$
$$= 2\pi\bigg(\sum_{\ell\geq 1} w\left(\frac{\ell}{L}\right) - \int_0^\infty w\left(\frac{\ell}{L}\right)d\ell\bigg) J_{k-1}\left(4\pi\sqrt{\frac{a_1^2b_1y_1\ell}{a_2b_2cbhg\gamma q}}\right) J_{k-1}\left(4\pi\sqrt{\frac{a_2^2b_2y_2\ell}{a_1b_1cbhg\gamma q}}\right).$$

For brevity, we set

$$\mathcal{A}_1 = 4\pi\sqrt{\frac{a_1^2b_1}{a_2b_2cbhg\gamma q}}, \qquad \mathcal{A}_2 = 4\pi\sqrt{\frac{a_2^2b_2}{a_1b_1cbhg\gamma q}}.$$

Returning to the definition of $\mathcal{I}_2$, the above analysis and change of variables $y_i \to y_i^2$ give

$$\begin{aligned}\mathcal{I}_2(l_1,l_2,j_1,j_2) = Q\sum_{h|u_1u_2}\frac{\mu(h)}{h}\sum_{\substack{b\geq 1\\(u_1u_2,b)=1}}\frac{\mu(b)}{b}\sum_{c\geq 1} cD_{-l_1}\left(\frac{u_2cb}{(a_1,u_2cb)}\right)D_{-l_2}\left(\frac{u_1cb}{(a_2,u_1cb)}\right)\\ \times\sum_{\gamma|c}\frac{1}{\gamma}\sum_{g|\frac{c}{\gamma}}\frac{\mu(g)}{g}\prod_{\substack{p|c\\p\nmid b\gamma h}}\left(1-\frac{1}{p}\right)\left(\sum_{\ell\geq 1} w\left(\frac{\ell}{L}\right) - \int_0^\infty w\left(\frac{\ell}{L}\right)d\ell\right)\\ \times\int_0^\infty(\log y_1)^{j_1}U\left(\frac{a_1^3b_1^2y_1}{q^{\frac{3}{2}}}\right)J_{k-1}\left(\mathcal{A}_1y_1\sqrt{\ell}\right)dy_1\\ \times\int_0^\infty(\log y_2)^{j_2}U\left(\frac{a_2^3b_2^2y_2}{q^{\frac{3}{2}}}\right)J_{k-1}\left(\mathcal{A}_2y_2\sqrt{\ell}\right)dy_2.\end{aligned} \tag{7.10}$$

Let $\mathcal{I}_3(\ell,\mathcal{A}_1,\mathcal{A}_2)$ denote the product of integrals on the last line. Here and throughout this section, we let Q denote a positive constant, not necessarily the same at each occurrence, depending at most on j_1, j_2. Opening the factor of U, the integral in y_1 is

$$\frac{Q}{2\pi i}\int_{(\alpha_1)}\gamma(s_1)^3G^3(s_1)\left(\frac{q^{\frac{3}{2}}}{a_1^3b_1^2}\right)^{s_1}\int_0^\infty(\log y_1)^{j_1}y_1^{-2s_1}J_{k-1}\left(\mathcal{A}_1y_1\sqrt{\ell}\right)dy_1\frac{ds_1}{s_1}, \tag{7.11}$$

where $\alpha_1 > 0$, and a similar expression holds for the integral in y_2. The inner integrals are

Hankel transforms which can be evaluated explicitly using equation 6.561.14 of [6], which is

$$\int_0^\infty x^\mu J_\nu(ax)dx = 2^\mu a^{-\mu-1}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)}, \qquad (-\mathrm{Re}\ \nu - 1 < \mathrm{Re}\ \mu < \tfrac{1}{2},\ a>0).$$

Differentiating this with respect to μ, we obtain

$$\int_0^\infty x^\mu (\log x)^j J_\nu(ax)dx = (-1)^j 2^\mu a^{-\mu-1}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)}\mathcal{P}_j(\log a,\mu,\nu), \tag{7.12}$$

for $j \geq 0$, where $\mathcal{P}_j(w,\mu,\nu)$ is a monic polynomial of degree j in w with coefficients involving polygamma functions and the parameters μ,ν. For instance,

$$\mathcal{P}_1(w) = w - \log 2 - \frac{1}{2}\frac{\Gamma'(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)} - \frac{1}{2}\frac{\Gamma'(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}.$$

In the present case, specifying $\nu = k-1$ and $\mu = -2s_i$, the coefficients will be holomorphic and of rapid decay on vertical lines so long as $\alpha_i > -k/2+\varepsilon$, say. Applying (7.12), we see that (7.11) is

$$\frac{(-1)^{j_1}Q}{2\pi i}\int\limits_{(\alpha_1)} \gamma(s_1)^3 G^3(s_1)\left(\frac{4\pi^2 q^{\frac{3}{2}}}{a_1^3 b_1^2}\right)^{s_1}\left(\sqrt{\frac{a_1^2 b_1 \ell}{a_2 b_2 cbhg\gamma q}}\right)^{2s_1-1}\frac{\Gamma(\frac{k}{2}-s_1)}{\Gamma(\frac{k}{2}+s_1)}\mathcal{P}_j\left(\log\left(\mathcal{A}_1\sqrt{\ell}\right),s_1\right)\frac{ds_1}{s_1}.$$

Here we have suppressed the dependence of $\mathcal{P}_j$ on k. A similar expression holds for the integral in y_2, and thus $\mathcal{I}_3(\ell,\mathcal{A}_1,\mathcal{A}_2)$ is

$$(-1)^{j_1+j_2}Q\frac{cbhg\gamma q}{\ell\sqrt{a_1a_2}}\left(\frac{1}{2\pi i}\right)^2\int\limits_{(\alpha_1)}\int\limits_{(\alpha_2)}\gamma(s_1)^3G^3(s_1)\gamma(s_2)^3G^3(s_2)\frac{\Gamma(\frac{k}{2}-s_1)}{\Gamma(\frac{k}{2}+s_1)}\frac{\Gamma(\frac{k}{2}-s_2)}{\Gamma(\frac{k}{2}+s_2)}$$
$$\times\left(\frac{4\pi^2 q^{\frac{1}{2}}\ell}{a_1b_1a_2b_2cbhg\gamma}\right)^{s_1+s_2}\mathcal{P}_{j_1}\left(\log\left(\mathcal{A}_1\sqrt{\ell}\right),s_1\right)\mathcal{P}_{j_2}\left(\log\left(\mathcal{A}_2\sqrt{\ell}\right),s_2\right)\frac{ds_2}{s_2}\frac{ds_1}{s_1}.$$

Let $P_{j_i}(n,s_i)$ denote the coefficient of w^n in $\mathcal{P}_{j_i}(w,s_i)$. We view the product of $\mathcal{P}_{j_1}$ and $\mathcal{P}_{j_2}$

as a polynomial $\mathcal{P}$ in $\log \ell$ of degree j_1+j_2, where the coefficient of $(\log \ell)^n$ is given by

$$P(n,s_1,s_2)=\frac{1}{2^n}\sum_{n=k_1+k_2}\sum_{n_1=k_1}^{j_1}\sum_{n_2=k_2}^{j_2}\binom{n_1}{k_1}\binom{n_2}{k_2}P_{j_1}(n_1,s_1)P_{j_2}(n_2,s_2)(\log\mathcal{A}_1)^{n_1-k_1}(\log\mathcal{A}_2)^{n_2-k_2}.$$

Applying (7.5), we see that

$$\sum_{\ell\geq 1} w\left(\frac{\ell}{L}\right)\frac{\mathcal{P}(\log \ell)}{\ell^{1-s_1-s_2}}-\int_0^\infty w\left(\frac{\ell}{L}\right)\frac{\mathcal{P}(\log \ell)}{\ell^{1-s_1-s_2}}d\ell=\sum_{n=0}^{j_1+j_2}P(n,s_1,s_2)\zeta^{(n)}(1-s_1-s_2)+O(q^{-20}).$$

