OBSTACLE PROBLEMS WITH ELLIPTIC OPERATORS IN DIVERGENCE FORM

by

ZHENG HAO

M.S., Kansas State University, 2012

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2014

Abstract

Under the guidance of Dr. Ivan Blank, I study the obstacle problem with an elliptic operator in divergence form. First, I give all of the nontrivial details needed to prove a mean value theorem, which was stated by Caffarelli in the Fermi lectures in 1998. In fact, in 1963, Littman, Stampacchia, and Weinberger proved a mean value theorem for elliptic operators in divergence form with bounded measurable coefficients. The formula stated by Caffarelli is much simpler, but he did not include the proof. Second, I study the obstacle problem with an elliptic operator in divergence form. I develop all of the basic theory of existence, uniqueness, optimal regularity, and nondegeneracy of the solutions. These results allow us to begin the study of the regularity of the free boundary in the case where the coefficients are in the space of vanishing mean oscillation (VMO).

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Chapter 1

Introduction

1.1 Notations and Assumptions

We will use the following basic notation throughout our work:

χ_D	the characteristic function of the set D
$\overline{D}$	the closure of the set D
∂D	the boundary of the set D
x	$(x_1, x_2, \dots, x_n)$
x'	$(x_1, x_2, \dots, x_{n-1}, 0)$
$B_r(x)$	the open ball with radius r centered at the point x
B_r	$B_r(0)$
$\Omega(w)$	$\{w > 0\}$
$\Lambda(w)$	$\{w = 0\}$
$FB(w)$	$\partial\Omega(w) \cap \partial\Lambda(w)$

We define the divergence form elliptic operator: (We will use Einstein summation notation throughout.)

$$L := D_j\, a^{ij}(x) D_i\ , \tag{1.1}$$

or, in other words, for a function $u \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$ we say "$Lu = f$ in Ω" if for any $\phi \in W^{1,2}_0(\Omega)$ we have:

$$-\int_\Omega a^{ij}(x) D_i u D_j \phi = \int_\Omega f\phi\ . \tag{1.2}$$

(Notice that with our sign conventions we can have $L = \Delta$ but not $L = -\Delta$.). We assume that at each $x \in B_1$, the matrix $\mathcal{A} = (a^{ij})$ is symmetric and strictly and uniformly elliptic, i.e.

$$\mathcal{A} \equiv \mathcal{A}^T \ \text{ and } \ 0 < \lambda I \leq \mathcal{A} \leq \Lambda I\ , \tag{1.3}$$

or, in coordinates:

$$a^{ij} \equiv a^{ji} \ \text{ and } \ 0 < \lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \ \text{ for all } \xi \in \mathbb{R}^n,\ \xi \neq 0\ .$$

Throughout the entire paper, n, λ, and Λ will remain fixed, and so we will omit all dependence on these constants in the statements of our theorems. We will typically work in the Sobolev spaces and the Hölder spaces, and we will follow all of the definitions and conventions found in the book by Gilbarg and Trudinger. (See[GT].) To simplify exposition slightly, for $u, v \in W^{1,2}(D)$ we will say that $u = v$ on ∂D if $u - v \in W^{1,2}_0(D)$.

1.2 Outline

Based on the ubiquitous nature of the mean value theorem in problems involving the Laplacian, it is clear that an analogous formula for a general divergence form elliptic operator would necessarily be very useful. In[LSW], Littman, Stampacchia, and Weinberger stated a mean value theorem for a general divergence form operator, L. If μ is a nonnegative measure on Ω and u is the solution to:

$$\begin{aligned} Lu = \mu \quad & \text{in} \;\; \Omega \\ 0 \quad & \text{on} \;\; \partial\Omega \,, \end{aligned} \tag{1.4}$$

and $G(x, y)$ is the Green's function for L on Ω then Equation 8.3 in their paper states that $u(y)$ is equal to

$$\lim_{a \to \infty} \frac{1}{2a} \int_{a \le G \le 3a} u(x) a^{ij}(x) D_{x_i} G(x, y) D_{x_j} G(x, y) \; dx \tag{1.5}$$

almost everywhere, and this limit is nondecreasing. The pointwise definition of u given by this equation is necessarily lower semi-continuous. There are a few reasons why this formula is not as nice as the basic mean value formulas for Laplace's equation. First, it is a weighted average and not a simple average. Second, it is not an average over a ball or something which is even obviously homeomorphic to a ball. Third, it requires knowledge of derivatives of the Green's function.

