ELASTIC BUCKLING OF TAPERED BEAM

by

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B. S., National Taiwan University, China, 1962

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Applied Mechanics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

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NOTATION

E Young's modulus.

I Moment of inertia of the cross section about the neutral axis through its centroid.

P External axial load.

X, Y Rectangular coordinates, X in the longitudinal direction, Y in the direction of deflection.

L Reference length for the tapering, or the distance from the origin, O, to the larger end of a beam.

a The ratio of the distance from O to smaller end and the distance from O to larger end.

k^2 A dimensionless quantity of $\frac{PL^2}{EI_o}$

α, β Parameters determine the cross section.

ψ, ϕ Parameters control the taper of the beam in width and thickness.

INTRODUCTION

The problem of finding the critical load of a beam-column is important in the analysis and design of modern structures such as airplanes or space vehicles. If the external load P, which is an axial force, is less than the critical value, the beam subject to load P remains straight and undergoes only axial compression. Only a small lateral deflection is produced if a lateral force is applied. The deflection disappears when the lateral force is removed, and the beam returns to its straight form. If P is gradually increased to a certain value, even a small lateral force will produce a large deflection which does not disappear when the lateral force is removed. This phenomenon is called buckling. Therefore, the critical load is defined as the axial force which is sufficient to keep the beam in a slightly bent condition.

The problem of buckling has been discussed for a long time and numerous methods to calculate the critical load of a beam have been developed. Euler's Column Formula (Euler Theory), Energy Method, Beam-column Theory, Rayleigh's method and Numerical Successive Approximation are some of the methods. All can be found in any standard text on elastic stability and most are convenient for solving the problem of a beam having either a uniform cross section or a cross section varying linearly.

For certain reasons, a column with a variable cross section is most practical. In this report, the case where the moment of inertia of the cross section varies according to the power n of the longitudinal coordinate coinciding with axis of the beam X is investigated. The assumption does not lose too much of the generality of the problems involved. The method of Frobenius[2] is applied here to solve the governing differential equation. Then the general solution in the form of Bessel function is obtained. Finally, the critical load is solved for by using the boundary conditions of the system. Several kinds of tapered beams are illustrated in this report.

PROFILES OF THE BEAM

Before investigating the buckling problems of tapered cantilever beams, the general expression of profiles of the beams will be studied. In this investigation, the moment of inertia of a beam varying according to an arbitrary power, n, of the longitudinal coordinate is considered. The relationship may be written as:

$$I = I_o(\frac{X}{L})^n \tag{1.1}$$

where I_o is the moment of inertia at the large end of the beam, L denotes the longitudinal coordinate of the end and X denotes the longitudinal coordinate.

The relation (1.1) can be applied to a general class of cross sections with varying thickness and width.(1) The cross section has two symmetrical axes which are perpendicular to each other; its first quandrant is bounded by the curve of the equation

$$(\frac{z}{b})^\beta + (\frac{y}{h})^\alpha = 1 \tag{1.2}$$

where b represents half of the width and h represents half of the thickness of the beam. These parameters vary according to the relation

$$b = b_o(\frac{X}{L})^\psi \qquad h = h_o(\frac{X}{L})^\phi \tag{1.3}$$

The constants ψ and ϕ are positive but not necessary integers.

The selection of different values for the parameters α and β

in Eq. (1.2) permits the cross section of the beam to be varied from the diamond shape, $\alpha=\beta=1$, through the elliptical shape, $\alpha=\beta=2$, to the rectangular shape, α and $\beta >> 1$. The moment of inertia of this group of cross sections may be expressed in terms of α and β which gives

$$I = \frac{4}{3} b_o h_o \left[\frac{\Gamma(\frac{1}{\alpha}+1)\Gamma(\frac{3}{\beta}+1)}{\Gamma(\frac{1}{\alpha}+\frac{3}{\beta}+1)}\right]\left(\frac{X}{L}\right)^{\psi+3\phi} \tag{1.4}$$

Comparison of Eq. (1.4) with Eq. (1.1) yields the relationship

$$n = \psi + 3\phi \tag{1.5}$$

If the constants ψ and ϕ are not zero, an important group of beam-shapes can be considered as shown in Fig. 1.

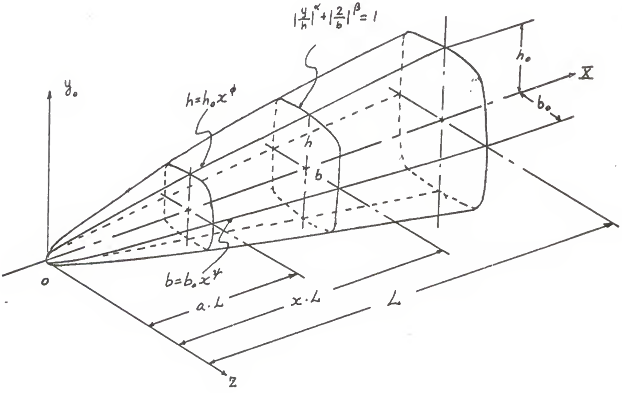

Fig. 1. Tapered beam $\psi \neq \phi \neq 0$.

If the constants are $\psi \neq 0$ and $\phi = 0$, the beam shape can be considered as shown in Fig. 2.

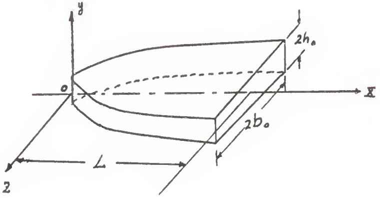

Fig. 2. Tapered beam with $\psi \neq 0$ and $\phi = 0$.

If the constants are $\psi = 0$ and $\phi \neq 0$, the beam shape can be considered as shown in Fig. 3.

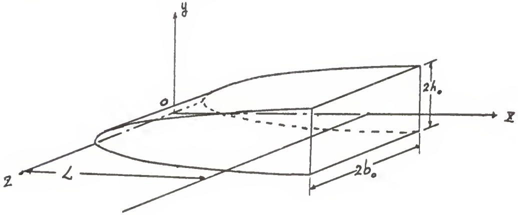

Fig. 3. Tapered beam with $\psi = 0$ and $\phi \neq 0$.

BASIC EQUATION

By using the Euler theory, the differential equation for a bending beam is

$$EI\,\frac{d^2y}{dx^2} = -M. \tag{2.1}$$

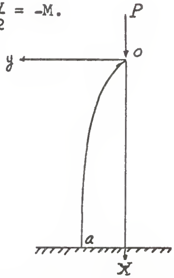

Fig. 4. A cantilever beam under an axial load.

If the coordinate axes are taken as indicated in Fig. 4, where the curve $\widehat{oa}$ represents the center line of the beam and if the relationship of Eq. (1.1) is used, Eq. (2.1) yields

$$EI_o\left(\frac{X}{L}\right)^n\frac{d^2y}{dX^2} = -Py \tag{2.2}$$

By letting x be a dimensionless parameter and defined as $x=X/L$, then Eq. (2.2) becomes

$$\frac{EI_o}{L^2}\,x^n\frac{d^2y}{dx^2} = -Py \tag{2.3}$$

Multiplying each term in Eq. (2.3) by x^{2-n} yields

$$x^2\frac{d^2y}{dx^2} + k^2x^{2-n}y = 0 \tag{2.4}$$

where

$$k^2 = \frac{PL^2}{EI_o}. \tag{2.5}$$

METHOD OF SOLUTION

A. Series Solution

The differential equation Eq. (2.4) has a general solution in the form of series.[2] For simplicity, a differential operator δ, which represents $x\,\frac{d}{dx}$, is introduced. Let the operator operate any function f(x). Then by definition, the following relations are obtained:

$$\delta f = x\,\frac{df}{dx}\ ,$$

$$\delta^2 f = x\,\frac{d}{dx}(x\,\frac{df}{dx}) = x^2\,\frac{d^2 f}{dx^2} + x\,\frac{df}{dx}\ .$$

It may be verified that

$$x^r\,\frac{d^r f}{dx^r} = \delta(\delta-1)(\delta-2)\ .\ .\ .\ (\delta-r+1)f.$$

Using the above results, it is a simple matter to express any linear homogeneous differential equation in terms of δ.

Now $$\delta x^m = x(\frac{dx^m}{dx}) = mx^m$$

and $$\delta(\delta-1)x^m = x^2\,\frac{d^2 x^m}{dx^2} = m(m-1)x^m\ .$$

In the case that $m \geqslant r$, an identity can be derived as follows:

$$\delta(\delta-1)(\delta-2)\ .\ .\ .\ (\delta-r+1)x^m = x^r\,\frac{d^r x^m}{dx^r}$$

$$= m(m-1)\ .\ .\ .\ (m-r+1)x^m \qquad (3.1)$$

By using this notation, Eq. (2.4) becomes

$$[\delta(\delta-1) + k^2x^q]y = 0 \tag{3.2}$$

where q = 2-n.

