

## Creating desired potentials by embedding small inhomogeneities

A. G. Ramm

Citation: *J. Math. Phys.* **50**, 123525 (2009); doi: 10.1063/1.3267887

View online: <http://dx.doi.org/10.1063/1.3267887>

View Table of Contents: <http://jmp.aip.org/resource/1/JMAPAQ/v50/i12>

Published by the [American Institute of Physics](#).

---

### Related Articles

Non-adiabatic molecular dynamics with complex quantum trajectories. II. The adiabatic representation  
*J. Chem. Phys.* **137**, 22A518 (2012)

An observation of quadratic algebra, dual family of nonlinear coherent states and their non-classical properties, in the generalized isotonic oscillator  
*J. Math. Phys.* **53**, 082102 (2012)

Singularities for solutions to Schrödinger equations with asymptotically constant magnetic fields  
*J. Math. Phys.* **53**, 073707 (2012)

Quantum Painlevé-Calogero correspondence for Painlevé VI  
*J. Math. Phys.* **53**, 073508 (2012)

Quantum Painlevé-Calogero correspondence  
*J. Math. Phys.* **53**, 073507 (2012)

---

### Additional information on *J. Math. Phys.*

Journal Homepage: <http://jmp.aip.org/>

Journal Information: [http://jmp.aip.org/about/about\\_the\\_journal](http://jmp.aip.org/about/about_the_journal)

Top downloads: [http://jmp.aip.org/features/most\\_downloaded](http://jmp.aip.org/features/most_downloaded)

Information for Authors: <http://jmp.aip.org/authors>

## ADVERTISEMENT

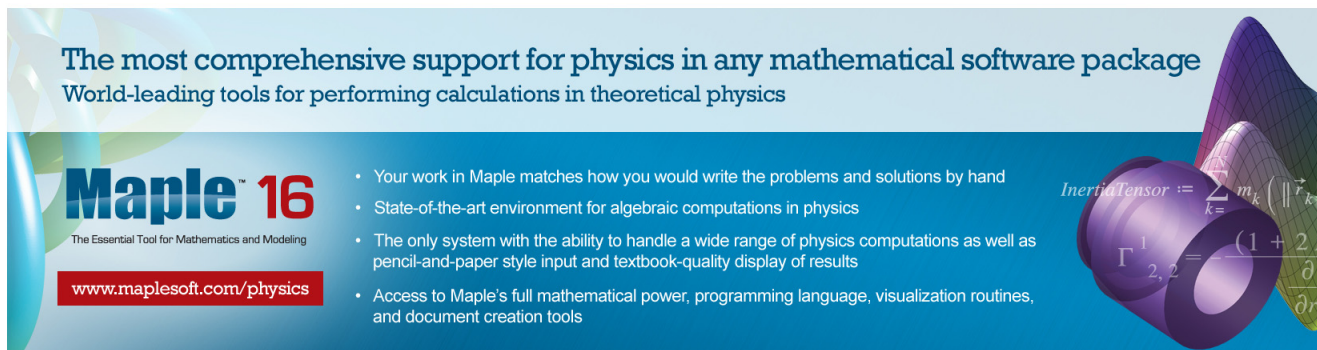
The most comprehensive support for physics in any mathematical software package  
World-leading tools for performing calculations in theoretical physics

**Maple™ 16**  
The Essential Tool for Mathematics and Modeling  
[www.maplesoft.com/physics](http://www.maplesoft.com/physics)

- Your work in Maple matches how you would write the problems and solutions by hand
- State-of-the-art environment for algebraic computations in physics
- The only system with the ability to handle a wide range of physics computations as well as pencil-and-paper style input and textbook-quality display of results
- Access to Maple's full mathematical power, programming language, visualization routines, and document creation tools

*InertiaTensor* :=  $\sum_{k=1}^n m_k \left( \|\vec{r}_k\|^2 \mathbf{1} - \vec{r}_k \vec{r}_k^T \right)$

$\Gamma_{2,2}^1 = \frac{(1 + 2r)}{\partial r}$



## Creating desired potentials by embedding small inhomogeneities

A. G. Ramm<sup>a)</sup>

Department of Mathematics, Kansas State University, Manhattan, Kansas 66506-2602, USA

(Received 7 June 2009; accepted 4 November 2009; published online 29 December 2009)

