PASSIVE TIME-DELAY NETWORKS

by

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INTRODUCTION

An ideal time-delay network has a response $f(t - \tau)$ to an excitation $f(t)$. Since

$$L\left[f(t)\right] = F(s) \qquad (1)$$

and

$$L\left[f(t - \tau)\right] = e^{-\tau s}F(s) \qquad (2)$$

the transfer function of an ideal delay network is $e^{-\tau s}$. This paper explores the synthesis of an unbalanced network with transfer function e^{-s}, the unit second delay operator. For purposes of simplicity, the time delay τ is normalized to one second.

The desired network has a voltage transfer function

$$\frac{E_2(s)}{E_1(s)} = T(s) = e^{-s} \qquad (3)$$

For sinusoidal inputs

$$s = j\omega \qquad (4)$$

and

$$T(j\omega) = e^{-j\omega} \qquad (5)$$

$$\left|T(j\omega)\right| = 1 \qquad (6)$$

and

$$\theta = \omega \qquad (7)$$

where θ is the network phase lag. The group time delay is

$$\tau = \frac{d\theta}{d\omega} = 1 \text{ second} \qquad (8)$$

Since no such network exists, with a finite number of elements, e^{-s} is approximated and networks are synthesized having the approximate e^{-s} as their transfer functions.

PADE APPROXIMATION

The Pade approximations are to be found for the unit delay operator, e^{-s}. These are rational fraction approximations of the Taylor series of e^{-s}. The (m, n) approximant has a numerator polynomial of degree m in s and a denominator of degree n in s. Of these, only the (n - 1, n) and (n, n) approximates will be investigated in this paper. These approximates are listed in Table 1.

SYNTHESIS OF THE LATTICE EQUIVALENTS OF THE PADE APPROXIMATES OF e^{-s}

The various Pade approximants will be synthesized in the form of normalized constant resistance lattices. A normalized constant resistance lattice has series impedance A(s), shunt impedance B(s), and B(s) is the inverse of A(s). This lattice will have a characteristic impedance, Z_o, of one ohm.

$$Z_o = \sqrt{A(s)\ B(s)} = 1 \text{ ohm} \tag{9}$$

The transmission matrix of a symmetric lattice is

$$\frac{1}{B - A}\begin{pmatrix} B + A & 2AB \\ 2 & B + A \end{pmatrix} \tag{10}$$

If this lattice is terminated in a 1-ohm resistance, its voltage transfer function is

$$T(s) = \frac{B - A}{B + A + 2\sqrt{AB}} \tag{11}$$

For the constant resistance lattice mentioned,

$$B = 1/A \tag{12}$$

Table 1. Table of the Pade. (n - 1, n) and (n, n) approximates of e^{-s}.

n :	$P_{n-1,\,n}(e^{-s})$	: $P_{n,\,n}(e^{-s})$
1	$\dfrac{1}{1 + s}$	$\dfrac{1 - s/2}{1 + s/2}$
2	$\dfrac{1 - 1/3\ s}{1 + 2/3\ s + 1/6\ s^2}$	$\dfrac{1 - \frac{1}{2}\ s + 1/12\ s^2}{1 + \frac{1}{2}\ s + 1/12\ s^2}$
3	$\dfrac{1 - \frac{2}{5} s + \frac{1}{20} s^2}{1 + \frac{3}{5} s + \frac{3}{20} s^2 + \frac{1}{60} s^3}$	$\dfrac{1 - \frac{1}{2} s + \frac{1}{10} s^2 - \frac{1}{120} s^3}{1 + \frac{1}{2} s + \frac{1}{10} s^2 + \frac{1}{120} s^3}$
4	$\dfrac{1 - \frac{3}{7} s + \frac{1}{14} s^2 - \frac{1}{210} s^3}{1 + \frac{4}{7} s + \frac{1}{7} s^2 + \frac{2}{105} s^3 + \frac{1}{840} s^4}$	$\dfrac{1 - \frac{s}{2} + \frac{3s^2}{28} - \frac{s^3}{84} + \frac{s^4}{1680}}{1 + \frac{s}{2} + \frac{3s^2}{28} + \frac{s^3}{84} + \frac{s^4}{1680}}$

and
$$T(s) = \frac{1 - A}{1 + A} \tag{13}$$

Solving equation (13) for A as a function of T(s) gives

$$A = \frac{1 - T(s)}{1 + T(s)} \tag{14}$$

Thus the process of synthesizing the desired lattice network is reduced to two steps. The first step is the synthesis of impedance A. Step two is the construction of impedance B, the inverse of A.