In general, Eq. (3.2) possesses a series solution in ascending powers of x in the form[2]

$$y = \sum_{r=0}^{\infty} a_r x^{s+r}.$$

In this investigation of a tapered beam, q is not necessarily an integer. Therefore, a more general series solution must be assumed which takes the form[4]

$$y = \sum_{r=0}^{\infty} a_r x^{s+rq}. \tag{3.3}$$

Substituting series Eq. (3.3) into Eq. (3.2), and using the identity of Eq. (3.1) yields

$$\sum_{r=0}^{\infty} a_r[(s+rq)(s+rq-1)+k^2x^q]x^{s+rq} = 0$$

or

$$a_0 s(s-1)x^s + \sum_{r=1}^{\infty}\{a_r(s+rq)(s+rq-1)+k^2a_{r-1}\}x^{s+rq} = 0 \tag{3.4}$$

Equating the coefficients of each power of x in Eq. (3.4) to zero, gives the recurrent relations

$$a_r(s+rq)(s+rq-1)+k^2a_{r-1} = 0 \qquad r \geq 1$$

or

$$a_r = \frac{-k^2}{(s+rq)(s+rq-1)} a_{r-1} \qquad r \geq 1 \tag{3.5}$$

The value of s is determined by equating the coefficient of the first term in Eq. (3.4) to zero, i.e., the coefficient of the term x^s. This gives the indicial equation

$$s(s-1) = 0.$$

$s_1=0$ and $s_2=1$ are two roots of the indicial equation. If s=0, the coefficients a_r in Eq. (3.5) can be written as

$$_0a_r = \frac{-k^2}{rq(rq-1)}\,{}_0a_{r-1} \quad , \qquad r \geq 1 \tag{3.6}$$

or

$$_0a_1 = \frac{-k^2}{q(q-1)}\,{}_0a_0 = \frac{-k^2}{qq(1-\frac{1}{q})}\,{}_0a_0 \quad ,$$

$$_0a_2 = \frac{-k^2}{2q(2q-1)}\,{}_0a_1 = \frac{(-k^2)^2}{2q^2q^2(2-\frac{1}{q})(1-\frac{1}{q})}\,{}_0a_0 \quad ,$$

$$_0a_3 = \frac{-k^2}{3q(3q-1)}\,{}_0a_2 = \frac{(-k^2)^3}{3\,!q^3q^3(3-\frac{1}{q})(2-\frac{1}{q})(1-\frac{1}{q})}\,{}_0a_0 \quad ,$$

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The general term is

$$_0a_r = \frac{(-1)^r\,(k^2)^r}{r!q^{2r}(r-\frac{1}{q})(r-\frac{1}{q}-1)\,\ldots\,(2-\frac{1}{q})(1-\frac{1}{q})}\,{}_0a_0 \quad . \tag{3.7}$$

Eq. (3.7) can be expressed in Gamma functions as

$$_0a_r = \frac{(-1)^r\,(k^2)^r}{r!q^{2r}\Gamma(r-\frac{1}{q}+1)}\,{}_0a_0 \quad .$$

Hence, Eq. (3.3) yields

$$y_o = \sum_{r=o}^{\infty} \frac{(-k^2)^r}{r!q^{2r}\Gamma(r - \frac{1}{q} + 1)} \, {}_oa_o \, x^{rq} \quad . \qquad (3.8)$$

By choosing ${}_oa_o=(\frac{k}{q})^{-\frac{1}{q}}$ and substituting it into Eq. (3.8), the result is

$$y_o = x^{\frac{1}{2}} \sum_{r=o}^{\infty} \frac{(-1)^r}{r!\Gamma(r - \frac{1}{q} + 1)} \left(\frac{k}{q} x^{\frac{q}{2}}\right)^{- \frac{1}{q} + 2r} \quad . \qquad (3.9)$$

For the case s=1, the coefficients a_r in Eq. (3.5) can be written as

$${}_1a_r = \frac{-k^2}{rq(rq+1)} \, {}_1a_{r-1} \qquad r \geqslant 1 \quad . \qquad (3.10)$$

Similarly, Eq. (3.10) can be expressed in terms of ${}_1a_o$ as

$${}_1a_r = \frac{(-1)^r \, (k^2)^r}{r!q^{2r}(r +\frac{1}{q})(r +\frac{1}{q} -1) \, \ldots \, (2 +\frac{1}{q})(1 +\frac{1}{q})} \, {}_1a_o \qquad (3.11)$$

which can be written in gamma functions as

$${}_1a_r = \frac{(-1)^r \, (k^2)^r}{r!q^{2r} \, \Gamma(r + \frac{1}{q} + 1)} \, {}_1a_o \quad .$$

Then Eq. (3.3) yields

$$y_1 = \sum_{r=o}^{\infty} \frac{(-1)^r \, (k^2)^r}{r!q^{2r} \, \Gamma(r + \frac{1}{q} + 1)} \, {}_1a_o \, x^{1+rq} \qquad (3.12)$$

Again choosing ${}_1a_r = (\frac{k}{q})^{\frac{1}{q}}$, Eq. (3.12) then yields

$$y_1 = x^{\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r + \frac{1}{q} + 1)} \left(\frac{k}{q}\, x^{\frac{q}{2}}\right)^{\frac{1}{q} + 2r} \tag{3.13}$$

A comparison of Eqs. (3.9) and (3.13) with the expression for the Bessel function of the order ν (6)

$$J_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2r}$$

$$J_{-\nu}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r-\nu+1)} \left(\frac{x}{2}\right)^{-\nu+2r} \tag{3.14}$$

indicates that Eqs. (3.9) and (3.13) may be expressed in the form

$$y_o = x^{\frac{1}{2}} J_{-\frac{1}{q}} \left(\frac{2k}{q}\, x^{\frac{q}{2}}\right)$$

$$y_1 = x^{\frac{1}{2}} J_{\frac{1}{q}} \left(\frac{2k}{q}\, x^{\frac{q}{2}}\right)$$

In general, s_1 and s_2 differ by an integer; these two series solutions y_o and y_1 are not always independent. Now y_o and y_1 are expressed in Bessel functions, and the two Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ are independent of each other if ν is not an integer. Hence, y_o and y_1 are independent when $\frac{1}{q}$ is not an integer. Then the general solution of Eq. (2.4) is the linear combination of these two independent convergent series (except for x=0) y_o and y_1. Thus

$$y = A_o y_o + A_1 y_1$$

or

$$y = x^{\frac{1}{2}} \left[A_1 J_{\frac{1}{q}}\left(\frac{2k}{q}\, x^{\frac{q}{2}}\right) + A_o J_{-\frac{1}{q}} \left(\frac{2k}{q}\, x^{\frac{q}{2}}\right)\right] \qquad \frac{1}{q} \text{ is not an integer} \tag{3.15}$$

where A_1 and A_o are constants of integration. The solution for the case when 1/q is an integer will be discussed later in this paper.

B. By Changing Variable

Another method for solving Eq. (2.4) is by assuming[5]

$$y = x^{\frac{1}{2}} Y. \tag{3.16}$$

The following relationships may then be derived

$$y' = \tfrac{1}{2}x^{-\frac{1}{2}}\, Y + x^{\frac{1}{2}}\, Y'$$

$$y'' = -\tfrac{1}{4}x^{-\frac{3}{2}}\, Y + x^{-\frac{1}{2}}\, Y' + x^{\frac{1}{2}}\, Y''$$

Here, the primes indicate differention with respect to the dimensionless coordinate x. Substituting those relationships into Eq. (2.4) results in the following equation:

$$x^{\frac{5}{2}}\, Y'' + x^{\frac{3}{2}}\, Y' + (k^2 x^{q+\frac{1}{2}} - \tfrac{1}{4}x^{\frac{1}{2}})\, Y = 0 \tag{3.17}$$

where q = 2-n. Multiplying each term in Eq. (3.17) by $x^{-\frac{1}{2}}$ yields the equation

$$x^2 Y'' + xY' + (k^2 x^q - \tfrac{1}{4})Y = 0 \tag{3.18}$$

Let

$$x = (\tfrac{q}{2k}\, X)^{\frac{2}{q}} \qquad \text{or} \qquad \tfrac{q}{2k}\, X = x^{\frac{q}{2}}$$

then

$$\tfrac{q}{2}\, x^{\frac{q}{2}-1}\, dx = \tfrac{q}{2k}\, dX\ ,$$

$$\frac{dY}{dx} = kx^{\frac{q}{2}-1} \frac{dY}{dX}$$

$$\frac{d^2Y}{dx^2} = k(\frac{q}{2}-1)x^{\frac{q}{2}-1} \frac{dY}{dX} + k^2x^{q-2} \frac{d^2Y}{dX^2} .$$

