ESTIMATING ELASTICITIES OF INPUT SUBSTITUTION USING DATA ENVELOPMENT ANALYSIS

by

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Abstract

The use of elasticities of substitution between inputs has become the standard method for addressing the effect of a change in the mix of input used for production from a technological or cost standpoint. (Chambers 1988) A researcher that wants to estimate this elasticity, or some other comparative static, typically would do so using parametric production or cost function (e.g. translog or normalized quadratic) with panel data. For a study with only cross-sectional data, the construction of such a function may be problematic. Using a dual approach, a nonparametric alternative in such a situation may be the use of Data Envelopment Analysis (DEA). Cooper et al. (2000) provided a methodology for estimating elasticities of substitution for the technical production problem using DEA. To our knowledge, this has not been extended to the cost efficiency problem, which would be equivalent to estimating Allen partial or Morishima elasticities of substitution between inputs using a cost function (or cost minimization framework). The purpose of this thesis is to show how elasticities of substitution can be derived and estimated for the technical production and cost (overall economic) efficiency DEA under variable returns to scale. In addition, an empirical example using Kansas Farm Management Association (KFMA) data is presented to illustrate the estimation of these elasticities. The results showed that input substitutability is relatively limited at the enterprise level.

Table of Contents

List of Figures

List of Tables

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Chapter 1 - Introduction

In agriculture, input productivity and input price have been frequently subject to change. For a farmer wanting to maximize yields and minimize costs of production, an understanding of the tradeoff, or substitutability, of one input for another is essential. The derivation of elasticities of substitution has become the standard method for addressing the effect of a change in the ratio of inputs used for production from a technological or cost perspective (Chambers 1988). From the technological perspective (using a production function), this measurement shows how a per unit change in the marginal rate of technical substitution will alter the ratio of inputs, while maintaining a fixed level of output. In the case of a cost function, the elasticity of substitution shows how a shift in input prices will shift the ratio of inputs. More generally, it relates a percentage change in the ratio of inputs being used to an incremental increase in the ratio of the marginal products of the inputs (the ratio of input prices).

When the elasticity of substitution is elastic, a small change in the ratio of the marginal products of the inputs (for the production function) or the ratio of input prices (for the cost function), results in a greater change in the ratio of inputs used. From a graphical perspective, the isoquant between the two inputs being examined is less curved. On the other hand, when the elasticity is inelastic, a small change in the ratio of the marginal products of the inputs or the ratio of input prices results in a change in the ratio of inputs that is less than the percentage change in the ratio of the marginal products of the inputs or the ratio of input prices. From a graphical perspective, the isoquant will have a more pronounced degree of curvature. The elasticity of input substitution is a relevant tool in analyzing comparative static questions regarding the relative mix of inputs used in relation to technological and cost efficiency.

Hicks (1932) originally derived the elasticity of substitution to describe the effect that a change in the ratio of capital to labor would have on income distribution (a two input case). Several attempts have been made to generalize the Hicksian elasticity of substitution to the case of more than two inputs (Allen and Hicks 1934; Allen 1938; Uzawa 1962; McFadden 1963; Morishima 1967). Of these, the McFadden, Allen-Uzawa (or Allen partials), and Morishima elasticities of substitution have been the most prominent.

The McFadden elasticity of substitution describes the substitutability of two inputs along an isoquant, with all other input quantities maintained at a constant level, for an n-input production function. The McFadden elasticity however does not allow for optimal adjustment of inputs in response to changes in input prices, and therefore has not been widely used (Mundra and Russell 2010).

The Allen-Uzawa (or the Allen partial) and Morsihima elasticities of substitution describe the substitutability of two inputs, along an isoquant, in an n-input production function, with all other input quantities free to adjust. However, the Allen partial elasticity of substitution has been criticized for not being able to directly measure the ease of substitution between inputs, in that it does not provide a direct measure of the curvature of the isoquant, thereby not offering insight into changes in relative input shares (Blackorby and Russell 1981; Blackorby and Russell 1989). In contrast, the Morishima elasticity of substitution is more flexible, being both a measure of substitutability and capturing the change to input shares from a change in price or input quantities (Blackorby and Russell 1989).

A researcher wishing to estimate any of these elasticities would ordinarily do so by collecting time series or panel data to estimate a parametric production or cost function. For a study with only cross-sectional data available, constructing such a function may prove problematic. In particular, using cross-sectional data to construct a cost function can result in errors when there is limited relative price variability present in the data (which may be the case across space) (Lusk, Featherstone, Marsh and Abdulkadri 2002). In such a situation, a nonparametric alternative to model the production process from a technological or cost perspective is Data Envelopment Analysis (DEA).

DEA is a linear programming technique designed to evaluate the efficiency of productive decision-making units (DMUs). DEA grew out of the work of M.J. Farrell (1957), who sought to determine how a DMU could optimize its production capabilities purely through the adoption of efficiency-increasing measures. Farrell (1957) offered an analytic approach that examined a DMU's level of outputs to its inputs vis-à-vis the performance of its peers. Based on Farrell's results, Charnes, Cooper and Rhodes formulated the DEA model, known as the Charnes-Cooper-Rhodes (CCR) model (Charnes, Cooper and Rhodes 1978). A modified version of the CCR-model, the Banker-Charnes-Cooper (BCC) model allows for production technology exhibiting variable returns to scale (VRS) (Banker, Charnes and Cooper 1984; Ahn, Charnes and Cooper 1988). Under the BCC model, a DMU is compared to its peers, and an efficiency score, θ, is generated. The efficiency measure θ represents the ratio of a DMU's virtual output (i.e. the weighted sum of the DMU's outputs) to that same DMU's virtual input (i.e. the weighted sum of the DMU's inputs). Theta provides a measure of the technological efficiency of a DMU relative to other DMUs being examined along the production frontier. That is, the DEA model estimates a piece-wise linear production frontier connecting the technically efficient DMUs and θ provides a measure of how far a particular DMU is from that frontier.

DEA has also been applied to the cost minimization problem to assess cost efficiency (Färe, Grosskopf and Lovell 1985; Ferrier and Lovell 1990). For DMUs that reside on the technically efficient frontier, further efficiency gains are possible. Firms can move along the frontier to a point at which cost is minimized. In doing so, technically efficient firms solve the problem of finding the optimal mix of inputs that minimizes cost. Cost efficiency is measured as the ratio of the cost minimizing level of input use to the actual total input cost achieved for a particular DMU (relative to all other DMUs being examined). Cost or overall economic efficiency is comprised of two components: (1) reaching the technically efficient frontier, and (2) moving along the frontier to a point where the allocation of inputs is optimized.

Cooper, Parks and Ciruana (2000) provided a methodology for estimating elasticities of substitution for a slacks-based technical efficiency problem using DEA, assuming VRS. However, this study presented only a general elasticity of substitution that does not take into direct account of changes in the ratio of marginal products or prices between two inputs. The authors did not derive the Hicksian and Morishima elasticities most commonly encountered in the literature. In addition, the estimation has was not sufficiently extended to the cost efficiency problem.

The purpose of this thesis is to provide a methodology for estimating specific elasticities of substitution for the technical production and cost efficiency DEA models assuming VRS. This will extend the work of Cooper, Park and Ciurana (2000). In addition, an empirical example using Kansas Farm Management Association (KFMA) data is presented to illustrate the estimation of these elasticities. The specific objectives of the thesis are to:

1) Derive equivalent Hicksian and Morishima elasticities of substitution for the technical production and cost efficiency DEA models assuming VRS; and
2) Illustrate the use of elasticities of substitution in an applied setting using farm enterprise data for corn production from KFMA farms.

The thesis is divided into five chapters. The next chapter is comprised of an overview of the literature concerning the theory and application of the elasticity of substitution. The third chapter lays out the technical production and cost efficiency DEA models, their associated dual models, and derivations of the Hicksian and Morishima elasticities of substitution. The fourth chapter applies the elasticities derived in the previous chapter to corn production in Kansas, and the final chapter provides some concluding remarks.

Chapter 2 - Literature Review

In neoclassical economics, an individual's choices in production are governed by a production possibility set that relates the quantity of output that can be achieved with a set of inputs (Varian 1992). Changes in the quantity of inputs applied affects the quantity of output produced - so it is important that a decision maker know the manner in which inputs can be combined and the ease by which they can be substituted when making production decisions.

Previous research has looked at the estimation of elasticities of substitution in a variety of settings. The substitutability of inputs for a given production technology has been applied to studies at the firm level, at the regional and national levels, and internationally, comparatively across nations. Methodologically, this research derives elasticities in one of two ways: either directly from a production function, or indirectly from a cost function. Shankar, Piesse and Thirtle (2003) used a production function to derive elasticities of substitution. Their study examined the overreliance of energy as an input in Hungarian agriculture and offered recommendations for policymakers interested in decreasing energy use. Historically, Hungarian farmers adopted energy-intensive production strategies, in response to artificially low energy prices set by the government. Shankar, Piesse and Thirtle's paper used farm-level panel data from 117 farms from the years 1985-1991, a key transition period for the country's economy. The researchers specified a production function and estimated Allen and Morishima elasticities. The magnitude of the Allen elasticity estimates involving energy were found to be the largest of the group, indicating that energy use was sensitive to fluctuations in input prices. The Morishima elasticities involving energy and capital indicated eliminating artificially low prices for energy (rather than subsidizing capital investment) would be the most effective policy for inducing a decrease in energy use by farmers.

A study by Squires and Tabor (1994) focused on the wetland rice-based agriculture sector of rural Java, Indonesia, which has shown a remarkable ability to absorb a rapidly expanding supply of labor. The authors were interested in calculating the capacity of this sector to absorb labor, the rate of labor substitutability, and relative changes in income shares between family and hired labor that would occur with an increased labor population. Using annual farm-level data collected by Indonesia's Ministry of Agriculture on wetland rice and secondary crops, they estimated a translog production function and Hicks elasticities of substitution between labor and non-labor inputs (capital, land, chemicals) for different regions in Java and surrounding islands. Hicks elasticities of substitution between family and non-family labor were also calculated. The Hicks elasticities demonstrated that for wetland rice production, increases in inputs would increase the demand for labor (i.e. a complementary relationship was found between labor and capital [in Central and East Java], and labor and chemicals [in West Java and surrounding islands]). Their results also indicated that increases in capital investment in secondary crops (dryland rice and corn on Java) would increase the demand for labor. Furthermore, family and non-family labor was found to be highly substitutable (especially in wet-rice production on Java), thus confirming Java's exceptionality in absorbing increases in family and non-family labor.

Other studies have calculated elasticities of substitution to consider questions regarding input-output use at the international level. Hayami and Ruttan (1970) used a cross-country production function to estimate elasticities of substitution to better understand the differences in agricultural productivity between the developed and the developing world. In the early 1970s, research on developing countries, showed that some of these countries had agricultural output per worker to be only 1/50th of the levels found in the United States. Cobb-Douglas production functions were estimated for three years (1955, 1960, and 1965) from an international agricultural production data set compiled by one of the study's authors (this included both per farm data and national aggregate data sets). The production functions used labor, land, livestock, fertilizer, machinery, education, and technical manpower as inputs with the composite gross output as the single output. Elasticities of substitution were calculated to examine the accuracy of using a Cobb-Douglas production function in cross-country analysis. The elasticities of substitution were found to be consistent with the Cobb-Douglas imposition of unitary elasticities of substitution among inputs. From this the authors concluded that the Cobb-Douglas function was an appropriate approach in conducting cross-country production analysis.

