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DSM for general nonlinear equations

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Abstract

If $F : H \rightarrow H$ is a map in a Hilbert space H , $F \in C_{loc}^2$, and there exists a solution y , possibly non-unique, such that $F(y) = 0$, $F'(y) \neq 0$, then equation $F(u) = 0$ can be solved by a DSM (Dynamical Systems Method) and the rate of convergence of the DSM is given provided that a source-type assumption holds. A discrete version of the DSM yields also a convergent iterative method for finding y . This method converges at the rate of a geometric series. Stable approximation to a solution of the equation $F(u) = f$ is constructed by a DSM when f is unknown but the noisy data f_δ are known, where $\|f_\delta - f\| \leq \delta$.

1 Introduction

In this paper a method for solving a general class of nonlinear operator equations $F(u) = 0$ in a Hilbert space is proposed, its convergence is proved, and an iterative method for solving the above equation is constructed. Convergence of the iterative method is proved. These results are based on the following assumptions: a) the above equation has a solution y , possibly non-unique, b) $F \in C_{loc}^2$, and c) $F'(y) \neq 0$. The last condition means that there exists a z such that $F'(y)z \neq 0$. This is a very weak assumption. It allows the null-space of the operator $F'(y)$ to be infinite-dimensional. No restrictions on the rate of growth of nonlinearity are made. The literature on the methods for solving nonlinear equations is large (see, e.g., [2] and references therein). Most of the known results are based on Newton-type methods and their modifications. There is a well-developed theory for equations with monotone operators and more general classes of operator equations ([4], [5]). The method used in this paper is a version of the Dynamical Systems Method (DSM). The general development of the DSM is presented in [3]–[8]. In this paper the ideas from [6] are used. The general idea of the DSM is described briefly below. In [1] a Newton-type DSM version is proposed under the assumption that the Fréchet derivative F' is a boundedly invertible linear operator. Under this assumption many classical numerical methods for

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solving operator equations $F(u) = 0$ are shown in [4] to be various versions of the DSM. These methods include Newton-type, modified Newton-type, Gauss-Newton-type, gradient method, simple iterations method, etc. But the DSM allows one to treat the problems in which F' is not boundedly invertible. In [5] the numerical efficiency of the DSM is demonstrated by many examples.

Let $F : H \rightarrow H$ be a map in a Hilbert space. One can also consider the case when $F : H_1 \rightarrow H_2$, where H_1 and H_2 are Hilbert spaces, but for simplicity of notations we restrict the presentation in this paper to the case of one Hilbert space $H_1 = H_2 = H$. The results and proofs can be rewritten for the case when $F : H_1 \rightarrow H_2$. Assume that equation

$$F(u) = 0 \tag{1}$$

has a solution y , possibly non-unique, and

$$F'(y) \neq 0, \tag{2}$$

where F' is the Frèchet derivative of F . This assumption means that $F'(y)$ is not equal to zero identically on H . Assume that $F \in C_{loc}^2$, i.e.,

$$\sup_{u \in B(u_0, R)} \|F^{(j)}(u)\| \leq M_j(R) \quad 0 \leq j \leq 2, \tag{3}$$

where $u_0 \in H$ is a given element, $R > 0$, and no restrictions on the growth of $M_j(R)$ as R grows are made. This means that the nonlinearity F can grow arbitrarily fast as $\|u - u_0\|$ grows. Under these assumptions equation (1) may have no solutions. Thus, we have assumed that a solution y to (1) exists. There are many results giving sufficient conditions for the existence of a solution to nonlinear equations, but we do not go into detail since it is not the topic of our paper.

We *do not assume* that $F'(u)$ has a bounded inverse operator, so the standard Newton-type methods are not applicable. The Dynamical Systems Method (DSM) consists of finding an operator Φ such that the problem

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0, \quad t \geq 0, \quad \dot{u} = \frac{du}{dt}, \tag{4}$$

has a unique global solution $u(t)$, (that is, the solution exists for all $t \geq 0$), there exists $u(\infty) := \lim_{t \rightarrow \infty} u(t)$, and $F(u(\infty)) = 0$. To ensure the unique *local* solvability of (4) we assume that

$$\|\Phi(t, u) - \Phi(t, v)\| \leq L(R)\|u - v\| \quad \forall u, v \in B(u_0, R).$$

Then the global existence of the unique local solution holds if $\sup_{t \geq 0} \|u(t)\| < \infty$.

