Modulus of edge covers, fractional edge covers, and stars

by

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B.S., University of Puerto Rico, Rio Piedras, 2018

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Abstract

Modulus on graphs is a very flexible and useful tool for measuring the richness of families of objects defined on a graph. It has been shown that the modulus of special families of objects generalizes classical network theoretic quantities such as shortest path, max flow/min cut, and effective resistance. When the family of graph objects is finite, the modulus problem can be viewed as a convex optimization problem where each object in the family induces an inequality constraint. This implies that calculating the modulus of large families can be difficult due to the large number of constraints. One focus of this dissertation is to establish an efficient way of calculating the modulus of fractional edge covers, a very large family, using the modulus of stars, a much smaller family.

In this dissertation, the main objects of interest are edge covers. An edge cover is a subset of edges with the property that each node of the graph sees at least one edge in the cover. The concept of edge covers can be relaxed by defining fractional edge covers, where edges are assigned weights and a node is said to be 'covered' if the weights at each node add to at least one. Direct calculations of both the edge cover modulus and fractional edge cover modulus can be expensive as these families tend to be exponentially large. Moreover, the modulus of these families are related; from the modulus of fractional edge covers we obtain upper and lower bounds for the modulus of edge covers.

Through the theory of Fulkerson blocking duality, every family of objects has a corresponding dual family whose modulus is closely related to the modulus of the original family. One of the main results in this dissertation is that the dual family of fractional edge covers is the family of stars, which greatly reduces the number of constraints for the modulus problem. This result allows an approximation for the modulus of edge covers using the modulus of stars. This dissertation studies the relationships of these three families—edge

covers, fractional edge covers, and stars—as well as the computational complexity involved in calculating the modulus of these families.

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Major Professor
Dr. Nathan Albin

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Dedication

This work is for all the people who showed me love and support while I was in graduate school.

Chapter 1

Introduction

Graph Theory is the study of mathematical structures that model relationships between objects. In its simplest form, a graph is a collection of *nodes* or *vertices* which are connected by *edges*. One type of *object* that can be defined on a graph is called a *covering*. There are different types of 'covers' that can be defined on the edges of a graph:

1. An **edge cover** is a subset of edges such that each node in the graph is incident to *at least* one edge of the set.

2. A **matching** is a subset of edges such that each node in the graph is incident to *at most* one edge of the set.

3. A **perfect matching** is a subset of edges such that each node in the graph is incident to *exactly* one edge of the set.

In this dissertation, we study edge covers and their relationship to other families of graph objects. In the literature, edge covers have been studied through the minimum edge cover problem, which is a very well-known covering problem [15, 18, 27, 36]. Edge covers are related to other problems studied in the field which include:

- Spanning trees: A spanning tree T of a graph G is a subgraph that is a tree that contains all of the vertices of G. If we consider the edges of a spanning tree, these edges are in fact an edge cover. Therefore, the family of spanning trees is a subset of the family of edge covers.

- Graph Coloring: There are many different ways of coloring the edges of a graph. One such coloring is analogous to the idea of edge covers where every vertex sees each color at least one time [37].

The idea of a *fractional* object has been introduced and studied in [12, 17, 22, 28, 29]. Specifically, given a non-negative vector $x \in \mathbb{R}^E$ we define the following:

1. x is a **fractional edge cover** if $x(\delta(v)) := \sum_{e\in\delta(v)} x(e) \geq 1$, where $\delta(v)$ is the set of edges incident to node v (called a *star*), for every $v \in V$.

2. x is a **fractional matching** if $x(\delta(v)) \leq 1$, for every $v \in V$.

3. x is a **fractional perfect matching** if $x(\delta(v)) = 1$, for every $v \in V$.

Therefore, a fractional edge cover is a generalization of an edge cover.

One way to study the family of edge covers and the family of fractional edge covers is through the *p-modulus problem*. The modulus problem originates in Complex Analysis as a conformally invariant way of studying families of curves [2]. This problem was translated to the field of Graph Theory to study families of graph objects [20, 30]. It has been shown that the modulus of special families of graph objects generalize known graph theoretic quantities and other concepts. For example, the modulus of all paths connecting two nodes in a graph is related to shortest path, max flow/min cut, and effective resistance [6, 32], the modulus of all cycles can be used for clustering and community detection [33], and the modulus of all spanning trees can be used to describe a hierarchical decomposition of a graph [8, 9, 23]. In general, modulus can be adapted to any family of graph objects and will measure the richness of that family.

The modulus problem is studied over a family of graph objects, and when that family is finite the problem can be viewed as a convex optimization problem where each object induces an inequality constraint. This implies that the larger the family, the more inequality constraints need to be satisfied. The family of edge covers is combinatorially large, so calculating the edge cover modulus can be difficult. Through the theory of Fulkerson duality, it has been shown that every family of objects has a corresponding dual family whose modulus is closely related to the modulus of the original family [7, 16]. In this dissertation, we prove that the dual family of fractional edge covers is the family of stars, which greatly reduces the number of constraints for the p-modulus problem. In this way, we can calculate the modulus of fractional edge covers using the modulus of stars, and then obtain a bound for the modulus of edge covers.

A probabilistic interpretation for the p-modulus problem has also proven valuable in some cases [3, 7, 10]. This interpretation allows us to reinterpret the modulus problem as an optimization problem related to random objects. Specifically, it was shown that solving the modulus problem is equivalent to finding a probability mass function (pmf) on the family of objects that minimizes a function of certain expectations on the edges. We use the probabilistic interpretation to obtain lower bounds for the modulus of each family of objects for specific graphs and prove more general results for the modulus of stars.

Organization. The remainder of this document is organized as follows. In Chapter 2, we introduce the p-modulus problem, as well as the theory of Fulkerson duality, the probabilistic interpretation of modulus, and the definition of equivalent families. In Chapter 3, we discuss the families of edge covers, fractional edge covers, and stars, and how these families are related. In Chapter 4, we discuss the modulus of the three families discussed in Chapter 3. Chapter 5 discusses numerical examples for each family and Chapter 6 closes with a conclusion of this dissertation and ideas for extensions of this work. The contributions in this document include:

- The definition of equivalent families with respect to modulus in Section 2.3.
- The definition of basic fractional edge covers in Section 3.2.
- A description of the extreme points of the convex hull of the family of fractional edge covers in Section 3.2.

- A proof that the dual family of fractional edge covers is the family of stars in Section 3.3.1.
- Bounds on the modulus of edge covers with respect to the modulus of fractional edge covers in Section 4.2.1. This also includes an improved bound in Theorem 4.7.
- An exploration of the tightness of the bounds for the modulus of edge covers in Section 5.3.

Chapter 2

The Modulus Problem

In this chapter, we define the p-modulus problem as it's been studied previously in [5, 6]. We also discuss the theory of Fulkerson duality and the probabilistic interpretation of modulus as described in [3, 7], respectively.

2.1 Definitions and Notation

We consider an undirected, weighted graph $G = (V, E, \sigma)$ with vertex set V, edge set E, and a positive vector $\sigma \in \mathbb{R}^E_{>0}$ that assigns to each edge a positive weight, $\sigma(e)$. In general, we let $n = |V|$ be the number of nodes and $m = |E|$ be the number of edges in the graph G. Let Γ be a *family of objects* on G. The concept of *object* is very flexible; in this dissertation an object will either refer to a subset of edges of a graph (for example, a path), or a vector in $\mathbb{R}^E$ that assigns each edge of the graph a value. As stated in the introduction, we are focused primarily on three specific families: the families of stars, edge covers, and fractional edge covers. Each of these families will be defined in Chapter 3.

Given a family Γ we can define it's nonnegative *usage matrix*, $\mathcal{N} \in \mathbb{R}^{\Gamma \times E}_{\geq 0}$, where $\mathcal{N}(\gamma, e)$ indicates the degree to which the object $\gamma \in \Gamma$ "uses" the edge $e \in E$. When Γ consists of subsets of E, a natural choice for $\mathcal{N}$ is the indicator function

$$\mathcal{N}(\gamma, e) = \mathbb{1}_\gamma(e) := \begin{cases} 1 & \text{if } e \in \gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

For the family of fractional edge covers, however, $\mathcal{N}$ will be allowed to take other real values. In this paper, we restrict our attention to families that are *nontrivial* in the sense that each row of $\mathcal{N}$ contains at least one nonzero entry. Note that we refer to $\mathcal{N}$ as a matrix, even if Γ is an infinite set.

The usage matrix provides a useful representation of the objects in γ; by associating each γ with the corresponding row vector $\mathcal{N}(\gamma, \cdot)$ in the usage matrix we may view a family of objects as a subset of the nonnegative vectors $\mathbb{R}^E_{\geq 0}$. Given a family of objects $\Gamma \subseteq \mathbb{R}^E_{\geq 0}$ it is often useful to consider its convex hull, $\mathrm{co}(\Gamma)$, as well as its *dominant*,

$$\mathrm{Dom}(\Gamma) = \mathrm{co}(\Gamma) + \mathbb{R}^E_{\geq 0}.$$

A nonnegative vector, $\rho \in \mathbb{R}^E_{\geq 0}$, is called a *density*. Each density induces a measure of length, called the *ρ-length*, on the objects in Γ. Given a density ρ and an object $\gamma \in \Gamma$, the ρ-length of γ, $\ell_\rho(\gamma)$, is defined as

$$\ell_\rho(\gamma) := \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) = \rho^T \gamma,$$

where the last equality arises from identifying γ with its row in the matrix $\mathcal{N}$.

A density ρ is called *admissible for* Γ, or simply *admissible*, if $\ell_\rho(\gamma) \geq 1$ for every $\gamma \in \Gamma$. The set of all admissible densities for a family Γ is denoted

$$\mathrm{Adm}(\Gamma) := \{\rho \in \mathbb{R}^E_{\geq 0} : \ell_\rho(\gamma) \geq 1, \text{ for all } \gamma \in \Gamma\}.$$

One lemma that we will use repeatedly is the following.

Lemma 2.1. *Suppose* Γ_1 *and* Γ_2 *are two families of objects satisfying* $\Gamma_1 \subseteq \Gamma_2$*, then* $\mathrm{Adm}(\Gamma_2) \subseteq \mathrm{Adm}(\Gamma_1)$.

Proof. This is evident from the definitions. Any density that is admissible for the larger family, Γ_2, is necessarily admissible for Γ_1. □

2.2 The p-Modulus Problem

For $1 \leq p \leq \infty$, the *p-energy* of a density ρ is defined as

$$\mathcal{E}_{p,\sigma}(\rho) := \begin{cases} \sum_{e\in E} \sigma(e)\rho(e)^p, & \text{if } 1 \leq p < \infty, \\ \max_{e\in E} \sigma(e)\rho(e), & \text{if } p = \infty. \end{cases}$$

The *p-modulus* of Γ is then defined as

$$\mathrm{Mod}_{p,\sigma}(\Gamma) := \inf_{\rho\in\mathrm{Adm}(\Gamma)} \mathcal{E}_{p,\sigma}(\rho). \tag{2.2}$$

If the graph is unweighted, we define $\sigma \equiv 1$ and adopt the simplified notation $\mathcal{E}_p(\rho)$ and $\mathrm{Mod}_p(\Gamma)$.

In the case that Γ is finite, modulus can be expressed as a convex optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & \mathcal{E}_{p,\sigma}(\rho) \\ \text{subject to} \quad & \rho \succeq \mathbf{0} \\ & \mathcal{N}\rho \succeq \mathbf{1}. \end{aligned} \tag{2.3}$$

The notation $\succeq$ indicates elementwise comparison and $\mathbf{0}$ and $\mathbf{1}$ indicate the appropriately shaped vectors of all zeros and ones, respectively.

A density $\rho \in \mathrm{Adm}(\Gamma)$ is said to be *extremal* or *optimal* if $\mathcal{E}_{p,\sigma}(\rho) = \mathrm{Mod}_{p,\sigma}(\Gamma)$. The notation ρ^* is commonly used to denote an extremal density. If Γ is finite, then [6, Theorem

4.1] implies that an extremal density exists and it is unique for $1 < p < \infty$. A useful property of modulus comes from Proposition 3.4 in [6], which is a direct consequence of Lemma 2.1.

Proposition 2.2 (Monotonicity)**.** *Let* Γ_1 *and* Γ_2 *be families of graph objects. If* $\Gamma_1 \subseteq \Gamma_2$*, then* $\mathrm{Mod}_{p,\sigma}(\Gamma_1) \le \mathrm{Mod}_{p,\sigma}(\Gamma_2)$*.*

2.3 Equivalent Families

As stated in the previous section, the modulus problem minimizes the p-energy over the admissible set of a family Γ. Note that the admissible set is what changes the value of the problem. This leads to the natural question of asking if there are multiple families that have the same admissible set, hence the same modulus? This leads to the definition of *equivalent* families given below, which can help simplify the modulus problem, as well as relate the modulus of one family to the modulus of another.

Definition 2.3. Two families of objects, Γ and Γ', are called *equivalent* (in the sense of modulus) if $\mathrm{Adm}(\Gamma) = \mathrm{Adm}(\Gamma')$. We write $\Gamma \simeq \Gamma'$ to denote that two families are equivalent.

In light of (2.2), this definition implies that $\mathrm{Mod}_{p,\sigma}(\Gamma) = \mathrm{Mod}_{p,\sigma}(\Gamma')$ for any choice of the parameter p and weights σ; equivalent families are indistinguishable in the context of modulus.

One straightforward example of equivalence comes from the following lemma.

Lemma 2.4. *Let* Γ *be a family of objects on* G*. Then* $\Gamma \simeq \mathrm{Dom}(\Gamma)$*.*

Proof. By definition, $\Gamma \subseteq \mathrm{Dom}(\Gamma)$ so by Lemma 2.1 $\mathrm{Adm}(\mathrm{Dom}(\Gamma)) \subseteq \mathrm{Adm}(\Gamma)$. Let $\rho \in \mathrm{Adm}(\Gamma)$ and let $\tilde{\gamma} \in \mathrm{Dom}(\Gamma)$. Then there must exist a collection of objects $\gamma_1, \gamma_2, \cdots, \gamma_r \in \Gamma$, a choice of weights $\mu_1, \mu_2, \ldots, \mu_r \ge 0$ summing to one, and a vector $\xi \in \mathbb{R}^E_{\ge 0}$ such that

$$\tilde{\gamma} = \sum_{i=1}^{r} \mu_i \gamma_i + \xi.$$

Since ρ and ξ are nonnegative, and since ρ is admissible for Γ,

$$\rho^T\tilde{\gamma} = \sum_{i=1}^{r} \mu_i \rho^T \gamma_i + \rho^T \xi \geq 1.$$

Thus, $\rho \in \mathrm{Adm}(\mathrm{Dom}(\Gamma))$. □

For a given family Γ, consider the set of extreme points $\mathrm{ext}(\mathrm{Dom}(\Gamma))$. The extreme points are nonnegative vectors in $\mathbb{R}^E$ and, therefore, can be viewed as another family of objects. Lemma 2.4 implies that $\Gamma \simeq \mathrm{ext}(\mathrm{Dom}(\Gamma))$, since both families share the same dominant.

As an application of this last equivalence, consider the family Γ'_{st} of all walks in G connecting two distinct vertices s and t and the family Γ_{st} of all simple paths connecting s and t. Then, since $\Gamma_{st} = \mathrm{ext}(\mathrm{Dom}(\Gamma'_{st}))$, it follows that $\Gamma_{st} \simeq \Gamma'_{st}$. This recovers the intuitive observation that a density is admissible for the family of st-walks if and only if it is admissible for the family of st-paths.

2.4 Fulkerson Duality

The theory of Fulkerson duality applied to modulus was developed in [7]. If Γ is a finite family, then its admissible set, $\mathrm{Adm}(\Gamma)$, has finitely many faces and finitely many extreme points. Since $\mathrm{Adm}(\Gamma)$ is a recessive closed convex set, it can be written as the dominant of its extreme points

$$\mathrm{Adm}(\Gamma) = \mathrm{Dom}(\mathrm{ext}(\mathrm{Adm}(\Gamma))).$$

We define

$$\hat{\Gamma} := \mathrm{ext}(\mathrm{Adm}(\Gamma)) = \{\hat{\gamma}_1, \ldots, \hat{\gamma}_r\} \subseteq \mathbb{R}^E_{\geq 0}$$

to be the *Fulkerson blocker of* Γ. These extreme points can be thought of as another family of graph objects with usage matrix $\hat{\mathcal{N}} \in \mathbb{R}^{\hat{\Gamma}\times E}_{\geq 0}$. This construction provides a duality among families of objects due to the fact that

$$\hat{\hat{\Gamma}} = \mathrm{ext}(\mathrm{Dom}(\Gamma)) \simeq \Gamma.$$

The relationship between Γ and $\hat{\Gamma}$ in terms of modulus is given by the following theorem.

Theorem 2.5 (Theorem 4 in [7]). *Let $G = (V, E)$ be a graph and let Γ be a non-trivial finite family of objects on G with Fulkerson blocker $\hat{\Gamma}$. Let the exponent $1 < p < \infty$ be given, with $q := p/(p-1)$ its Hölder conjugate. For any set of weights $\sigma \in \mathbb{R}^E_{>0}$, define the dual set of weights $\hat{\sigma}$ as $\hat{\sigma}(e) := \sigma(e)^{-\frac{q}{p}}$, for all $e \in E$. Then,*

$$\text{Mod}_{p,\sigma}(\Gamma)^{1/p}\,\text{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})^{1/q} = 1. \tag{2.4}$$

Moreover, the optimal $\rho^* \in \text{Adm}(\Gamma)$ and $\eta^* \in \text{Adm}(\hat{\Gamma})$ are unique and are related as follows:

$$\eta^*(e) = \frac{\sigma(e)\rho^*(e)^{p-1}}{\text{Mod}_{p,\sigma}(\Gamma)} \qquad \forall e \in E. \tag{2.5}$$

The relationship of the modulus of the families when $p = 1$ and $p = \infty$ are described using the following theorem.

Theorem 2.6 (Theorem 5 in [7]). *Under the assumptions of Theorem 2.5,*

$$\text{Mod}_{1,\sigma}(\Gamma)\,\text{Mod}_{\infty,\sigma^{-1}}(\hat{\Gamma}) = 1,$$

where $\sigma^{-1}(e) = \sigma(e)^{-1}$.

2.5 Probabilistic Interpretation

The optimization problem in (2.3) is a convex problem for which, for $1 < p < \infty$, the minimizer is unique. A Lagrangian dual problem was developed in [5] and endowed with a probabilistic interpretation in [3].

Let $\mathcal{P}(\Gamma) \subset \mathbb{R}^{\Gamma}_{\geq 0}$ represent the set of probability mass functions (pmfs) on the set Γ. That is, $\mu \in \mathcal{P}(\Gamma)$ if and only if μ is a nonnegative vector with $\mu^T\mathbf{1} = 1$. For a given μ, we can define a random variable $\underline{\gamma}$ with distribution given by μ where $\mu(\gamma) = \mathbb{P}_\mu(\underline{\gamma} = \gamma)$. For an edge $e \in E$, the value $\mathcal{N}(\underline{\gamma}, e)$ is a random variable as well and we denote its expectation as $\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)]$. The probabilistic interpretation is given by the following theorem.

Theorem 2.7 (Theorem 2 in [7]). *Let $G = (V, E)$ be a finite graph with edge weights σ, and let Γ be a non-trivial finite family of objects on G with usage matrix $\mathcal{N}$. Then, for any $1 < p < \infty$, letting $q := p/(p-1)$ be the conjugate exponent to p, we have*

$$\mathrm{Mod}_{p,\sigma}(\Gamma)^{-\frac{1}{p}} = \left(\min_{\mu \in \mathcal{P}(\Gamma)} \sum_{e \in E} \sigma(e)^{-\frac{q}{p}} \mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)]^q \right)^{\frac{1}{q}}. \tag{2.6}$$

Moreover, $\mu \in \mathcal{P}(\Gamma)$ is optimal for the right-hand side of (2.6) if and only if

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = \frac{\sigma(e)\rho^*(e)^{\frac{p}{q}}}{\mathrm{Mod}_{p,\sigma}(\Gamma)} \qquad \forall e \in E. \tag{2.7}$$

where ρ^* is the unique extremal density for $\mathrm{Mod}_{p,\sigma}(\Gamma)$.

As stated in [7], the probabilistic interpretation is particularly informative when $p = 2$, $\sigma \equiv 1$, and Γ is a collection of subsets of edges, so that the usage matrix $\mathcal{N}$ is as defined in (2.1). In this case, the duality relation from Theorem 2.7 can be written as

$$\mathrm{Mod}_2(\Gamma)^{-1} = \min_{\mu \in \mathcal{P}(\Gamma)} \mathbb{E}_\mu |\underline{\gamma} \cap \underline{\gamma}'| \tag{2.8}$$

where $\underline{\gamma}$ and $\underline{\gamma}'$ are two independent random variables chosen according to the pmf μ, and $|\underline{\gamma} \cap \underline{\gamma}'|$ is their *overlap*, which is also a random variable. This implies that computing the 2-modulus is equivalent to finding a pmf μ that minimizes the expected overlap of two independent, identically distributed random objects.

Chapter 3

Edge Covers, Fractional Edge Covers, and Stars

In this chapter we discuss the families of *edge covers*, *fractional edge covers* and *stars*. We study properties of the three families, define the family of basic fractional edge covers, and state equivalence results to be used in the next chapter.

3.1 Edge Covers

An *edge cover* of G is a set of edges $C \subseteq E$ such that each vertex in G is incident to at least one edge in C. The family of all edge covers is denoted by Γ_{ec}. Figure 3.1 shows all the edge covers for the square graph.