There has also been much research with elasticities of substitution that were estimated from the cost function. Vincent (1977) explored the relative usage of land, labor, and capital in Australian agriculture over the span of fifty years. Vincent's study highlighted the advantages of examining elasticities using a cost minimization approach. A translog functional form was assumed and elasticities were estimated using time series data. These elasticities showed statistical significance and led him to conclude that substitution between inputs was highly inelastic. His findings confirmed his original assumptions about the low degree of labor mobility, irrespective of changes to input prices.

Nieswiadomy (1988) similarly explored input substitution at the farm level. His study looked at the changing exploitation of five inputs on irrigated farmland in Texas over the 1970s, a period when the center pivot system came into use. In the wake of the adoption of the center pivot, Nieswiadomy used time series data to estimate a translog cost function and elasticities of substitution between inputs. The majority of Nieswiadomy's elasticities showed statistical significance and confirmed his underlying assumptions regarding the impact of technological innovation and changes to input price on the substitutability of inputs.

Dalton, Masters and Foster (1997) used farm level data collected from 65 smallholder farms over two years, to estimate a translog cost function. Their paper was concerned with the ability of Zimbabwean farms to absorb a rapidly increasing rural labor force. Morishima elasticities between three inputs (labor, capital, and biochemicals) were derived from the translog cost function. The results indicated there was moderate substitutability between the three inputs, with the greatest substitutability occurring between labor and biochemical inputs. This led the authors to conclude that an increase in the labor population could occur, dependent on input prices, in conjunction with a substitution of other inputs.

The literature on the application of DEA to the estimation of elasticities of substitution is limited. Cooper, Park and Ciurana (2000) presented a slacks-based additive DEA models and described the similarity of these models' efficient frontiers to the production and cost frontiers found in microeconomics. The study built a conceptual framework surrounding the use of elasticities as a means to measure movement along the efficient frontiers. Several studies focused on the energy sector have considered the substitutability of inputs using DEA. Reinhard, Lovell and Thijssen (2000) described the use of conventional inputs and environmentally unfriendly inputs in Dutch dairy farming. Their study considered the relationship between environmental efficiency to energy efficiency, but stopped short of estimating elasticities. Lee and Zhang (2012) assessed the substitutability of capital for fossil fuels in reducing carbon dioxide emissions in China. Using DEA they examined the technical efficiency of the Chinese manufacturing industry, but then calculated Morishima elasticities separately, based off of an input-distance function.

Chapter 3 - Theory

The importance that elasticities of substitution have in production efficiency analysis makes it advantageous to show how derivation of these elasticities can occur in the absence of a parametric production or cost function. The purpose of this chapter is to illustrate how this can be done using the BCC technical and cost efficiency DEA models, and derive elasticities from both of these models.

3.1 – The Technical Efficiency Problem

Data Envelopment Analysis (DEA) is a linear programming method to estimate the relative efficiency of a group of DMUs. This methodology is used to evaluate technical efficiency of DMUs (Cooper, Seiford and Tone 2007). One of the first models proposed was the CCR model (Charnes, Cooper and Rhodes 1978). In economics, the dual to this model (CCR-DLP) is commonly utilized and is given by:

CCR-DLP(min): $\min_{\theta_o,\lambda} \theta_o$ (3.1)

Subject to: $\boldsymbol{\lambda}'\boldsymbol{x_k} \leq \theta_o x_{k,o}$ ∀ k inputs $\rightarrow \boldsymbol{v}$

$\boldsymbol{\lambda}'\boldsymbol{y_m} \geq y_{m,o}$ ∀ m outputs $\rightarrow \boldsymbol{u}$

$\boldsymbol{\lambda} \geq 0$

The objective of the problem (3.1) is to estimate the technical efficiency of a DMU, θ_o, relative to all the other DMUs in the sample. This is done by choosing weights, $\boldsymbol{\lambda}$, associated with each DMU in the sample, that puts it on the technological or production frontier. Theta therefore is a measurement of how far a firm is from the efficient frontier (with $\theta_o = 1$ characterizing a firm that is technically efficient, and θ_o bounded by 0 and 1 in value). From the dual problem (the CCR model), the parameter θ_o is equal to the virtual output (that achieved on the frontier) divided by the virtual input (that achieved on the frontier) for the firm being examined, i.e. $\theta_o = \frac{\sum_{m=1}^{n} u_m y_{m,o}}{\sum_{w=1}^{n} v_k x_{k,o}}$, where y_m is the firm's

mth output, x_k is the firm's kth input, u_m is the firm's mth output weight corresponding to y_m, and v_k is the firm's kth input weight corresponding to x_k. The first constraint forces composite inputs, $\boldsymbol{\lambda' x_k}$**,** to be less than or equal to the technically efficient input level. The second constraint forces composite outputs, $\boldsymbol{\lambda' y_m}$, to be greater to or equal to the technically efficient output level**.** Here the input and output vectors of weights, $\boldsymbol{u}$ and $\boldsymbol{v}$ represent the shadow prices (dual variables) to the first and second constraints. These weights can be used to identify the relative importance of the inputs and outputs that affect a firm's technical efficiency.

The BCC Model is an extension of the CCR-DLP model through the addition of a convexity constraint ($\boldsymbol{e'\lambda} = 1,$ where $\boldsymbol{e}$ is a column vector with all elements summing to one). The convexity constraint allows the model to exhibit variable returns to scale (such that an identical adjustment in the amount of inputs applied will not necessarily affect output(s) by the same amount). The effect is a transformation of the CCR Model's linear efficient frontier into a convex hull. Because agricultural production is not likely characterized by constant returns to scale technologies, it is this model that we turn to in deriving elasticities of substitution. For the sake of clarity, the BCC Model is explicitly stated below:

$$\textit{BCC(min):} \qquad \min_{\theta_o, \lambda} \theta_o \tag{3.2}$$

$$\begin{aligned} \textit{Subject to:} \qquad & \boldsymbol{\lambda' x_k} \le \theta_o x_{k,o} \ \forall \text{ k inputs} && \rightarrow \boldsymbol{v} \\ & \boldsymbol{\lambda' y_m} \ge y_{m,o} \quad \forall \text{ m outputs} && \rightarrow \boldsymbol{u} \\ & \boldsymbol{e'\lambda} = 1 && \rightarrow u_0 \end{aligned}$$

$\boldsymbol{\lambda} \geq 0$

The shadow price for the convexity constraint, u_0, is described as a "free variable" (Cooper, Seiford and Tone 2007) to allow for variable-returns-to-scale in the corresponding CCR model. For firms on the technically efficient frontier, their shadow prices will necessarily be equal to zero - there can be no further increase in technical efficiency for these firms.

3.2 – The Dual to the Technical Efficiency Problem

Deriving the dual problem to the BCC minimization illustrates where the shadow prices, described in the previous section, come from. One must first begin by restating the BCC Model as a maximization problem:

BCC(max): $\max_{\theta_o,\lambda} -\theta_o$ (3.3)

Subject to: $\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{k,o} \leq 0$ ∀ k inputs $\rightarrow \boldsymbol{v}$

$-\boldsymbol{\lambda}'\boldsymbol{y_m} \leq -y_{m,o}$ ∀ m outputs $\rightarrow \boldsymbol{u}$

$\boldsymbol{e}'\boldsymbol{\lambda} = 1$ $\rightarrow u_0$

$\boldsymbol{\lambda} \geq 0$

Applying the rules for deriving the dual of a linear program (Samuelson 1953; Shephard 1953; Uzawa 1964), the dual to the maximization problem can be stated as:

Dual Problem(min): $\min_{\boldsymbol{v},\boldsymbol{u},u_0} -\boldsymbol{u}'\boldsymbol{y_o} + u_0$ (3.4)

Subject to: $\boldsymbol{v}'\boldsymbol{x_o} = 1$ $\rightarrow \theta_o$

$\boldsymbol{v}'\boldsymbol{x_n} - \boldsymbol{u}'\boldsymbol{y_n} + u_0 \geq 0$ ∀ n firms $\rightarrow \boldsymbol{\lambda}$

$v, u \geq 0$, with u_0 unrestricted in sign

Restating the dual problem as a maximization problem gives:

Dual Problem(max): $\max_{\boldsymbol{v},\boldsymbol{u},u_0} \boldsymbol{u}'\boldsymbol{y_o} - u_0$ (3.5)

Subject to: $\boldsymbol{v}'\boldsymbol{x_o} = 1$ $\rightarrow \theta_o$

$\boldsymbol{u}'\boldsymbol{y_n} - \boldsymbol{v}'\boldsymbol{x_n} - u_0 \leq 0$ $\forall$ n firms $\rightarrow \boldsymbol{\lambda}$

$\boldsymbol{v}, \boldsymbol{u} \geq 0$, with u_0 unrestricted in sign

One can observe the shadow prices of the primal problem as variables in the objective function of the dual problem. Similarly, the variables in the objective of the primal problem are transformed into the shadow prices of the dual problem.

3.3 – The Cost Efficiency Problem

Alternatively, in the situation where input prices and costs are known, DEA can be applied to assess cost or overall economic efficiency. Cost efficient DMU's are defined as those that are technically efficient, and also exhibit allocative efficiency (Cooper, Seiford and Tone 2007). Allocative efficiency is the degree to which a DMU minimizes cost along the technically efficient frontier. This model general cost efficiency models is given by:

Cost(min): $\min_{z,\lambda} \boldsymbol{w}'\boldsymbol{z}$ (3.6)

Subject to: $z_k - \boldsymbol{\lambda}'\boldsymbol{x_k} \geq 0$ $\forall$ k inputs $\rightarrow \boldsymbol{v_k}$

$\boldsymbol{\lambda}'\boldsymbol{y_m} - y_{m,o} \geq 0$ $\forall$ m outputs $\rightarrow \boldsymbol{u_m}$

$\boldsymbol{e}'\boldsymbol{\lambda} = 1$ $\rightarrow u_0$

$\boldsymbol{\lambda}, \boldsymbol{z} \geq 0$

The objective is for the DMU to choose $\boldsymbol{z}$ and $\boldsymbol{\lambda}$ that minimizes cost, where $\boldsymbol{w}$ is a column vector of input unit costs, $\boldsymbol{z}$ is column vector of cost minimizing levels of input quantities, $\boldsymbol{y_m}$ is a row vector of the mth output for all DMUs, $y_{m,o}$ is an element within $\boldsymbol{y_m}$ representing the firm of interest's mth output, and $\boldsymbol{\lambda}$ is a non-negative column vector of weights. Cost efficiency is equal to $\frac{\boldsymbol{w'z}}{\boldsymbol{w'x_o}}$ (where $\frac{\boldsymbol{w'z}}{\boldsymbol{w'x_o}} \leq 1$), or the amount of separation between the DMU's actual choice of inputs, $\boldsymbol{x_o}$, and the cost minimizing level, $\boldsymbol{z}$ (Cooper, Seiford and Tone 2007). The first constraint causes the kth cost minimizing input to be less than or equal to the composite kth input. The second constraint causes the composite output to be greater than or equal to the mth output of the firm of interest.

