

FREE HARMONIC MODERATELY LARGE AMPLITUDE VIBRATIONS
OF AXISYMMETRIC ORTHOTROPIC VARIABLE THICKNESS
SOLID CIRCULAR PLATES CLAMPED AT THE EDGES

by 557

VAN MAN-MING LEE

B.S. in Applied Mathematics, Taiwan Provincial Chung-Hsing University, 1968

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

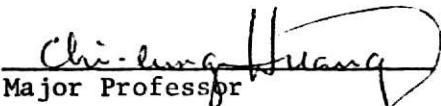
MASTER OF SCIENCE

Department of Applied Mechanics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1970

Approved by:


Major Professor

LD
2668
R4
1971
L4
C.2

TABLE OF CONTENTS

	Page
Introduction.....	1
Derivation Of The Governing Equations.....	3
Boundary Conditions For Clamped Circular Plates.....	13
Thickness Variation.....	14
Method Of Solution - The Shooting Method.....	16
Evaluations Of Maximum Stresses.....	28
Cases Studied In This Report.....	30
General Conclusions.....	31
Nomenclature.....	35
Acknowledgment.....	38
Bibliography.....	39
References.....	41
Appendix I.....	42
Appendix II.....	50

INTRODUCTION

In a variety of situations, motions may be generated in modern structures involving thin circular plates which lead to vibrations with moderately large amplitudes of the order of magnitude of the plate's thickness. The study of such motion is greatly complicated by the mathematical complexity connected with the non-linearity of the governing field equations. Explicit solutions to the set of governing non-linear partial differential equations of motion are not available in the literature. Various approximation methods were employed in specific cases to improve the understanding of these motions in the non-linear range.

In 1954, H.M. Berger [21] formulated and solved the problem of large deflection of circular plates in his paper under the assumption that the strain energy due to the second invariant of the strains in the middle surface of the plate is negligible. J. Nowinski [19] in 1962 derived and solved the same problem using an orthogonalization procedure. In 1963, A.N. Sherbourne [17] transformed the static case of the two-point boundary-value problem into an initial-value problem and obtained solutions by an iterative method. A.V. Srinivasan [12] in 1966 approximated the dynamic case using the Ritz method and N. Gajender [9] in 1967 using Berger's assumption and Galerkin's method solved the problem for elastic foundations. All the aforementioned studies were concerned with isotropic circular plates of uniform cross-section.

In recent years problems in large deflections of circular plates were investigated either for orthotropic with constant thickness or isotropic with variable thickness. Different approximations such as

the dynamic relaxation method used by K.R. Rushton [6], the Ritz method used by B.Ya. Kantor and L.M. Afanaseva [5] for varying thickness circular plates in 1968, and the asymptotic integration used by O.E. Widera [1] for anisotropic plates in 1969 were used. But relatively few investigations have been made to study the problem of moderately large deflection of an anisotropic solid circular plate with variable thickness.

The present investigation is concerned with harmonic, free vibrations of orthotropic axisymmetric, thin solid circular plates with variable thickness of the form $\bar{h} = h_0(1 - m\zeta^n)$ clamped at the edges. The derivation of the governing equations leads to a set of two coupled non-linear differential equations; one describing the transverse motion and the other describing the in-plane motion.

The shooting method is then employed to obtain frequency response curves. The effect of the ratio of elastic constants of the material in the radial direction to that in the circumferential direction on frequency responses of the plates are first separately studied, then their combined effect is studied. Results of these effects and the bending and membrane stresses are presented in graphical forms. Graphs are also presented to visualize the effect of moderately large amplitude on shape functions of harmonic vibration and on stress distributions.

DERIVATION OF THE GOVERNING EQUATIONS

The following assumptions are made:

1. the maximum thickness of the plate is small in comparison with the radius of the plate^{*},
2. middle plane is the plane of symmetry,
3. an element of the plate along a normal to the middle plane in the undeformed plate remains straight and normal to the deformed middle plane and its extension is negligible,
4. transverse shear deformations are not considered [15], and
5. within the elastic limit.

The above assumptions lead to the strain-displacement relations:

$$e_r = u_{,r} - zw_{,rr} + \frac{1}{2} w_{,r}^2$$

$$e_\theta = \frac{u}{r} - \frac{z}{r} w_{,r} \quad (1)$$

in cylindrical co-ordinates (Fig. 3), where e_r , e_θ are radial and circumferential normal strains and u, w are radial and lateral displacements respectively.

Next, it is assumed that the plates are made of cylindrically orthotropic materials, i.e. the elastic properties of the plate in the radial and circumferential directions are different. In view of this, the pertinent stress-strain relation may be written as:

$$e_\theta = a_{11} \sigma_\theta + a_{12} \sigma_r$$

$$e_r = a_{12} \sigma_\theta + a_{22} \sigma_r \quad (2a)$$

*Von Karman equations are not good approximations for non-linear bending of clamped circular plate of moderate thickness with large load [12].

$$\sigma_r = \frac{a_{11}}{a_{11}a_{22} - a_{12}^2} \left(e_r - \frac{a_{12}}{a_{11}} e_\theta \right)$$

$$\sigma_\theta = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2} \left(e_\theta - \frac{a_{12}}{a_{22}} e_r \right) \quad (2b)$$

where a_{11} , a_{12} , a_{22} are elastic constants and σ_r , σ_θ are normal radial and circumferential stresses.

I. DISPLACEMENT FORMULATION

The stress-strain relations together with the strain-displacement relations of Eq'n (1) may now be used to derive expressions for in-plane forces per unit length N_r , N_θ and bending moments per unit length M_r , M_θ :

$$N_r = \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_r dz = \frac{\bar{h}}{a_{22}(c-v^2)} \left[c(u_{,r} + \frac{1}{2} w_{,r}^2) + v \frac{u}{r} \right]$$

$$N_\theta = \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_\theta dz = \frac{\bar{h}}{a_{22}(c-v^2)} \left[\frac{u}{r} + v u_{,r} + \frac{v}{2} w_{,r}^2 \right] \quad (3)$$

$$M_r = \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_r z dz = -D \left(c w_{,rr} + \frac{v}{r} w_{,r} \right)$$

$$M_\theta = \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_\theta z dz = -D \left(\frac{1}{r} w_{,r} + v w_{,rr} \right) \quad (4)$$

where

$$v = -\frac{a_{12}}{a_{22}} \quad c = \frac{a_{11}}{a_{22}}$$

$$D = \frac{a^3 h^3}{12 \begin{pmatrix} a_{22} & & \\ & a_{11} & \\ & & a_{22} - a_{12}^2 \end{pmatrix}}$$

and

$$\bar{h} = h_0 (1 - m \xi^n)$$

is the local thickness of the plate.