A simpler formula was stated by Caffarelli in[C5] and[CR]. That formula provides an increasing family of sets, $D_R(x_0)$, which are each comparable to B_R and such that for a supersolution to $Lu = 0$ the average:

$$\frac{1}{|D_R(x_0)|} \int_{D_R(x_0)} u(x) \; dx$$

is nondecreasing as $R \to 0$. On the other hand, Caffarelli did not provide any details about showing the existence of an important test function used in the proof of this result, and showing the existence of this function turns out to be nontrivial. The first part of my dissertation grew out of an effort to prove rigorously all of the details of the mean value theorem that Caffarelli asserted in[C5] and[CR].

In order to get the existence of the key test function, one must be able to solve the variational inequality or obstacle type problem:

$$D_i a^{ij} D_j V_R = \frac{1}{R^n} \chi_{\{V_R > 0\}} - \delta_{x_0} \tag{1.6}$$

where δ_{x_0} denotes the Dirac mass at x_0. In[CR], the book by Kinderlehrer and Stampacchia is cited (see[KS]) for the mean value theorem. Although many of the techniques in that book are used in the current work, an exact theorem to give the existence of a solution to Equation (1.6) was not found in[KS] by myself, my advisor, or by Kinderlehrer. We were also unable to find a suitable theorem in other standard sources for the obstacle problem. (See[F] and[R].) Indeed, we believe that without the nondegeneracy theorem stated in this paper there is a gap in the proof.

To understand the difficulty inherent in proving a nondegeneracy theorem in the divergence form case it helps to review the proof of nondegeneracy for the Laplacian and/or in the nondivergence form case. (See[B],[BT], and[C5].) In those cases good use is made of the barrier function $|x - x_0|^2$. The relevant properties are that this function is nonnegative and vanishing at x_0, it grows quadratically, and most of all, for a nondivergence form elliptic operator L, there exists a constant $\gamma > 0$ such that $L(|x - x_0|^2) \geq \gamma$. On the other hand, when L is a divergence form operator with only bounded measurable coefficients, it is clear that $L(|x - x_0|^2)$ does not make sense in general.

In section 2.1, we almost get the existence of a solution to a PDE formulation of the obstacle problem. In section 2.2 we first show the basic quadratic regularity and nondegeneracy result for our functions which are only "almost" solutions, and then we use these results to show that our "almost" solutions are true solutions. In section 2.3 we get existence and uniqueness of solutions of a variational formulation of the obstacle problem, and then show that the two formulations are equivalent. In section 2.4 we show the existence of a function which we then use in the sixth section to prove the mean value theorem stated in[C5] and[CR], and give some corollaries.

The results in this chapter 2 are used in chapter 3 where we establish some weak regularity results for the free boundary in the case where the coefficients are assumed to belong to the space of vanishing mean oscillation. We will discuss the space VMO in chapter 3. The methods rely on stability, flatness, and compactness arguments. In the case where the coefficients are assumed to be Lipschitz continuous, recent work of Focardi, Gelli, and Spadaro establishes stronger regularity results of the free boundary. The methods of that work have a more "energetic" flavor: They generalize some important monotonicity formulas, and use these formulas along with the epiperimetric inequality due to Weiss and a generalization of Rellich and Nĕcas' identity to prove their regularity results. (See[FGS].)