Using the above relations, Eq. (3.18) now yields

$$(\frac{qX}{2})^2 \frac{d^2Y}{dX^2} + (\frac{q}{2}-1)(\frac{qX}{2}) \frac{dY}{dX} + (\frac{qX}{2}) \frac{dY}{dX} + [k^2(\frac{qX}{2k})^2 - \frac{1}{4}]Y = 0 . \quad (3.19)$$

Simplifying Eq. (3.19) gives

$$X^2 \frac{d^2Y}{dX^2} + X\frac{dY}{dX} + [X^2 - \frac{1}{q^2}]Y = 0 \quad (3.20)$$

which is known simply as Bessel's equation of order 1/q[5] and has as a complete solution

$$Y = A_1J_{\frac{1}{q}}(X) + A_2J_{-\frac{1}{q}}(X) \qquad \frac{1}{q} \text{ is not an integer} \quad (3.21)$$

where A_1 and A_2 are constants of integration. Substituting Eq. (3.21) into Eq. (3.16) results in

$$y = x^{\frac{1}{2}}[A_1J_{\frac{1}{q}}(X) + A_2J_{-\frac{1}{q}}(X)]$$

$$= x^{\frac{1}{2}}[A_1J_{\frac{1}{q}}(\frac{2k}{q} x^{\frac{q}{2}}) + A_2J_{-\frac{1}{q}}(\frac{2k}{q} x^{\frac{q}{2}})] \quad (3.22)$$

Eq. (3.22) has the same form as Eq. (3.15). It checks that either Eq. (3.15) or Eq. (3.22) is the solution for the differential equation Eq. (2.4) when 1/q is not an integer, or, in other words, when 1/(n-2) is not an integer. If 1/q is an integer, the solutions y_0 and y_1 are not independent of each other and Eqs. (3.15) or (3.22) are no longer the complete solution of Eq. (2.4). The solution for such case will be discussed next.

C. The Complete Solution for 1/q as an Integer

In Eq. (3.22), the term $J_{-\frac{1}{q}}(X)$ can be changed to the Bessel function of the second kind of order $\frac{1}{q}$, $Y_{\frac{1}{q}}(X)$ by the following relation:

$$Y_{\frac{1}{q}}(X) = \frac{\cos\frac{\pi}{q}\cdot J_{\frac{1}{q}}(X) - J_{-\frac{1}{q}}(X)}{\sin\frac{\pi}{q}} \qquad (3.23)$$

Thus, the complete solution of Eq. (2.4) can be written as:

$$y = x^{\frac{1}{2}}\left[A_1 J_{\frac{1}{q}}\left(\frac{2k}{q}\,x^{\frac{q}{2}}\right) + A_2 Y_{\frac{1}{q}}\left(\frac{2k}{q}\,x^{\frac{q}{2}}\right)\right]. \qquad (3.24)$$

If $\frac{1}{q} = m$, $m = 0, 1, 2, \ldots$, the right hand side of Eq. (3.23) has an indeterminate form. In this case, $Y_{\frac{1}{q}}(X)$ is interpreted as[2]

$$Y_m(X) = \lim_{\frac{1}{q}\to m} \frac{\cos\frac{\pi}{q}\cdot J_{\frac{1}{q}}(X) - J_{-\frac{1}{q}}(X)}{\sin\frac{\pi}{q}}$$

$$= \left[\frac{\partial}{\partial(\frac{1}{q})}\left\{\cos\frac{\pi}{q}\cdot J_{\frac{1}{q}}(X) - J_{-\frac{1}{q}}(X)\right\} \Big/ \frac{\partial}{\partial(\frac{1}{q})}\sin\frac{\pi}{q}\right]_{\frac{1}{q}=m}$$

by L'Hopital's rule. That is,

$$Y_m(X) = \left(\frac{1}{\pi}\right)\left(\frac{\partial J_{\frac{1}{q}}(X)}{\partial(\frac{1}{q})} - (-1)^{\frac{1}{q}} \frac{\partial J_{-\frac{1}{q}}(X)}{\partial(\frac{1}{q})}\right)_{\frac{1}{q} = m} \qquad (3.26)$$

Now, for any case, the complete solution of Eq. (2.4) is

$$y = x^{\frac{1}{2}}\left[A_1 J_{\frac{1}{q}}\left(\frac{2k}{q} x^{\frac{q}{2}}\right) + A_2 Y_{\frac{1}{q}}\left(\frac{2k}{q} x^{\frac{q}{2}}\right)\right] \qquad (3.27)$$

where A_1 and A_2 are constants of integration,

$J_{\frac{1}{q}}(X)$ is defined as Eq. (3.14),

$Y_{\frac{1}{q}}(X)$ is defined as Eq. (3.23), for $\frac{1}{q}$ is not an integer, and

$Y_{\frac{1}{q}}(X)$ is defined as Eq. (3.26), for $\frac{1}{q} = m$, $m = 0, 1, 2, \ldots$.

In practical use, Eq. (3.26) is equal to[2]

$$Y_m(X) = \left(\frac{2}{\pi}\right)\left[\left\{\log X - \log 2 + \nu\right\} \cdot J_{\frac{1}{q}}(X) - \frac{1}{2}\sum_{s=0}^{m-1}\frac{(m-s-1)!\left(\frac{X}{2}\right)^{-m+2s}}{s!}\right.$$

$$\left. - \frac{1}{2}\sum_{s=0}^{\infty}(-1)^s \frac{(X/2)^{m+2s}}{s!(s+m)!}\left\{\phi(s) + \phi(s+m)\right\}\right] \qquad (3.26')$$

where $\nu = \lim_{s\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{s} - \log s) = 0.57726$, is Euler's constant: and $\phi(s) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{s}$ with $\phi(0) = 0$.

Up to this stage, the general solution of the differential equation Eq. (2.4) has been established for any parameter "n" except for n=2, because when n equals 2, q is zero and 1/q is undefined. But when n=2, Eq. (2.4) is the Euler Differential

Equation and can be solved directly.[3] In this case (n=2), Eq. (2.4) has the form

$$x^2 \frac{d^2y}{dx^2} + k^2 y = 0 \tag{3.28}$$

which can be reduced to an equation with constant coefficients by the substitution

$$x = e^z . \tag{3.29}$$

From Eq. (3.29) $\frac{dz}{dx} = \frac{1}{x}$ is obtained. Therefore,

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}.\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{1}{x}\frac{dy}{dz}) = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x}.\frac{dy}{dz}$$

Substituting the above relation into Eq. (3.28) gives the following differential equation with constant coefficients:

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + k^2 y = 0. \tag{3.30}$$

The general solution of Eq. (3.30) is

$$y = (e^z)^{\frac{1}{2}} (A \sin\beta z + B \cos \beta z) \tag{3.31}$$

where A and B are constants of integration and the quantity

$$\beta = \sqrt{k^2 - \frac{1}{4}} \tag{3.32}$$

is assumed to be real and positive. Using Eq. (3.29), the solution of Eq. (3.31) is expressed in the form

$$y = x^{\frac{1}{2}}[A \sin(\beta \log x) + B \cos (\beta \log x)] . \tag{3.33}$$

Therefore,

$$y' = \frac{1}{2}x^{-\frac{1}{2}}\left[A \sin(\beta \log x) + B \cos(\beta \log x)\right]$$
$$+ x^{-\frac{1}{2}}\left[\beta A \cos(\beta \log x) - B\beta \sin(\beta \log x)\right] \qquad (3.34)$$

Note that there is a singularity point at origin, i.e., x=0, thus the solution (3.34) does not hold for x=0.