3.4 – The Dual to the Cost Efficiency Problem

Formulating the dual problem of the cost efficiency problem, in the same manner as applied to the technical efficiency problem, provides the link to the shadow prices stated for the primal cost efficiency problem. Restating the cost efficiency problem as a maximization problem gives:

$$\textit{Cost(max):} \quad \max_{z,\lambda} -\boldsymbol{w'z} \tag{3.7}$$

$$\begin{aligned} \textit{Subject to:} \quad & \boldsymbol{\lambda' x_k} - z_k \leq 0 && \forall \text{ k inputs} && \rightarrow \boldsymbol{v_k} \\ & -\boldsymbol{\lambda' y_m} \leq -y_{m,o} && \forall \text{ m outputs} && \rightarrow \boldsymbol{u_m} \\ & \boldsymbol{e'\lambda} = 1 && && \rightarrow u_0 \\ & \boldsymbol{\lambda}, \boldsymbol{z} \geq 0 \end{aligned}$$

The dual to the maximization problem is the following minimization problem:

$$\textit{Dual Problem(min):} \quad \min_{\boldsymbol{v},\boldsymbol{u},u_0} -\boldsymbol{y_o' u} + u_o \tag{3.8}$$

$$\textit{Subject to:} \quad -\boldsymbol{v} \geq -\boldsymbol{w} \qquad \rightarrow \boldsymbol{z}$$

$$\boldsymbol{v}'\boldsymbol{x_n} - \boldsymbol{u}'\boldsymbol{y_n} + u_o \geq 0 \qquad \forall\, n \text{ DMUs} \qquad \rightarrow \boldsymbol{\lambda}$$

$$\boldsymbol{v}, \boldsymbol{u} \ \geq 0, \text{ with } u_0 \text{ unrestricted in sign.}$$

Restating the dual problem as a maximization problem yields:

Dual Problem(max): $\quad \max_{\boldsymbol{v},\boldsymbol{u},u_0} \boldsymbol{y_o'}\boldsymbol{u} - u_o \qquad (3.9)$

Subject to: $\quad \boldsymbol{v} \leq \boldsymbol{w} \qquad \rightarrow \boldsymbol{z}$

$$\boldsymbol{u}'\boldsymbol{y_n} - \boldsymbol{v}'\boldsymbol{x_n} - u_o \leq 0 \qquad \forall\, n \text{ DMUs} \qquad \rightarrow \boldsymbol{\lambda}$$

$$\boldsymbol{v}, \boldsymbol{u} \ \geq 0, \text{ with } u_0 \text{ unrestricted in sign.}$$

As in the dual to the technical efficiency problem, one can observe that the shadow prices of the primal problem appear as variables in the objective function of the dual problem, and the variables in the objective statement of the primal problem appear as shadow prices in the dual problem.

3.5 – The Hicksian Elasticity for Technical Efficiency

Using the total derivation of a linearly homogenous two-input production function, $y = f(x_1, x_2)$, in an economy that possesses constant returns to scale technology, Hicks presented the elasticity of input substitution ($\sigma_{i,j}^H$) (Hicks 1932) as:

$$\sigma_{i,j}^H \equiv \frac{d(x_2/x_1)}{d(f_1/f_2)} \frac{f_1/f_2}{x_2/x_1} \qquad (3.10)$$

This can alternatively be written in logarithmic form as,

$$\sigma_{i,j}^H \equiv \frac{dln\left({}^{x_2}/_{x_1}\right)}{dln\left({}^{f_1}/_{f_2}\right)} \qquad (3.11)$$

where f_1/f_2 represents the marginal rate of substitution of x_2 for x_1. In this instance, the elasticity of substitution is shown to be the rate of change of the ratio of inputs divided by the rate of change to the marginal rate of substitution (Chambers 2007). The Hicksian elasticity is symmetric, such that $\sigma_{i,j}^{H} = \sigma_{j,i}^{H}$.

Using this measure, the Hicksian elasticity for technical efficiency can be derived directly from the BCC minimization problem (3.2) stated earlier. The Lagrangian function for the constrained optimization BCC minimization problem is:

$$L = \theta_o + \sum_k v_k\left(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{k,o}\right) - \sum_m u_m\left(\boldsymbol{\lambda}'\boldsymbol{y_m} - y_{m,o}\right) + u_o(\boldsymbol{e}'\boldsymbol{\lambda} - 1) \tag{3.12}$$

Using 3.12, the following first order derivatives can be derived,

$$f_k = \frac{\partial L}{\partial \boldsymbol{\lambda}'\boldsymbol{x_k}} = v_k, \text{and } \frac{\partial L}{\partial v_k} = \boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{k,o}. \tag{3.13}$$

And the Hicksian elasticity for technical efficiency can be shown to be equal to:

$$\sigma_{i,j}^{H} = \frac{dln\left({}^{x_2}/_{x_1}\right)}{dln\left({}^{f_1}/_{f_2}\right)} = \left[\frac{\partial\left({}^{\boldsymbol{\lambda}'\boldsymbol{x_j}}/_{\boldsymbol{\lambda}'\boldsymbol{x_i}}\right)}{\partial\left({}^{f_i}/_{f_j}\right)}\right]\left[\frac{\left({}^{f_i}/_{f_j}\right)}{\left({}^{\boldsymbol{\lambda}'\boldsymbol{x_j}}/_{\boldsymbol{\lambda}'\boldsymbol{x_i}}\right)}\right] =$$

$$= \left[\frac{\left(\frac{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j} - \boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}{\boldsymbol{\lambda}'\boldsymbol{x}_i^2}\right)}{\left(\frac{f_j\partial f_i - f_i\partial f_j}{f_j^2}\right)}\right] = \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\frac{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j} - \boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}{f_j\partial f_i - f_i\partial f_j}\right]$$

$$\ldots = \left[\frac{\left(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j(\boldsymbol{\lambda}'\boldsymbol{x_j})\right)\left(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\right)\left(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\right)}{\left(\boldsymbol{\lambda}'\boldsymbol{x_i}\boldsymbol{\lambda}'\boldsymbol{x_j}\right)\left(v_j\left(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\right) - v_i\left(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\right)\right)}\right]. \tag{3.14}$$

This elasticity shows the degree of substitutability an inefficient firm (at optimality) can make to its inputs and remain on the technically efficient frontier. The formula is only relevant for inefficient firms (See Appendix A.1 for the full derivation). For firms already on the frontier (existing at vertex points), continuous derivatives cannot be derived. This would require the use of directional derivatives, which would make the elasticities of substitution for these firms non-unique. Podinovski and Førsund (2010) lay out a methodology using directional derivatives to find elasticities for firms on the efficient frontier. The exploration of these elasticities, however is beyond the scope of this study and will be explored in future research.

3.6 – The Hicksian Elasticity for Cost Efficiency

Hicks' elasticity of substitution for the cost minimization problem is analogous to the technical efficiency problem. For a two-input cost function, defined as, $C = C(w, z)$, where w is defined as the price (or cost) of the input z, the Hicks elasticity of substitution between two inputs *($\sigma_{i,j}^{CH}$)* can be expressed in logarithmic form as:

$$\sigma_{i,j}^{CH} \equiv \frac{\mathrm{dln}\left({}^{w_j}/_{w_i}\right)}{dln\left({}^{C_i}/_{C_j}\right)} = \frac{\mathrm{dln}\left({}^{z_j}/_{z_i}\right)}{dln\left({}^{w_i}/_{w_j}\right)}, \tag{3.15}$$

where $C_i = dc/dw_i = z_i$ (Shephard 1981). Thus, $\sigma_{i,j}^{CH}$ is equal to the logarithmic ratio of input quantities to input prices. Using this result, $\sigma_{i,j}^{CH}$ can be derived from the cost efficiency problem, 3.16. The Lagrangian function for the cost efficiency problem, where w_k refers to the kth input's price and z_k refers to the kth cost-minimizing level of input for the firm, is given by:

$$P = w'z + \sum_k v_k(\boldsymbol{\lambda}'\boldsymbol{x_k} - z_k) - \sum_m u_m\left(\boldsymbol{\lambda}'\boldsymbol{y_m} - y_{m,o}\right) + u_o(\boldsymbol{e}'\boldsymbol{\lambda} - 1), \tag{3.16}$$

Using 3.16, the following first order derivatives can be derived,

$$P_k = \frac{\partial P}{\partial w_k} = z_k;\ \ \frac{\partial P}{\partial z_k} = w_k - v_k \tag{3.17}$$

Thus the Hicksian elasticity for cost efficiency can be shown to be equal to:

$$\sigma_{i,j}^{HC} = \frac{\partial ln\left({}^{w_j}/_{w_i}\right)}{\partial ln\left({}^{P_i}/_{P_j}\right)} = \frac{\partial ln\left({}^{z_j}/_{z_i}\right)}{\partial ln\left({}^{w_i}/_{w_j}\right)} = \frac{\partial lnz_j - \partial lnz_i}{\partial lnw_i - \partial lnw_j}$$

$$= \left(\frac{z_j(w_j - v_j)}{w_i z_i} - \frac{z_j(w_j - v_j)}{w_j z_j}\right)^{-1} - \left(\frac{z_i(w_i - v_i)}{w_i z_i} - \frac{z_i(w_i - v_i)}{w_j z_j}\right)^{-1}. \quad (3.18)$$

This elasticity shows the degree of substitutability an inefficient firm (at optimality) can make to its inputs and remain on the cost efficient frontier. Again, this elasticity can only be derived for inefficient firms, as efficient firms exist at vertex points on the frontier, areas where continuous derivatives cannot be derived. (See Appendix B.1 for the full derivation).

3.7 – The Morishima Elasticity for Cost Efficiency

The Morishima formulation for the cost problem provides a more easily intuitive measurement of elasticity then the alternatives presented in Chapter 2. Following Chambers (1988), the Morishima elasticity of substitution for cost efficiency can be seen to be equal to the natural log of the ratio of the i^{th} and j^{th} input price divided by the log of the j^{th} input. This elasticity can be estimated from the cost efficiency model:

$$\sigma_{i,j}^{MC} = \frac{dln\left({}^{z_i}/_{z_j}\right)}{dln(w_j)} \quad (3.19)$$

$$= \frac{dlnz_i - dlnz_j}{dlnw_j}$$

$$\dots = \frac{w_j z_j}{(w_i - v_i) z_i} - \frac{w_j z_j}{(w_j - v_j) z_j} \quad (3.20)$$

The Morishima elasticity for cost efficiency is valuable to estimate along side the Hicksian elasticity. Recall from the previous discussion that the Morishima elasticity allows for changes in all other inputs from a change in input price. Because it has the ability to generalize and retain most of the features of the Hicksian model, it is preferred to alternate approaches. However, it must be noted that, in the case of more than two inputs, the assumption of symmetry is no longer reliable (Blackorby and Russell 1981) for the Morishima elasticities.