In Chapter 3 we study minimizers of

$$\int_{B_1} a^{ij} D_i u D_j u \tag{1.7}$$

among u in the Hilbert space $W_0^{1,2}(B_1)$ which are constrained to lie above a fixed obstacle $\varphi \in C^0(\overline{B}_1)$. We assume that our obstacle $\varphi < 0$ on ∂B_1, and to avoid triviality we will assume that $\max \varphi > 0$.

If we let $Lv := D_i a^{ij} D_j v$ in the usual weak sense for a divergence form operator and we consider the case where $L\varphi \in L^\infty(B_1)$, then by letting $w := u - \varphi$ and by letting $f := -L\varphi$, the study of the minimizers above leads us to look at weak solutions of the obstacle-type problem:

$$Lw := D_i a^{ij} D_j w = \chi_{\{w>0\}} f \quad \text{in} \quad B_1 \ , \tag{1.8}$$

where χ_S denotes the characteristic function of the set S, and where we look for $w \geq 0$.

Our motivations for studying this type of problem are primarily theoretical. Indeed, the obstacle problem is possibly the most fundamental and important free boundary problem, and it originally motivated the study of variational inequalities. On the other hand, the obstacle problem has well-established connections to the Stefan problem and the Hele-Shaw problem. (See[C1] and[BKM] for example.) Furthermore, as observed in[MPS] the mathematical modeling of numerous physical and engineering phenomena can lead to elliptic problems with discontinuous coefficients, and so the current case seems to allow some of the weakest possible solutions.

Our main result is the following:

1.2.1 Theorem (Free Boundary Regularity)**.** *We assume*

1. $w \geq 0$ *satisfies Equation (1.8),*
2. a^{ij} *satisfies Equation (1.3),*
3. $0 < \lambda^* \leq f \leq \Lambda^*$, *and*
4. a^{ij} *and* f *belong to the space of vanishing mean oscillation (VMO).*

We let S_r *denote the set of regular points of the free boundary within* B_r, *and assume* $K \subset\subset S_{1/2}$. *Then* K *is a relatively Reifenberg vanishing set.*

The definition of Reifenberg vanishing is found at the beginning of section 3.4.

As a corollary of this result we will conclude that blowup limits at regular points will be rotations and scalings of the function $(x_n^+)^2$. In terms of the fact that this function is homogeneous of degree 2, it is quite usual to use Weiss's celebrated monotonicity formula to prove this type of result. (See[W].) On the other hand, the weak nature of our equation, together with the weak $W^{1,2}$ convergence to blowup solutions make it difficult to estimate differences of the values of the Dirichlet integrals which appear in Weiss's formula. So, instead of using homogeneity to prove Reifenberg flatness, we will have to prove things in the opposite direction.

In section 3.1, we introduce and recall some basic results. In section 3.2, we show a Measure Stability. In section 3.3, we prove the exstence of blowup limits, and establish the Caffaralli's Alternative. Then in section 3.4, we provide the Reigenberg Flatness of the free boundary of our problem and some corollaries.

Chapter 2

The Mean Value Theorem

2.1 The PDE Obstacle Problem with a Gap

We wish to establish the existence of weak solutions to an obstacle type problem, that is, we want to find a nonnegative function $w \in W^{1,2}(B_1)$ which is a weak solution of:

$$\begin{aligned} Lw = \chi_{\{w>0\}} f \quad &\text{in} \quad B_1 \\ w = g \quad &\text{on} \quad \partial B_1 \,. \end{aligned} \tag{2.1}$$

In addition to the assumptions in Chapter 1, we assume that

$$f, a^{ij} \in L^{\infty}(B_1) \quad \text{and} \quad g \in W^{1,2}(B_1) \,\cap\, L^{\infty}(B_1), \tag{2.2}$$

which satisfy:

$$\begin{aligned} &0 < \bar{\lambda} \leq f \leq \bar{\Lambda} \,, \\ &g \not\equiv 0 \text{ on } \partial B_1, \; g \geq 0. \end{aligned} \tag{2.3}$$