CHARACTERISTIC EQUATION

Eq. (3.27) has two constants of integration which can be determined by boundary conditions. Considering a tapered cantilever beam truncated at the location x=a as shown in Fig. 1, the total length of the beam is (1-a)L where L is the reference length for the tapered beam or is the longitudinal coordinate of the far end at which the moment of inertia is maximum. Now the boundary conditions of the cantilever beam are:

$$\begin{aligned} y &= 0 \qquad \text{at } x = a \qquad 0<a<1 \\ y' &= 0 \qquad \text{at } x = 1 \end{aligned} \tag{4.1}$$

For simplicity, Eq. (3.27) can be written as

$$y = A\left(\frac{2k}{q}\,x^{\frac{q}{2}}\right)^{\frac{1}{q}} \cdot J_{\frac{1}{q}}\left(\frac{2k}{q}\,x^{\frac{q}{2}}\right) + B\left(\frac{2k}{q}\,x^{\frac{q}{2}}\right)^{\frac{1}{q}} Y_{\frac{1}{q}}\left(\frac{2k}{q}\,x^{\frac{q}{2}}\right)$$

and

$$y' = A\left(\frac{2k}{q}\,x^{\frac{q}{2}}\right)^{\frac{1}{q}} \cdot J_{\frac{1}{q}-1}\left(\frac{2k}{q}\,x^{\frac{q}{2}}\right) + B\left(\frac{2k}{q}\,x^{\frac{q}{2}}\right)^{\frac{1}{q}} Y_{\frac{1}{q}-1}\left(\frac{2k}{q}\,x^{\frac{q}{2}}\right) \tag{4.2}$$

Applying the boundary conditions of Eq. (4.1) to Eq. (4.2) yields

$$\begin{aligned} 0 &= A\left(\frac{2k}{q}\right)^{\frac{1}{q}} a^{\frac{1}{2}} \cdot J_{\frac{1}{q}}\left(\frac{2k}{q}\,a^{\frac{q}{2}}\right) + B\left(\frac{2k}{q}\right)^{\frac{1}{q}} a^{\frac{1}{2}}\, Y_{\frac{1}{q}}\left(\frac{2k}{q}\,a^{\frac{q}{2}}\right) \\ 0 &= A\left(\frac{2k}{q}\right)^{\frac{1}{q}} \cdot J_{\frac{1}{q}-1}\left(\frac{2k}{q}\right) + B\left(\frac{2k}{q}\right)^{\frac{1}{q}} Y_{\frac{1}{q}-1}\left(\frac{2k}{q}\right) \end{aligned} \tag{4.3}$$

Eq. (4.3) can be rewritten in matrix form as

$$\begin{pmatrix} a^{\frac{1}{2}} J_{\frac{1}{q}}\left(\frac{2k}{q} a^{\frac{q}{2}}\right) & a^{\frac{1}{2}} Y_{\frac{1}{q}}\left(\frac{2k}{q} a^{\frac{q}{2}}\right) \\ J_{\frac{1}{q}-1}\left(\frac{2k}{q}\right) & Y_{\frac{1}{q}-1}\left(\frac{2k}{q}\right) \end{pmatrix} \begin{Bmatrix} A\left(\frac{2k}{q}\right)^{\frac{1}{q}} \\ B\left(\frac{2k}{q}\right)^{\frac{1}{q}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \qquad (4.4)$$

If solutions A and B in Eq. (4.4) are not both equal to zero, the determinant of coefficient of A and B vanishes; that is

$$\begin{vmatrix} a^{\frac{1}{2}} J_{\frac{1}{q}}\left(\frac{2k}{q} a^{\frac{q}{2}}\right) & a^{\frac{1}{2}} Y_{\frac{1}{q}}\left(\frac{2k}{q} a^{\frac{q}{2}}\right) \\ J_{\frac{1}{q}-1}\left(\frac{2k}{q}\right) & Y_{\frac{1}{q}-1}\left(\frac{2k}{q}\right) \end{vmatrix} = 0 \qquad (4.5)$$

Expanding the determinant and dividing the result by the factor $a^{\frac{1}{2}}$ yields

$$J_{\frac{1}{q}}\left(\frac{2k}{q} a^{\frac{q}{2}}\right) \cdot Y_{\frac{1}{q}-1}\left(\frac{2k}{q}\right) - Y_{\frac{1}{q}}\left(\frac{2k}{q} a^{\frac{q}{2}}\right) \cdot J_{\frac{1}{q}-1}\left(\frac{2k}{q}\right) = 0 \qquad (4.6)$$

This is the characteristic equation of this problem, the roots of which give the characteristic values. From these characteristic values the critical load can be calculated.

Eq. (4.6) does hold either when $1/q$ is an integer or when it is not. If $1/q$ is not an integer, the characteristic equation can be expressed as:

$$J_{\frac{1}{q}}\left(\frac{2k}{q} a^{\frac{q}{2}}\right) \cdot J_{-\left(\frac{1}{q}-1\right)}\left(\frac{2k}{q}\right) + J_{-\frac{1}{q}}\left(\frac{2k}{q} a^{\frac{q}{2}}\right) \cdot J_{\frac{1}{q}-1}\left(\frac{2k}{q}\right) = 0 \qquad (4.7)$$

EXAMPLES

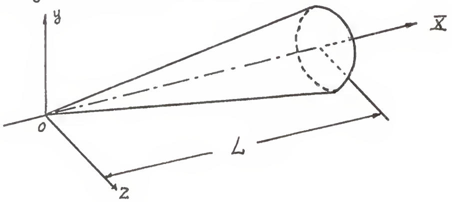
In this section, several values of "n" are taken to illustrate how to evaluate the critical loads.

A. For the Case N = 4

For the first example, a truncated cone as shown in Fig. 5 is used. In this case,

$$b = b_o(\frac{X}{L}) \qquad \text{and} \qquad h = h_o(\frac{X}{L})$$

where b_o and h_o are equal, and $\psi=\phi=1$.



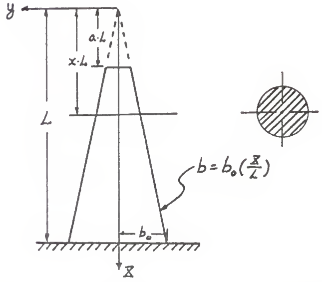


Fig. 5. A truncated cone column, $\psi=\phi=1$.

No matter what the values of b_0, h_0 are, the value of n depends only on ψ and ϕ. From Eq. (1.5),

$$n = \psi + 3\phi = 1 + 3 = 4.$$

Then substituting q=2-n=-2 into the characteristic equation Eq. (4.7) gives

$$J_{-\frac{1}{2}}(-ka^{-1})\ J_{\frac{3}{2}}(-k) + J_{\frac{1}{2}}(-ka^{-1})\ J_{-\frac{3}{2}}(-k) = 0. \qquad (5.1)$$

All of those Bessel's functions can be expressed in a series of trigonometric functions.[7]

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x$$

$$J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x$$

$$J_{\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left(\frac{\sin x}{x} - \cos x\right)$$

$$J_{-\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left(-\sin x - \frac{\cos x}{x}\right) \quad .$$

From the above relationships, Eq. (5.1) can be simplified to

$$\cos\left(\frac{k}{a}\right)\left(\frac{\sin k}{k} - \cos k\right) - \sin\left(\frac{k}{a}\right)\left(\sin k + \frac{\cos k}{k}\right) = 0. \qquad (5.2)$$

Dividing each term in Eq. (5.2) by $[\cos(\frac{k}{a})\cos k]/k$, Eq. (5.2) yields

$$\tan k - k\cdot\tan\left(\frac{k}{a}\right)\tan k - k - \tan\left(\frac{k}{a}\right) = 0$$

or

$$\frac{\tan k-\tan(\frac{k}{a})}{1+\tan(\frac{k}{a})\tan k} = \tan(k-\frac{k}{a}) = k \qquad (5.3)$$

Let $\theta = \frac{k}{a} - k = (\frac{1-a}{a})k$, then Eq. (5.3) becomes

$$\tan\theta = -k = \frac{a}{a-1}\theta \ . \qquad (5.4)$$

The value of θ in Eq. (5.4) can be solved easily (see page 36) if any particular value of "a" is given. Knowing θ, the critical load can be found from Eq. (2.5).

$$\frac{PL^2}{EI_o} = k^2 = (\frac{a}{a-1})^2\theta^2 \ . \qquad (5.a)$$

Eq. (5.a) can be rewritten as

$$P_{cr} = (\frac{a}{a-1})^2\theta^2 \frac{EIo}{L^2} = M\frac{EIo}{L^2} \qquad (5.b)$$

where $M = (\frac{a}{a-1})^2\theta^2$. Table 1 shows the values of M with different "a".

Table 1. The value of θ and M at first mode for n=4.

a	θ	M
0.1	2.836	0.099
0.2	2.570	0.413
0.3	2.352	1.016
0.4	2.175	2.102
0.5	2.029	4.116
0.6	1.908	8.191
0.7	1.804	17.718
0.8	1.716	47.114
0.9	1.638	217.326

Values of M of higher mode are listed in Table 2.

Table 2. Value of M of higher mode for n = 4.

