# Heat transfer in a medium in which many small particles are embedded

A.G. Ramm



**Published Version Information**

**Citation**: Ramm, A.G. (2013). Heat transfer in a medium in which many small particles are embedded. Mathematical Modelling of Natural Phenomena, 8(1), 193-199.



**Digital Object Identifier (DOI)**: doi:10.1051/mmnp/20138114

**Publisher's Link**: http://www.mmnp-journal.org/action/displayJournal?jid=MNP

A. G. Ramm, Heat transfer in a medium in which many small particles are embedded, Math. Model. Nat. Phenom., Vol. 8, N1, (2013), pp. 193-199.

# Heat transfer in a medium in which many small particles are embedded

A G Ramm
Department of Mathematics
Kansas State University, Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

**Abstract**

The heat equation is considered in the complex system consisting of many small bodies (particles) embedded in a given material. On the surfaces of the small bodies a Newton-type boundary condition is imposed. An equation for the limiting field is derived when the characteristic size $a$ of the small bodies tends to zero, their total number $\mathcal{N}(a)$ tends to infinity at a suitable rate, and the distance $d = d(a)$ between neighboring small bodies tends to zero $a << d$. No periodicity is assumed about the distribution of the small bodies.

Keywords:
heat transfer; many-body problem
MSC 80M40; 80A20
PACS 65.80.-g

## 1 Introduction

Let many small bodies (particles) $\mathcal{D}_m$, $1 \leq m \leq M$, be distributed in a bounded domain $\mathcal{D} \subset \mathbb{R}^3$, diam$\mathcal{D}_m = 2a$. The small bodies are distributed according to the law

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_{\Delta} N(x)dx[1 + o(1)], a \to 0. \tag{1}$$

Here $\Delta \subset \mathcal{D}$ is an arbitrary open subdomain of $\mathcal{D}$, $\kappa \in (0, 1)$ is a constant, $N(x) \geq 0$ is a continuous function, and $\mathcal{N}(\Delta)$ is the number of the small bodies $\mathcal{D}_m$ in $\Delta$. The heat equation can be stated as follows:

$$u_t = \nabla^2 u + f(x) \text{ in } \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} \mathcal{D}_m := \Omega, \, u|_{t=0} = 0, \tag{2}$$

$$u_N = \zeta_m u \text{ on } \mathcal{S}_m, 1 \leq m \leq M. \tag{3}$$

Here $N$ is the outer unit normal to $\mathcal{S} := \bigcup_{m=1}^{M} \mathcal{S}_m$, $\zeta_m = \frac{h(x_m)}{a^\kappa}, x_m \in \mathcal{D}_m, 1 \leq m \leq M$, where $h(x)$ is a continuous function in $\mathcal{D}$.
Denote $\mathcal{U} := \mathcal{U}(x,\lambda) = \int_0^\infty e^{-\lambda t} u(x,t)dt$. Then, (2) - (3) imply

$$-\nabla^2 \mathcal{U} + \lambda \mathcal{U} = \lambda^{-1} f(x) \text{ in } \Omega, \tag{4}$$

$$\mathcal{U}_N = \zeta_m \mathcal{U} \text{ on } \mathcal{S}_m, 1 \leq m \leq M. \tag{5}$$

Let

$$g(x,y) := g(x,y,\lambda) = \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|}, \ \frac{1}{\lambda}\int_{\mathbb{R}^3} g(x,y) f(y)dy = F(x,\lambda). \tag{6}$$

Look for the solution to (4) - (5) of the form

$$\mathcal{U}(x,\lambda) = F(x,\lambda) + \sum_{m=1}^{M} \int_{\mathcal{S}_m} g(x,s)\sigma_m(s)ds, \quad \mathcal{U}(x,\lambda) := \mathcal{U}(x) := \mathcal{U}, \tag{7}$$

where $\sigma_m$ are unknown and should be found from the boundary conditions (5). Equation (4) is satisfied by $\mathcal{U}$ of the form (7) for any $\sigma_m$. To satisfy (5) one has to solve equation

$$\frac{\partial \mathcal{U}_e(x)}{\partial N} + \frac{A_m\sigma_m - \sigma_m}{2} - \zeta_m \mathcal{U}_e - \zeta_m T_m \sigma_m = 0 \text{ on } \mathcal{S}_m, 1 \leq m \leq M. \tag{8}$$

Here

$$\mathcal{U}_e(x) := \mathcal{U}_{e,m}(x) := \mathcal{U}(x) - \int_{\mathcal{S}_m} g(x,s)\sigma_m(s)ds, \tag{9}$$

$$T_m\sigma_m = \int_{\mathcal{S}_m} g(s,s')\sigma_m(s')ds', \ A_m\sigma_m = 2\int_{\mathcal{S}_m} \frac{\partial g(s,s')}{\partial N_S}\sigma_m(s')ds', \tag{10}$$

and the known formula for the outer limiting value on $\mathcal{S}_m$ of the normal derivative of a simple layer potential was used. We now apply the ideas and methods for solving many-body scattering problems developed in [3] - [4].