The results of this paper are summarized in several theorems. Let us denote

$$A := F'(u(t)), \quad T := A^*A, \quad T_a := T + aI; \quad \tilde{A} := F'(y), \quad \tilde{T} = \tilde{A}^*\tilde{A}. \tag{5}$$

Assume that $a(t)$ is a positive monotonically decaying function,

$$a(t) > 0, \quad \lim_{t \rightarrow \infty} a(t) = 0, \quad \dot{a} < 0, \quad \frac{|\dot{a}|}{a} \leq 0.4. \tag{6}$$

Theorem 1. *Assume that a solution to equation (1) exists, (possibly non-unique), that assumptions (2) and (3) hold, and that*

$$y - u_0 = \tilde{T}v, \quad \|v\| \leq (20M_1M_2)^{-1}, \quad \|y - u_0\| \leq 5M_1M_2^{-1}. \quad (7)$$

Let $a(0) = (5M_1)^{2/3}M_2^{4/3}\|y - u_0\|^{4/3}$. Then there exists $\lim_{t \rightarrow \infty} u(t) := u(\infty) := y$, $F(y) = 0$, and

$$\|u(t) - y\| \leq C_I a^{1/2}(t), \quad C_I := \frac{\|y - u_0\|^{1/3}}{(5M_1M_2^2)^{1/3}}, \quad (8)$$

where $u(t)$ solves the DSM Cauchy problem:

$$\dot{u} = -T_{a(t)}^{-1}[A^*F(u(t)) + a(t)(u(t) - u(0))], \quad u(0) = u_0, \quad (9)$$

the solution of which exists globally and is unique.

Theorem 2. *Under the assumptions of Theorem 1, the iterative process*

$$u_{n+1} = u_n - h_n T_{a_n}^{-1}[A^*(u_n)F(u_n) + a_n(u_n - u(0))], \quad u_0 = u(0), \quad (10)$$

where $h_n > 0$ and $a_n > 0$ are suitably chosen, generates the sequence u_n converging to y .

Remark 1. *The suitable choices of a_n and h_n are made in the proof of Theorem 2.*

Remark 2. *Theorem 1 says that any solvable operator equation with C_{loc}^2 operator, satisfying weak assumptions, stated in Theorem 1, can be solved by the DSM (9). Condition (2) means that the range of the linear operator $F'(y)$ contains at least one non-zero element. This condition allows $F'(y)$ to have an infinite-dimensional null-space.*

In Section 2 we prove Theorems 1 and 2. In Section 3 and Section 4 we study the stability of the solution. In the proofs we use the following lemmas.

Lemma 1. *Assume that $g(t) \geq 0$ is a $C_{loc}^1([0, \infty))$ function satisfying the inequality*

$$\dot{g}(t) \leq -\gamma(t)g + \alpha(t)g^2 + \beta(t), \quad t \geq 0, \quad \dot{g} := \frac{dg}{dt}, \quad (11)$$

where $\gamma(t)$, $\alpha(t)$ and $\beta(t)$ are nonnegative continuous functions defined on $[0, \infty)$. Assume that there exists $\mu \in C^1([0, \infty))$, $\mu > 0$, such that

$$\beta(t) + \alpha(t)\mu^{-2} \leq \frac{1}{\mu(t)} \left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)} \right), \quad g(0)\mu(0) \leq 1. \quad (12)$$

Then any non-negative solution $g(t)$ to (11) exists globally, that is, on $[0, \infty)$, and

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad t \in [0, \infty). \quad (13)$$

A generalized version of Lemma 1 is proved in [7].

Lemma 2. Let $g_{n+1} \leq \gamma g_n + p g_n^2$, $g_0 := m > 0$, $0 < \gamma < 1$, $p > 0$. If $m < \frac{q-\gamma}{p}$, where $\gamma < q < 1$, then $\lim_{n \rightarrow \infty} g_n = 0$, and $g_n \leq g_0 q^n$.