To calculate the modulus of edge covers in Chapter 4, one would first have to find all of the edge covers of a graph. It has been shown that the problem of counting the total number of edge covers of a graph is #P-Complete, and generating all possible edge covers of a graph is an exponential combinatorial problem [25]. Table 3.1 summarizes known results for the total number of edge covers of various families of graphs.

To calculate the modulus problem, recall that each edge cover induces an inequality constraint in the optimization problem (2.3). Therefore, calculating the edge cover modulus

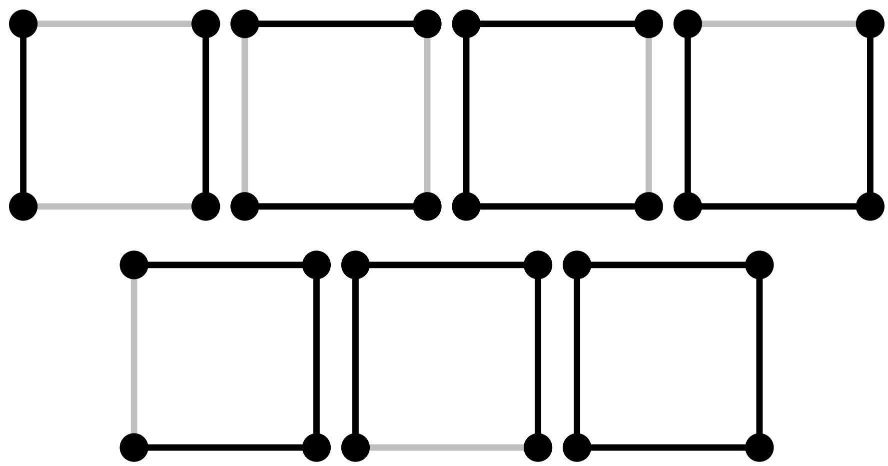


Figure 3.1: Edge covers of the square graph. Black edges make up the edge cover.

Graph	Number of edge covers, $\lvert\Gamma_{\text{ec}}\rvert$	Reference
Path graph, P_n	F_{n-1}	[11]
Cycle graph, C_n	$\sum_{k=1}^{n} \frac{n}{k}\binom{k}{n-k} = L_n = F_{n+2} - F_{n-2}$	[11, 34]
Complete graph, K_n	$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{\binom{n-k}{2}}$	[34]
Complete Bipartite graph, $K_{n,n}$	$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(2^{n-k}-1)^n$	[34]

Table 3.1: Number of edge covers for different graphs. The n-th Fibonacci number is denoted as F_n and the n-th Lucas number is denoted as L_n.

of a graph with a very large family of edge covers is hard and inefficient. Instead of generating all possible edge covers, we can find an equivalent family (in the sense of Section 2.3) to simplify the computations. One natural family is that of minimal edge covers. A minimal edge cover is an edge cover that is not a proper subset of any other edge cover, which implies that the family of minimal edge covers, Γ_{mec}, satisfies $|\Gamma_{\text{mec}}| \leq |\Gamma_{\text{ec}}|$.

Lemma 3.1. *The family of edge covers and the family of minimal edge covers are equivalent.*

Proof. Let Γ_{ec} be the family of edge covers and Γ_{mec} be the family of minimal edge covers. By definition, $\Gamma_{\text{mec}} \subseteq \Gamma_{\text{ec}}$ so by Lemma 2.1, $\text{Adm}(\Gamma_{\text{ec}}) \subseteq \text{Adm}(\Gamma_{\text{mec}})$. To show the reverse inclusion, let $\rho \in \text{Adm}(\Gamma_{\text{mec}})$ and let $\gamma \in \Gamma_{\text{ec}}$. Note that $\gamma = \gamma' \cup \bar{\gamma}$ where $\gamma' \in \Gamma_{\text{mec}}$ and $\bar{\gamma} \subset E$. Since γ' is a minimal edge cover and ρ is admissible for Γ_{mec}, then $\rho^T\gamma' \geq 1$. Moreover, since ρ and $\bar{\gamma}$ are both nonnegative vector, we have that $\rho^T\bar{\gamma} \geq 0$. Then,

$$\rho^T\gamma = \rho^T\gamma' + \rho^T\bar{\gamma} \geq 1.$$

This gives $\text{Adm}(\Gamma_{\text{mec}}) \subseteq \text{Adm}(\Gamma_{\text{ec}})$. □

This equivalence between edge covers and minimal edge covers allows the modulus problem to be slightly more feasible. Counting the number of minimal edge covers of a particular graph can still be challenging, but we can obtain a lower bound for the number of minimal edge covers by counting the number of *minimum edge covers*. A minimum edge cover is an edge cover of minimum cardinality, therefore minimum edge covers are also minimal, but the converse is not always true. For example, for the cycle graph C_n on an even number of nodes there are 2 minimum edge covers, so $|\Gamma_{\text{mec}}| \geq 2$. When n is odd, the cycle graph has n minimum edge covers, therefore $|\Gamma_{\text{mec}}| \geq n$. Comparing these values to the number of elements in Γ_{ec} from Table 3.1, we see a decrease in the number of objects, hence a decrease in the number of inequality constraints in the optimization problem. On general graphs, this can still lead to an exponentially large number of elements in the set. For example, when n is even the complete graph K_n has $(n-1)!!$ minimum edge covers, and when n is odd, there are $\frac{n!!(n-1)}{2}$ minimum edge covers (see Appendix A), where $n!! = n(n-2)(n-4)\ldots$. One

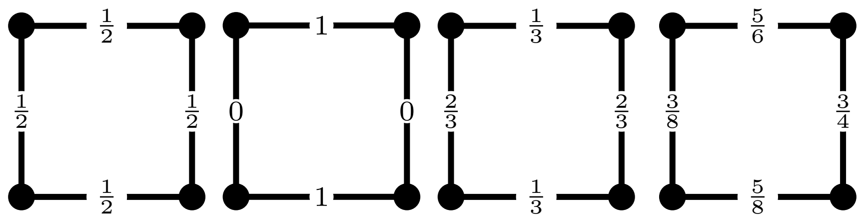


Figure 3.2: Some fractional edge covers of the square graph.

way to circumvent this combinatorial complexity is to introduce a relaxed problem. This can be accomplished using *fractional edge covers*.

3.2 Fractional Edge Covers

The concept of an edge cover can be generalized as follows (see [31]). To each vertex, $v \in V$, we associate the *star*, $\delta(v) \subset E$, comprised of the set of edges incident to v. Let $\gamma \in \mathbb{R}^E_{\geq 0}$ be a nonnegative vector on E. If

$$\gamma(\delta(v)) := \sum_{e \in \delta(v)} \gamma(e) \geq 1 \quad \text{for all } v \in V,$$

then γ is called a *fractional edge cover*. Each such γ can be considered to be an object on the graph G with corresponding edge usage

$$\mathcal{N}(\gamma, \cdot) := \gamma(\cdot),$$

yielding the (uncountably infinite) family of fractional edge covers, Γ_{fec}. Each edge cover, $\gamma \in \Gamma_{\text{ec}}$, can be associated with its incidence vector $\mathbb{1}_\gamma$, which provides a natural inclusion $\Gamma_{\text{ec}} \subset \Gamma_{\text{fec}}$ (the reverse inclusion is only true on the trivial graph). See Figure 3.2 for some examples of fractional edge covers on the square graph.

Since the family Γ_{fec} is uncountably infinite, the modulus problem (2.3) on this family cannot be immediately viewed as a standard convex optimization problem. However, it is possible to find an equivalent finite family. This comes from the observations that Γ_{fec} is convex and *recessive*, in the sense that $\Gamma_{\text{fec}} = \text{Dom}(\Gamma_{\text{fec}})$, which implies that $\Gamma_{\text{fec}} \simeq \text{ext}(\Gamma_{\text{fec}})$. The extreme points of Γ_{fec} have a relatively simple structure.

The following definition, lemmas, and proofs where inspired by the work done in [29] with fractional perfect matchings.

Definition 3.2. A vector $\gamma \in \mathbb{R}^E_{\geq 0}$ is called a *basic fractional edge cover* if

- γ is a fractional edge cover taking only values in $\{0, 1/2, 1\}$,
- the support, $\text{supp}\,\gamma$, is a vertex-disjoint union of odd cycles and substars, and
- $\gamma(e) = 1/2$ if and only if e belongs to an odd cycle in $\text{supp}\,\gamma$.

The family of all basic fractional edge covers is denoted $\Gamma_{\overline{\text{fec}}}$.

The remainder of this section is devoted to showing that the extreme points of Γ_{fec} are basic fractional edge covers. The proof depends on what the elements of $\text{supp}\,\gamma$ look like, for $\gamma \in \text{ext}(\Gamma_{\text{fec}})$, and will go as follows:

1. We first show $\text{supp}\,\gamma$ contains no even cycles (Lemma 3.3).
2. We then show that any connected component of $\text{supp}\,\gamma$ that contains a pendant edge is a substar (Lemma 3.4).
3. We then show that any connected component of $\text{supp}\,\gamma$ that does not contain a pendant edge is an odd cycle (Lemma 3.5).
4. Finally, we show that every extreme point of Γ_{fec} is a basic fractional edge cover (Lemma 3.6).

With these results, we show that the family of fractional edge covers and the family of basic fractional edge covers are equivalent (Theorem 3.7) and we make a note of what happens when the graph G is bipartite (Corollary 3.8).

Lemma 3.3. *If $\gamma \in \text{ext}(\Gamma_{\text{fec}})$ then $\text{supp}\,\gamma$ contains no even cycles.*

Proof. Suppose, to the contrary, that C is an even cycle in $\operatorname{supp}\gamma$. Define $\tau \in \mathbb{R}^E$ to be a function that assigns 1 and -1 alternately to the edges of C (starting from an arbitrary edge), and that assigns 0 to all other edges of G. Note that, for any number α and any vertex v,

$$(\gamma + \alpha\tau)(\delta(v)) = \gamma(\delta(v)) + \alpha\tau(\delta(v)) = \gamma(\delta(v)) \geq 1.$$

Thus, for sufficiently small $|\alpha|$, $\gamma + \alpha\tau$ is non-negative and, therefore, a fractional edge cover. This implies that the extreme point γ lies in an open line segment in Γ_{fec}, which is a contradiction. □

Lemma 3.4. *Let $\gamma \in \operatorname{ext}(\Gamma_{\text{fec}})$. Then, any connected component of $\operatorname{supp}\gamma$ that contains a pendant edge is a substar.*

Proof. Consider a connected component of $H := \operatorname{supp}\gamma$ containing a pendant edge. We shall show that this component contains no path of length greater than 2, which implies that the component is a substar. Let $e = \{u, v\}$ be the pendant edge, with $\deg_H(u) = 1$. If $\deg_H(v) = 1$, then the component consists solely of the edge e and we are done. Now, suppose $\deg_H(v) > 1$ and that there is a path of length greater than 2. Starting at node u, trace a maximum-length path in H. This path, which is assumed to have length at least 3, must either end at another pendant edge in H or cannot be extended further without creating a cycle. We consider these two possibilities in turn.

(a) Assume the path ends in another pendant edge, denoted $\{x, y\}$, with $\deg_H(y) = 1$. See Figure 3.3. Since the path has length at least 3, $x \neq v$. Let $\tau \in \mathbb{R}^E$ be a function that assigns 0 to the two pendant edges, assigns 1 and -1 alternately to the other edges in the path, and assigns 0 to all other edges in E (see Figure 3.3). Again, if we show that for sufficiently small $|\alpha|$, $\gamma + \alpha\tau \in \Gamma_{\text{fec}}$, we arrive at a contradiction. Consider the vertex v. The path in H connecting u to y passes from u to v and then on to a third vertex w (it is possible that $w = x$). Since $\gamma \in \Gamma_{\text{fec}}$, we know that $\gamma(\{u, v\}) \geq 1$ and since $\{v, w\} \in \operatorname{supp}\gamma$, we know that $\gamma(\{v, w\}) > 0$. So,

$$(\gamma + \alpha\tau)(\delta(v)) = \gamma(\delta(v)) + \alpha\tau(\delta(v)) = \gamma(\{u, v\}) + \gamma(\{v, w\}) - \alpha \geq 1 + \gamma(\{v, w\}) - \alpha,$$

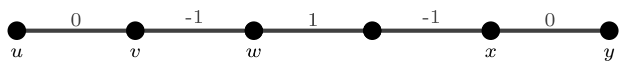


Figure 3.3: Values of τ for pendant edges connected by a path.

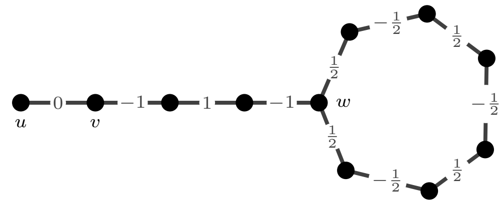


Figure 3.4: Values of τ for a pendant edge connected to a cycle.

which is greater than or equal to 1 for sufficiently small $|\alpha|$. The vertex x is similar. For all other vertices, z, $\tau(\delta(z)) = 0$, arriving at the contradiction.

(b) Assume instead that the path ends in a cycle. See Figure 3.4. That is, one can trace a path in H starting from u, passing through v, and continuing until the path eventually circles back to repeat a vertex w. (It is possible that $w = v$.) From Lemma 3.3, the cycle through w must be odd. Let $\tau \in \mathbb{R}^E$ be a function, shown in Figure 3.4, that assigns 0 to the pendant edge, assigns $\pm 1/2$ alternately on the odd cycle with the two edges through w sharing the same value, and assigns 0 to all other edge in E. If $w \neq v$, then τ should alternately assign ± 1 to the edges connecting v and w in such a way that $\tau(\delta(w)) = 0$. (The value of τ is set to zero on all other edges.)

It can be seen that this once again leads to a contradiction of the assumption that γ is an extreme point. The function τ sums to zero on all stars other than $\delta(v)$ and $\gamma(\delta(v)) > 1$, which makes $\gamma + \alpha\tau \in \Gamma_{\text{fec}}$ for sufficiently small $|\alpha|$.

□

Lemma 3.5. *Let $\gamma \in \operatorname{ext}(\Gamma_{\text{fec}})$. Then, any connected component of* $\operatorname{supp} \gamma$ *that does not contain a pendant edge is an odd cycle.*

Proof. Let H' be a connected component of $H := \operatorname{supp} \gamma$ that does not contain a pendant edge. The H' must contain at least one cycle, C. By Lemma 3.3, any such cycle must be odd. As before, we proceed by contradiction. Suppose that $H' \setminus C \neq \emptyset$. Then H' must contain a

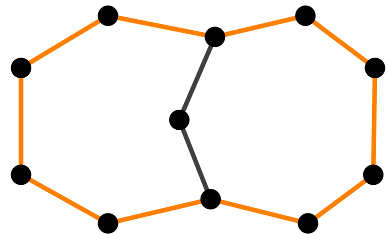
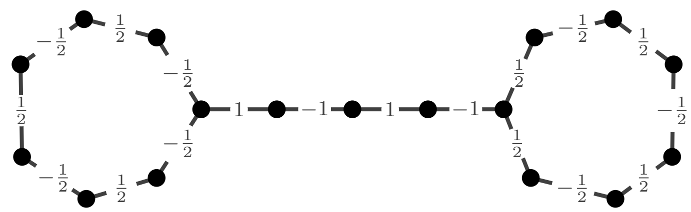
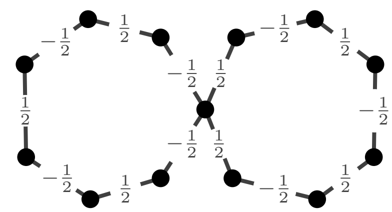


Figure 3.5: A path that returns to an odd cycle C and creates an even cycle, i.e. the "snowman" graph. Orange edges make up the even cycle.



(a)



(b)

Figure 3.6: Values of τ for two connected cycles.

vertex v such that $\deg_H(v) \geq 3$. That is, there must be an edge of H that is incident to v but does not lie in the cycle C. Consider extending this edge to a maximum-length path in H. Since the component is assumed to contain no pendant edges, this path must eventually create a cycle. There are a few cases to consider.

One possibility is that the path leaving v returns to a different vertex w in C, producing a "snowman" as shown in Figure 3.5. The vertices v and w are connected by two paths in C, and since C is an odd cycle, one of these paths has an even number of edges while the other has an odd number. Removing the path with even length results in an even cycle, which contradicts Lemma 3.3.

So, the path leaving C from v does not return to any other vertex of C (and does not end in a pendant edge). Thus, H must contain a graph K with 2 (necessarily odd) cycles connected by a path (possibly of length 0) as in Figure 3.6. Let $\tau \in \mathbb{R}^E$ be a function that assigns 0 to edges not in K, ± 1 alternately on the path connecting the two cycles of K, and $\pm 1/2$ alternately around the cycles of K so that τ sums to zero on all stars of G. (See Figure 3.6a and Figure 3.6b for examples.) This again yields a contradiction, since $\gamma + \alpha\tau \in \Gamma_{\text{fec}}$ for sufficiently small $|\alpha|$.

□

Lemma 3.6. *Every extreme point in* $\text{ext}(\Gamma_{\text{fec}})$ *is a basic fractional edge cover.*

Proof. Lemmas 3.4 and 3.5 show that each connected component of $H := \operatorname{supp}\gamma$ is either a substar or an odd cycle in G. To complete the proof, we must show that γ only takes values in $\{0, 1/2, 1\}$, with the value $1/2$ occurring exactly on the odd cycles of H.

First, consider a substar component, S, of H. All edges in this case are pendant edges and, therefore, γ must be at least 1 on each edge. On the other hand, suppose $\gamma(e') > 1$ for some $e' \in S$, and define

$$\tilde{\gamma}(e) = \begin{cases} 1 & \text{if } e = e', \\ \gamma(e) & \text{otherwise.} \end{cases}$$

Then $\tilde{\gamma} \in \Gamma_{\text{fec}}$ and $\tilde{\gamma} \preceq \gamma$. Since Γ_{fec} is recessive, this implies that γ lies on the relative interior of a ray in Γ_{fec} emanating from $\tilde{\gamma}$ and, therefore, that γ cannot be an extreme point.

Next, consider an odd cycle C which is a component of H. We wish to show that $\gamma(e) = 1/2$ on all edges of C. We begin by observing that $\gamma(\delta(v)) = 1$ for every vertex v in the cycle. Suppose to the contrary that $\gamma(\delta(v)) > 1$ for some vertex v, and consider the vector $\tau \in \mathbb{R}^E$, supported on C, alternately taking the values ± 1 around the cycle in such a way that $+1$ is assigned to both edges incident on v. Then, $\gamma + \alpha\tau \in \Gamma_{\text{fec}}$ for sufficiently small $|\alpha|$, contradicting the extremality of γ.

Next, we argue that if $\gamma(\delta(v)) = 1$ for all $v \in C$ then $\gamma(e) = 1/2$ for all edges in C. To see this, define

$$\tilde{\gamma}(e) = \begin{cases} 1/2 & \text{if } e \in C, \\ \gamma(e) & \text{otherwise.} \end{cases}$$

Then, $\tilde{\gamma} \in \Gamma_{\text{fec}}$ and $\tilde{\gamma}(\delta(v)) = 1$ for every vertex $v \in C$. Define $\tau := \gamma - \tilde{\gamma}$. Then $\operatorname{supp}\tau \subseteq C$ and $\tau(\delta(v)) = 0$ for every $v \in C$. Choose an adjacent pair of edges $e_1, e_2 \in C$. Then $\tau(e_2) = -\tau(e_1)$. Continuing around the cycle we find that the next edge, e_3, must satisfy $\tau(e_3) = -\tau(e_2) = \tau(e_1)$ and so on. Since the cycle is odd, the last edge we cross, e_r, that completes the cycle must have $\tau(e_r) = \tau(e_1)$. But, since τ must sum to zero on the star

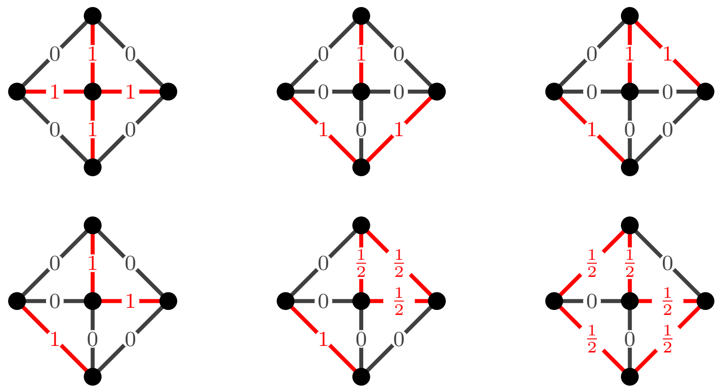


Figure 3.7: Extreme points of Γ_{fec} (up to rotation and reflection) for $G = W_5$. There are 25 extreme points in total.

including these two edges, $\tau(e_r) = \tau(e_1) = 0$, implying that $\tau = 0$. Thus, $\gamma = \tilde{\gamma}$, implying that $\gamma(e) = 1/2$ on all edges $e \in C$. □

Theorem 3.7. *The families* Γ_{fec} *and* $\Gamma_{\overline{\text{fec}}}$ *are equivalent;* $\Gamma_{\text{fec}} \simeq \Gamma_{\overline{\text{fec}}}$.