As an example of this process, the network with transfer function of $P_{(3,3)}(e^{-s})$ will be synthesized.

$$T(s) = P_{(3,3)}(e^{-s}) = \frac{1 - \frac{1}{2} s + \frac{1}{10} s^2 - \frac{1}{120} s^3}{1 + \frac{1}{2} s + \frac{1}{10} s^2 + \frac{1}{120} s^3} \tag{15}$$

Then
$$A = \frac{1 - T(s)}{1 + T(s)} = \frac{s(1 + \frac{1}{60} s^2)}{2 + \frac{1}{5} s^2} \tag{16}$$

A is a pure reactance with an existing admittance Cauer form $Y_A(s)$.

$$Y_A(s) = \frac{2}{s} + \cfrac{1}{\cfrac{6}{s} + \cfrac{1}{10/s}} \tag{17}$$

A and B, the inverse of A, are shown in Fig. 1 and Fig. 2, respectively.

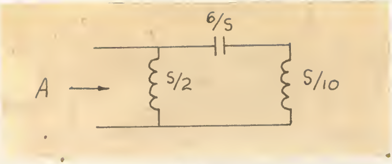

Fig. 1.

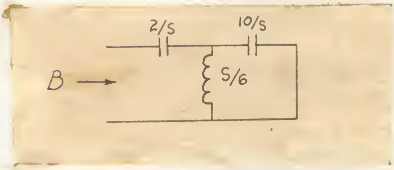

Fig. 2.

The constant resistance lattice network for the (3,3) Pade approximant is shown in Fig. 3.

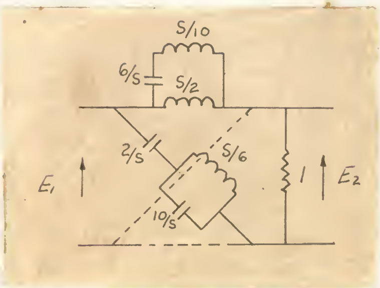

Fig. 3.

The procedure of synthesizing the networks of the (n - 1, n) approximants differs in one respect with that of the (n, n) networks. The impedance A is still the inverse of B, but A is no longer a pure reactance. Therefore its Cauer forms do not exist and other methods of synthesis must be employed. If the (n - 1, n) network impedances, A, are synthesized one by one, a pattern is immediately established. The $A_{(n-1,\ n)}$ impedance appears as the corresponding $A_{(n,\ n)}$ impedance terminated with a one-ohm resistance. The network corresponding to the (2, 3) approximant will be synthesized as an example of obtaining the (n - 1, n) network.

$$T_{(3,3)}(s) = \frac{1 - \frac{2}{5}s + \frac{1}{20}s^2}{1 + \frac{3}{5}s + \frac{3}{20}s^2 + \frac{1}{60}s^3} \tag{18}$$

$$A = \frac{1 - T}{1 + T} = \frac{s + \frac{1}{10}s^2 + \frac{1}{60}s^3}{2 + \frac{1}{5}s + \frac{1}{5}s^2 + \frac{1}{60}s^3} \tag{19}$$

An admittance 2/s is first subtracted out of Y_A, leaving Y_1.

$$Y_1 = Y_A - \frac{2}{s} = \frac{\frac{1}{6}s + \frac{1}{60}s^2}{1 + \frac{1}{10}s + \frac{1}{60}s^2} \tag{20}$$

Next, an impedance $\frac{6}{s}$ is subtracted from the impedance Z_1, $1/Y_1$, which leaves impedance Z_2.

$$Z_2 = Z_1 - \frac{6}{s} = \frac{s}{10 + s} \tag{21}$$

$$Y_2 = \frac{10 + s}{s} = \frac{10}{s} + 1 \qquad (22)$$

Then impedance $A_{(2,\ 3)}$ is as shown in Fig. 4, and the network for the (2, 3) approximant is shown in Fig. 5.

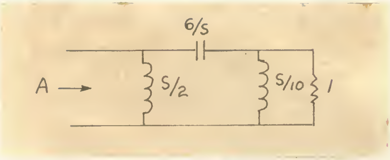

Fig. 4.