The governing equation is  $[\nabla^2 + k^2 - q(x)]u = 0$  in  $\mathbb{R}^3$ . It is shown that any desired potential  $q(x)$ , vanishing outside a bounded domain  $D$ , bounded in  $D$ , Riemann integrable, can be obtained if one embeds into  $D$  many small scatterers  $q_m(x)$ , vanishing outside balls  $B_m := \{x: |x - x_m| < a\}$ , such that  $q_m = A_m$  in  $B_m$ ,  $q_m = 0$  outside  $B_m$ ,  $1 \leq m \leq M$ ,  $M = M(a)$ . It is proven that if the number of small scatterers in any subdomain  $\Delta$  is defined as  $N(\Delta) := \sum_{x_m \in \Delta} 1$  and is given by the formula  $N(\Delta) = |V(a)|^{-1} \int_{\Delta} n(x) dx [1 + o(1)]$  as  $a \rightarrow 0$ , where  $V(a) = 4\pi a^3/3$ , then the limit of the function  $u_M(x)$ ,  $\lim_{a \rightarrow 0} u_M = u_e(x)$ , does exist and solves the equation  $[\nabla^2 + k^2 - q(x)]u = 0$  in  $\mathbb{R}^3$ , where  $q(x) = n(x)A(x)$ ,  $A(x_m) = A_m$ , and  $u_M(x)$  is a solution to the equation  $[\nabla^2 + k^2 - p(x)]u = 0$ , where  $p(x) := p_M(x)$  is some piecewise-constant potential. The total number  $M$  of small inhomogeneities is equal to  $N(D)$  and is of the order  $O(a^{-3})$  as  $a \rightarrow 0$ . A similar result is derived in the one-dimensional case. © 2009 American Institute of Physics. [doi:10.1063/1.3267887]

### I. INTRODUCTION

Consider the scattering problem,

$$[\nabla^2 + k^2 - q(x)]u = 0 \text{ in } \mathbb{R}^3, \quad k = \text{const} > 0, \quad (1)$$

$$u = e^{ik\alpha \cdot x} + A(\beta, \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty, \quad \beta = \frac{x}{r}, \quad \alpha \in S^2, \quad (2)$$

where  $S^2$  is the unit sphere in  $\mathbb{R}^3$ , and  $A(\beta, \alpha, k) = A_q(\beta, \alpha, k)$  is the scattering amplitude corresponding to the potential  $q(x)$ ,  $\alpha$  is the direction of the incident plane wave,  $\beta$  is a direction of the scattered wave, and  $k^2$  is the energy.

Let us assume that  $p = p_M(x)$  is a real-valued compactly supported bounded function, which is a sum of small inhomogeneities:  $p = \sum_{m=1}^M q_m(x)$ , where  $q_m(x)$  vanishes outside the ball  $B_m := \{x: |x - x_m| < a\}$  and  $q_m = A_m$  inside  $B_m$ ,  $1 \leq m \leq M$ ,  $M = M(a)$ .

The problem, we are studying in this paper, is the following.

*Problem:* Under what conditions the field  $u_M$ , which solves the Schrödinger equation with the potential  $p_M(x)$ , has a limit  $u_e(x)$  as  $a \rightarrow 0$ , and this limit  $u_e(x)$  solves the Schrödinger equation with a desired potential  $q(x)$ ?

We give a complete answer to this question. Theorem 1 (see below) is our basic result.

The class of potentials  $q$ , that can be obtained by our method, consists of bounded, compactly supported, Riemann-integrable functions. It is known that the set of Riemann-integrable functions is precisely the set of almost everywhere continuous functions, that is, the set of bounded functions with the set of discontinuities of Lebesgue measure zero in  $\mathbb{R}^3$ . These assumptions on  $q$  are not repeated but are always valid when we write, e.g., “an arbitrary potential.”

<sup>a)</sup>Electronic mail: ramm@math.ksu.edu.

In fact, a more general set of potentials can be constructed by the method of this paper. We do not go into detail, but mention that for some class of unbounded potentials, having local singularities, which are absolutely integrable, our theory can be generalized.

Our answer is as follows.

Assume that  $q(x)$  is an arbitrary Riemann integrable in  $D$  potential, vanishing outside  $D$ , where  $D$  is an arbitrary large but finite domain, and functions  $A(x)$  and  $n(x) \geq 0$  are such that  $A(x_m) = A_m$  and  $A(x)n(x) = q(x)$ . Then the limit  $u_e(x)$  of  $u_M(x)$  as  $a \rightarrow 0$  does exist and solves problem (1) and (2).