Proof of Lemma 2. Estimate $g_1 \leq \gamma m + p m^2 \leq q m$ holds if $m \leq \frac{q-\gamma}{p}$, $\gamma < q < 1$. Assume that $g_n \leq g_0 q^n$. Then

$$g_{n+1} \leq \gamma g_0 q^n + p (g_0 q^n)^2 = g_0 q^n (\gamma + p g_0 q^n) < g_0 q^{n+1},$$

because $\gamma + p g_0 q^n < \gamma + p g_0 q \leq \gamma + p m \leq q$. Lemma 2 is proved. \square

The following lemma is borrowed from [7].

Lemma 3. Assume that $g_n \geq 0$, $\alpha(n, g_n) \geq 0$,

$$g_{n+1} \leq (1 - h_n \gamma_n) g_n + h_n \alpha(n, g_n) + h_n \beta_n; \quad h_n > 0, \quad 0 < h_n \gamma_n < 1, \quad (14)$$

and $\alpha(n, g_n) \geq \alpha(n, p_n)$ if $g_n \geq p_n$. If there exists a sequence $\mu_n > 0$ such that

$$\alpha(n, \frac{1}{\mu_n}) + \beta_n \leq \frac{1}{\mu_n} (\gamma_n - \frac{\mu_{n+1} - \mu_n}{h_n \mu_n}), \quad (15)$$

and

$$g_0 \leq \frac{1}{\mu_0}, \quad (16)$$

then

$$0 \leq g_n \leq \frac{1}{\mu_n}, \quad \forall n \geq 0. \quad (17)$$

Proof. For $n = 0$ inequality (17) holds because of (16). Assume that it holds for all $n \leq m$ and let us check that then it holds for $n = m + 1$. If this is done, the lemma is proved. Using the inductive assumption, one gets:

$$g_{m+1} \leq (1 - h_m \gamma_m) \frac{1}{\mu_m} + h_m \alpha(m, \frac{1}{\mu_m}) + h_m \beta_m.$$

This and inequality (15) imply:

$$\begin{aligned} g_{m+1} &\leq (1 - h_m \gamma_m) \frac{1}{\mu_m} + h_m \frac{1}{\mu_m} (\gamma_m - \frac{\mu_{m+1} - \mu_m}{h_m \mu_m}) \\ &= \mu_m^{-1} - \frac{\mu_{m+1} - \mu_m}{\mu_m^2} \leq \mu_{m+1}^{-1}. \end{aligned}$$

The last inequality is obvious since it can be written as $-(\mu_m - \mu_{m+1})^2 \leq 0$. Lemma 3 is proved. \square

2 Proofs of Theorems 1 and 2

Proof of Theorem 1. Since $\tilde{T} \neq 0$, for a suitable choice of u_0 there exists a v such that $y - u(0) = \tilde{T}v$, and we assume that $\|v\| \leq c_1 := (20M_1M_2)^{-1}$. Let

$$u(t) - y := w(t), \quad \|w(t)\| := g(t).$$

Write equation (9) as

$$\dot{w} = -T_{a(t)}^{-1} [A^*(F(u) - F(y)) + a(t)w + a(t)(y - u(0))], \quad (18)$$

and use the formula $F(u) - F(y) = Aw + K$, where $\|K\| \leq \frac{M_2g^2}{2}$. Then (18) yields

$$\dot{w} = -w - T_{a(t)}^{-1} A^*K - a(t)T_{a(t)}^{-1}\tilde{T}v. \quad (19)$$

Multiply this equation by w in H and use the estimate $\|T_a^{-1}A^*\| \leq \frac{1}{2\sqrt{a}}$, $a > 0$, which follows from the spectral theorem (see [4]), to get

$$g\dot{g} \leq -g^2 + \frac{1}{2\sqrt{a(t)}} \frac{M_2g^3}{2} + a(t) \left\| \left(T_{a(t)}^{-1} - \tilde{T}_{a(t)}^{-1} + \tilde{T}_{a(t)}^{-1} \right) \tilde{T} \right\| \|v\| g.$$

If $a > 0$ then, by the spectral theorem, $\|(\tilde{T}_a)^{-1}\tilde{T}\| \leq 1$, $a\|T_a^{-1}\| \leq 1$, and

$$a\|(T_a^{-1} - \tilde{T}_a^{-1})\tilde{T}\| = a\|T_a^{-1}(A^*A - \tilde{A}^*\tilde{A})\tilde{T}_a^{-1}\tilde{T}\| \leq 2M_1M_2g.$$