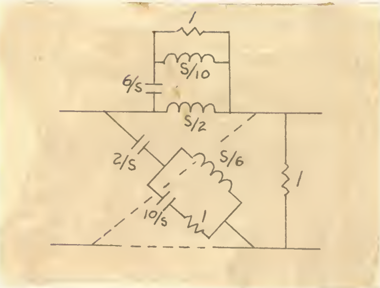

Fig. 5.

BODE'S SYNTHESIS PROCEDURE

The (n, n) networks may be found using Storer's (5) approach. The same idea, a network with transfer function e^{-s}, is still foremost. The constant resistance lattice is chosen as a canonic form for the network and the synthesis procedure is as follows.

$$T = \frac{1 - A}{1 + A} \qquad A = \frac{1 - T}{1 + T} \tag{23}$$

Therefore $$A = \frac{1 - e^{-s}}{1 + e^{-s}} = \tanh \frac{s}{2} \tag{24}$$

The final step is to synthesize A which is obviously a pure reactance. (A is the ratio of an odd to even polynomial,

$\tanh \frac{s}{2} = \frac{\sinh s/2}{\cosh s/2}$.)

$$A(s) = \tanh s/2 = \cfrac{1}{\cfrac{2 \cdot 1}{s} + \cfrac{1}{\cfrac{2 \cdot 3}{s} + \cfrac{1}{\cfrac{2 \cdot 5}{s} + \cfrac{1}{\cfrac{2 \cdot 7}{s} + \cdots}}}} \tag{25}$$

A(s) is then a ladder network with an infinite number of elements. Therefore an approximation is necessary if the lattice is to be composed of a finite number of elements. The approximation chosen is to truncate the expansion with n elements. The corresponding lattices terminated with one-ohm resistors turn out to have the $P_{(n,\ n)}(e^{-s})$ as their transfer functions.

THE CONTINUED FRACTION APPROXIMATION OF e^{-s}

The Maclaurin series of e^{ps} divided by the Maclaurin series of e^{-qs}, $p + q = 1$, will be expanded to form a continued fraction equivalent of e^{s}.

$$e^s = \frac{e^{ps}}{e^{-qs}} = \frac{1 + ps + \frac{p^2s^2}{2} + \frac{p^3s^3}{6} + \dots}{1 - qs + \frac{q^2s^2}{2} - \frac{q^3s^3}{6} + \dots} \tag{26}$$

Routh's algorithm is used in Table 2 to find the coefficients of the continued fraction. The ratios of consecutive elements in the left-hand column are: 1, 1/s, -2, -3/s, 2, 5/s, -2, ... Therefore the continued fraction equivalent is

$$e^s = 1 + \cfrac{1}{\cfrac{1}{s} + \cfrac{1}{-2 + \cfrac{1}{-\cfrac{3}{s} + \cfrac{1}{2 + \cfrac{1}{\cfrac{5}{s} + \cfrac{1}{-2 + \cfrac{1}{-\cfrac{7}{s} + \dots}}}}}}} \tag{27}$$

or

$$e^s = 1 + \cfrac{1}{\cfrac{1}{s} - \cfrac{1}{2 + \cfrac{1}{\cfrac{3}{s} - \cfrac{1}{2 + \cfrac{1}{\cfrac{5}{s} - \cfrac{1}{2 + \cfrac{1}{\cfrac{7}{s} - \dots}}}}}}} \tag{28}$$

Table 2. Routh's algorithm for e^{ps}/e^{-qs} in continued fraction form.

:	:	:	:	:	:
1	p	$\frac{p^2}{2}$	$\frac{p^3}{6}$	$\frac{p^4}{24}$	$\frac{p^5}{120}$
1	$-q$	$\frac{q^2}{2}$	$-\frac{q^3}{6}$	$\frac{q^4}{24}$	$-\frac{q^5}{120}$
1	$\frac{1-2q}{2}$	$\frac{1-3q+3q^2}{6}$	$\frac{1-4q+6q^2-4q^3}{24}$	$\frac{1-5q+10q^2-10q^3+5q^4}{120}$	...
$-\frac{1}{2}$	$\frac{3q-1}{6}$	$-\frac{1+4q-6q^2}{24}$	$-\frac{1+5q-10q^2+10q^3}{120}$	...	...
$\frac{1}{6}$	$\frac{1-2q}{12}$	$\frac{3-10q+10q^2}{120}$	...	...	...
$\frac{1}{12}$	$\frac{4-10q}{120}$	...	...	...	...
$\frac{1}{60}$	...	...	...	...	...