In this section we will content ourselves to produce a nonnegative function $w \in W^{1,2}(B_1)$ which is a weak solution of:

$$\begin{aligned} Lw = h \quad &\text{in} \ \ B_1 \\ w = g \quad &\text{on} \ \ \partial B_1 \ , \end{aligned} \tag{2.4}$$

where we know that h is a nonnegative function satisfying:

$$\begin{aligned} h(x) \ &= 0 \qquad &&\text{for } x \in \{w = 0\}^{\text{o}} \\ h(x) \ &= f(x) \qquad &&\text{for } x \in \{w > 0\}^{\text{o}} \\ h(x) \ &\leq \bar{\Lambda} \qquad &&\text{for } x \in \partial\{w = 0\} \cup \partial\{w > 0\} \ , \end{aligned} \tag{2.5}$$

where for any set $S \subset \mathbb{R}^n$, we use S^{o} to denote its interior. Thus h agrees with $\chi_{\{w>0\}} f$ everywhere except possibly the free boundary. (The "gap" mentioned in the title to this section is the fact that we won't know that $h = \chi_{\{w>0\}} f$ *a.e.* until we show that the free boundary (that is $\partial\{w = 0\} \cup \partial\{w > 0\}$) has measure zero.) We will show that such a w exists by obtaining it as a limit of functions w_s which are solutions to the semilinear PDE:

$$\begin{aligned} Lw = \Phi_s(w) f \quad &\text{in} \ \ B_1 \\ w = g \quad &\text{on} \ \ \partial B_1 \ , \end{aligned} \tag{2.6}$$

where for $s > 0$, $\Phi_s(x) := \Phi_1(x/s)$ and $\Phi_1(x)$ is a function which satisfies

1. $\Phi_1 \in C^\infty(\mathbb{R})$,

2. $0 \leq \Phi_1 \leq 1$,

3. $\Phi_1 \equiv 0$ for $x < 0$, $\quad \Phi_1 \equiv 1$ for $x > 1$, and

4. $\Phi_1'(x) \geq 0$ for all x.

The function Φ_s has a derivative which is supported in the interval $[0, s]$ and notice that for a fixed x, $\Phi_s(x)$ is a nonincreasing function of s.

If we let H denote the standard Heaviside function, but make the convention that $H(0) := 0$ then we can rewrite the PDE in Equation (2.1) as

$$Lw = H(w)f$$

to see that it is formally the limit of the PDEs in Equation (2.6). We also define

$$\Phi_{-s}(x) := \Phi_s(x + s)$$

so that we will be able to "surround" our solutions to our obstacle problem with solutions to our semilinear PDEs.

The following theorem seems like it should be stated somewhere, but without further smoothness assumptions on the a^{ij} we could not find it within[GT],[HL], or[LU]. The proof is a fairly standard application of the method of continuity, so we will only sketch it.

2.1.1 Theorem (Existence of Solutions to a Semilinear PDE)**.** *Given the assumptions above, for any $s \in [-1, 1] \setminus \{0\}$ there exists a w_s that satisfies Equation (2.6).*

Proof. We provide only a sketch. Fix $s \in [-1, 1] \setminus \{0\}$. Let T be the set of $t \in [0, 1]$ such that there is a unique solution to the problem

$$\begin{aligned} Lw = t\Phi_s(w)f \quad &\text{in} \;\; B_1 \\ w = g \quad &\text{on} \;\; \partial B_1 \; . \end{aligned} \tag{2.7}$$

We know immediately that T is nonempty by observing that Theorem 8.3 of[GT] shows us that $0 \in T$. Now we need to show that T is both open and closed.