Now that the forces and moments per unit length are known at each point of the plate we easily obtain the strain energy due to stretching of the middle plane of the plate,

$$V_1 = \int_c^a \left(\frac{N_r e_r^0}{2} + \frac{N_\theta e_\theta^0}{2} \right) 2\pi r dr$$

where

$$e_r^0 = u_{,r} + \frac{1}{2} w_{,r}^2$$

$$e_\theta^0 = \frac{u}{r}$$

are the normal strain components at the middle plane.

Denoting

$$d = \frac{\begin{matrix} a & a & - a^2 \\ 11 & 22 & 12 \\ & a & 22 \end{matrix}}{a}$$

and the mass density of the plate by ρ ,

upon substituting the above equations and Eq'n(3)

$$V_1 = \frac{\pi}{d} \int_c^a \left[cu_{,r}^2 + cu_{,r} w_{,r}^2 + \frac{c}{4} w_{,r}^4 + 2\nu \frac{u}{r} u_{,r} + \nu \frac{u}{r} w_{,r}^2 + \left(\frac{u}{r}\right) \right] \bar{h} r dr \quad (5)$$

and the strain energy due to bending of the plate

$$dV_2 = -\frac{1}{2} \left(M_r w_{,rr} + \frac{M_\theta}{r} w_{,r} \right) r dr d\theta$$

substituting Eq'n (4)

$$V_2 = \pi \int_c^a D \left[\left(c^{\frac{1}{2}} w_{,rr} + \frac{w_{,r}}{r} \right)^2 + \frac{2(\nu - c^{\frac{1}{2}})}{r} w_{,rr} w_{,r} \right] r dr \quad (6)$$

Since we consider only moderately large amplitude of the order of maximum magnitude of $2h_0$, and h_0 is small in comparison with other dimensions, the radial displacement and the time rate of radial displacement $u_{,t}$ is small, therefore we can safely neglect the kinetic energy due to the time derivative of radial displacement, thus the kinetic energy will be

$$\begin{aligned} T &= \frac{1}{2} \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \int_0^{2\pi} \int_c^a \rho w_{,t}^2 r dr d\theta dz \\ &= \pi \int_c^a \rho \bar{h} \dot{w}_{,t}^2 r dr \end{aligned} \quad (7)$$

Define the Lagrangian by $L = T - (V_1 + V_2)$, substituting Eq'ns

(5), (6) and (7) we obtain

$$\begin{aligned}
L = & \pi \int_c^a \rho \bar{h} w_{,t}^2 r dr - \frac{\pi}{d} \int_c^a \left[cu_{,r}^2 + cu_{,r} w_{,r}^2 + \frac{c}{4} w_{,r}^4 \right. \\
& + 2v \frac{u}{r} u_{,r} + v \frac{u}{r} w_{,r}^2 + \left. \left(\frac{u}{r} \right)^2 \right] \bar{h}(r) r dr - \pi \int_c^a D \left[cw_{,rr}^2 \right. \\
& \left. + \frac{2v}{r} w_{,rr} w_{,r} + \frac{1}{r^2} w_{,r}^2 \right] r dr \quad (8)
\end{aligned}$$

Since Hamilton's Principle states that the true motion of a system within a certain arbitrary chosen interval of time is characterized by the fact that the increment of the integral

$$\int_{t_1}^{t_2} L' dt$$

vanishes for any continuously varying virtual displacement provided the displacement vanishes at the limits t_1 and t_2 of the chosen interval i.e.

$$\delta \int_{t_1}^{t_2} L' dt = 0$$

therefore, substituting Eq. (8)

$$\begin{aligned}
\bar{H} = & \delta \int_{t_1}^{t_2} L dt \\
= & \delta \int_{t_1}^{t_2} \int_c^a \pi \left\{ \rho \bar{h}(r) w_{,t}^2 - \frac{\bar{h}(r)}{d} \left[cu_{,r}^2 + cu_{,r} w_{,r}^2 + \frac{c}{4} w_{,r}^4 \right. \right. \\
& \left. \left. + 2v \frac{u}{r} u_{,r} + v \frac{u}{r} w_{,r}^2 + \left(\frac{u}{r} \right)^2 \right] - \frac{\bar{h}^3(r)}{12d} \left[cw_{,rr}^2 + \frac{2v}{r} w_{,rr} w_{,r} + \frac{1}{r^2} w_{,r}^2 \right] \right\} r dr dt \\
= & \delta \int_{t_1}^{t_2} \int_c^a \int (r; w, u; w_{,t}, w_{,r}, u_{,r}, w_{,rr}) dr dt = 0 \quad (9)
\end{aligned}$$

Introducing dimensionless quantities:

$$r = a\xi \quad w = ax \quad u = a\eta$$

$$t = \sqrt{\frac{\rho h_0 a^4}{D_0}} \tau \quad \bar{h} = h_0 h$$

and orthogonal harmonic functions of transverse and radial vibrations:

$$x = Ag(\xi) \sin \gamma \tau$$

$$\eta = Af(\xi) \sin^2 \gamma \tau$$

letting

$$\bar{H} = \frac{H}{\pi(D_0 \rho h_0 a)^{\frac{1}{2}}}$$

we obtain

$$\begin{aligned} \bar{H} = \frac{\pi A^2}{r} \int_{c/a}^1 \delta \left\{ \gamma^2 \xi h g^2 - 9 \left(A \frac{a}{h_0} \right)^2 h \left[c \xi \frac{f^2}{\xi} \right. \right. \\ \left. \left. + c \xi \frac{f}{\xi} g^2 + \frac{c}{4} \xi g^4 + 2\nu \int \frac{f}{\xi} + \nu \int \frac{g^2}{\xi} + \frac{f^2}{\xi} \right] \right. \\ \left. - h^3 \left[c \xi \frac{g^2}{\xi \xi} + 2\nu g \frac{g}{\xi \xi} + \frac{g^2}{\xi} \right] \right\} d\xi = 0 \quad \text{----} \quad (10) \end{aligned}$$

where the derivatives are now ordinary ones.

