ANALYSIS OF AN ELONGATED SPLIT-RING DYNAMOMETER

by

SRIKISHEN N. BINDIGANAVALE

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Approved by:

F. C. Appl
Major Professor

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NOMENCLATURE

A	area of cross-section of the ring
α	numerical factor used to obtain the shearing stress; = 1.5 for rectangular cross-section
b	width of the ring
β	dimensionless quantity, $= \frac{h}{r}$
χ	sensitivity ratio
δ_V, δ_H	vertical and horizontal deflections respectively of a quadrant of the ring
$\delta(y)$	increase in fiber length after application of load
Δ_V, Δ_H	vertical and horizontal deflections respectively of the whole ring
Δ_ψ, Δ_γ	angle between end sections of the curved beam before and after application of load respectively
e	distance of the neutral axis from the centroidal axis
E, G	modulus of elasticity and modulus of rigidity respectively
η	dimensionless quantity, $= \frac{\ell}{r}$
$\varepsilon(y)$	strain or unit elongation of the fiber
h	thickness of the ring
H	horizontal component of cutting force
I	area moment of Inertia about the centroidal axis in the plane of the width of the ring
ℓ	length of the straight beam
L	$= \dfrac{[\frac{\ell r}{I} + \frac{\ell^2}{2I} - \frac{r}{Ae} + \frac{1}{A} + \frac{\pi r}{2Ae}]}{[\frac{\ell}{I} + \frac{\pi}{2Ae}]}$

λ	function of β, $= 1 - \dfrac{\beta}{\ln\left(\dfrac{2+\beta}{2-\beta}\right)}$
m	dimensionless parameter; $= \dfrac{e}{r - e}$
M_c, M_A, M_B	bending moments at the fixed end of the cross section
N, V,M	normal force, shear force, and bending moment respectively at any section
ω_1, ω_2	change in angle per unit of angle
ψ	stiffness ratio
P, Q, M_o	normal force, shear force, and bending moment respectively, at the section indicated in figure
r, ϕ	polar co-ordinates to refer points on the centroidal axis of the curved portion of the ring
r_2, r_1	outer and inner radius of the curved portion of the ring
R	radius of curvature after application of load
s	distance along the centroidal axis of the ring
σ_H	stress at the outer fiber of the ring subjected to the horizontal component of the cutting force
σ_{Vo}, σ_{Vi}	stress at the outer and inner fiber respectively of the ring subjected to the vertical component of the cutting force
θ_V, θ_{HR}	angular position of the cross-section in the curved portion of the ring
θ_{Vo}	angular position of the cross-section in the curved portion of the ring for $\sigma_{Vo} = 0$
U	total strain energy
U_1, U_2	strain energy of the straight and curved portions respectively of the ring
V	vertical component of cutting force
V_{max}, H_{max}	maximum vertical and horizontal component of cutting force

y normal distances of the stressed fiber from the centroidal axis

INTRODUCTION

Accurate measurement of the forces generated in cutting metals or refractories is needed for rational design of machine tools and cutting tools. Also, valuable information concerning machinability and tool wear may be obtained from the force data. The cutting forces are usually measured by means of a tool dynamometer.

Machine tool dynamometers typically contain an elastic member such as a metal column or beam, which deforms under applied force. The deformation is transmitted to strain gauges, located at suitable places on the elastic member. The strain gauges are usually connected in a four-arm wheatstone bridge to produce an output signal, which when calibrated, is an accurate indication of the applied force.

A particularly popular type of dynamometer used in practice is a split-ring dynamometer, which utilizes load rings as shown schematically in Figure 1. For analysis purposes, the circular ring is substituted for the octagonal ring as shown in Figure 2a. The active part of the dynamometer is shown schematically in Figure 2b. A split-ring dynamometer with strain gauge locations and four-arm wheatstone bridge wiring diagrams is shown schematically in Figures 3a and 3b. A three-dimensional version of the split-ring dynamometer using the type of load ring discussed, was used by B. King and R. O. Foschi [1]* to obtain the three orthogonal components of a cutting force.

As yet, a single dynamometer has not been designed which can be used for the measurement of desired forces on all machines (lathes, milling machines, drilling machines, grinding machines, etc.), as the forces to be

*Numbers in brackets designate references at the end of report.