Let $\mathcal{P}^*(\log\mathcal{A}_1,\log\mathcal{A}_2,s_1,s_2)$ denote the sum on the right. Then

$$\begin{aligned}&\left(\sum_{\ell\geq 1} w\left(\frac{\ell}{L}\right)-\int_0^\infty w\left(\frac{\ell}{L}\right)d\ell\right)\mathcal{I}_3(\ell,\mathcal{A}_1,\mathcal{A}_2)\\ &\qquad=Q\frac{cbhg\gamma q}{\sqrt{a_1a_2}}\left(\frac{1}{2\pi i}\right)^2\int_{(\alpha_1)}\int_{(\alpha_2)}\gamma(s_1)^3G^3(s_1)\gamma(s_2)^3G^3(s_2)\frac{\Gamma(\frac{k}{2}-s_1)}{\Gamma(\frac{k}{2}+s_1)}\frac{\Gamma(\frac{k}{2}-s_2)}{\Gamma(\frac{k}{2}+s_2)}\\ &\qquad\times\left(\frac{4\pi^2q^{\frac12}}{a_1b_1a_2b_2cbhg\gamma}\right)^{s_1+s_2}\mathcal{P}^*(\log\mathcal{A}_1,\log\mathcal{A}_2,s_1,s_2)\frac{ds_2}{s_2}\frac{ds_1}{s_1}.\end{aligned}$$

say. Note that in $\mathcal{P}^*$, the coefficient of $(\log\mathcal{A}_1)^{j_1}(\log\mathcal{A}_2)^{j_2}$ is $\zeta(1-s_1-s_2)$. We deal only with the contribution of this term, as it will be clear from our analysis that the other terms of $\mathcal{P}^*$ can be treated similarly. Returning to (7.10), we see that the representative term of $\mathcal{I}_2(l_1,l_2,j_1,j_2)$ is

$$\begin{aligned}&Q\frac{q}{\sqrt{a_1a_2}}\sum_{h|u_1u_2}\mu(h)\sum_{\substack{b\geq 1\\(u_1u_2,b)=1}}\mu(b)\sum_{c\geq 1}cD_{-l_1}\left(\frac{u_2cb}{(a_1,u_2cb)}\right)D_{-l_2}\left(\frac{u_1cb}{(a_2,u_1cb)}\right)\\ &\quad\times\sum_{\gamma|c}\sum_{g|\frac{c}{\gamma}}\mu(g)\prod_{\substack{p|c\\p\nmid b\gamma h}}\left(1-\frac{1}{p}\right)(\log\mathcal{A}_1)^{j_1}(\log\mathcal{A}_2)^{j_2}\mathcal{Y}(cbhg\gamma),\end{aligned}\tag{7.13}$$

where

$$\mathcal{Y}(w) = \left(\frac{1}{2\pi i}\right)^2 \int\limits_{(1)}\int\limits_{(1)} \gamma(s_1)^3 G^3(s_1)\gamma(s_2)^3 G^3(s_2) \frac{\Gamma(\frac{k}{2}-s_1)}{\Gamma(\frac{k}{2}+s_1)}\frac{\Gamma(\frac{k}{2}-s_2)}{\Gamma(\frac{k}{2}+s_2)} \left(\frac{4\pi^2 q^{\frac{1}{2}}}{a_1b_1a_2b_2w}\right)^{s_1+s_2}$$
$$\times \zeta(1-s_1-s_2)\frac{ds_2}{s_2}\frac{ds_1}{s_1}$$
$$= \left(\frac{1}{2\pi i}\right)^2 \int\limits_{(1)}\int\limits_{(1)} \mathcal{G}(s_1,s_2)\left(\frac{4\pi^2 q^{\frac{1}{2}}}{a_1b_1a_2b_2w}\right)^{s_1+s_2} \zeta(1-s_1-s_2)\frac{ds_2}{s_2}\frac{ds_1}{s_1},$$

say, and we have taken the lines of integration to 1. Here $\mathcal{G}$ is holomorphic and decays rapidly on vertical lines so long as Re s_1, Re $s_1 < \frac{k}{2}$. Changing variables, we obtain

$$\mathcal{Y}(w) = \left(\frac{1}{2\pi i}\right)^2 \int\limits_{(1)}\int\limits_{(2)} \mathcal{G}(s,z-s)\left(\frac{4\pi^2 q^{\frac{1}{2}}}{a_1b_1a_2b_2w}\right)^{z} \zeta(1-z)\frac{du}{z-s}\frac{ds}{s}. \tag{7.14}$$

We deal first with the case $cbhg\gamma > q$. For $w > q$, we take the line of integration in z to Re $u = \frac{9}{2}$. Applying the functional equation for ζ and Stirling's formula, we see that in this case

$$\mathcal{Y}(w) \ll q^{\frac{9}{4}}\frac{1}{(a_1b_1a_2b_2w)^{\frac{9}{2}}}.$$

Recalling the definition of $\mathcal{A}_1, \mathcal{A}_2$ and noting the ranges of summations for the variables in the expression $\mathcal{I}_2$, we have

$$\log\mathcal{A}_1, \log\mathcal{A}_2 \ll \log(bcq) \ll (cbq)^{\varepsilon}.$$

Using the trivial bound $D_{-i}(\eta) \ll 1$, we find that

$$\sum_{c>q} cD_{-l_1}\left(\frac{u_2cb}{(a_1,u_2cb)}\right)D_{-l_2}\left(\frac{u_1cb}{(a_2,u_1cb)}\right)\sum_{\gamma|c}\sum_{g|\frac{c}{\gamma}}\mu(g)\prod_{\substack{p|c\\ p\nmid b\gamma h}}\left(1-\frac{1}{p}\right)\mathcal{Y}(j_1,j_2,cbhg\gamma)$$
$$\ll \frac{q^{\frac{9}{4}+\varepsilon}b^{\varepsilon}}{(a_1b_1a_2b_2bh)^{\frac{9}{2}}}\sum_{c>q}c^{-\frac{7}{2}+\varepsilon} \ll q^{-\frac{1}{4}+\varepsilon}\frac{b^{\varepsilon}}{(a_1b_1a_2b_2bh)^{\frac{9}{2}}}.$$

Using (7.6) and (7.10), we see that the contribution to $\mathcal{R}(q)$ from those terms with $c > q$ is $O(q^{-\frac{1}{4}+\varepsilon})$ (the error term can be improved here, but this suffices for our purposes).

3.7.3. The Remaining Dirichlet Series

Let $\mathcal{H}(c)$ denote the product of c and the factors of D_{-l_i}. We consider the sum over c,

$$\sum_{c\le q}\mathcal{H}(c)\sum_{\gamma|c}\sum_{g|\frac{c}{\gamma}}\mu(g)\prod_{\substack{p|c\\ p\nmid\gamma bh}}\left(1-\frac{1}{p}\right)(\log\mathcal{A}_1)^{j_1}(\log\mathcal{A}_2)^{j_2}\mathcal{Y}(cbhg\gamma).$$

Since b and h are squarefree, we may rewrite this as

$$\sum_{d|bh}\sum_{\substack{\lambda c_d\ge 1\\ (\lambda,bh)=1\\ p|c_d\iff p|d}}\mathcal{H}(\lambda c_d)\sum_{\gamma|\lambda c_d}\prod_{\substack{p|\lambda c_d\\ p\nmid\gamma bh}}\left(1-\frac{1}{p}\right)\sum_{g|\frac{\lambda c_d}{\gamma}}\mu(g)(\log\mathcal{A}_1)^{j_1}(\log\mathcal{A}_2)^{j_2}\mathcal{Y}(\lambda c_d bhg\gamma),$$

and the variable c has been modified in the definitions of $\mathcal{A}_1,\mathcal{A}_2$. For fixed $d \mid bh$ and $\gamma \mid \lambda c_d$, we write $\gamma=\gamma_1\gamma_2$, where $\gamma_1 \mid \lambda$ and $\gamma_2 \mid c_d$ so that

$$\prod_{\substack{p|\lambda c_d\\ p\nmid\gamma bh}}\left(1-\frac{1}{p}\right)=\frac{\varphi(\lambda c_d)}{\lambda c_d}\prod_{p|(\lambda c_d,\gamma bh)}\left(1-\frac{1}{p}\right)^{-1}=\frac{\varphi(\lambda c_d)}{\lambda c_d}\prod_{p|\gamma_1}\left(1-\frac{1}{p}\right)^{-1}\prod_{p|c_d}\left(1-\frac{1}{p}\right)^{-1}=\frac{\gamma_1\varphi(\lambda)}{\lambda\varphi(\gamma_1)}.$$

The sum over γ becomes

$$\frac{\varphi(\lambda)}{\lambda}\sum_{\gamma_1|\lambda}\frac{\gamma_1}{\varphi(\gamma_1)}\sum_{g_1|\frac{\lambda}{\gamma_1}}\mu(g)\sum_{\gamma_2|c_d}\sum_{g_2|\frac{c_d}{\gamma_2}}\mu(g_2)(\log\mathcal{A}_1)^{j_1}(\log\mathcal{A}_2)^{j_2}\mathcal{Y}(\lambda c_d bhg_1g_2\gamma_1\gamma_2),$$

where again, the variables in the definitions of $\mathcal{A}_1,\mathcal{A}_2$ have been appropriately modified. Moving the sums over γ_1,γ_2 inside the integral, we are led to consider the functions

$$\begin{aligned}\mathscr{C}_1(n)=\mathscr{C}_1(n,z;j_1,j_2)&=\frac{\varphi(n)}{n}\sum_{\gamma|n}\frac{\gamma}{\varphi(\gamma)}\frac{(\log\gamma)^{j_1}}{\gamma^z}\sum_{g|\frac{n}{\gamma}}\frac{\mu(g)(\log g)^{j_2}}{g^z},\\ \mathscr{C}_2(n)=\mathscr{C}_2(n,z;j_1,j_2)&=\sum_{\gamma|n}\frac{(\log\gamma)^{j_1}}{\gamma^z}\sum_{g|\frac{n}{\gamma}}\frac{\mu(g)(\log g)^{j_2}}{g^z},\end{aligned}\tag{7.15}$$

where now the j_1, j_2 are arbitrary nonnegative integers.