Set $\alpha = \left(A \frac{a}{h_0} \right)^2$, we hence obtain the Lagrangian in the final form

$$\begin{aligned} L = \gamma^2 \xi h g^2 - 9\alpha h \left[c \xi \frac{f^2}{\xi} + c \xi \frac{f}{\xi} g^2 + \frac{c}{4} \xi g^4 \right. \\ \left. + 2\nu \int \frac{f}{\xi} + \nu \int \frac{g^2}{\xi} + \frac{f^2}{\xi} \right] - h^3 \left[c \xi \frac{g^2}{\xi \xi} + 2\nu g \frac{g}{\xi \xi} \right. \\ \left. + \frac{g^2}{\xi} \right] \end{aligned}$$

Using Euler's equations for

$$\delta \int_{t_1}^{t_1} L' dt = 0$$

$$L_{,w} - (L_{,w,t})_{,t} - (L_{,w,r})_{,r} + (L_{,w,rr})_{,rr} = 0 \quad (A)$$

$$L_{,u} - (L_{,u,r})_{,r} = 0 \quad (B)$$

With equations (11), (A) and (B), we easily arrive at the desired set of governing differential equations:

$$\begin{aligned} & h^3 \left[c g_{\xi\xi\xi\xi} + \frac{2c}{\xi} g_{\xi\xi\xi} - \frac{1}{\xi^2} g_{\xi\xi} + \frac{1}{\xi^3} g_{\xi} \right] \\ & + h^3 \left[2c g_{\xi\xi\xi} + \frac{(2c+\nu)}{\xi} g_{\xi\xi} - \frac{1}{\xi^2} g_{\xi} \right] \\ & + h^3 \left[c g_{\xi\xi} + \frac{\nu}{\xi} g_{\xi} \right] \\ & - \gamma^2 h g \\ & = \frac{9\alpha}{\xi} \left[\xi h g_{\xi} \left(c \frac{f}{\xi} + \frac{c}{2} g_{\xi}^2 + \frac{\nu f}{\xi} \right)_{,\xi} \right] \end{aligned} \quad (I)$$

and

$$\begin{aligned} & h \left[c \frac{f}{\xi\xi} + \frac{c}{\xi} \frac{f}{\xi} - \frac{1}{\xi^2} f + c g_{\xi} g_{\xi\xi} + \frac{c-\nu}{2\xi} g_{\xi}^2 \right] \\ & + h \left[c \frac{f}{\xi} + \frac{c}{2} g_{\xi}^2 + \frac{\nu f}{\xi} \right] \\ & = 0 \end{aligned} \quad (II)$$

II. STRESS FORMULATION

From the strain-displacement relations

$$e_r = u_{,r} - z w_{,rr} + \frac{1}{2} w_{,r}^2$$

$$e_{\theta} = \frac{u}{r} - \frac{z}{r} w_{,r}$$

we have strains at the middle plane which can be regarded as the average values of strains throughout the thickness

$$e_r^o = u_{,r} + \frac{1}{2} w_{,r}^2$$

$$e_\theta^o = \frac{u}{r}$$

Using the stress-strain relations

$$e_\theta = a_{11} \sigma_\theta + a_{12} \sigma_r$$

$$e_r = a_{12} \sigma_\theta + a_{22} \sigma_r$$

and integrating throughout the thickness we obtain the total deformations as follows:

$$\bar{e}_\theta = \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} e_\theta dz = a_{11} \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_\theta dz + a_{12} \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_r dz$$

$$\bar{e}_r = \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} e_r dz = a_{12} \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_\theta dz + a_{22} \int_{-\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}} \sigma_r dz$$

or

$$\bar{e}_\theta = e_\theta^o \bar{h} = a_{11} N_\theta + a_{12} N_r$$

$$\bar{e}_r = e_r^o \bar{h} = a_{12} N_\theta + a_{22} N_r$$

substituting Eqs. (1a):

$$\frac{u}{r} \bar{h} = a_{11} N_\theta + a_{12} N_r$$

$$\left[u_{,r} + \frac{1}{2} w_{,r}^2 \right] \bar{h} = a_{12} N_\theta + a_{22} N_r \quad (12)$$

Refer to Fig. 2 and assuming radial displacement and time derivative of radial displacement are negligible with the same reasoning as that in the displacement formulation, we have

$$rN_{r,r} + N_r - N_\theta = 0$$

or

$$\left(r N_r \right)_{,r} = N_\theta \quad (13)$$

Define a stress function Ψ such that

$$\begin{aligned} N_\theta &= \Psi_{,r} \\ N_r &= \frac{\Psi}{r} \end{aligned} \quad (14)$$

obviously the function Ψ satisfied Eq'n (13). By substituting Eq'ns (14) into Eq'n (12) we derive the following set of equations:

$$\begin{aligned} u_{,r} + \frac{1}{2} w_{,r}^2 &= \frac{a}{h(r)} \left(\frac{\Psi}{r} - \nu \Psi_{,r} \right) \\ \frac{u}{r} &= \frac{a}{h(r)} \left(c \Psi_{,r} - \nu \frac{\Psi}{r} \right) \end{aligned} \quad (15)$$

Introducing dimensionless quantities and orthogonal harmonic functions as before

$$\begin{aligned} w &= ax = aAg(\xi) \sin \gamma \tau \\ u &= a\eta = aA^2 \int (\xi) \sin^2 \gamma \tau \\ r &= a\xi \\ \bar{h} &= h_o h \end{aligned}$$

and

$$\Psi = aA^2 \varphi(\xi) \sin^2 \gamma \tau$$

Eq'ns (15) take the form

$$\begin{aligned} \int_{\xi} + \frac{1}{2} g_{\xi}^2 &= \frac{a}{h_o h} \left(\frac{\varphi(\xi)}{\xi} - \nu \varphi_{,\xi} \right) \\ \int_{\xi} &= \frac{a}{h_o h} \left(c \varphi_{,\xi} - \nu \frac{\varphi}{\xi} \right) \end{aligned} \quad (16)$$

Hence

$$c \int_{\xi} + \frac{c}{2} g_{\xi}^2 + \frac{\nu \int_{\xi}}{\xi} = \frac{a}{h_o h} (c - \nu^2) \varphi \quad (17)$$

Let

$$F(\xi) = \frac{a}{h_0} (c - v^2)\varphi(\xi) \quad (18)$$

then

$$c \int_{\xi} + \frac{c}{2} g_{\xi}^2 + \frac{v}{\xi} \int = \frac{F}{\xi h} \quad (19)$$

Eq'n (I) becomes:

$$\begin{aligned} & h^3 \left[c g_{\xi\xi\xi\xi\xi} + \frac{2c}{\xi} g_{\xi\xi\xi\xi} - \frac{1}{\xi^2} g_{\xi\xi\xi} + \frac{1}{\xi^3} g_{\xi\xi} \right] \\ & + h^3 \left[2c g_{\xi\xi\xi} + \frac{2c+v}{\xi} g_{\xi\xi\xi} - \frac{1}{\xi^2} g_{\xi\xi} \right] \\ & + h^3 \left[c g_{\xi\xi} + \frac{v}{\xi} g_{\xi\xi} \right] \\ & - v^2 h g \\ & = \frac{qQ}{\xi} \left[g_{\xi} F \right]_{,\xi} \end{aligned} \quad (I')$$

and Eq'n (II) becomes:

$$\begin{aligned} & c F_{\xi\xi} + \frac{c}{\xi} F_{\xi} - \frac{F}{\xi^2} \\ & - \frac{h}{\xi} \left(c F_{\xi} - v \frac{F}{\xi} \right) \\ & = - (c - v^2) \frac{h}{2\xi} g_{\xi}^2 \end{aligned} \quad (II')$$