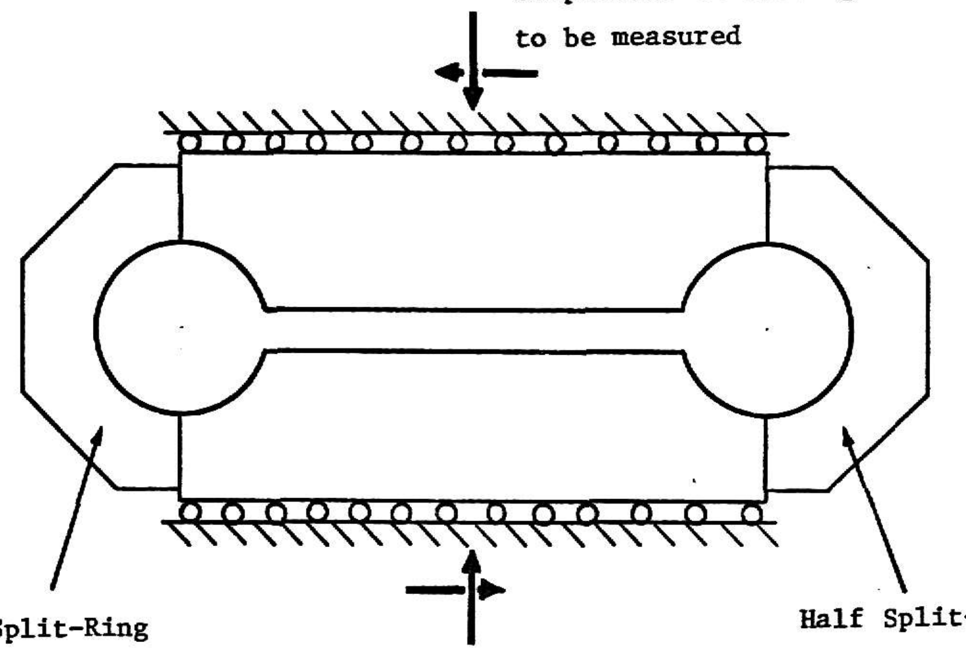

Components of Cutting Force
to be measured
Half Split-Ring
Half Split-Ring

Fig. 1

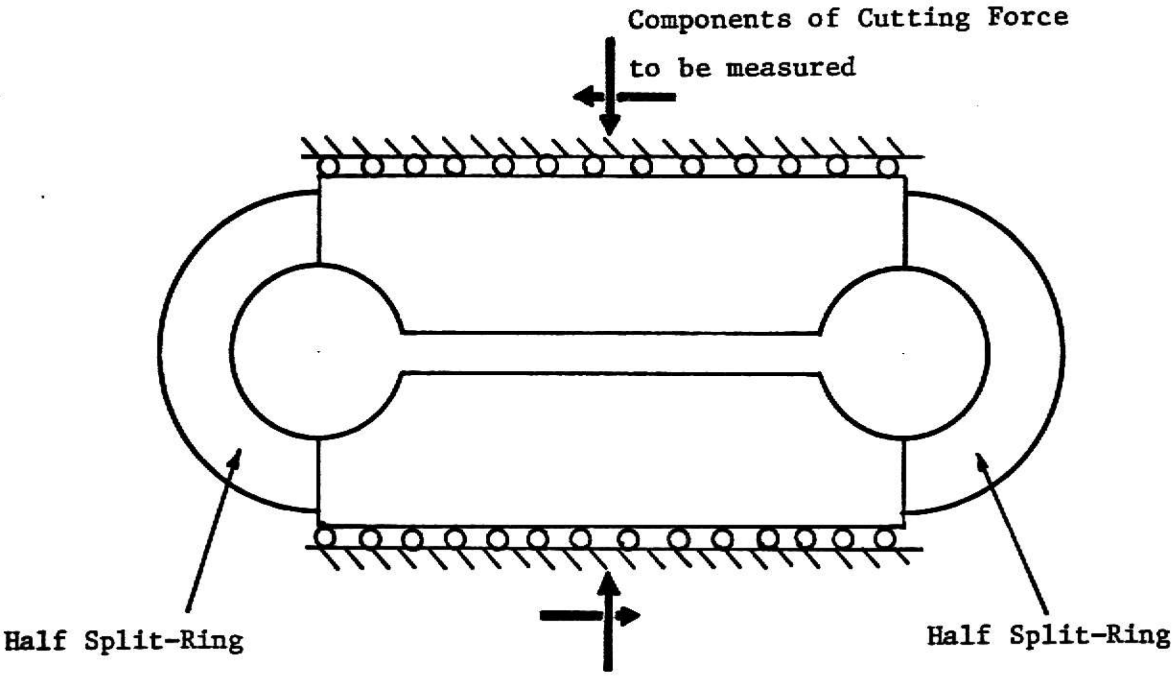

Components of Cutting Force
to be measured
Half Split-Ring
Half Split-Ring

Fig. 2a

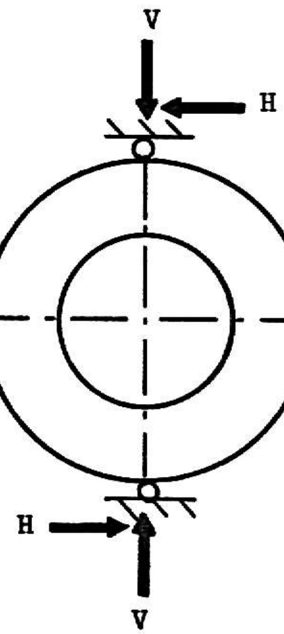

V
H
H
V

Fig. 2b

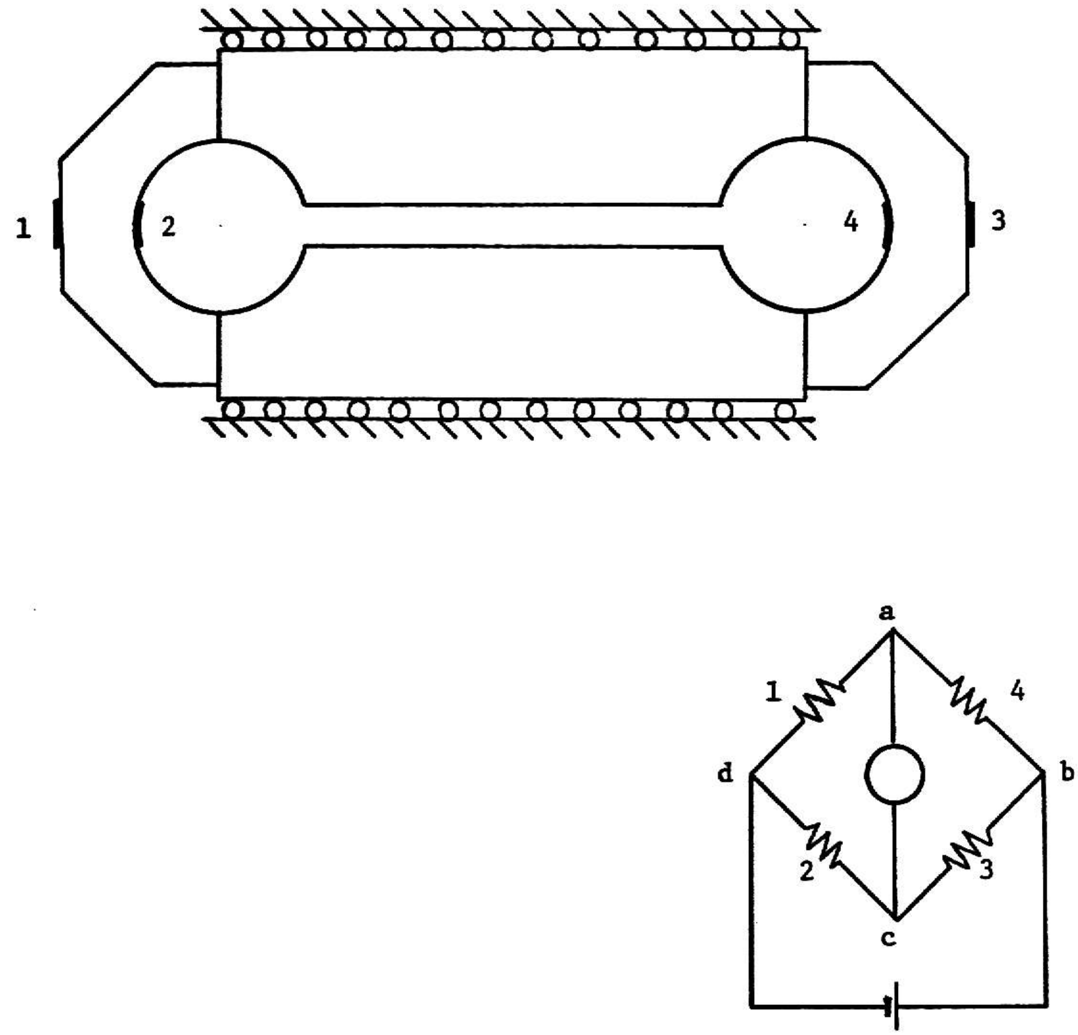


Fig. 3a Wiring Diagrams for Vertical Force Gages

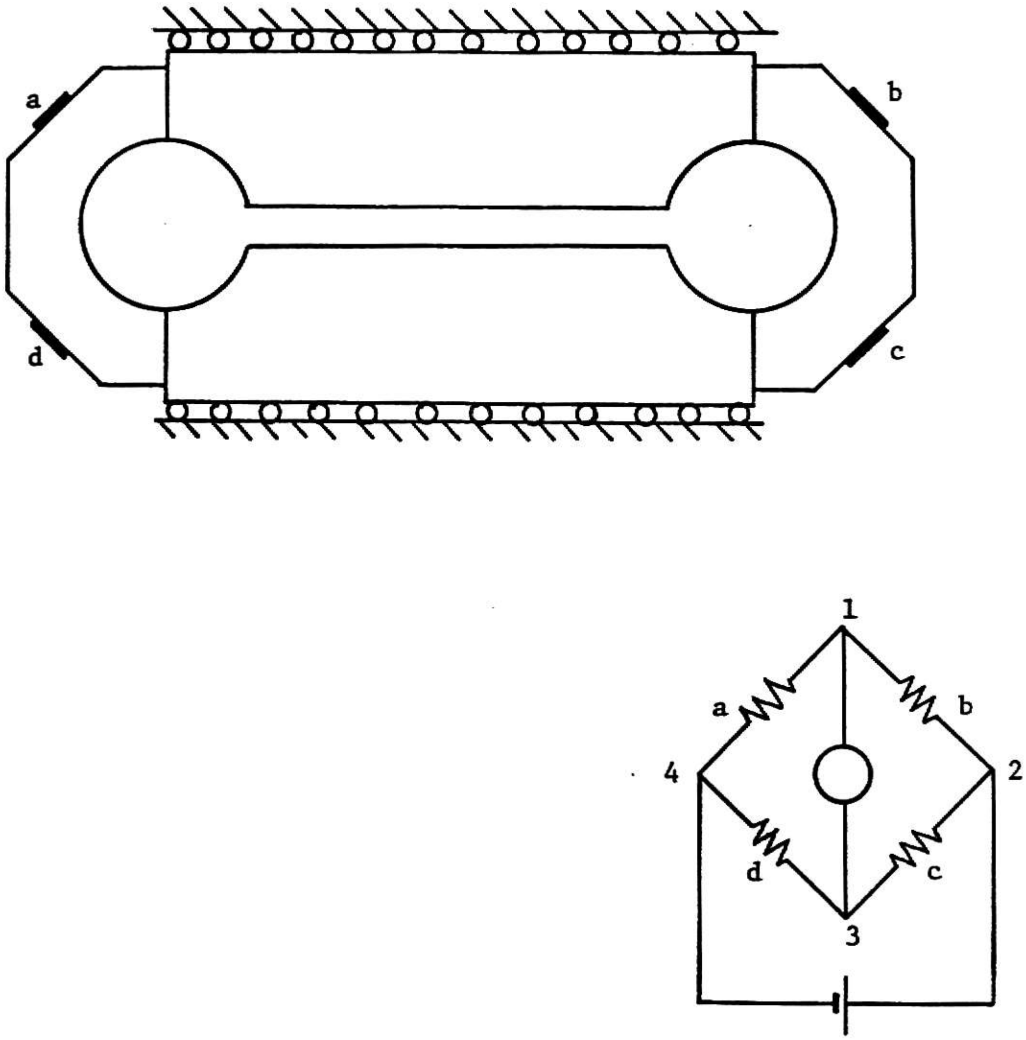


Fig. 3b Wiring Diagrams for Horizontal Force Gages

measured vary in magnitude and direction from one machine to another. Therefore, the type of machine tool and the magnitude of the forces to be measured must be considered in the design of a dynamometer for a specific application. Thus each dynamometer should be individually designed to meet the requirements of the situation.

In some applications, design requirements for tool dynamometers are unusually severe. Maximum stiffness is required to avoid interfering with normal operation of the cutting tool, but this tends to reduce the sensitivity of the dynamometer which makes it difficult to accurately measure small forces. Therefore in designing dynamometers, a compromise must be made between stiffness and sensitivity. This is controlled by the geometrical design of the dynamometer in conjunction with the sensitivity of the strain gauges. Thus there is a need for general design criteria which may be used in determining the most suitable dynamometer design.

The split-ring dynamometer does not satisfy the requirements for some applications since, as will be seen later, the range of the stiffness and sensitivity ratios of vertical and horizontal loading is restricted between approximately 0.2-0.5 and 0.55-1.3 respectively. For some applications a higher stiffness or sensitivity ratio is desired. To improve the split-ring dynamometer, an elongated split-ring dynamometer is considered in this report. The elongated split-ring dynamometer, as shown in Figure 4, differs from the split-ring dynamometer in that it has straight beams included with the split-rings. Within practical limits for the dimensions of the ring, a higher range of stiffness and sensitivity ratio can be obtained. This means that there is more flexibility in the design.

In the analysis of the elongated split-ring, equations for the deflections,

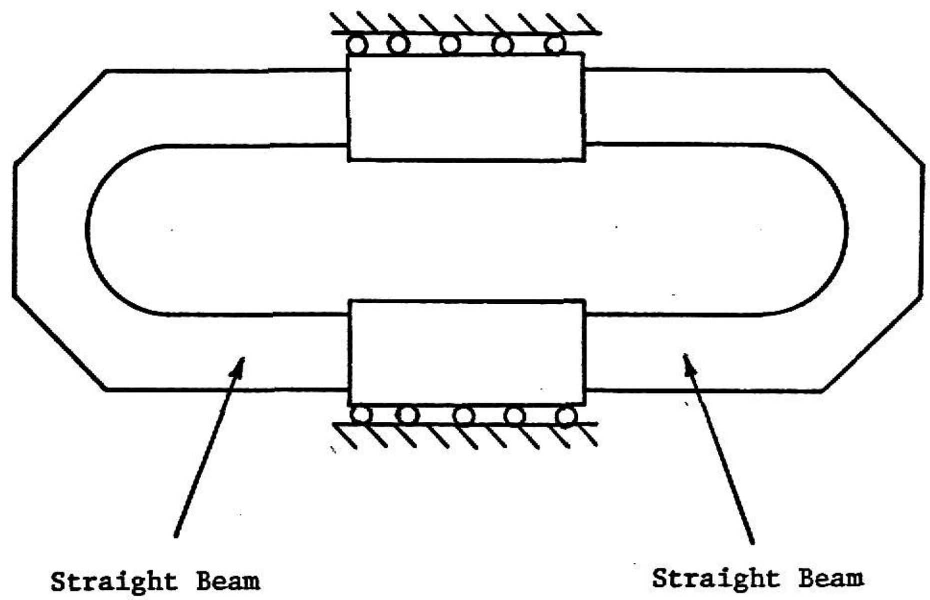


Fig. 4

stresses, stiffness, and sensitivity are set up for each component of force on the ring. The equations derived are based on curved beam theory and Castigliano's theorem for deflection. Graphs are presented for different stiffness and sensitivity ratios between horizontal and vertical loading with respect to two non-dimensional quantities, β and η. A computer program is written to solve the equation set up for the dynamometer design. Finally a brief discussion of the results is presented.

ANALYSIS OF ELONGATED SPLIT-RING DYNAMOMETER

Formulation of the Problem

For the elongated split-ring dynamometer shown in Figure 5, V and H are the vertical and horizontal components of a cutting force to be measured. It is assumed that members E and F are rigid and that relative motion can occur in the vertical or horizontal direction but that no relative angular displacement occurs.

In designing the dynamometer it is proposed to find an expression for stiffness and sensitivity, for each of the loads V and H, in terms of the dimensions of the load ring. The vertical stiffness is defined as the vertical load required for a unit vertical deflection and the vertical sensitivity is defined as the strain output measured due to a unit vertical load.

Load Ring Analysis

Consider one quarter of the load ring as shown in Figure 6a. The ring is assumed to be made of an elastic material and is subjected to forces P, Q and M_o. s represents the distance along the centroidal axis of the ring. The length of the straight portion is ℓ. A polar co-ordinate system together with the length ℓ will be used to denote s for the curved portion of the ring.

To find an expression for the stiffness it is required to find the deflections Δ_V and Δ_H as indicated in Figure 6a. Castigliano's theorem for deflection is used to find Δ_V and Δ_H. To use this theorem it is necessary to determine the total strain energy for the load ring.