Proof. Lemma 3.6 implies that $\text{ext}(\Gamma_{\text{fec}}) \subseteq \Gamma_{\overline{\text{fec}}} \subseteq \Gamma_{\text{fec}}$, which shows that

$$\text{Adm}(\Gamma_{\text{fec}}) \subseteq \text{Adm}(\Gamma_{\overline{\text{fec}}}) \subseteq \text{Adm}(\text{ext}(\Gamma_{\text{fec}})) = \text{Adm}(\Gamma_{\text{fec}}).$$

□

Figures 3.7 and 3.8 demonstrate the implication of Theorem 3.7. These figures show all extreme points (up to rotation and reflection) of Γ_{fec} for the wheel graphs W_5 and W_6 respectively. Note that (as guaranteed by the theorem) all are basic fractional edge covers. Any density that is admissible for one of these sets is admissible for all fractional edge covers on the corresponding graph.

Corollary 3.8. *If G is a bipartite graph, then* $\Gamma_{\text{fec}} \simeq \Gamma_{\text{ec}}$.

Proof. Since $\Gamma_{\text{ec}} \subseteq \Gamma_{\text{fec}}$, by Lemma 2.1, $\text{Adm}(\Gamma_{\text{fec}}) \subseteq \text{Adm}(\Gamma_{\text{ec}})$. To show the reverse inclusion, recall that $\Gamma_{\text{fec}} \simeq \Gamma_{\overline{\text{fec}}}$, so we prove the result using $\Gamma_{\overline{\text{fec}}}$. Let $\rho \in \text{Adm}(\Gamma_{\text{ec}})$ and $\gamma \in \Gamma_{\overline{\text{fec}}}$. Since a bipartite graph has no odd cycles, Definition 3.2 shows that γ is a 0/1-vector supported on a disjoint union of substars. Since γ is a fractional edge cover, it must sum to 1

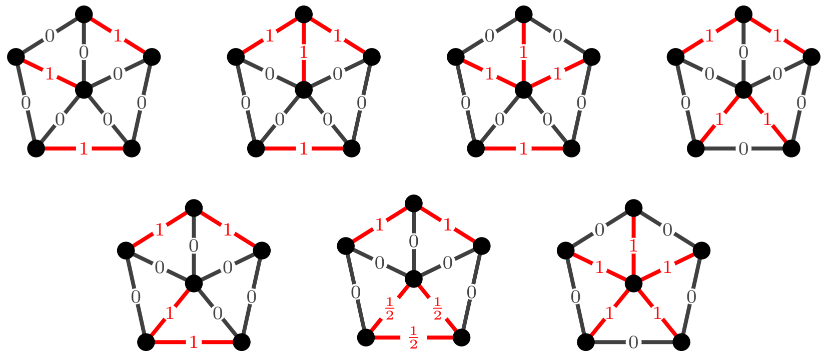


Figure 3.8: Extreme points of Γ_{fec} (up to rotation and reflection) for $G = W_6$. There are 36 extreme points in total.

on every star, implying that γ is the incidence vector of an edge cover. Therefore, since ρ is admissible for edge covers,

$$\rho^T \gamma \geq 1,$$

implying that $\rho \in \text{Adm}(\Gamma_{\overline{\text{fec}}})$. This shows that $\text{Adm}(\Gamma_{\text{ec}}) \subseteq \text{Adm}(\Gamma_{\overline{\text{fec}}}) = \text{Adm}(\Gamma_{\text{fec}})$. □

With these results, we've made a connection between four families of objects: Γ_{ec}, Γ_{fec}, $\Gamma_{\overline{\text{fec}}}$, and $\text{ext}(\Gamma_{\text{fec}})$. In the next section, we define the family of stars and show that this family is also related to Γ_{fec}, hence it is related to all other families studied in this chapter.

3.3 Stars

As defined at the beginning of Section 3.2, the star centered at node v, denoted $\delta(v)$, is the set of edges incident to v. Figure 3.9 shows the four stars of the square graph. These objects are relatively simple and can be useful when studying the modulus of other families of graph objects. This is because a connected graph will always have $|V|$ stars, and each star is made up of the edges connected to a node. Note that stars appear naturally in the definition of an edge cover (and similarly in the definition of a fractional edge cover as seen in Section 3.2). Specifically, we can restate the definition of an edge cover using stars as follows: $C \subseteq E$ is an edge cover if and only if $|C \cap \delta(v)| \geq 1$ for every $v \in V$. Therefore, there is a natural connection between these families.

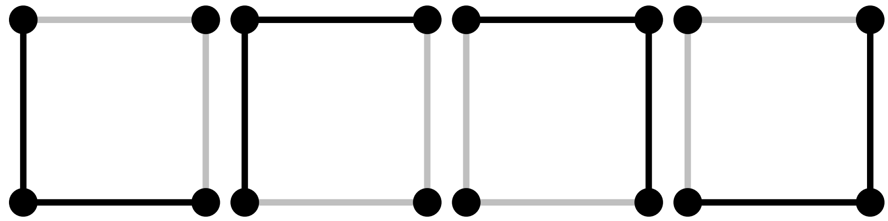


Figure 3.9: Stars of the square graph. Black edges make up the stars.

3.3.1 Dual Family of Fractional Edge Covers

Recall from Section 2.4 that if the family Γ is finite, then there is a dual family $\hat{\Gamma}$ whose modulus is related to the original family.

The relationship between dual families of objects has several useful implications. Most important in the present setting is the fact that upper bounds on the modulus of the dual family provide lower bounds for the original (primal) family. In particular, the following lemma shows that the family of stars can be used to provide lower bounds for the modulus of fractional edge covers.

Lemma 3.9. *The families of stars and fractional edge covers on G are dual in the sense that*

$$\hat{\Gamma}_{star} \simeq \Gamma_{\text{fec}}.$$

Proof. Consider a vector $\gamma \in \mathbb{R}^E_{\geq 0}$. Then, $\gamma \in \Gamma_{\text{fec}}$ if and only if

$$\sum_{e \in \delta(v)} \gamma(e) \geq 1 \quad \text{for every } v \text{ in } V,$$

which is equivalent to saying that $\gamma \in \text{Adm}(\Gamma_{\text{star}})$. By Lemma 2.4, then,

$$\hat{\Gamma}_{\text{star}} = \text{ext}(\text{Adm}(\Gamma_{\text{star}})) \simeq \text{Adm}(\Gamma_{\text{star}}) = \Gamma_{\text{fec}}.$$

□

3.4 Summary of Families and Their Relationships

As a summary of this chapter, in this section we restate the definitions of the objects of interest in this dissertation, as well as their relationships. Table 3.2 summarizes the notation used.

Γ_{ec}	The family of all edge covers
Γ_{mec}	The family of all minimal edge covers
Γ_{fec}	The family of all fractional edge covers
$\Gamma_{\overline{\text{fec}}}$	The family of all basic fractional edge covers
Γ_{star}	The family of all stars
$\hat{\Gamma}$	The dual family of Γ
$\text{ext}(\Gamma)$	The set of extreme points of Γ

Table 3.2: Notation for each family of objects.

3.4.1 Summary of definitions

edge cover: A set $C \subseteq E$ such that each vertex in G is incident to at least one edge in C.

minimal edge cover: An edge cover that is not a proper subset of any other edge cover.

star: A set, $\delta(v) \subset E$, of all edges incident upon a given vertex v.

fractional edge cover: A vector $\gamma \in \mathbb{R}^E_{\geq 0}$ such that

$$\gamma(\delta(v)) := \sum_{e \in \delta(v)} \gamma(e) \geq 1 \quad \text{for all } v \in V.$$

basic fractional edge cover: A fractional edge cover with the following properties.

- γ takes values only in $\{0, 1/2, 1\}$.
- The support, $\operatorname{supp} \gamma$, is a vertex-disjoint union of odd cycles and substars.
- $\gamma(e) = 1/2$ if and only if e belongs to an odd cycle in $\operatorname{supp} \gamma$.

3.4.2 Summary of relationships

Set inclusions. Any set $E' \subseteq E$ can be represented by its 0/1 incidence vector

$$\mathbb{1}_{E'}(e) = \begin{cases} 1 & \text{if } e \in E', \\ 0 & \text{otherwise.} \end{cases}$$

With this representation, the families are viewed as subsets of $\mathbb{R}^E$ and are related as follows

- $\Gamma_{\text{mec}} \subseteq \Gamma_{\text{ec}} \subseteq \Gamma_{\text{fec}}$.
- $\Gamma_{\text{mec}} \subseteq \Gamma_{\overline{\text{fec}}} \subseteq \Gamma_{\text{fec}}$.
- $\text{ext}(\Gamma_{\text{fec}}) \subseteq \Gamma_{\overline{\text{fec}}}$.

Equivalences. The equivalence results from this chapter can be summarized as follows:

- $\Gamma_{\text{ec}} \simeq \Gamma_{\text{mec}}$.
- $\Gamma_{\text{fec}} \simeq \Gamma_{\overline{\text{fec}}}$.
- $\hat{\Gamma}_{\text{star}} \simeq \Gamma_{\text{fec}}$.
- If G is bipartite, then $\Gamma_{\text{fec}} \simeq \Gamma_{\text{ec}}$.

In the next chapter, we show how to use these results to calculate the modulus of each family.

Chapter 4

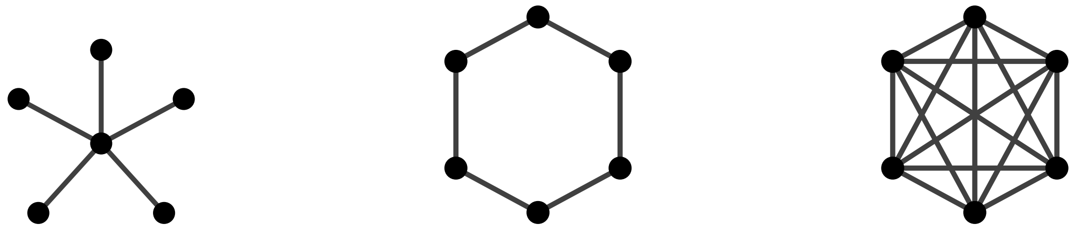
Modulus of Edge Covers, Fractional Edge Covers, and Stars

In this chapter we focus on studying the modulus of the families of edge covers, fractional edge covers, and stars. For simplicity, in all examples presented we assume the graphs are unweighted.

Recall from Chapter 2 that to calculate the modulus of a family Γ, we have to generate all of the objects in that family and find a density ρ that is admissible. The approach in each case is similar, we first establish an upper bound on the modulus by finding an admissible density. In some cases, a lower bound is established using the probabilistic interpretation of modulus, and in other cases we use duality results, thus establishing the value of the modulus.

4.1 Modulus of Edge Covers

Recall that an edge cover is a subset of edges such that each node in the graph sees at least one edge in the cover. This implies that for a graph with n nodes, where n is even, an edge cover will have at least $\frac{n}{2}$ edges. In the case when n is odd, an edge cover will have at least $\frac{n+1}{2}$ edges. This is because we can think of covering $n-3$ nodes using disjoint edges and



(a) The Star Graph, S_6. (b) The Cycle Graph, C_6. (c) The Complete Graph, K_6.

Figure 4.1: A standard set of graphs for modulus examples.

the remaining 3 nodes can be covered with 2 edges. Therefore, every edge cover will have at least $\lceil \frac{n}{2} \rceil$ edges. The following lemma gives an upper bound for the edge cover modulus using this fact.

Lemma 4.1. *If* $\rho \equiv \lceil \frac{n}{2} \rceil^{-1}$*, then* $\rho \in \mathrm{Adm}(\Gamma_{\mathrm{ec}})$ *and* $\mathrm{Mod}_p(\Gamma_{\mathrm{ec}}) \leq m \lceil \frac{n}{2} \rceil^{-p}$*.*

Proof. Let $\rho \equiv \lceil \frac{n}{2} \rceil^{-1}$. This density is admissible since every edge cover will have at least $\lceil \frac{n}{2} \rceil$ edges, then for $\gamma \in \Gamma_{\mathrm{ec}}$

$$\ell_\rho(\gamma) = \left\lceil \frac{n}{2} \right\rceil^{-1} |\gamma| \geq 1.$$

Then,

$$\mathrm{Mod}_p(\Gamma_{\mathrm{ec}}) \leq \sum_{e \in E} \rho(e)^p = m \left(\left\lceil \frac{n}{2} \right\rceil^{-1} \right)^p = \frac{m}{\lceil \frac{n}{2} \rceil^p}.$$

□

To get a better understanding of the modulus of edge covers, we start by calculating the p-modulus of some common graphs shown in Figure 4.1.

Example 4.1 (Star graph). Let $G = S_n$ be the unweighted star graph with $|V| = n \geq 3$ and $|E| = n - 1$. Note that all edges of S_n are *pendant edges*—they are incident to at least one vertex of degree one (see Figure 4.1a). It follows that there is a single edge cover: $\Gamma_{\mathrm{ec}} = \{E\}$. The symmetries of the graph and the uniqueness of the extremal density suggest that we restrict our search to constant densities. Since the single edge cover has $n - 1$ edges, the density $\rho_0 \equiv \frac{1}{n-1}$ is admissible. This provides an upper bound on modulus:

$$\mathrm{Mod}_p(\Gamma_{\mathrm{ec}}) \leq \mathcal{E}_p(\rho_0) = \sum_{e \in E} \rho_0(e)^p = (n-1) \cdot \frac{1}{(n-1)^p} = \frac{1}{(n-1)^{p-1}}.$$

To obtain the lower bound we use the probabilistic interpretation and Theorem 2.7. Since Γ_{ec} only has one object, E, we can define $\mu(E) = 1$. This implies that the expected edge usage is given by

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = 1.$$

Using (2.6),

$$\text{Mod}_p(\Gamma_{\text{ec}})^{-\frac{q}{p}} \leq \sum_{e \in E} \mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)]^q = n - 1$$
$$\text{Mod}_p(\Gamma_{\text{ec}}) \geq (n-1)^{-\frac{p}{q}} = \frac{1}{(n-1)^{p-1}}$$

where the last equality comes from the fact that $\frac{p}{q} = p - 1$. Thus we have that $\text{Mod}_p(\Gamma_{\text{ec}}) = \frac{1}{(n-1)^{p-1}}$, ρ_0 is the extremal density, and μ is an optimal pmf.

Example 4.2 (Cycle graph). Let $G = C_n$ be the unweighted cycle graph with $|V| = n$ and $|E| = n$. Again, symmetry suggests that the extremal density is constant. To calculate the edge cover modulus, we need to consider the cases when n is even and odd.

(a) Let n be even. Lemma 4.1 gives an upper bound for modulus with $\rho \equiv \frac{2}{n}$:

$$\text{Mod}_p(\Gamma_{\text{ec}}) \leq \frac{m}{\lceil \frac{n}{2} \rceil^p} = \frac{2^p}{n^{p-1}}.$$

To obtain the lower bound we use the probabilistic interpretation and Theorem 2.7 as in the previous example. We first note that there are 2 edge covers of C_n that have exactly $\frac{n}{2}$ edges (the 2 perfect matchings of the graph). We define $\mu(\gamma) = \frac{1}{2}$ for each of the 2 objects γ with $\frac{n}{2}$ edges and every other object γ' in Γ_{ec} is assigned $\mu(\gamma') = 0$. This implies that the expected edge usage is

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = \frac{1}{2}.$$

Using (2.6),

$$\begin{aligned}\text{Mod}_p(\Gamma_{\text{ec}})^{-\frac{q}{p}} &\leq \sum_{e\in E}\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)]^q = \frac{n}{2^q}\\ \text{Mod}_p(\Gamma_{\text{ec}}) &\geq \left(\frac{n}{2^q}\right)^{-\frac{p}{q}} = \frac{2^p}{n^{p-1}}.\end{aligned}$$

Thus we have that $\text{Mod}_p(\Gamma_{\text{ec}}) = \frac{2^p}{n^{p-1}}$, ρ is the extremal density, and μ is an optimal pmf.

(b) Let n be odd. Lemma 4.1 again gives an upper bound for modulus with $\rho \equiv \frac{2}{n+1}$:

$$\text{Mod}_p(\Gamma_{\text{ec}}) \leq \frac{m}{\lceil \frac{n}{2} \rceil^p} = \frac{2^p n}{(n+1)^p}.$$

To obtain the lower bound, first note that there are n edge covers that have $\frac{n+1}{2}$ edges (covers like in Figure 4.2 which are the minimum edge covers). Define $\mu(\gamma) = \frac{1}{n}$ for each of the n objects γ with $\frac{n+1}{2}$ edges and every other object $\gamma' \in \Gamma_{\text{ec}}$ is assigned $\mu(\gamma') = 0$. This implies that

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = \frac{n+1}{2}\cdot\frac{1}{n} = \frac{n+1}{2n}$$

since every edge appears in $\frac{n+1}{2}$ of the minimum edge covers. Using (2.6),

$$\begin{aligned}\text{Mod}_p(\Gamma_{\text{ec}})^{-\frac{q}{p}} &\leq \sum_{e\in E}\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)]^q = n\cdot\left(\frac{n+1}{2n}\right)^q\\ \text{Mod}_p(\Gamma_{\text{ec}}) &\geq \left(\frac{(n+1)^q}{2^q n^{q-1}}\right)^{-\frac{p}{q}} = \frac{2^p n}{(n+1)^p}\end{aligned}$$

where the last equality comes from the fact that $p = q/(q-1)$. Thus we have that $\text{Mod}_p(\Gamma_{\text{ec}}) = \frac{2^p n}{(n+1)^p}$, ρ is the extremal density, and μ is an optimal pmf.

Example 4.3 (Complete graph). Let $G = K_n$ be the unweighted complete graph with $|V| = n$ and $|E| = \frac{n(n-1)}{2}$. To calculate the modulus of edge covers, we again consider the cases when n is even and odd.

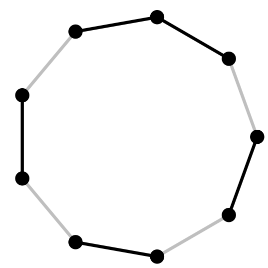


Figure 4.2: A minimum edge cover for an odd cycle C. All other minimum edge covers are obtained by rotating this picture.

(a) If n is even, then all minimum edge covers have $\frac{n}{2}$ edges. Lemma 4.1 gives an upper bound for modulus with $\rho \equiv \frac{2}{n}$:

$$\mathrm{Mod}_p(\Gamma_{\mathrm{ec}}) \leq \frac{m}{\lceil \frac{n}{2} \rceil^p} = \frac{n(n-1)}{2} \left(\frac{2}{n}\right)^p = \frac{2^{p-1}(n-1)}{n^{p-1}}.$$

To obtain the lower bound, recall that K_n has $(n-1)!!$ minimum edge covers, each with $\frac{n}{2}$ edges. Define $\mu(\gamma) = \frac{1}{(n-1)!!}$ for each of the minimum edge covers and every other object $\gamma' \in \Gamma_{\mathrm{ec}}$ is assigned $\mu(\gamma') = 0$. This implies that

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = (n-3)!! \cdot \frac{1}{(n-1)!!} = \frac{1}{n-1}$$

since every edge appears in $(n-3)!!$ of the minimum edge covers. Using (2.6),

$$\mathrm{Mod}_p(\Gamma_{\mathrm{ec}})^{-\frac{q}{p}} \leq \sum_{e \in E} \mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)]^q = \frac{n(n-1)}{2} \cdot \left(\frac{1}{n-1}\right)^q$$

$$\mathrm{Mod}_p(\Gamma_{\mathrm{ec}}) \geq \left(\frac{n}{2(n-1)^{q-1}}\right)^{-\frac{p}{q}} = \frac{2^{p-1}(n-1)}{n^{p-1}}.$$

Thus we have that $\mathrm{Mod}_p(\Gamma_{\mathrm{ec}}) = \frac{2^{p-1}(n-1)}{n^{p-1}}$, ρ is the extremal density, and μ is an optimal pmf.

(b) If n is odd, then all minimum edge covers have $\frac{n+1}{2}$ edges. Lemma 4.1 gives an upper bound for modulus with $\rho \equiv \frac{2}{n+1}$:

$$\mathrm{Mod}_p(\Gamma) \leq \frac{m}{\lceil \frac{n}{2} \rceil^p} = \frac{n(n-1)}{2} \cdot \left(\frac{2}{n+1}\right)^p = \frac{2^{p-1}n(n-1)}{(n+1)^p}.$$

To obtain the lower bound, recall that K_n has $\frac{n!!(n-1)}{2}$ minimum edge covers when n is odd, each with $\frac{n+1}{2}$ edges. Define $\mu(\gamma) = \frac{2}{n!!(n-1)}$ for each of the minimum edge covers and every other object $\gamma' \in \Gamma_{\text{ec}}$ is assigned $\mu(\gamma') = 0$. This implies that

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = \frac{(n-2)!!(n+1)}{2} \cdot \frac{2}{n!!(n-1)} = \frac{n+1}{n(n-1)}$$

since every edge appears in $\frac{(n-2)!!(n+1)}{2}$ of the minimum edge covers. Using (2.6),

$$\mathrm{Mod}_p(\Gamma_{\text{ec}})^{-\frac{q}{p}} \leq \sum_{e \in E} \mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)]^q = \frac{n(n-1)}{2} \cdot \left(\frac{n+1}{n(n-1)}\right)^q$$

$$\mathrm{Mod}_p(\Gamma_{\text{ec}}) \geq \left(\frac{(n+1)^q}{2n^{q-1}(n-1)^{q-1}}\right)^{-\frac{p}{q}} = \frac{2^{p-1}n(n-1)}{(n+1)^p}.$$

Thus we have that $\mathrm{Mod}_p(\Gamma_{\text{ec}}) = \frac{2^{p-1}n(n-1)}{(n+1)^p}$, ρ is the extremal density, and μ is an optimal pmf.