M \ mode a	1st	2nd	3rd	4th	5th
0.1	0.099	0.404	0.926	1.676	2.663
0.2	0.413	1.729	4.308	8.030	12.794
0.3	1.016	4.849	12.150	23.044	37.554
0.4	2.102	11.127	28.719	55.052	90.150
0.5	4.116	24.139	63.660	122.890	201.852

B. The Case When $N = \frac{4}{3}$

In this example, a truncated pyramid is considered, but both the thickness and width of the pyramid vary according to $\frac{1}{3}$ power of the longitudinal coordinate, i.e., two constants $\psi=\phi=\frac{1}{3}$. The shape is shown in Fig. 6.

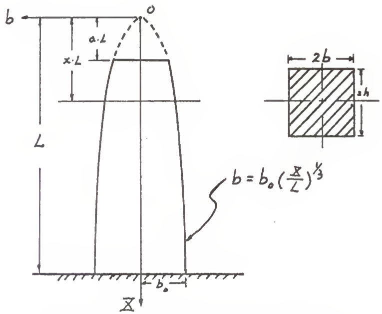


Fig. 6. Pyramid with $\psi=\phi=\frac{1}{3}$.

From Eq. (1.5), $n = \nu + 3\phi = \frac{4}{3}$. Substituting $q = 2 - n = \frac{2}{3}$ into characteristic equation (4.7) gives

$$J_{\frac{3}{2}}(3ka^{\frac{1}{3}}) \cdot J_{-\frac{1}{2}}(3k) + J_{-\frac{3}{2}}(3ka^{\frac{1}{3}}) \cdot J_{\frac{1}{2}}(3k) = 0 \qquad (5.5)$$

Rewriting Eq. (5.5) in trigonometric function and simplifying it results in

$$\cos 3k\left[\frac{\sin 3ka^{\frac{1}{3}}}{3ka^{\frac{1}{3}}} - \cos 3ka^{\frac{1}{3}}\right] - \sin 3k\left[\sin 3ka^{\frac{1}{3}} + \frac{\cos 3ka^{\frac{1}{3}}}{3ka^{\frac{1}{3}}}\right] = 0 \qquad (5.6)$$

Dividing each term in Eq. (5.6) by factor $(\cos 3k \cdot \cos 3ka^{\frac{1}{3}})/3ka^{\frac{1}{3}}$, yields

$$\tan(3ka^{\frac{1}{3}}) - 3ka^{\frac{1}{3}} - 3ka^{\frac{1}{3}}\tan(3ka^{\frac{1}{3}})\tan 3k - \tan 3k = 0$$

or

$$\frac{\tan(3ka^{\frac{1}{3}}) - \tan 3k}{1 + \tan(3ka^{\frac{1}{3}})\tan(3k)} = \tan(3ka^{\frac{1}{3}} - 3k) = 3ka^{\frac{1}{3}} \qquad (5.7)$$

Let $\theta = 3k - 3ka^{\frac{1}{3}} = 3k(1-a^{\frac{1}{3}})$, then Eq. (5.7) becomes

$$\tan\theta = \left(\frac{a^{\frac{1}{3}}}{a^{\frac{1}{3}}-1}\right)\theta \qquad (5.8)$$

Eq. (5.8) can be solved for θ for any particular value of "a" (see page 37). The critical load is obtained from the following relation:

$$\frac{PL^2}{EI_o} = \frac{\theta^2}{9(a^{\frac{1}{3}}-1)^2} = M \qquad \text{or} \qquad P_{cr} = M\,\frac{EI_o}{L^2}$$

The value of M of several values of "a" are listed in Table 3.

Table 3. The values of M for n = 4/3.

M \ mode / a	1st	2nd	3rd	4th	5th
0.1	1.671	9.451	24.750	47.673	78.230
0.2	2.387	15.208	40.666	78.838	129.731
0.3	3.419	23.570	63.721	123.936	204.222
0.4	5.025	36.755	100.397	195.073	321.722
0.5	7.730	59.324	162.397	317.000	523.136

C. The Case for $N = \frac{1}{2}$

In the third example, a plate beam of uniform thickness is considered. The width of the plate varies according to $\frac{1}{2}$ power of the longitudinal coordinate. For a uniform thickness, $\phi = 0$. The other constant is $\psi = \frac{1}{2}$. The shape of this beam is shown in Fig. 7 and also in Fig. 2.

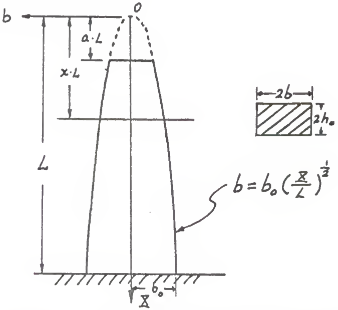


Fig. 7. Uniform thickness beam with $\psi = \frac{1}{2}$.

From Eq. (1.5), $n = \frac{1}{2}$ and $q = \frac{3}{2}$. Substituting this into characteristic equation (4.7) yields

$$J_{\frac{2}{3}}(ka^{\frac{3}{4}}) \cdot J_{\frac{1}{3}}(\tfrac{4}{3}k) + J_{-\frac{2}{3}}(\tfrac{4}{3}ka^{\frac{3}{4}}) \cdot J_{-\frac{1}{3}}(\tfrac{4}{3}k) = 0 \tag{5.9}$$

Where $J_{\frac{2}{3}}(x)$ and $J_{-\frac{2}{3}}(x)$ cannot be expressed in a trigonometric function for small k, the relationships

$$J_{\nu}(x) = \sum_{r=o}^{\infty} \frac{(-1)^r (x/2)^{\nu+2r}}{r!\,\Gamma(r+\nu+1)} ,$$

$$J_{-\nu}(x) = \sum_{r=o}^{\infty} \frac{(-1)^r (x/2)^{-\nu+2r}}{r!\,\Gamma(r-\nu+1)} \tag{5.10}$$

are used. Then the values of k in Eq. (5.9) can be solved by a computer if particular value of "a" is given. (The computer program is given in the Appendix, pp. 38-40.) The results are listed in Table 4.

$$k^2 = \frac{PL^2}{EI_o} \qquad \text{or} \qquad P_{cr} = k^2 \frac{EI_o}{L^2}$$

Table 4. Critical load $\frac{PL^2}{EI_o}$ for $n = \frac{1}{2}$.

k^2 mode / a	1st	2nd	3rd	4th	5th
0.1	2.556	19.691	44.305	66.381	92.340
0.2	3.399	26.910	70.796	128.680	225.938
0.3	4.785	33.785	85.562	183.517	318.399
0.4	6.566	39.062	127.972	257.001	437.211
0.5	9.765	74.121	189.062	412.597	648.657

D. The Case When $N = \frac{3}{2}$.

A beam of uniform width but whose thickness varies according to $\frac{1}{2}$ power of the longitudinal coordinate is considered here. The two constants are $\gamma = 0$ and $\phi = 1/2$. From Eq. (1.5), the value of n is $\frac{3}{2}$, and $q = 2 - n = \frac{1}{2}$. Now $\frac{1}{q} = 2$ is an integer and the characteristic equation (4.6) is used. A beam of this type is shown in Figs. 3 and 8.

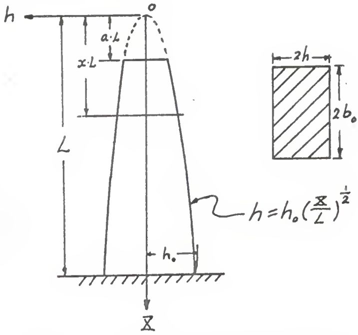


Fig. 8. Beam with uniform width when $\phi = 1/2$.

When $q = \frac{1}{2}$, Eq. (4.6) yields

$$J_2(4ka^{\frac{1}{4}}) \cdot Y_1(4k) - Y_2(4ka^{\frac{1}{4}}) \cdot J_1(4k) = 0 \qquad (5.11)$$

Eq. (5.11) cannot be expressed in a trigonometric function for small argument, so the relationships of Eqs. (3.27') and (5.10) are used. Also mathematical table[8] can be used to solve Eq. (5.11). (The detail processes are shown in the Appendix, pp. 41-43.) If the argument is sufficiently large, the following relationships can be used:[9]

$$J_m(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{\pi}{4} - \frac{m\pi}{2}\right)$$

$$Y_m(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - \frac{\pi}{4} - \frac{m\pi}{2}\right).$$

The computer program is shown on page 44. Results obtained are shown in Table 5.