BOUNDARY CONDITIONS FOR
CLAMPED CIRCULAR PLATES

At the edge where $r = a$, the lateral displacement and its partial derivatives with respect to r vanishes, also the in-plane displacement is zero. Hence, we have for:

(I) Displacement Formulation:

$$\begin{aligned} & g = 0 \\ \text{at } \xi = 1 & \quad g, \xi = 0 \\ & u = 0 \end{aligned} \tag{III}$$

and

(II) Stress Formulation:

$$\begin{aligned} & g = 0 \\ \text{at } \xi = 1 & \quad g, \xi = 0 \\ & cF, \xi - \nu F = 0 \end{aligned} \tag{III'}$$

The third of this set of equations is derived from the second of Eq'ns (15) together with $\psi = aA^2 \omega(\xi) \sin \gamma \tau$, Equation (18), and the condition $u = 0$ at $\xi = 1$.

THICKNESS VARIATION

The form

$$\bar{h} = h_0 \left[1 - \frac{h_0 - h_a}{h_0} \left(\frac{r}{a} \right)^n \right] = h_0 (1 - m\xi^n) \quad \begin{array}{l} 1 > m \geq 0 \\ n \geq 0 \end{array}$$

has been selected for thickness variation along r .

The following functions of ξ are studied for later use in removing singularity:

$$h = 1 - m\xi^n$$

$$h,_{\xi} = -nm\xi^{n-1}$$

$$h^3 = (1 - m\xi^n)^3$$

$$h^3,_{\xi} = -3nm\xi^{n-1} (1 - m\xi^n)^2$$

$$h^3,_{\xi\xi} = -3n(n-1)m\xi^{n-2} (1 - m\xi^n)^2 + 6n^2 m^2 \xi^{2(n-1)} (1 - m\xi^n)$$

for $n \neq 1$,

$$h(0) = 1$$

$$h,_{\xi}(0) = 0$$

$$h^3(0) = 1$$

$$h^3,_{\xi}(0) = 0$$

$$h^3,_{\xi\xi}(0) = 0$$

for $n = 1$,

$$h(0) = 1$$

$$h,_{\xi}(0) = -m$$

$$h^3(0) = 1$$

$$h^3,_{\xi}(0) = -3m$$

$$h^3,_{\xi\xi}(0) = 6m^2$$

$$u_1 = \frac{h\xi^3}{h^3} = \frac{-3n m \xi^{n-1}}{1 - m \xi^n}$$

$$u_2 = \frac{h^3 \xi \xi}{h^3} = \frac{-3n(n-1)m \xi^{n-2}(1-m \xi^n) + 6m^2 n^2 \xi^2(n-1)}{(1-m \xi^n)^2}$$

$$u_3 = \frac{h\xi}{h} = \frac{-n m \xi^{n-1}}{1 - m \xi^n}$$

if $n \neq 1$

$$u_1^0 = 0$$

$$u_2^0 = 0$$

$$u_3^0 = 0$$

if $n = 1$

$$u_1^0 = -3m$$

$$u_2^0 = 6m^2$$

$$u_3^0 = -m$$

METHOD OF SOLUTION - THE SHOOTING TECHNIQUE

(I) Reduction of the Eq'ns (I') and (II') into first degree equations -

The set of two coupled non-linear differential equations with variable coefficients will be transformed by introducing

$$\begin{aligned}
 y_1 &= g \\
 y_2 &= g, \xi \\
 y_3 &= g, \xi \xi \\
 y_4 &= g, \xi \xi \xi \\
 y_5 &= F \\
 y_6 &= F, \xi
 \end{aligned}
 \tag{20}$$

and their derivatives:

$$\begin{aligned}
 Dy_1 &= y_2 \\
 Dy_2 &= y_3 \\
 Dy_3 &= y_4 \\
 Dy_4 &= g, \xi \xi \xi \xi \\
 Dy_5 &= y_6 \\
 Dy_6 &= F, \xi \xi
 \end{aligned}
 \tag{21}$$

into a system of 6 simultaneous first degree differential equations:

$$\begin{aligned}
Dy_1 &= y_2 \\
Dy_2 &= y_3 \\
Dy_3 &= y_4 \\
Dy_4 &= V_1 y_1 + \frac{V_2}{\xi^3} y_2 + \frac{V_3}{\xi^2} y_3 + \frac{V_4}{\xi} y_4 + \frac{V_5}{\xi} (y_3 y_5 + y_2 y_6) \\
Dy_5 &= y_6 \\
Dy_6 &= \frac{V_6}{\xi^2} y_5 + \frac{V_7}{\xi} y_6 + \frac{V_8}{\xi} y_2^2
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
V_1 &= \frac{\gamma^2}{ch^2} \\
V_2 &= -\frac{1}{c} + \frac{\xi}{c} u_1 - \frac{v\xi^2}{c} u_2 \\
V_3 &= \frac{1}{c} - \frac{(2c+v)\xi}{c} u_1 - \xi^2 u_2 \\
V_4 &= -2(1+\xi u_1) \\
V_5 &= \frac{9\alpha}{ch^3} \\
V_6 &= \frac{1}{c} - \frac{v\xi}{c} u_3 \\
V_7 &= \xi u_3 - 1 \\
V_8 &= -\frac{(c-v)^2 h}{2c}
\end{aligned} \tag{23}$$

and boundary conditions at $\xi = 1$:

$$\begin{aligned}
y_1 &= 0 \\
y_2 &= 0 \\
cy_6 - vy_5 &= 0
\end{aligned} \tag{24}$$