As in[LSW] we let $\tau^{1,2}$ denote the Hilbert space formed as the quotient space $W^{1,2}(B_1)/W_0^{1,2}(B_1)$ and then we define the Hilbert space

$$H := W_0^{1,2}(B_1)^* \oplus \tau^{1,2} \; , \tag{2.8}$$

where $W_0^{1,2}(B_1)^*$ denotes the dual space to $W_0^{1,2}(B_1)$. Next we define the nonlinear operator $L^t : W^{1,2}(B_1) \to H$. For a function $w \in W^{1,2}(B_1)$, we set

$$L^t(w) = \ell^t(w) \oplus \mathcal{R}(w) , \tag{2.9}$$

where $\mathcal{R}(w)$ is simply the restriction from w to its boundary values in $\tau^{1,2}$, and for any $\phi \in W_0^{1,2}(B_1)$ we let

$$[\ell^t(w)](\phi) := \int_{B_1} \left(a^{ij}(x) D_i w D_j \phi + t\Phi_s(w) f \phi \right) \, dx \ . \tag{2.10}$$

In order to show that T is open we need the implicit function theorem in Hilbert space. In order to use that theorem we need to show that the Gateaux derivative of L^t is invertible. The relevent part of that computation is simply the observation that the Gateaux derivative of ℓ^t, which we denote by $D\ell^t$, is invertible. Letting $v \in W^{1,2}(B_1)$ we have

$$\Big[[D\ell^t(w)](\phi) \Big] (v) = \int_{B_1} \left(a^{ij}(x) D_i v D_j \phi + t\Phi_s'(w) f v \phi \right) \, dx \ . \tag{2.11}$$

The function $d(x) := t\Phi_s'(w(x)) f(x)$ is a nonnegative bounded function of x and so we can apply Theorem 8.3 of[GT] again in order to verify that $D\ell^t$ is invertible. Then by the discussion above, this fact leads to the openness of T.

In order to show that T is closed we let $t_n \to \tilde{t}$, and assume that $\{t_n\} \subset T$. We let w_n solve

$$\begin{aligned} Lw = t_n \Phi_s(w) f \quad &\text{in} \ \ B_1 \\ w = g \quad &\text{on} \ \ \partial B_1 \ , \end{aligned} \tag{2.12}$$

and observe that the right hand side of our PDE is bounded by $\bar{\Lambda}$. Knowing this information we can use Corollary 8.7 of[GT] to conclude $||w_n||_{W^{1,2}(B_1)} \leq C$, and we can use the theorems of De Giorgi, Nash, and Moser to conclude that for any $r < 1$ we have $||w_n||_{C^{\alpha}(\overline{B_r})} \leq C$. Elementary functional analysis allows us to conclude that a subsequence of our w_n will converge weakly in $W^{1,2}(B_r)$ and strongly in $C^{\alpha/2}(\overline{B}_r)$ to a function $\tilde{w}$. Using a simple diagonalization argument we can show that $\tilde{w}$ satisfies

$$\begin{aligned} Lw = \tilde{t}\Phi_s(w)f \quad &\text{in} \ \ B_1 \\ w = g \quad &\text{on} \ \ \partial B_1 \ , \end{aligned} \tag{2.13}$$

and this fact show us that $\tilde{t} \in T$. ■

We will also need the following comparison results:

2.1.2 Proposition (Basic Comparisons)**.** *Under the assumptions of the previous theorem and letting w_s denote the solution to Equation (2.6), we have the following comparison results:*

1. $s > 0 \ \Rightarrow w_s \geq 0$,
2. $s < 0 \ \Rightarrow w_s \geq s$,
3. $t < s \ \Rightarrow w_t \geq w_s$,
4. $t < 0 < s \ \Rightarrow w_s \leq w_t + s - t$, *and*
5. *For a fixed $s \in [-1, 1] \setminus \{0\}$ the solution, w_s is unique.*

Proof. All five statements are proved in very similar ways, and their proofs are fairly standard, but for the convenience of the reader, we will prove the fourth statement. We assume that it is false, and we let

$$\Omega^- := \{w_s - w_t > s - t\} \ . \tag{2.14}$$

Obviously $w_s - w_t = s - t$ on $\partial\Omega^-$. Next, observe that by the second statement we know that Ω^- is a subset of $\{w_s > s\}$. Thus, within Ω^- we have $L(w_s - w_t) = 1 - \Phi_t(w_t) \geq 0$ and so if Ω^- is not empty, then we contradict the weak maximum principle. ∎

We are now ready to give our existence theorem for our "problem with the gap."