3.8 – Concluding Remarks

Estimating elasticities using the DEA method presented above yields several advantages over the parametric approach. Because DEA relies only on cross-sectional data, a researcher can estimate elasticities without needing to gather a more complex dataset (i.e. time series data). Additionally, DEA makes minimal assumptions about the underlying production technology of the DMUs under observation, and allows for individual estimates for each DMU. Consequently, the DEA approach can produce potentially more information, without some of the burdens of traditional methods.

Chapter 4 - Empirical Applications in Kansas

The empirical application illustrating the elasticity measures derived in Chapter 3 will use data from the Kansas Farm Management Association (KFMA) to examine dryland corn production at the enterprise level under different tillage practices (e.g. no-tillage, reduced tillage and conventional tillage). The KFMA is an organization that provides financial data and planning for farmers and is affiliated with Kansas State University (KFMA 2014). The KFMA maintains an enterprise-level database of annual production, financial, and cost data for Kansas farms.

For the empirical application, the efficiency of dryland corn production under different tillage regimes was examined for farms in Kansas planting corn in 2014. The data used for the analysis was for 119 farms. KFMA input data included enterprise level expenses for fuel, fertilizer, herbicide, seed, labor (including both hired and unpaid labor), machinery (including machinery rentals and repairs), and land (i.e. total acres used). Output was measured using total value of dryland corn produced. Input variables for the DEA analyses were measured using a quantity index (except for the land variable, which was given as quantity used), with total input expenses divided by input cost per acre. Input cost per acre values were obtained from the KFMA's 2014 State of Kansas Enterprise Summary Report for non-irrigated corn (KFMA 2014). The output variable was not transformed, since corn price was assumed to remain constant across the farms in 2014. Deriving a quantity index for output would result in a scaled version of the total value of dryland corn production, with the relative differences between farms remaining the same. Given DEA analysis is scale invariant, transformation of output using output price should yield the same results as if no transformation was used. Table 4.1 contains input and output prices as well as mean, minimum, and maximum values and the standard deviation of the quantity indices across the 119 farms.

Technical and cost efficiency models were estimated following equations 3.2 and 3.6 for each farm using the General Algebraic Modeling System (GAMS). The results from the GAMS model were used to compute Hicksian production and cost elasticities as well as Morishima cost elasticities in MATLAB for each farm using equations 3.14, 3.18, and 3.20. Elasticity estimates are presented in Tables 4.2 to 4.4. The estimates are averages of the individual elasticity estimates across the sample of farms. The 90 percent confidence intervals of the elasticity measures across farms were estimated and are presented below the mean estimates in parentheses, as well.

The results of the estimation of Hicksian and Morishima elasticities (Tables 4.2 to 4.4) show only slight substitutability or complementarity between inputs. The mean values indicate that, at least for this set of farms for the year 2014, the response to changes in an input's relative marginal productivity or price does not dramatically alter the proportion of inputs applied. The 90% confidence interval is much more pronounced than the mean values, indicating a diversity in responses to input substitutability across the farms examined. Such diversity between farms may be due to a number of factors, such as relative variability in farm size, environmental factors, tillage methods, or management practices. Similarly, the elasticity results for the different DEA models vary. For example, the Hicksian elasticity for technical efficiency indicate that the majority of inputs act as complements, while the Hicksian elasticity for cost efficiency indicate that the majority of inputs are substitutes (Table 4.2, 4.3). The limited substitutability of inputs is consistent with previous work on the estimation of elasticities of substitution in agricultural production (Ray 1982; Hertel 1989).

The results of the estimation of the Hicksian elasticities for technical efficiency show that, on average across farms, inputs behave as complements with one another, except for several which behave as substitutes (many substitutions involving machinery or land have negative mean values) (Table 4.2). The degree of complementarity varies from one input to another, and from farm to farm, with mean values for the elasticities ranging from -2.44 to 0.66 in magnitude. Again, the amount of variability from farm to farm is large and for each of the estimated elasticities, there are some farms that report negative elasticities. That is, for some specific farms the inputs being compared are substitutes.

The Hicksian elasticity of substitution of fertilizer for seed has a mean value of 0.28, a lower confidence bound of -0.0028, and an upper confidence bound of 1.21. In this case, an increase in the ratio of the marginal products of fertilizer and seed leads to substitution between the inputs. Figure 4.1 illustrates this with an estimate of the empirical cumulative distribution function (ecdf) of the Hicksian technical efficiency elasticities of fertilizer for seed across farms. These estimates indicate that fertilizer and seed are substitutes for several farms, but for the majority of farms, fertilizer and seed are complements. One of the Hicksian technical efficiency elasticities that presents two of the inputs as substitutes is the estimate involving labor and machinery, which has a negative mean value of -2.44 with a lower confidence bound of -14.64 and an upper confidence bound of 0.85. Figure 4.2 shows the estimated ecdf of the Hicksian technical efficiency elasticity of labor for machinery across farms.

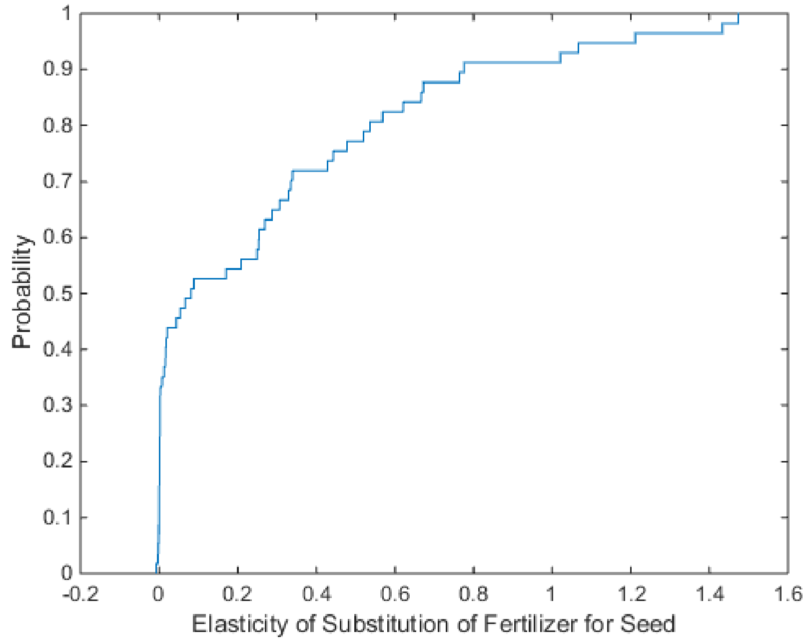

Figure 4.1: ECDF of the Hicksian Technical Efficiency Elasticities of Fertilizer for Seed

These estimates show that an increase in the ratio of the marginal product of labor and the marginal product of machinery will lead to a large degree of substitution between the two inputs.

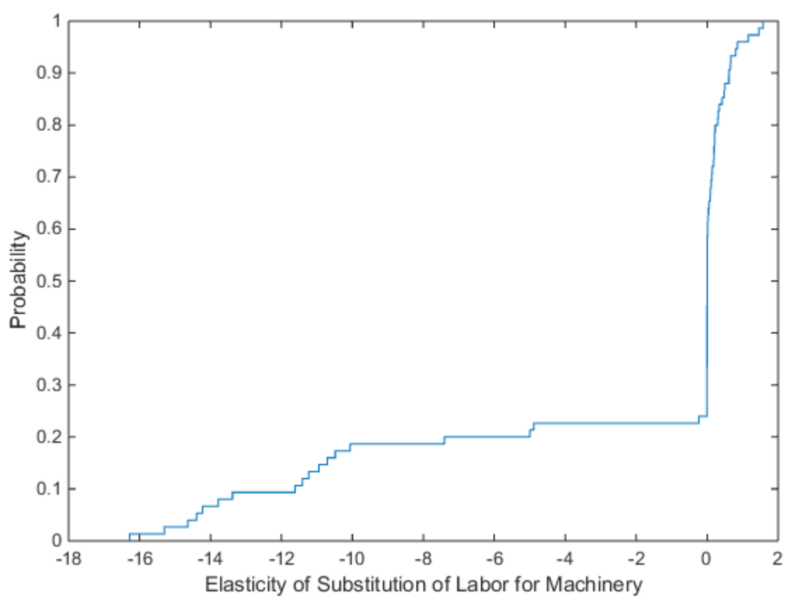

Figure 4.2: ECDF of the Hicksian Technical Efficiency Elasticities of Labor for Machinery

The results for the Hicksian elasticities for cost efficiency show the majority to be smaller in value then the technical efficiency elasticities (Table 4.3), with many appearing as substitutes (with negative signs). In addition, many of the mean estimates are close to 0. This suggests that changes in the ratio of input costs may not have a strong impact on substitutability. The Hicksian cost efficiency elasticity of seed for land has a mean value of -0.035. The lower and upper confidence bounds for the sample are -0.036 and -0.035. Figure 4.3 shows the estimated ecdf of the Hicksian cost efficiency elasticity of seed for land across farms. The Hicksian cost efficiency elasticity for fuel for machinery is another example. It has a mean value of 0.0079 and lower and upper confidence bounds of -0.37 and 0.15, with values on either side of zero, indicating that an increase in the ratio of input prices will make these inputs behave as complements (Figures 4.4). However, the mean value is close to zero and is smaller in value compared with its technical efficiency counterpart (where the mean value is 0.051).