(II) Singularity at the origin of the co-ordinate system -

An examination of the Equations (22) immediately reveals that the origin $\xi = 0$ is a singularity point and thus needs further exploration. MacLaurin Series Expansion is employed to remove this singularity, thus the fourth and the sixth equations of the system can be written as:

$$\begin{aligned}
(y_4^0)_{,\xi} + \dots &= V_1 \left(y_1^0 + \xi y_2^0 + \frac{y_3^0}{2!} \xi^2 + \frac{y_4^0}{3!} \xi^3 + \frac{(y_4')^0}{4!} \xi^4 + \dots \right) \\
&+ \frac{V_2}{\xi^3} \left(y_2^0 + y_3^0 \xi + \frac{y_4^0}{2!} \xi^2 + \frac{(y_4')^0}{3!} \xi^3 + \dots \right) \\
&+ \frac{V_3}{\xi^2} \left(y_3^0 + y_4^0 \xi + \frac{(y_4')^0}{2!} \xi^2 + \dots \right) \\
&+ \frac{V_4}{\xi} \left(y_4^0 + (y_4')^0 \xi + \dots \right) \\
&+ \frac{V_5}{\xi} \left[\left(y_3^0 + y_4^0 \xi + \frac{(y_4')^0}{2!} \xi^2 + \dots \right) \left(y_5^0 + y_6^0 \xi + \frac{(y_6')^0}{2!} \xi^2 + \dots \right) \right. \\
&\quad \left. + \left(y_2^0 + y_3^0 \xi + \frac{y_4^0}{2!} \xi^2 + \frac{(y_4')^0}{3!} \xi^3 + \dots \right) \left(y_6^0 + (y_6')^0 \xi + \dots \right) \right]
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
(y_6^0)_{,\xi} + \dots &= \frac{V_6}{\xi^3} \left(y_5^0 + y_6^0 \xi + \frac{(y_6')^0}{2!} \xi^2 + \dots \right) \\
&+ \frac{V_2}{\xi} \left(y_6^0 + (y_6')^0 \xi + \dots \right) \\
&+ \frac{V_5}{\xi} \left(y_2^0 + y_3^0 \xi + \frac{y_4^0}{2!} \xi^2 + \frac{(y_4')^0}{3!} \xi^3 + \dots \right)^2
\end{aligned} \tag{26}$$

Substituting Equations (23) and after regrouping, Equations (25)

and (26) become:

$$\begin{aligned}
(y_4')^0_{,\xi} &= \frac{\gamma^2}{ch^2} y_1^0 \\
&+ \left(-\frac{1}{c\xi^3} + \frac{1}{c\xi^2} u_1^0 - \frac{\nu}{c\xi} u_2^0 \right) y_2^0 \\
&+ \left(\frac{1-2c-\nu}{c\xi} u_1^0 - \frac{\nu+c}{c} u_2^0 \right) y_3^0 \\
&+ \left(\frac{1-4c}{2c\xi} + \frac{1-2\nu-8c}{2c} u_1^0 - \frac{\nu\xi+2c\xi}{2c} u_2^0 \right) y_4^0 \\
&+ \left(\frac{1-6c}{3c} + \frac{\xi-3\nu\xi-18c\xi}{6c} u_1^0 - \frac{-\nu-3c}{6c} \xi^2 u_2^0 \right) (y_4')^0 \\
&+ \frac{9\alpha}{ch^2} \left(y_3^0 y_5^0 + y_4^0 y_5^0 \xi + 2y_3^0 y_6^0 \xi + y_2^0 y_6^0 + y_2^0 (y_6')^0_{,\xi} \right) \\
&+ \dots
\end{aligned} \tag{27}$$

$$\begin{aligned}
(y_6')^0_{,\xi} &= \left(\frac{1}{c\xi^2} - \frac{\nu}{c\xi} u_3^0 \right) y_5^0 \\
&+ \left(\frac{1}{c\xi} - \frac{1}{\xi} + \frac{c-\nu}{c} u_3^0 \right) y_6^0 \\
&+ \frac{-(c-\nu^2)h}{2c\xi} (y_2^0)^2 - \frac{2(c-\nu^2)h}{2c} y_2^0 y_3^0 \\
&+ \dots
\end{aligned} \tag{28}$$

Boundedness and Continuity require that

$$\text{at } \xi = 0, \quad c = 1, \quad y_2^0 = y_4^0 = y_5^0 = 0 \tag{29}$$

and $u_1^0 = 0$ which is equivalent to the demand that $n \neq 1$.

With these conditions at $\xi = 0$, Equations (25) and (26) finally become

$$\begin{aligned}
(y_4)_{,\xi} &= \frac{3}{8} \gamma y_1^0 + \frac{27}{4} \alpha y_3^0 y_6^0 \\
(y_6)_{,\xi} &= 0
\end{aligned} \tag{30}$$

as a result to remove the singularity point.

(III) The Shooting Method - Frequency Search:

In order to ensure a unique relation between α , the adjusted amplitude parameter, and γ , the frequency parameter, we enforce the transverse displacement at the origin to take the value of unity. Thus by fixing the amplitude parameter, we are able to search for its corresponding frequency parameter in the frequency domain. Hence an additional condition is introduced

$$y_1^0 = 1 \quad (31)$$

At this point the problem can be completely described as a two-point boundary-value problem represented by the system of Equations (22) with boundary conditions Equations (29), (31) and (24); in a more compact form we have

$$\bar{Y}_{,\xi} = \bar{H}(\xi, \bar{Y}; \alpha_0, \lambda) \quad 0 \leq \xi \leq 1 \quad (32)$$

where \bar{Y}, \bar{H} are vector functions of six-dimensions, and $\lambda = \gamma^2$ is an amplitude parameter, and boundary conditions:

$$b \bar{Y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (33)$$

$$B \bar{Y}(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (34)$$

where

$$b = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix} \quad (35)$$

$$B = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu & c \end{vmatrix}$$

Eq'ns (33) and (34) serve to determine 7 unknown quantities y_1, \dots, y_6 , and the particular frequency parameter λ_0 corresponding to a given entity α_0 .

Instead of solving this non-linear two point boundary-value problem we proceed to solve the much easier equivalent initial-value problem as follows:

with initial conditions at $\xi = 0$

$$\bar{Y}_i = \begin{vmatrix} \bar{1} \\ 0 \\ K_i \\ 0 \\ 0 \\ L_i \end{vmatrix} \quad \text{and} \quad \lambda = \lambda_i \quad (36)$$

where K_i , L_i and λ_i are guessed values and the rest of the initial conditions are those corresponding to the boundary values at $\xi = 0$. Substituting Eq'ns (36) into (32) and carrying out the integration throughout the interval (0,1), should the outcome $\bar{Y}_i(\xi, \lambda_i, \alpha_i)$ at $\xi = 1$ satisfy the condition (34), that is, if

$$B \bar{Y}_i(1, \lambda_i, \alpha_i) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (37)$$

then a solution to the problem is obtained and $\lambda_i^{\frac{1}{2}}$ is the frequency parameter corresponding to α_i . The problem is now apparently equivalent to that of finding a functional relation

$$\lambda = \gamma^2 = G(\alpha) \quad (38)$$

which satisfies Eq. (37) for every α .