2.1.3 Theorem (Existence Theorem). *Given the assumptions above, there exists a pair (w, h) such that $w \geq 0$ satisfies Equation (2.4) with an $h \geq 0$ which satisfies Equation (2.5) .*

Proof. Using the last proposition, we can find a sequence $s_n \to 0$, and a function w such that (with w_n used as an abbreviation for w_{s_n}) we have strong convergence of the w_n to w in $C^\alpha(\overline{B}_r)$ for any $r < 1$ and weak convergence of the w_n to w in $W^{1,2}(B_1)$. Elementary functional analysis allows us to conclude that the functions $\chi_{\{w_n>0\}} f$ converge weak-$*$ in $L^\infty(B_1)$ to a function h which automatically satisfies $0 \leq h \leq \bar{\Lambda}$. By looking at the equations satisfied by the w_n's and using the convergences, it then follows very easily that the function w satisfies Equation (2.4), but it remains to verify that the function h is equal to $\chi_{\{w>0\}} f$ away from the free boundary.

Since the limit is continuous, the set $\{w > 0\}$ is already open, and by the uniform convergence of the w_n's we can say that on any set of the form $\{w > \gamma\}$ (where $\gamma > 0$) we will have $\Phi_{s_n}(w_n) \equiv 1$ once n is sufficiently large. Thus we must have $h = f$ on this set. On the other hand, in the interior of the set $\{w = 0\}$ we have $\nabla w \equiv 0$, and so it is clear that in that set $h \equiv 0$ *a.e.* ∎

2.2 Regularity, Nondegeneracy, and Closing the Gap

Now we begin with a pair (w, h) like the pair given by Theorem (2.1.3), except that we do not insist that it have any particular boundary data on ∂B_1. In other words, in this section w will always satisfy

$$L(w) = h \quad \text{in } B_1, \tag{2.15}$$

for a function h which satisfies Equation (2.5). In addition we will assume Equations (2.2) and (2.3) hold. By the end of this section we will know that the set $\partial\{w = 0\}$ has Lebesgue measure zero and so w actually satisfies:

$$L(w) = \chi_{\{w>0\}} f \quad \text{in } B_1, \tag{2.16}$$

which will allow us to forget about h afterward. Before we eliminate h, we have two main results: First, w enjoys a parabolic bound from above at any free boundary point, and second, w has a quadratic nondegenerate growth from such points. It turns out that these properties are already enough to ensure that the free boundary has measure zero.

2.2.1 Lemma. *Assume that w satisfies everything described above, but in addition, assume that $w(0) = 0$. Then there exists a $\tilde{C}$ such that*

$$\| w \|_{L^\infty(B_{1/2})} \leq \tilde{C}. \tag{2.17}$$

Proof. Let u solve the following PDE:

$$\begin{cases} Lu = h & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 . \end{cases} \tag{2.18}$$

Then Theorem 8.16 of[GT] gives

$$\| u \|_{L^\infty(B_1)} \leq C_1. \tag{2.19}$$

Now, consider the solution to:

$$\begin{cases} Lv = 0 & \text{in } \ B_1 \\ \\ v = w & \text{on } \ \partial B_1 \ . \end{cases} \tag{2.20}$$

Notice that $u(x) + v(x) = w(x)$, and in particular $0 = w(0) = u(0) + v(0)$. Then by the Weak Maximum Principle and the Harnack Inequality, we have

$$\sup_{B_{1/2}} |v| = \sup_{B_{1/2}} v \leq C_2 \inf_{B_{1/2}} v \leq C_2 v(0) \leq C_2(-u(0)) \leq C_2 \cdot C_1. \tag{2.21}$$

