ON THE SOLUTION OF A LINEAR DIFFERENTIAL EQUATION WHOSE COEFFICIENTS HAVE A REGULAR SINGULAR POINT

by

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INTRODUCTION

The differential equation

$$x^n\frac{d^nz}{dx^n} + x^{n-1}P_1(x)\frac{d^{n-1}z}{dx^{n-1}} + \cdots + xP_{n-1}(x)\frac{dz}{dx} + P_n(x)z = 0$$

was considered by Frobenius and Fuchs as early as 1866. Since that time various other writers have added to the works of these men. Solutions for the general case and for $n = 2$ appear in many text-books. In this paper the author employs the method used by Frobenius in his solution of the general case (1, 2, 3, 4).

The differential equation

$$x^3\frac{d^3z}{dx^3} + x^2P_1(x)\frac{d^2z}{dx^2} + xP_2(x)\frac{dz}{dx} + P_3(x)z = 0$$

in which $P_1(x)$, $P_2(x)$, and $P_3(x)$ are holomorphic at $x = 0$, is said to have a regular point at $x = 0$. This is the most general differential equation whose solutions contain a finite number of terms with negative exponents. Solutions of this kind are called regular solutions. The singular point in question is called the regular singular point.

Any set of three linearly independent solutions of the differential equation is called a fundamental set of solutions because it has the property that any solution of the equation can be expressed linearly in terms of that set.

The problem to be solved here is that of finding fundamental sets of solutions of the differential equation under all possible hypotheses with regard to the coefficients $P_1(x)$, $P_2(x)$, and $P_3(x)$.

METHOD OF SOLUTION

The differential equation under consideration is

$$x^3\frac{d^3z}{dx^3} + x^2P_1(x)\frac{d^2z}{dx^2} + xP_2(x)\frac{dz}{dx} + P_3(x)z = 0, \tag{1}$$

in which the $P_j(x)$ are holomorphic; i. e., they may be expanded in the power series

$$P_j(x) = \sum_{k=0}^{\infty} a_{j,k}x^k, \qquad (j = 1, 2, 3).$$

Let the question be raised whether or not there exists a solution in the form of the power series

$$z = x^r \sum_{n=0}^{\infty} b_n x^n$$

$$= \sum_{n=0}^{\infty} b_n x^{r+n}. \tag{2}$$

The substitution of the values for the P_j and z in (1), transforms (1) into

$$\sum_{n=0}^{\infty}\Big[b_n x^3(n + r)(n + r - 1)(n + r - 2)x^{n+r-3}$$

$$+ b_n x^2 \sum_{k=0}^{\infty} a_{1,k}x^k(n + r)(n + r - 1)x^{n+r-2}$$

$$+ b_n x \sum_{k=0}^{\infty} a_{2,k}x^k(n + r)x^{n+r-1}$$

$$+ b_n \sum_{k=0}^{\infty} a_{3,k}x^k x^{n+r}\Big] = 0.$$

This may be rewritten in the form

$$(3) \quad \sum_{n=0}^{\infty}\Big[(n+r)(n+r-1)(n+r-2)b_n x^{n+r}$$

$$+ (n+r)(n+r-1)b_n x^{n+r}\sum_{k=0}^{\infty} a_{1,k}x^k$$

$$+ (n+r)b_n x^{n+r}\sum_{k=0}^{\infty} a_{2,k}x^k$$

$$+ b_n x^{n+r}\sum_{k=0}^{\infty} a_{3,k}x^k\Big] = 0.$$

Upon equating to zero the coefficients of the various powers of x in (3), there results for the determination of the coefficients b_k the set of equations

$$(4)\begin{cases} b_0 f(r) = 0, \\ b_1 f(r+1) + b_0 f_1(r) = 0, \\ b_2 f(r+2) + b_1 f_1(r+1) + b_0 f_2(r) = 0, \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ b_n f(r+n) + b_{n-1} f_1(r+n-1) + b_{n-2} f_2(r+n-2) \\ \quad \cdots + b_1 f_{n-1}(r+1) + b_0 f_n(r) = 0, \end{cases}$$

where

$$f(r) = r(r-1)(r-2) + r(r-1)a_{1,0} + ra_{2,0} + a_{3,0},$$

$$f_1(r) = r(r-1)a_{1,1} + ra_{2,1} + a_{3,1},$$

$$f_2(r) = r(r-1)a_{1,2} + ra_{2,2} + a_{3,2},$$

$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$f_{n-1}(r) = r(r-1)a_{1,n-1} + ra_{2,n-1} + a_{3,n-1},$$

$$f_n(r) = r(r-1)a_{1,n} + ra_{2,n} + a_{3,n}.$$

The equation

$$f(r) = 0 \tag{5}$$

is called the indicial equation. For each value of r which satisfies this equation the values of the b_k in terms of b_0 are uniquely determined by (4). The nature of the roots determines the character of the solution. There are six cases to be considered.