Proposition 3.7.5. *For all integers $n \geq 2$, $j_1, j_2 \geq 0$, and z with* $\operatorname{Re} z \geq 0$*, we have*

$$\mathscr{C}_1(n), \mathscr{C}_2(n) \leq (\log n)^{j_1+j_2}.$$

Proof. Let $v(n)$ be either 1 or $\frac{\varphi(n)}{n}$ and consider the function

$$\mathscr{C}(n,z,s) = v(n) \sum_{\gamma | n} \frac{1}{v(\gamma)\gamma^z} \sum_{g | \frac{n}{\gamma}} \frac{\mu(g)}{g^s}.$$

We will show that

$$\left[\frac{\partial^{j_1}}{\partial z^{j_1}} \frac{\partial^{j_2}}{\partial s^{j_2}} \mathscr{C}(n,z,s)\right]_{s=u} \leq (\log n)^{j_1+j_2}$$

for all $n \geq 2$ and z with $\operatorname{Re} z \geq 0$. We have

$$\begin{aligned}
\mathscr{C}(n,z,s) &= \frac{v(n)}{n^u} \prod_{p^r || n} \left(\frac{1}{v(p^r)} + \left(1 - \frac{1}{p^s}\right)\left(\frac{p^z}{v(p^{r-1})} + \cdots + \frac{p^{(r-1)z}}{v(p)} + p^{ru}\right)\right) \\
&= \prod_{p^r || n} \left(\frac{1}{p^{rz}} + \left(1 - \frac{1}{p^s}\right)\left(v(p) + \frac{1}{p^{rz}}\left(\frac{p^{rz} - p^z}{p^z - 1}\right)\right)\right) \\
&= \prod_{p^r || n} \mathscr{C}_p(z,s).
\end{aligned}$$

say, since $v(p^r) = v(p)$ for all p and $r \geq 1$. For $s = z$, we have

$$\mathscr{C}(n,z,z) = \prod_{p^r || n} \left(v(p) + \frac{v(p)-1}{p^z}\right),$$

and specifying $v(p) = 1$ and $v(p) = 1 - \frac{1}{p}$, it follows that $|\mathscr{C}(n,z,z)| \leq 1$. This gives the case $j_1 = j_2 = 0$.

To produce the logarithmic factors, we differentiate in z and s. Writing $n = p_1^{r_1} \cdots p_{\omega(n)}^{r_{\omega(n)}}$, we have

$$\frac{\partial^j}{\partial s^j} \mathscr{C}(n,z,s) = \sum_{j_1 + \cdots + j_{\omega(n)} = j} \binom{j}{j_1, \ldots, j_{\omega(n)}} \prod_{i=1}^{\omega(n)} \frac{\partial^{j_i}}{\partial s^{j_i}} \mathscr{C}_p(z,s),$$

where $\binom{j}{j_1,\ldots,j_{\omega(n)}}$ denotes the multinomial coefficient. A similar expression holds for the jth partial derivative with respect to z. For $j \geq 1$, we have

$$\frac{\partial^j}{\partial s^j}\mathscr{C}_p(z,s) = (-1)^{j-1}\left(\frac{(\log p)^j}{p^s}\right)\left(v(p) + \frac{1}{p^{rz}}\left(\frac{p^{rz}-p^z}{p^z-1}\right)\right),$$
$$\frac{\partial^j}{\partial z^j}\mathscr{C}_p(z,s) = (-\log p)^j\left(\frac{r^j}{p^{rz}} + \left(1-\frac{1}{p^s}\right)\left(\frac{1}{p^z} + \frac{2^j}{p^{2z}} + \cdots + \frac{(r-1)^j}{p^{(r-1)z}}\right)\right).$$

Since $v(p) \leq 1$, we have

$$\left|v(p) + \frac{1}{p^{rz}}\left(\frac{p^{rz}-p^z}{p^z-1}\right)\right| \leq r,$$

and thus

$$\left|\frac{\partial^j}{\partial s^j}\mathscr{C}_p(z,s)\right| \leq r(\log p)^j$$

for $\operatorname{Re} z, \operatorname{Re} s \geq 0$. Likewise, if we set $s = z$, then

$$\left[\frac{\partial^j}{\partial z^j}\mathscr{C}_p(z,s)\right]_{s=u} = (-\log p)^j\left(\frac{1}{p^{rz}} + \frac{2^j-1}{p^{2z}} + \cdots \frac{r^j-(r-1)^j}{p^{rz}}\right),$$

and thus

$$\left|\left[\frac{\partial^j}{\partial z^j}\mathscr{C}_p(u,s)\right]_{s=u}\right| \leq (\log p)^j(1 + (2^j-1) + \cdots + (r^j-(r-1)^j)) = (r\log p)^j$$

so long as $\operatorname{Re} z \geq 0$. Since we have already shown that $|\mathscr{C}_p(z,s)| \leq 1$, we deduce that

$$\left|\left[\frac{\partial^j}{\partial s^j}\mathscr{C}(n,z,s)\right]_{s=z}\right| \leq \sum_{j_1+\cdots+j_{\omega(n)}=j}\binom{j}{j_1,\ldots,j_{\omega(n)}}\prod_{\substack{i=1\\ j_i>0}}^{\omega(n)} r_i(\log p_i)^{j_i} \leq (\log n)^j,$$

and the same estimate holds for the jth partial with respect to u evaluated when $s = u$. It remains to deal with the case when both j_1, j_2 are nonzero. In this case, we have

$$\frac{\partial^{j_1}}{\partial u^{j_1}}\frac{\partial^{j_2}}{\partial s^{j_2}}\mathscr{C}_p(z,s) = \frac{-(-\log p)^{j_1+j_2}}{p^s}\left(\frac{1}{p^z} + \frac{2^{j_1}}{p^{2z}} + \cdots + \frac{(r-1)^{j_1}}{p^{(r-1)z}}\right),$$

and as before, we find that

$$\begin{aligned}\left|\left[\frac{\partial^{j_1}}{\partial z^{j_1}}\mathscr{C}(n,z,s)\right]_{s=z}\right| &= \left|\sum_{\substack{k_1+\cdots+k_{\omega(n)}=j_1\\ l_1+\cdots+l_{\omega(n)}=j_2}}\binom{j_1}{k_1,\ldots,k_{\omega(n)}}\binom{j_2}{l_1,\ldots,l_{\omega(n)}}\prod_{i=1}^{\omega(n)}\frac{\partial^{k_i}}{\partial z^{k_i}}\frac{\partial^{l_i}}{\partial s^{l_i}}\mathscr{C}_p(z,s)\right| \\ &\leq \sum_{\substack{k_1+\cdots+k_{\omega(n)}=j_1\\ l_1+\cdots+l_{\omega(n)}=j_2}}\binom{j_1}{k_1,\ldots,k_{\omega(n)}}\binom{j_2}{l_1,\ldots,l_{\omega(n)}}\prod_{\substack{i=1\\ (k_i,l_i)\neq(0,0)}}^{\omega(n)}(r_i\log p_i)^{k_i+l_i} \\ &\leq (\log n)^{j_1+j_2}.\end{aligned}$$

□

Returning to our analysis, we now study the sum over $c \leq q$ with the additional assumption that $cbhg\gamma \leq q$, so $\log \mathcal{A}_1, \log \mathcal{A}_2 \ll \log q$. We decompose the sum over c as in the beginning of this subsection, move the sums over g, γ inside the integral, take the line of integration in $\mathcal{Y}$ to $\operatorname{Re} z = \frac{1}{\log q}$, and apply Proposition 3.7.5. The factor $(\log \mathcal{A}_1)^{j_1}(\log \mathcal{A}_2)^{j_2}$ in (7.13) produces products of logarithms of various combinations of the summation variables, but no matter how they are arranged, their boundedness by $\log q$ combined with Proposition 3.7.5 shows that we obtain a power of $(\log q)^{j_1+j_2}$ that may be factored through the entire sum after applying the triangle inequality. From (7.6) and (7.13), we deduce that the representative term of $\mathcal{R}(q)$ is bounded by

$$\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\ (a_1a_2b_1b_2,q)=1\\ a_i^3b_i^2\ll q^{3/2+\varepsilon}}}\frac{\tau_3(b_1)\tau_3(b_2)\tau_2(u_1u_2)}{a_1^2b_1a_2^2b_2}\sum_{\substack{1\leq l_1,l_2\leq 3\\ 0\leq j_1,j_2\leq 2\\ j_i-l_i=-1}}(\log q)^{j_1+j_2+1}$$
$$\times\sum_{b\geq 1}\sum_{c\leq q}c\left|D_{-l_1}\left(\frac{u_2cb}{(a_1,u_2cb)}\right)D_{-l_2}\left(\frac{u_1cb}{(a_2,u_1cb)}\right)\right|,$$

where the extra $\log q$ comes from the factor of ζ in $\mathcal{Y}$ and we have ignored the contribution from $c > q$. Each variable in the summation is bounded a power of q, and thus so are the arguments of D_{-l_i}. We note at this point that if one considers a term other than the leading term in $\mathcal{P}^*$, one obtains a higher power of $\log q$ from the zeta factor (which will have been differentiated some additional number of times), but the total powers of logarithms of the other variables are smaller, and so we still obtain the same estimate as above. Thus after applying Lemma 3.7.4, we obtain

$$\mathcal{R}(q) \ll (\log q)^5 \sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\ (a_1a_2b_1b_2,q)=1\\ a_i^3b_i^2\ll q^{3/2+\varepsilon}}} \frac{\tau_3(b_1)\tau_3(b_2)\tau_2(u_1u_2)}{a_1^2b_1a_2^2b_2}$$
$$\times \sum_{b\geq 1}\frac{1}{b^2}\sum_{c\leq q}\frac{(a_1,u_2cb)(a_2,u_1cb)}{c}\tau\left(\frac{u_2cb}{(a_1,u_2cb)}\right)\tau\left(\frac{u_1cb}{(a_2,u_1cb)}\right).$$