Therefore

$$\| \ w \ \|_{L^\infty(B_{1/2})} \leq C \tag{2.22}$$

■

2.2.2 Theorem (Optimal Regularity). *If* $0 \in \partial\{w > 0\}$, *then for any* $x \in B_{1/2}$ *we have*

$$w(x) \leq 4\tilde{C}|x|^2 \tag{2.23}$$

where $\tilde{C}$ *is the same constant as in the statement of Lemma (2.2.1).*

Proof. By the previous lemma, we know $\| \ w \ \|_{L^\infty(B_{1/2})} \leq \tilde{C}$. Notice that for any $\gamma > 1$,

$$u_\gamma(x) := \gamma^2 w\left(\frac{x}{\gamma}\right) \tag{2.24}$$

is also a solution to the same type of problem on B_1, but with a new operator $\tilde{L}$, and with a new function $\tilde{f}$ multiplying the characteristic function on the right hand side. On the other hand, the new operator has the same ellipticity as the old operator, and the new function

$\tilde{f}$ has the same bounds that f had. Suppose there exist some point $x_1 \in B_{1/2}$ such that

$$w(x_1) > 4\tilde{C}|x_1|^2. \tag{2.25}$$

Then since $\frac{1}{2|x_1|} > 1$ and since $\frac{x_1}{2|x_1|} \in \partial B_{\frac{1}{2}}$, we have

$$u_{\left(\frac{1}{2|x_1|}\right)}\left(\frac{x_1}{2|x_1|}\right) = \frac{1}{4|x_1|^2} w(x_1) > \tilde{C}\ , \tag{2.26}$$

which contradicts Lemma (2.2.1). ∎

Now we turn to the nondegeneracy statement. The first thing we need is a variant of the following result from[LSW]:

2.2.3 Lemma (Corollary 7.1 of[LSW]). *Suppose μ is a nonnegative measure supported in C which we assume is a compact subset of B_1. Suppose L and $\tilde{L}$ are divergence form elliptic operators exactly of the type considered in this work, and assume that their constants of ellipticity are all contained in the interval of positive numbers: $[\bar{\lambda}, \bar{\Lambda}]$. If*

$$\begin{aligned} Lu = \tilde{L}\tilde{u} = \mu \quad &\text{in} \ \ B_1 \\ u = \tilde{u} = 0 \quad &\text{on} \ \ \partial B_1\ , \end{aligned} \tag{2.27}$$

then there exists a constant $K = K(n, C, \bar{\lambda}, \bar{\Lambda})$ such that for all $x \in C$ we have

$$K^{-1}u(x) \leq \tilde{u}(x) \leq Ku(x)\ .$$

We need to do away with the restriction that μ be supported on a compact subset of B_1, but we can restrict our attention to much simpler nonnegative measures. In fact, the following lemma is good enough for our purposes:

2.2.4 Lemma. *Assume L and $\tilde{L}$ are taken exactly as in Lemma (2.2.3), and assume*

$$\begin{aligned} Lw = \tilde{L}\tilde{w} = 1 \quad & \textit{in} \ \ B_1 \\ w = \tilde{w} = 0 \quad & \textit{on} \ \ \partial B_1 \ . \end{aligned} \tag{2.28}$$

Then there exists a postive constant $C_0 = C_0(n, \bar{\lambda}, \bar{\Lambda})$ such that for all $x \in B_{1/4}$ we have

$$C_0^{-1} w(x) \leq \tilde{w}(x) \leq C_0 w(x) \ . \tag{2.29}$$

Proof. Without loss of generality we can assume that $\tilde{L}$ is the Laplacian, and we can also replace the assumption $Lw = \Delta\tilde{w} = 1$ with the assumption $Lw = \Delta\tilde{w} = -1$ so that w and $\tilde{w}$ are positive functions. In fact, $\tilde{w}(x) = \Theta(x)$ where we define

$$\Theta(x) := \frac{1 - |x|^2}{2n} \ .$$

It will be convenient to define the following positive universal constants:

$$\theta_1 := \int_{B_1} |\nabla\Theta|^2 \qquad \text{and} \qquad \theta_2 := \int_{B_{1/2}} \Theta \ . \tag{2.30}$$