The notation  $u_e(x)$  stands for the effective field, which is the limiting field in the medium as  $M \rightarrow \infty$  or, equivalently,  $a \rightarrow 0$ . Under our assumptions (see Lemma 1 below) one has  $M = O(1/a^3)$ .

The field  $u_M$  is the unique solution to the integral equation,

$$u_M(x) = u_0(x) - \sum_{m=1}^M \int_D g(x, y, k) q_m(y) u_M(y) dy, \quad g(x, y, k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{3}$$

where  $u_0(x)$  is the incident field, which one may take as the plane wave, for example,  $u_0 = e^{ik\alpha \cdot x}$ , where  $\alpha \in S^2$  is the direction of the propagation of the incident wave.

We assume that the scatterers are small in the sense  $ka \ll 1$ . Parameter  $k > 0$  is assumed fixed, so the limits below are designated as limits  $a \rightarrow 0$ , and condition  $ka \ll 1$  is valid as  $a \rightarrow 0$ .

If  $ka \ll 1$ , then the following transformation of (3) is valid:

$$u_M(x) = u_0(x) - \sum_{m=1}^M \frac{e^{ik|x-x_m|}}{4\pi} A_m u_M(x_m) \int_{|y-x_m| < a} \frac{dy}{|x-y|} [1 + o(1)]. \tag{4}$$

To get (4) we have used the following estimates:

$$|x - x_m| - a \leq |x - y| \leq |x - x_m| + a, \quad |y - x_m| \leq a.$$

These estimates imply

$$e^{ik|x-y|} = e^{ik|x-x_m|} [1 + o(1)],$$

provided that  $|y - x_m| < a$  and  $a \rightarrow 0$ .

We also have taken into account that, as  $a \rightarrow 0$ , one has

$$\max_{x \in B_m} |u_M(x) - u_M(x_m)| = o(1).$$

A proof of this statement is given in Ref. 2.

One can also argue that Eq. (3) has a unique solution because it is a Lippmann–Schwinger equation with a compactly supported bounded uniformly with respect to  $M$  potential  $q_m(x)$ . Therefore, its solution is uniformly (with respect to  $M$ ) continuous in  $D$ .

An argument, different from the one, given in Ref. 2, can be outlined as follows. The limiting function  $u_e(x)$  is in  $H^2(D)$  by the standard elliptic regularity results, so it is continuous in  $D$ . The function  $u_M$  converges to  $u_e$  uniformly. Therefore, it satisfies the above inequality as  $M \rightarrow \infty$ , or, equivalently, as  $a \rightarrow 0$ .

A justification of a collocation method for solving the Lippmann–Schwinger equation (16) for the limiting field  $u_e$  is given recently in Ref. 9. From the arguments, given in Ref. 9, one obtains again the uniform convergence of  $u_M$  to  $u_e$ ,

$$\lim_{M \rightarrow \infty} \max_{x \in D} |u_M(x) - u_e(x)| = 0.$$

We want to prove that the sum in (4) has a limit as  $a \rightarrow 0$ , and to calculate this limit assuming that the distribution of small inhomogeneities or, equivalently, the points  $x_m$ , is given by formula (5),

see below, and  $M=N(D)$ , where  $N(\Delta)$  is defined in (5) for any subdomain  $\Delta \subset D$ , and  $N(D)$  is  $N(\Delta)$  for  $D=\Delta$ .

Our basic new tool is the following lemma.

*Lemma 1: If the points  $x_m$  are distributed in a bounded domain  $D \subset \mathbb{R}^3$  so that their number in any subdomain  $\Delta \subset D$  is given by the formula*

$$N(\Delta) = |V(a)|^{-1} \int_{\Delta} n(x) dx [1 + o(1)], \quad a \rightarrow 0, \quad (5)$$

where  $V(a) = 4\pi a^3/3$  and  $n(x) \geq 0$  is an arbitrary given continuous in the closure of  $D$  function, and if  $f(x)$  is an arbitrary Riemann integrable in  $D$  function, then the following limit exists:

$$\lim_{a \rightarrow 0} \sum_{m=1}^M f(x_m) V(a) = \int_D f(x) n(x) dx. \quad (6)$$