At the section located at s, the tensile force N, the shear force V, and the bending moment M are as shown in Figure 6b. Referring to (69) in

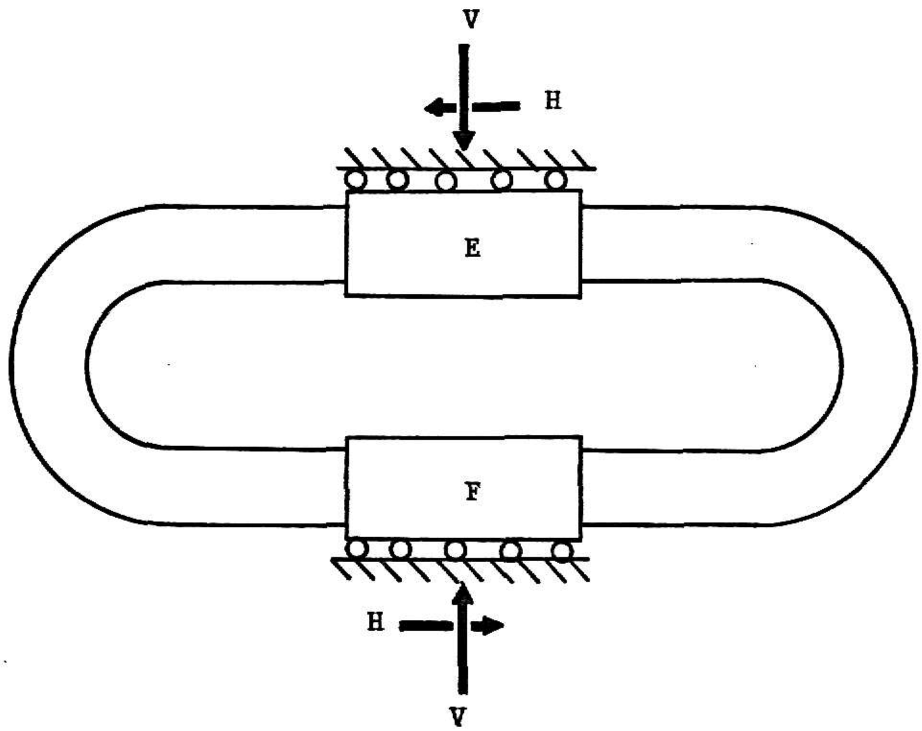


Fig. 5

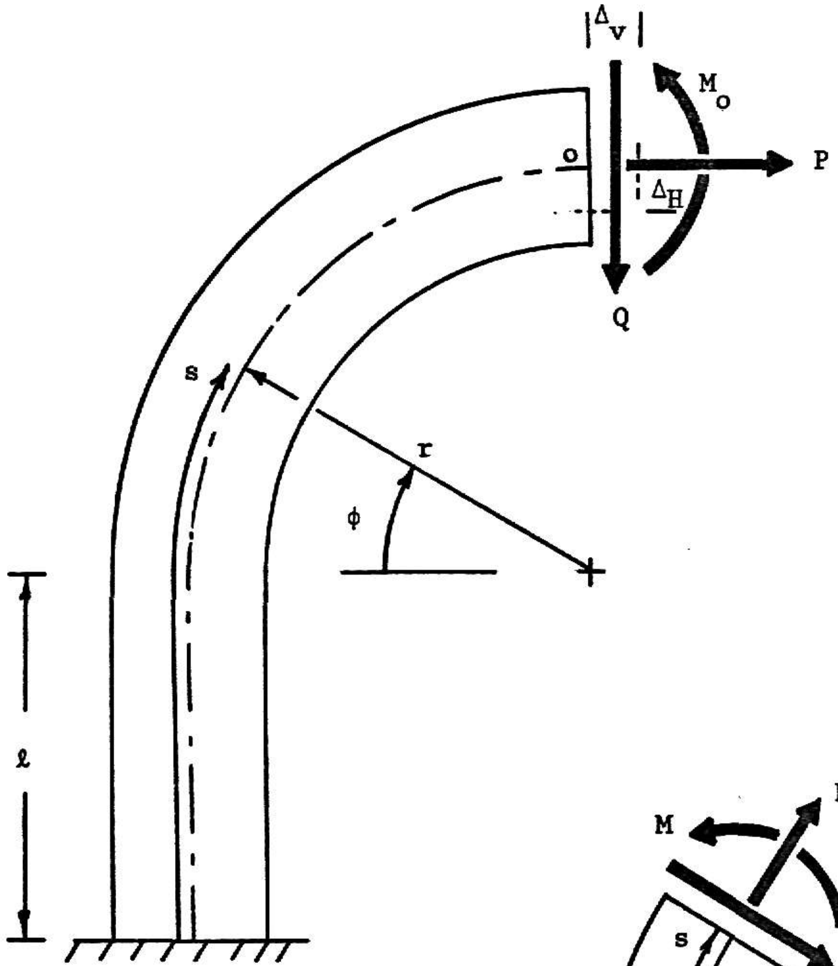


Fig. 6a

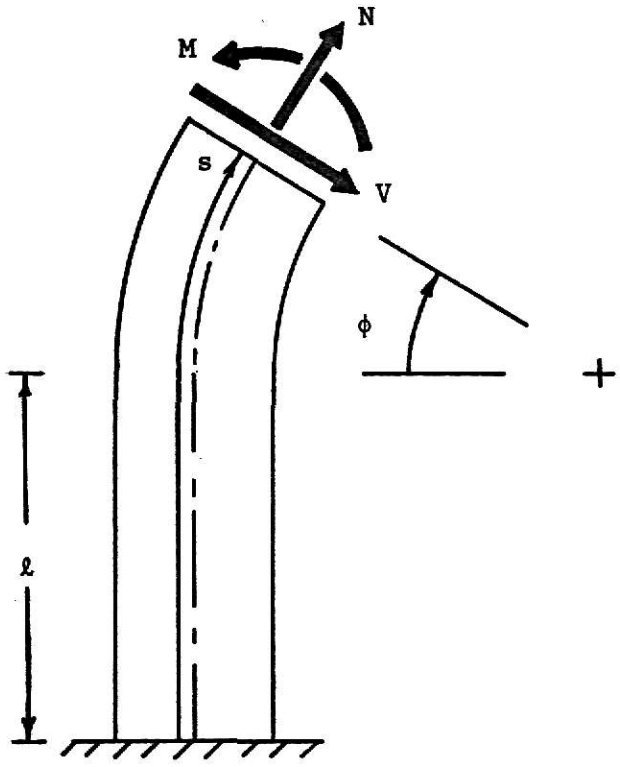


Fig. 6b

Appendix A, for an increment $r\Delta\Psi$ of a curved beam the strain energy has been shown to be

$$dU = \left(\frac{N^2}{2AE} - \frac{MN}{rAE} + \frac{M^2}{2reAE} + \frac{\alpha V^2}{2AG}\right) r\Delta\Psi \tag{1}$$

This form is directly applicable to the curved section of the ring, but must be modified for the straight portion where the radius of curvature becomes infinite.

It is necessary to determine the limit of the quantity [reA] as $r \to \infty$.

Referring to (59) in Appendix A,

$$e = \frac{\int_A \frac{y}{r - y}\, dA}{\int_A \frac{1}{r - y}\, dA} \tag{2}$$

and,

$$r \int_A \frac{y}{r - y}\, dA = \int_A \frac{y^2}{r - y}\, dA \tag{3}$$

Combining (2) and (3), gives,

$$rAe = \frac{rA \int_A \frac{y}{r - y}\, dA}{\int_A \frac{1}{r - y}\, dA} \tag{4a}$$

Substituting (3) into (4a),

$$[rAe] = \frac{A \int_A \frac{y^2}{r - y}\, dA}{\int_A \frac{1}{r - y}\, dA} \tag{4b}$$

Multiplying numerator and denominator by r,

$$[\,rAe\,] = \frac{A\displaystyle\int_A \frac{y^2}{1-\frac{y}{r}}\,dA}{\displaystyle\int_A \frac{1}{1-\frac{y}{r}}\,dA} \tag{4c}$$

Taking the limit of each side as $r \to \infty$,

$$\lim_{r\to\infty}[\,rAe\,] = \frac{A\displaystyle\int_A y^2 dA}{\displaystyle\int_A dA} \tag{5}$$

By the definition of area and moment of inertia of the area,

$$\int_A dA = A \tag{6a}$$

and,

$$\int_A y^2 dA = I \tag{6b}$$

Therefore,

$$\lim_{r\to\infty}[rAe] = I \tag{7}$$

This is in agreement with the straight beam theory. Using (1) and (7) it is possible to write the expression for the total strain energy of the straight portion of the ring as,

$$U_1 = \int_0^{\ell}\left(\frac{N^2}{2AE} + \frac{M^2}{2EI} + \frac{\alpha V^2}{2AG}\right) ds \tag{8a}$$

The total strain energy for the curved portion of the ring is

$$U_2 = \int_o^{\pi/2} \left(\frac{N^2}{2AE} - \frac{MN}{rAE} + \frac{M^2}{2reAE} + \frac{\alpha V^2}{2AG}\right) rd\phi \tag{8b}$$

The total strain energy then is,

$$U = U_1 + U_2 \tag{9}$$

From equilibrium, the forces N, V, and M at any cross-section are given as follows,

For the straight portion

$$M(s) = M_o - P(\ell + r - s) - Qr \qquad o \leq s \leq \ell \tag{10a}$$

$$N(s) = - Q \qquad o \leq s \leq \ell \tag{10b}$$

$$V(s) = P \qquad o \leq s \leq \ell \tag{10c}$$

$$r(s) = \infty \qquad o \leq s \leq \ell \tag{10d}$$

For the curved portion

$$M(\phi) = M_o - P(r - r \text{ Sin } \phi) - Qr \text{ Cos } \phi \qquad o \leq \phi \leq \frac{\pi}{2} \tag{11a}$$

$$N(\phi) = - Q \text{ Cos } \phi + P \text{ Sin } \phi \qquad o \leq \phi \leq \frac{\pi}{2} \tag{11b}$$

$$V(\phi) = Q \text{ Sin } \phi + P \text{ Cos } \phi \qquad o \leq \phi \leq \frac{\pi}{2} \tag{11c}$$

$$r(\phi) = r \qquad o \leq \phi \leq \frac{\pi}{2} . \tag{11d}$$

Using (8a),

$$U_1 = \int_o^{\ell} \left(\frac{N^2(s)}{2AE} + \frac{M^2(s)}{2EI} + \frac{\alpha V^2(s)}{2AG}\right) ds \tag{12}$$