4.2 Modulus of Fractional Edge Covers

We now revisit the graphs we studied in the previous section and calculate the unweighted p-modulus of fractional edge covers. By Theorem 3.7, we have that $\Gamma_{\text{fec}} \simeq \Gamma_{\overline{\text{fec}}}$. Therefore, calculating $\mathrm{Mod}_p(\Gamma_{\text{fec}})$ is equivalent to calculating $\mathrm{Mod}_p(\Gamma_{\overline{\text{fec}}})$.

Example 4.4 (Star graph). Let $G = S_n$ be the star graph. Note that S_n is a bipartite graph, so by Corollary 3.8, $\Gamma_{\text{ec}} \simeq \Gamma_{\text{fec}}$ and

$$\mathrm{Mod}_p(\Gamma_{\text{fec}}) = \mathrm{Mod}_p(\Gamma_{\text{ec}}) = \frac{1}{(n-1)^{p-1}}.$$

Example 4.5 (Cycle graph). Let $G = C_n$ be the cycle graph. To calculate the fractional edge cover modulus, we again consider the cases when n is even and odd.

(a) If n is even, then the graph is bipartite. Again, by Corollary 3.8, $\Gamma_{\text{ec}} \simeq \Gamma_{\text{fec}}$ and

$$\text{Mod}_p(\Gamma_{\text{fec}}) = \text{Mod}_p(\Gamma_{\text{ec}}) = \frac{2^p}{n^{p-1}}.$$

(b) If n is odd, then $\Gamma_{\overline{\text{fec}}}$ is the union of all the minimal edge covers and the constant vector $\gamma \equiv 1/2$. Let $\rho_0 \equiv \frac{2}{n}$. To verify that this density is admissible, we first note that a minimal edge cover will have at least $\frac{n+1}{2}$ edges. Then, for a minimal edge cover γ',

$$\ell_{\rho_0}(\gamma') = \sum_{e \in \gamma'} \frac{2}{n} = |\gamma'| \frac{2}{n} \geq \frac{n+1}{2} \cdot \frac{2}{n} = \frac{n+1}{n} \geq 1.$$

It remains to show that this density is admissible for the object $\gamma \equiv 1/2$:

$$\ell_{\rho_0}(\gamma) = \rho_0^T \gamma = \frac{1}{2} \sum_{e \in E} \frac{2}{n} = \frac{n}{2} \cdot \frac{2}{n} = 1.$$

Thus, ρ_0 is admissible for this family and

$$\text{Mod}_p(\Gamma_{\text{fec}}) \leq \mathcal{E}_p(\rho_0) = \sum_{e \in E} \rho_0(e)^p = n \cdot \left(\frac{2}{n}\right)^p = \frac{2^p}{n^{p-1}}.$$

The corresponding lower bound is established in Example 4.8 of Section 4.3.

Example 4.6 (Complete graph). Let $G = K_n$ be the complete graph. In this example, the number of basic fractional edge covers grows quickly with n, and is difficult to visualize. Nevertheless, we can show that $\rho_0 = \frac{2}{n}$ is the extremal density and that the modulus is

$$\text{Mod}_p(\Gamma_{\text{fec}}) = \frac{2^{p-1}(n-1)}{n^{p-1}}.$$

This is proved in Example 4.9 in Section 4.3.

Note that when calculating $\text{Mod}_p(\Gamma_{\text{fec}})$ for the cycle graph and the complete graph, the value of the modulus is the same for even or odd n. That is, modulus does not differentiate between the 2 cases like the modulus of edge cover does.

4.2.1 Bounds on the Modulus of Edge Covers

The examples discussed in the previous two sections show that the modulus of edge covers and the modulus of fractional edge covers seem to be closely related. The following theorem states a relationship between the modulus of both families.

Theorem 4.2. *For all $p \in [1, \infty)$*

$$\left(\frac{3}{4}\right)^p \text{Mod}_{p,\sigma}(\Gamma_{\text{fec}}) \leq \text{Mod}_{p,\sigma}(\Gamma_{\text{ec}}) \leq \text{Mod}_{p,\sigma}(\Gamma_{\text{fec}}).$$

If G is bipartite then $\text{Mod}_{p,\sigma}(\Gamma_{\text{ec}}) = \text{Mod}_{p,\sigma}(\Gamma_{\text{fec}})$.

The second inequality in these bounds follows from the inclusion $\Gamma_{\text{ec}} \subseteq \Gamma_{\text{fec}}$. The other bound follows from a more careful look at cycles. First, we establish an upper bound on the number of edges in a minimal edge cover for a cycle.

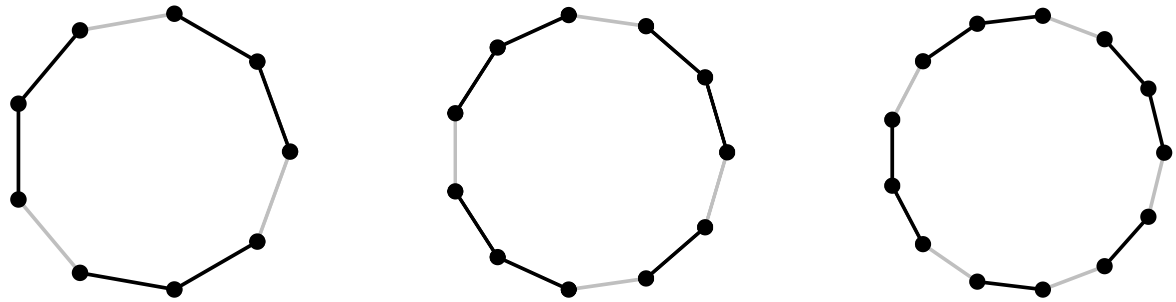
Lemma 4.3. *Let C be a cycle with length $|C| = k$ and let γ be a minimal edge cover of C. Then $|\gamma| \leq \frac{2k}{3}$.*

Proof. Each vertex of C has degree 1 or 2 in γ. Let k_1 be the number of vertices with degree 1 and $k_2 = k - k_1$ be the vertices with degree 2. Since γ is minimal, each degree-2 vertex is connected in γ to two degree-1 neighbors. (An edge connecting two degree-2 vertices could be removed from γ leaving an edge cover and violating the minimality of γ.) This implies that $k_2 \leq \frac{k}{3}$. The handshaking lemma then implies that

$$|\gamma| = \frac{k_1 + 2k_2}{2} = \frac{k + k_2}{2} \leq \frac{2k}{3}.$$

□

Figure 4.3 shows examples of the largest minimal edge covers for different odd cycles. In each case we see the result from Lemma 4.3.



(a) $k = 9$, $|\gamma| = 6$ (b) $k = 11$, $|\gamma| = 7$ (c) $k = 13$, $|\gamma| = 8$

Figure 4.3: Maximal minimal edge cover γ for cycles of length k.

Lemma 4.4. *Let $C \subset E$ be a cycle, let $\Gamma_{\mathrm{ec}}(C)$ be the set of edge covers for C, and let $\rho \in \mathbb{R}^C_{\geq 0}$. Then*

$$\frac{1}{2}\sum_{e\in C}\rho(e) \geq \frac{3}{4}\min_{\gamma\in\Gamma_{\mathrm{ec}}(C)} \ell_\rho(\gamma)$$

Proof. Let C be a cycle with length $|C| = k$. Since ρ is non-negative, the minimum value on the right must be attained by a minimal edge cover. Let $\gamma' = \arg\min_{\gamma\in\Gamma_{\mathrm{ec}}(C)} \ell_\rho(\gamma)$, and assume γ' is minimal. By Lemma 4.3, $|\gamma'| \leq \frac{2k}{3}$. By considering all rotations of the cycle, we obtain a set of k rotations of γ', which we may enumerate $\{\gamma'_1, \gamma'_2, \ldots, \gamma'_k\}$. Moreover, each edge of C lies in exactly $|\gamma'|$ of these minimal edge covers, which implies that

$$\frac{3}{4k}\sum_{i=1}^{k}\gamma_i = \frac{3}{4k}|\gamma'|\mathbf{1} \leq \frac{1}{2}\mathbf{1}$$

From this, it follows that

$$\frac{1}{2}\sum_{e\in C}\rho(e) = \frac{1}{2}\rho^T\mathbf{1} \geq \frac{3}{4k}\sum_{i=1}^{k}\ell_\rho(\gamma_i) \geq \frac{3}{4}\min_{\gamma\in\Gamma_{\mathrm{ec}}(C)} \ell_\rho(\gamma).$$

□

Lemma 4.5. *Let $\rho \in \mathrm{Adm}(\Gamma_{\mathrm{ec}})$, then $\frac{4}{3}\rho \in \mathrm{Adm}(\Gamma_{\overline{\mathrm{fec}}})$.*

Proof. Let $\rho \in \mathrm{Adm}(\Gamma_{\mathrm{ec}})$ and let $\gamma \in \Gamma_{\overline{\mathrm{fec}}}$. From Definition 3.2, we know that the support of γ is a vertex-disjoint union of substars, $\mathscr{S} = \{S_1, \ldots, S_t\}$, and odd cycles, $\mathscr{C} = \{C_1, \ldots, C_r\}$. Moreover, γ takes the value 1 on each substar edge and 1/2 on each cycle edge. From γ, we construct an edge cover, $\tilde{\gamma} \in \Gamma_{\mathrm{ec}}$ as follows.

For edges $e \notin \operatorname{supp} \gamma$, we set $\tilde{\gamma}(e) = 0$. Similarly, we set $\tilde{\gamma}(e) = \gamma(e) = 1$ for each edge $e \in \bigcup_{i=1}^{t} S_i$. The remaining edges lie on the disjoint union of the odd cycles in $\mathscr{C}$. On each of these odd cycles, we choose $\tilde{\gamma}$ to be the (incidence vector of the) minimal edge cover that has the smallest ρ-length. By construction, $\tilde{\gamma}$ is an edge cover for G. By admissibility, then, it follows that

$$1 \leq \ell_\rho(\tilde{\gamma}) = \sum_{C \in \mathscr{C}} \sum_{e \in C} \tilde{\gamma}(e)\rho(e) + \sum_{S \in \mathscr{S}} \sum_{e \in S} \rho(e). \tag{4.1}$$

Now consider the ρ-length of the basic fractional edge cover γ,

$$\ell_\rho(\gamma) = \sum_{C \in \mathscr{C}} \sum_{e \in C} \frac{1}{2}\rho(e) + \sum_{S \in \mathscr{S}} \sum_{e \in S} \rho(e). \tag{4.2}$$

By the construction of $\tilde{\gamma}$, Lemma 4.4 implies that for every $C \in \mathscr{C}$,

$$\frac{1}{2} \sum_{e \in C} \rho(e) \geq \frac{3}{4} \sum_{e \in C} \tilde{\gamma}(e)\rho(e). \tag{4.3}$$

Combining (4.1)–(4.3) shows that

$$\begin{aligned}
\ell_\rho(\gamma) &= \sum_{C \in \mathscr{C}} \sum_{e \in C} \frac{1}{2}\rho(e) + \sum_{S \in \mathscr{S}} \sum_{e \in S} \rho(e) \\
&\geq \sum_{C \in \mathscr{C}} \frac{3}{4} \sum_{e \in C} \tilde{\gamma}(e)\rho(e) + \sum_{S \in \mathscr{S}} \sum_{e \in S} \rho(e) \\
&\geq \frac{3}{4} \left(\sum_{C \in \mathscr{C}} \sum_{e \in C} \tilde{\gamma}(e)\rho(e) + \sum_{S \in \mathscr{S}} \sum_{e \in S} \rho(e) \right) \\
&= \frac{3}{4} \ell_\rho(\tilde{\gamma}) \\
&\geq \frac{3}{4}.
\end{aligned}$$

Since $\gamma \in \Gamma_{\overline{\mathrm{fec}}}$ was arbitrary, it follows that $\frac{4}{3}\rho \in \operatorname{Adm}(\Gamma_{\overline{\mathrm{fec}}})$.

□

Proof of Theorem 4.2. If G is bipartite, then the result follows from Corollary 3.8. Otherwise, as stated earlier, the second inequality follows from the inclusion $\Gamma_{\text{ec}} \subseteq \Gamma_{\text{fec}}$. The first inequality is a consequence of Lemma 4.5. To see this, let ρ^* be an extremal density for $\text{Mod}_{p,\sigma}(\Gamma_{\text{ec}})$. By Lemma 4.5, $\rho = \frac{4}{3}\rho^* \in \text{Adm}(\Gamma_{\overline{\text{fec}}}) = \text{Adm}(\Gamma_{\text{fec}})$. Thus, we have

$$\text{Mod}_{p,\sigma}(\Gamma_{\text{fec}}) = \text{Mod}_{p,\sigma}(\Gamma_{\overline{\text{fec}}}) \leq \mathcal{E}_{p,\sigma}(\rho) = \left(\frac{4}{3}\right)^p \mathcal{E}_{p,\sigma}(\rho^*) = \left(\frac{4}{3}\right)^p \text{Mod}_{p,\sigma}(\Gamma_{\text{ec}}).$$

□

Remark 4.1. A similar theorem is true for the case $p = \infty$. Using the same proof technique, one finds that

$$\frac{3}{4}\text{Mod}_{\infty,\sigma}(\Gamma_{\text{fec}}) \leq \text{Mod}_{\infty,\sigma}(\Gamma_{\text{ec}}) \leq \text{Mod}_{\infty,\sigma}(\Gamma_{\text{fec}}).$$

Constant extremal densities. In the case when the extremal density is constant, the lower bound can be further generalized using the following lemma.

Lemma 4.6. *Let $\rho \in \text{Adm}(\Gamma_{\text{ec}})$ be constant. If the graph, G, contains no odd cycles shorter than $k = 2\ell + 1$, then $\dfrac{k+1}{k}\rho \in \text{Adm}(\Gamma_{\overline{\text{fec}}})$.*

Proof. Let $\rho \in \text{Adm}(\Gamma_{\text{ec}})$, where $\rho \equiv \omega$ is constant, and let $\gamma \in \Gamma_{\overline{\text{fec}}}$. As stated in the proof of Lemma 4.5, we know that the support of γ is a vertex-disjoint union of substars, $\mathscr{S} = \{S_1, \ldots, S_t\}$, and odd cycles, $\mathscr{C} = \{C_1, \ldots, C_r\}$ where γ takes the value 1 on each substar edge and 1/2 on each cycle edge. From γ, we construct an edge cover, $\tilde{\gamma} \in \Gamma_{\text{ec}}$ as before.

For edges $e \notin \operatorname{supp}\gamma$, we set $\tilde{\gamma}(e) = 0$. Similarly, we set $\tilde{\gamma}(e) = \gamma(e) = 1$ for each edge $e \in \bigcup_{i=1}^t S_i$. The remaining edges lie on the disjoint union of the odd cycles in $\mathscr{C}$. On each of these odd cycles, we choose $\tilde{\gamma}$ to be the (incidence vector of the) minimal edge cover that has the smallest ρ-length. Since ρ is constant, this will be attained by a minimum edge cover of the cycle, which has $\frac{|C|+1}{2}$ edges. By construction, $\tilde{\gamma}$ is an edge cover for G. By admissibility, it follows that

$$1 \leq \ell_\rho(\tilde{\gamma}) = \sum_{C\in\mathscr{C}}\sum_{e\in C}\tilde{\gamma}(e)\rho(e) + \sum_{S\in\mathscr{S}}\sum_{e\in S}\rho(e) = \sum_{C\in\mathscr{C}}\omega\frac{|C|+1}{2} + \sum_{S\in\mathscr{S}}\omega|S|. \tag{4.4}$$

Now consider the ρ-length of the basic fractional edge cover γ,

$$\ell_\rho(\gamma) = \sum_{C\in\mathscr{C}} \sum_{e\in C} \frac{1}{2}\rho(e) + \sum_{S\in\mathscr{S}} \sum_{e\in S} \rho(e) = \sum_{C\in\mathscr{C}} \frac{\omega}{2}|C| + \sum_{S\in\mathscr{S}} \omega|S|. \tag{4.5}$$

Using the fact that the smallest odd cycle in the graph is of length k implies that for every $C \in \mathscr{C}$,

$$|C| = (|C|+1)\frac{|C|}{|C|+1} \geq (|C|+1)\frac{k}{k+1} \tag{4.6}$$

Combining (4.4)–(4.6) shows that

$$\begin{aligned}
\ell_\rho(\gamma) &= \sum_{C\in\mathscr{C}} \frac{\omega}{2}|C| + \sum_{S\in\mathscr{S}} \omega|S| \\
&\geq \sum_{C\in\mathscr{C}} \frac{\omega}{2}(|C|+1)\frac{k}{k+1} + \sum_{S\in\mathscr{S}} \omega|S| \\
&\geq \frac{k}{k+1}\left(\sum_{C\in\mathscr{C}} \frac{\omega}{2}(|C|+1) + \sum_{S\in\mathscr{S}} \omega|S|\right) \\
&= \frac{k}{k+1}\ell_\rho(\tilde{\gamma}) \\
&\geq \frac{k}{k+1}.
\end{aligned}$$

Since $\gamma \in \Gamma_{\overline{\text{fec}}}$ was arbitrary, it follows that $\frac{k+1}{k}\rho \in \text{Adm}(\Gamma_{\overline{\text{fec}}})$. □

Theorem 4.7. *Let k be the length of the shortest odd cycle in G. For all $p \in [1,\infty)$ where the extremal density, ρ^*, for $\text{Mod}_{p,\sigma}(\Gamma_{\text{ec}})$ is constant, then*

$$\left(\frac{k}{k+1}\right)^p \text{Mod}_{p,\sigma}(\Gamma_{\text{fec}}) \leq \text{Mod}_{p,\sigma}(\Gamma_{\text{ec}}) \leq \text{Mod}_{p,\sigma}(\Gamma_{\text{fec}}).$$

Proof. By Lemma 4.6, $\rho = \frac{k+1}{k}\rho^* \in \text{Adm}(\Gamma_{\overline{\text{fec}}}) = \text{Adm}(\Gamma_{\text{fec}})$. Thus, we have

$$\text{Mod}_{p,\sigma}(\Gamma_{\text{fec}}) \leq \mathcal{E}_{p,\sigma}(\rho) = \left(\frac{k+1}{k}\right)^p \mathcal{E}_{p,\sigma}(\rho^*) = \left(\frac{k+1}{k}\right)^p \text{Mod}_{p,\sigma}(\Gamma_{\text{ec}}).$$

□

Remark 4.2. A similar theorem is true for the case $p = \infty$. Using the same proof technique, one finds that when the extremal density $\rho^* \in \mathrm{Adm}(\Gamma_{\mathrm{ec}})$ is constant, then

$$\frac{k}{k+1} \mathrm{Mod}_{\infty,\sigma}(\Gamma_{\mathrm{fec}}) \leq \mathrm{Mod}_{\infty,\sigma}(\Gamma_{\mathrm{ec}}) \leq \mathrm{Mod}_{\infty,\sigma}(\Gamma_{\mathrm{fec}}).$$

4.3 Modulus of Stars

Lemma 3.9 implies that the duality theorems (Theorems 2.5 and 2.6) hold with $\Gamma = \Gamma_{\mathrm{star}}$ and $\hat{\Gamma} = \Gamma_{\mathrm{fec}}$. This means that the modulus and extremal densities for fractional edge covers can be understood through the modulus of stars. Specifically, we have that

$$\mathrm{Mod}_{p,\sigma}(\Gamma_{\mathrm{star}})^{1/p} \, \mathrm{Mod}_{q,\hat{\sigma}}(\Gamma_{\mathrm{fec}})^{1/q} = 1. \tag{4.7}$$

where $q = p/(p-1)$ and $\hat{\sigma}(e) = \sigma(e)^{-\frac{q}{p}}$. If $p = 2$, then the moduli are reciprocals of each other

$$\mathrm{Mod}_{2,\sigma}(\Gamma_{\mathrm{star}}) \, \mathrm{Mod}_{2,\sigma^{-1}}(\Gamma_{\mathrm{fec}}) = 1.$$

Calculating the star modulus turns out to be computationally simpler than calculating the modulus of fractional edge covers based on the number of constraints in the minimization problem. Specifically, the number of stars in a graph is equal to the number of vertices $|V|$, whereas the family of basic fractional edge covers in a graph is at least as big as the family of minimal edge covers. In this section, we prove simple results for the modulus of stars, as well as studying examples of well-known graphs.

The following lemma states a basic estimate for the star modulus by restating a result from [6], along with a lower bound.

Lemma 4.8. *Let $G = (V, E, \sigma)$ be a finite graph and let Γ be the family of all stars in G. Let $\delta(G) := \min_{v \in V} \deg(v)$, then*

$$\frac{\sigma_{min}}{\delta(G)^p} \leq \mathrm{Mod}_{p,\sigma}(\Gamma) \leq \frac{\sigma(E)}{\delta(G)^p}$$

where $\sigma_{min} = \min_{e\in E} \sigma(e)$ *and* $\sigma(E) = \sum_{e\in E} \sigma(e)$.