Inverting both sides, gives the equivalent

$$e^{-s} = \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{s} - \cfrac{1}{2 + \cfrac{1}{\cfrac{3}{s} - \cfrac{1}{2 + \cfrac{1}{\cfrac{5}{s} - \cfrac{1}{2 + \cfrac{1}{\cfrac{7}{s} - \cdots}}}}}}}} \tag{29}$$

If this expansion is truncated after the first n elements, an approximate of e^{-s} is formed. This process yields the $P_{((n-1)/2,\ (n-1)/2)}(e^{-s})$ for n odd (n = 1, 3, 5, ...) and the $P_{((n-2)/2,\ n/2)}(e^{-s})$ for n even (n = 2, 4, 6, 8, ...). As an example for n odd, consider the third order approximant.

$$e^{-s}\Big|_3 = \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{s} - \cfrac{1}{2}}} = \cfrac{1}{1 + \cfrac{2\,s}{2 - s}} \tag{30}$$

$$= \frac{2 - s}{2 + s} = \frac{1 - \frac{s}{2}}{1 + \frac{s}{2}} = P_{(1,1)}(e^{-s}) \tag{31}$$

Choosing n = 4 should give the $P_{(1,2)}(e^{-s})$.

$$e^{-s}\Big|_4 = \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{s} - \cfrac{1}{2 + \cfrac{1}{\cfrac{3}{s}}}}} \tag{32}$$

$$= \frac{1 - \frac{1}{3}s}{1 + \frac{2}{3}s + \frac{1}{6}s^2} = P_{(1,2)}(e^{-s}) \tag{33}$$

REDUCTION OF THE LATTICES TO UNBALANCED FORM

Methods of reducing lattices to unbalanced forms appear in various network texts. The methods apply when the series and shunt lattice impedances have common series, shunt, or series and shunt elements. These common elements are then factored out of the lattice, leaving a simpler lattice. A successive number of these factorizations may reduce the lattice to the form of a balanced ladder or bridged-tee network. This network may then be replaced by its unbalanced equivalent. Time scale changes, impedance level changes, and m-derivations may be used as tools when employing these methods.

Since A(s) and B(s) are inverse, the common methods of attack seem to fail. Inverse impedances have none of the common qualities required by the normal methods of reduction.

It should be noted, however, that all of the (n - 1, n) and (n, n) approximate lattices have a series impedance, A(s), composed of an inductor of 1/2 henry in parallel with another impedance $A_1(s)$, as shown in Fig. 6. B(s) may now be altered to include a series inductor of 1/2 henry, as in Fig. 7, and one

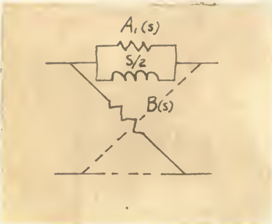

Fig. 6.

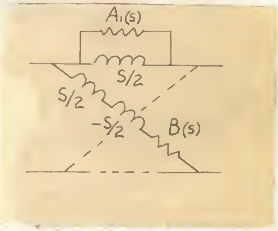

Fig. 7.

of the above methods of reduction now applies. The 1/2-henry inductors are factored from the inner lattice, as shown in Figs. 8, 9, and 10. The negative inductance has been treated as though its physical counterpart exists. Since no such element

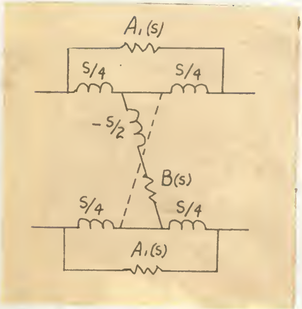

Fig. 8.

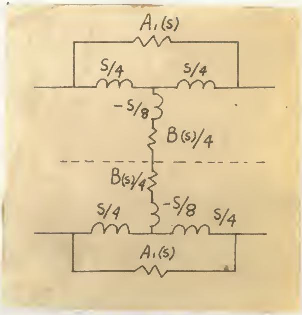

Fig. 9.

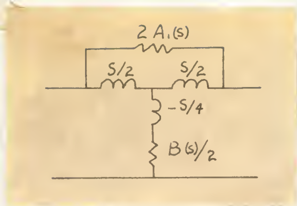

Fig. 10.