Using (10a), (10b), (10c),

$$M^2(s) = \{M_o - P(\ell + r) - Qr\}^2$$

$$+ 2Ps\{M_o - P(\ell + r) - Qr\} + P^2 s^2 \qquad o \leq s \leq \ell \qquad (13a)$$

$$N^2(s) = Q^2 \qquad o \leq s \leq \ell \qquad (13b)$$

$$V^2(s) = P^2 \qquad o \leq s \leq \ell \qquad (13c)$$

Substituting (13a), (13b), (13c) in (12),

$$U_1 = \int_o^{\ell} [\frac{Q^2}{2AE} + \frac{\{M_o - P(\ell + r) - Qr\}^2}{2EI} + \frac{P\{M_o - P(\ell + r) - Qr\}}{EI} s + \frac{P^2}{2EI} s^2 + \frac{\alpha P^2}{2AG}] \, ds \qquad (14a)$$

Solving (14a), gives,

$$U_1 = \frac{\ell}{2E} [\frac{Q^2}{A} + \alpha \frac{E}{G} \frac{P^2}{A} + \frac{P^2 \ell^2}{3I} + \frac{\{M_o - r(P + Q)\}\{M_o - P(\ell + r) - Qr\}}{I}] \qquad (14b)$$

using (8b),

$$U_2 = \int_o^{\pi/2} \{ \frac{N^2(\phi)}{2AE} - \frac{M(\phi)\,N(\phi)}{rAE} + \frac{M^2(\phi)}{2reAE} + \frac{\alpha V^2(\phi)}{2AG}\} \, r d\phi \qquad (15)$$

Using (11a), (11b), and (11c) gives,

$$\int_0^{\pi/2} M^2(\phi)\,d\phi = \frac{\pi}{2}(M_o - Pr)^2 + 2(M_o - Pr)r[P-Q] + \frac{\pi}{4} r^2P^2 - PQr^2 + \frac{\pi}{4}r^2Q^2 \quad (16a)$$

$$\int_0^{\pi/2} N^2(\phi)\,d\phi = \frac{\pi}{4}Q^2 - PQ + \frac{\pi}{4} P^2 \quad (16b)$$

$$\int_0^{\pi/2} M(\phi)N(\phi)\,d\phi = M_o(P-Q) - \left(\frac{4-\pi}{4}\right)P^2r + \frac{\pi}{4}Q^2r \quad (16c)$$

$$\int_0^{\pi/2} V^2(\phi)\,d\phi = \frac{\pi}{4} Q^2 + PQ + \frac{\pi}{4} P^2 \quad (16d)$$

Substituting (16a),(16b), (16c), and (16d) in (15) gives,

$$U_2' = \frac{r}{2AE}\Big\{\frac{\pi}{4}(P^2 + Q^2)\left(-1 + \frac{r}{e} + \frac{\alpha E}{G}\right) + P^2\left(\frac{\pi r}{2e} - \frac{2r}{e} + 2\right) - PQ\left(1 - \frac{r}{e} - \frac{\alpha E}{G}\right) - 2(P-Q)\frac{M_o}{r} + (2P - \pi P - 2Q)\frac{M_o}{e} + \frac{\pi}{2re}M_o^2\Big\} \quad (17)$$

Adding (14b) and (17), and rearranging terms one obtains the expression for the total strain energy for one quarter of the elongated ring, subjected to forces P,Q, and M_o as,

$$U = \frac{Q^2\ell}{2AE} + \frac{\ell}{2EI}\{M_o - r(P + Q)\}\{M_o - r(P + Q) - P\ell\}$$
$$+ \frac{P^2\ell^3}{6EI} + \frac{\alpha P^2\ell}{2AG} + \frac{\pi r(P^2 + Q^2)}{8AE}\left(-1 + \frac{\alpha E}{G} + \frac{r}{e}\right)$$
$$+ \frac{PQr}{2AE}\left(1 + \frac{\alpha E}{G} - \frac{r}{e}\right)$$
$$+ \frac{(M_o - Pr)}{AE}\left\{\left(\frac{r}{e} - 1\right)(P - Q) + \frac{\pi}{4e}(M_o - Pr)\right\} \qquad (18)$$

The above expression for strain energy is used for two specific cases, one with the load ring subjected to vertical load only and the other with the load ring subjected to horizontal load only. The two cases are discussed below.

Case 1: Load Ring Subjected to a Vertical Load V

From the condition of symmetry, the elongated load ring subjected to a vertical load V as shown in Figure 7a, can be split into four quadrants as shown in Figure 7b. The distribution of stress in any one quadrant is the same as in the other quadrants. Since the problem is statically indeterminate it is required to find the unknown moment M_o.

Considering one quadrant of the load ring, the forces acting are as shown in Figure 7c.

From statics,

$$M_c = \frac{V}{2}(\ell + r) - M_o \qquad (19)$$

Putting $P = -\frac{V}{2}$, $Q = 0$, and $M_o = -M_o$ in equation (18) gives the total strain energy for the present case as

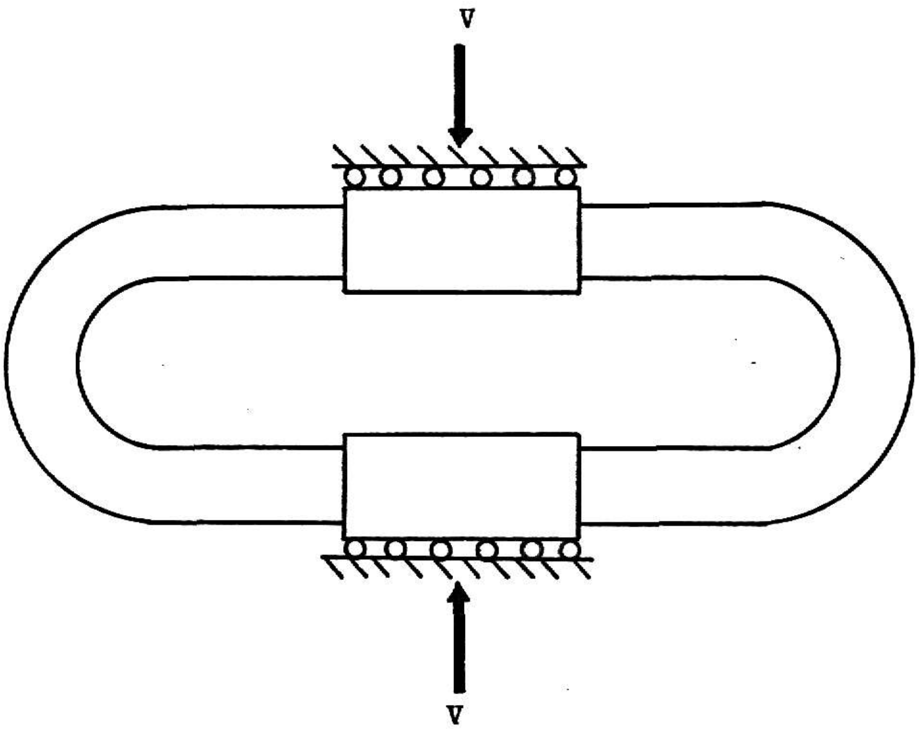


Fig. 7a

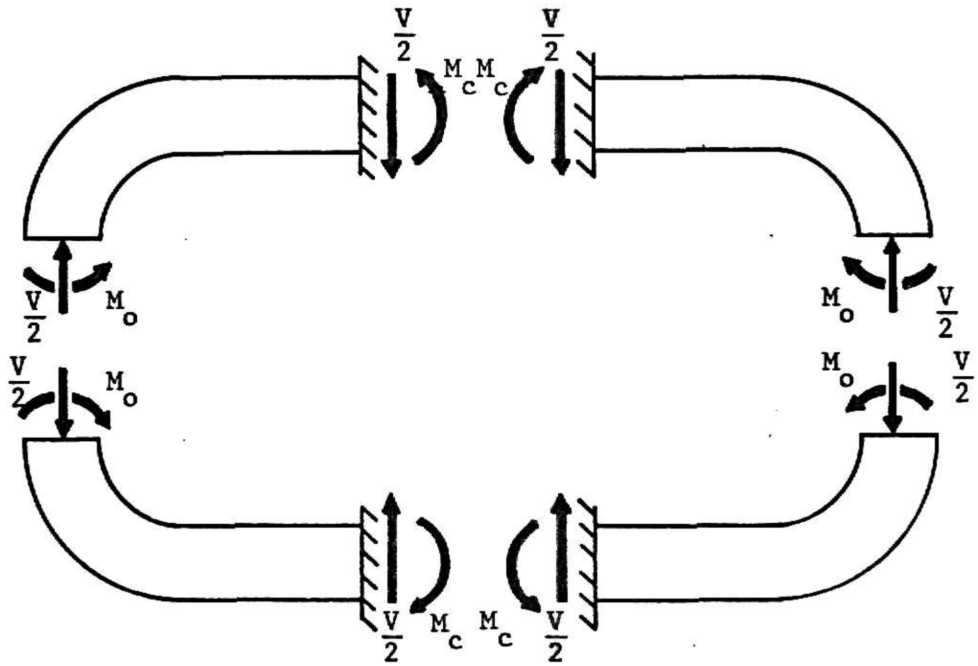


Fig. 7b

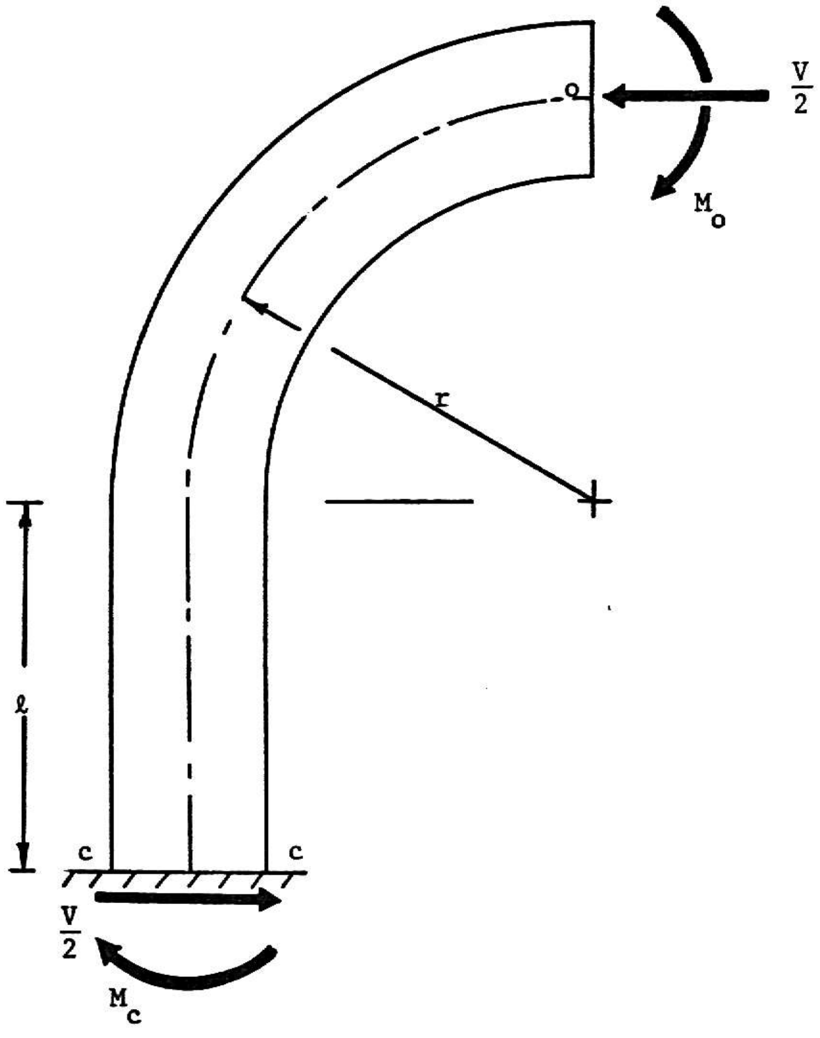


Fig. 7c

$$U = \frac{\ell}{2EI}\left(-M_o + \frac{Vr}{2}\right)\left(-M_o + \frac{Vr}{2} + \frac{V\ell}{2}\right) + \frac{V^2\ell^3}{4(6EI)} \tag{20}$$

$$+ \frac{\alpha V^2 \ell}{4(2AG)} + \frac{\pi r\left(\frac{V^2}{4}\right)}{8AE}\left(-1 + \frac{\alpha E}{G} + \frac{r}{e}\right)$$

$$+ \frac{\left(-M_o + \frac{Vr}{2}\right)}{AE}\left\{\left(\frac{r}{e} - 1\right)\left(-\frac{V}{2}\right) + \frac{\pi}{4e}\left(-M_o + \frac{Vr}{2}\right)\right\}$$