Using the bounds

$$\tau_j(ab)\leq \tau_j(a)\tau_j(b), \qquad \tau_j\left(\frac{a}{d}\right)\leq \tau_j(a) \text{ if } d\mid a, \qquad \tau_j(a)\ll a^{\varepsilon}, \qquad (a,bc)\leq (a,b)(a,c),$$

and neglecting several summation conditions, we find that

$$\mathcal{R}(q) \ll (\log q)^5 \sum\sum\sum\sum_{a_1,b_1,a_2,b_2\geq 1} \frac{(a_1b_1a_2b_2)^{\varepsilon}(a_1b_1,a_2b_2)^2(a_1,u_2)(a_2,u_1)}{(a_1a_2)^3(b_1b_2)^2}$$
$$\times\left(\sum_{b\geq 1}\frac{(a_1a_2,b)\tau_2(b)^2}{b^2}\right)\left(\sum_{c\leq q}\frac{(a_1,c)(a_2,c)\tau_2(c)^2}{c}\right).$$

The sum over b is

$$\sum_{d\mid a_1a_2}\frac{1}{d}\sum_{\substack{b\geq 1\\ (\frac{a_1a_2}{d},b)=1}}\frac{\tau_2(bd)^2}{b^2}\leq \sum_{d\mid a_1a_2}\frac{\tau_2(d)^2}{d}\sum_{b\geq 1}\frac{\tau_2(b)^2}{b^2}\ll \tau_2(a_1a_2)\ll (a_1a_2)^{\varepsilon},$$

so we are left to consider

$$(\log q)^5 \sum\sum\sum\sum_{a_1,b_1,a_2,b_2\geq 1} \frac{(a_1b_1a_2b_2)^{\varepsilon}(a_1b_1,a_2b_2)^2(a_1,u_2)(a_2,u_1)}{(a_1a_2)^3(b_1b_2)^2}\sum_{c\leq q}\frac{(a_1,c)(a_2,c)\tau_2(c)^2}{c}.$$

Let $\delta = (a_1, a_2)$ and write $a_1 = \delta\lambda_1$, $a_2 = \delta\lambda_2$ with $(\lambda_1, \lambda_2) = 1$. Then the sum over c is

$$\begin{aligned}\sum_{d|\delta} \sum_{\substack{c\le q\\(c,\delta)=d}} \frac{(\delta\lambda_1, c)(\delta\lambda_2, c)\tau_2(c)^2}{c} &\le \sum_{d|\delta} \tau(d)^2 d \sum_{\substack{c\le q/d\\(c,\frac{\delta}{d})=1}} \frac{(\lambda_1\lambda_2, c)\tau_2(c)^2}{c}\\ \le \sum_{d|\delta} \tau(d)^2 d \sum_{c\le q} \frac{(\lambda_1\lambda_2, c)\tau_2(c)^2}{c} &\le \sum_{d|\delta} \tau(d)^2 d \sum_{g|\lambda_1\lambda_2} \sum_{c\le q} \frac{\tau_2(cg)^2}{c}\\ \le \sum_{d|\delta} \tau(d)^2 d \sum_{g|\lambda_1\lambda_2} \tau_2(g)^2 \sum_{c\le q} \frac{\tau_2(c)^2}{c}&.\end{aligned}$$

The inner sum over c on the right is bounded by $(\log q)^4$, the sum over g by $(a_1a_2)^\varepsilon$, and the sum over d by $(a_1a_2)^\varepsilon(a_1, a_2)$. Thus

$$\mathcal{R}(q) \ll (\log q)^9 \sum\sum\sum\sum_{a_1,b_1,a_2,b_2\ge1} \frac{(a_1b_1a_2b_2)^\varepsilon (a_1b_1, a_2b_2)^2 (a_1, u_2)(a_2, u_1)(a_1, a_2)}{(a_1a_2)^3(b_1b_2)^2}.$$

Let $\mathscr{D}$ denote the sum on the right. To see that $\mathscr{D}$ converges, we take $\varepsilon \le \frac{1}{4}$ and express $\mathscr{D}$ as an Euler product $\mathscr{D} = \prod_p \mathscr{D}_p$ with

$$\mathscr{D}_p = \sum\sum\sum\sum_{a_1,b_1,a_2,b_2\ge0} p^{y(a_1,b_1,a_2,b_2)},$$

and

$$\begin{aligned}y(a_1, b_1, a_2, b_2) = \varepsilon(a_1 + b_1 + a_2 + b_2) + 2\min(a_1 + b_1, a_2 + b_2) + \min(a_1, u_2) + \min(a_2, u_1)\\ + \min(a_1, a_2) - 3(a_1 + a_2) - 2(b_1 + b_2).\end{aligned}$$

Here we have written $u_i = a_i + b_i - \min(a_1 + b_1, a_2 + b_2)$. It suffices to show that for a_i, b_i not all 0, we have $y(a_1, b_1, a_2, b_2) \le -\frac{3}{2}$, say. We have trivially that

$$\begin{aligned}y(a_1, b_1, a_2, b_2) &\le \tfrac{1}{4}(a_1 + b_1 + a_2 + b_2) + (a_1 + b_1 + a_2 + b_2) + a_1 + a_2\\ &\quad + \tfrac{1}{2}(a_1 + a_2) - 3(a_1 + a_2) - 2(b_1 + b_2)\\ &= -\tfrac{1}{4}(a_1 + a_2) - \tfrac{3}{4}(b_1 + b_2).\end{aligned}$$

Thus we may assume that $a_1+a_2 \le 5$ and $b_1+b_2 \le 1$. This leaves only a few cases to check, and one may verify by direct computation that we indeed have $y(a_1,b_1,a_2,b_2) \le -\frac{3}{2}$ unless all the a_i, b_i are 0. Therefore the sum converges, and we have

$$\mathcal{R}(q) \ll (\log q)^9.$$

3.8. Proof of Proposition 3.6.3

The analysis of the other 8 terms coming from Voronoi summation adheres closely to the analysis in Section 8 of [2]. Recall that by the decay of U, we may assume $a_i^3 b_i^2 N_i \ll q^{3/2+\varepsilon}$. For $j = 2, \ldots, 8$, let

$$E_j(\boldsymbol{a},\boldsymbol{b},\boldsymbol{N}) = \sum_{c\le C} \frac{1}{c} \sum_{x(c)}^{*} \mathcal{T}_j(c,x).$$

Changing variables in the sum over c as in (7.7) – (7.9), we have

$$E_j(\boldsymbol{a},\boldsymbol{b},\boldsymbol{N}) = \sum_{\delta\le C} \frac{1}{\delta} \sum_{c\le C/\delta} \frac{1}{c} \sum_{\substack{x(c\delta)\\(u_2x-u_1,c\delta)=\delta}}^{*} \mathcal{T}_j(c\delta,x),$$

where u_1, u_2 are as in (7.7).

3.8.1. The Sums $\mathcal{T}_2, \ldots, \mathcal{T}_5$

Since each of these sums has the same form and behavior, we treat only $\mathcal{T}_2$. The residue in the definition of $\mathcal{T}_2$ gives

$$\begin{aligned}
&\sum_{\substack{x(c\delta)\\(u_2x-u_1,c\delta)=\delta}}^{*} \mathcal{T}_j(c\delta,x)\\
&\quad= \frac{\pi^{\frac{3}{2}}}{\eta_2^3} \sum_{m\ge1} A_3^+\left(m,\frac{\lambda_2}{\eta_2}\right) \int_0^\infty F_1(y_1)\left(D_{-1}(\eta_1)+D_{-2}(\eta_1)\log y_1+\tfrac{1}{2}D_{-3}(\eta_1)(\log y_1)^2\right)\\
&\qquad\times \int_0^\infty F_2(y_2)U_3\left(\frac{\pi^3 m y_2}{\eta_2^3}\right) J_{k-1}\left(\frac{4\pi}{c\delta q}\sqrt{a_1a_2y_1y_2}\right)dy_2\,dy_1 \sum_{\substack{x(c\delta)\\(u_2x-u_1,c\delta)=\delta}}^{*} 1
\end{aligned}$$

where as before,

$$F_1(y) = y^{-\frac{1}{2}} f\left(\frac{y}{N}\right) e\left(\frac{a_1^2 b_1 y}{c\delta q a_2 b_2}\right) U\left(\frac{a_1^3 b_1^2 y}{q^{\frac{3}{2}}}\right),$$
$$F_2(y) = y^{-\frac{1}{2}} f\left(\frac{y}{M}\right) e\left(\frac{a_2^2 b_2 y}{c\delta q a_1 b_1}\right) U\left(\frac{a_2^3 b_2^2 y}{q^{\frac{3}{2}}}\right).$$

By Lemma 3.7.4, we have $D_{-i}(\eta_1) \ll q^\varepsilon \eta_1^{-1}$, and from (8.9) of [10], we have

$$A_3^{\pm}\left(m, \frac{\lambda}{\eta}\right) \ll (\eta m)^\varepsilon \eta^{\frac{3}{2}} m^{\frac{1}{4}}. \tag{8.1}$$

We analyze the term coming from D_{-1}, as the analysis of the other two terms is nearly identitcal. Thus

$$E_2(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{N}) \ll q^\varepsilon \sum_{\delta \le C} \sum_{c \le C/\delta} \frac{1}{\eta_1 \eta_2^{\frac{3}{2}}} \sum_{m \ge 1} m^{\frac{1}{4}+\varepsilon} \left|I(m)\right|,$$

where

$$\begin{aligned} I(m) &= \int_0^\infty F_1(y_1) \int_0^\infty F_2(y_2) U_3\left(\frac{\pi^3 m y_2}{\eta_2^3}\right) J_{k-1}\left(\frac{4\pi}{c\delta q}\sqrt{a_1 a_2 y_1 y_2}\right) dy_2 \; dy_1, \\ &= \int_0^\infty F_1(y_1) I_1(m, y_1) dy_1, \end{aligned}$$

say, and we have bounded the sum over x trivially by $c\delta$. To estimate E_2, we write

$$E_2(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{N}) = H_1 + H_2,$$

where H_1 is the contribution to E_2 from $m \le q^\varepsilon \eta_2^3/M$, and H_2 is the rest.