Let us call $\mathcal{U}_{e,m}$ the effective (self-consistent) value of $\mathcal{U}$, acting on $m$-th body. As $a \to 0$, the dependence on $m$ disappears, since $\int_{\mathcal{S}_m} g(x,s)\sigma_m(s)ds \to 0$ as $a \to 0$. One has

$$\mathcal{U}(x,\lambda) = F(x,\lambda) + \sum_{m=1}^{M} g(x,x_m)Q_m + \mathcal{J}_2, \quad x_m \in \mathcal{D}_m, \tag{11}$$

where $Q_m := \int_{\mathcal{S}_m} \sigma_m(s)ds$,

$$\mathcal{J}_2 := \sum_{m=1}^{M} \int_{\mathcal{S}_m} [g(x,s') - g(x,x_m)]\sigma_m(s')ds', \ \mathcal{J}_1 := \sum_{m=1}^{M} g(x,x_m)Q_m. \tag{12}$$

We prove (in Section 3) that

$$|\mathcal{J}_2| << |\mathcal{J}_1| \text{ as } a \to 0 \tag{13}$$

provided that

$$\lim_{a\to 0} \frac{a}{d(a)} = 0, \tag{14}$$

where $d(a) = d$ is the minimal distance between neighboring particles. If (13) holds, then problem (4) - (5) is solved asymptotically by the formula

$$\mathcal{U}(x, \lambda) = F(x, \lambda) + \sum_{m=1}^{M} g(x, x_m) Q_m, \ a \to 0, \tag{15}$$

provided that asymptotic formulas for $Q_m$, as $a \to 0$, are found. To find formulas for $Q_m$, let us integrate (8) over $\mathcal{S}_m$ and estimate the order of the terms in the resulting equation as $a \to 0$. We get

$$\int_{\mathcal{S}_m} \frac{\partial \mathcal{U}_e}{\partial N} ds = \int_{\mathcal{D}_m} \nabla^2 \mathcal{U}_e dx = O(a^3) \tag{16}$$

Here we assumed that $|\nabla^2 \mathcal{U}_e| = O(1), a \to 0$. This assumption will be justified in Section 2.

$$\int_{\mathcal{S}_m} \frac{A_m \sigma_m - \sigma_m}{2} ds = -Q_m[1 + o(1)], \ a \to 0. \tag{17}$$

This relation is justified in Section 2. Furthermore,

$$-\zeta_m \int_{\mathcal{S}_m} \mathcal{U}_e ds = -\zeta_m |\mathcal{S}_m| \mathcal{U}_e(x_m) = O(a^{2-\kappa}), \ a \to 0, \tag{18}$$

where $|\mathcal{S}_m| = O(a^2)$ is the surface area of $\mathcal{S}_m$. Finally,

$$-\zeta_m \int_{\mathcal{S}_m} ds \int_{\mathcal{S}_m} g(s, s') \sigma_m(s') ds' = -\zeta_m \int_{\mathcal{S}_m} ds' \sigma_m(s') \int_{\mathcal{S}_m} ds g(s, s')$$
$$= Q_m O(a^{1-\kappa}), \quad a \to 0. \tag{19}$$

Thus, the main term of the asymptotics of $Q_m$ is

$$Q_m = -\zeta_m |\mathcal{S}_m| \mathcal{U}_e(x_m). \tag{20}$$

Formulas (20) and (15) yield

$$\mathcal{U}(x, \lambda) = F(x, \lambda) - \sum_{m=1}^{M} g(x, x_m) \zeta_m |\mathcal{S}_m| \mathcal{U}_e(x_m, \lambda), \tag{21}$$

and

$$\mathcal{U}_e(x_m, \lambda) = F(x_m, \lambda) - \sum_{m' \neq m, m'=1}^{M} g(x_m, x_{m'}) \zeta_{m'} |\mathcal{S}_{m'}| \mathcal{U}_e(x_{m'}, \lambda). \tag{22}$$