Proof. Define $\rho_0 \equiv \frac{1}{\delta(G)}$, then ρ_0 is admissible since for every $v \in V$

$$\ell_{\rho_0}(\delta(v)) = \sum_{e\in\delta(v)} \rho_0(e) = \frac{\deg(v)}{\delta(G)} \geq 1,$$

where the last inequality is true since every star will have at least $\delta(G)$ edges in it. So,

$$\text{Mod}_{p,\sigma}(\Gamma) \leq \mathcal{E}_{p,\sigma}(\rho_0) = \sum_{e\in E} \sigma(e)\rho_0(e)^p = \frac{\sigma(E)}{\delta(G)^p}.$$

Let $v_0 \in V$ be the node with smallest degree ($\deg(v_0) = \delta(G)$) and let $\rho \in \text{Adm}(\Gamma)$. Then,

$$1 \leq \ell_\rho(\delta(v_0)) = \sum_{e\in\delta(v_0)} \rho(e) \leq \max_{e\in E} \rho(e) \sum_{e\in\delta(v_0)} 1 = \max_{e\in E} \rho(e)\delta(G)$$

This implies that there exists an edge e_0 such that $\rho(e_0) \geq \frac{1}{\delta(G)}$ and

$$\mathcal{E}_{p,\sigma}(\rho) = \sum_{e\in E} \sigma(e)\rho(e)^p \geq \sigma(e_0)\rho(e_0)^p \geq \frac{\sigma(e_0)}{\delta(G)^p} \geq \frac{\sigma_{\min}}{\delta(G)^p}.$$

Taking the infimum over all $\rho \in \text{Adm}(\Gamma)$ we get the result

$$\text{Mod}_{p,\sigma}(\Gamma) \geq \frac{\sigma_{\min}}{\delta(G)^p}.$$

□

Example 4.7 (Star graph). Let $G = S_n$ be the star graph. The density $\rho_0 \equiv 1$ is admissible for Γ_{star}. So,

$$\text{Mod}_p(\Gamma_{\text{star}}) \leq \mathcal{E}_p(\rho_0) = \sum_{e\in E} \rho_0(e)^p = (n-1)\cdot 1^p = n-1.$$

The lower bound can be shown in multiple ways. We could use the probabilistic interpretation and a choice of μ as shown in previous examples, but we can also use duality results about the modulus of fractional edge covers and stars. Specifically, Example 4.4, shows that

$\text{Mod}_q(\Gamma_{\text{fec}})$ for this graph is $\frac{1}{(n-1)^{q-1}}$. Theorem 2.5 says that $\text{Mod}_p(\Gamma_{\text{star}})^{1/p}\,\text{Mod}_q(\Gamma_{\text{fec}})^{1/q} = 1$, hence showing that $\text{Mod}_p(\Gamma_{\text{star}}) = n-1$.

Lemma 4.9. *Let G be an unweighted d-regular graph and let Γ be the family of all stars in G. Then, for $1 < p < \infty$, the density $\rho \equiv \frac{1}{d}$ is extremal and $\text{Mod}_p(\Gamma) = \frac{m}{d^p}$.*

The proof of Lemma 4.9 is given in Section 4.3.1 once the probabilistic interpretation of the star modulus has been developed.

Example 4.8 (Cycle graph). Let $G = C_n$ be the cycle graph. Since G is a 2-regular graph, by Lemma 4.9, $\rho \equiv \frac{1}{2}$ is extremal and

$$\text{Mod}_p(\Gamma_{\text{star}}) = \frac{n}{2^p}.$$

Example 4.9 (Complete graph). Let $G = K_n$ be the complete graph. Since G is a $(n-1)$-regular graph, by Lemma 4.9, $\rho \equiv \frac{1}{n-1}$ is extremal and

$$\text{Mod}_p(\Gamma_{\text{star}}) = \frac{\frac{n(n-1)}{2}}{(n-1)^p} = \frac{n}{2(n-1)^{p-1}}.$$

By duality, Examples 4.8 and 4.9 complete Examples 4.5 and 4.6.

Remark 4.3. Since Γ_{star} and Γ_{fec} are dual families, Lemma 4.9 gives a result about the modulus of fractional edge covers for d-regular graphs. More specifically, using equation (4.7) we get that for a d-regular graph,

$$\text{Mod}_p(\Gamma_{\text{fec}}) = d\left(\frac{2}{n}\right)^{p-1}$$

and the extremal density is given by $\rho^* \equiv \frac{2}{n}$. Note that this recovers the values for the extremal densities and the p-modulus of Γ_{fec} for the cycle graph and complete graph in Section 4.2

4.3.1 Probabilistic Interpretation of $\mathrm{Mod}_{p,\sigma}(\Gamma_{\text{star}})$

In Section 2.5 we reviewed the probabilistic interpretation of modulus. The expectations that appear in this section have special forms in the case of Γ_{star}. In particular, if $e = \{x, y\} \in E$, then

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = \mu(\delta(x)) + \mu(\delta(y)). \tag{4.8}$$

Moreover, the expected overlap can be written as

$$\mathbb{E}_\mu|\underline{\gamma}\cap\underline{\gamma}'| = \sum_{v,v'\in V} |\delta(v)\cap\delta(v')|\mu(\delta(v))\mu(\delta(v')) = \sum_{v\in V}\deg(v)\mu(\delta(v))^2 + \sum_{v\in V}\sum_{v'\sim v}\mu(\delta(v))\mu(\delta(v')), \tag{4.9}$$

which expresses the expected overlap as the sum of two terms. In the second term, the notation $v' \sim v$ in the inner summation indicates that the sum is taken over all neighbors v' of v. Since the expected overlap is being minimized, the first term suggests that stars with more edges, will be assigned smaller μ values than stars with a smaller number of edges. The second term acts to minimize the probabilities of neighboring vertices.

In the case of Γ_{star}, selecting the uniform pmf yields a simple lower bound on the modulus.

Lemma 4.10. *Let $\Gamma = \Gamma_{\text{star}}$ and $1 < p < \infty$. Then,*

$$\mathrm{Mod}_{p,\sigma}(\Gamma) \geq \left(\frac{|V|}{2}\right)^p \left(\sum_{e\in E}\sigma(e)^{-\frac{q}{p}}\right)^{-\frac{p}{q}}.$$

In particular, if $\sigma \equiv 1$, then

$$\mathrm{Mod}_p(\Gamma) \geq \left(\frac{|V|}{2|E|^{\frac{1}{q}}}\right)^p. \tag{4.10}$$

Proof. Define μ to be the uniform pmf on Γ. That is, $\mu(\delta(v)) = \frac{1}{|V|}$ for every $v \in V$. Then, the expected edge usage for every edge e is

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = \frac{2}{|V|}.$$

Substituting this into equation (2.6) we get

$$\begin{aligned}
\mathrm{Mod}_{p,\sigma}(\Gamma)^{-\frac{1}{p}} &\le \left(\sum_{e\in E}\sigma(e)^{-\frac{q}{p}}\left(\frac{2}{|V|}\right)^{q}\right)^{\frac{1}{q}} \\
&\le \frac{2}{|V|}\left(\sum_{e\in E}\sigma(e)^{-\frac{q}{p}}\right)^{\frac{1}{q}} \\
\mathrm{Mod}_{p,\sigma}(\Gamma) &\ge \left(\frac{2}{|V|}\right)^{-p}\left(\sum_{e\in E}\sigma(e)^{-\frac{q}{p}}\right)^{-\frac{p}{q}}.
\end{aligned}$$

which gives an upper bound to the modulus. Letting $\sigma \equiv 1$ yields the result from equation (4.10). □

This allows us to prove Lemma 4.9.

Proof of Lemma 4.9. Lemma 4.8 provides an upper bound. The lower bound follows from Lemma 4.10. Since G is assumed to be d-regular, $|V| = 2|E|/d$. Substituting into (4.10) yields the bound

$$\mathrm{Mod}_p(\Gamma) \ge \left(\frac{|E|^{1-\frac{1}{q}}}{d}\right)^{p} = \frac{|E|}{d^p}.$$

□

The following examples show optimal pmf's for several types of graphs. These examples highlight the balance of terms in (4.9).

Example 4.10 (Star graph). Let $G = S_n$ be the star graph and define $\mu = 0$ on the center star and $\mu = \frac{1}{n-1}$ on the remaining $n-1$ stars. Then $\mu \in \mathcal{P}(\Gamma_{\text{star}})$. Moreover, for any edge $e \in E$, the expected edge usage is

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = \frac{1}{n-1},$$

since each edge sees exactly one of the stars of degree one.

Using (2.6), we obtain

$$\begin{aligned} \mathrm{Mod}_p(\Gamma_{\text{star}})^{-\frac{q}{p}} &\leq \sum_{e\in E}\left(\frac{1}{n-1}\right)^q = (n-1)\left(\frac{1}{n-1}\right)^q = \frac{1}{(n-1)^{q-1}} \\ \mathrm{Mod}_p(\Gamma_{\text{star}}) &\geq \left(\frac{1}{(n-1)^{q-1}}\right)^{-\frac{p}{q}} = (n-1)^{\frac{p}{q(q-1)}} = n-1. \end{aligned}$$

From Example 4.7 we know that $\mathrm{Mod}_p(\Gamma_{\text{star}}) = n-1$, so μ is an optimal pmf.

Example 4.11 (Cycle graph). Let $G = C_n$ be the cycle graph. By choosing the uniform pmf as in the proof of Lemma 4.10, we obtain the lower bound

$$\mathrm{Mod}_p(\Gamma_{\text{star}}) \geq \left(\frac{|V|}{2|E|^{1/q}}\right)^p = \frac{n^p}{2^p n^{p/q}} = \frac{n}{2^p}$$

where the last equality is from the fact that $p/q = p-1$. From Example 4.8 we know that $\mathrm{Mod}_p(\Gamma_{\text{star}}) = \frac{n}{2^p}$, so μ is an optimal pmf.

Example 4.12 (Complete graph). Let $G = K_n$ be the complete graph. As in the previous example, the uniform pmf yields a lower bound,

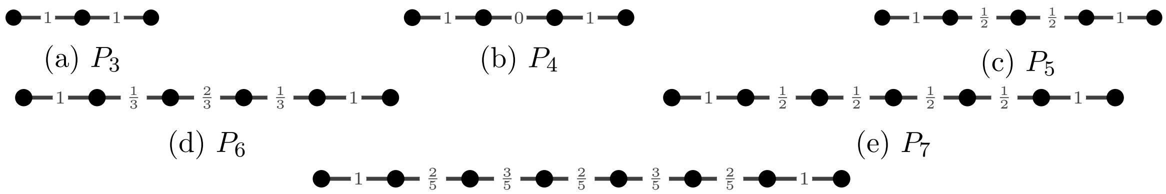
$$\mathrm{Mod}_p(\Gamma_{\text{star}}) \geq \left(\frac{|V|}{2|E|^{1/q}}\right)^p = \frac{n^p}{2^p}\cdot\frac{2^{p/q}}{[n(n-1)]^{p/q}} = \frac{n}{2(n-1)^{p-1}},$$

which coincides with the modulus as computed in Example 4.9, showing that the uniform pmf is optimal.

4.3.2 More Examples

In the following examples we compute $\mathrm{Mod}_2(\Gamma_{\text{star}})$ on two unweighted, undirected graphs: the path graph and the wheel graph.

Example 4.13 (Path graphs). Let $G = P_n$ be the unweighted path graph with $|V| = n \geq 3$ and $|E| = n-1$.



(a) P_3 (b) P_4 (c) P_5 (d) P_6 (e) P_7 (f) P_8

Figure 4.4: The extremal density ρ_0 for path graphs when $p = 2$.

a) For $n = 3$, the density $\rho_0 \equiv 1$ is admissible for Γ_{star} (see Figure 4.4a). So,

$$\text{Mod}_2(\Gamma_{\text{star}}) \leq 2.$$

For the lower bound, define $\mu(\delta(v)) = \frac{1}{2}$ on the two nodes with degree 1 and $\mu(\delta(u)) = 0$ on the node with degree 2. Using (2.8) and (4.9) we get

$$\text{Mod}_2(\Gamma_{\text{star}})^{-1} \leq 2\left(\frac{1}{2}\right)^2 = \frac{1}{2}.$$

Thus we have that $\text{Mod}_2(\Gamma_{\text{star}}) = 2$, ρ_0 is the extremal density, and μ is an optimal pmf.

b) For $n = 4$, define ρ_0 as

$$\rho_0(e) = \begin{cases} 1 & \text{if } e \text{ is a pendant edge,} \\ 0 & \text{otherwise} \end{cases}$$

(see Figure 4.4b). There are two types of stars for this path graph. Specifically, the star centered at the node with degree 1 and the star whose node is adjacent to both pendant edges (see Figure 4.4b). For both of these stars, this density is admissible. So,

$$\text{Mod}_2(\Gamma_{\text{star}}) \leq 2.$$

For the lower bound, similar to the $n = 3$ case, define $\mu(\delta(v)) = \frac{1}{2}$ on the nodes with degree 1 and $\mu(\delta(u)) = 0$ for the nodes with degree 2. By (2.8) and (4.9),

$$\text{Mod}_2(\Gamma_{\text{star}})^{-1} \leq 2\left(\frac{1}{2}\right)^2 = \frac{1}{2}.$$

Thus we have that $\text{Mod}_2(\Gamma_{\text{star}}) = 2$, ρ_0 is the extremal density, and μ is an optimal pmf.

c) For $n \geq 5$ and n odd, define ρ_0 as

$$\rho_0(e) = \begin{cases} 1 & \text{if } e \text{ is a pendant edge,} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

(See Figures 4.4c and 4.4e.) To verify this density is admissible for Γ_{star}, we consider the three different types of stars:

(i) Let v be a node with $\deg(v) = 1$. The edge incident to v is a pendant edge and

$$\ell_{\rho_0}(\delta(v)) = \sum_{e \in \delta(v)} \rho_0(e) = 1.$$

(ii) Let u be a node with $\deg(u) = 2$ and u is incident to the pendant edge. Then

$$\ell_{\rho_0}(\delta(u)) = \sum_{e \in \delta(u)} \rho_0(e) = 1 + \frac{1}{2} \geq 1.$$

(iii) Let w be a node with $\deg(u) = 2$ and u is not incident to the pendant edge. Then

$$\ell_{\rho_0}(\delta(w)) = \sum_{e \in \delta(w)} \rho_0(e) = 2 \cdot \frac{1}{2} = 1.$$

Thus, ρ_0 is admissible for Γ_{star}, so

$$\text{Mod}_2(\Gamma_{\text{star}}) \leq \mathcal{E}_2(\rho_0) = 2 \cdot 1^2 + (n-3) \cdot \left(\frac{1}{2}\right)^2 = \frac{n+5}{4}.$$

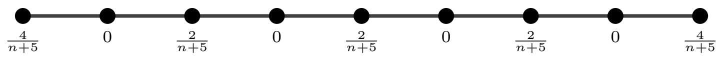


Figure 4.5: The values of μ for a path graph with $n \geq 5$ and odd.

For the lower bound define $\mu = \frac{4}{n+5}$ on the stars with degree 1. On the stars with degree 2, assign μ to be 0 and $\frac{2}{n+5}$ alternately. There will be $\frac{n-1}{2}$ stars assigned a value of 0 and $\frac{n-3}{2}$ stars assigned a value of $\frac{2}{n+5}$ (see Figure 4.5). This is a pmf on the stars since

$$\sum_{v \in V} \mu(\delta(v)) = 2 \cdot \frac{4}{n+5} + \frac{n-3}{2} \cdot \frac{2}{n+5} = \frac{8+n-3}{n+5} = 1.$$

Since no two stars with positive probability share an edge, (2.8) and (4.9) show that

$$\text{Mod}_2(\Gamma_{\text{star}})^{-1} \leq 2\left(\frac{4}{n+5}\right)^2 + \frac{n-3}{2} \cdot 2\left(\frac{2}{n+5}\right)^2 = \frac{4n+20}{(n+5)^2} = \frac{4}{n+5}.$$

Thus we have that $\text{Mod}_2(\Gamma_{\text{star}}) = \frac{n+5}{4}$, ρ_0 is the extremal density and μ is an optimal pmf.

d) For $n \geq 6$ and even, define ρ_0 to be 1 on the pendant edges, and alternating between the values ρ_1 and ρ_2 on the remaining edges, where

$$\rho_1 = \frac{n-4}{2n-6} \qquad \rho_2 = \frac{n-2}{2n-6}.$$

(See Figures 4.4d and 4.4f.) To verify this density is admissible for Γ_{star}, we consider the three different types of stars:

(i) Let v be a node with $\deg(v) = 1$. The edge incident to v is a pendant edge and

$$\ell_{\rho_0}(\delta(v)) = \sum_{e \in \delta(v)} \rho_0(e) = 1.$$

(ii) Let u be a node with $\deg(u) = 2$ and u is incident to the pendant edge. Then

$$\ell_{\rho_0}(\delta(u)) = \sum_{e \in \delta(u)} \rho_0(e) = 1 + \rho_1 = 1 + \frac{n-4}{2n-6} \geq 1.$$

(iii) Let w be a node with $\deg(w) = 2$ and w is not incident to the pendant edge. Then

$$\ell_{\rho_0}(\delta(w)) = \sum_{e \in \delta(w)} \rho_0(e) = \rho_1 + \rho_2 = \frac{n-4}{2n-6} + \frac{n-2}{2n-6} = 1.$$

Thus, ρ_0 is admissible for Γ_{star} so

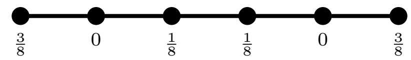
$$\text{Mod}_2(\Gamma_{\text{star}}) \leq 2 \cdot 1^2 + \left(\frac{n}{2} - 1\right)\left(\frac{n-4}{2n-6}\right)^2 + \left(\frac{n}{2} - 2\right)\left(\frac{n-2}{2n-6}\right)^2 = \frac{n^2 + 2n - 16}{2\,(2n-6)}.$$

To obtain a lower bound, first enumerate the first half of the vertices $v_1, v_2, \ldots, v_k$, where $k = \frac{n}{2}$. For the stars centered at v_i, $1 \leq i \leq k$, define μ as follows:

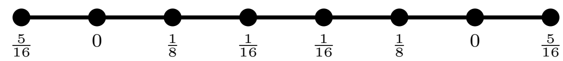
$$\mu(\delta(v_i)) = \begin{cases} \frac{2(2n-6)}{n^2+2n-16}, & i = 1 \\ 0, & i = 2 \\ \frac{2n-2(i+1)}{n^2+2n-16}, & 2 < i \leq k, i \text{ odd} \\ \frac{2(i-2)}{n^2+2n-16}, & 2 < i \leq k, i \text{ even} \end{cases}$$

Once the first k stars have been assigned a μ value, assign the values of μ to the following k vertices by mirroring the values: $\mu(\delta(v_{n+1-i})) = \mu(\delta(v_i))$ for $i = 1, 2, \ldots, k$. See Figure 4.6. A straightforward (but long) calculation shows that the corresponding lower bound on modulus agrees with the upper bound above. See Appendix B for the details of this calculation.

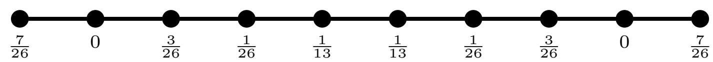
Example 4.14 (Wheel graphs). Let $G = W_n$ be the wheel graph with $|V| = n \geq 4$ and $|E| = 2(n-1)$. Specifically, G has a special center node $v \in V$ with $\deg(v) = n - 1$, and every other node $u \neq v$ has $\deg(u) = 3$.



(a) P_6



(b) P_8



(c) P_{10}

Figure 4.6: Values of μ for different path graphs for $n \geq 6$ and even.

a) For $n \in \{4, 5\}$, define ρ_0 as

$$\rho_0(e) = \begin{cases} \frac{1}{n-1}, & \text{if } e \text{ is incident to } v \\ \frac{n-2}{2(n-1)}, & \text{otherwise.} \end{cases}$$

(See Figure 4.7.) To verify this density is admissible for Γ_{star}, we consider the two different types of stars:

(i) Let v be the center node, then

$$\ell_{\rho_0}(\delta(v)) = \sum_{e \in \delta(v)} \rho_0(e) = (n-1) \cdot \frac{1}{n-1} = 1.$$

(ii) Let u be a node with $\deg(u) = 3$. Then

$$\ell_{\rho_0}(\delta(u)) = \sum_{e \in \delta(u)} \rho_0(e) = \frac{1}{n-1} + 2 \cdot \frac{n-2}{2(n-1)} = \frac{2n-2}{2(n-1)} = 1.$$

Thus, ρ_0 is admissible for Γ_{star}, so

$$\text{Mod}_2(\Gamma_{\text{star}}) \leq (n-1) \left(\frac{1}{n-1}\right)^2 + (n-1) \left(\frac{n-2}{2(n-1)}\right)^2 = \frac{4 + (n-2)^2}{4(n-1)}.$$

To obtain a lower bound, define $\mu(\delta(v)) = \frac{6-n}{4+(n-2)^2}$ and define $\mu(\delta(u)) = \frac{n-2}{4+(n-2)^2}$ for

$u \neq v$. To verify that $\mu \in \mathcal{P}(\Gamma_{\text{star}})$ we sum μ over all stars:

$$\sum_{v \in V} \mu(\delta(v)) = \frac{6-n}{4+(n-2)^2} + (n-1) \cdot \frac{n-2}{4+(n-2)^2} = \frac{n^2-4n+8}{4+(n-2)^2} = 1.$$

Using (4.8) and Theorem 2.7,

$$\begin{aligned} \text{Mod}_2(\Gamma_{\text{star}})^{-1} \leq & \left(\frac{6-n}{4+(n-2)^2}\right)^2 + 5(n-1)\left(\frac{n-2}{4+(n-2)^2}\right)^2 \\ & + 2(n-1)\left(\frac{6-n}{4+(n-2)^2}\right)\left(\frac{n-2}{4+(n-2)^2}\right) \\ & = \frac{4(n-1)(n^2-4n+8)}{(4+(n-2)^2)^2} = \frac{4(n-1)}{4+(n-2)^2} \end{aligned}$$

Thus we have that $\text{Mod}_2(\Gamma_{\text{star}}) = \frac{4+(n-2)^2}{4(n-1)}$, ρ_0 is the extremal density and μ is an optimal pmf.

b) For $n \geq 6$, define ρ_0 as

$$\rho_0(e) = \begin{cases} \frac{1}{5}, & \text{if } e \text{ is incident to } v \\ \frac{2}{5}, & \text{otherwise.} \end{cases}$$

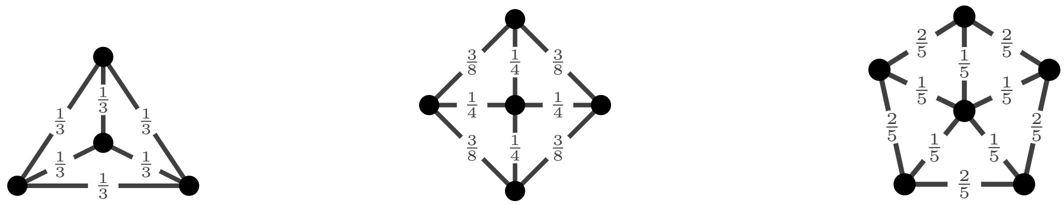
To verify this density is admissible for Γ_{star}, we consider the two different types of stars:

(i) Let v be the center node, then

$$\ell_{\rho_0}(\delta(v)) = \sum_{e \in \delta(v)} \rho_0(e) = (n-1) \cdot \frac{1}{5} = \frac{n-1}{5} \geq 1.$$

(ii) Let u be a node with $\deg(u) = 3$. Then

$$\ell_{\rho_0}(\delta(u)) = \sum_{e \in \delta(u)} \rho_0(e) = \frac{1}{5} + 2 \cdot \frac{2}{5} = 1.$$



(a) W_4 (b) W_5 (c) W_6.