The Contribution of H_1

For brevity, put $C_1 = q^{-1}\sqrt{a_1 a_2 N M}$. Using (3.17) of [10], which is

$$U_3(x) \ll x^\varepsilon, \tag{8.2}$$

and (5.5), we have

$$I(m) \ll q^{\varepsilon}(NM)^{\frac{1}{2}} \min\left(\left(\frac{C_1}{c\delta}\right)^{k-1}, \left(\frac{C_1}{c\delta}\right)^{-\frac{1}{2}}\right),$$

and so

$$\begin{aligned}
H_1 &\ll q^{\varepsilon} \sum_{\delta \le C} \sum_{c \le C/\delta} \frac{1}{\eta_1 \eta_2^{\frac{3}{2}}} \sum_{m \le \eta_2^3 q^{\varepsilon}/M} m^{\frac{1}{4}} (NM)^{\frac{1}{2}} \min\left(\left(\frac{C_1}{c\delta}\right)^{k-1}, \left(\frac{C_1}{c\delta}\right)^{-\frac{1}{2}}\right) \\
&\ll q^{\varepsilon} \frac{N^{\frac{1}{2}}}{M^{\frac{3}{4}}} \sum_{\delta \le C} \sum_{c \le C/\delta} \frac{\eta_2^{\frac{9}{4}}}{\eta_1} \min\left(\left(\frac{C_1}{c\delta}\right)^{k-1}, \left(\frac{C_1}{c\delta}\right)^{-\frac{1}{2}}\right) \\
&\ll q^{\varepsilon} \frac{N^{\frac{1}{2}} a_1 u_1^{\frac{9}{4}}}{M^{\frac{3}{4}} u_2} \sum_{\delta \le C} \sum_{c \le C/\delta} c^{\frac{5}{4}} \min\left(\left(\frac{C_1}{c\delta}\right)^{k-1}, \left(\frac{C_1}{c\delta}\right)^{-\frac{1}{2}}\right) \\
&\ll q^{\varepsilon} \frac{N^{\frac{1}{2}} a_1 u_1^{\frac{9}{4}}}{M^{\frac{3}{4}} u_2} C_1^{\frac{9}{4}} \\
&\ll (a_1^3 b_1^2 N)^{\frac{13}{8}} (a_2^3 b_2^2 M)^{\frac{3}{8}} q^{-\frac{9}{4}+\varepsilon} \\
&= q^{\frac{3}{4}+\varepsilon}.
\end{aligned}$$

Here we have used the estimates

$$\eta_2 \le u_1 c, \qquad \eta_1 \ge \frac{u_2 c}{a_1}.$$

Summing the above estimate over $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{N}$ gives the desired result. All of the estimates that follow will be sufficient when summed over these variables, so we omit this sort of remark in what follows.

The Contribution of H_2

To handle H_2, we use the following identity for U_3, given by (3.12) of [10]. For some suitable constants c_j, d_j, we have

$$U_3(\pi^3 x) = \sum_{j=1}^{K} \frac{1}{x^{\frac{j}{3}}} \left(c_j e\left(3x^{\frac{1}{3}}\right) + d_j e\left(-3x^{\frac{1}{3}}\right) \right) + O\left(\frac{1}{x^{(K+1)/3}} \right). \tag{8.3}$$

In the present case, we have $\frac{\pi^3 m y_2}{\eta_1^3} \gg q^{\varepsilon}$. Thus

$$\begin{aligned} I_1(m, y_1) = \sum_{j=1}^{K} \left(\frac{\eta_2}{M^{\frac{1}{3}} m^{\frac{1}{3}}} \right)^j \int_0^\infty F_2(y_2) \left(\frac{M}{y_2} \right)^{\frac{j}{3}} \left(c_j e\left(\frac{3m^{\frac{1}{3}} y_2^{\frac{1}{3}}}{\eta_2} \right) + d_j e\left(-\frac{3m^{\frac{1}{3}} y_2^{\frac{1}{3}}}{\eta_2} \right) \right) \\ \times J_{k-1}\left(\frac{4\pi}{c\delta q} \sqrt{a_1 a_2 y_1 y_2} \right) dy_2 + O\left(q^{-2022}\right), \end{aligned} \tag{8.4}$$

for K sufficiently large in terms of ε. This also gives the trivial bound

$$I(m) \ll \frac{\eta_2}{m^{\frac{1}{3}}} N^{\frac{1}{2}} M^{\frac{1}{6}} \ll q^{\varepsilon} (NM)^{\frac{1}{2}}. \tag{8.5}$$

Let $C_2 = 8\pi(q\delta)^{-1}\sqrt{a_1 a_2 NM}$. We divide into two cases depending as $c \le C_2$ and $c > C_2$.

Case 1: $c > C_2$**.** Using (5.4), we can write I_1 as

$$\begin{aligned} I_1(m, y_1) = \sum_{j=1}^{K} \left(\frac{\eta_2}{M^{\frac{1}{3}} m^{\frac{1}{3}}} \right)^j \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!(\ell+k-1)!} \\ \times \int_0^\infty \mathcal{F}_j(y_1, y_2, \ell) \left(c_j e\left(\frac{3m^{\frac{1}{3}} y_2^{\frac{1}{3}}}{\eta_2} \right) + d_j e\left(-\frac{3m^{\frac{1}{3}} y_2^{\frac{1}{3}}}{\eta_2} \right) \right) e\left(\frac{a_2^2 b_2 y_2}{c\delta q a_1 b_1} \right) dy_2, \end{aligned}$$

where

$$\mathcal{F}_j(y_1, y_2, \ell) = y_2^{-\frac{1}{2}} \left(\frac{y_2}{M} \right)^{-\frac{j}{3}} f\left(\frac{y_2}{M} \right) \left(\frac{2}{c\delta q} \sqrt{a_1 a_2 y_1 y_2} \right)^{2\ell+k-1} U\left(\frac{a_2^3 b_2^2 y_2}{q^{\frac{3}{2}}} \right).$$

We now analyze the integrals

$$\int_0^\infty \mathcal{F}_j(y_1, y_2, \ell)e(\omega_\pm(m, y_2))dy_2,$$

where

$$\omega_\pm(m, y_2) = \pm\frac{3m^{\frac{1}{3}}}{\eta_2}y_2^{\frac{1}{3}} + By_2, \qquad B = \frac{a_2^2 b_2}{c\delta q a_1 b_1}.$$

We have

$$\omega'_\pm(m, y_2) = \pm\frac{m^{\frac{1}{3}}}{y_2^{\frac{2}{3}}\eta_2} + B$$

If $m > 64(B\eta_2)^3 M^2$ or $m < \frac{1}{64}(B\eta_2)^3 M^2$, we have $\omega'_\pm(m, y_2) \gg \frac{m^{\frac{1}{3}}}{y_2^{\frac{2}{3}}\eta_2} \gg \frac{q^\varepsilon}{M}$. Thus the contribution of these terms is negligible by integrating by parts many times. Thus we need only consider those m for which $m \asymp (B\eta_2)^3 M^2$. But since

$$(B\eta_2)^3 M^2 \ll \frac{q^\varepsilon}{\delta^3},$$

there are no terms of this form unless $M \gg \frac{q^{\frac{3}{2}}}{(a_2^2 b_2)^{\frac{3}{2}}}$ and $\delta \ll q^\varepsilon$. Using the trivial bound (8.5) we see that the contribution to H_2 of these terms is bounded by

$$\begin{aligned}
q^\varepsilon N^{\frac{1}{2}} M^{\frac{1}{6}} \sum_{\delta \ll q^\varepsilon} \sum_{c > C_2} \frac{1}{\eta_1 \eta_2^{\frac{1}{2}}} \sum_{m \ll q^\varepsilon} m^{-\frac{1}{12}+\varepsilon} &\ll q^\varepsilon N^{\frac{1}{2}} M^{\frac{1}{6}} a_1^{\frac{5}{4}} b_1^{\frac{1}{4}} a_2^{\frac{1}{4}} b_2^{-\frac{1}{4}} \sum_{\delta \ll q^\varepsilon} \sum_{c > C_2} c^{-\frac{3}{2}} \\
&\ll q^\varepsilon N^{\frac{1}{2}} M^{\frac{1}{6}} a_1^{\frac{5}{4}} b_1^{\frac{1}{4}} a_2^{\frac{1}{4}} b_2^{-\frac{1}{4}} \left(\frac{q}{\sqrt{NMa_1a_2}}\right)^{\frac{1}{2}} \\
&\ll q^{\frac{1}{2}+\varepsilon} a_1 b_1^{\frac{1}{4}} N^{\frac{1}{4}} b_2^{-\frac{1}{4}} M^{-\frac{1}{12}} \\
&\ll q^{\frac{1}{2}+\varepsilon} b_1^{\frac{1}{4}} N^{\frac{1}{4}} b_2^{-\frac{1}{4}} \left(\frac{(a_2^2 b_2)^{\frac{3}{2}}}{q^{\frac{3}{2}}}\right)^{\frac{1}{12}} \\
&\ll q^{\frac{3}{8}+\varepsilon} (a_1 a_2)^{\frac{1}{4}} \left(a_1^3 b_1^2 N\right)^{\frac{1}{4}} \\
&\ll (a_1 a_2)^{\frac{1}{4}} q^{\frac{3}{4}+\varepsilon}.
\end{aligned}$$

