

FREE HARMONIC MODERATELY LARGE AMPLITUDE VIBRATIONS  
OF AXISYMMETRIC ORTHOTROPIC VARIABLE THICKNESS  
SOLID CIRCULAR PLATES CLAMPED AT THE EDGES

by 557

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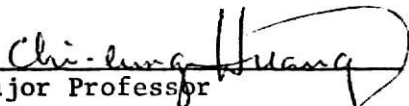
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## INTRODUCTION

In a variety of situations, motions may be generated in modern structures involving thin circular plates which lead to vibrations with moderately large amplitudes of the order of magnitude of the plate's thickness. The study of such motion is greatly complicated by the mathematical complexity connected with the non-linearity of the governing field equations. Explicit solutions to the set of governing non-linear partial differential equations of motion are not available in the literature. Various approximation methods were employed in specific cases to improve the understanding of these motions in the non-linear range.

In 1954, H.M. Berger [21] formulated and solved the problem of large deflection of circular plates in his paper under the assumption that the strain energy due to the second invariant of the strains in the middle surface of the plate is negligible. J. Nowinski [19] in 1962 derived and solved the same problem using an orthogonalization procedure. In 1963, A.N. Sherbourne [17] transformed the static case of the two-point boundary-value problem into an initial-value problem and obtained solutions by an iterative method. A.V. Srinivasan [12] in 1966 approximated the dynamic case using the Ritz method and N. Gajender [9] in 1967 using Berger's assumption and Galerkin's method solved the problem for elastic foundations. All the aforementioned studies were concerned with isotropic circular plates of uniform cross-section.

In recent years problems in large deflections of circular plates were investigated either for orthotropic with constant thickness or isotropic with variable thickness. Different approximations such as

the dynamic relaxation method used by K.R. Rushton [6], the Ritz method used by B.Ya. Kantor and L.M. Afanaseva [5] for varying thickness circular plates in 1968, and the asymptotic integration used by O.E. Widera [1] for anisotropic plates in 1969 were used. But relatively few investigations have been made to study the problem of moderately large deflection of an anisotropic solid circular plate with variable thickness.

The present investigation is concerned with harmonic, free vibrations of orthotropic axisymmetric, thin solid circular plates with variable thickness of the form  $\bar{h} = h_0(1 - m\zeta^n)$  clamped at the edges. The derivation of the governing equations leads to a set of two coupled non-linear differential equations; one describing the transverse motion and the other describing the in-plane motion.

The shooting method is then employed to obtain frequency response curves. The effect of the ratio of elastic constants of the material in the radial direction to that in the circumferential direction on frequency responses of the plates are first separately studied, then their combined effect is studied. Results of these effects and the bending and membrane stresses are presented in graphical forms. Graphs are also presented to visualize the effect of moderately large amplitude on shape functions of harmonic vibration and on stress distributions.

DERIVATION OF THE GOVERNING EQUATIONS

The following assumptions are made:

1. the maximum thickness of the plate is small in comparison with the radius of the plate<sup>\*</sup>,
2. middle plane is the plane of symmetry,
3. an element of the plate along a normal to the middle plane in the undeformed plate remains straight and normal to the deformed middle plane and its extension is negligible,
4. transverse shear deformations are not considered [15], and
5. within the elastic limit.

The above assumptions lead to the strain-displacement relations:

$$e_r = u_{,r} - zw_{,rr} + \frac{1}{2} w_{,r}^2$$

$$e_\theta = \frac{u}{r} - \frac{z}{r} w_{,r} \quad (1)$$

in cylindrical co-ordinates (Fig. 3), where  $e_r$ ,  $e_\theta$  are radial and circumferential normal strains and  $u, w$  are radial and lateral displacements respectively.

Next, it is assumed that the plates are made of cylindrically orthotropic materials, i.e. the elastic properties of the plate in the radial and circumferential directions are different. In view of this, the pertinent stress-strain relation may be written as:

$$e_\theta = a_{11} \sigma_\theta + a_{12} \sigma_r$$

$$e_r = a_{12} \sigma_\theta + a_{22} \sigma_r \quad (2a)$$

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\*Von Karman equations are not good approximations for non-linear bending of clamped circular plate of moderate thickness with large load [12].

$$\begin{aligned}\sigma_r &= \frac{a_{11}}{a_{11}a_{22} - a_{12}^2} \left( e_r - \frac{a_{12}}{a_{11}} e_\theta \right) \\ \sigma_\theta &= \frac{a_{22}}{a_{11}a_{22} - a_{12}^2} \left( e_\theta - \frac{a_{12}}{a_{22}} e_r \right)\end{aligned}\quad (2b)$$

where  $a_{11}$ ,  $a_{12}$ ,  $a_{22}$  are elastic constants and  $\sigma_r$ ,  $\sigma_\theta$  are normal radial and circumferential stresses.

### I. DISPLACEMENT FORMULATION

The stress-strain relations together with the strain-displacement relations of Eq'n (1) may now be used to derive expressions for in-plane forces per unit length  $N_r$ ,  $N_\theta$  and bending moments per unit length  $M_r$ ,  $M_\theta$ :

$$\begin{aligned}N_r &= \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_r dz = \frac{\bar{h}}{a_{22}(c-v^2)} \left[ c(u_{,r} + \frac{1}{2} w_{,r}^2) + v \frac{u}{r} \right] \\ N_\theta &= \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_\theta dz = \frac{\bar{h}}{a_{22}(c-v^2)} \left[ \frac{u}{r} + v u_{,r} + \frac{v}{2} w_{,r}^2 \right]\end{aligned}\quad (3)$$

$$\begin{aligned}M_r &= \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_r z dz = -D \left( c w_{,rr} + \frac{v}{r} w_{,r} \right) \\ M_\theta &= \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_\theta z dz = -D \left( \frac{1}{r} w_{,r} + v w_{,rr} \right)\end{aligned}\quad (4)$$

where

$$v = -\frac{a_{12}}{a_{22}} \quad c = \frac{a_{11}}{a_{22}}$$