The statically indeterminate moment M_o is determined from the condition that the cross-section on which this moment acts does not rotate. That is, $\frac{dU}{dM_o} = 0$.

Differentiating (20) with respect to M_o and equating to zero gives

$$\frac{dU}{dM_o} = \frac{\ell}{2EI}\left\{(-1)\left(-M_o + \frac{Vr}{2} + \frac{V\ell}{2}\right) + \left(-M_o + \frac{Vr}{2}\right)(-1)\right\} \tag{21}$$

$$+ \frac{1}{AE}\left[(-1)\left\{\left(\frac{r}{e} - 1\right)\left(-\frac{V}{2}\right) + \frac{\pi}{4e}\left(-M_o + \frac{Vr}{2}\right)\right\}\right.$$

$$\left. + \left(-M_o + \frac{Vr}{2}\right)\left(-\frac{\pi}{4e}\right)\right] = 0$$

Solving for M_o gives

$$M_o = \frac{\frac{V}{2}\left(\frac{\ell r}{I} + \frac{\ell^2}{2I} - \frac{r}{Ae} + \frac{1}{A} + \frac{\pi r}{2Ae}\right)}{\left(\frac{\ell}{I} + \frac{\pi}{2Ae}\right)} \tag{22}$$

$$\text{Let} \quad \frac{\left(\frac{\ell r}{I} + \frac{\ell^2}{2I} - \frac{r}{Ae} + \frac{1}{A} + \frac{\pi r}{2Ae}\right)}{\left(\frac{\ell}{I} + \frac{\pi}{2Ae}\right)} = L \tag{23}$$

Substituting the value of M_o in (20) gives

$$U = \frac{\ell}{2EI}\left(-\frac{VL}{2} + \frac{Vr}{2}\right)\left(-\frac{VL}{2} + \frac{Vr}{2} + \frac{V\ell}{2}\right) + \frac{V^2\ell^3}{4(6EI)} \tag{24}$$

$$+ \frac{\alpha V^2\ell}{4(2AG)} + \frac{\pi r\left(\frac{V^2}{4}\right)}{8AE}\left(-1 + \frac{\alpha E}{G} + \frac{r}{e}\right)$$

$$+ \frac{\left(-\frac{VL}{2} + \frac{Vr}{2}\right)}{AE}\left\{\left(\frac{r}{e} - 1\right)\left(-\frac{V}{2}\right) + \frac{\pi}{4e}\left(-\frac{VL}{2} + \frac{Vr}{2}\right)\right\}$$

According to Castigliano's theorem of deflection, the deflection δ_V, in the direction of the force $\frac{V}{2}$, of the point of application of the force at point o, is equal to the partial derivative, with respect to the force, of the total internal strain energy in the member.

Thus, differentiating (24) with respect to $\frac{V}{2}$ gives,

$$\delta_V = \frac{V\ell^3}{6EI} + \frac{\alpha V\ell}{2AG} + \frac{V\pi r}{8AE}\left(-1 + \frac{\alpha E}{G} + \frac{r}{e}\right) \tag{25a}$$

$$+ (r - L)\left\{\frac{V\ell^2}{2EI} + \frac{V}{AE}\left(1 - \frac{r}{e}\right)\right\} + (r - L)^2\left(\frac{V\ell}{2EI} + \frac{V\pi}{4AEe}\right)$$

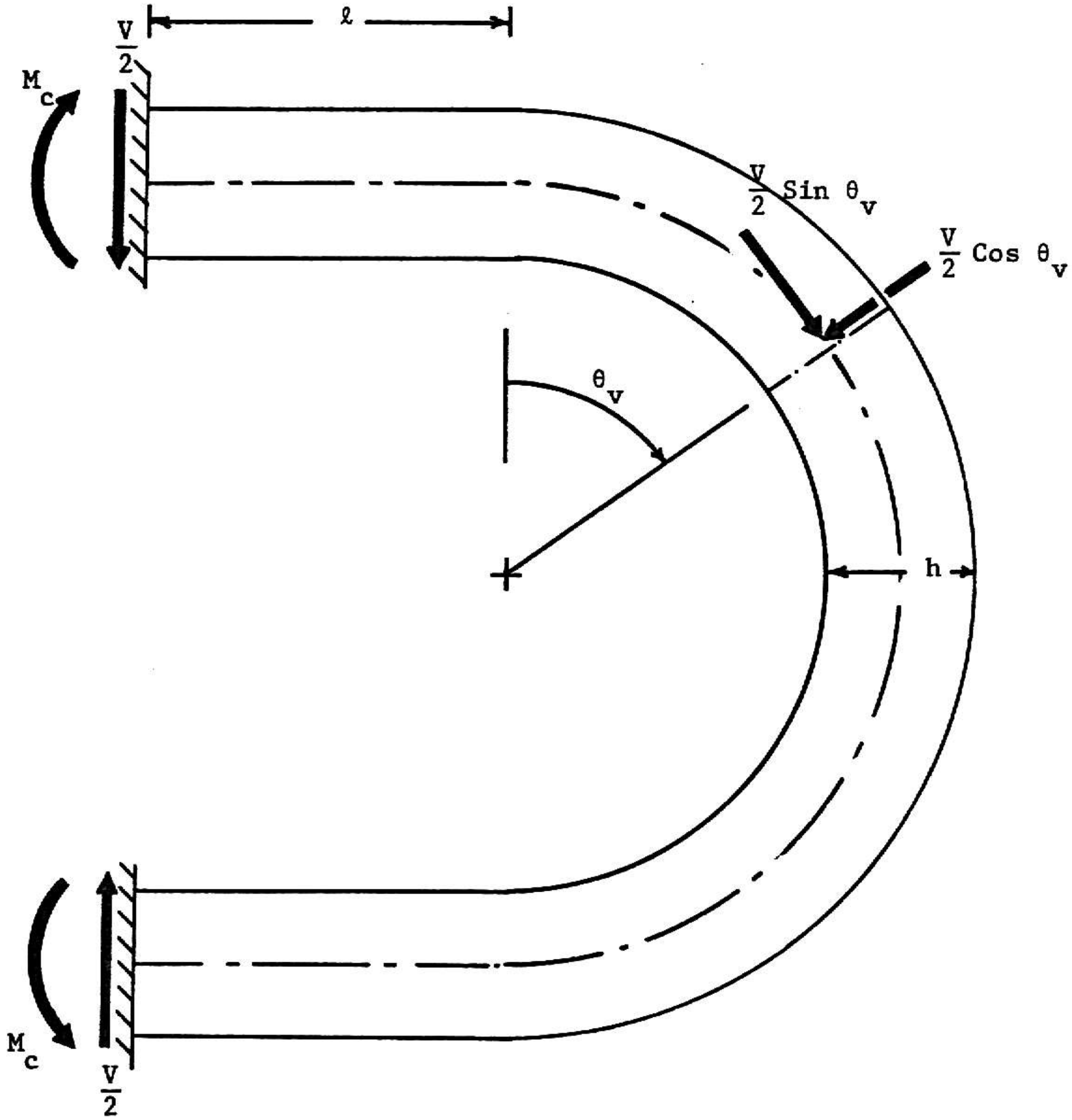
Equation (25a) gives deflection of only one side of the load ring under the load V in Figure 7a. Therefore, the total deflection for the whole ring subjected to a vertical load V is given by

$$\Delta_V = 2\delta_V = \frac{V}{AE}\left[\frac{A\ell^3}{3I} + \frac{\alpha\ell E}{G} + \frac{\pi r}{4}\left(-1 + \frac{\alpha E}{G} + \frac{r}{e}\right)\right. \tag{25b}$$

$$\left. + (r - L)\left\{\frac{A\ell^2}{I} + 2\left(1 - \frac{r}{e}\right)\right\} + (r - L)^2\left(\frac{A\ell}{I} + \frac{\pi}{2e}\right)\right]$$

Determination of stress on the Inside and Outside Fiber of the Load Ring

Consider the half ring with the forces as shown in Figure 8. Since



ℓ
$\frac{V}{2}$
M_c
$\frac{V}{2}$ Sin θ_v
$\frac{V}{2}$ Cos θ_v
θ_v
h
M_c
$\frac{V}{2}$

Fig. 8

strain gauges will be placed only in the curved portion of the ring for the measurement of vertical force, an expression for the stress will be found only for this section of the ring.

From equilibrium, the bending moment M, normal force N, and shear force V at any cross-section in the curved portion of the ring are given as follows,

$$M(\theta_V) = M_c - \frac{V}{2}(\ell + r\,\mathrm{Sin}\theta_V) \qquad 0 \leq \theta_V \leq \pi \qquad (26a)$$

$$N(\theta_V) = -\frac{V}{2}\mathrm{Sin}\,\theta_V \qquad 0 \leq \theta_V \leq \pi \qquad (26b)$$

$$V(\theta_V) = -\frac{V}{2}\mathrm{Cos}\,\theta_V \qquad 0 \leq \theta_V \leq \pi \qquad (26c)$$

Using (19), (26a) becomes

$$M(\theta_V) = \frac{V}{2}\,r(1 - \mathrm{Sin}\theta_V) - M_o \qquad (26d)$$

The shear stress does not enter into the calculation of the stress since its value is zero at the outer and inner fibers.

