METHOD OF LAGRANGE MULTIPLIERS AND THE KUHN-TUCKER CONDITIONS

by

PRAMOD KUMAR GUPTA

B.S., BIRLA INSTITUTE OF TECHNOLOGY, INDIA, 1969

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Approved by:

Ching-Lai Hwang

Major Professor

TABLE OF CONTENTS

0. INTRODUCTION

The general programming problem consists of determining values of n variables $x_1, \ldots, x_n$ which optimize the function, $S = f(x_1, \ldots, x_n)$, and satisfy m constraints given by $g_j(x_1, \ldots, x_n) \{\leq, =, \geq\} b_j$, $j = 1, \ldots, m$. A particular case of the programming problem is a linear programming problem which seeks to determine non-negative values of n variables, $x_i \geq 0$, $i = 1, \ldots, n$, which optimize the linear function, $S = \sum_i c_i x_i$, and satisfy the m linear constraints, $\sum_i a_{ji} x_i \{\leq, =, \geq\} b_j$, $j = 1, \ldots, m$. All programming problems that are not linear are called nonlinear programming problem. Although the class of nonlinear programming problems which has been studied most extensively is that where the constraints are linear and the objective function is nonlinear, the other classes of nonlinear programming problems are that the constraints are nonlinear and the objective function is linear, and that both the constraints and the objective function are nonlinear.

A number of algorithms have been proposed for the solution of the general nonlinear programming problem. However, only a few have been demonstrated to be effective when applied to large-scale nonlinear programming problems, and none of the algorithms has proved to be superior that it can be classified as a universal algorithm for general nonlinear programming problems (Himmelblau, 1972). No general method exists to solve nonlinear programming problems in the sense that the simplex algorithm exists to solve linear programming problems. We may recall that linear programming problems have the following properties (Hadley, 1964).

(1) The set of feasible solutions which satisfies the constraints and the non-negativity restrictions is a convex set. This convex set has a finite number of corners which are referred to as extreme points. (2) The set of all n variables (x_1, ..., x_n), which yield a specific value of the objective function is a hyper-plane, and the hyperplanes corresponding to different values of the objective function are parallel (3) A local maximum or minimum is also the global maximum or minimum of the objective function over the set of feasible solutions. (4) If the optimal value of the objective function is bounded, the optimal solution will be one of the extreme points of the convex set of the feasible solution. Furthermore, if we start the search of the optimal solution at any extreme point of the convex set of feasible solutions, we will reach the optimal extreme point in a series of steps such that at each step we moves only to an adjacent extreme point. No efficient superior algorithm for nonlinear programming problems exists because contrary to linear programming problems for any given nonlinear programming problems, some or all of these features which characterize linear programming problems may be violated.

When the general nonlinear programming problem has that (1) no inequalities appear in the constraints, (2) there are no non-negativity or discreteness restrictions on the variables, (3) the number of equality constraints, m, is less than the number of variables, n, that is, $m < n$, and (4) the objective function, $f(x)$, and the functions in the equality constraints, $g_j(x)$, $j = 1, \ldots, m$ are continuous and possess partial derivatives at least through second order, the problem can be solved by the method of Lagrange multiplier. This classical method is of use mainly in theoretical analyses, and is not, in general, well suited for numerical calculation. The method of Lagrange multipliers can be generalized to handle problems involving inequality constraints and non-negative variables. The necessary conditions for optimizing these problems are the Kuhn-Tucker conditions. Although the theory related to the method of the Lagrange multipliers and the Kuhn-Tucker conditions is not directly concerned with computational techniques, it has been of fundamental importance in developing a numerical procedures for solving nonlinear programming problems, for example, quadratic programming problem.

The method of Lagrange multipliers and the Kuhn-Tucker conditions have been studied extensively by many investigators. In 1951, Kuhn and Tucker published an important paper "Nonlinear programming" dealing with necessary and sufficient conditions for optimal solutions to programming problems, which laid the foundations for a great deal of later work in nonlinear programming. These conditions are known as the Kuhn-Tucker conditions in their honour. They introduced the concept of constrained qualification and rejects certain stationary points as possible optima. Generalization of their theoretical work by other authors appeared later in different books [Arrow, Hurwicz, and Uzawa (1958), Mangasarian (1969)] and papers. Samuelson (1955) described the necessary and sufficient conditions for local optimum of a nonlinear objective function subject to nonlinear equality constraints. He derived these conditions from Taylor's series expansion for single and multi equality constraints. Tucker (1957) presented the use of Lagrange multiplier technique for minimizing a convex