Denote $\mathcal{U}_e(x_m,\lambda) := \mathcal{U}_m, F(x_m,\lambda) := F_m, g(x_m,x_{m'}) := g_{mm'}$, and write (22) as a linear algebraic system

$$\mathcal{U}_m = F_m - a^{2-\kappa}\sum_{m'\neq m} g_{mm'}h_{m'}c_{m'}\mathcal{U}_{m'}, \quad 1\leq m\leq M, \tag{23}$$

where $h_{m'} = h(x_{m'}), \zeta_{m'} = \frac{h_{m'}}{a^\kappa}, c_{m'} := |S_{m'}|a^{-2}$.
Consider a partition of the bounded domain $\mathcal{D}$, in which the small bodies are distributed, into a union of $P << M$ small nonintersecting cubes $\Delta_p$, $1\leq p\leq P$, of side $b >> d$, $b = b(a)\to 0$ as $a\to 0$. Let $x_p\in\Delta_p$, $|\Delta_p|$ = volume of $\Delta_p$. One has

$$a^{2-\kappa}\sum_{m'=1,m'\neq m}^{M} g_{mm'}h_{m'}c_{m'}\mathcal{U}_{m'} = a^{2-\kappa}\sum_{p'=1,p'\neq p}^{P} g_{pp'}h_{p'}c_{p'}\mathcal{U}_{p'}\sum_{x_{m'}\in\Delta_{p'}} 1 =$$

$$= \sum_{p'\neq p} g_{pp'}h_{p'}c_{p'}\mathcal{U}_{p'}N(x_{p'})|\Delta_{p'}|[1+o(1)], \quad a\to 0. \tag{24}$$

Thus, (23) yields

$$\mathcal{U}_p = F_p - \sum_{p'\neq p,p'=1}^{P} g_{pp'}h_{p'}c_{p'}N_{p'}\mathcal{U}_{p'}|\Delta_{p'}|, \quad 1\leq p\leq P \tag{25}$$

We have assumed that

$$h_{m'} = h_{p'}[1+o(1)], c_{m'} = c_{p'}[1+o(1)], \mathcal{U}_{m'} = \mathcal{U}_{p'}[1+o(1)],\ a\to 0, \tag{26}$$

for $x_{m'}\in\Delta_{p'}$. This assumption is justified if the functions $h(x)$,$\mathcal{U}(x,\lambda)$, $c(x) = \lim_{x_{m'}\in\Delta_x, a\to 0}\dfrac{|S_{m'}|}{a^2}$, and $N(x)$ are continuous. The function $h(x)$ and $N(x)$ are continuous by the assumption. The continuity of the $\mathcal{U}(x,\lambda)$ is proved in Section 3, and the continuity of $c(x)$ is assumed. If all the small bodies are identical, then $c(x) = c =$ const.
The sum in the right-hand side of (25) is the Riemannian sum for the integral

$$lim_{a\to 0}\sum_{p'=1,p'\neq p}^{P} g_{pp'}h_{p'}c_{p'}N(x_{p'})\mathcal{U}_{p'} = \int_{\mathcal{D}} g(x,y)h(y)c(y)N(y)\mathcal{U}(y,\lambda)dy \tag{27}$$

Therefore, linear algebraic system (25) is a collocation method for solving integral equation

$$\mathcal{U}(x,\lambda) = F(x,\lambda) - \int_{\mathcal{D}} g(x,y)h(y)c(y)N(y)\mathcal{U}(y,\lambda)dy. \tag{28}$$

Convergence of this method for solving equations with weakly singular kernels is proved in [5].

Applying the operator $-\nabla^2 + \lambda$ and then taking the inverse Laplace transform of (28) yields

$$u_t = \Delta u + f(x) - q(x)u, \; q(x) := h(x)c(x)N(x). \tag{29}$$

One concludes that the limiting equation for the temperature contains the term $q(x)u$. Thus, the embedding of many small particles creates a distribution of source and sink terms in the medium, the distribution of which is described by the term $q(x)u$.
If one solves equation (28) for $\mathcal{U}(x, \lambda)$, or linear algebraic system (25) for $\mathcal{U}_p(\lambda)$, then one can Laplace-invert $\mathcal{U}(x, \lambda)$ for $\mathcal{U}(x, t)$. Numerical methods for Laplace inversion from the real axis are discussed in [6] - [7].
If one is interested only in the average temperature, one can use the relation

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T u(x, t)dt = \lim_{\lambda\to 0} \lambda\mathcal{U}(x, \lambda) := \psi(x). \tag{30}$$