The above limit exists if  $f(x)n(x)$  is a Riemann-integrable function. The class of such functions is precisely the class of bounded almost everywhere continuous functions, i.e., bounded functions with the set of discontinuities of Lebesgue measure zero in  $\mathbb{R}^3$ . The class of such functions is precisely the class of functions for which Riemannian sums converge to the integral of the function. The result (6) can be generalized to the class of functions for which the integral  $\int_D f(x)n(x)dx$  exists as an improper integral. In this case  $f(x)$  may be unbounded at some points  $y$ , but the limit  $\lim_{\delta \rightarrow 0} \int_{D_\delta} f(x)n(x)dx$  exists and

$$\lim_{\delta \rightarrow 0} \int_{D_\delta} f(x)n(x)dx := \int_D f(x)n(x)dx,$$

where  $D_\delta := D \setminus B(y, \delta)$  and  $B(y, \delta)$  is the ball centered at  $y \in D$  and of radius  $\delta$ . In this case the sum in (6) is defined as

$$\lim_{a \rightarrow 0} \sum_{m=1}^M f(x_m) V(a) := \lim_{\delta \rightarrow 0} \lim_{a \rightarrow 0} \sum_{x_m \in D_\delta} f(x_m) V(a).$$

The same remark is valid for the conclusion of Theorem 1, which is our basic result.

**Theorem 1:** *If the small inhomogeneities are distributed so that (5) holds, and  $q_m(x)=0$  if  $x \notin B_m$ ,  $q_m(x)=A_m$  if  $x \in B_m$ , where  $B_m = \{x: |x-x_m| < a\}$ ,  $A_m := A(x_m)$ , and  $A(x)$  is a given in  $D$  function, such that the function  $q(x) := A(x)n(x)$  is Riemann integrable, then the limit*

$$\lim_{a \rightarrow 0} u_M(x) = u_e(x) \quad (7)$$

does exist and solves problem (1) and (2) with

$$q(x) = A(x)n(x). \quad (8)$$

There is a large literature on wave scattering by small inhomogeneities. A recent paper is Ref. 1. Our approach is new. Some of the ideas of this approach were earlier applied by the author to scattering by small particles embedded in an inhomogeneous medium.<sup>2-8</sup>

In Sec. II proofs are given and the one-dimensional version of the result is formulated and proven.

## II. PROOFS

*Proof of Lemma 1:* Let  $\{\Delta_p\}_{p=1}^P$  be a partition of  $D$  into a union of small cubes  $\Delta_p$  with centers  $y_p$ , without common interior points, and

$$\lim_{a \rightarrow 0} \max_p \text{diam} \Delta_p = 0. \tag{9}$$

One has

$$\sum_{m=1}^M f(x_m)V(a) = \sum_{p=1}^P f(y_p)V(a) \sum_{x_m \in \Delta_p} 1[1 + o(1)]. \tag{10}$$

We use formula (5) and the assumption (9) and get

$$\sum_{x_m \in \Delta_p} 1 = V(a)n(y_p)|\Delta_p|[1 + o(1)], \tag{11}$$

where  $|\Delta_p|$  is the volume of the cube  $\Delta_p$ .

It follows from (10) and (11) that

$$\sum_{m=1}^M f(x_m)V(a) = \sum_{p=1}^P f(y_p)n(y_p)|\Delta_p|[1 + o(1)], \tag{12}$$

which is the Riemannian sum for the integral in the right-hand side of (6), and the assumption (9) allows one to write

$$f(x_m) = f(y_p)[1 + o(1)] \quad \forall x_m \in \Delta_p, \tag{13}$$

if  $f$  is Riemann integrable.

The Riemannian sum in (12) converges to the integral in the right-hand side of (6) since the function  $f(x)n(x)$  is Riemann integrable.

Lemma 1 is proven. ■

*Proof of Theorem 1:* We apply Lemma 1 to the sum in (4), in which we choose  $A_m := A(x_m)$ , where  $A(x)$  is an arbitrary continuous in  $D$  function which we may choose as we wish. A simple calculation yields the following formula:

$$\int_{|y-x_m| < a} |x-y|^{-1} dy = V(a)|x-x_m|^{-1}, \quad |x-x_m| \geq a, \tag{14}$$

and

$$\int_{|y-x_m| < a} |x-y|^{-1} dy = 2\pi \left( a^2 - \frac{|x-x_m|^2}{3} \right), \quad |x-x_m| \leq a. \tag{15}$$

Therefore, the sum in (4) is of the form (6) with

$$f(x_m) = \frac{e^{ik|x-x_m|}}{4\pi|x-x_m|} A(x_m)u_M(x_m)[1 + o(1)].$$

Applying Lemma 1, one concludes that the limit  $u_e(x)$  in (7) does exist and solves the integral equation

$$u_e(x) = u_0(x) - \int_D \frac{e^{ik|x-y|}}{4\pi|x-y|} q(y)u_e(y)dy, \tag{16}$$

where  $q(x)$  is defined by formula (8).