Here we have used the estimate

$$\eta_1\eta_2^{\frac{1}{2}} \geq \frac{u_2 c}{a_1}\left(\frac{u_1 c}{a_2}\right)^{\frac{1}{2}} = c^{\frac{3}{2}}\frac{b_1^{\frac{1}{2}}a_2^{\frac{1}{2}}b_2}{a_1^{\frac{1}{2}}(a_1b_1, a_2b_2)^{\frac{3}{2}}} \geq c^{\frac{3}{2}}\frac{b_1^{\frac{1}{2}}a_2^{\frac{1}{2}}b_2}{a_1^{\frac{1}{2}}(a_1b_1a_2b_2)^{\frac{3}{4}}} = c^{\frac{3}{2}}a_1^{-\frac{5}{4}}b_1^{-\frac{1}{4}}a_2^{-\frac{1}{4}}b_2^{\frac{1}{4}}.$$

Case 2: $c \leq C_2$**.** We return to (8.4), but instead use (5.3) in place of (5.4) to write I_1 as

$$I_1(m, y_1) = \sum_{j=1}^{K}\left(\frac{\eta_2}{M^{\frac{1}{3}}m^{\frac{1}{3}}}\right)^j \int_0^\infty F_2(y_2)\left(\frac{M}{y_2}\right)^{\frac{j}{3}}\left(c_j e\left(\frac{3m^{\frac{1}{3}}y_2^{\frac{1}{3}}}{\eta_2}\right) + d_j e\left(-\frac{3m^{\frac{1}{3}}y_2^{\frac{1}{3}}}{\eta_2}\right)\right)$$
$$\times\left(\frac{c\delta q}{2\sqrt{a_1a_2y_1y_2}}\right)^{\frac{1}{2}}\left(2\mathrm{Re}\, W\left(\frac{4\pi}{c\delta q}\sqrt{a_1a_2y_1y_2}\right)e\left(\frac{2}{c\delta q}\sqrt{a_1a_2y_1y_2} - \frac{k}{4} + \frac{1}{8}\right)\right)dy_2$$
$$+ O\left(q^{-2022}\right).$$

Note that this gives the trivial bound

$$I(m) \ll \frac{\eta_2}{m^{\frac{1}{3}}}\left(\frac{c\delta q}{\sqrt{a_1a_2}}\right)^{\frac{1}{2}} N^{\frac{1}{4}}M^{-\frac{1}{12}}. \tag{8.6}$$

Define

$$\mathcal{H}_j(y_1, y_2) = y_1^{-\frac{1}{4}}y_2^{-\frac{3}{4}}\left(\frac{M}{y_2}\right)^{\frac{j}{3}} f\left(\frac{y_2}{M}\right) W_1\left(\frac{4\pi}{c\delta q}\sqrt{a_1a_2y_1y_2}\right) U\left(\frac{a_2^3b_2^2y_2}{q^{\frac{3}{2}}}\right),$$

where W_1 is either W or $\overline{W}$. For some absolute constants b_j, we find that I_1 is (up to a negligible error term) a sum of expressions of the form

$$I_1(m, y_1) = \left(\frac{\sqrt{c\delta q}}{(a_1a_2)^{\frac{1}{4}}}\right)\sum_{j=1}^{K} b_j\left(\frac{\eta_2}{M^{\frac{1}{3}}m^{\frac{1}{3}}}\right)^j \int_0^\infty \mathcal{H}_j(y_1, y_2)e(\omega(y_1, y_2))\, dy_2,$$

where

$$\omega(y_1, y_2) = \pm\frac{3m^{\frac{1}{3}}}{\eta_2}y_2^{\frac{1}{3}} \pm 2Ay_2^{\frac{1}{2}} + By_2, \qquad A = \frac{(a_1a_2y_1)^{\frac{1}{2}}}{c\delta q}, \quad B = \frac{a_2^2b_2}{c\delta q a_1 b_1}.$$

Differentiating with respect to y_2 gives

$$\omega'(y_1, y_2) = \pm\frac{m^{\frac{1}{3}}}{\eta_2} y_2^{-\frac{2}{3}} \pm A y_2^{-\frac{1}{2}} + B y_2.$$

We divide into several cases.

Case 1.1: $a_1^{\frac{3}{2}} b_1 N^{\frac{1}{2}} \geq 4 a_2^{\frac{3}{2}} b_2 M^{\frac{1}{2}}$. Then it is easily checked that

$$\frac{1}{2}\frac{A}{y_2^{\frac{1}{2}}} \leq \left|\pm A y_2^{-\frac{1}{2}} + B y_2\right| \leq 2\frac{A}{y_2^{\frac{1}{2}}}.$$

If $m \geq 64(A\eta_2)^3 M^{\frac{1}{2}}$ or $m \leq \frac{1}{64}(A\eta_2)^3 M^{\frac{1}{2}}$, then $|\omega'(y_1, y_2)| \gg \frac{m^{\frac{1}{3}}}{y_2^{\frac{2}{3}}\eta_2} \gg \frac{q^\varepsilon}{M}$, since $n \gg \frac{\eta_2^3 q^\varepsilon}{M}$. Thus we may integrate by parts many times to see that the contribution of these terms is negligible. For the terms with $\frac{1}{64}(A\eta_2)^3 M^{\frac{1}{2}} \leq m \leq 64(A\eta_2)^3 M^{\frac{1}{2}}$, note that

$$(A\eta_2)^3 M^{\frac{1}{2}} \ll \frac{(a_1^3 b_1^2 N)^{\frac{3}{2}} (a_2^3 M)^{\frac{1}{2}}}{q^3 \delta^3} \ll \frac{q^\varepsilon}{\delta^3}.$$

Moreover, the left side is only $\gg 1$ if $M \gg q^{\frac{3}{2}}/a_2^3$ and $\delta \ll q^\varepsilon$. By (8.6), the contribution from these terms is bounded by

$$\begin{aligned}
q^{\frac{1}{2}+\varepsilon} \frac{(NM)^{\frac{1}{4}}}{M^{\frac{1}{3}}(a_1 a_2)^{\frac{1}{4}}} \sum_{\delta \leq q^\varepsilon} \delta^{\frac{1}{2}} \sum_{c \leq C_2} \frac{c^{\frac{1}{2}}}{\eta_1 \eta_2^{\frac{1}{2}}} \sum_{m \ll q^\varepsilon} m^{-\frac{1}{12}+\varepsilon}
&\ll q^{\frac{1}{2}+\varepsilon} \frac{(NM)^{\frac{1}{4}}}{M^{\frac{1}{3}}(a_1 a_2)^{\frac{1}{4}}} a_1 a_2^{\frac{1}{2}} \\
&= a_2^{\frac{1}{2}} q^{\frac{1}{2}+\varepsilon} (a_1^3 N)^{\frac{1}{4}} (a_2^3 M)^{\frac{1}{4}} (a_2^3 M)^{-\frac{1}{3}} \\
&\ll a_2^{\frac{1}{2}} q^{\frac{3}{4}+\varepsilon},
\end{aligned}$$

where we have used the trivial bound $\eta_i \geq \frac{c}{a_i}$.

Case 1.2: $a_2^{\frac{3}{2}} b_2 M^{\frac{1}{2}} \geq 4 a_1^{\frac{3}{2}} b_1 N^{\frac{1}{2}}$. Then as before, one checks that $\frac{1}{2}B \leq \left|\pm A y_2^{-\frac{1}{2}} + B\right| \leq \frac{3}{2}B$. By the same arguments as in the previous case, the range of m that should be considered is of size $(B\eta_2)^3 M$, and the contribution to H_2 of this range is bounded by $a_2^{\frac{1}{2}} q^{\frac{3}{4}+\varepsilon}$.

Case 1.3: $\frac{1}{4} a_2^{\frac{3}{2}} b_2 M^{\frac{1}{2}} < a_1^{\frac{3}{2}} b_1 N^{\frac{1}{2}} < 4 a_2^{\frac{3}{2}} b_2 M^{\frac{1}{2}}$. In this case $Ay_2^{-\frac{1}{2}} \asymp B$, and so the range of m that should be considered is of size $(A\eta_2)^3 M^{\frac{1}{2}}$ by the same arguments above, and the contribution from this range is also bounded by $a_2^{\frac{1}{2}} q^{\frac{3}{4}+\varepsilon}$.