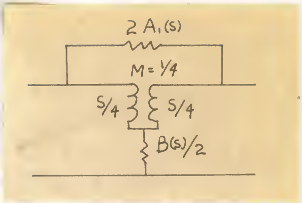

Fig. 11.

is available, the three inductors are now incorporated into a non-ideal, center-tapped inductor, as shown in Fig. 11. All of the approximates may now be constructed of passive elements in unbalanced form.

PERFORMANCE OF THE (n, n) APPROXIMATE NETWORKS

The transfer function of the (n, n) network is written as a ratio of polynomials in s.

$$\left.\frac{E_2}{E_1}\right|_{(n,n)} = T_{(n,n)}(s) = \frac{D_n(-s)}{D_n(s)} \tag{34}$$

Storer (5) shows that $D_n(s)$ may be expressed as

$$D_n(s) = \frac{n!\ 2^n}{(2\ n)!} \left(\frac{2s}{\pi}\right)^{\frac{1}{2}} s^n e^s K_{n+\frac{1}{2}}(s) \tag{35}$$

where the $K_{n+\frac{1}{2}}(s)$ are the half integer Bessel function, or it may be expressed as

$$D_n(s) = \frac{n!}{(2n)!} \sum_{0}^{n} \frac{(2n - N)!\ s^N}{N!(n - N)!} \tag{36}$$

The sinusoidal response is determined through examination of $T_{n,n}(j\omega)$.

$$T_{(n,n)}(j\omega) = \frac{D_n(-j\omega)}{D_n(j\omega)} = 1 \qquad (37)$$

This type of amplitude response shows that no amplitude distortion is introduced by the network. Storer (5) also shows that $\mathcal{T}_n$, the group time delay of the (n, n) network, may be written as

$$\mathcal{T}_n(\omega) = 1 - \frac{n!}{(2n)!}\left[(\omega)^{2n} - \frac{(\omega)^{2n+2}}{4(2n-1)} + \dots\right] \qquad (38)$$

The first (2n - 1) derivatives of $\mathcal{T}_n(\omega)$ are zero, and the network then has a time delay with maximal flatness of order (2n - 1). Storer (5) also shows that a lossy (n, n) network having

$$T_{(n,n)}(s) = e^{-s-\alpha} = e^{-s}e^{-\alpha} \qquad (39)$$

can be synthesized directly from the lossless (n, n) network. Inductors Ls are changed to inductors Ls in series with resistors of $L\alpha$ ohms. Capacitors $\frac{1}{cs}$ are paralleled with resistors of $\frac{1}{c\alpha}$ $(s + \alpha)$ for s in the (n, n) network. α must be small $(\alpha << 1)$ if these networks are to have performance comparable with the lossless networks.

PERFORMANCE OF THE $P_{(n-1,n)}(e^{-s})$ NETWORKS

The transfer function of the $P_{(n-1,n)}(e^{-s})$ will be written as

$$\frac{E_2(s)}{E_1(s)} = T_{(n-1,n)}(s) = \frac{A_n(s)}{B_n(s)} \qquad (40)$$

Careful scrutiny of the Pade table produces the following differential equations.

$$\text{(a)} \quad A_n(s) = -2 \frac{d}{ds} \left[D_n(-s) \right] \tag{41}$$

$$\text{(b)} \quad \frac{2n-1}{n} \frac{d}{ds} \left[B_n(s) \right] = D_{n-1}(s) \tag{42}$$

Using the series equivalent for $D_n(-s)$ and $D_n(s)$, the equations (a) and (b) can be solved for $A_n(s)$ and $B_n(s)$, as follows.

$$\text{(a)} \quad A_n(s) = -2 \frac{d}{ds} \left[D_n(-s) \right]$$

$$= -2 \frac{n!}{(2n)!} \sum_0^n \frac{d}{ds} \frac{(2n-N)!(-s)^N}{N!(n-N)!}$$

$$= 2 \frac{n!}{(2n)!} \sum_0^n \frac{N(2n-N)!(-s)^{N-1}}{N!(n-N)!} \tag{43}$$

$$\text{(b)} \quad B_n(s) = \frac{n}{2n-1} \int D_{n-1}(s)\, ds + C$$

$$= \frac{n}{(2n-1)} \frac{(n-1)!}{(2n-2)!} \sum_0^{n-1} \int \frac{(2n-2-N)!s^N}{N!(n-1-N)!} ds + C$$

$$= \frac{n!}{(2n-1)!} \sum_0^{n-1} \frac{(2n-2-N)!s^{N+1}}{(N+1)!(n-1-N)!} + 1 \tag{44}$$