Figure 4.7: The extremal density ρ_0 for wheel graphs when $p = 2$.

Thus, ρ_0 is admissible for Γ_{star}, so

$$\text{Mod}_2(\Gamma_{\text{star}}) \leq (n-1)\left(\frac{1}{5}\right)^2 + (n-1)\left(\frac{2}{5}\right)^2 = \frac{n-1}{5}.$$

To obtain a lower bound, define $\mu(\delta(v)) = 0$ and $\mu(\delta(u)) = \frac{1}{n-1}$ for $u \neq v$. We see that $\mu \in \mathcal{P}(\Gamma_{\text{star}})$ since

$$\sum_{v\in V} \mu(\delta(v)) = (n-1)\cdot\frac{1}{n-1} = 1.$$

Using (4.8) and Theorem 2.7,

$$\text{Mod}_2(\Gamma_{\text{star}})^{-1} \leq 3(n-1)\left(\frac{1}{n-1}\right)^2 + 2(n-1)\left(\frac{1}{n-1}\right)^2 = \frac{5}{n-1}.$$

Thus we have that $\text{Mod}_2(\Gamma_{\text{star}}) = \frac{n-1}{5}$, ρ_0 is the extremal density, and μ is an optimal pmf.

The two examples calculated in this section highlight the complexity of calculating the star modulus, as well as the values of the extremal densities, ρ^*, and the optimal pmfs, μ^*. From the path graph we see that non-pendant edges alternate values of ρ^* and the ρ^*-length of these stars that don't contain any pendant edges is exactly 1. We also see that an optimal pmf will assign the heaviest star (star with 2 edges where one edge is the pendant edge) a value of 0. This agrees with the observations made in Section 4.3.1 about equation (4.9) and how heavier stars will be assigned smaller μ^* values. From the wheel graph, we again make the same observation that the star centered in the center node with degree $n-1$ is assigned a smaller μ^* value and when $n \geq 6$, it is assigned a value of 0. In terms of the extremal density, we see that as n becomes larger, the values of ρ^* stabilize and become $\frac{1}{5}$ for edges incident to the center, and $\frac{2}{5}$ for the others. Even though the center star becomes larger as n gets bigger, these values of the extremal density and optimal pmf indicate that modulus seems to ignore the center star.

Using the results found in these examples and equation (4.7) we can find the value of the 2-modulus of fractional edge covers. Specifically, we obtain the following results.

Path graphs. Since the path graph is bipartite, by Corollary 3.8, $\Gamma_{\text{ec}} \simeq \Gamma_{\text{fec}}$ and the modulus of the two families are equal to each other. Moreover, we know $\text{Mod}_2(\Gamma_{\text{fec}}) = \text{Mod}_2(\Gamma_{\text{star}})^{-1}$, therefore we know the exact value of $\text{Mod}_2(\Gamma_{\text{ec}})$. From Example 4.13 we have that:

a) For $n \in \{3, 4\}$,

$$\text{Mod}_2(\Gamma_{\text{fec}}) = \text{Mod}_2(\Gamma_{\text{ec}}) = \frac{1}{2}.$$

b) For $n \geq 5$ and n odd,

$$\text{Mod}_2(\Gamma_{\text{fec}}) = \text{Mod}_2(\Gamma_{\text{ec}}) = \frac{4}{n+5}.$$

c) For $n \geq 6$ and n even,

$$\text{Mod}_2(\Gamma_{\text{fec}}) = \text{Mod}_2(\Gamma_{\text{ec}}) = \frac{2(2n-6)}{n^2+2n-16}.$$

Wheel graphs. Since wheel graphs are not bipartite, we are not able to obtain the exact value of the edge cover modulus. Instead, we use the bounds from Theorem 4.2 and approximate the value of $\text{Mod}_2(\Gamma_{\text{ec}})$. From Example 4.14 we have that:

a) For $n \in \{4, 5\}$, $\text{Mod}_2(\Gamma_{\text{fec}}) = \frac{4(n-1)}{4+(n-2)^2}$, and

$$\frac{9(n-1)}{4(4+(n-2)^2)} \leq \text{Mod}_2(\Gamma_{\text{ec}}) \leq \frac{4(n-1)}{4+(n-2)^2}.$$

b) For $n \geq 6$, $\mathrm{Mod}_2(\Gamma_{\mathrm{fec}}) = \frac{5}{n-1}$, and

$$\frac{45}{16(n-1)} \leq \mathrm{Mod}_2(\Gamma_{\mathrm{ec}}) \leq \frac{5}{n-1}.$$

In Section 5.3 we verify these bounds by numerically calculating the edge cover modulus of the wheel graph.

Chapter 5

Numerical Examples

For an unweighted graph, G, we can use the results from Lemma 3.9 and Theorem 2.5 to restate the upper and lower bounds relating $\mathrm{Mod}_p(\Gamma_{\mathrm{ec}})$, $\mathrm{Mod}_p(\Gamma_{\mathrm{fec}})$ and $\mathrm{Mod}_q(\Gamma_{\mathrm{star}})$ as

$$\left(\frac{3}{4}\right)^p \leq \mathrm{Mod}_p(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_p(\Gamma_{\mathrm{fec}})^{-1} = \mathrm{Mod}_p(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_q(\Gamma_{\mathrm{star}})^{p/q} \leq 1. \tag{5.1}$$

In the case when $p = 2$ and $\sigma \equiv 1$, we have

$$\left(\frac{3}{4}\right)^2 \leq \mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})^{-1} = \mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{star}}) \leq 1, \tag{5.2}$$

(If G is bipartite, then $\Gamma_{\mathrm{fec}} \simeq \Gamma_{\mathrm{ec}}$, so the moduli of the two families are equal.) In this chapter we present the methods used to numerically calculate the modulus of the different families, as well as multiple examples that explore the tightness of the bounds in (5.1) on nonbipartite graphs.

5.1 Numerical Methods

One natural way to solve the modulus problem numerically is to apply a convex optimization solver to (2.3). For sufficiently small families, Γ, it is not difficult to form the full usage matrix $\mathcal{N} \in \mathbb{R}^{\Gamma\times E}$. For example, the usage matrix for Γ_{star} is a $|V| \times |E|$ matrix.

For larger families (e.g., Γ_{ec} and $\Gamma_{\overline{\text{fec}}}$), it is often not feasible even to enumerate all constraints. A possible option for such cases is the *basic algorithm* described in [6]. This algorithm proceeds by maintaining a relatively small set of active constraints, thus eliminating the need to fully construct $\mathcal{N}$. Instead, the algorithm iteratively improves an estimate of the optimal density. In each round of the iteration, the algorithm requires a subroutine `shortest(`ρ`)`, that produces a ρ-shortest object in Γ, that is

$$\gamma = \texttt{shortest}(\rho) \Longrightarrow \forall\gamma', \ell_\rho(\gamma) \leq \ell_\rho(\gamma').$$

As long as an efficient implementation of `shortest` exists for Γ, the basic algorithm can be used to numerically approximate modulus. For example, Dijkstra's algorithm can be used to compute the modulus of *st*-paths, and Kruskal's algorithm can be used to compute the modulus of spanning trees. This method can also be applied to the families in the present paper.

Stars. For Γ_{star}, it is possible to fully compute $\mathcal{N}$ (there is one row per vertex). There is also an efficient `shortest` method; one simply loops over all vertices to find the one with minimum ρ-degree.

Edge covers. For the family of edge covers, there is a minimum weight edge cover (MWEC) algorithm given by Schrijver in [31]. This algorithm reduces the minimum weight edge cover problem to the minimum weight perfect matching (MWPM) problem, for which there are several polynomial-time algorithms [14, 24, 26]. The description of Shrijver's algorithm is given below.

Let $G = (V, E, \sigma)$ be a graph without isolated vertices and $\sigma : E \to \mathbb{Q}^+$ be the edge weights. Let $G' = (V', E', \sigma')$ be the graph obtained by adding a disjoint copy $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{\sigma})$ of G and for each vertex $v \in V$ adding an edge $\{v, \tilde{v}\}$ connecting v with its copy $\tilde{v}$. The new edge weights σ' will be assigned as follows:

1. For each $e \in E$ and its corresponding copy $\tilde{e} \in \tilde{E}$ assign $\sigma'(e) := \sigma'(\tilde{e}) := \sigma(e)$.

2. For each new edge $\{v, \tilde{v}\} \in E'$ assign $\sigma'(\{v, \tilde{v}\}) := 2m(v)$, where $m(v)$ is the minimum weight of the edges of G incident to v.

Using the blossom algorithm (or the Micali and Vazirani method) we can find a minimum weight perfect matching M in G'. This then yields a minimum weight edge cover F in G as follows: replace any edge $\{v, \tilde{v}\}$ in M by an edge e_v of minimum weight of G incident with v, and delete all edges in $M \cap E'$. Then $\sum_{e \in F} \sigma(e) = \frac{1}{2} \sum_{e \in M} \sigma'(e)$. To ensure that F is a minimum weight edge cover, notice that any edge cover F' of G gives by a reverse construction a perfect matching M' in G' with $\sigma'(M') \leq 2\sigma(F)$, where $\sigma(X) = \sum_{x \in X} \sigma(x)$.

Fractional edge covers. At the time of the publication of this dissertation, we are not aware of an efficient method for computing the minimum weight fractional edge cover. (Considering the method described above for the MWEC problem, it is possible that an analogous reduction to the minimum weight fractional perfect matching problem exists.) Fortunately, it is possible to compute the modulus of this family anyway, using Theorem 2.5 and Lemma 3.9. This allows us to repurpose the star modulus code to compute the modulus of fractional edge covers.

5.2 Implementation Details

For the computations presented in this paper, we used the Python implementation of the basic algorithm found in [4]. The graphs are represented using NetworkX [19], NumPy [21], and SciPy [35], and the convex optimization problem is solved numerically using CVXPY [1, 13].

When calculating the modulus of stars, it is efficient to compute the full usage matrix $\mathcal{N}$ using the NetworkX built-in function `incidence_matrix`. Since the family of edge covers tends to be large, it is not feasible to generate $\mathcal{N}$. Instead, we use the basic algorithm and implement the `shortest` subroutine with Schrijver's MWEC algorithm, using the NetworkX function `min_weight_matching` to solve the MWPM problem. It is also generally infeasible to generate the full usage matrix for fractional edge covers. For fractional edge covers, we calculate the modulus by using the results from Section 2.4 and the code that computes modulus of stars.

5.3 Examples

The following section goes over the details of calculating the modulus of the three families of interest in this dissertation and studies the comparison of the edge cover modulus and the fractional edge cover modulus.

5.3.1 Cycle Graphs

The modulus values for C_n were found in Examples 4.2, 4.5, and 4.8. Summarizing these examples, we found that

$$\mathrm{Mod}_2(\Gamma_{\text{ec}}) = \begin{cases} \frac{4}{n} & \text{if } n \text{ is even,} \\ \frac{4n}{(n+1)^2} & \text{if } n \text{ is odd,} \end{cases} \qquad \text{and} \quad \mathrm{Mod}_2(\Gamma_{\text{fec}}) = \frac{4}{n} = \mathrm{Mod}_2(\Gamma_{\text{star}})^{-1}.$$

In the case of edge covers, the value of the modulus splits into two cases depending on whether n is even or odd. Note that since these graphs are symmetric, the optimal density is constant and the difference between these two cases is due to the structure of the graph objects. Specifically, we look at the minimum edge covers of C_n. Recall that these are the edge covers that use the least number of edges to cover the graph. When n is even, the minimum edge covers are also perfect matchings, therefore every node is covered by exactly one edge. When n is odd, a minimum edge cover will have one node covered by two edges, and the rest covered by exactly one edge (as seen in Figure 4.2). Therefore, there will always be a "heavier" node in the edge cover.

The modulus of fractional edge covers does not have this same dependence on the evenness or oddness of n; there is a single formula for all cases. This suggests that, when n is odd, the fractional edge covers using odd cycles become important. For this class of graphs, we

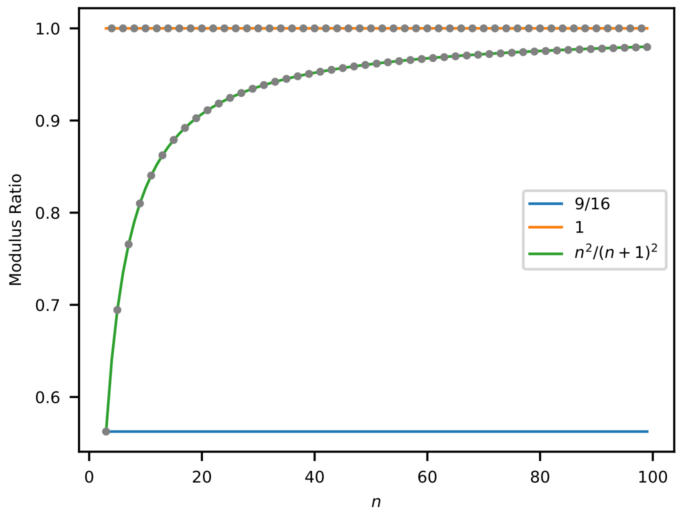


Figure 5.1: The value of $\text{Mod}_2(\Gamma_{\text{ec}})\,\text{Mod}_2(\Gamma_{\text{fec}})^{-1}$ as a function of n for the cycle graph. The orange and blue lines represent the upper and lower bounds in equation (5.2) and the green line represents the lower bound from Theorem 4.7.

have

$$\text{Mod}_2(\Gamma_{\text{ec}})\,\text{Mod}_2(\Gamma_{\text{fec}})^{-1} = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \frac{n^2}{(n+1)^2} & \text{if } n \text{ is odd.} \end{cases}$$

Since the graph is bipartite when n is even, then $\Gamma_{\text{ec}} \simeq \Gamma_{\text{fec}}$ and we see that the modulus of the two families are equal to each other. In the case when n is odd, we can obtain an upper and lower bound using Theorem 4.7 since the extremal density for the edge cover modulus is constant. This implies that the ratio of edge cover modulus to fractional edge cover modulus is bounded from below by $\left(\frac{k}{k+1}\right)^2$ where k is the length of the shortest odd cycle. For cycle graphs, $k = n$, therefore the modulus is bounded by $\left(\frac{n}{n+1}\right)^2$ which is exactly the value of the product of the moduli. This shows an example of a graph where the lower bound in Corollary 4.7 is sharp. The ratio approaches 1 in the limit as $n \to \infty$.

Numerical computations of the edge cover modulus for both even and odd values of n verify that the lower bound is met when n is odd, and the upper bound is met when n is even. See Figure 5.1.

5.3.2 Complete Graphs

The modulus values for K_n were found in Examples 4.3, 4.6, and 4.9. The values obtained were

$$\mathrm{Mod}_2(\Gamma_{\mathrm{ec}}) = \begin{cases} \frac{2(n-1)}{n} & \text{if } n \text{ is even,} \\ \frac{2n(n-1)}{(n+1)^2} & \text{if } n \text{ is odd,} \end{cases} \quad \text{and} \quad \mathrm{Mod}_2(\Gamma_{\mathrm{fec}}) = \frac{2(n-1)}{n} = \mathrm{Mod}_2(\Gamma_{\mathrm{star}})^{-1}.$$

For the complete graph, we see a similar behavior as discussed in the cycle graph example above. There is a difference in the edge cover modulus when n is even or odd, this is again due to the difference in minimum edge covers. The modulus of fractional edge covers does not have this same dependence on the evenness or oddness of n and there is a single formula for all cases. For this class of graphs, we again have

$$\mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})^{-1} = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \frac{n^2}{(n+1)^2} & \text{if } n \text{ is odd.} \end{cases}$$

Equation (5.1) shows that the ratio of edge cover modulus to fractional edge cover modulus is bounded below by $\frac{9}{16}$. The actual ratio approaches 1 in the limit as $n \to \infty$, showing that the lower bound is not tight. Even though the extremal density for the edge cover modulus is constant, every complete graph contains a triangle. Therefore, the lower bound from Theorem 4.7 gives the same lower bound as equation (5.1). The value of the edge cover modulus and fractional edge cover modulus were numerically calculated for n ranging from 3 to 50 and plotted in Figure 5.2.

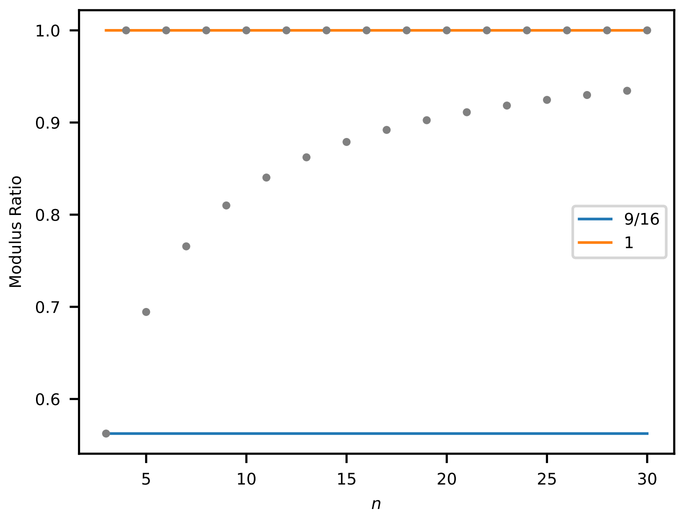


Figure 5.2: The value of $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})^{-1}$ as a function of n for the complete graph. The lines represent the upper and lower bounds in equation (5.2).

5.3.3 Wheel Graphs

In Section 4.3.2 we calculated $\mathrm{Mod}_2(\Gamma_{\mathrm{star}})$ for the wheel graph and stated the upper and lower bounds for $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}})$. Specifically, we found that when $n \in \{4, 5\}$,

$$\frac{9(n-1)}{4(4+(n-2)^2)} \leq \mathrm{Mod}_2(\Gamma_{\mathrm{ec}}) \leq \frac{4(n-1)}{4+(n-2)^2}.$$

We verify the value of the edge cover modulus by using the numerical method described in Section 5.1. From this we got that

$$\mathrm{Mod}_2(\Gamma_{\mathrm{ec}}) = \begin{cases} 1.5 & \text{if } n = 4 \\ 0.889 & \text{if } n = 5. \end{cases}$$

In the case when $n = 4$, the upper bound is $\frac{3}{2} = 1.5$ and the lower bound is $\frac{27}{32} \approx 0.844$. The value of the modulus is equal to the upper bound, therefore that bound is sharp. When $n = 5$, the upper bound is $\frac{16}{13} \approx 1.231$ and the lower bound is $\frac{9}{13} \approx 0.692$. The value of the modulus is between the bounds, and it is closer to the lower bound.

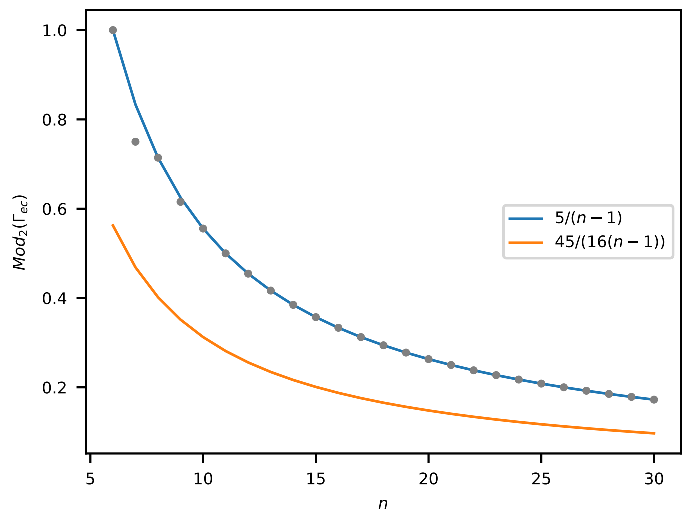


Figure 5.3: The value of $\text{Mod}_2(\Gamma_{\text{ec}})$ as a function of n for the wheel graph.