Newton's Method is employed to obtain a solution to the initial-value problem satisfying Eq'n (37); that is, if

$$B \bar{Y}_i(1, G^s(\alpha_i), \alpha_i) \neq \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

we replace the guessed values with

$$\begin{Bmatrix} K \\ L \\ \lambda \end{Bmatrix}_{i+1} = \begin{Bmatrix} K \\ L \\ \lambda \end{Bmatrix}_i + \begin{Bmatrix} E \\ \\ \end{Bmatrix}_i \quad (39)$$

where $[E]_i$ is the correction matrix given by

$$[E]_i = - (BJ_i)^{-1} B \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}_{i, \xi=1} \quad (40)$$

$$J_i = \left(\frac{\partial (y_1, y_2, y_3, y_4, y_5, y_6)}{\partial (K, L, \lambda)} \right)_{\xi=1}^i$$

Repetition of this procedure until the boundary conditions are met to within a required degree of accuracy will lead to a solution⁽⁸⁾.

Starting with $\alpha = 0$, which corresponds to the linear problem of free vibration possessing well known solutions in the isotropic, constant thickness case, this method converges rapidly to solutions in the linear orthotropic, variable thickness cases. By gradually increasing and incorporating parabolic extrapolation techniques, a functional relation between the frequency parameter and the adjusted amplitude parameter can easily be found in each case.

(IV) Construction of J_i :

Accordingly, taking derivatives of Eqs. (22) with respect to K , L and λ and setting

$$\begin{array}{ll}
 Y(1) = y_1 & Y(7) = \frac{\partial y_1}{\partial K} \\
 Y(2) = y_2 & Y(8) = \frac{\partial y_2}{\partial K} \\
 Y(3) = y_3 & Y(9) = \frac{\partial y_3}{\partial K} \\
 Y(4) = y_4 & Y(10) = \frac{\partial y_4}{\partial K} \\
 Y(5) = y_5 & Y(11) = \frac{\partial y_5}{\partial K} \\
 Y(6) = y_6 & Y(12) = \frac{\partial y_6}{\partial K}
 \end{array} \tag{41}$$

$$Y(13) = \frac{\partial y_1}{\partial L}$$

$$Y(19) = \frac{\partial y_1}{\partial \lambda}$$

$$Y(14) = \frac{\partial y_2}{\partial L}$$

$$Y(20) = \frac{\partial y_2}{\partial \lambda}$$

$$Y(15) = \frac{\partial y_3}{\partial L}$$

$$Y(21) = \frac{\partial y_3}{\partial \lambda}$$

$$Y(16) = \frac{\partial y_4}{\partial L}$$

$$Y(22) = \frac{\partial y_4}{\partial \lambda}$$

$$Y(17) = \frac{\partial y_5}{\partial L}$$

$$Y(23) = \frac{\partial y_5}{\partial \lambda}$$

$$Y(18) = \frac{\partial y_6}{\partial L}$$

$$Y(24) = \frac{\partial y_6}{\partial \lambda}$$

we have for $0 < \xi \leq 1$

$$DY(1) = Y(2)$$

$$DY(2) = Y(3)$$

$$DY(3) = Y(4)$$

$$DY(4) = v_1 Y(1) + \frac{V_2}{\xi^3} Y(2) + \frac{V_3}{\xi^2} Y(3) + \frac{V_4}{\xi} Y(4) + \frac{V_5}{\xi} [Y(3) Y(5) + Y(2) Y(6)]$$

$$DY(5) = Y(6)$$

(42)

$$DY(6) = \frac{V_6}{\xi^3} Y(5) + \frac{V_7}{\xi} Y(6) + \frac{V_8}{\xi} Y(2) Y(2)$$

$$DY(7) = Y(8)$$

$$DY(8) = Y(9)$$

$$DY(9) = Y(10)$$

$$DY(10) = v_1 Y(7) + \frac{V_2}{\xi^3} Y(8) + \frac{V_3}{\xi^2} Y(9) + \frac{V_4}{\xi} Y(10) + \frac{V_5}{\xi} [Y(9) Y(5) + Y(3) Y(11) + Y(8) Y(6) + Y(2) Y(12)]$$

$$DY(11) = Y(12)$$

$$DY(12) = \frac{V_6}{\xi^3} Y(11) + \frac{V_7}{\xi} Y(12) + \frac{2V_8}{\xi} Y(2) Y(8)$$

$$DY(13) = Y(14)$$

$$DY(14) = Y(15)$$

$$DY(15) = Y(16)$$

$$DY(16) = V_1 Y(13) + \frac{V_2}{\xi^3} Y(14) + \frac{V_3}{\xi^2} Y(15) + \frac{V_4}{\xi} Y(16) \\ + \frac{V_5}{\xi} [Y(15) Y(5) + Y(3) Y(17) + Y(2) Y(18) + Y(14) Y(6)]$$

$$DY(17) = Y(18)$$

$$DY(18) = \frac{V_6}{\xi^3} Y(17) + \frac{V_7}{\xi} Y(18) + \frac{2V_8}{\xi} Y(2) Y(14)$$

$$DY(19) = Y(20)$$

$$DY(20) = Y(21)$$

$$DY(21) = Y(22)$$

$$DY(22) = V_1 Y(19) + \frac{V_2}{\xi^3} Y(20) + \frac{V_3}{\xi^2} Y(21) + \frac{V_4}{\xi} Y(22) \\ + \frac{V_5}{\xi} [Y(21) Y(5) + Y(3) Y(23) + Y(2) Y(24) + Y(6) Y(20)]$$

$$DY(23) = Y(24)$$

$$DY(24) = \frac{V_6}{\xi^3} Y(23) + \frac{V_7}{\xi} Y(24) + \frac{2V_8}{\xi} Y(2) Y(20)$$

with initial values at $\xi = 0$

$$Y(1) = 1.0$$

$$Y(3) = K$$

$$Y(6) = L$$

$$Y(9) = 1.0$$

$$Y(18) = 1.0$$

$$Y(I) = 0$$

for $I=1, \dots, 24$ except for
 $I=1, 3, 6, 9,$ and $18.$

(43)

and corresponding equations at $\xi = 0$

$$DY(1) = Y(2)$$

$$DY(2) = Y(3)$$

$$DY(3) = Y(4)$$

$$DY(4) = \frac{3\lambda}{8h_o^2} Y(1) + \frac{27\alpha}{4h_o^3} Y(3) Y(6)$$

$$DY(5) = Y(6)$$

$$DY(6) = 0.0$$

$$DY(7) = Y(8)$$

$$DY(8) = Y(9)$$

$$DY(9) = Y(10)$$

$$DY(10) = \frac{3\lambda}{8h_o^2} Y(7) + \frac{27\alpha}{4h_o^3} [Y(3) Y(12) + Y(9) Y(6)]$$