Applying (62) in Appendix A, and using (26b), (26c), and (26d), the stress at the outer fiber is

$$\sigma_{Vo} = -\frac{V}{2A}\left[\mathrm{Sin}\,\theta_V + \frac{\{r(1 - \mathrm{Sin}^{\theta}V) - L\}\,(\frac{h}{2} + e)}{e(r + \frac{h}{2})}\right] \qquad (27a)$$

and at the inner fiber is

$$\sigma_{Vi} = -\frac{V}{2A}\left[\mathrm{Sin}\,\theta_V - \frac{\{r(1 - \mathrm{Sin}^{\theta}V) - L\}\,(\frac{h}{2} - e)}{e(r - \frac{h}{2})}\right] \qquad (27b)$$

Case 2: Load Ring Subjected to a Horizontal Load H

The elongated load ring subjected to horizontal load H as shown in Figure 9a, can be split into two halfs as shown in Figure 9b. The problem is a

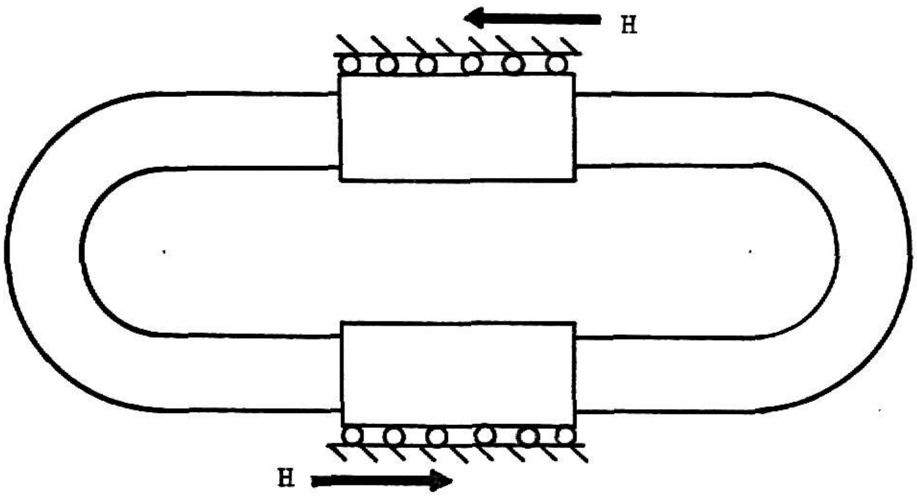


Fig. 9a

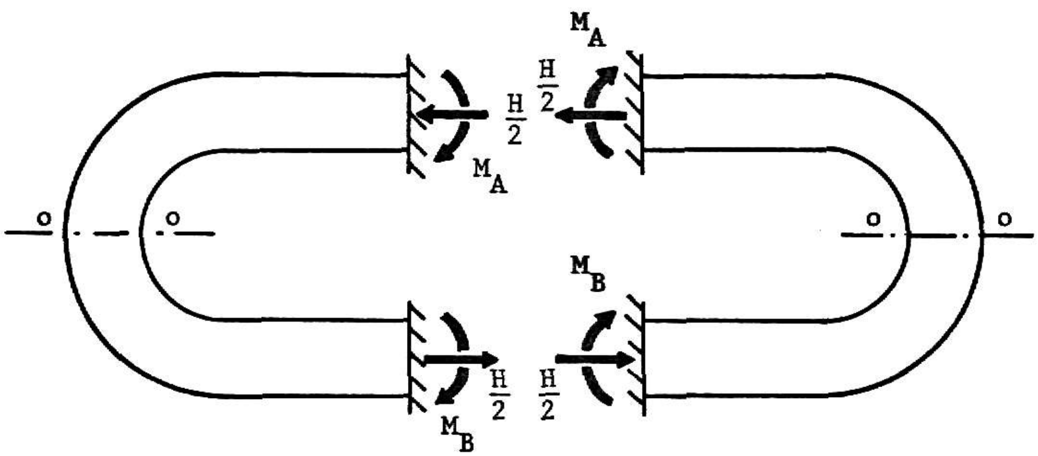


Fig. 9b

statically indeterminate one, and in order to apply (18), it is required to find the moment M_o at section oo.

Determination of M_o

Considering one half of the load ring, the forces acting are as shown in Figure 9c. From statics, taking moments about point A gives

$$M_A + M_B = Hr \tag{28}$$

Again from equilibrium, the bending moment M, normal force N, and shear force V at any cross-section in the three portions of the ring, namely the two straight beams and the curved beam are given as follows,

$$M(s) = M_A \qquad 0 \le s \le \ell \tag{29a}$$

$$N(s) = \frac{H}{2} \qquad 0 \le s \le \ell \tag{29b}$$

$$V(s) = 0 \qquad 0 \le s \le \ell \tag{29c}$$

$$M(\theta_{HR}) = M_A - \frac{Hr}{2}(1 - \text{Cos}\,\theta_{HR}) \qquad 0 \le \theta_{HR} \le \pi \tag{30a}$$

$$N(\theta_{HR}) = \frac{H}{2}\,\text{Cos}\,\theta_{HR} \qquad 0 \le \theta_{HR} \le \pi \tag{30b}$$

$$V(\theta_{HR}) = -\frac{H}{2}\,\text{Sin}\,\theta_{HR} \qquad 0 \le \theta_{HR} \le \pi \tag{30c}$$

$$M(s) = M_A - Hr \qquad \ell + \pi r \le s \le 2\ell + \pi r \tag{31a}$$

$$N(s) = \frac{H}{2} \qquad \ell + \pi r \le s \le 2\ell + \pi r \tag{31b}$$

$$V(s) = 0 \qquad \ell + \pi r \le s \le 2\ell + \pi r \tag{31c}$$

The statically indeterminate moment M_o is determined from the condition set forth by the constraints which allows for no change in angle between the planes at A and B.

That is,

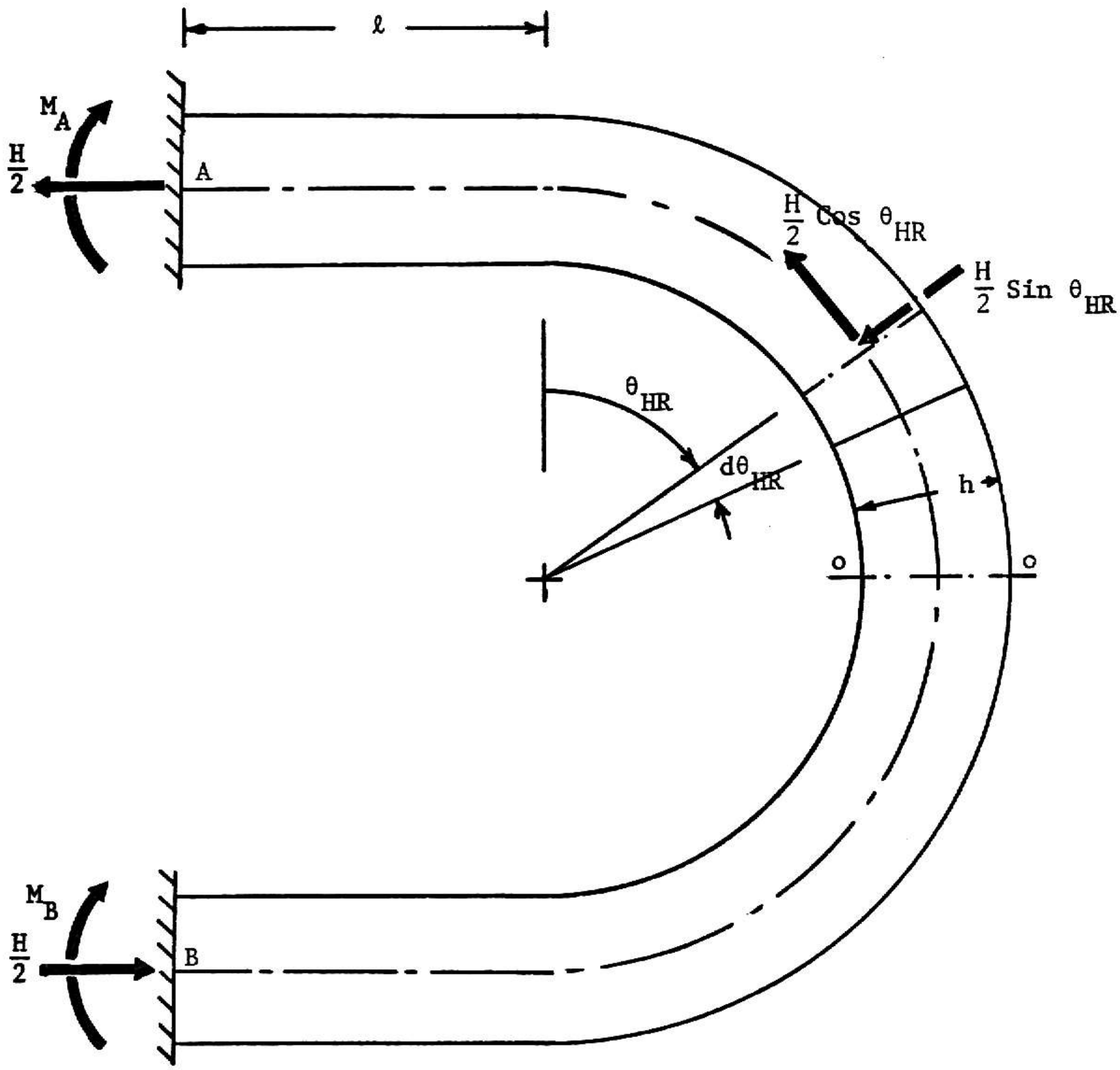

ℓ
M_A
$\frac{H}{2}$
A
$\frac{H}{2}$ Cos θ_{HR}
$\frac{H}{2}$ Sin θ_{HR}
θ_{HR}
$d\theta_{HR}$
h
o
o
M_B
$\frac{H}{2}$
B

Fig. 9c

$$\int_0^{\ell} \frac{M}{EI}\, ds + \int_0^{\pi} \omega \, d\theta_{HR} + \int_{\ell+\pi r}^{2\ell+\pi r} \frac{M}{EI}\, ds = 0 \tag{32}$$

where ω is the change in angle per unit of angle, and

where for an initially straight beam, the rate of change of slope of the elastic curve of deformation is given by,

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{33a}$$

Using (65) in Appendix A, one obtains the equation for ω as

$$\omega = \frac{M}{AeE} - \frac{N}{AE} \tag{33b}$$

Substituting (33a) and (33b) in (32), one obtains

$$\int_0^{\ell} \frac{M(s)}{EI}\, ds + \int_0^{\pi} \left\{\frac{M(\theta_{HR})}{AeE} - \frac{N(\theta_{HR})}{AE}\right\} d\theta_{HR} + \int_{\ell+\pi r}^{2\ell+\pi r} \frac{M(s)}{EI}\, ds = 0 \tag{34a}$$