Relation (30) is proved in Section 3, which holds if the limit on one of its sides exists. The limit on the right-hand side of (30) can be calculated by the formula

$$\psi(x) = (I + B)^{-1}\varphi, \; \varphi = \int \frac{1}{4\pi|x - y|} f(y)dy. \tag{31}$$

Here, $B$ is the operator $B\psi := \int \frac{q(y)\psi(y)}{4\pi|x - y|}dy, \; q(x) := h(x)c(x)N(x)$. From the physical point of view the function $h(x)$ is non-positive because the flux $-\nabla u$ of the heat flow is proportional to the temperature $u$ and is directed along the outer normal $N$: $-u_N = h_1 u$, where $h_1 = -h > 0$. Thus, $q \le 0$. It is proved in [8] - [9] that zero is not an eigenvalue of the operator $-\nabla^2 + q(x)$ provided that $q(x) \ge 0$ and $q = O(\frac{1}{|x|^{2+\epsilon}})$ as $|x| \to \infty, \epsilon > 0$. In our case, $q(x) = 0$ outside $\mathcal{D}$, so the operator $(I + B)^{-1}$ exists and is bounded in $C(\mathcal{D})$. Let us formulate our basic result.

**Theorem 1.** *Assume (1), (14), and $h \le 0$. Then, there exists the limit $\mathcal{U}(x, \lambda)$ of $\mathcal{U}_e(x, \lambda)$ as $a \to 0$, $\mathcal{U}(x, \lambda)$ solves equation (28), and there exists the limit (30), where $\psi(x)$ is given by formula (31).*

## 2 Proofs

**Lemma 1.** *Assume (14). Then relation (13) holds.*

*Proof.* One has

$$\mathcal{J}_{1,m} := |g(x, x_m)Q| = O\Big(\frac{|Q_m|e^{-\sqrt{x}|x-x_m|}}{4\pi|x - x_m|}\Big) \le \frac{e^{-1}|Q_m|}{|x - x_m|}, |x - x_m| \ge d. \tag{32}$$

$$\mathcal{J}_{2,m} := \int_{\mathcal{S}_m} \frac{e^{-\sqrt{\lambda}|x-x_m|}}{4\pi|x - x_m|} \max\left(\sqrt{\lambda}a, \frac{a}{|x - x_m|}\right)|\sigma_m(s')|ds' \le O\Big(\frac{|Q_m|a}{|x - x_m|^2}\Big), \tag{33}$$

where $|x-x_m|\geq 2$, and the inequality $\max_{\lambda\geq 0}(\sqrt{\lambda}e^{-\sqrt{\lambda}|x-x_m|})\leq \frac{e^{-1}}{|x-x_m|}$ was used. The $|Q_m|\neq 0$. In fact, $\sigma_m$ keeps sign on $\mathcal{S}_m$, as follows from equation (8) as $a\to 0$.
It follows from (32) - (33) that

$$\left|\frac{\mathcal{J}_{2,m}}{\mathcal{J}_{1,m}}\right|\leq O\left(\left|\frac{a}{x-x_m}\right|\right)\leq O\left(\frac{a}{d}\right)<<1. \tag{34}$$

From (34) by the arguments similar to the given in [10] one obtains (13). □

Let us justify relation (17). As $a\to 0$, one has

$$\frac{\partial}{\partial N_s}\frac{e^{-\sqrt{\lambda}|s-s'|}}{4\pi|s-s'|}=\frac{\partial}{\partial N_s}\frac{1}{4\pi|s-s'|}+\frac{\partial}{\partial N_s}\frac{e^{-\sqrt{\lambda}|s-s'|}-1}{4\pi|s-s'|}. \tag{35}$$

It is known (see [3]) that

$$\int_{\mathcal{S}_m}ds\int_{\mathcal{S}_m}\frac{\partial}{\partial N_s}\frac{1}{4\pi|s-s'|}\sigma_m(s')ds'=-\int_{\mathcal{S}_m}\sigma_m(s')ds'=-Q_m. \tag{36}$$

On the other hand, as $a\to 0$, one has

$$\left|\int_{\mathcal{S}_m}ds\int_{\mathcal{S}_m}\frac{e^{-\sqrt{\lambda}|s-s'|}-1}{4\pi|s-s'|}\sigma_m(s')ds'\right|\leq |Q_m|\int_{\mathcal{S}_m}ds\frac{1-e^{-\sqrt{\lambda}|s-s'|}}{4\pi|s-s'|}=o(Q_m). \tag{37}$$

The relations (36) and (37) justify (17).