(44)

$$DY(11) = Y(12)$$

$$DY(12) = 0$$

$$DY(13) = Y(14)$$

$$DY(14) = Y(15)$$

$$DY(15) = Y(16)$$

$$DY(16) = \frac{3\lambda}{8h_o^2} Y(13) + \frac{27\alpha}{4h_o^3} [Y(3) Y(18) + Y(15) Y(16)]$$

$$DY(17) = Y(18)$$

$$DY(18) = 0$$

$$DY(19) = Y(20)$$

$$DY(20) = Y(21)$$

$$DY(21) = Y(22)$$

$$\begin{aligned} DY(22) &= \frac{3\lambda}{8h_o^2} Y(19) + \frac{3\lambda}{8h_o^2} Y(1) \\ &\quad + \frac{27\alpha}{4h_o^3} [Y(21) Y(6) + Y(24) Y(3)] \end{aligned}$$

$$DY(23) = Y(24)$$

$$DY(24) = 0$$

Simultaneous integration with initial conditions will yield values necessary to construct the Jacobian at every step.

EVALUATION OF MAXIMUM BENDING AND MEMBRANE STRESSES

From the theory of pure bending, we deduce that the maximum bending stress is

$$\sigma = \frac{6M}{h^2}$$

hence we have maximum bending stresses in radial and circumferential directions in the following respective forms:

$$\sigma_r^B = \frac{6M}{h^2} r$$

$$\sigma_\theta^B = \frac{6M}{h^2} \theta$$

with equations (4) they assume the alternate forms

$$\begin{aligned} \sigma_r^B &= -\frac{6D}{h^2} \left(w_{,rr} + \frac{\nu}{r} w_{,r} \right) \\ \sigma_\theta^B &= -\frac{6D}{h^2} \left(\frac{1}{r} w_{,r} + \nu w_{,rr} \right) \end{aligned} \tag{45}$$

where

$$D = \frac{\bar{h}^3}{12a_{22}(c-\nu^2)} = \frac{h_o^3 h^3}{12a_{22}(c-\nu^2)}$$

$$\bar{h}(r) = h_o \left[1 - m \left(\frac{r}{a} \right)^n \right] = h_o (1 - m \xi^n) = h_o h$$

and

$$w = ax = aAg(\xi) \sin \gamma \tau \quad \tau = \frac{2\pi}{\gamma}$$

Since

$$w_{\max} = aAg(\xi)$$

$$w_r = Ag_\xi$$

and

$$w_{rr} = \frac{A}{a} g_{\xi\xi}$$

substituting these into (45) and upon simplifying we arrive at the following equations for evaluating maximum bending stresses at every point ξ :

$$\frac{a_{22}\sigma_r^B a^2}{h_o^2} = - \frac{h \sqrt{\alpha}}{2(c-v^2)} \left(g_{\xi\xi} + \frac{v}{\xi} g_{\xi} \right) \quad (46)$$

$$\frac{a_{22}\sigma_{\theta}^B a^2}{h_o^2} = - \frac{h \sqrt{\alpha}}{2(c-v^2)} \left(\frac{g_{\xi}}{\xi} + v g_{\xi\xi} \right)$$

Similarly we have for maximum membrane stresses

$$\sigma_r^M = \frac{N_r}{h} \quad (47)$$

$$\sigma_{\theta}^M = \frac{N_{\theta}}{h}$$

and by equs. (14), (47) becomes

$$\sigma_r^M = \frac{\psi}{rh} \quad (48)$$

$$\sigma_{\theta}^M = \frac{\psi, r}{h}$$

now since

$$\psi = aA^2 \varphi(\xi) \sin \gamma T$$

and

$$\psi_{\max} = aA^2 \varphi(\xi) \quad (49)$$

where

$$\varphi(\xi) = \frac{h_o F(\xi)}{a_{22} (c - v^2)} \quad (50)$$

substituting equs. (49), (50) and its derivative into equs. (48), and simplifying, we have for maximum membrane stresses

$$\frac{a_{22} \sigma_r^M a^2}{h_o^2} = \frac{\alpha}{(c - v^2)} \frac{F}{\xi h}$$

$$\frac{a_{22} \sigma_\theta^M a^2}{h_o^2} = \frac{\alpha}{(c - v^2)} \frac{F'}{\xi h}$$
(51)

Obviously the origin of the polar cylindrical co-ordinates represents removable singularity point, upon taking limiting process, we come up with the following equations for evaluating stresses at the origins:

$$\frac{a_{22} \sigma_r^B a^2}{h_o^2} = - \frac{h_o \sqrt{\alpha}}{2(1-\nu)} g_{\xi\xi}(0) = \frac{a_{22} \sigma_\theta^B a^2}{h_o^2}$$

$$\frac{a_{22} \sigma_r^M a^2}{h_o^2} = \frac{\alpha}{c - v^2} \frac{F_\xi(0)}{h_o} = \frac{a_{22} \sigma_\theta^M a^2}{h_o^2}$$
(52)

CASES STUDIED IN THIS REPORT

In this paper, the following axisymmetric solid circular plates are investigated by this method and the results are studied:

a) Anisotropic Constant Cross-section Plates of

$$c = 0.5, \quad c = 1, \text{ and } c = 1.5$$

b) Isotropic Variable Thickness Plates for

$$m = 0.1 \text{ and } m = 0.3$$

c) Anisotropic Variable Thickness Plates

$$m = 0.1$$

$$c = 0.5$$

$$m = 0.3$$

$$m = 0.1$$

$$c = 1.5$$

$$m = 0.3$$

GENERAL CONCLUSIONS

In studying the free harmonic vibrations of axisymmetric orthotropic variable thickness solid circular plates clamped at the edges - their frequency response curves, mode shapes, stress distributions and amplitude - stress relations, quite a few conclusions can be drawn. And due to the amazing regularities in the results, considering the non-linear nature of the problem, these conclusions can be safely generalized to problems of this class concerning anisotropic variable thickness plates which have not been investigated in the present work.

(A) Frequency Response Curves:

In general, the frequency response curves for moderately large amplitude deflections are smooth parabolic deviations from that of small deflection theory. They exhibit the anticipated behavior that the rate of increases in frequencies is increasingly greater for larger amplitudes. Some relationships between frequency and the material property constant c , frequency and the plate geometry parameter m , and frequency and the combined effect of c and m can be established.