Table 5. Critical load $k^2 = \frac{PL^2}{EI_o}$ for $n = \frac{3}{2}$.

k^2 \ mode / a	1st	2nd	3rd	4th	5th
0.1	1.562	7.362	20.135	39.454	65.225
0.2	2.281	12.656	35.161	68.903	113.889
0.3	3.1546	20.567	57.073	111.896	185.003
0.4	4.622	33.152	92.040	180.356	298.102
0.5	7.426	54.853	152.368	298.641	493.673

E. The Case When N = 2

In the previous examples, the cases where $\frac{1}{q}$ is an integer and where it is not were discussed. Now the case when $q = 0$ is considered. Several combinations of ψ and ϕ yields $n = 2$, for example $\psi = \emptyset = 1/2$. If $\psi = \emptyset = 1/2$, the shape is as shown in Fig. 6 except that the width and thickness vary according to $\frac{1}{2}$ power of the longitudinal axis.

When n = 2, the solution of Eq. (2.4) is expressed in Eqs. (3.34) and (3.35). Using the boundary conditions of Eq. (4.1) in Eqs. (3.34) and (3.25), results in the following characteristic equation:

$$\tan \theta = \frac{2\theta}{\log a} \qquad \theta = \beta \log a \tag{5.12}$$

where $\beta = \sqrt{k^2 - \frac{1}{4}}$. If any particular value of "a" is given, θ and β can be solved from Eq. (5.12). Knowing β, the critical loads $\frac{PL^2}{EI_o}$ are found from the following relation:

$$\frac{PL^2}{EI_o} = \beta^2 + \frac{1}{4} = \frac{\theta^2}{(\log a)^2} + \frac{1}{4} = M$$

or

$$P_{cr} = M \frac{EI_o}{L^2}$$

The results M are listed in Table 6.

Table 6. The values of M for n = 2.

M mode / a	1st	2nd	3rd	4th	5th
0.1	1.064	4.855	12.312	23.485	38.378
0.2	1.734	9.428	24.676	47.543	78.027
0.3	2.690	16.385	43.630	84.485	138.950
0.4	4.185	27.777	74.807	145.341	239.385
0.5	6.732	47.899	130.077	253.334	417.674

There is another particular and very useful structure whose n equals 2. It is a built-up column consisting of four angles connected by diagonals. In this case, the cross sectional area of the column remains constant and the moment of inertia is approximately proportional to the square of the distance of the centroids of the angles from the axes of symmetry of the cross section.[3]

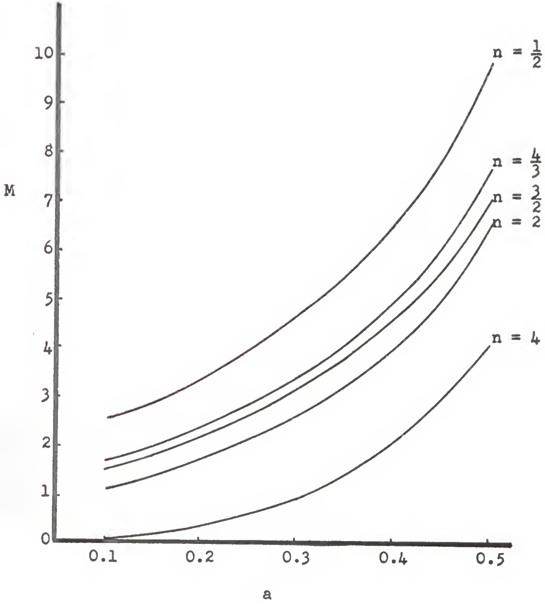


Fig. 9. Relation between a and M for several n (First Mode).

DISCUSSION AND CONCLUSION

In general, the critical load of a tapered or a tapered truncated beam with various cross sections can be determined from the characteristic equation (4.6). The characteristic equation is in terms of the Bessel function, in general, and the order of those functions depends on $\frac{1}{q}$, where $q = 2 - \psi - 3\phi$. Hence, the coefficients of the series as well as the order of the Bessel function depend directly on ψ and ϕ which control the taper of the beam in thickness and in width, respectively.

Characteristic equation (4.6) can be used for any value of "q"; that is, the value which is any combination of the two parameters ψ and ϕ, except the following two cases:

(1) The first case is that q equals zero. If $q = 0$, $\frac{1}{q}$ is undefined and Eq. (4.6) does not hold. However, for the particular case $\psi + 3\phi = 2$ and differential equation (2.4) yields the Euler differential equation which can be solved easily.

(2) The second limitation is when q equals infinity, or $\frac{1}{q} = 0$. The argument of the Bessel function $(\frac{2k}{q} a^{\frac{q}{2}})$ is always zero no matter what the finite value of "k". Actually, this does not occur in practice. If q equals infinity, either ψ or ϕ or both must equal infinity. Then it is not a beam or column.

Previous examples show that the smaller the value of "n", the higher critical load is obtained if the moment of inertia of the base section Io and reference length L are the same.

ACKNOWLEDGMENT

The author wishes to express his sincere appreciation and thanks to his major professor, Dr. Chi-lung Huang, for suggesting this problem and for his encouragement and direction through the course of this work.

BIBLIOGRAPHY

(1) Worley, W. J., and Breuer, F. D. "Areas, Centroids and Inertias for a Family of Elliptic-Type Closed Curve," Product Engineering, V. 28, N. 8, August, 1957, pp. 141-144.

(2) Irving, J., and Mullineux, N. Mathematics in Physics and Engineering, Academic Press, N. Y., 1959.

(3) Timoshenko, S. Theory of Elastic Stability, 2nd ed., McGraw-Hill Book Company, Inc., N. Y., 1961.

(4) Wang, H. A General Treatment of the Transverse Vibration of Tapered Beams, Thesis for the degree of Doctor of Philosophy in Theoretical and Applied Mechanics in the Graduate College of the University of Illinois, Urbana, Illinois, 1966.

(5) Wylie, C. R., Jr. Advanced Engineering Mathematics, 2nd ed., McGraw-Hill Book Company, Inc., N. Y., 1960.

(6) Johnson, D. E., and Johnson, J. R. Mathematical Methods in Engineering and Physics, The Ronald Press Company, N. Y., 1965.

(7) Dwight, H. B. Tables of Integrals and Other Mathematical Data, 3rd ed., The Macmillan Company, 1957.

(8) Jahnke, Emde, Losch. Table of Higher Function, 6th ed., McGraw-Hill Book Company, Inc., N. Y., 1960.

(9) Watson, G. N. A Treatise on the Theory of Bessel Functions, 2nd ed., Cambridge University Press, London, 1944.

APPENDIX

(I)

```
C  (  FORGO PROGRAM OF CRITICAL LOADS OF N=4.0
      DIMENSION Y(5),Y1(5),Y2(8),Y3(8),Z1(5),B(5) ., P(10)
   40 FORMAT(3F10.5)
      A=0.1
    2 DO 51 I=1,5
      X=A/(A-1.0)
      Z=I-1
      Y1(I)=1.6+Z*3.1416
      Y2(I)=Y1(I)+1.5
    1 Y3(I)=(Y1(I)+Y2(I))/2.0
      AA  =COS(Y3(I)-Z*3.1416)
      BB  =SIN(Y3(I)-Z*3.1416)
      Y(I)=BB/AA
      Z1(I)=X*Y3(I)
      B(I)=ABS(Y(I)-Z1(I))
      IF(B(I)-0.001)50,50,10
   10 IF(Y(I)-Z1(I))20,50,30
   20 Y1(I)=Y3(I)
      GO TO 1
   30 Y2(I)=Y3(I)
      GO TO 1
   50 P(I)=(X**2)*(Y3(I)**2)
   51 PUNCH 40,A,Y3(I),P(I)
      A=A+0.1
      IF(A-0.5)2,2,3
    3 STOP
      END
C  C  RESULTS OF N=4.0
       A        THETA       PL**2/EI
     .10000    2.83633      .09932
     .10000    5.71719      .40353
     .10000    8.65884      .92562
     .10000   11.65322     1.67651
     .10000   14.68715     2.66312
     .20000    2.57046      .41295
     .20000    5.35391     1.79152
     .20000    8.30288     4.30861
     .20000   11.33480     8.02985
     .20000   14.40801    12.97442
     .30000    2.35220     1.01623
     .30000    5.13857     4.84989
     .30000    8.13332    12.15017
     .30000   11.20095    23.04391
     .30000   14.29897    37.55398
     .40000    2.17459     2.10170
     .40000    5.00362    11.12723
     .40000    8.03847    28.71870
     .40000   11.12956    55.05265
     .40000   14.24214    90.15045
     .50000    2.02883     4.11616
     .50000    4.91317    24.13924
     .50000    7.97869    63.65949
     .50000   11.08556   122.88962
     .50000   14.20746   201.85198
```