3.8.2. **The Sums $\mathcal{T}_6, \ldots, \mathcal{T}_9$**

Each of the four sums $\mathcal{T}_6, \ldots, \mathcal{T}_9$ has essentially the same form and behavior, so we deal only with $\mathcal{T}_6$. The treatment these sums is very similar to that of $\mathcal{T}_2, \ldots, \mathcal{T}_5$, so we shall be somewhat brief. Recall that

$$E_6(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{N}) = \sum_{\delta \le C} \frac{1}{\delta} \sum_{c \le C/\delta} \frac{1}{c} \sideset{}{^*}\sum_{\substack{x(c\delta) \\ (u_2 x - u_1, c\delta) = \delta}} \frac{\pi^3}{\eta_1^3 \eta_2^3} \sum_{n \ge 1} \sum_{m \ge 1} A_3^+\left(n, \frac{\lambda_1}{\eta_1}\right) A_3^+\left(m, \frac{\lambda_2}{\eta_2}\right)$$
$$\times \int_0^\infty \int_0^\infty F_1(y_1) F_2(y_2) U_3\left(\frac{\pi^3 n y_1}{\eta_1^3}\right) U_3\left(\frac{\pi^3 m y_2}{\eta_2^3}\right) J_{k-1}\left(\frac{4\pi}{c\delta q}\sqrt{a_1 y_1 a_2 y_2}\right) dy_1\, dy_2,$$

where

$$\eta_1 = \frac{u_2 c}{(a_1, u_2 c)}, \qquad \eta_2 = \frac{u_1 c}{(a_2, u_1 c)}.$$

As before, we use (8.1) and estimate the sum over x trivially by $c\delta$ to see that

$$E_6(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{N}) \ll \sum_{\delta \le C} \sum_{c \le C/\delta} \frac{1}{(\eta_1 \eta_2)^{\frac{3}{2}}} \sum_{n \ge 1} \sum_{m \ge 1} (nm)^{\frac{1}{4}+\varepsilon} \left| I(n,m) \right|,$$

where

$$I(n,m) = \int_0^\infty \int_0^\infty F_1(y_1) F_2(y_2) U_3\left(\frac{\pi^3 n y_1}{\eta_1^3}\right) U_3\left(\frac{\pi^3 m y_2}{\eta_2^3}\right) J_{k-1}\left(\frac{4\pi}{c\delta q}\sqrt{a_1 y_1 a_2 y_2}\right) dy_1\, dy_2.$$

Recall that it suffices to show that $E_6 \ll (a_1 a_2)^{\frac{1}{2}} q^{\frac{3}{4}+\varepsilon}$. Following our previous analysis, we write

$$E_6 = \sum_{i=1}^{4} E_{6,i},$$

where $E_{6,i}$ is the contribution to E_6 from case i below.

(1) $n \ll \frac{\eta_1^3 q^\varepsilon}{N}$ and $m \ll \frac{\eta_2^3 q^\varepsilon}{M}$;

(2) $n \gg \frac{\eta_1^3 q^\varepsilon}{N}$ and $m \ll \frac{\eta_2^3 q^\varepsilon}{M}$;

(3) $n \ll \frac{\eta_1^3 q^\varepsilon}{N}$ and $m \gg \frac{\eta_2^3 q^\varepsilon}{M}$;

(4) $n \gg \frac{\eta_1^3 q^\varepsilon}{N}$ and $m \gg \frac{\eta_2^3 q^\varepsilon}{M}$.

By symmetry, the treatment of cases (2) and (3) is the same, so we treat only the second case.

The Contribution of $E_{6,1}$

For this case, we use the estimate

$$U_3\left(\frac{\pi^3 n_i y_i}{\eta_i^3}\right) \ll q^\varepsilon$$

along with (5.5) to see that

$$I(n,m) \ll q^\varepsilon (NM)^{\frac{1}{2}} \min\left(\left(\frac{\sqrt{a_1 a_2 NM}}{c\delta q}\right)^{-\frac{1}{2}}, \left(\frac{\sqrt{a_1 a_2 NM}}{c\delta q}\right)^{k-1}\right),$$

and so

$$\begin{aligned}
E_{6,1}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{N}) &\ll q^{\varepsilon}(NM)^{\frac{1}{2}}\sum_{\delta\leq C}\sum_{c\leq C/\delta}\frac{1}{(\eta_1\eta_2)^{\frac{3}{2}}}\min\left(\left(\frac{\sqrt{a_1a_2NM}}{c\delta q}\right)^{-\frac{1}{2}},\left(\frac{\sqrt{a_1a_2NM}}{c\delta q}\right)^{k-1}\right)\\
&\times\sum_{n\ll\frac{\eta_1^3q^{\varepsilon}}{N}}\sum_{m\ll\frac{\eta_2^3q^{\varepsilon}}{M}}(nm)^{\frac{1}{4}+\varepsilon}\\
&\ll q^{\varepsilon}\frac{(u_1u_2)^{\frac{9}{4}}}{(NM)^{-\frac{3}{4}}}\sum_{\delta\leq C}\left(\frac{\sqrt{a_1a_2NM}}{\delta q}\right)^{\frac{13}{4}}\\
&\ll q^{-\frac{13}{4}+\varepsilon}(a_1a_2)^{\frac{31}{8}}(b_1b_2)^{\frac{9}{4}}(NM)^{\frac{7}{8}}\\
&\ll q^{-\frac{13}{4}+\varepsilon}(a_1a_2)^{\frac{1}{2}}(a_1^3b_1^2N)^{\frac{9}{8}}(a_2^3b_2^2M)^{\frac{9}{8}}\\
&\ll (a_1a_2)^{\frac{1}{2}}q^{\frac{1}{8}+\varepsilon}.
\end{aligned}$$

The Contribution of $E_{6,2}$

We write

$$I(n,m)=\int_0^{\infty}F_2(y_2)U_3\left(\frac{\pi^3my_2}{\eta_2^3}\right)I_1(n,y_2)dy_1\ dy_2.$$

The integration in y_2 can be bounded trivially and the sum over m can be treated as in the previous subsection. The integral $I_1(n,y_2)$ can be handled in the same way as cases 1 and 2 in the Section 3.8.1, and we obtain $E_{6,2}\ll a_1^{\frac{1}{2}}q^{\frac{1}{2}+\varepsilon}$.

The Contribution of $E_{6,4}$

As in Section 3.8.1, we let $C_2=8\pi(q\delta)^{-1}\sqrt{a_1a_2NM}$ and divide into two cases depending as $c\leq C_2$ and $c>C_2$.

Case 1: $c>C_2$. We again use (5.4) and (8.3) and consider integrals of the form

$$\int_0^{\infty}\int_0^{\infty}\mathcal{H}(y_1,y_2)e\left(\frac{a_1^2b_1y_1}{c\delta qa_2b_2}+\frac{a_2^2b_2y_2}{c\delta qa_1b_1}\pm\frac{3n^{\frac{1}{3}}y_1^{\frac{1}{3}}}{\eta_1}\pm\frac{3m^{\frac{1}{3}}y_2^{\frac{1}{3}}}{\eta_2}\right)dy_1\ dy_2,$$

where $\frac{\partial^j \partial^k \mathcal{H}(y_1,y_2)}{\partial y_1^j \partial y_2^k}\mathcal{H}(y_1, y_2) \ll N^{-j}M^{-k}$, $\mathcal{H}(y_1, y_2) \ll 1$, and is supported on $[N, 2N] \times [M, 2M]$. Thus the range of integration is $O(NM)$.

By the same arguments as in Case 1 of Section 3.8.1, it suffices to consider the case when $c_1(B_1\eta_1)^3N^2 \ll n \ll c_2(B_1\eta_1)^3N^2$ and $c_1(B_2\eta_2)^3M^2 \ll m \ll c_2(B_2\eta_2)^3M^2$, where c_1, c_2 are some constants,

$$B_1 = \frac{a_1^2 b_1}{c\delta q a_2 b_2} \qquad \text{and} \qquad B_2 = \frac{a_2^2 b_2}{c\delta q a_1 b_1}.$$

The terms outside these ranges give negligible contribution from integration by parts many times. As before, we have

$$(B_1\eta_1)^3N^2 \ll \frac{(a_1^2b_1)^3N^2}{\delta^3q^3} \ll \frac{q^\varepsilon}{\delta^3}, \quad (B_2\eta_2)^3M^2 \ll \frac{(a_2^2b_2)^3M^2}{\delta^3q^3} \ll \frac{q^\varepsilon}{\delta^3}.$$

There are no terms of this form unless

$$N \gg \frac{q^{\frac{3}{2}}}{(a_1^2b_1)^{\frac{3}{2}}}, \qquad M \gg \frac{q^{\frac{3}{2}}}{(a_2^2b_2)^{\frac{3}{2}}}, \qquad \delta \ll q^\varepsilon.$$

The contribution to $E_{6,4}$ of the terms with $c > C_2$ is bounded by

$$\frac{q^\varepsilon (NM)^{\frac{1}{2}}}{(u_1u_2)^{\frac{3}{2}}} \sum_{\delta \ll q^\varepsilon} \sum_{c > C_2} \frac{((a_1, u_2c)(a_2, u_1c))^{\frac{3}{2}}}{c^3}. \tag{8.7}$$