1) Effect of c on frequency:

As c increases, i.e., the numerical value of the elastic constant in the circumferential direction becomes greater than that of radial direction; the frequency parameter increases, and vice versa. But it is not a linearly proportional relation; it appears that the rate of change in frequency due to changes in c is greater for $c < 1$ than for $c > 1$. However, the rate of change in frequency due to amplitude increment is greater for $c > 1$ than for $c < 1$.

2) Effect of plate geometry on frequency:

The study conducted on thickness variation has been limited, results on isotropic plates with different thickness variations are readily accessible in literature. Nevertheless, it does reveal that clamped plates with $m < 1$ have lower frequencies than that of constant thickness and plates with $m > 1$ would have higher frequencies as would be expected. The sensitivity of frequency to thickness variation is quite noticeable.

3) Combined effect of material property and thickness variation on frequency:

It is quite amazing that in spite of the complexity and non-linear nature of the problem, the frequency responses come very close to the superposition of orthotropic constant thickness plates and isotropic variable thickness plates. In other words, the combined effect of c and m can roughly be approximated by the sum of individual effects. It is also interesting to note that for thin plates in common practice, the effect of material property has a more dominant influence on frequency than the effect of thickness variation.

(B) Stress Distributions and Stress-Amplitude relations:

The condition necessitated in the removal of singularity that c has to be equal to unity at the center introduces discontinuities at $r = 0$ for orthotropic plate stress distributions. In fact they assume the corresponding values of the isotropic case at the center while in the immediate neighborhoods the continuities resume. The author chose to consider the physically more probable continuous stress

distributions while keeping in mind the fictitious mathematical singular point where the actual nature of the ratio of elastic constants is unknown.

In moderately large deflection of plate, stretching had been coupled with bending and thus produced the complicated non-linear nature of the problem. However, as the results manifest, for small magnitude, membrane stresses are usually insignificant in comparison to bending stresses, especially the circumferential membrane stresses. As the magnitude of amplitude increases, they gradually gain their place with respect to bending stresses; this is particularly so for $c = 1.5$ at the immediate neighborhood of the center of the plate. It is worth noting that the shapes of stress distributions are pretty much preserved as amplitude increases while the numerical values are magnified. In all cases, maximum stresses are those of radial bending and they occur at the clamped edges of the plates.

1) Orthotropic constant thickness plates:

The magnitudes of stresses increase rapidly with larger values of amplitude. The stress distributions are well ordered with smallest c having the greatest bending stresses at the edge and at the center. Membrane stresses are relatively small except for $c = 1.5$ in which comparable membrane stresses developed at the center.

2) Isotropic variable thickness plates:

Plates with larger values of m develop higher stresses at the edge and lower stresses at the center than those of plates with smaller values of m . In all instances, membrane stresses are small

as compared to bending stresses and radial stresses are more dominant than circumferential stresses.

3) Orthotropic variable thickness plates:

The stress distributions of these plates seem to exhibit the superposed characteristics of orthotropic constant thickness plates in that the stresses are greater in magnitude for small values of c and larger values of m .

For $c = 0.5$, the stresses are considerably higher than those of $c = 1.5$ and radial stresses are much more pronounced than circumferential stresses. And plates with larger m have higher stresses at the edge and lower stresses at the center than the corresponding stresses for smaller m . For $c = 1.5$, the phenomenon of the increasingly significant membrane stresses best illustrates the need for considering the stretching of plate in those deflections of the plate exceeding the plate's thickness.

NOMENCLATURE

A	Amplitude
a	Plate radius
a_{11}, a_{12}, a_{22}	Elastic constants
B	Edge boundary-value selection matrix
b	Center initial-value selection matrix
c	$= \frac{a_{11}}{a_{22}}$ ratio of elastic constants
D	$= \frac{a_{22} \bar{h}^3(r)}{12(a_{11} a_{22} - a_{12}^2)} = \frac{\bar{h}^3(r)}{12d}$ plate flexural rigidity
D_0	$= \frac{a_{22} h_0^3}{12(a_{11} a_{22} - a_{12}^2)}$
d	$= \frac{a_{11} a_{22} - a_{12}^2}{a_{22}}$
e_r, e_θ	Radial and circumferential normal strains
e_r^0, e_θ^0	Normal strain components at the middle plane
$(E)_i$	Correction or error matrix
F	$= \frac{a}{h_0} (c - \nu^2) \varphi$ normalized in-plane stress function
f, g	Shape functions of ξ
\bar{h}	$= h_0 \left[1 - \frac{h_0 - h}{h_0} a \left(\frac{r}{a} \right)^n \right]$ plate thickness
h_a, h_0	Plate thickness at the edge and at the center
h	$= 1 - m \xi^n$ normalized thickness variation
J_i	Jacobian
L	Lagrangian function
M_r, M_θ	Radial and circumferential bending moments per unit length

m	$= \frac{h_0 - h_a}{h_0}$ non-dimensional thickness coefficient
N_r, N_θ	In-plane radial and circumferential forces per unit length
n	Exponent of thickness expression
T	Kinetic energy
t	Time variable
u	Radial displacement
V_1	Strain energy due to stretching
V_2	Strain energy due to pure bending
w	Lateral displacement
r, θ, z	Cylindrical co-ordinate in the undeformed configuration
α	$= \left(A \frac{a}{h_0} \right)^2$ revised amplitude parameter
γ	$= \left(\frac{\rho h_0 a^4}{D_0} \right)^{\frac{1}{2}} \omega$ frequency parameter
ρ	Mass density of the plate
ω	Circular frequency in rad/sec
σ_r, σ_θ	Normal radial and circumferential stresses
σ^B, σ^M	Bending and membrane stresses respectively
ν	$= \frac{12}{a^2}$ Poisson's ratio
ξ	$= \frac{r}{a}$ dimensionless radial co-ordinate
τ	$= \left(\frac{D_0}{\rho h_0 a^4} \right)^{\frac{1}{2}} t$ dimensionless time variable
η, x	Dimensionless displacement functions associated with u, w respectively

Ψ	$= aA^2\varphi(\xi)\sin^2\gamma\tau$ in-plane harmonic stress function
φ	Shape function for stress function
λ	$= \gamma^2$
$'_r \ 't$	Partial derivatives with respect to r and t
ξ	Ordinary derivative with respect to ξ

ACKNOWLEDGMENT

The author wishes to express his deep gratitude to his advisor, Dr. C.L. Huang, for his initiation of the problem and his kind constant direction, advice and review of the material. I am greatly indebted to Mr. Bruce Sandman for the assistance and interesting discussions about the problem, in particular those pertaining to the application of the Shooting Method to this type of boundary-value problem. Much thanks to Dr. K.K. Hu, whose enthusiasm and patience materialized the computer program and led the author to a broad scope of understanding. Finally, the author would like to thank all those who helped to develop and to accomplish this study.