(II)

```
C  CFORGO PROGRAM OF CRITICAL LOADS FOR N=4.0/3.0
      DIMENSION Y(5),Y1(5),Y2(8),Y3(8),Z1(5),B(5) , P(10)
   40 FORMAT(3F10.5)
    2 DO 51 I=1,5
      A=0.1
      X=A**(1.0/3.0)/(A**(1.0/3.0)-1.0)
      Z=I-1
      Y1(I)=1.5708+Z*3.1416
      Y2(I)=Y1(I)+1.5
    1 Y3(I)=(Y1(I)+Y2(I))/2.0
      AA  =COS(Y3(I)-Z*3.1416)
      BB  =SIN(Y3(I)-Z*3.1416)
      Y(I)=BB/AA
      Z1(I)=X*Y3(I)
      B(I)=ABS(Y(I)-Z1(I))
      IF(B(I)-0.001)50,50,10
   10 IF(Y(I)-Z1(I))20,50,30
   20 Y1(I)=Y3(I)
      GO TO 1
   30 Y2(I)=Y3(I)
      GO TO 1
   50 P(I)=Y3(I)**2/(9.0*(A**(1.0/3.0)-1.0)**2)
   51 PUNCH 40,A,Y3(I),P(I)
      A=A+0.1
      IF(A-0.5)2,2,3
    3 STOP
      END
C  C  RESULTS /F N=4.0/3.0
```

A	THETA	PL**2/EI
.10000	2.07791	1.67085
.10000	4.94192	9.45097
.10000	7.99737	24.75024
.10000	11.09923	47.67289
.10000	14.21821	78.23035
.20000	1.92428	2.38664
.20000	4.85753	15.20831
.20000	7.94315	40.66636
.20000	11.05971	78.83834
.20000	14.18720	129.73093
.30000	1.83383	3.41943
.30000	4.81460	23.56997
.30000	7.91629	63.72089
.30000	11.04029	123.93654
.30000	14.17203	204.22180
.40000	1.76993	5.02478
.40000	4.78688	36.75457
.40000	7.89919	100.08556
.40000	11.02798	195.07348
.40000	14.16241	321.72219
.50000	1.72072	7.73001
.50000	4.76687	59.32353
.50000	7.88694	162.39692
.50000	11.01918	317.00001
.50000	14.15556	523.13597

(III) SOLUTION FOR N = $\frac{1}{2}$.

i) If the argument $(\frac{4k}{3}a^{\frac{3}{4}})$ is small, the following program is used.

```
C  C  FORGO PROGRAM FOR CRITICAL LOAD WHEN N=1/2 (LOWER MODE)
      DIMENSION P(N),P1(N),P2(N),PP1(N),PP2(N)
      DIMENSION B1(N),B2(N),B3(N),B4(N)
      DIMENSION C1(N),C2(N),C3(N),C4(N)
  101 FORMAT(1I2)
  102 FORMAT(E20.10)
  103 FORMAT(3F15.5)
   15 FORMAT(3F10.5)
      READ 101 N
      READ 102(P(I) I=1,N)
      READ102(P1(I) I=1,N)
      READ102(P2(I)I=1,N)
      READ102(PP1(I) I=1,N)
      READ 102(PP2(I) I=1,N)
      DO 1 I=1,10
      Z=I
      B1(I)=0.67+2.0*(Z-1.0)
    1 C1(I)=2.0**B1(I)*P(I)*P2(I)
      DO 2 I=1,10
      Z=I
      B2(I)=0.333+2.0*(Z-1.0)
    2 C2(I)=2.0**B2(I)*P(I)*P1(I)
      DO 3 I=1,10
      Z=I
      B3(I)=2.0*(Z-1.0)-0.667
    3 C3(I)=2.0**B3(I)*P(I)*PP2(I)
      DO 4 I=1,10
      Z=I
      B4(I)=2.0*(Z-1.0)-0.333
    4 C4(I)=2.0**B4(I)*P(I)*PP1(I)
      DELX=0.5
      READ 103 A1,W1,X
   30 A=A1**0.75
      T=4.0*X/3.0

      AT=A*T
   10 Z1=0.0
      ZZ1=0.0
      DO11 I=1,5
      M=2*I-1
      Z1=Z1+(AT)**B1(M)/C1(M)
      N=2*I
   11 ZZ1=ZZ1+(AT)**B1(N)/C1(N)
      Z1=Z1-ZZ1
```

```
      Z2=0.0
      ZZ2=0.0
      DO21 I=1,5
      M=2*I-1
      Z2=Z2+T**B2(M)/C2(M)
      N=2*I
   21 ZZ2=ZZ2+T**B2(N)/C2(N)
      Z2=Z2-ZZ2
      Z3=0.0
      ZZ3=0.0
      DO31 I=1,5
      M=2*I-1
      Z3=Z3+(AT)**B3(M)/C3(M)
      N=2*I
   31 ZZ3=ZZ3+(AT)**B3(N)/C3(N)
      Z3=Z3-ZZ3
      Z4=0.0
      ZZ4=0.0
      DO41 I=1,5
      M=2*I-1
      Z4=Z4+T**B4(M)/C4(M)
      N=2*I
   41 ZZ4=ZZ4+T**B4(N)/C4(N)
      Z4=Z4-ZZ4
      Y=Z1*Z2+Z3*Z4
      IF(W1*Y)130,150,125
  125 W1=Y
      X=X+DELX
      GO TO 30
  130 W=ABS(Y)
      IF(W-0.01)150,150,120
  120 X=X-DELX
      DELX=0.5*DELX
      X=X+DELX
      GO TO 30
  150 XX=X*X
      PUNCH 15,A1,X,XX
      STOP
      END
C   C
    .10000    1.59961    2.55872
    .10000    4.43750   19.69140
    .10000    6.65625   44.30499
    .10000    8.08594   65.38177


    .20000    1.84375    3.39922
    .20000    5.18750   26.91015
    .30000    2.18750    4.78515
    .30000    5.81250   33.78515
    .40000    2.56250    6.56640
    .40000    6.25000   39.06250
    .50000    3.12500    9.76562
```

ii) If the argument $(\frac{4k}{3}a^{\frac{3}{4}})$ is sufficiently large, the fol-owing relationship is used.

$$J_m(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{\pi}{4} - \frac{m\pi}{2}\right),$$

and the computer program of this example is written below.

```
C   C   FORGO PROGRAM OF CRITICAL LOAD FOR N=1/2 (HIGHER MODE)
        DELX=0.5
        W=0.01
        A1=0.5
        X=25.0
        A=A1**0.75
      1 T=4.0*X/3.0
        AT=A*T
        Z1=(2.0/(3.14159*AT))**0.5*COS(AT-0.5833*3.14159)
        Z2=(2.0/(3.14159*T))**0.5*COS(T-0.41667*3.14159)
        Z3=(2.0/(3.14159*AT))**0.5*COS(AT+0.0833*3.14159)
        Z4=(2.0/(3.14159*T))**0.5*COS(T-0.08333*3.14159)
        Z=Z1*Z2+Z3*Z4
        IF(W*Z)30,50,25
     25 W=Z
        X=X+DELX
        GO TO 1
     30 W1=ABS(Z)
        IF(W1-0.001)50,50,20
     20 X=X-DELX
        DELX=0.5*DELX
        X=X+DELX
        GO TO 1
     50 XX=X**2
        PUNCH 40,A1,XX
        PRINT40,A1,XX
     40 FORMAT(2F10.5)
        STOP
        END
C   C
       .10000   92.34009
       .20000   70.79645
       .20000  128.68066
       .20000  225.93846
       .30000   85.56250
       .30000  163.51782
       .30000  316.59941
       .40000  127.97266
       .40000  257.00098
       .40000  431.21118
       .50000   74.12134
       .50000  169.[illegible]6250
       .50000  417.59700
       .50000  648.65723
```

(IV) SOLUTION FOR $N = \frac{3}{2}$.

When $n = 3/2$, $\frac{1}{q} = 2$ is an integer. The characteristic equation (4.6) yields

$$J_2(4ka^{\frac{1}{4}}) \cdot Y_1(4k) - J_1(4k) \cdot Y_2(4ka^{\frac{1}{4}}) = 0. \qquad \text{(a)}$$

The values J_o, J_1, Y_o, and Y_1 can be found from table, and the values J_2 and Y_2 can be calculated by the following relationships:

$$\frac{2n}{x} Y_n(x) = Y_{n-1}(x) + Y_{n+1}(x)$$

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x),$$

and let $n = 1$.