These expressions for $A_n(s)$ and $B_n(s)$ may be used to find the various (n - 1, n) approximates, but unfortunately no simple general expression for $T_{(n-1,n)}(j\omega)$ or $\mathcal{T}_{(n-1,n)g}(\omega)$ has been found. These quantities must be computed individually as in the following example for n = 2.

$$T_{(1,2)}(s) = \frac{1 - \frac{1}{3} s}{1 + \frac{2}{3} s + \frac{1}{6} s^2} \tag{45}$$

$$T_{(1,2)}(j\omega) = \frac{1 - j \frac{\omega}{3}}{1 - \frac{\omega^2}{6} + j \frac{2}{3} \omega} = \frac{6 - j\, 2\, \omega}{6 - \omega^2 + j\, 4\, \omega} \tag{46}$$

$$\left| T_{(1,2)}(j\omega) \right|^2 = \frac{36 + 4\, \omega^2}{36 - 12\, \omega^2 + \omega^4 + 16\, \omega^2}$$

$$= \frac{36 + 4\, \omega^2}{36 + 4\, \omega^2 + \omega^4} \tag{47}$$

$$\theta_L\, (\omega) = \tan^{-1} \frac{\omega}{3} + \tan^{-1} \frac{4\, \omega}{6 - \omega^2} \tag{48}$$

$$\mathcal{T}_g(\omega) = \frac{d\theta_L}{d\omega} = \frac{3}{9 + \omega^2} + \frac{24 + 4\, \omega^2}{36 + 4\, \omega^2 + \omega^4}$$

$$= \frac{324 + 72\, \omega^2 + 7\, \omega^4}{324 + 72\, \omega^2 + 13\, \omega^4 + \omega^6} \tag{49}$$

$\left| T_{(n-1,n)}(j\omega) \right|^2$ and $\mathcal{T}_g(\omega)$ for n = 1, 2, and 3 are given in Table 3.

Inspection of Table 3 shows that the (n - 1, n) networks have both time-delay and magnitude response with flatness of order (2 n - 1). The (n - 1, n) networks have time delays identical with those of the (n, n) network except for $\pi/2$ radians less phase lag, but they have an inferior magnitude response. However, the (n - 1, n) network could be utilized as a low-pass filter. This filter has the highly coveted maximally flat magnitude

Table 3. $\left[|T(j\omega)|^2 \text{ and } \tau_g \text{ for } P_{(n-1,n)}(e^{-s}) \text{ network} \right]$.

	$\lvert T(j\omega) \rvert^2$	$\tau_g(\omega)$
$P_{0,1}(e^{-s})$	$\frac{1}{1 + \omega^2}$	$\frac{1}{1 + \omega^2}$
$P_{1,2}(e^{-s})$	$\frac{36 + 4\,\omega^2}{36 + 4\,\omega^2 + \omega^4}$	$\frac{324 + 72\,\omega^2 + 7\,\omega^4}{324 + 72\,\omega^2 + 13\,\omega^4 + \omega^6}$
$P_{2,3}(e^{-s})$	$\frac{3,600+216\,\omega^2+9\,\omega^4}{3,600+216\,\omega^2+9\,\omega^4+\omega^6}$	$\frac{1,440,000+172,800\,\omega^2+12,384\,\omega^4+592\,\omega^6+17\,\omega^8}{1,440,000+172,800\,\omega^2+12,384\,\omega^4+80,832\,\omega^6+33\,\omega^8+\omega^{10}}$

characteristic, and as a bonus it would have maximally flat time delay.

THE DIFFERENTIAL EQUATION OF $P_{(u,v)}(e^{-s})$

The unit delay operator satisfies the differential equation

$$e^{-s} + \frac{d}{ds}(e^{-s}) = 0 \qquad (50)$$

The Pade approximants satisfy the equation

$$P_{(u,v)}(e^{-s}) + \frac{d}{ds} P_{(u,v)}(e^{-s}) = E_{(u,v)}(s) \qquad (51)$$

Ideally, $E_{(u,v)}(s)$ should be zero, and its response to any finite excitation would be zero. Even though $E_{(u,v)}(s)$ is not zero it has the properties of an annihilator in a certain sense. The $E_{(u,v)}(s)$ are tabulated in Table 4.