Here we have used estimate (3) and the estimate of the type $\|A^*[A(u) - \tilde{A}(y)]\| \leq M_1M_2g$. Collecting the above estimates and the estimate $\|v\| \leq c_1$, one gets

$$\dot{g} \leq -\frac{9g}{10} + \frac{c_0g^2}{\sqrt{a(t)}} + c_1a(t), \quad c_0 := \frac{M_2}{4}, \quad c_1 := (20M_1M_2)^{-1}. \quad (20)$$

Apply Lemma 1 to (20). Here $\gamma = \frac{9}{10}$, $\alpha(t) = \frac{c_0}{\sqrt{a(t)}}$, $\beta(t) = c_1a(t)$. Choose

$$\mu(t) = \frac{\lambda}{\sqrt{a(t)}}, \quad \lambda = \text{const} > 0, \quad \frac{\dot{\mu}}{\mu} = 0.5 \frac{|\dot{a}(t)|}{a(t)} \leq 0.4.$$

Conditions of Lemma 1 hold if

$$\lambda a^{-0.5}(t)[c_0a^{0.5}(t)\lambda^{-2} + c_1a(t)] \leq (0.9 - 0.4) = 0.5, \quad (21)$$

and $g(0)\lambda \leq a^{0.5}(0)$. Choose $\lambda = a^{0.5}(0)/g(0)$. Then (21) holds if

$$c_0\lambda^{-1} + c_1a^{0.5}(0)\lambda \leq 0.5. \quad (22)$$

Consider the problem

$$m(s) := c_0s^{-1} + c_1a^{0.5}(0)s = \min := m,$$

where the minimization is over $s > 0$. The minimum is attained at $s = s_m := (\frac{c_0}{c_1})^{0.5} a(0)^{-0.25}$ and $m = 2(c_0 c_1)^{1/2} a^{1/4}(0)$. Note that $2(c_0 c_1)^{1/2} = (20M_1)^{-0.5}$. Thus, if $\lambda = a^{0.5}(0)/g(0)$ and $2(c_0 c_1)^{1/2} a^{1/4}(0) \leq 0.5$, that is, $a(0) \leq 25M_1^2$, then, by Lemma 1, the solution to (9) exists for all $t \geq 0$ and

$$\|u(t) - y\| \leq \frac{\sqrt{a(t)}\|u(0) - y\|}{a^{0.5}(0)}. \quad (23)$$

In the proof of Theorem 1 we satisfied inequality (22) by taking $a(0) \leq 25M_1^2$ and $\lambda = a^{1/2}(0)g^{-1}(0) = (\frac{c_0}{c_1})^{0.5} a(0)^{-0.25}$. The last relation implies $g(0) = a^{3/4}(0)(5M_1 M_2^2)^{-0.5}$. Therefore, $a^{0.5}(0) = g^{2/3}(0)(5M_1 M_2^2)^{1/3}$. Consequently, the right-hand side of the estimate (23) is $C_I a^{1/2}(t)$, where C_I is defined in (8). Since $a(0) \leq 25M_1^2$, it follows that $g(0) \leq 5M_1 M_2^{-1}$. Thus, the initial approximation u_0 should be not too far from the solution y , namely, $\|y - u_0\| \leq 5M_1 M_2^{-1}$, as in (7).

Theorem 1 is proved. \square

Proof of Theorem 2. Let $w_n := u_n - y$, $g_n := \|w_n\|$. We assume that $2M_1 M_2 \|v\| \leq \frac{1}{2}$ and rewrite (10) as

$$w_{n+1} = w_n - h_n T_{a_n}^{-1} [A^*(u_n)(F(u_n) - F(y)) + a_n w_n + a_n (y - u_0)], \quad w_0 = \|u_0 - y\|.$$