CASE I

The roots of (5) are all distinct and no two of them differ by an integer. Each of the expressions

$$z_1 = x^{r_1}\sum_{k=0}^{\infty} b_{1,k}x^k = x^{r_1}\phi_1(x),$$

$$z_2 = x^{r_2}\sum_{k=0}^{\infty} b_{2,k}x^k = x^{r_2}\phi_2(x),$$

$$z_3 = x^{r_3}\sum_{k=0}^{\infty} b_{3,k}x^k = x^{r_3}\phi_3(x),$$

is a solution of (1). It will be shown that these solutions constitute a fundamental set.

It is to be shown that zero is the only value for the C_j such that

$$C_1z_1 + C_2z_2 + C_3z_3 = 0. \tag{6}$$

If the number x be represented in the complex plane, then

(7) $$x^{r_1} = e^{r_1(\log \rho + i\theta)}.$$

If the amplitude is increased by $2n\pi$, (7) becomes

$$x^{r_1} = e^{r_1(\log \rho + i\theta)} \cdot e^{2\pi i r_1}, \qquad n = 1;$$

and

$$x^{r_1} = e^{r_1(\log \rho + i\theta)} \cdot e^{4\pi i r_1}, \qquad n = 2.$$

Hence each of the equations

$$C_1 S_1 z_1 + C_2 S_2 z_2 + C_3 S_3 z_3 = 0,$$

and

$$C_1 S_1^2 z_1 + C_2 S_2^2 z_2 + C_3 S_3^2 z_3 = 0,$$

where

$$S_j = e^{2\pi i r_j},$$

is true if (6) is true.

The determinant of the coefficients is

$$\begin{vmatrix} 1 & 1 & 1 \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{vmatrix} = (S_1 - S_2)(S_2 - S_3)(S_3 - S_1) = 0.$$

Therefore, in order for the equation

$$C_1 z_1 + C_2 z_2 + C_3 z_3 = 0,$$

to be true, the following relation must exist:

$$C_1 = C_2 = C_3 = 0.$$

CASE II

(5) has a double root and one which is not different by an integer.

Let b_0 be an arbitrary function of r, where r is in a region C of the complex plane from which are excluded by small circles the points congruent on the left to r_1, r_1, r_2.

Consider the equation

$$D(z) = b_0(r)f(r)x^r, \tag{8}$$

where

$$D(z) \equiv x^3\frac{d^3z}{dx^3} + x^2P_1(x)\frac{d^2z}{dx^2} + xP_2(x)\frac{dz}{dx} + P_3(x)z.$$

This equation is not the same as (1), but when $r = r_1$, r_1, r_2, it reduces to (1).

Since r_1 is a double root of (5), the derivative of the right member, and consequently the derivative of the left member vanish for $r = r_1$; i. e.,

$$\frac{\partial}{\partial r}\Big[D(z)\Big]_{r=r_1} = 0,$$

whence

$$x^3\frac{\partial z'''}{\partial r} + x^2P_1(x)\frac{\partial z''}{\partial r} + xP_2(x)\frac{\partial z'}{\partial r} + P_3(x)\frac{\partial z}{\partial r} = 0, \qquad r = r_1.$$

Since $\frac{\partial z'}{\partial r} = \left(\frac{\partial z}{\partial r}\right)'$,

$$x^3\left(\frac{\partial z}{\partial r}\right)''' + x^2P_1(x)\left(\frac{\partial z}{\partial r}\right)'' + xP_2(x)\left(\frac{\partial z}{\partial r}\right)' + P_3(x)\left(\frac{\partial z}{\partial r}\right) = 0,$$

from which it is seen that $\frac{\partial z}{\partial r}$ is a solution of (1) if $r = r_1$.