The method of using mathematical table to solve Eq. (a) is illustrated below. Let $x = 4k$, $z = xa^{\frac{1}{4}}$ and

$$F = J_2(z) \cdot Y_1(x) - J_1(x) \cdot Y_2(z).$$

If $a = 0.1$,

x	$J_1(x)$	$Y_1(x)$	z	$J_2(z)$	$Y_2(z)$	F
4	-0.0660	0.3979	2.24	0.4090	-0.3230	0.1414
5	-0.3276	0.1479	2.81	0.4783	-0.2423	-0.0086
.	.	.	.	.	.	.
.	.	.	.	.	.	.
8	0.2346	-0.1581	4.50	0.2177	0.3280	-0.0932
9	0.2453	0.1043	5.05	0.0292	0.3674	-0.0807
10	0.0435	0.2490	5.61	-0.15	0.3563	-0.0528
11	-0.1768	0.1637	6.17	-0.1753	0.1841	-0.0297
10.5	-0.0789	0.2337	5.89	-0.2198	0.2568	-0.0311
10.7	-0.1224	0.2114	6.00	-0.2414	0.2299	-0.0229
10.8	-0.1422	0.1973	6.05	-0.2532	0.2170	-0.0114
10.9	-0.1603	0.1813	6.12	-0.2610	0.2014	0.01725

From the above table, the result is that x = 5, and x = 10.85, the value of F equals to zero, or k = 1.25 and 2.7134. Now

$$k^2 = \frac{PL^2}{EI_o} = 1.5625 \text{ and } 7.3625.$$

If a = 0.2

x	$J_1(x)$	$Y_1(x)$	z	$J_2(z)$	$Y_2(z)$	F
5	-0.3276	0.1479	3.35	0.4758	-0.0182	0.0644
6	-0.2767	-0.1750	4.01	0.2052	0.2178	0.0243
7	-0.0047	-0.3025	4.68	0.1566	0.3524	-0.0457
6.1	-0.2559	-0.1998	4.08	0.3427	0.2384	-0.0206

From the above table, F equals to zero at x = 6.05. Then k = 1.51.

$$k^2 = \frac{PL^2}{EI_o} = 2.2801$$

If a = 0.3.

x	$J_1(x)$	$Y_1(x)$	z	$J_2(z)$	$Y_2(z)$	F
6	-0.2767	-0.1750	4.44	0.2373	0.3187	0.04665
7	-0.0047	-0.3027	5.18	-0.0015	0.3630	0.0058
8	0.2346	-0.1581	5.92	-0.2274	0.2485	-0.02234

x = 7.1, F approximates to zero. k = x/4 = 1.775

$$k^2 = \frac{PL^2}{EI_o} = 3.1546$$

If a = 0.4.

x	$J_1(x)$	$Y_1(x)$	z	$J_2(z)$	$Y_2(z)$	F
7	-0.0047	-0.3027	5.56	-0.1350	0.3217	0.0423
8	0.2346	-0.1581	6.35	-0.2952	0.1327	0.0155
8.5	0.2731	-0.0262	6.75	-0.2656	0.0136	0.0102
8.7	-0.0125	0.0280	6.91	-0.3074	-0.0304	-0.0089

F = 0 at x equals to 8.6 or k = 2.15.

$$k^2 = \frac{PL^2}{EI_o} = 4.6225.$$

If a = 0.5.

x	$J_1(x)$	$Y_1(x)$	z	$J_2(z)$	$Y_2(z)$	F
9	0.2545	0.1043	7.64	-0.2018	-0.2138	0.03288
10	0.0435	0.2490	8.50	0.0224	-0.2772	0.01763
11	-0.1768	0.0579	9.34	0.2494	-0.0665	-0.00504

F = 0 at x = 10.9, or k = 2.725.

$$k^2 = \frac{PL^2}{EI_o} = 7.4256.$$

When x is larger and larger, the following relationships can be used:

$$J_m(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{\pi}{4} - \frac{m\pi}{2}\right)$$

$$Y_m(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - \frac{\pi}{4} - \frac{m\pi}{2}\right).$$

Computer program is typed next page.

```
C   C   FORGO PROGRAM OF CRITICAL LOADS FOR N=3.0/2.0
        READ 1,A1,W1,XK
      1 FORMAT(3F5.2)
        DELX=0.5
        A=A1**0.25
     10 X=4.0*XK
        AX=A*X
        Z2=(2.0/(3.14159*AX))**0.5*COS(AX-3.927)
        Y1=(2.0/(3.14159*X))**0.5*SIN(X-2.3565)
        Y2=(2.0/(3.14159*AX))**0.5*SIN(AX-3.927)
        Z1=(2.0/(3.14159*X))**0.5*COS(X-2.3565)
        W=Z2*Y1-Z1*Y2
        IF(W1*W)30,50,25
     25 W1=W
        XK=XK+DELX
        GO TO 10
     30 W2=ABS(W)
        IF(W2-0.0001)50,50,20
     20 XK=XK-DELX
        DELX=0.5*DELX
        XK-XK+DELX
        GO TO 10
     50 XX=XK*XK
        PUNCH 40,A1,XX
     40 FORMAT(2F15.5)
        PRINT 40,A1,XX
        STOP
        END
C   C   RESULT OF N=3/2 (HIGHER MODE)
            .10000         20.13590
            .10000         39.45410
            .10000         65.22455
            .20000         12.65664
            .20000         35.16119
            .20000         66.90297
            .20000        113.88892
            .30000         20.56764
            .30000         57.07330
            .30000        111.89673
            .30000        185.00249
            .40000         33.15241
            .40000         92.[illegible]
            .40000        180.35649
            .40000        298.10181
            .50000         54.85254
            .50000        152.36816
            .50000        296.64166
            .50000        493.67285
```

(V)

```
C  C  FORGO PROGRAM OF CRITICAL LOADS FOR N=2.0
      DIMENSION Y(5),Y1(5),Y2(8),Y3(8),Z1(5),B(5) , P(10)
   40 FORMAT(3F10.5)
      A=0.1
    2 DO 51 I=1,5
      X=2.0/LOG(A)
      Z=I-1
      Y1(I)=1.5708+Z*3.1416
      Y2(I)=Y1(I)+1.5
    1 Y3(I)=(Y1(I)+Y2(I))/2.0
      AA  =COS(Y3(I)-Z*3.1416)
      BB  =SIN(Y3(I)-Z*3.1416)
      Y(I)=BB/AA
      Z1(I)=X*Y3(I)
      B(I)=ABS(Y(I)-Z1(I))
      IF(B(I)-0.001)50,50,10
   10 IF(Y(I)-Z1(I))20,50,30
   20 Y1(I)=Y3(I)
      GO TO 1
   30 Y2(I)=Y3(I)
      GO TO 1
   50 P(I)=Y3(I)**2/(LOG(A))**2+0.25
   51 PUNCH 40,A,Y3(I),P(I)
      A=A+0.1
      IF(A-0.5)2,2,3
    3 STOP
      END
C  C  RESULT OF N=2.0
```

A	THETA	PL**2/EI
.10000	2.07681	1.06351
.10000	4.94128	4.85518
.10000	7.99698	12.31203
.10000	11.09896	23.48449
.10000	14.21799	38.37811
.20000	1.96035	1.73361
.20000	4.87595	9.42848
.20000	7.95482	24.67936
.20000	11.06818	47.54375
.20000	14.19383	78.02693
.30000	1.88052	2.68962
.30000	4.83622	16.38536
.30000	7.92977	43.62978
.30000	11.05002	84.48501
.30000	14.17963	138.95624
.40000	1.81771	4.18535
.40000	4.80740	27.77673
.40000	7.91184	74.80700
.40000	11.03708	145.34148
.40000	14.16952	239.38528
.50000	1.76471	6.73179
.50000	4.78470	47.89954
.50000	7.89785	130.07734
.50000	11.02702	253.33592
.50000	14.16166	417.67372

ELASTIC BUCKLING OF TAPERED BEAM

by

HAN-CHOU WANG

B. S., National Taiwan University, China, 1962

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Applied Mechanics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

Many methods to solve for the critical load on a column with either a uniform cross section or with the cross section varying linearly are presented in texts on elastic stability. In this report, a tapered cantilever column with the moment of inertia of the cross section varying according to a power of the longitudinal coordinate coinciding with the beam axis is investigated. A general differential equation of a deflection curve of a buckling bar is derived. The method of Frobenius and the change of variable method are used to solve the governing equation. A general solution is obtained in terms of Bessel functions. A characteristic equation is found by applying the boundary conditions to the solution. By using a computer and a mathematical table, the critical loads of the first five modes for several kinds of tapered columns are obtained.