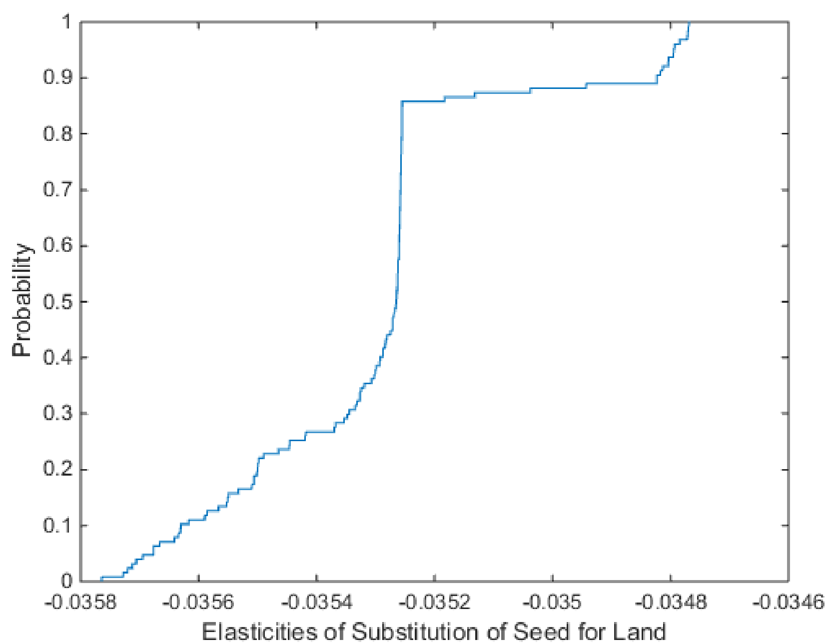

Figure 4.3: ECDF of the Hicksian Cost Efficiency Elasticities of Seed for Land

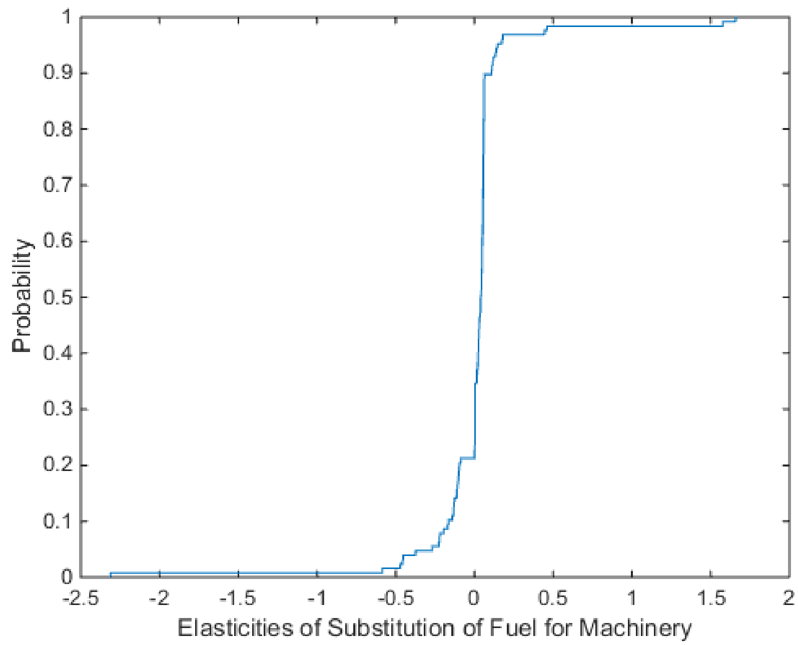

Figure 4.4: ECDF of the Hicksian Cost Efficiency Elasticities of Fuel for Machinery

This suggests that the degree of complementarity is slight.

While most of the Hicksian cost efficiency elasticities are smaller than their technical efficiency counterparts, the Hicksian cost efficiency elasticity involving labor and machinery is larger than the technical efficiency measure. Whereas the technical efficiency results indicate that labor and machinery are substitutes, the cost efficiency results indicate that they are complements, with a mean value of 0.0063, and lower and upper bounds of the confidence interval of -0.0034 and 0.017. The mean value is quite close to zero, but examining the ecdf of the Hicksian cost efficiency elasticity of labor machinery (Figures 4.5) shows that the majority of farms reside in the positive interval. For most of the farms, labor and machinery are complements.

The results from the estimation of the Morishima cost efficiency elasticities display mean values that indicate a more even division between complementarity and substitutability among inputs. In addition, for many of this group of elasticities, the lower

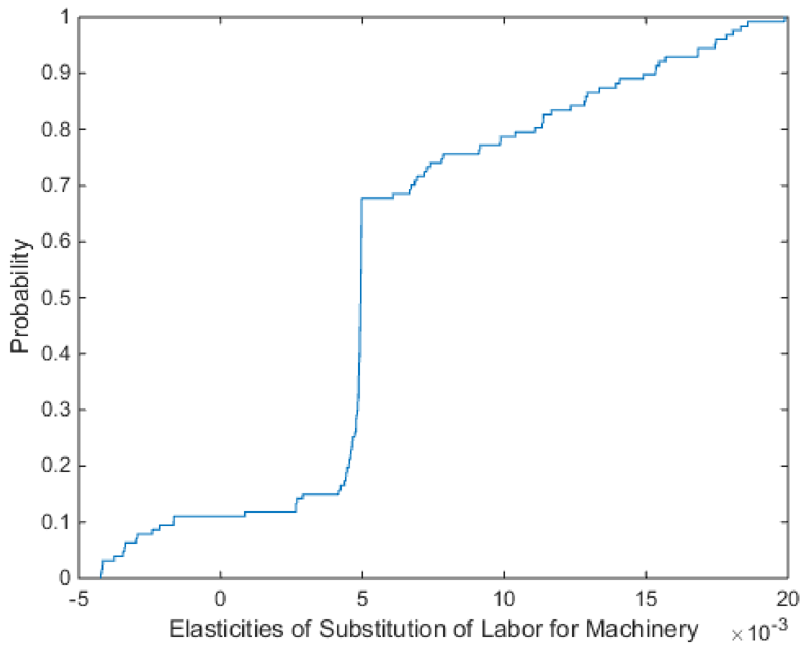

Figure 4.5: ECDF of the Hicksian Cost Efficiency Elasticities of Labor for Machinery

and upper bounds of the confidence interval are much closer to one another. The elasticity estimates involving the substitution of machinery for fuel has a mean of -0.034,

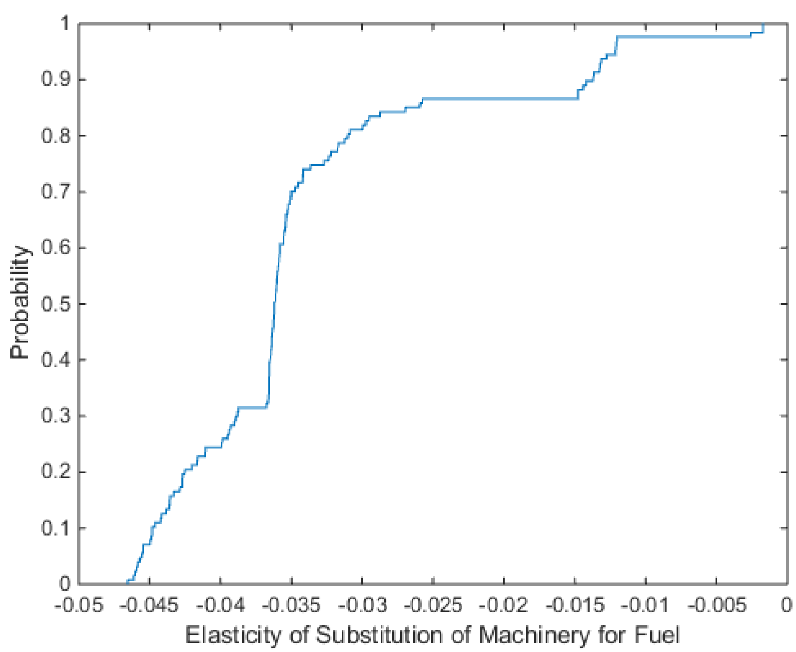

Figure 4.6: ECDF of the Morishima Cost Efficiency Elasticities of Machinery for Fuel

and lower and upper bounds of the confidence interval of -0.046 and -0.012, signaling slight substitutability. The elasticity involving the substitution of fuel for machinery however shows slight complementarity, with a mean of 0.028, and lower and upper bounds of the confidence interval of 0.0028 and 0.079 (Figures 4.6 and 4.7). The difference in the pair of elasticities highlight the non-symmetric aspect of the Morishima elasticities. Changes to the price of one input will have a different effect on ease of substitutability, then changes to the price of the other input. A substitution towards a particular input may therefore not be the same or have the same effect as a substitution away from that input.

The Morishima cost efficiency elasticity for the substitution of machinery for land and the elasticity for the substitution of land for machinery show this same characteristic (Figures 4.8 and 4.9). The mean value of the elasticity of machinery for land is -0.033,

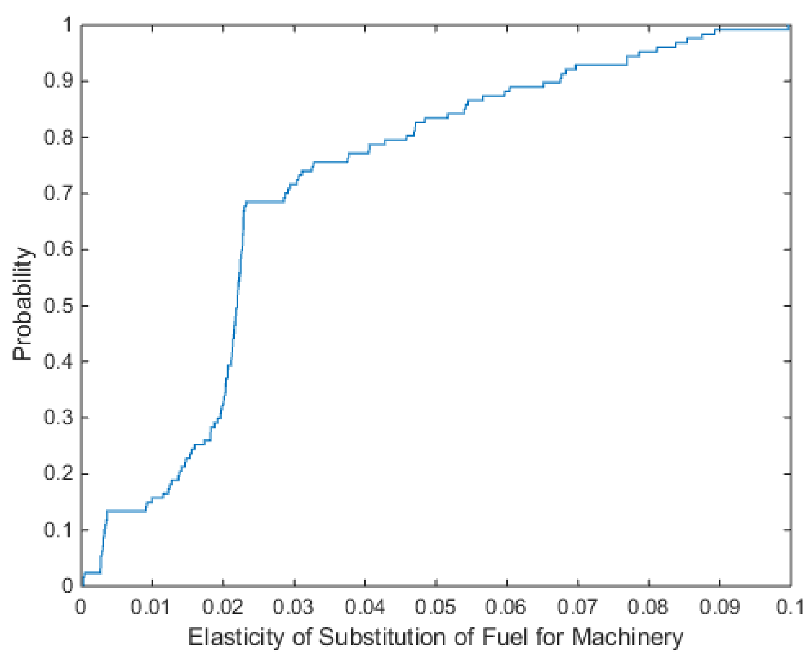

Figure 4.7: ECDF of the Morishima Cost Efficiency Elasticities of Fuel for Machinery

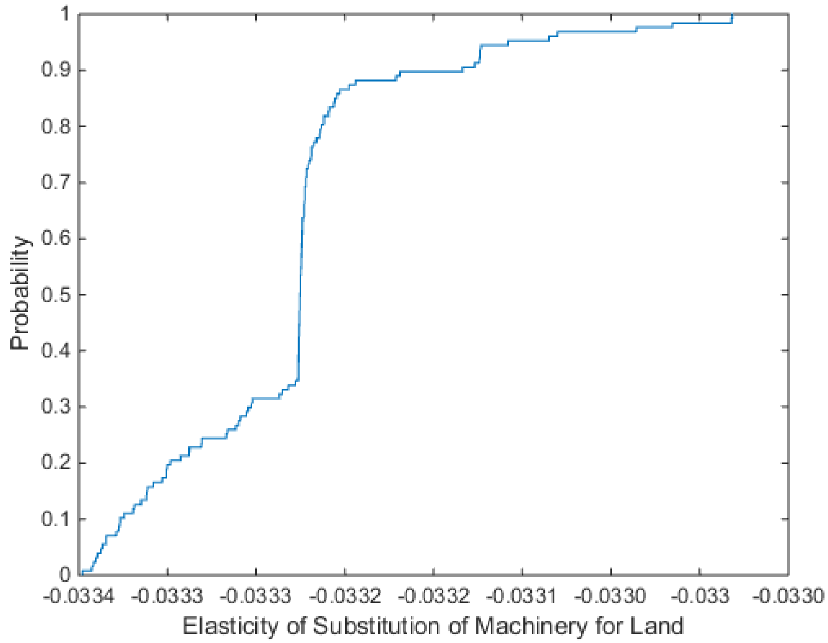

Figure 4.8: ECDF of the Morishima Cost Efficiency Elasticities of Machinery for Land

while the mean value for the elasticity of land for machinery is 0.30. The lower and upper bounds of the confidence interval for the elasticity of land for machinery are 0.26 and