*Proof of (30).* Denote

$$\frac{1}{t}\int_0^t u(t)dt:=v(t),\quad \bar{u}(\sigma):=\int_0^\infty e^{-\sigma t}u(t)dt.$$

Then $\bar{v}(\lambda)=\int_\lambda^\infty\frac{\bar{u}(\sigma)}{\sigma}d\sigma$ by the properties of the Laplace transform. Assume that the limit $v(\infty):=v_\infty$ exists:

$$\lim_{t\to\infty}v(t)=v_\infty. \tag{38}$$

Then,

$$v_\infty=\lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}v(t)dt=\lim_{\lambda\to 0}\lambda\bar{v}(\lambda).$$

Indeed $\lambda\int_0^\infty e^{-\lambda t}dt=1$, so $\lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}(v(t)-v_\infty)dt=0$, and (38) is verified. One has

$$\lim_{\lambda\to 0}\lambda\bar{v}(\lambda)=\lim_{\lambda\to 0}\int_\lambda^\infty\frac{\lambda}{\sigma}\bar{u}(\sigma)d\sigma=\lim_{\lambda\to 0}\lambda\bar{u}(\lambda). \tag{39}$$

Let us check this:

$$\lim_{\lambda\to 0}\int_\lambda^\infty \frac{\lambda}{\sigma}\bar{u}(\sigma)d\sigma = \lim_{\lambda\to 0}\int_\lambda^\infty \frac{\lambda}{\sigma^2}\sigma\bar{u}(\sigma)d\sigma = \lim_{\sigma\to 0}\sigma\bar{u}(\sigma), \tag{40}$$

where we have used the relation $\int_\lambda^\infty \frac{\lambda}{\sigma^2}d\sigma = 1$.
Alternatively, let $\sigma^{-1} = \gamma$. Then,

$$\int_\lambda^\infty \frac{\lambda}{\sigma^2}\sigma\bar{u}(\sigma)d\sigma = \frac{1}{1/\lambda}\int_0^{1/\lambda}\frac{1}{\gamma}\bar{u}(\frac{1}{\gamma})d\gamma = \frac{1}{\omega}\int_0^\omega \frac{1}{\gamma}\bar{u}(\frac{1}{\gamma})d\gamma. \tag{41}$$

If $\lambda \to 0$, then $\omega = \lambda^{-1} \to \infty$, and if $\psi := \gamma^{-1}\bar{u}(\gamma^{-1})$, then

$$\lim_{\omega\to\infty}\frac{1}{\omega}\int_0^\omega \psi d\gamma = \psi(\infty) = \lim_{\gamma\to\infty}\gamma^{-1}\bar{u}(\gamma^{-1}) = \lim_{\sigma\to 0}\sigma\bar{u}(\sigma). \tag{42}$$

□

Methods of our proof of Theorem 1 are quite different from the proof of homogenization theory results in [1] and [2].

# References


[1] V.Jikov, S. Kozlov, O.Oleinik, *Homogenization of differential operators and integral functionals*, Springer, Berlin, 1994.

[2] V. A. Marchenko, E. Ya. Khruslov, *Homogenization of partial differential equations*, Birkhäuser, Boston, 2006.

[3] A. G. Ramm, *Wave scattering by small bodies of arbitrary shapes*, World Sci. Publishers, Singapore, 2005.

[4] A.G.Ramm, Wave scattering by many small bodies and creating materials with a desired refraction coefficient, Afrika Matematika, 22, N1, (2011), 33-55.

[5] A.G.Ramm, A collocation method for solving integral equations, Internat. Journ. Comp. Sci and Math., 3, N2, (2009), 222-228.

[6] A.G.Ramm, Inversion of the Laplace transform from the real axis, Inverse problems, 2, (1986), L55-59.

[7] A.G.Ramm, S.Indratno, Inversion of the Laplace transform from the real axis using an adaptive iterative method, Internat. Jour. Math. Math. Sci (IJMMS), Vol. 2009, Article 898195, 38 pages.

[8] A.G.Ramm, Sufficient conditions for zero not to be an eigenvalue of the Schrödinger operator, J. Math Phys., 28, (1987), 1341-1343.

[9] A.G.Ramm, Conditions for zero not to be an eigenvalue of the Schrödinger operator, J. Math. Phys. 29, (1988), 1431-1432.

[10] A.G.Ramm, Many-body wave scattering by small bodies and applications, J. Math. Phys., 48, N10, (2007), 103511.