Hence each of the expressions

$$z_1 = x^{r_1} \sum_{k=0}^{\infty} b_k(r_1)x^k,$$

$$z_2 = x^{r_1} \sum_{k=0}^{\infty} b_k'(r_1)x^k + x^{r_1} \sum_{k=0}^{\infty} b_k(r_1)x^k \log x,$$

$$z_3 = x^{r_2} \sum_{k=0}^{\infty} b_k(r_2)x^k,$$

is a solution of (1). It will be shown that they constitute a fundamental set.

Again it is to be shown that zero is the only value for the C_j such that

$$C_1z_1 + C_2z_2 + C_3z_3 = 0.$$

Since the solutions of Case I are linearly independent, it will suffice to show that

$$C_1z_1 + C_2z_2 = 0$$

only if the $C_j = 0$. This will be done through the use of the theorem: "If the $u_j(x)$ are single valued, the relation

(9) $$u_1(x) + u_2(x)\log x + u_3(x)\log^2 x = 0$$

is true only if the $u_i(x) \equiv 0$".

Since $x = \rho e^{i\theta}$, (9) may be written

(10) $$u_1(x) + u_2(x)\left[\log \rho + i\theta\right] + u_3(x)\left[\log \rho + i\theta\right]^2 = 0.$$

If the amplitude is increased by $2n\pi$, (10) becomes

$$u_1(x) + u_2(x)\left[\log \rho + i\theta + 2in\pi\right]$$
$$+ u_3(x)\left[\log \rho + i\theta + 2in\pi\right]^2 = 0$$

which may be rewritten

(11) $$u_1(x) + u_2(x)\log \rho + u_2(x)i\theta + u_2(x)2in\pi$$
$$+ u_3(x)\log^2 \rho - u_3(x)4n^2\pi^2 - u_3(x)\theta^2$$
$$+ 4u_3(x)\log \rho\ (\pi n i) + 2u_3(x)(i\theta)\log \rho$$
$$- 4u_3(x)(\pi n)(\theta) = 0.$$

If (11) is to be true, the coefficients of the various powers of n must be identically zero. Hence

$$- 4u_3(x)\pi^2 \equiv 0,$$

and therefore

$$u_3(x) \equiv 0.$$

Likewise $u_2(x) \equiv 0$, and consequently $u_1(x) \equiv 0$.

$$C_1 z_1 + C_2 z_2 = 0,$$

may be written in the form

$$C_1 g_1(x) + C_2 g_1(x)\log x + C_2 g_1'(x) = 0.$$

The preceding discussion shows that if this relation is true

$$C_2 g_1(x) = 0;$$

but since $g_1(x) \not\equiv 0$,

$$C_2 = 0,$$

and therefore

$$C_1 = 0.$$

Hence in order for the equation

$$C_1 z_1 + C_2 z_2 + C_3 z_3 = 0$$

to be true, the following relation must exist:

$$C_1 = C_2 = C_3 = 0.$$

CASE III

(5) has a triple root.

By a slight extension of the discussion of Case II it is found that each of the expressions

$$z_1 = x^{r_1} \sum_{k=0}^{\infty} b_k(r_1)x^k,$$

$$z_2 = x^{r_1} \sum_{k=0}^{\infty} b_k'(r_1)x^k + x^{r_1} \sum_{k=0}^{\infty} b_k(r_1)x^k \log x,$$

$$z_3 = x^{r_1} \sum_{k=0}^{\infty} b_k''(r_1)x^k + 2x^{r_1} \sum_{k=0}^{\infty} b_k'(r_1)x^k \log x$$

$$+ x^{r_1} \sum_{k=0}^{\infty} b_k(r_1)x^k \log^2 x,$$

is a solution of (1). The linear independence of the solutions follows directly from the theorem mentioned in Case II. Hence, the solutions constitute a fundamental set.

CASE IV

(5) has the roots r_1, $r_2 = r_1 + q$, and r_3 which is not different by an integer.

Let

(12) $$b_0(r) = f(r+1)(r+2)\ .\ .\ .\ (r+n)g(r)$$

where $g(r) \neq 0$ for r_1, r_2, r_3, then all the b_k are finite, if $n \geqq q$.

Consider the equation

(13) $$D(z) = b_0(r)f(r)x^r,$$

where

$$D(z) = x^3\frac{d^3z}{dx^3} + x^2P_1(x)\frac{d^2z}{dx^2} + xP_2(x)\frac{dz}{dx} + P_3(x)z.$$

This equation is not the same as (1), but when $r = r_1$, r_2, r_3, it reduces to (1).