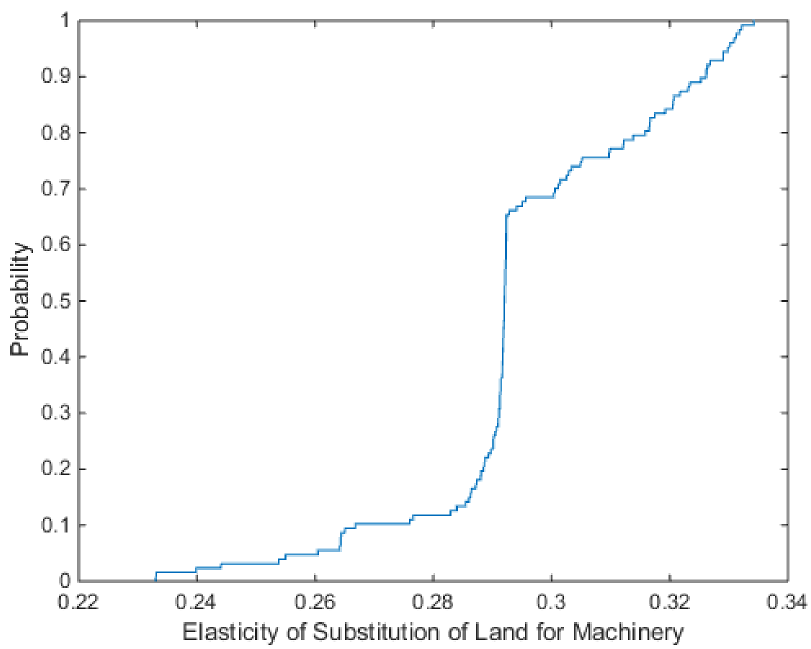

Figure 4.9: ECDF of the Morishima Cost Efficiency Elasticities of Land for Machinery

0.33, and for the elasticity of machinery for land are -0.0333 and -0.0331. Furthermore, all of the mean values for the Morishima elasticity estimates are close to zero, with the largest being the substitution of land for labor at 1.75. Most mean values report in the hundredth decimal place or lower. This may be indicative of a non-substitutionary relationship among the inputs.

This chapter presented an application of the theoretical derivations of the previous chapter to a set of Kansas farms at the enterprise level examining dryland corn production under different tillage techniques. Hicksian and Morishima production elasticities, and Morishima cost elasticities were estimated for each of the 119 farms. This application shows that DEA can provide useful elasticity estimates for a sample of DMUs and provide individual estimates, for the set of DMUs, along the production and cost frontiers. This is an advantage not always available with traditional parametric approaches. In providing a range of elasticity estimates, DEA can help farmers manage their inputs by examining the different conditions under which two inputs are classified as substitutes and as complements.

Additionally, the results of these estimations demonstrate that the particular methodology used has a significant impact on the extent of an input's substitutability across farms. In general, the elasticities derived from the technical efficiency problem were larger than the elasticities derived from the cost function. The numerical implications reveal a unique point of divergence between cost efficiency elasticities and technical efficiency elasticities. Changes in cost efficiency account for changes in input price and productivity, whereas changes in technical efficiency only account for changes in input productivity. This implies that changes in the ratio of input prices have a greater effect on the substitutability of inputs than do changes in the ratio of marginal products of two inputs.

Table 4.1: Price and Expenditure Data for the Sample Farms

	Input Data							Output Data
	Fuel	Fertilizer	Herbicide	Seed	Labor	Machinery	Land	Total Corn Value
Price ($/acre)	18.75	86.39	35.74	64.64	12.56	113	28.22	376.42
Mean*	498.36	498.48	498.99	496.53	2019.73	157.92	496.37	591.88
Min*	12.59	4.42	7.73	4.47	180.29	2.26	8	42.40
Max*	3694.19	2946.58	2620.23	2427.88	2499.65	2157.52	3123.10	750.62
Std. Dev.*	3681.60	2942.16	2612.50	2423.41	2319.36	2155.26	3115.10	708.22

Note: Input and output means, minimum and maximum values, and standard deviations are quantity/price across 119 farms, with input price constant across the sample.

Source: KFMA website (http://www.agmanager.info/kfma

Table 4.2: Mean Estimates of the Hicksian Elasticity of Substitution (Technical Efficiency)

	Fuel	Fertilizer	Herbicide	Seed	Labor	Machinery	Land
Fuel	-	0.38 (-0.0047, 1.61)	0.50 (-0.012, 1.77)	0.41 (-0.0040, 1.11)	0.56 (-0.018, 1.72)	0.051 (-1.86, 0.94)	0.27 (-0.0036, 0.93)
Fertilizer	-	-	0.55 (-0.012, 1.55)	0.28 (-0.0028, 1.21)	0.66 (-0.010, 1.70)	-0.0035 (-2.24, 1.02)	0.24 (-0.0054, 0.81)
Herbicide	-	-	-	0.47 (-0.0024, 1.30)	0.63 (-0.0095, 2.34)	0.15 (-2.47, 2.14)	0.49 (-0.0016, 1.54)
Seed	-	-	-	-	0.52 (-0.094, 1.47)	-0.31 (-2.56, 0.76)	0.14 (-0.041, 0.69)
Labor	-	-	-	-	-	-2..44 (-14.64, 0.85)	-0.1303 (-3.72, 0.99)
Machinery	-	-	-	-	-	-	0.37 (-0.0098, 1.087)

Note: Numbers in parentheses represent the lower and upper bounds of the 90% confidence interval.

Table 4.3: Mean Estimates of the Hicksian Elasticity of Substitution (Cost Efficiency)

	Fuel	Fertilizer	Herbicide	Seed	Labor	Machinery	Land
Fuel	-	0.0021 (-0.42, 0.30)	-0.094 (-0.32, 0.066)	-0.20 (-0.96, 0.38)	-0.039 (-0.046, -0.035)	0.0079 (-0.37, 0.15)	-0.033 (-0.034, -0.031)
Fertilizer	-	-	-0.0062 (-0.13, 0.0061)	0.00063 (-0.011, 0.045)	-0.00011 (-0.0081, 0.028)	-0.014 (-0.021, -0.0076)	-0.042 (-0.049, -0.035)
Herbicide	-	-	-	-0.021 (-0.16, 0.094)	-0.0057 (-0.022, 0.023)	-0.048 (-0.21, 0.085)	-0.035 (-0.036, -0.035)
Seed	-	-	-	-	0.021 (0.0094, 0.027)	-0.0020 (-0.0069, 0.0064)	-0.035 (-0.036, -0.035)
Labor	-	-	-	-	-	0.0063 (-0.0034, 0.017)	-0.033 (-0.034, -0.033)
Machinery	-	-	-	-	-	-	-0.038 (-0.038, -0.037)

Note: Numbers in parentheses represent the lower and upper bounds of the 90% confidence interval.

Table 4.4: Mean Estimates of the Morishima Elasticity of Substitution (Cost Efficiency)

	Fuel	Fertilizer	Herbicide	Seed	Labor	Machinery	Land
Fuel	-	0.015 (-0.0021, 0.058)	0.021 (-0.0095, 0.13)	0.057 (0.029, 0.11)	0.11 (0.047, 0.23)	0.028 (0.0028, 0.079)	-0.030 (-0.033, -0.025)
Fertilizer	-0.013 (-0.043, 0.012)	-	-0.0047 (-0.018, 0.0016)	0.034 (-0.0035, 0.070)	0.055 (-0.044, 0.15)	0.015 (-0.0060, 0.030)	-0.032 (-0.034, -0.030)
Herbicide	-0.0054 (-0.042, 0.027)	0.0039 (-0.00070, 0.021)	-	.0.044 (0.0057, 0.091)	0.082 (-0.022, 0.20)	0.019 (-0.0013, 0.040)	-0.031 (-0.033, -0.029)
Seed	-0.039 (-0.045, -0.034)	-0.0059 (-0.0094, 0.0033)	-0.017 (-0.023, -0.0077)	-	-0.033 (-0.039, -0.023)	-0.0015 (-0.0053, 0.0019)	-0.033 (-0.034, -0.033)
Labor	-0.029 (-0.039, -0.021)	-0.0012 (-0.0077, 0.017)	-0.0091 (-0.020, 0.011)	0.013 (0.0066, 0.016)	-	0.0048 (-0.0032, 0.012)	-0.033 (-0.034, -0.032)
Machinery	-0.0341 (-0.046, -0.012)	-0.0031 (-0.0089, 0.022)	-0.015 (-0.022, 0.0040)	0.0047 (-0.0029, 0.022)	-0.0203 (-0.043, 0.041)	-	-0.033 (-0.0333, -0.0331)
Land	0.50 (0.14, 1.27)	0.26 (0.077, 1.042)	0.40 (0.14, 0.87)	0.66 (0.46, 1.16)	1.75 (1.10, 3.50)	0.30 (0.26, 0.33)	-

Note: Numbers in parentheses represent the lower and upper bounds of the 90% confidence interval.

Chapter 5 - Conclusion

Farmers make rational choices regarding production based on the information that they have available. A farmer's allocation of inputs is based on an understanding of the marginal productivity or the cost of an input. However, this information may be incomplete. Because agriculture is an area of production in which input use is not fixed, but varies across time and location, a farmer may not be able to identify how changes in an input mix alter the amount of output produced, or the cost of production. It is therefore useful, from a farm management perspective, for a farmer to know the ease with which one input can be substituted for another. The elasticity of input substitution is an essential metric for understanding the ease by which inputs can be substituted for one another. Traditionally, elasticities of substitution have been obtained from parametric estimates of production and cost functions, using panel or time series data. In the absence of such data, a parametric approach may be difficult. A solution to this issue is the estimation of elasticities using nonparametric techniques, such as DEA.

This paper developed procedures by which this task can be accomplished. Hicksian production and cost, as well as Morishima cost elasticities for inefficient firms were derived using traditional technical and cost efficiency DEA frameworks. The derivation of these elasticities expands on the usefulness of DEA as a tool in economic analysis and provides a novel contribution to the literature.

An empirical example involving Kansas famers' corn enterprises under reduced tillage served as an illustration of estimating these elasticities. Making use of KFMA data, technical and cost efficiency DEA models were estimated for the 119 farms in the sample. Fuel, fertilizer, herbicide, seed, labor, machinery, and land were used as inputs, with total crop value used as output. Hicksian production elasticities were estimated for each of the inefficient farms not residing on the production frontier. Similarly, Hicksian and Morishima elasticities were estimated for each of the inefficient farms not residing on the cost frontier. Comparisons among the differing results between farms were made for each set of elasticities. The empirical example showed different outcomes among the different elasticity estimation models. The mean values of the Hicksian production elasticities suggested that the inputs exhibited a complementary effect, while the mean values of the Hicksian and Morishima cost elasticities suggested that the inputs exhibited a substitute effect. For both sets of production and cost elasticities, the degree of complementarity or substitution was low, with wide ranges across the sample of farms examined.