Table 4. The functions $E_{(u,v)}(s)$.

n	$E_{(n-1,n)}(s)$	$E_{(n,n)}(s)$
1	$\frac{s}{(1+s)^2}$	$\frac{-s^2}{(2+s)^2}$
2	$\frac{-2\,s^3}{(6+4\,s+s^2)^2}$	$\frac{s^4}{(12+6\,s+s^2)^2}$
3	$\frac{3\,s^5}{(60+36\,s+9\,s^2+s^3)^2}$	$\frac{-s^6}{(120+60\,s+12\,s^2+s^3)^2}$

The general form for $E_{(u,v)}(s)$ is

$$E_{(u,v)}(s) = \frac{K\ s^{u+v}}{(a_o + a_1 s + \ldots + a_v s^v)^2} = \frac{s^{u+v}}{\phi_v(s)} \qquad (52)$$

The response of $E_{(u,v)}(s)$ to an excitation $F(s)$ is

$$\theta(s) = F(s)\ E_{(u,v)}(s) = F(s)\ \frac{s^{u+v}}{\phi_v(s)} \qquad (53)$$

The Laplace transform equation for $\theta(t)$ is

$$\theta(s) = \int_o^{\infty} \theta(t) e^{-st}\ dt = F(s)\ \frac{s^{u+v}}{\phi_v(s)} \qquad (54)$$

Taking the limit, as s approaches zero, of the equation, yields

$$\int_o^{\infty} \theta(t)dt = \lim_{s \to 0} \theta(s) = \lim_{s \to 0} F(s)\ \frac{s^{u+v}}{\phi_v(s)} \qquad (55)$$

The average value of $\theta(t)$ will be zero unless the denominator of $F(s)$ contains a factor s^q where

$$q \geq u + v \qquad (56)$$

Thus each $E_{(u,v)}(s)$ will annihilate, in an average sense, a certain class of functions, the polynomials in t whose degree does not exceed $u + v - 2$. The class of functions grows as $u + v$ increases or higher order networks are used.

THE CUT-PRODUCT APPROXIMANT

Warfield (11) developed the cut-product approximant of the unit delay operator from the infinite product expansion of $\tanh \frac{s}{2}$.

$$\tanh \frac{s}{2} = \frac{s}{2} \; \frac{\prod_{1}^{\infty} \left(1 + \frac{s^2}{4 n^2 \pi^2}\right)}{\prod_{1}^{\infty} \left(1 + \frac{s^2}{(2n - 1)^2 \pi^2}\right)} \tag{57}$$

The approximants are formed by retaining the first term in the denominator expansion, the first terms in the denominator and numerator expansion, etc. The second order approximant is

$$T_c(s) = \frac{1 - \frac{1}{2} s + \frac{1}{\pi^2} s^2}{1 + \frac{1}{2} s + \frac{1}{\pi^2} s^2} \tag{58}$$

The corresponding time delay is

$$\tau_c(\omega) = \frac{1 + \frac{\omega^2}{\pi^2}}{1 + \frac{\omega^2}{\pi^2}\left(\frac{\pi^2}{4} - 2\right) + \frac{\omega^4}{\pi^4}} \tag{59}$$

$\tau_c(\omega)$ is greater than one second for

$$0 < \omega < .55 \tag{60}$$

and is greater than the (2, 2) Pade time delay for $\omega < 1$ second. Therefore $\tau_c(\omega)$ is less desirable for small ω but more desirable than the Pade time delay characteristic for higher values of ω within the usable range of both networks.

Note the similarity of the cut-product networks to the Pade (n, n) networks. Examination shows that the cut-product can be reduced to unbalanced form by the method used to unbalance the Pade networks.

THE LADDER NETWORKS

Storch (6) derived a class of asymmetric filter networks having (2 n - 1) order flat time delay. These networks are composed of n reactive elements with no mutual coupling and are doubly terminated with resistors.