Using (29a) - (31c) in (34a) gives,

$$\int_0^{\ell} \left(\frac{M_A}{EI}\right) ds + \int_0^{\pi} \left\{\frac{M_A - \frac{Hr}{2}(1 - \cos\theta_{HR})}{AeE} - \frac{\frac{H}{2}\cos\theta_{HR}}{AE}\right\} d\theta_{HR} \tag{34b}$$

$$+ \int_{\ell+\pi r}^{2\ell+\pi r} \left(\frac{M_A - Hr}{EI}\right) ds = 0$$

Solving the above equation, one obtains,

$$M_A = \frac{H}{2}\, r \tag{35a}$$

Substituting (35a) in (28) gives

$$M_B = \frac{H}{2}\, r \tag{35b}$$

Using (30a), moment M_o at section oo is given by

$$M_o = 0 \tag{36}$$

With $M_o = 0$, the load ring can now be split as shown in Figure 10a. Figure 10b shows one quarter of the ring which corresponds to the right top half of Figure 10a. Putting $P = 0$, $Q = -\frac{H}{2}$, and $M_o = 0$ in equation (18) gives the total strain energy for the present case as

$$U = \frac{(\frac{H^2}{4})\ell}{2AE} + \frac{(\frac{H^2}{4})\ r^2\ell}{2EI} + \frac{\pi r(\frac{H^2}{4})}{8AE}\{-1 + \frac{\alpha E}{G} + \frac{r}{e}\} \tag{37}$$

The deflection in the direction of the force $\frac{H}{2}$, of the point of application of the force at point o is given by,

$$\delta_H = \frac{dU}{d\frac{H}{2}} = \frac{(\frac{H}{2})\ell}{AE} + \frac{(\frac{H}{2})\ r^2\ell}{EI} + \frac{\pi r(\frac{H}{2})}{4AE}\{-1 + \frac{\alpha E}{G} + \frac{r}{e}\} \tag{38a}$$

The total horizontal deflection Δ_H for the whole ring is given by,

$$\Delta_H = 2\delta_H = \frac{H}{AE}\{\ell(1 + \frac{Ar^2}{I}) + \frac{\pi r}{4}(-1 + \frac{\alpha E}{G} + \frac{r}{e})\} \tag{38b}$$

Determination of Stress on the Outside Fiber of the Load Ring

The strain gauges used to measure horizontal force will be placed only on the outside fiber in the curved portion of the ring. Therefore, an expression for stress will be found only for the outside fiber of the curved portion of the ring. The shear stress is zero at the inner and outer fibers of the load ring.

Applying (62) in Appendix A, and using (30a), (30b), and (30c), the stress at the outer fiber is given as

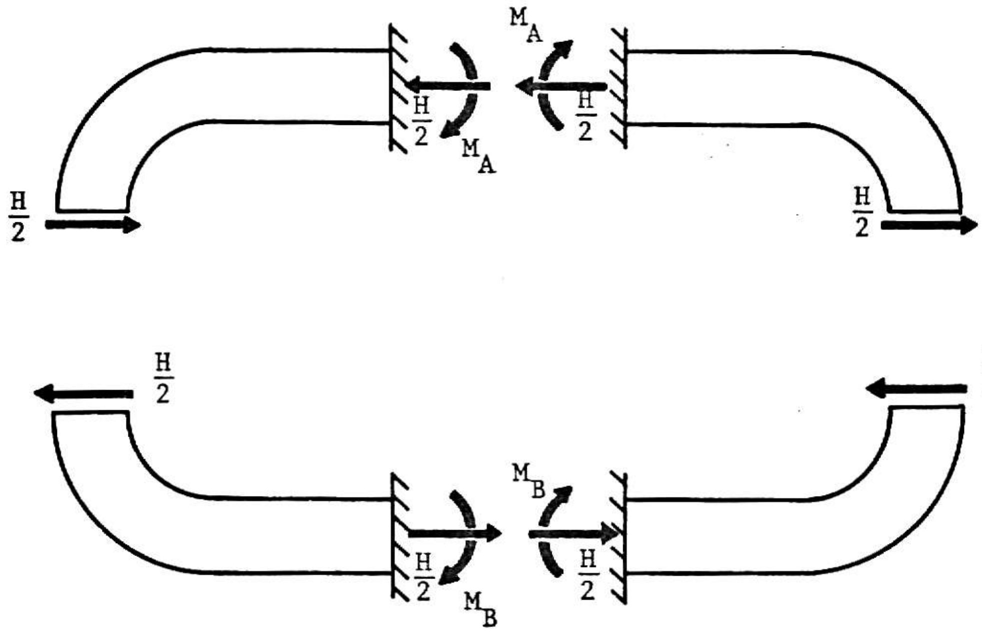

M_A
$\frac{H}{2}$
$\frac{H}{2}$
M_A
$\frac{H}{2}$
$\frac{H}{2}$
$\frac{H}{2}$
$\frac{H}{2}$
M_B
$\frac{H}{2}$
$\frac{H}{2}$
M_B

Fig. 10a

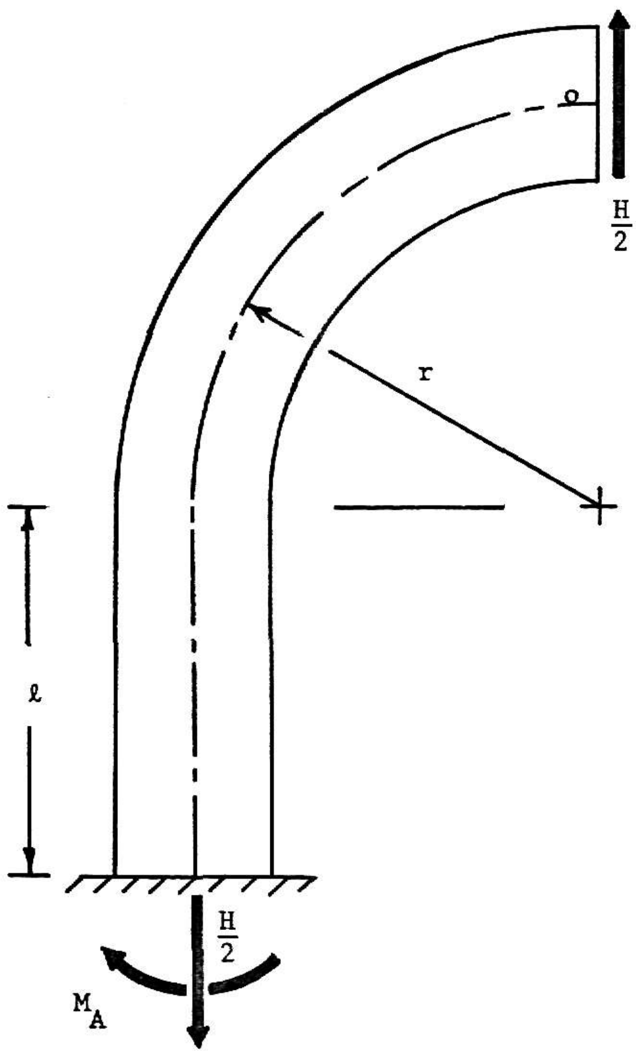


Fig. 10b

$$\sigma_H = \frac{H \cos\theta_{HR}}{2A} \{1 - \frac{r(\frac{h}{2} + e)}{e(r + \frac{h}{2})}\} \quad (39)$$

FORMULATION OF DESIGN EQUATIONS

In the analysis of an elongated split-ring dynamometer, expressions for deflection and stress, for the vertical and horizontal loading of the load ring, in terms of its dimensions are obtained. In this section design equations for stiffness and sensitivity ratios will be derived. Before embarking on the derivation of these equations, a particular type of cross-section of the ring, namely rectangular, will be analyzed. The rectangular cross-section of the ring is shown in Figure 11.

Using (60a) in Appendix A gives

$$mA = b\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{y\,dy}{r-y} \tag{40a}$$

Solving (40a) and using (61a) in Appendix A gives,

$$e = r - \frac{h}{\log_e(\frac{r_2}{r_1})} \tag{40b}$$

Stiffness Ratio

In this analysis, stiffness is defined as the load required to produce a unit deflection at the point of application of the load. Thus, from (25b), one obtains an expression for the vertical stiffness as,

$$\frac{V}{\Delta_V} = \frac{AE}{[\frac{A\ell^3}{3I} + \frac{\alpha\ell E}{G} + \frac{\pi r}{4}(-1 + \frac{\alpha E}{G} + \frac{r}{e}) + (r - L)\{\frac{A\ell^2}{I} + 2(1 - \frac{r}{e})\} + (r - L)^2(\frac{A\ell}{I} + \frac{\pi}{2e})]} \tag{41a}$$

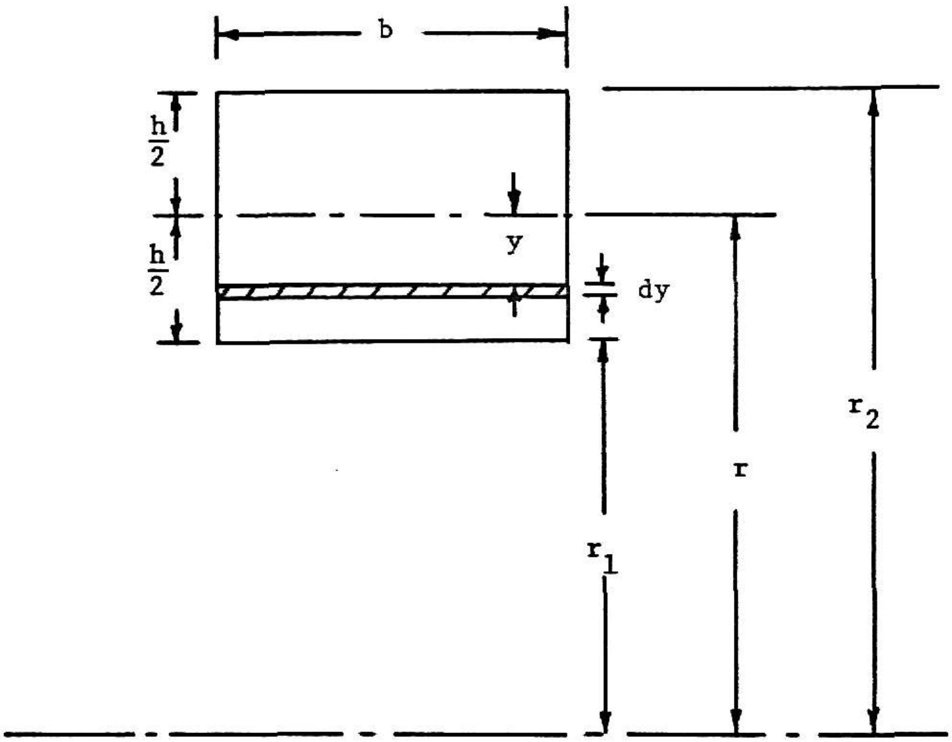


Fig. 11

and using (38b), the horizontal stiffness is given as,

$$\frac{H}{\Delta_H} = \frac{AE}{\{\ell(1 + \frac{Ar^2}{I}) + \frac{\pi r}{4}(-1 + \frac{\alpha E}{G} + \frac{r}{e})\}} \tag{41b}$$