Using the Taylor formula $F(u_n) - F(y) = A(u_n)w_n + K(w_n)$, $\|K\| \leq \frac{M_2 g_n^2}{2}$, the estimate $\|T_{a_n}^{-1} A^*(u_n)\| \leq \frac{1}{2\sqrt{a_n}}$, and the formula $y - u_0 = \tilde{T}v$, we get

$$w_{n+1} = (1 - h_n)w_n - h_n T_{a_n}^{-1} A^*(u_n)K(w_n) - h_n a_n T_{a_n}^{-1} \tilde{T}v. \quad (24)$$

Taking into account that $\|\tilde{T}_a^{-1} \tilde{T}\| \leq 1$, and $a\|T_a^{-1}\| \leq 1$ if $a > 0$, we obtain $\|T_{a_n}^{-1} \tilde{T}v\| \leq \|(T_{a_n}^{-1} - \tilde{T}_{a_n}^{-1})\tilde{T}\|\|v\| + \|v\|$, and $\|(T_{a_n}^{-1} - \tilde{T}_{a_n}^{-1})\tilde{T}\| = \|T_{a_n}^{-1}(\tilde{T}_{a_n} - T_{a_n})\tilde{T}_{a_n}^{-1} \tilde{T}\| \leq \frac{2M_1 M_2 g_n}{a_n} := \frac{C_1 g_n}{a_n}$. Let $c_0 := \frac{M_2}{4}$. Then it follows from (24) that

$$g_{n+1} \leq (1 - h_n)g_n + \frac{c_0 h_n g_n^2}{\sqrt{a_n}} + C_1 h_n \|v\| g_n + h_n a_n \|v\|.$$

Let us assume that $C_1 \|v\| \leq \frac{1}{2}$. Then

$$g_{n+1} \leq (1 - \frac{h_n}{2})g_n + \frac{c_0 h_n}{\sqrt{a_n}} g_n^2 + h_n a_n \|v\|.$$

Choose $a_n = 16c_0^2 g_n^2$, so that $\frac{c_0 g_n}{\sqrt{a_n}} = \frac{1}{4}$, and get

$$g_{n+1} \leq (1 - \frac{h_n}{4})g_n + 16c_0^2 h_n \|v\| g_n^2, \quad g_0 = \|u_0 - y\| \leq R, \quad (25)$$

where $R > 0$ is defined in (3). Take $h_n = h \in (0, 1)$ and choose $g_0 := m$, where

$$m < \frac{q + \frac{h}{4} - 1}{16c_0^2 h \|v\|}, \quad q \in (0, 1), \quad 1 > q > 1 - \frac{h}{4} > 0.$$

Then Lemma 2 with $\gamma = 1 - \frac{h}{4}$ and $p = 16c_0^2 h \|v\|$ implies

$$\|u_n - y\| \leq g_0 q^n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Theorem 2 is proved. \square

3 Stability of the solution

Assume that $F(y) = f$, where the exact data f are not known but the noisy data f_δ are given, $\|f_\delta - f\| \leq \delta$. Then the DSM yields a stable approximation of the solution y if the stopping time t_δ is properly chosen. The DSM is similar to (9):

$$\dot{u}_\delta(t) = -T_{a(t)}^{-1}[A^*(F(u_\delta(t)) - f_\delta) + a(t)(u_\delta(t) - u_0)], \quad u_\delta(0) = u_0, \quad (26)$$

Let

$$w_\delta := u_\delta(t) - y, \quad g_\delta(t) := \|w_\delta\|.$$

As in the proof of Theorem 1 we derive the inequality similar to (14):

$$\dot{g}_\delta \leq -\frac{g_\delta}{2} + \frac{c_0 g_\delta^2}{\sqrt{a(t)}} + a(t)\|v\| + \frac{\delta}{2\sqrt{a(t)}}, \quad c_0 := \frac{M_2}{4}, \quad (27)$$

and apply Lemma 1. Rather than to repeat the arguments, given in the proof of Theorem 1, we will use the results obtained in this proof. The constant c_1 in Theorem 1 is now replaced by $c_\delta := c_1 + 0.5\delta a^{-1.5}(t_\delta)$. As in the proof of Theorem 1 one gets the estimate $\|u_\delta(t) - y\| \leq a^{0.5}(t)\lambda^{-1}$, $t \in [0, t_\delta]$. Let us define the stopping time t_δ from the equation

$$0.5\delta a^{-1.5}(t_\delta) = c_1. \quad (28)$$

This equation has a unique solution t_δ , because $a(t)$ is decaying monotonically. Clearly, $\lim_{\delta \rightarrow 0} t_\delta = \infty$. Since $\lim_{t \rightarrow \infty} a(t) = 0$ and λ in our argument does not depend on t_δ , the estimate $\|u_\delta(t_\delta) - y\| \leq a^{0.5}(t_\delta)\lambda^{-1}$ shows that $\lim_{\delta \rightarrow 0} \|u_\delta(t_\delta) - y\| = 0$. Thus, we have proved the following theorem.