If $f(r)$ contains the factors $(r - r_1)$, $(r - [r_1 + q])$, and $(r - r_3)$; the q th factor of (12) contains the factors $(r - [r_1 + q])$, $(r - [r_1 + 2q])$, and $(r - [r_3 + q])$. Since $(r - [r_1 + q])$ is a double root of (13), the derivative of the right member, and consequently the derivative of the left member vanish for $r = r_1 + q$, i. e.,

$$\frac{\partial}{\partial r}\Big[D(z)\Big]_{r=r_2} = 0,$$

whence

$$x^3\frac{\partial z'''}{\partial r} + x^2P_1(x)\frac{\partial z''}{\partial r} + xP_2(x)\frac{\partial z'}{\partial r} + P_3(x)\frac{\partial z}{\partial r} = 0, \qquad r = r_2.$$

Since $\dfrac{\partial z'}{\partial r} = \left(\dfrac{\partial z}{\partial r}\right)'$,

$$x^3\left(\frac{\partial z}{\partial r}\right)''' + x^2P_1(x)\left(\frac{\partial z}{\partial r}\right)'' + xP_2(x)\left(\frac{\partial z}{\partial r}\right)' + P_3(x)\left(\frac{\partial z}{\partial r}\right) = 0,$$

from which it is seen that $\dfrac{\partial z}{\partial r}$ is a solution of (1) if $r = r_2$.

Hence each of the expressions

$$z_1 = x^{r_1}\sum_{k=0}^{\infty} b_k(r_1)x^k,$$

$$z_2 = x^{r_2}\sum_{k=0}^{\infty} b_k'(r_2)x^k + x^{r_2}\sum_{k=0}^{\infty} b_k(r_2)x^k\log x,$$

$$z_3 = x^{r_3}\sum_{k=0}^{\infty} b_k(r_3)x^k,$$

is a solution of (1). They can be shown to constitute a fundamental set by the method employed in Case II.

CASE V

(5) has the roots r_1, $r_2 \equiv r_1 + q$, and $r_3 \equiv r_1 + p$.

By a slight extension of the discussion of Case IV it is found that each of the expressions

$$z_1 = x^{r_1}\sum_{k=0}^{\infty} b_k(r_1)x^k,$$

$$z_2 = x^{r_2} \sum_{k=0}^{\infty} b_k'(r_2)x^k + x^{r_2} \sum_{k=0}^{\infty} b_k(r_2)x^k \log x,$$

$$z_3 = x^{r_3} \sum_{k=0}^{\infty} b_k''(r_3)x^k + 2x^{r_3} \sum_{k=0}^{\infty} b_k'(r_3)x^k \log x + x^{r_3} \sum_{k=0}^{\infty} b_k(r_3)x^k \log^2 x,$$

is a solution of (1). They can be shown to constitute a fundamental set by the method employed in Case II.

CASE VI

(5) has the roots r_1, $r_2 = r_1$, and $r_3 = r_1 + q$.

Again by a slight extension of the discussion of Case IV it is found that each of the expressions

$$z_1 = x^{r_1} \sum_{k=0}^{\infty} b_k(r_1)x^k,$$

$$z_2 = x^{r_1} \sum_{k=0}^{\infty} b_k'(r_1)x^k + x^{r_1} \sum_{k=0}^{\infty} b_k(r_1)x^k \log x,$$

$$z_3 = x^{r_3} \sum_{k=0}^{\infty} b_k''(r_3)x^k + 2x^{r_3} \sum_{k=0}^{\infty} b_k'(r_3)x^k \log x + x^{r_3} \sum_{k=0}^{\infty} b_k(r_3)x^k \log^2 x,$$

is a solution of (1). Here again the method employed in Case II will show that they constitute a fundamental set.

CONVERGENCE OF THE SOLUTIONS

The substitution

$$z = ye^{-1/3\int \frac{P_1(x)}{x}dx}$$

transforms (1) into an equation in which $\frac{d^2z}{dx^2}$ is missing. Thus, without loss of generality, $P_1(x)$ may be set equal to zero.