To estimate the sum over c, let

$$g_1 = (a_1, u_2), \qquad a_1 = \lambda_1 g_1, \qquad u_2 = \gamma_1 g_1,$$

$$g_2 = (a_2, u_1), \qquad a_2 = \lambda_2 g_2, \qquad u_1 = \gamma_2 g_2,$$

and

$$d = (\lambda_1, \lambda_2), \quad \lambda_1 = \alpha_1 d, \quad \lambda_2 = \alpha_2 d,$$

where $(\lambda_1, \gamma_1) = (\lambda_2, \gamma_2) = (\alpha_1, \alpha_2) = 1$. Then the sum over c is

$$(g_1g_2)^{\frac{3}{2}} \sum_{c>C_2} \frac{((\lambda_1, c)(\lambda_2, c))^{\frac{3}{2}}}{c^3} = (g_1g_2)^{\frac{3}{2}} \sum_{\ell|d} \sum_{\substack{c>C_2 \\ (c,d)=\ell}} \frac{((\alpha_1 d, c)(\alpha_2 d, c))^{\frac{3}{2}}}{c^3}$$

$$= (g_1g_2)^{\frac{3}{2}} \sum_{\ell|d} \sum_{\substack{c>\frac{C_2}{\ell} \\ (c,\frac{d}{\ell})=1}} \frac{(\alpha_1\alpha_2, c)^{\frac{3}{2}}}{c^3} = (g_1g_2)^{\frac{3}{2}} \sum_{\ell|d} \sum_{k|\alpha_1\alpha_2} k^{\frac{3}{2}} \sum_{\substack{c>\frac{C_2}{\ell} \\ (c,\frac{d}{\ell})=1 \\ (\alpha_1\alpha_2,c)=k}} \frac{1}{c^3}$$

$$= (g_1g_2)^{\frac{3}{2}} \sum_{\ell|d} \sum_{k|\alpha_1\alpha_2} k^{-\frac{3}{2}} \sum_{\substack{c>\frac{C_2}{\ell k} \\ (c,\frac{d}{\ell})=1 \\ (\frac{\alpha_1\alpha_2}{k},c)=1}} \frac{1}{c^3} \ll \frac{(g_1g_2)^{\frac{3}{2}}}{C_2^2} \sum_{\ell|d} \ell^2 \sum_{k|\alpha_1\alpha_2} k^{\frac{1}{2}}$$

$$\ll \delta^2 q^{2+\varepsilon} \frac{(g_1g_2)^{\frac{3}{2}}}{a_1a_2NM} d^2 (\alpha_1\alpha_2)^{\frac{1}{2}} \ll \delta^2 q^{2+\varepsilon} \frac{(a_1,u_2)(a_2,u_1)(a_1,a_2)}{NM(a_1a_2)^{\frac{1}{2}}}.$$

Thus the total contribution is bounded by

$$q^{2+\varepsilon} \frac{(a_1,u_2)(a_2,u_1)(a_1,a_2)}{(u_1u_2)^{\frac{3}{2}}(a_1a_2)^{\frac{1}{2}}(NM)^{\frac{1}{2}}} \ll q^{\frac{1}{2}+\varepsilon} \frac{(a_1,u_2)(a_2,u_1)(a_1,a_2)}{(u_1u_2)^{\frac{3}{2}}(a_1a_2)^{\frac{1}{2}}} \left(a_1^{\frac{3}{2}} b_1^{\frac{3}{4}} a_2^{\frac{3}{2}} b_2^{\frac{3}{4}}\right)$$

$$\ll q^{\frac{1}{2}+\varepsilon} \frac{(a_1,a_2)}{(u_1u_2a_1a_2)^{\frac{1}{2}}} \left(a_1^{\frac{3}{2}} b_1^{\frac{3}{4}} a_2^{\frac{3}{2}} b_2^{\frac{3}{4}}\right) = q^{\frac{1}{2}+\varepsilon}(a_1,a_2)(a_1b_1,a_2b_2)(b_1b_2)^{\frac{1}{4}}$$

$$\ll q^{\frac{7}{8}+\varepsilon}(a_1,a_2)(a_1b_1,a_2b_2).$$

Summing this over a_1, a_2, b_1, b_2 produces several factors of $\log q$, and thus the contribution from these terms is sufficiently small.

Case 2: $c \le C_2$. We proceed as in the last case, except that we use (5.3) in place of (5.4). The integrals we consider have the form

$$\int_0^\infty \int_0^\infty G(y_1,y_2) e\left(\varphi(m,n,y_1,y_2)\right) dy_1 \, dy_2,$$

where

$$\varphi(m,n,y_1,y_2) = \frac{a_1^2 b_1 y_1}{c\delta q a_2 b_2} + \frac{a_2^2 b_2 y_2}{c\delta q a_1 b_1} \pm \frac{3n^{\frac{1}{3}} y_1^{\frac{1}{3}}}{\eta_1} \pm \frac{3m^{\frac{1}{3}} y_2^{\frac{1}{3}}}{\eta_2} \pm \frac{2\sqrt{a_1a_2y_1y_2}}{c\delta q}$$

$\frac{\partial^j \partial^k G(y_1,y_2)}{\partial y_1^j \partial y_2^k} G(y_1, y_2) \ll N^{-j} M^{-k}$, $G(y_1, y_2) \ll 1$. As before, the range of integration is $O(NM)$. Then

$$\frac{\partial \varphi(m,n,y_1,y_2)}{\partial y_1} = B_1 \pm \frac{n^{\frac{1}{3}}}{y_1^{\frac{2}{3}} \eta_1} \pm \frac{A_1}{y_1^{\frac{1}{2}}},$$
$$\frac{\partial \varphi(m,n,y_1,y_2)}{\partial y_2} = B_2 \pm \frac{m^{\frac{1}{3}}}{y_1^{\frac{2}{3}} \eta_2} \pm \frac{A_2}{y_2^{\frac{1}{2}}},$$

where B_1, B_2 are as above and $A_1 = \frac{\sqrt{a_1 a_2 y_2}}{c\delta q}$ and $A_2 = \frac{\sqrt{a_1 a_2 y_1}}{c\delta q}$. We now follow closely the analysis for Case 2 of Section 3.8.1, dividing into several subcases.

Case 2.1: $a_2^{\frac{3}{2}} b_2 M^{\frac{1}{2}} \geq 4 a_1^{\frac{3}{2}} b_1 N^{\frac{1}{2}}$. For this case, we have $\left| \frac{A_1}{y_1^{\frac{1}{2}}} \pm B_1 \right| \asymp \frac{A_1}{y_1}$ and $\left| \frac{A_2}{y_2^{\frac{1}{2}}} \pm B_2 \right| \asymp B_2$. By similar arguments to Case 2 of Section 3.8.1, we consider the ranges $n \asymp (A_1 \eta_1)^3 N^{\frac{1}{2}}$ and $m \asymp (B_2 \eta_2)^3 M^2$ and note that

$$(A_1 \eta_1)^3 N^{\frac{1}{2}} \ll \left(\frac{\sqrt{a_1}}{\delta q^{\frac{1}{2}}} \right)^3 N^{\frac{1}{2}} \ll \frac{q^\varepsilon}{\delta^3}, \qquad (B_2 \eta_2)^3 M^2 \ll \left(\frac{(a_2^2 b_2)^3 M^2}{\delta^3 q^3} \right) \ll \frac{q^\varepsilon}{\delta^3},$$

and thus there are no terms of this form unless $N \gg \frac{q^{\frac{3}{2}}}{a_1^3}$, $M \gg \frac{q^{\frac{3}{2}}}{(a_2^2 b_2)^{\frac{3}{2}}}$ and $\delta \ll q^\varepsilon$. The contribution from these terms to $E_{6,4}$ is $O(a_1^{\frac{1}{2}} a_2^{\frac{1}{2}} q^{\frac{1}{2}+\varepsilon})$.

Case 2.2: $a_1^{\frac{3}{2}} b_1 N^{\frac{1}{2}} \geq 4 a_2^{\frac{3}{2}} b_2 N^2$. The calculation as in Case 2.1 gives and error of $O(a_1^{\frac{1}{2}} a_2^{\frac{1}{2}} q^{\frac{1}{2}+\varepsilon})$.

Case 2.3: $\frac{1}{4} a_1^{\frac{3}{2}} b_1 N^{\frac{1}{2}} < a_2^{\frac{3}{2}} b_2 M^{\frac{1}{2}} < 4 a_1^{\frac{3}{2}} b_1 N^{\frac{1}{2}}$. For this case, we have $\frac{A_1}{y_1^{\frac{1}{2}}} \asymp B_1$ and $\frac{A}{y_2^{\frac{1}{2}}} \asymp B_2$. By similar arguments to Cases 1.2 and 1.3, we need only consider the ranges $n \asymp (A_1 \eta_1)^3 N^{\frac{1}{2}}$ and $m \asymp (A_1 \eta_1)^3 N^{\frac{1}{2}}$. The contribution from these terms is also $O\left(a_1^{\frac{1}{2}} a_2^{\frac{1}{2}} q^{\frac{1}{2}+\varepsilon} \right)$.

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