These networks have transfer functions of

$$T_n(s) = \frac{H}{h_n(s)} \tag{61}$$

where $h_n(s) = s^n Y_n(1/s)$ and the $Y_n(1/s)$ are the Bessell polynomials of (1/s).

$$Y_n(1/s) = \sum_{n=0}^{m} \frac{(n + N)!}{(n - N)!N!(2\ s)^N} \tag{62}$$

These $T_n(s)$ are directly related to the transfer functions of the (n, n) Pade networks. If the summation for $Y_n(1/s)$ is reversed, N is replaced by (n - N). Then

$$h_n(s) = s^n Y_n(1/s) = \frac{1}{2^n} \sum_{N=0}^{n} \frac{(2\ n - N)!(2\ s)^N}{(n - N)!N!} \tag{63}$$

and

$$h_n(s) = \frac{(2\ n)!}{2^n\ n!} D_n(2\ s) \tag{64}$$

From this relationship it is obvious that Storch's filters have a time delay characteristic with (2 n - 1) order flatness. However, examination shows that the amplitude response has only first order flatness.

Weinburg (9, 10) published tables for the design of these filters.

Storch's filters obviously have an inferior filtering action compared to that of the (n - 1, n) Pade networks. The networks have identical time delay characteristics. The Pade (n - 1, n) networks have a constant resistive input impedance while the Storch filters have a widely varying input impedance. The possibility of cascading a number of networks to form a tapped delay line is then very practical if the (n - 1, n) networks are used, but not very practical if Storch's networks are considered. From the preceding discussion, it is apparent that the Pade (n - 1, n) networks are more desirable than Storch's network.

CONCLUSION

Approximation of the continued fraction expansion for e^{-s} was shown to result in the (n, n) and (n - 1, n) Pade approximants. The (n, n) and (n - 1, n) approximants were realized as networks with unbalanced form. These networks are passive, symmetric, and have resistive characteristic impedances of one ohm.

The (n, n) networks are all pass and have a time-delay characteristic with (2n - 1) order flatness. The (n, n) networks have a more desirable time-delay characteristic than the cut-product networks for small values of ω. For large values of ω, within the usable range of both networks, the cut-product networks are more desirable. Both the (n, n) Pade and the cut-product networks are ideally suited for cascading to form tapped-delay lines.

The response of an (n, n) network to a delta function tends to the gaussian form of a delayed delta function as n is increased. The (n, n) Pade networks have poor pulse response, but this could be improved by cascading with one of the (n - 1, n) networks.

The transfer function of the Pade (n - 1, n) network was related to the transfer functions of the (n, n) networks through two differential equations. These equations were used to find the general form of the transfer function of the (n - 1, n) networks. These (n - 1, n) networks were shown to have both amplitude and time-delay characteristics with (2n - 1) order flatness. Although they have application as time-delay networks, they can be used more wisely as filters. In terms of filters the (n - 1, n) network amplitude response is of the maximally flat, or Butterworth type, and in addition has maximally flat time delay.

Therefore these networks are important in any application where good filtering action, constant time delay, and constant characteristic impedance are necessary. Frequency transformations can be used to convert these low-pass filters to band-pass, band-elimination, or high-pass networks having these same characteristics within the pass band.

Since the (n - 1, n) filters have a resistive characteristic impedance, they are perfectly suited for use in cascaded networks or could be paralleled with each other and used as band separation networks.

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This thesis is concerned with the analysis and reduction to unbalanced form of resistor-terminated constant resistance lattice networks whose transfer functions are the (n, n) and (n - 1, n) Pade approximants of the unit delay operator, e^{-s}. The equivalent unbalanced networks contain tapped inductors, are symmetric, and have resistive characteristic impedances.

The networks of the (n, n) approximants are all pass and have an almost constant time-delay characteristic with (2n - 1) order flatness. These (n, n) networks compare favorably with the cut-product approximant networks.

The general form of the (n - 1, n) network transfer function is derived from differential equations which relate the (n, n) and (n - 1, n) network transfer functions. Using these equations and intuitive reasoning, the (n - 1, n) networks are shown to have both amplitude and time-delay characteristics with (2n - 1) order flatness. Since these are characteristics desired in filter design, the (n - 1, n) networks are compared with and are shown to out perform Storch's ladder filters.

Since the function, e^{-s}, satisfies the differential equation

$$f(s) + \frac{d}{ds} f(s) = 0$$

the Pade approximants are substituted into this differential equation, resulting in

$$P_{(m,n)} (e^{-s}) + \frac{d}{ds} P_{(m,n)} (e^{-s}) = E_{(m,n)} (s)$$

$E_{m,n}$ is then investigated and shown to be an approximate annihilator of certain polynomial time functions of prescribed degree.