Now

$$P_2(x) = a_{2,0} + a_{2,1}x + a_{2,2}x^2 + \cdots$$

and

$$P_3(x) = a_{3,0} + a_{3,1}x + a_{3,2}x^2 + \cdots$$

whence

$$f(r) = r(r - 1)(r - 2) + a_{2,0}r + a_{3,0}$$

and

$$f_v(r) = a_{2,v}r + a_{3,v}, \qquad (v = 1, 2, \ldots).$$

Assume that the P's converge absolutely for $|x| = \rho$. Then

$$\sum_{v=0}^{\infty} \left|a_{2,v}\rho^v\right| < M_2,$$

whence

$$\left|a_{2,v}\rho^v\right| < M_2,$$

and therefore

$$\left|a_{2,v}\right| < \frac{M_2}{\rho^v}, \qquad (v = 1, 2, \cdots);$$

also

$$\left|a_{3,v}\right| < \frac{M_3}{\rho^v}, \qquad (v = 1, 2, \cdots).$$

Choose M the larger of M_2 and M_3, then

$$\left|a_{2,v}\right| < \frac{M}{\rho^v}, \qquad (v = 1, 2, \cdots);$$

and

$$\left|a_{3,v}\right| < \frac{M}{\rho^v}, \qquad (v = 1, 2, \cdots).$$

Choose $|r| < R$.

Then

$$\left|f_\lambda(r)\right| \leqq \left|a_{2,\lambda}\right|\left|r + a_{3,\lambda}\right|$$

$$< \frac{M}{\rho^\lambda}R + \frac{M}{\rho^\lambda}$$

$$f_\lambda(r) < \frac{M}{\rho^\lambda}(R + 1).$$

For $v > n$, $f(r + v)$ vanishes for no value of r in C. $f(r + v)$ is of the 3 rd degree in v and the coefficient of v^3 is 1, therefore

$$\lim_{v \to \infty} \frac{v^3}{f(r + v)} = 1.$$

For $v > n$

$$\frac{v^3}{f(r + v)} < \Gamma$$

where Γ is a finite positive quanity.

Therefore

$$\frac{1}{f(r + v)} < \frac{\Gamma}{v^3}.$$

It will be shown that there exists a quantity independent of r such that for all values of r in C

(14) $$|b_u| \leqq \frac{G}{\rho^u}, \qquad (u = 0, 1, 2, \ldots).$$

Let N be a positive integer not less than n. Then G can be chosen large enough so that (14) is true for u = 1, 2, . . . N. Assume that this can be done for u = 0, 1, 2, . . . v - 1, when v > N. Then

$$b_v = -\frac{f_1(r + v - 1)}{f(r + v)} b_{v-1} \ldots -\frac{f_{v-1}(r + 1)}{f(r + v)} b_1 - \frac{f_v(r)}{f(r + v)} b_0$$

and hence

$$|b_v| \leqq \frac{\Gamma}{v^3}\left\{\frac{M}{\rho}(R + 1)\frac{G}{\rho^{v-1}} + \ldots + \frac{M}{\rho^{v-1}}(R + 1)\frac{G}{\rho} + \frac{M}{\rho^v}(R + 1)G\right\}$$

$$\leqq \frac{\Gamma MG(R + 1)}{v^3\rho^v}.v.$$

Therefore

$$|b_v| \leqq \frac{\Gamma M(R + 1)}{v^2} \frac{G}{\rho^v}.$$

Choose now $N > \Gamma M(R + 1)$. Since $v > N$,

$$\left|\frac{\Gamma M(R + 1)}{v^2}\right| < 1.$$

Hence

$$|b_v| < \frac{G}{\rho^v}.$$

and therefore (14) is true.

Thus for $|x| < \rho$ and all r in C

$$|b_v x^v| \leq G\left(\frac{|x|}{\rho}\right)^v.$$

Since the series

$$\sum_{v=0}^{\infty} G\left(\frac{|x|}{\rho}\right)^v$$

converges uniformly for $|x| < \rho'$ when $\rho' < \rho$, the series

$$\sum_{v=0}^{\infty} b_v x^v$$

is absolutely and uniformly convergent for $|x| < \rho'$ and all r in C.

CONCLUSION

The differential equation

$$x^3\frac{d^3z}{dx^3} + x^2P_1(x)\frac{d^2z}{dx^2} + xP_2(x)\frac{dz}{dx} + P_3(x)z = 0$$

does have a solution in the form of the power series

$$z = x^r \sum_{n=0}^{\infty} b_n x^n.$$

In every case the fundamental set of solutions depends upon the nature of the roots of the indicial equation.

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