Theorem 3. *Let $u_\delta := u_\delta(t_\delta)$, where $u_\delta(t)$ solves problem (27) and t_δ is chosen in (28). Then $\lim_{\delta \rightarrow 0} \|u_\delta - y\| = 0$.*

4 Stability of the iterative solution

Assume that the equation is $F(u) = f$, f is unknown, but the ‘‘noisy datum’’ f_δ is known, such that $\|f_\delta - f\| \leq \delta$. Consider the iterative process similar to (10):

$$v_{n+1} = v_n - h_n T_{a_n}^{-1}[A^*(v_n)(F(v_n) - f_\delta) + a_n(v_n - u_0)], \quad v_0 = u_0, \quad (29)$$

Let

$$w_n := v_n - y, \quad \|w_n\| := \psi_n,$$

and choose $h_n = h$ independent of n , $h \in (0, 1)$. A positive lower bound on h is imposed in formula (35) below. An inequality similar to (25) takes the form:

$$\psi_{n+1} \leq \gamma \psi_n + p \psi_n^2 + \frac{h\delta}{2\sqrt{a_n}}, \quad \psi_0 = \|u_0 - y\|, \quad (30)$$

where

$$\gamma := 1 - \frac{h}{4}, \quad p := 16c_0^2 \|v\|, \quad a_n = 16c_0^2 \psi_n^2. \quad (31)$$

We stop iterations in formula (29) when $n = n(\delta)$, where $n(\delta)$ is the largest integer for which the inequality

$$\frac{h\delta}{2\sqrt{a_n}} \leq \kappa\gamma\psi_n, \quad \kappa \in (0, \frac{1}{3}), \quad (32)$$

holds. One can use formula (31) for a_n and rewrite this inequality as

$$\frac{h\delta}{8c_0\kappa\gamma} \leq \psi_n^2, \quad \kappa \in (0, \frac{1}{3}). \quad (33)$$

If (32) holds, then (30) implies:

$$\psi_{n+1} \leq (1 + \kappa)\gamma\psi_n + p\psi_n^2, \quad (1 + \kappa)\gamma < 1, \quad (34)$$

and the conditions

$$\gamma = 1 - \frac{h}{4}, \quad 0 < \kappa < \frac{1}{3}, \quad h \in (\frac{4\kappa}{1 + \kappa}, 1), \quad (35)$$

imply that $(1 + \kappa)\gamma < 1$ and $\frac{4\kappa}{1 + \kappa} < 1$. If

$$\psi_0 < \frac{q - (1 + \kappa)\gamma}{p}, \quad \text{where } (1 + \kappa)\gamma < q < 1, \quad \gamma = 1 - \frac{h}{4}, \quad (36)$$

then inequality (34) and Lemma 2 imply

$$\psi_n \leq \psi_0 q^n, \quad (37)$$

provided that

$$n < n(\delta), \quad (1 + \kappa)\gamma < q < 1, \quad 0 < \kappa < \frac{1}{3}, \quad (38)$$

where $n(\delta)$ is the largest integer for which inequality (32) holds. Clearly, $\lim_{\delta \rightarrow 0} n(\delta) = \infty$. Thus

$$\lim_{\delta \rightarrow 0} \psi_{n(\delta)} = 0. \quad (39)$$

We have proved the following result.

Theorem 4. *Let the assumptions of Theorem 1 hold and $(1 + \kappa)\gamma < 1$. Assume that conditions (35) and (36) hold, and $\psi_n = \|v_n - y\|$, where v_n is defined by equation (29). Then relations (37) and (39) hold, and $\lim_{\delta \rightarrow 0} \|v_{n(\delta)} - y\| = 0$.*

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