Applying the operator  $\nabla^2 + k^2$  to (16), one verifies that the function  $u_e(x)$  solves problem (1) and (2).

Theorem 1 is proven. ■

*Remark 1:* Our method can be applied to the one-dimensional scattering problem. The role of

the balls  $B_m$  is now played by the segments:  $B_m := \{x: x \in \mathbb{R}^1, |x - x_m| < a\}$ , the role of  $D$  is played by an interval  $(c, d)$ , the  $V(a) = 2a$  in the one-dimensional case, an analog of formula (5) for the number of small inhomogeneities  $N(\Delta) = \sum_{x_m \in \Delta} 1$  is

$$N(\Delta) = (2a)^{-1} \int_{\Delta} n(x) dx [1 + o(1)], \quad (17)$$

and  $\Delta$  is now any interval on the line. The total number  $M$  of small inhomogeneities is now of the order of  $O(a^{-1})$ .

In the one-dimensional case an analog of the function  $g(x, y, k)$  is

$$g(x, y, k) = - \frac{e^{ik|x-y|}}{2ik}. \quad (18)$$

An analog of the potential  $q_m$  is  $q_m(x) = A_m$  inside the interval  $B_m$ ,  $q_m(x) = 0$  outside  $B_m$ , and we assume that  $A_m = A(x_m)$ , where  $A(x)$  is a continuous function which we can choose at will. With these notations one can use Eq. (4) without any change, but remember that  $g(x, y, k)$  is now defined as in (18). An analog of (4) now is

$$u_M(x) = u_0(x) + \sum_{m=1}^M \frac{e^{ik|x-x_m|}}{2ik} A(x_m) u_M(x_m) 2a [1 + o(1)]. \quad (19)$$

An analog of Theorem 1 can be stated as follows.

**Theorem 2:** *If the small inhomogeneities are distributed so that (5) holds, and  $q_m(x) = 0$  if  $x \notin B_m$ ,  $q_m(x) = A_m$  if  $x \in B_m$ , where  $B_m = \{x: |x - x_m| < a\}$ ,  $A_m := A(x_m)$ , and  $A(x)$  is a given continuous in  $D$  function, then the limit  $u_\epsilon(x)$  in (7) does exist and solves problem (1) and (2) with  $q(x)$  defined in (8),  $\nabla^2 u$  replaced by  $u''$ , and the radiation condition (2) modified to fit the one-dimensional problem.*

<sup>1</sup>Froufe-Pérez, L., Yezpez, M., Mello, P., and Saenz, J., "Statistical scattering of waves in disordered waveguides: From microscopic potentials to limiting macroscopic statistics," *Phys. Rev. E* **75**, 031113 (2007).

<sup>2</sup>Ramm, A. G., "Many-body wave scattering by small bodies and applications," *J. Math. Phys.* **48**, 103511 (2007).

<sup>3</sup>Ramm, A. G., "Distribution of particles which produces a "smart" material," *J. Stat. Phys.* **127**, 915 (2007).

<sup>4</sup>Ramm, A. G., "Wave scattering by many small particles embedded in a medium," *Phys. Lett. A* **372**, 3064 (2008).

<sup>5</sup>Ramm, A. G., "Electromagnetic wave scattering by small bodies," *Phys. Lett. A* **372**, 4298 (2008).

<sup>6</sup>Ramm, A. G., *Wave Scattering by Small Bodies of Arbitrary Shapes* (World Scientific, Singapore, 2005).

<sup>7</sup>Ramm, A. G., *Inverse Problems* (Springer, New York, 2005).

<sup>8</sup>Ramm, A. G., *Scattering by Obstacles* (Reidel, Dordrecht, 1986).

<sup>9</sup>Ramm, A. G., "A collocation method for solving integral equations," *Int. J. Comp. Sci. Math.* **2**, 222 (2009).