For $n \geq 6$, we found that

$$\frac{45}{16(n-1)} \leq \text{Mod}_2(\Gamma_{\text{ec}}) \leq \frac{5}{n-1}.$$

Figure 5.3 shows the value of the edge cover modulus compared to these bounds. We note that for all n except 7, the value of the modulus agrees with the upper bound. From these computations we can inspect the value of the extremal density ρ^* for each n. We note that in general, there will be 2 distinct values of ρ^*: one value for the edges incident to the center node and one value for the rest of the edges. When $n = 7$, the value of the extremal density is equal to $\frac{1}{4}$ for every edge, which may indicate why the value of the modulus for $n = 7$ does not lie on the curve corresponding to the upper bound.

5.3.4 Barbell Graphs

In the case of the n-barbell graphs (two disjoint copies of K_n connected by a single edge), we see a similar behavior to the cycle and complete graph examples discussed above. Using the modulus code described in Section 5.1, we compute $\text{Mod}_2(\Gamma_{\text{ec}})$ and $\text{Mod}_2(\Gamma_{\text{fec}})$ on the n-barbell graph for n ranging from 3 to 30. The ratio of these two moduli is plotted in Figure 5.4. As in the cycle graph and complete graph example, we observe two types of

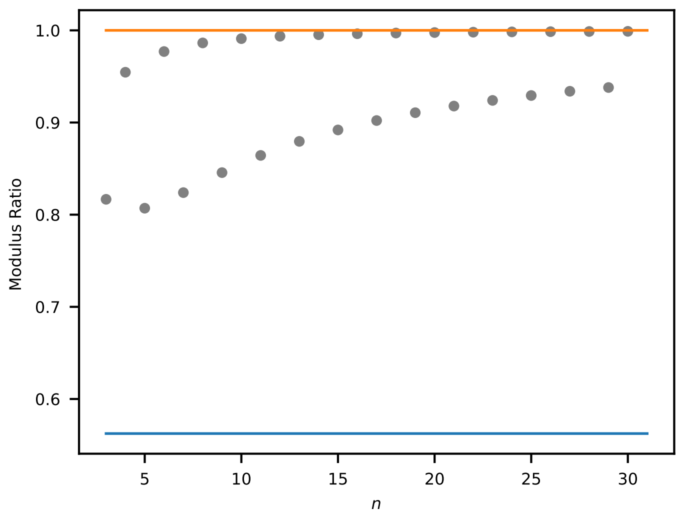


Figure 5.4: The value of $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})^{-1}$ as a function of n for the n-barbell graph. The lines represent the upper and lower bounds in equation (5.2).

behavior depending on the parity of n. In both cases, it appears that the ratio of moduli approaches 1 as $n \to \infty$. However, the convergence is much faster for even n than for odd. Again, this suggests that the odd cycles play an important role in $\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})$ for odd n.

The difference can be further understood by considering the family of minimum edge covers on the n-barbell. When n is even, no minimum edge cover uses the "bridge" of the n-barbell; these edge covers are built independently from edge covers of the two copies of K_n. From the perspective of the probabilistic interpretation, it appears that the optimal pmfs avoid using the bridge.

When n is odd, however, all minimum edge covers use the bridge. If the optimal pmfs were to concentrate only on minimum edge covers, then the bridge would be "overloaded." For these graphs, the optimal pmfs appear to balance betwen minimum edge covers and slightly larger edge covers that avoid the bridge. Choosing the smaller edge covers is beneficial since it tends to reduce the overlap with other edge covers; however, since these edge covers share a common edge (the bridge) it is also beneficial to choose the larger edge covers at times. An optimal pmf will balance these two competing preferences.

This can be seen in Figure 5.5. When n is odd, the bridge is more likely to appear than any other edge in a random edge cover chosen by an optimal pmf (Figure 5.5 top left). On the other hand, when n is even, the optimal edge usage probability of the bridge is zero (Figure 5.5 top right). For fractional edge covers, the bridge always has the lowest edge usage probability, followed by adjacent edges, and then all other edges.

Figure 5.6 compares the optimal expected edge usage, $\mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, b)] = \frac{\rho^*(b)}{\mathrm{Mod}_2(\Gamma)}$, of the bridge, b, for both the edge cover and fractional edge cover modulus. While the expected usage for the fractional edge cover modulus follows a single smooth curve

$$\mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, b)] = \frac{2n+2}{n^2-n+4},$$

the expected edge usage for edge cover modulus oscillates between two behaviors:

$$\mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, b)] = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{n-2}{n^2-n-1} & \text{if } n \text{ is odd.} \end{cases}$$

(There is a special case for $n = 3$.)

From this example, we see that fractional edge cover modulus approximates edge cover modulus for large n. Again, the lower bound in (5.1) is overly pessimistic. However, for smaller n, the additional flexibility in Γ_{fec} arising from the ability to use odd cycles, fails to capture the parity-dependence of the edge cover modulus.

The $n = 3$ case. The 3-barbell graph can be seen in Figure 5.7, which also shows the values of $\mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, e)]$ on the edges. This case is studied separately because the expected edge usage of the bridge does not follow the same curve as the case when $n > 3$. In terms of the edge cover modulus, we see that $\mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, b)] = 0.57$, whereas for fractional edge covers, $\mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, b)] = 0.2$. This implies that it is more likely to choose the bridge when building an edge cover, than it is when building a fractional edge cover. Note that for this graph, a minimum edge cover uses the bridge. In contrast, when building a basic fractional edge cover, it is more likely to choose the edges in the triangles to make up a 3-cycle, than it is to choose the bridge.

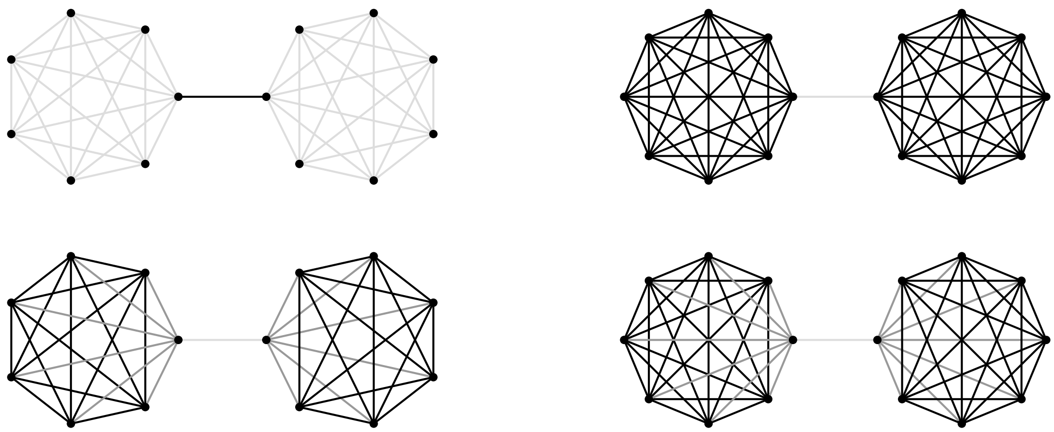


Figure 5.5: n-Barbell graphs with $n = 7$ and $n = 8$. The edges are colored using the expected edge usage, $\mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, e)] = \rho^*(e)/\operatorname{Mod}_2(\Gamma)$, for $\Gamma = \Gamma_{\text{ec}}$ (top row) and $\Gamma = \Gamma_{\text{fec}}$ (bottom row). Lighter colors represent smaller values.

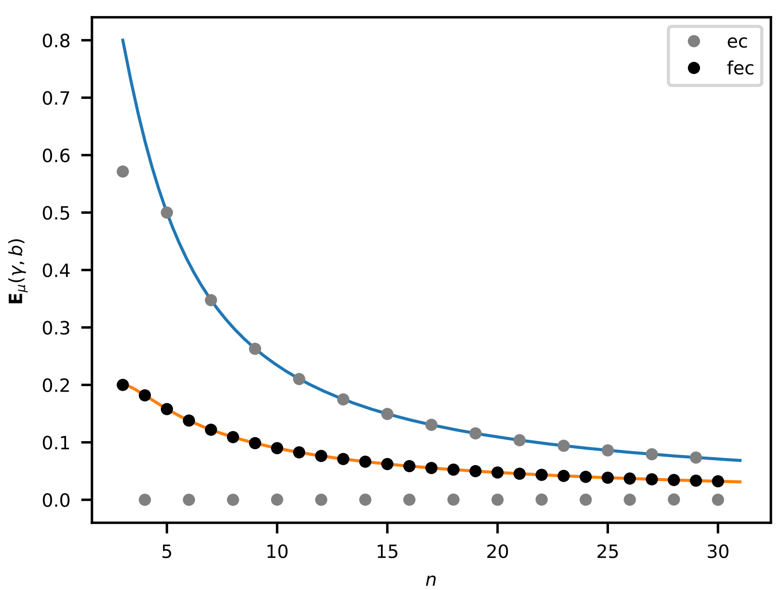


Figure 5.6: The graph above shows the value of $\mathbb{E}_{\mu}[\mathcal{N}(\underline{\gamma}, b)]$ in the y-axis and the value n on the x-axis, where b is the bridge connecting the two copies of K_n in the n-barbell graph.

(a) $\text{Mod}_2(\Gamma_{\text{ec}}) = 0.583$ (b) $\text{Mod}_2(\Gamma_{\text{fec}}) = 0.714$

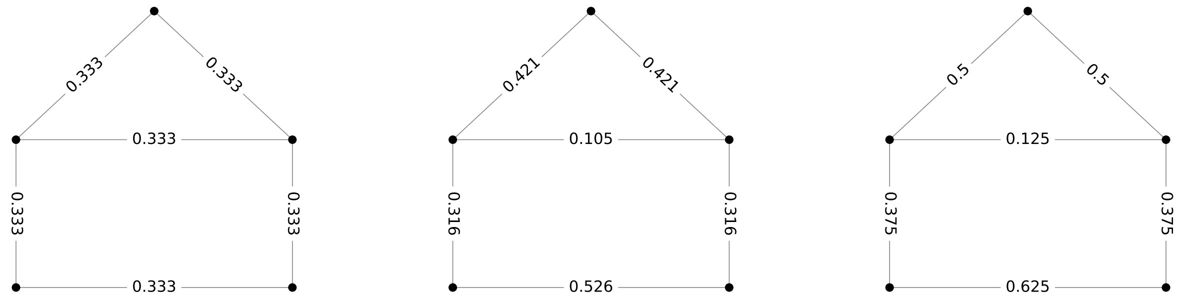
Figure 5.7: Modulus of the 3-barbell graph. Values of the edges correspond to the expected edge usage, $\mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, e)] = \rho^*(e)/\,\text{Mod}_2(\Gamma)$. The product of the moduli is $\text{Mod}_2(\Gamma_{\text{ec}})\,\text{Mod}_2(\Gamma_{\text{fec}})^{-1} = 0.8162$.

5.3.5 Small Graphs

In this section we discuss the modulus and extremal densities for some small, well-known graphs. The modulus of each graph is calculated using the implementation details discussed above. In each case, the graphs are unweighted and we use $p = 2$. We write ρ^*_{ec}, ρ^*_{fec}, and ρ^*_{star} to denote the extremal density for edge cover modulus, fractional edge cover modulus, and star modulus, respectively.

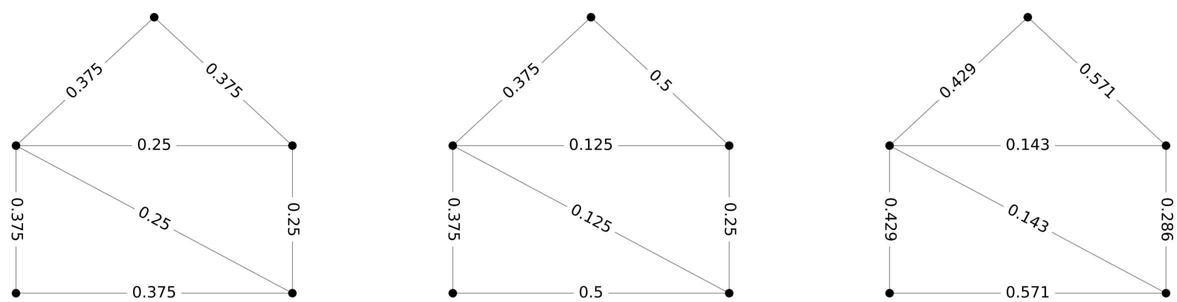
House graph. We start by comparing three variations of the house graph. Specifically, we look at the house graph, the slashed house, and the house X-graph (see Figures 5.8, 5.9 and 5.10). In this case, we can discuss what happens to the modulus of each family when we add one more edge. The lower bound in equation (5.2) is $\frac{9}{16} = 0.5625$. As expected, we can see that the value of $\text{Mod}_2(\Gamma_{\text{ec}})$ and $\text{Mod}_2(\Gamma_{\text{fec}})$ both increase when we add an edge. For the house graph shown in Figure 5.8, we see that the ρ^*_{ec} assigns the same value to each edge, but ρ^*_{fec} makes a distinction between the types of edges. The product of the moduli is equal to 0.792, compared to the bounds from equation (5.2), this value is slightly closer to the upper bound than the lower bound.

Adding the slashed edge to the house, we see in Figure 5.9 that ρ^*_{ec} now makes the distinction between the 'roof' of the house and the lower left corner of the house with value 0.375, and the inside triangle with value 0.25. Compared to ρ^*_{fec}, this assigns different values



(a) $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}}) = 0.667$ (b) $\mathrm{Mod}_2(\Gamma_{\mathrm{fec}}) = 0.842$ (c) $\mathrm{Mod}_2(\Gamma_{\mathrm{star}}) = 1.188$

Figure 5.8: Modulus of the house graph. Values of the edges correspond to the extremal density ρ^* for each family. The product of the moduli is $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})^{-1} = 0.792$.

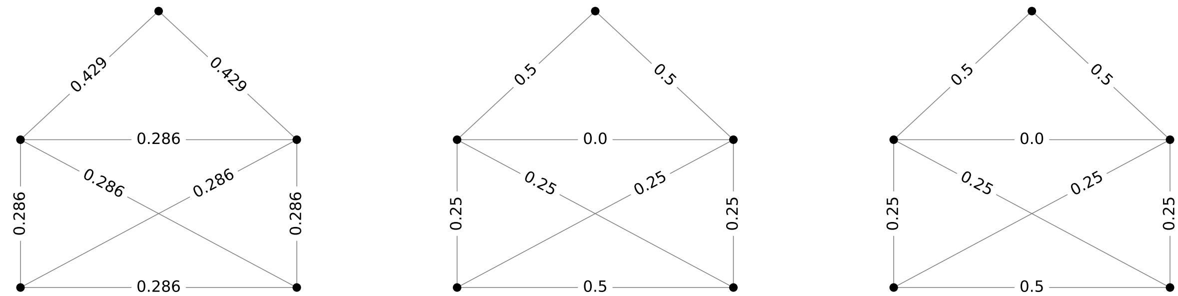


(a) $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}}) = 0.75$ (b) $\mathrm{Mod}_2(\Gamma_{\mathrm{fec}}) = 0.875$ (c) $\mathrm{Mod}_2(\Gamma_{\mathrm{star}}) = 1.143$

Figure 5.9: Modulus of the slashed house graph. Values of the edges correspond to the extremal density ρ^* for each family. The product of the moduli is $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})^{-1} = 0.857$.

to edges in different sections of the house. The product of the moduli is equal to 0.857 and we see that this value is now closer to the upper bound than in the case of the house graph.

Adding one more edge to create the house X-graph, we see in Figure 5.10 that ρ^*_{ec} again assigns the same value to the roof edges of 0.429, and every other edge has the value 0.286. An interesting thing happens for the fractional edge cover modulus as now ρ^*_{fec} assigns a value of 0 to an edge. This implies that this edge is never used when creating a basic fractional edge cover. The product of the moduli is 0.857, which is equal to the value obtained for the slashed house. In this case, adding this extra edge does not change the ratio of the modulus.

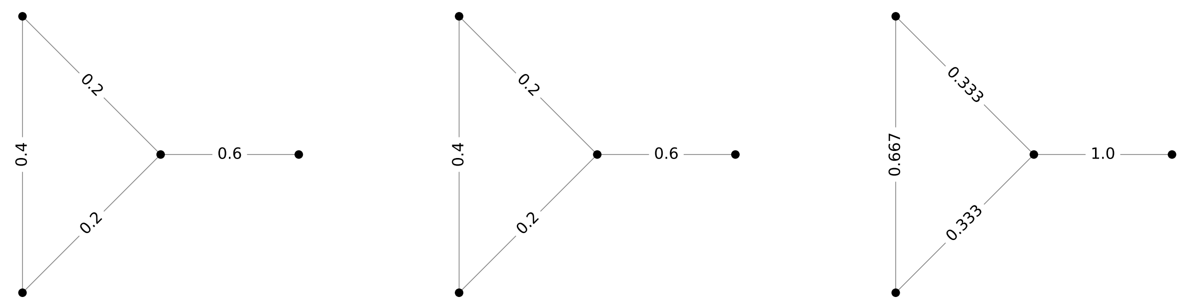


(a) $\text{Mod}_2(\Gamma_{\text{ec}}) = 0.857$ (b) $\text{Mod}_2(\Gamma_{\text{fec}}) = 1.0$ (c) $\text{Mod}_2(\Gamma_{\text{star}}) = 1.0$

Figure 5.10: Modulus of the house X-graph. Values of the edges correspond to the extremal density ρ^* for each family. The product of the moduli is $\text{Mod}_2(\Gamma_{\text{ec}})\,\text{Mod}_2(\Gamma_{\text{fec}})^{-1} = 0.857$.

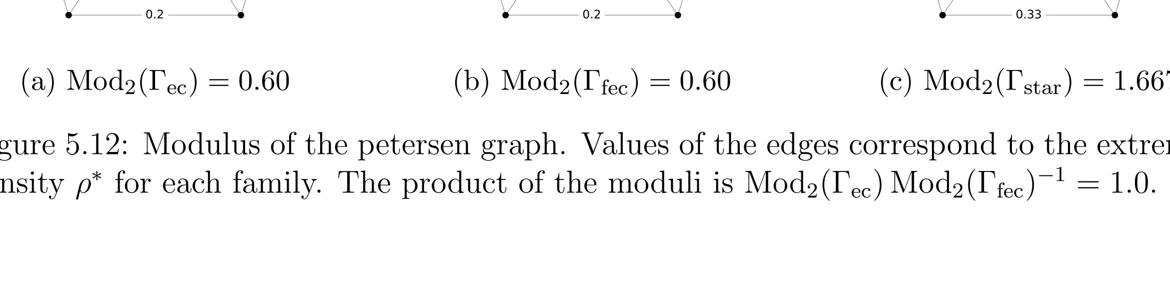
Paw graph. The paw graph is made up of a triangle with a pendant edge attached to the triangle. Figure 5.11 shows the values of the extremal densities and the modulus of each family for this graph. An interesting observation is that the value of the edge cover modulus is equal to the fractional edge cover modulus and the extremal densities are also the same. This is especially interesting since this graph is not bipartite, but it shows the same properties that a bipartite graph shows; that is, $\Gamma_{\text{ec}} \simeq \Gamma_{\text{fec}}$. This can be attributed to the fact that the paw graph is *almost bipartite*. A graph is said to be *(edge) almost bipartite* if it has only one odd cycle.

Petersen graph. Recall that the petersen graph is the 3-regular graph with 10 nodes and 15 edges shown in Figure 5.12. Note that for each modulus, ρ^* is constant, and again we see that $\text{Mod}_2(\Gamma_{\text{ec}}) = \text{Mod}_2(\Gamma_{\text{fec}})$ and $\rho^*_{\text{ec}} = \rho^*_{\text{fec}}$. This recovers the result shown in Lemma 4.9 and in Remark 4.3 about the modulus of regular graphs. We also note that the modulus values are equal to the modulus values of the paw graph, even though these are very different graphs. Since the extremal densities are constant the edge cover modulus can be approximated using Theorem 4.7, which implies that the product of the modulus is bounded below by $\left(\frac{k}{k+1}\right)^2$ where k is the length of the shortest odd cycle. For this graph, note that $k = 5$, then the lower bound improves slightly to be $\left(\frac{5}{6}\right)^2 \approx 0.694$. From our calculations we see that product of the moduli is equal to 1, showing again that the upper bound is tight.



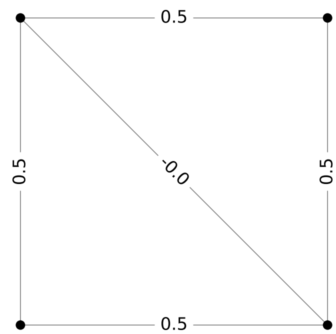
(a) $\text{Mod}_2(\Gamma_{\text{ec}}) = 0.60$ (b) $\text{Mod}_2(\Gamma_{\text{fec}}) = 0.60$ (c) $\text{Mod}_2(\Gamma_{\text{star}}) = 1.667$

Figure 5.11: Modulus of the paw graph. Values of the edges correspond to the extremal density ρ^* for each family. The product of the moduli is $\text{Mod}_2(\Gamma_{\text{ec}})\,\text{Mod}_2(\Gamma_{\text{fec}})^{-1} = 1.0$.