A possible avenue to explore in future research is the estimation of elasticities for efficient DMUs residing at the vertices on the production and cost frontiers. In Chapter 3, it was shown that the derivation of elasticities depended on successfully differentiating the Lagrangian technical and cost efficiency equations. Due to the piecewise linear nature of the DEA models, however, traditional differentiation methods are ineffective at the vertices of the frontier. Therefore, estimating elasticities at the vertices, following the methodology presented in this paper, is inadequate– a new estimation strategy is needed. As touched on earlier, this new strategy might include the use of numerical directional derivatives. The vertices of the frontier exist at the intersection of multiple hyper-planes defining the frontier border. By deriving directional derivatives, one could obtain multiple and non-unique elasticities, that represent multiple substitution-possibilities for the efficient farm under observation on the frontier (Podinovski and Førsund 2010).

A second area worth exploring is the estimation of output supply elasticities. Output supply elasticities show the response of a DMU in terms of output, given a change in the output price. Using the output-oriented BCC model, one could show how a change to price effects the production decisions of the DMU. As with the estimation of elasticities of input substitution this is a worthwhile field of inquiry from a farm management perspective. In reality, farmers consider the price of output as well as the price or marginal productivity of inputs when making judgments regarding the application of farm inputs. Such a study would nicely complement this one, giving a more holistic and complete depiction of producer decision-making.

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Appendix A - Derivation of the Elasticity of Substitution for Inefficient Firms (Production Problem)

A.1 – The Hicksian Elasticity of Substitution

BCC(min): $\min_{\theta_o,\lambda} \theta_o$

Subject to: $\boldsymbol{\lambda}'\boldsymbol{x_k} \leq \theta_o x_{k,o}$ ∀ k inputs $\rightarrow \boldsymbol{v}$

$\boldsymbol{\lambda}'\boldsymbol{y_m} \geq y_{m,o}$ ∀ m outputs $\rightarrow \boldsymbol{u}$

$\boldsymbol{e}'\boldsymbol{\lambda} = 1$ $\rightarrow u_0$

$\boldsymbol{\lambda} \geq 0$

Let L denote the Langrangian function for the technical efficiency problem, where θ_o refers to the objective value of firm 0.

$$L = \theta_o + \sum_k v_k\left(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{k,o}\right) - \sum_m u_m\left(\boldsymbol{\lambda}'\boldsymbol{y_m} - y_{m,o}\right) + u_o(\boldsymbol{e}'\boldsymbol{\lambda} - 1),$$

$$f_k = \frac{\partial L}{\partial \boldsymbol{\lambda}'\boldsymbol{x_k}} = v_k\ , and\ \frac{\partial L}{\partial v_k} = \boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{k,o}.$$

$$\sigma_{i,j}^H = \left[\frac{\partial\left({}^{\boldsymbol{\lambda}'\boldsymbol{x_j}}\big/_{\boldsymbol{\lambda}'\boldsymbol{x_i}}\right)}{\partial\left({}^{f_i}\big/_{f_j}\right)}\right]\left[\frac{\left({}^{f_i}\big/_{f_j}\right)}{\left({}^{\boldsymbol{\lambda}'\boldsymbol{x_j}}\big/_{\boldsymbol{\lambda}'\boldsymbol{x_i}}\right)}\right]$$

$$\frac{\partial\left({}^{\boldsymbol{\lambda}'\boldsymbol{x_j}}\big/_{\boldsymbol{\lambda}'\boldsymbol{x_i}}\right)}{\partial\left({}^{f_i}\big/_{f_j}\right)} = \left[\frac{\left(\frac{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j} - \boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}{\boldsymbol{\lambda}'\boldsymbol{x}_i^2}\right)}{\left(\frac{f_j\partial f_i - f_i\partial f_j}{f_j^2}\right)}\right] = \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\frac{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j} - \boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}{f_j\partial f_i - f_i\partial f_j}\right]$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\frac{f_j\partial f_i - f_i\partial f_j}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j} - \boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\frac{f_j\partial f_i}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j} - \boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}} - \frac{f_i\partial f_j}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j} - \boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\left(\frac{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j} - \boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}{f_j\partial f_i}\right)^{-1} - \left(\frac{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j} - \boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}{f_i\partial f_j}\right)^{-1}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\left(\frac{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}{f_j\partial f_i} - \frac{\boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}{f_j\partial f_i}\right)^{-1} - \left(\frac{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}{f_i\partial f_j} - \frac{\boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}{f_i\partial f_j}\right)^{-1}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\left(\left[\frac{f_j\partial f_i}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}\right]^{-1} - \left[\frac{f_j\partial f_i}{\boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}\right]^{-1}\right)^{-1} - \left(\left[\frac{f_i\partial f_j}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}\right]^{-1} - \left[\frac{f_i\partial f_j}{\boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}\right]^{-1}\right)^{-1}\right]^{-1}.$$

$$\left[\frac{f_j\partial f_i}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}\right] = \left[\frac{v_j\partial v_i}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}\right] = \left[\left(\frac{\partial L}{\partial L}\right)\left(\frac{v_j\partial v_i}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}\right)\right] = \left[\frac{(v_j)\left({}^{\partial L}/_{\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}\right)}{(\boldsymbol{\lambda}'\boldsymbol{x_i})\left({}^{\partial L}/_{\partial v_i}\right)}\right] = \left[\frac{v_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})\left({}^{\partial L}/_{\partial v_i}\right)}\right]$$

$$= \left[\frac{v_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})}\right]$$

$$\left[\frac{f_j\partial f_i}{\boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}\right] = \left[\frac{v_j\partial v_i}{\boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}\right] = \left[\left(\frac{\partial L}{\partial L}\right)\left(\frac{v_j\partial v_i}{\boldsymbol{\lambda}'\boldsymbol{x_j}\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}\right)\right] = \left[\frac{(v_j)\left({}^{\partial L}/_{\partial\boldsymbol{\lambda}'\boldsymbol{x_i}}\right)}{(\boldsymbol{\lambda}'\boldsymbol{x_j})\left({}^{\partial L}/_{\partial v_i}\right)}\right] = \left[\frac{v_i v_j}{(\boldsymbol{\lambda}'\boldsymbol{x_j})\left({}^{\partial L}/_{\partial v_i}\right)}\right]$$

$$= \left[\frac{v_i v_j}{(\boldsymbol{\lambda}'\boldsymbol{x_j})(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})}\right]$$

$$\left[\frac{f_i\partial f_j}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}\right] = \left[\frac{v_i\partial v_j}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}\right] = \left[\left(\frac{\partial L}{\partial L}\right)\left(\frac{v_i\partial v_j}{\boldsymbol{\lambda}'\boldsymbol{x_i}\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}\right)\right] = \left[\frac{(v_i)\left({}^{\partial L}/_{\partial\boldsymbol{\lambda}'\boldsymbol{x_j}}\right)}{(\boldsymbol{\lambda}'\boldsymbol{x_i})\left({}^{\partial L}/_{\partial v_j}\right)}\right] = \left[\frac{v_i v_j}{(\boldsymbol{\lambda}'\boldsymbol{x_i})\left({}^{\partial L}/_{\partial v_j}\right)}\right]$$

$$= \left[\frac{v_i v_j}{(\boldsymbol{\lambda}'\boldsymbol{x_i})(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o})}\right]$$

$$\left[\frac{f_i \partial f_j}{\boldsymbol{\lambda' x_j} \partial \boldsymbol{\lambda' x_i}}\right] = \left[\frac{v_i \partial v_j}{\boldsymbol{\lambda' x_j} \partial \boldsymbol{\lambda' x_i}}\right] = \left[\left(\frac{\partial L}{\partial L}\right)\left(\frac{v_i \partial v_j}{\boldsymbol{\lambda' x_j} \partial \boldsymbol{\lambda' x_i}}\right)\right] = \left[\frac{(v_i)\left(\partial L / \partial \boldsymbol{\lambda' x_i}\right)}{(\boldsymbol{\lambda' x_j})\left(\partial L / \partial v_j\right)}\right] = \left[\frac{v_i^2}{(\boldsymbol{\lambda' x_j})\left(\partial L / \partial v_j\right)}\right]$$

$$= \left[\frac{v_i^2}{(\boldsymbol{\lambda' x_j})(\boldsymbol{\lambda' x_k} - \theta_o x_{j,o})}\right]$$

$$\left[\frac{f_j^2}{(\boldsymbol{\lambda' x_i})^2}\right]\left[\left(\left[\frac{f_j \partial f_i}{\boldsymbol{\lambda' x_i} \partial \boldsymbol{\lambda' x_j}}\right]^{-1} - \left[\frac{f_j \partial f_i}{\boldsymbol{\lambda' x_j} \partial \boldsymbol{\lambda' x_i}}\right]^{-1}\right)^{-1} - \left(\left[\frac{f_i \partial f_j}{\boldsymbol{\lambda' x_i} \partial \boldsymbol{\lambda' x_j}}\right]^{-1} - \left[\frac{f_i \partial f_j}{\boldsymbol{\lambda' x_j} \partial \boldsymbol{\lambda' x_i}}\right]^{-1}\right)^{-1}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda' x_i})^2}\right]\left[\left(\left[\frac{v_j^2}{(\boldsymbol{\lambda' x_i})(\boldsymbol{\lambda' x_k} - \theta_o x_{i,o})}\right]^{-1} - \left[\frac{v_i v_j}{(\boldsymbol{\lambda' x_j})(\boldsymbol{\lambda' x_k} - \theta_o x_{i,o})}\right]^{-1}\right)^{-1}\right.$$

$$\left. - \left(\left[\frac{v_i v_j}{(\boldsymbol{\lambda' x_i})(\boldsymbol{\lambda' x_k} - \theta_o x_{j,o})}\right]^{-1} - \left[\frac{v_i^2}{(\boldsymbol{\lambda' x_j})(\boldsymbol{\lambda' x_k} - \theta_o x_{j,o})}\right]^{-1}\right)^{-1}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda' x_i})^2}\right]\left[\left(\left[\frac{(\boldsymbol{\lambda' x_i})(\boldsymbol{\lambda' x_k} - \theta_o x_{i,o})}{v_j^2}\right] - \left[\frac{(\boldsymbol{\lambda' x_j})(\boldsymbol{\lambda' x_k} - \theta_o x_{i,o})}{v_i v_j}\right]\right)^{-1}\right.$$