Let the two stiffnesses be related by the stiffness ratio ψ, as given by

$$(\frac{H}{\Delta_H}) = \psi(\frac{V}{\Delta_V}) \tag{42}$$

Normally for a short beam or no beam, ψ will have a value less than 1, since the ring tends to be stiffer in the vertical direction than in the horizontal direction. Putting (41a) and (41b) in (42) one obtains,

$$\frac{AE}{\{\ell(1 + \frac{Ar^2}{I}) + \frac{\pi r}{4}(-1 + \frac{\alpha E}{G} + \frac{r}{e})\}} = \tag{43}$$

$$\psi \frac{AE}{[\frac{A\ell^3}{3I} + \frac{\alpha\ell E}{G} + \frac{\pi r}{4}(-1 + \frac{\alpha E}{G} + \frac{r}{e}) + (r - L)\{\frac{A\ell^2}{I} + 2(1 - \frac{r}{e})\} + (r - L)^2(\frac{A\ell}{I} + \frac{\pi}{2e})]}$$

Now,

$$A = bh \quad \text{and } I = \frac{bh^3}{12} \tag{44}$$

Substituting these into equation (43) yields

$$\begin{aligned}
&144\,\ell^5 e^3 + 30\,\pi\ell^4 h^2 e^2 + (\frac{\alpha E}{G} - \psi - 1)\,144\,\ell^3 h^2 e^3 + \pi^2 \ell^3 h^4 e \\
&+ 144\,\ell^3 h^2 e^2 r - 1728\,\psi\,\ell^3 e^3 r^2 + (\frac{\alpha E}{G} - \psi - 0.5)\,12\,\pi\ell^2 h^4 e^2 \\
&+ (1 - \frac{\alpha E}{G})(\psi - 1)\,36\,\pi\ell^2 h^2 e^3 r + 6\,\pi\ell^2 h^4 e r + (1 - 5\psi)\,36\,\pi\ell^2 h^2 e^2 r^2 \\
&- 12\,\ell h^4 e^3 + (\frac{\alpha E}{G} - \psi)\,0.25\,\pi^2 \ell h^6 e + \{\frac{\pi^2}{4}(\psi - 1)(1 - \frac{\alpha E}{G}) + 2\}\,12\ell h^4 e^2 r \\
&+ \{\frac{\pi^2}{4}(1 - 2\psi) - 1\}\,12\,\ell h^4 e r^2 + \{\frac{\pi^3}{16}(\psi - 1)(1 - \frac{\alpha E}{G}) + \pi\}\,r h^6 e \\
&+ \{\frac{\pi^3}{16}(1 - \psi) - \frac{\pi}{2}\}\,r^2 h^6 - 0.5\,\pi h^6 e^2 = 0
\end{aligned} \tag{45}$$

The width of the ring b, cancels out and therefore has no effect on the final result. To solve (45), by plotting a graph, dimensionless quantities will be introduced.

Let $\beta = \frac{h}{r}$ and $\eta = \frac{\ell}{r}$ (46a)

then,

$$h = \beta r$$
$$\ell = \eta r \qquad (46b)$$
$$e = \lambda r$$

where,

$$\lambda = 1 - \frac{\beta}{\ell n \left(\frac{2 + \beta}{2 - \beta}\right)}$$

Substituting (46b) in (45) gives,

$$\begin{aligned}
&144\ \lambda^3\eta^5 + 30\ \pi\lambda^2\beta^2\eta^4 + \left(\frac{\alpha E}{G} - \psi - 1\right) 144\ \lambda^3\beta^2\eta^3 \\
&+ \pi^2\ \lambda\beta^4\eta^3 + 144\ \lambda^2\beta^2\eta^3 - 1728\ \psi\lambda^3\eta^3 \\
&+ \left(\frac{\alpha E}{G} - \psi - 0.5\right) 12\ \pi\lambda^2\beta^4\eta^2 + \left(1 - \frac{\alpha E}{G}\right)(\psi-1)\ 36\ \pi\lambda^3\beta^2\eta^2 \\
&+ 6\ \pi\lambda\beta^4\eta^2 + (1-5\psi)\ 36\ \pi\lambda^2\beta^2\eta^2 - 12\ \lambda^2\beta^4\eta \qquad (47) \\
&+ \left(\frac{\alpha E}{G} - \psi\right) 0.25\ \pi^2\lambda\beta^6\eta + \left\{\frac{\pi^2}{4}(\psi-1)\left(1- \frac{\alpha E}{G}\right) + 2\right\}\ 12\ \lambda^2\beta^4\eta \\
&+ \left\{\frac{\pi^2}{4}(1 - 2\psi) - 1\right\} 12\ \lambda\beta^4\eta + \left\{\frac{\pi^3}{16}(\psi-1)\left(1 - \frac{\alpha E}{G}\right) + \pi\right\} \lambda\beta^6 \\
&+ \left\{\frac{\pi^3}{16}(1-\psi) - \frac{\pi}{2}\right\}\beta^6 - 0.5\ \pi\lambda^2\beta^6 = 0
\end{aligned}$$

Equation (47) helps to design a load ring for a given stiffness ratio. For numerical calculations it will be assumed that the load ring is made of steel and has a rectangular cross-section. For this case

$E = 30 \times 10^6$ lb/in^2, $G = 11.5 \times 10^6$ lb/in^2 and $\alpha = 1.5$.

Sensitivity Ratio

In the practical application of the dynamometer, both the vertical and horizontal components of cutting force will be acting on the ring simultaneously. Therefore, in order to obtain the sensitivity ratio equation, the proper location of the strain gauges should be investigated. The location of the strain gauges should be such that the stress (or strain) measured is due to one component of load only.

For the location of the strain gauges shown in Figures 12a, 12b, it is required to find the angular position of the gauges, that is, θ_V and θ_{HR} for which the stresses σ_{Vo} and σ_{HR} respectively will be zero. The angles θ_V and θ_{HR} will be called the best place for separation.

Determination of Best Place for Separation

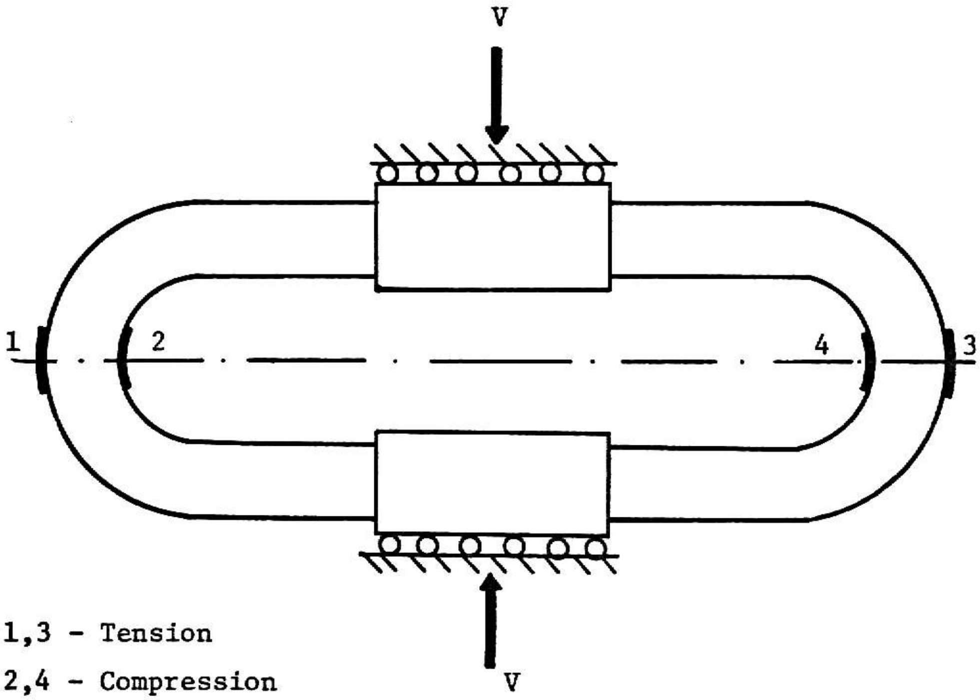
Equating σ_{Vo} given by (27a) to zero, one obtains,

$$- \frac{V}{2A} \left[\sin\theta_V + \frac{\{ r(1 - \sin\theta_V) - L \} (\frac{h}{2} + e)}{e(r + \frac{h}{2})}\right] = 0 \qquad (48a)$$

Solving (48a) for θ_V, and denoting θ_V by θ_{Vo},

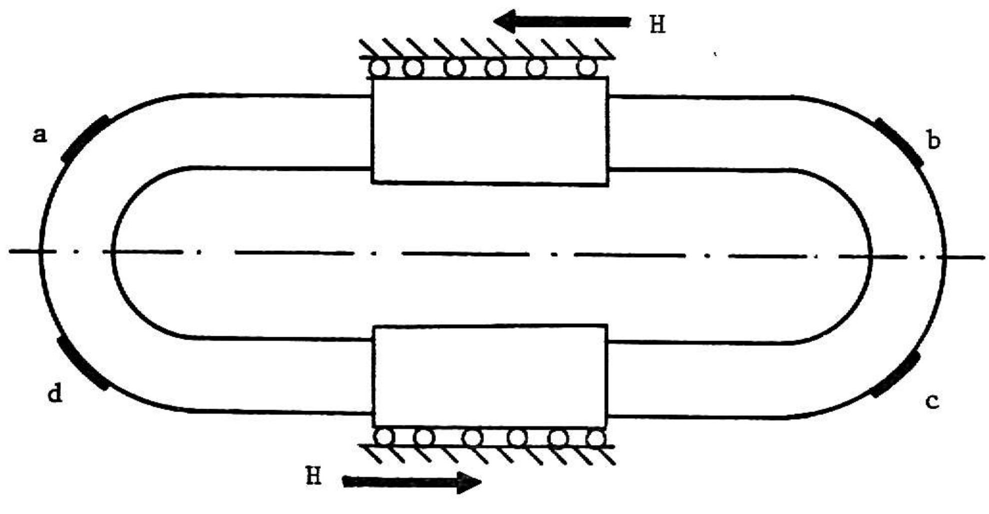
$$\theta_{Vo} = \sin^{-1} \frac{(L - r)(\frac{h}{2} + e)}{[e(r + \frac{h}{2}) - r(\frac{h}{2} + e)]} \qquad (48b)$$

Using dimensionless quantities, as before, in (46a) and (46b), gives



1,3 - Tension
2,4 - Compression

Fig. 12a



a,c - Tension
b,d - Compression

Fig. 12b