(a) $\text{Mod}_2(\Gamma_{\text{ec}}) = 0.60$ (b) $\text{Mod}_2(\Gamma_{\text{fec}}) = 0.60$ (c) $\text{Mod}_2(\Gamma_{\text{star}}) = 1.667$

Figure 5.12: Modulus of the petersen graph. Values of the edges correspond to the extremal density ρ^* for each family. The product of the moduli is $\text{Mod}_2(\Gamma_{\text{ec}})\,\text{Mod}_2(\Gamma_{\text{fec}})^{-1} = 1.0$.



(a) $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}}) = \mathrm{Mod}_2(\Gamma_{\mathrm{fec}}) = \mathrm{Mod}_2(\Gamma_{\mathrm{star}}) = 1$

Figure 5.13: Modulus of the diamond graph. Values on the edges correspond to the extremal density ρ^* for each family.

Diamond graph. The diamond graph shown in Figure 5.13 is of particular interest because it is the one example shown in this dissertation where $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}}) = \mathrm{Mod}_2(\Gamma_{\mathrm{fec}}) = \mathrm{Mod}_2(\Gamma_{\mathrm{star}}) = 1$. This also implies that $\rho^*_{\mathrm{ec}} = \rho^*_{\mathrm{fec}} = \rho^*_{\mathrm{star}}$. We also see the behavior of the extremal density assigning a value of 0. Again this means that the objects in a family do not make use of the diagonal edge.

5.3.6 Random Graphs

Erdős-Rényi graphs. Using the implementation details highighted at the beginning of this chapter, we calculated the modulus of edge covers and the modulus of stars for random graphs. Specifically, we used the NetworkX function `gnp_random_graph` to generate Erdős-Rényi (ER) graphs with $n \in \{49, 50, 99, 100, 149, 150\}$ and $p = 2\ln(n)/n$. For each n, there were 10 graphs randomly generated using fixed seeds. This choice of p has been shown to guarantee (as $n \to \infty$) that the graph is connected. We generate graphs for even and odd n and we compare the values of the product $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{star}}) = \mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})^{-1}$ for these graphs. As seen in Figure 5.14, there is a slight difference between even and odd values of n: for even n, the values seem to be 1 and for odd n, the values are still close to 1, but vary slightly. As n gets larger in the odd case, the value seems to be approaching 1.

From Theorem 4.2, we know that if a graph is bipartite, then $\Gamma_{\mathrm{ec}} \simeq \Gamma_{\mathrm{fec}}$ and the value

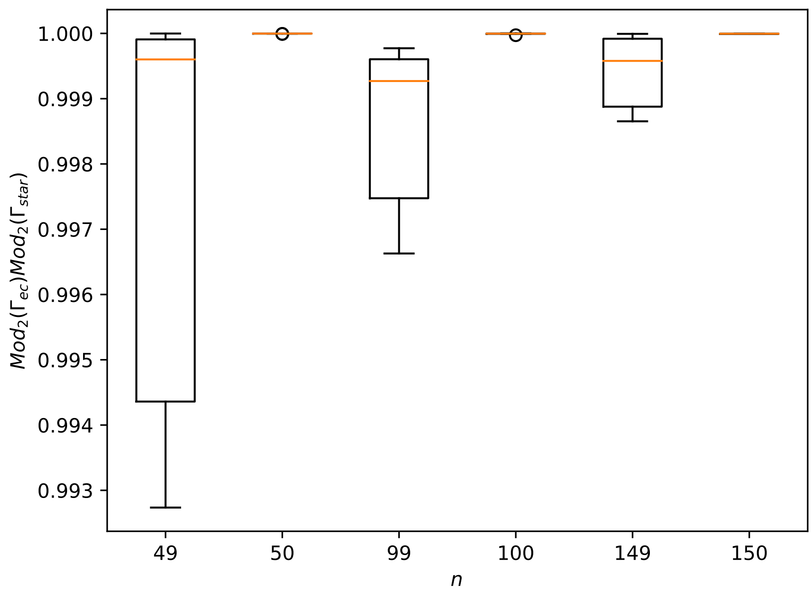


Figure 5.14: Boxplots of the values of $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{star}}) = \mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})^{-1}$ for Erdős-Rényi graphs where n is the number of nodes.

of the modulus of the 2 families are equal. For the case of the ER graphs generated, they are not bipartite and the shortest odd cycle for each value of n is a triangle. The fact that the value of $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})^{-1}$ goes to 1 as n gets larger seems to indicate that, even though the graphs are not bipartite, the value of the modulus of edge covers and fractional edge covers are equal to each other. This implies that fractional edge covers that contain odd cycles don't make a significant difference for $\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})$.

Barabasi-Albert graphs. The Barabasi-Albert model uses a preferential attachment model to sequentially add nodes and edges to a graph. Two parameters n and m, $m < n$, are specified for this model. At each step, a new node is added and m existing nodes are chosen (with a probability proportional to the number of edges that already exist in the graph) and connect them to the new node. This is repeated until the graph has n nodes. For this implementation, we use the NetworkX function `barabasi_albert_graph` to generate Barabasi-Albert random graphs for $n \in \{9, 10\}$ and 10 graphs are randomly generated for each $m \in \{1, 2, 3, 4, 5\}$ (this gives a total of 100 graphs generated).

When $m = 1$, note that the graph generated is in fact a tree. Therefore, this graph is bipartite and $\Gamma_{\text{ec}} \simeq \Gamma_{\text{fec}}$ and the product of their moduli is exactly 1. For $m \in \{2, 3, 4, 5\}$, the graphs generated are no longer bipartite, but we again see a distinct behavior between even and odd n, as well as differences in the values for different m. When n is even ($n = 10$), we see that the product of the moduli tends to 1 as m becomes larger. For the cases when $m = 2$ and $m = 3$, we see that there are some outliers, but again these values are very close to 1. When n is odd ($n = 9$), we see a different behavior:

1. for $m = 1$, the product of the moduli is exaclty 1.

2. for $m = 2$, the product of the moduli is close to 1, but there are two outliers where the product is 0.977 and 0.912.

3. for $m = 3$, the product of the moduli is again close to 1, but the values are a bit more spread out. There is one outlier where the product is 0.904.

4. for $m = 4$, the product of the moduli is close to 1, but there is one outlier where the poduct is 0.974.

5. for $m = 5$, the product of the moduli is close to 1.

Similarly to what we discussed for the ER graphs, even though the Barabasi-Albert graphs are not bipartite (for $m \neq 1$), the values of the modulus of edge covers and fractional edge covers seem to be equal or very close to each other. This again indicates that fractional edge covers that contain odd cycles with weights 1/2 are not that important.

Regular graphs. Recall that a graph with n nodes is called d-regular if $\deg(v) = d$ for every $v \in V$ and nd is even. Using the NetworkX function `random_regular_graph`, we generated random regular graphs by fixing a seed and calculating the 2-modulus of edge covers and the 2-modulus of stars as described above. There is a slight difference in the behavior of the moduli depending on the parity of n. Figure 5.16 shows the values of $\text{Mod}_2(\Gamma_{\text{ec}})$, $\text{Mod}_2(\Gamma_{\text{star}})$, and their product for different values of n and d. The upper and lower bounds shown in equation (5.2) are plotted in red. Specifically, Figures 5.16a and 5.16c

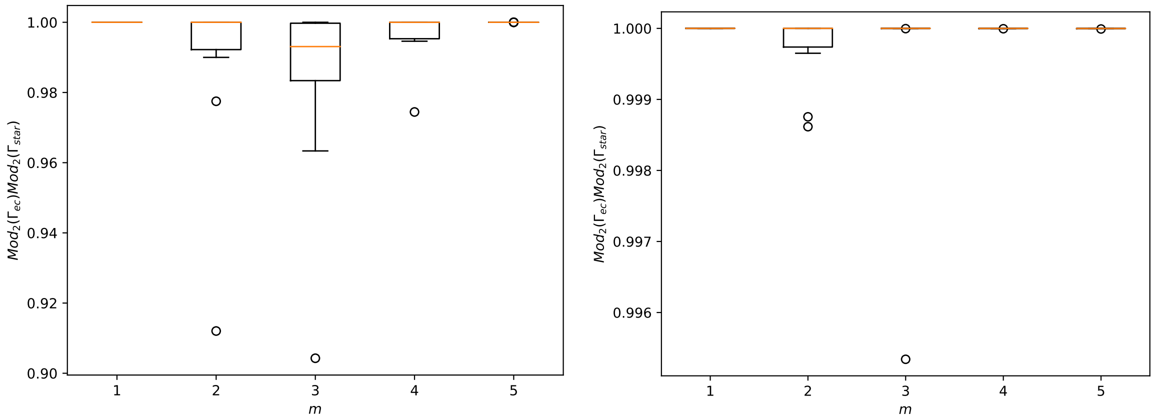


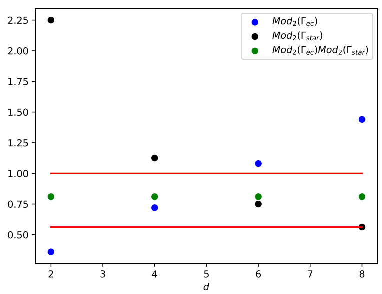
Figure 5.15: Boxplots of the values of $\mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{star}}) = \mathrm{Mod}_2(\Gamma_{\mathrm{ec}})\,\mathrm{Mod}_2(\Gamma_{\mathrm{fec}})^{-1}$ for the Barabasi-Albert graphs generated with $n = 9$ (left) and $n = 10$ (right) with $m \in \{1, 2, 3, 4, 5\}$.

show the values when n is odd and Figures 5.16b and 5.16d show the values when n is even. In the case when n is even, we see that the product of the moduli reaches the upper bound, whereas when n is odd, the product is distinctly < 1.

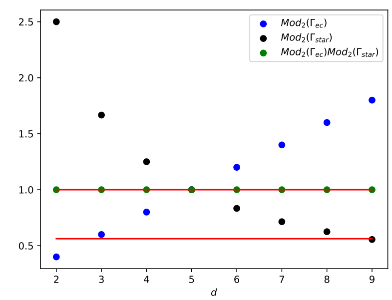
From Section 4.3, we know the exact value of the modulus of stars and the modulus of fractional edge covers. Specifically, from Remark 4.3, we have that $\mathrm{Mod}_2(\Gamma_{\mathrm{fec}}) = \frac{2d}{n}$, which implies that

$$\frac{9}{16} \leq \mathrm{Mod}_2(\Gamma_{\mathrm{ec}}) \cdot \frac{n}{2d} \leq 1.$$

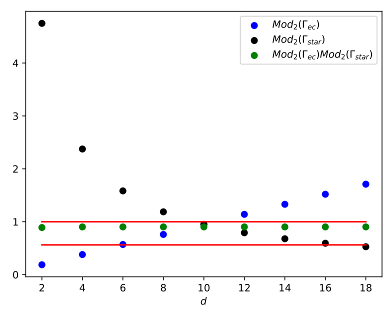
It is interesting to note that even though the product of the moduli depends on n and d, the values obtained numerically in Figure 5.16 show that the product does not depend on d, but it does depend on n.



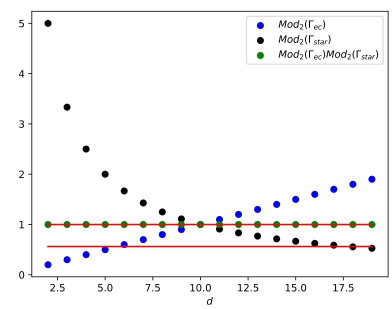
(a) $n = 9$



(b) $n = 10$



(c) $n = 19$



(d) $n = 20$

Figure 5.16: Modulus values for a d-regular graph with n nodes and $d \in \{2, \ldots, n-1\}$ with the condition that nd is even. The red lines correspond to the upper bound of 1 and lower bound of 9/16. When $n = 9$, the product of the moduli is 0.81 and when $n = 19$, the product is 0.9025. For $n = 10$ and $n = 20$, the product is 1.0.

Chapter 6

Conclusion

As mentioned in the introduction, one of the main motivations for studying the modulus of edge covers is to develop a deeper understanding of what properties of the underlying graph structure this modulus can expose. In this dissertation, we have laid the theoretical groundwork for the study of edge cover modulus. Moreover, by connecting it to the modulus of fractional edge covers and, ultimately, to that of stars, we have made it computationally feasible to approximate edge cover modulus on large graphs. A recurring theme in this dissertation is that the edge cover modulus appears to be more sensitive to certain structural properties of the graph, such as the parity of n, than the fractional edge cover modulus. We see this in symmetric graphs, like the cycle graph, complete graph, and n-barbell graphs, but we believe this idea could be generalized to more complicated graphs that are not necessarilly symmetric. Specifically, the modulus of edge covers might differ based on the structural properties of a certain subgraph of the graph, whereas the modulus of fractional edge covers would not. In addition to deepening the connection between edge cover modulus and graph structure, there are several other interesting research directions open to pursuit.

One path involves further developing the relationship between the moduli of edge covers and fractional edge covers. For bipartite graphs, these two families are equivalent (in the sense of modulus), which gives a starting point. Moreover, we see that in the case when the extremal density for the edge cover modulus is constant, the bounds established in

Theorem 4.7 depend on the length of the shortest odd cycle. This implies that in these cases of constant extremal densities, if a graph has very long odd cycles (a sort of "nearly-bipartite" property), then the fractional edge cover modulus is a good approximation for the edge cover modulus.

The examples shown in Section 5.3 highlight different properties of the edge cover modulus, as well as the fractional edge cover modulus and how the two quantities differ from each other. The results show that for some non-bipartite graphs, the 2-modulus of the two families are equal (or almost equal) to each other. We see this in Erdős-Rényi graphs, Barabasi-Albert graphs for larger values of m, as well as in smaller graphs like the paw graph, the petersen graph, and the diamond graph. Moreover, in randomly generated regular graphs we see that when the number of nodes is even, the 2-modulus of edge covers and fractional edge covers are equal to each other, but when the number of nodes is odd, the 2-modulus differ.

Another question that arises naturally is that of the dual family to Γ_{ec}. As shown in this paper, $\hat{\Gamma}_{\text{fec}} \simeq \Gamma_{\text{star}}$, which leads to a computationally efficient method for computing the modulus of Γ_{fec}. Even though there is an algorithm we can use to numerically approximate the edge cover modulus, computing it tends to be slower because of the need to repeatedly construct minimum edge covers as part of the basic algorithm. The edge covers may have a simpler dual family that could similarly aid in efficiently computing edge cover modulus.

Finally, we hope that a better understanding of the properties of the edge cover modulus will lead to similar insights into the modulus of related (and more complex) families, particularly the families of maximal matchings and perfect matchings. Both of these families have related relaxed (fractional) families that may be useful in their study just as the fractional edge covers can be used to understand edge covers.

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Appendix A

The Number of Minimum Edge Covers of K_n

In Chapter 3 we discussed edge covers and minimal edge covers. The following lemma counts the number of minimum edge covers in the complete graph K_n, which is a lower bound for the number of minimal edge covers of the complete graph.

Lemma A.1. *Let $G = K_n$ be the complete graph on n nodes. Then,*

- *if n is even, the number of minimum edge covers is $(n-1)!!$, and*
- *if n is odd, the number of minimum edge covers is $\frac{n!!(n-1)}{2}$.*

Proof. If n is even, note that a minimum edge cover will have $\frac{n}{2}$ edges. To count the number of minimum edge covers, we can use the formula for partitioning a set of n elements into r unordered partitions of size $n_1, n_2, \ldots, n_r$ given by : $\frac{n!}{n_1!n_2!\ldots n_r!r!}$. In this case, $n_1 = n_2 = \ldots = n_r = 2$ and $r = \frac{n}{2}$, so the number of minimum edge covers is

$$\frac{n!}{(2!)^{\frac{n}{2}}\frac{n}{2}!} = \frac{n!}{(2 \cdot \frac{n}{2})!!} = (n-1)!!.$$

In the case when n is odd, a minimum edge cover will have $\frac{n+1}{2}$ edges. Specifically, 3 nodes will be covered by 2 edges and the remaining $n-3$ nodes (an even number of nodes) are covered with disjoint edges. To count the number of minimum edge covers, consider choosing the 3 nodes and covering them with 2 edges. This can be done in $\binom{n}{3} \cdot 3 = \frac{n!}{2(n-3)!}$ ways. From the previous case, the remaining $n-3$ nodes will have a total of $(n-4)!!$ minimum edge covers. The total becomes

$$\frac{n!}{2(n-3)!} \cdot (n-4)!! = \frac{n!!(n-1)}{2}.$$

□

Appendix B

Verifying μ^* for Certain Path Graphs

In Chapter 4 we went over multiple examples to show the values of the modulus of the three families of interest. In Section 4.3.2, Example 4.13 goes over the modulus of stars for the path graph. We include the details omitted for case (d) when $n \geq 6$ and even. Specifically, we show that $\mu \in \mathcal{P}(\Gamma_{\text{star}})$ and establish the lower bound for $\text{Mod}_2(\Gamma_{\text{star}})$. Recall that we define μ as

$$\mu(\delta(v_i)) = \begin{cases} \frac{2(2n-6)}{n^2+2n-16}, & i = 1 \\ 0, & i = 2 \\ \frac{2n-2(i+1)}{n^2+2n-16}, & 2 < i \leq k, i \text{ odd} \\ \frac{2(i-2)}{n^2+2n-16}, & 2 < i \leq k, i \text{ even.} \end{cases}$$

Once the first k stars have been assigned a μ value, assign the values of μ to the following k vertices by mirroring the values: $\mu(\delta(v_{n+1-i})) = \mu(\delta(v_i))$ for $i = 1, 2, \ldots, k$. (See Figure 4.6.) To check that $\mu \in \mathcal{P}(\Gamma_{\text{star}})$ we consider 2 cases of n:

- If $n = 4j$, with $j \geq 2$

$$\begin{aligned} \sum_{v \in V} \mu(\delta(v)) &= 2 \cdot \frac{2(2n-6)}{n^2+2n-16} + 2\sum_{r=1}^{\frac{n-4}{4}} \frac{2n-2(2r+2)}{n^2+2n-16} + 2\sum_{l=1}^{\frac{n-4}{4}} \frac{2(2(l+1)-2)}{n^2+2n-16} \\ &= \frac{8n-24}{n^2+2n-16} + 2\sum_{r=1}^{\frac{n-4}{4}} \frac{2n-4r-4}{n^2+2n-16} + 2\sum_{l=1}^{\frac{n-4}{4}} \frac{4l}{n^2+2n-16} \end{aligned}$$

$$= \frac{8n-24}{n^2+2n-16} + \frac{n-4}{2} \cdot \frac{2n-4}{n^2+2n-16} - 2\sum_{r=1}^{\frac{n-4}{4}} \frac{4r}{n^2+2n-16} + 2\sum_{l=1}^{\frac{n-4}{4}} \frac{4l}{n^2+2n-16}$$
$$= \frac{8n-24+(n-4)(n-2)}{n^2+2n-16}$$
$$= \frac{8n-24+n^2-6n+8}{n^2+2n-16} = 1.$$

- If $n = 4j+2$, with $j \geq 1$ we have that

$$\sum_{v\in V} \mu(\delta(v)) = 2 \cdot \frac{2(2n-6)}{n^2+2n-16} + 2\sum_{j=1}^{\frac{n-2}{4}} \frac{2n-2(2j+2)}{n^2+2n-16} + 2\sum_{l=1}^{\frac{n-6}{4}} \frac{2(2(l+1)-2)}{n^2+2n-16}$$
$$= \frac{8n-24}{n^2+2n-16} + \frac{n-2}{2} \cdot \frac{2n-4}{n^2+2n-16} - 2\sum_{j=1}^{\frac{n-2}{4}} \frac{4r}{n^2+2n-16} + 2\sum_{l=1}^{\frac{n-6}{4}} \frac{4l}{n^2+2n-16}$$
$$= \frac{8n-24+(n-2)^2}{n^2+2n-16} - \frac{8(\frac{n-2}{4})}{n^2+2n-16}$$
$$= \frac{n^2+4n-20}{n^2+2n-16} - \frac{2n-4}{n^2+2n-16} = 1.$$

We now label the edges as $e_1, e_2, \ldots, e_{n-1}$ where e_1 and e_{n-1} are the pendant edges, and we find that the expected edge usage is given by:

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e_s)] = \begin{cases} \frac{2(2n-6)}{n^2+2n-16}, & s = 1 \text{ or } n-1 \\ \frac{2n-8}{n^2+2n-16}, & s \text{ even} \\ \frac{2n-4}{n^2+2n-16}, & \text{else.} \end{cases}.$$

Using (2.6), we have

$$\text{Mod}_2(\Gamma_{\text{star}})^{-1} \leq 2\left(\frac{2(2n-6)}{n^2+2n-16}\right)^2 + \frac{n-4}{2}\left(\frac{2n-4}{n^2+2n-16}\right)^2 + \frac{n-2}{2}\left(\frac{2n-8}{n^2+2n-16}\right)^2$$
$$= \frac{8(2n-6)^2+2(n-4)(n-2)^2+2(n-2)(n-4)^2}{(n^2+2n-16)^2}$$
$$= \frac{4n^3-4n^2-88n+192}{(n^2+2n-16)^2}$$

$$= \frac{(4n-12)(n^2+2n-16)}{(n^2+2n-16)^2}$$
$$= \frac{4n-12}{n^2+2n-16}.$$

From this, we obtain $\text{Mod}_2(\Gamma_{\text{star}}) \geq \frac{n^2+2n-16}{4n-12}$, which coincides with the upper bound found in Example 4.13 (d).