$$\left. - \left(\left[\frac{(\boldsymbol{\lambda' x_i})(\boldsymbol{\lambda' x_k} - \theta_o x_{j,o})}{v_i v_j}\right] - \left[\frac{(\boldsymbol{\lambda' x_j})(\boldsymbol{\lambda' x_k} - \theta_o x_{j,o})}{v_i^2}\right]\right)^{-1}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda' x_i})^2}\right]\left[\left(\left[\frac{v_i(\boldsymbol{\lambda' x_i})(\boldsymbol{\lambda' x_k} - \theta_o x_{i,o})}{v_i v_j^2}\right] - \left[\frac{v_j(\boldsymbol{\lambda' x_j})(\boldsymbol{\lambda' x_k} - \theta_o x_{i,o})}{v_i v_j^2}\right]\right)^{-1}\right.$$

$$\left. - \left(\left[\frac{v_i(\boldsymbol{\lambda' x_i})(\boldsymbol{\lambda' x_k} - \theta_o x_{j,o})}{v_i^2 v_j}\right] - \left[\frac{v_j(\boldsymbol{\lambda' x_j})(\boldsymbol{\lambda' x_k} - \theta_o x_{j,o})}{v_i^2 v_j}\right]\right)^{-1}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\left(\left[\frac{v_i(\boldsymbol{\lambda}'\boldsymbol{x_i})\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big) - v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_j}\big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big)}{v_i v_j^2}\right]\right)^{-1}\right.$$

$$\left. - \left(\left[\frac{v_i(\boldsymbol{\lambda}'\boldsymbol{x_i})\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big) - v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_j}\big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big)}{v_i^2 v_j}\right]\right)^{-1}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\left(\left[\frac{\Big(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_j}\big)\Big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big)}{v_i v_j^2}\right]\right)^{-1}\right.$$

$$\left. - \left(\left[\frac{\Big(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_j}\big)\Big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big)}{v_i^2 v_j}\right]\right)^{-1}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\left(\frac{v_i v_j^2}{\Big(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_j}\big)\Big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big)}\right)\right.$$

$$\left. - \left(\frac{v_i^2 v_j}{\Big(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_j}\big)\Big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big)}\right)\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\left(\frac{v_i v_j^2\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big)}{\Big(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_j}\big)\Big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big)}\right)\right.$$

$$\left. - \left(\frac{v_i^2 v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big)}{\Big(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_j}\big)\Big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big)}\right)\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\frac{\Big(v_i v_j^2\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big) - v_i^2 v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big)\Big)}{\Big(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_j}\big)\Big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big)}\right]^{-1}$$

$$= \left[\frac{f_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\frac{\big(v_i v_j\big)\Big(v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big) - v_i\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big)\Big)}{\Big(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j\big(\boldsymbol{\lambda}'\boldsymbol{x_j}\big)\Big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o}\big)\big(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}\big)}\right]^{-1}$$

$$= \left[\frac{v_j^2}{(\boldsymbol{\lambda}'\boldsymbol{x_i})^2}\right]\left[\frac{\left(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j(\boldsymbol{\lambda}'\boldsymbol{x_j})\right)(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o})}{(v_i v_j)\left(v_j(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}) - v_i(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})\right)}\right]$$

$$= \left[\frac{v_j\left(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j(\boldsymbol{\lambda}'\boldsymbol{x_j})\right)(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o})}{v_i(\boldsymbol{\lambda}'\boldsymbol{x_i})^2\left(v_j(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}) - v_i(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})\right)}\right]$$

$$\sigma_{i,j}^H = \left[\frac{\left({}^{v_i}\!/_{v_j}\right)}{\left({}^{\boldsymbol{\lambda}'\boldsymbol{x_j}}\!/_{\boldsymbol{\lambda}'\boldsymbol{x_i}}\right)}\right]\left[\frac{v_j\left(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j(\boldsymbol{\lambda}'\boldsymbol{x_j})\right)(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o})}{v_i(\boldsymbol{\lambda}'\boldsymbol{x_i})^2\left(v_j(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}) - v_i(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})\right)}\right]$$

$$\sigma_{i,j}^H = \left[\frac{(\boldsymbol{\lambda}'\boldsymbol{x_i} v_i v_j)\left(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j(\boldsymbol{\lambda}'\boldsymbol{x_j})\right)(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o})}{(\boldsymbol{\lambda}'\boldsymbol{x_j} v_j v_i(\boldsymbol{\lambda}'\boldsymbol{x_i})^2)\left(v_j(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}) - v_i(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})\right)}\right]$$

$$\sigma_{i,j}^H = \left[\frac{(v_i v_j)\left(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j(\boldsymbol{\lambda}'\boldsymbol{x_j})\right)(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o})}{(v_i v_j \boldsymbol{\lambda}'\boldsymbol{x_i}\boldsymbol{\lambda}'\boldsymbol{x_j})\left(v_j(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}) - v_i(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})\right)}\right]$$

$$\boldsymbol{\sigma_{i,j}^H} = \left[\frac{\left(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j(\boldsymbol{\lambda}'\boldsymbol{x_j})\right)(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o})}{(\boldsymbol{\lambda}'\boldsymbol{x_i}\boldsymbol{\lambda}'\boldsymbol{x_j})\left(v_j(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}) - v_i(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})\right)}\right]$$

Appendix B - Derivation of the Elasticity of Substitution for Inefficient Firms (Cost Problem)

B.1 – The Hicksian Elasticity of Substitution

Cost(min): $\min_{z,\lambda} \boldsymbol{w'z}$ (3.6)

Subject to: $z_k - \boldsymbol{\lambda' x_k} \geq 0$ $\forall$ k inputs $\rightarrow \boldsymbol{v_k}$

$\boldsymbol{\lambda' y_m} - y_{m,o} \geq 0$ $\forall$ m outputs $\rightarrow \boldsymbol{u_m}$

$\boldsymbol{e'\lambda} = 1$

$\boldsymbol{\lambda} \geq 0$

Let L denote the Lagrangian function for the cost efficiency problem, where w_k refers to the k^th^ input's price and z_k refers to the k^th^ cost-minimizing level of input for the DMU.

$$L = w'z + \sum_k v_k(\boldsymbol{\lambda' x_k} - z_k) - \sum_m u_m\left(\boldsymbol{\lambda' y_m} - y_{m,o}\right) + u_o(\boldsymbol{e'\lambda} - 1),$$

$$L_k = \frac{\partial L}{\partial w_k} = z_k \text{ and } \frac{\partial L}{\partial z_k} = w_k - v_k$$

$$\sigma_{i,j}^{HC} = \frac{\partial ln\left({}^{w_j}/_{w_i}\right)}{\partial ln\left({}^{L_i}/_{L_j}\right)} = \frac{\partial ln\left({}^{z_j}/_{z_i}\right)}{\partial ln\left({}^{w_i}/_{w_j}\right)} = \frac{\partial lnz_j - \partial lnz_i}{\partial lnw_i - \partial lnw_j} = \frac{\partial lnz_j}{\partial lnw_i - \partial lnw_j} - \frac{\partial lnz_i}{\partial lnw_i - \partial lnw_j}$$

$$= \left(\frac{\partial lnw_i - \partial lnw_j}{\partial lnz_j}\right)^{-1} - \left(\frac{\partial lnw_i - \partial lnw_j}{\partial lnz_i}\right)^{-1}$$

$$= \left(\frac{\partial lnw_i}{\partial lnz_j} - \frac{\partial lnw_j}{\partial lnz_j}\right)^{-1} - \left(\frac{\partial lnw_i}{\partial lnz_i} - \frac{\partial lnw_j}{\partial lnz_i}\right)^{-1}$$

$$= \left(\frac{z_j \partial w_i}{w_i \partial z_j} - \frac{z_j \partial w_j}{w_j \partial z_j}\right)^{-1} - \left(\frac{z_i \partial w_i}{w_i \partial z_i} - \frac{z_i \partial w_j}{w_j \partial z_i}\right)^{-1}$$

$$= \left(\left(\frac{\partial L}{\partial L} \right) \left(\frac{z_j \partial w_i}{w_i \partial z_j} - \frac{z_j \partial w_j}{w_j \partial z_j} \right) \right)^{-1} - \left(\left(\frac{\partial L}{\partial L} \right) \left(\frac{z_i \partial w_i}{w_i \partial z_i} - \frac{z_i \partial w_j}{w_j \partial z_i} \right) \right)^{-1}$$

$$= \left(\frac{z_j \left({\partial L}/{\partial z_j} \right)}{w_i \left({\partial L}/{\partial w_i} \right)} - \frac{z_j \left({\partial L}/{\partial z_j} \right)}{w_j \left({\partial L}/{\partial w_j} \right)} \right)^{-1} - \left(\frac{z_i \left({\partial L}/{\partial z_i} \right)}{w_i \left({\partial L}/{\partial w_i} \right)} - \frac{z_i \left({\partial L}/{\partial z_i} \right)}{w_j \left({\partial L}/{\partial w_j} \right)} \right)^{-1}$$

$$= \left(\frac{z_j \left({\partial L}/{\partial z_j} \right)}{w_i z_i} - \frac{z_j \left({\partial L}/{\partial z_j} \right)}{w_j z_j} \right)^{-1} - \left(\frac{z_i \left({\partial L}/{\partial z_i} \right)}{w_i z_i} - \frac{z_i \left({\partial L}/{\partial z_i} \right)}{w_j z_j} \right)^{-1}$$

$$\boldsymbol{\sigma_{i,j}^{HC}} = \left(\frac{z_j (w_j - v_j)}{w_i z_i} - \frac{z_j (w_j - v_j)}{w_j z_j} \right)^{-1} - \left(\frac{z_i (w_i - v_i)}{w_i z_i} - \frac{z_i (w_i - v_i)}{w_j z_j} \right)^{-1}$$

B.2 – The Morishima Elasticity of Substitution

Let L denote the Langrangian function for the cost efficiency problem, where w_k refers to the k[th] input's price, and z_k refers to the k[th] cost-minimizing input level for the firm.

$$L = \boldsymbol{w}'\boldsymbol{z} + \sum_k v_k (\boldsymbol{\lambda}' \boldsymbol{x_k} - z_k) - \sum_m u_m (\boldsymbol{\lambda}' \boldsymbol{y_m} - y_{m,o}) + u_o (\boldsymbol{e}' \boldsymbol{\lambda} - 1),$$

$$L_k = \frac{\partial L}{\partial w_k} = z_k \text{ and } \frac{\partial L}{\partial z_k} = w_k - v_k$$

$$\sigma_{i,j}^{MC} = \frac{dln \left({z_i}/{z_j} \right)}{dln(w_j)} = \frac{dlnz_i - dlnz_j}{dlnw_j} = \frac{dz_i}{dw_j} \frac{w_j}{z_i} - \frac{dz_j}{dw_j} \frac{w_j}{z_j} = \frac{\left({dL}/{dw_j} \right)}{\left({dL}/{dz_i} \right)} \frac{w_j}{z_i} - \frac{\left({dL}/{dw_j} \right)}{\left({dL}/{dz_j} \right)} \frac{w_j}{z_j}$$

$$\boldsymbol{\sigma_{i,j}^{MC}} = \frac{w_j z_j}{(w_i - v_i) z_i} - \frac{w_j z_j}{(w_j - v_j) z_j}$$