Let u solve

$$\begin{aligned} Lu = -\chi_{\{B_{1/2}\}} \quad & \text{in} \ \ B_1 \\ u = 0 \quad & \text{on} \ \ \partial B_1 \end{aligned} \tag{2.31}$$

and let v solve

$$\begin{aligned} Lv = -1 + \chi_{\{B_{1/2}\}} \quad & \text{in} \ \ B_1 \\ v = 0 \quad & \text{on} \ \ \partial B_1 \ . \end{aligned} \tag{2.32}$$

By the strong maximum principle, both u and v are positive in B_1, and since $w = u + v$ in

B_1, we have $w > u$ in B_1. By Theorem 8.18 of[GT]

$$\left(\frac{1}{4}\right)^{-n} ||u||_{L^1(B_{1/2})} \leq C \inf_{B_{1/4}} u \ . \tag{2.33}$$

By basic facts from the Calculus of Variations, u is characterized as the unique minimizer of the functional:

$$J(\phi; r) := \int_{B_1} \nabla\phi A(x) \nabla\phi - 2\int_{B_r} \phi \ , \tag{2.34}$$

when r is taken to be $1/2$. (We are letting $A(x)$ be the matrix of coefficients for the operator L.) Now we observe that for any $t > 0$, we have

$$\begin{aligned} J(t\Theta; 1/2) &= t^2 \int_{B_1} \nabla\Theta A(x) \nabla\Theta - 2t \int_{B_{1/2}} \Theta \\ &\leq t^2 \Lambda \theta_1 - 2t\theta_2 \ . \end{aligned}$$

(Recall that θ_1 and θ_2 are the positive universal constants defined in Equation (2.30) above.) Now by taking

$$t := \frac{\theta_2}{\Lambda\theta_1}$$

we can conclude

$$J(u; 1/2) \leq J(t\Theta; 1/2) \leq -\frac{\theta_2^2}{\Lambda\theta_1} =: -C_1 < 0 \ . \tag{2.35}$$

Since

$$J(u; 1/2) \geq -2\int_{B_1/2} u = -2||u||_{L^1(B_{1/2})} \ ,$$

we can conclude that

$$||u||_{L^1(B_{1/2})} \geq C_1/2 \ ,$$

which can be combined with Equation (2.33) to get

$$\inf_{B_{1/4}} w \geq \inf_{B_{1/4}} u \geq C \tag{2.36}$$

which is half of what we need.

By Theorem 8.17 of[GT] we know

$$\sup_{B_{1/2}} w \le C(||w||_{L^2(B_1)} + 1) \ . \tag{2.37}$$

Using the fact that w is the unique minimizer of $J(\cdot;1)$ and reasoning in a fashion almost identical to what we did above we get:

$$\begin{aligned} 0 &\ge J(w;1) \\ &\ge \lambda \int_{B_1} |\nabla w|^2 - 2\int_{B_1} w \\ &= \lambda ||\nabla w||^2_{L^2(B_1)} - 2||w||_{L^1(B_1)} \\ &\ge C\lambda ||w||^2_{L^2(B_1)} - 2||w||_{L^1(B_1)} \qquad \text{by Poincaré's inequality} \\ &\ge C\lambda ||w||^2_{L^2(B_1)} - 2(||w||_{L^2(B_1)} + |B_1|) \end{aligned}$$

which forces $||w||_{L^2(B_1)} \le C_0$ for some universal C_0. Combining this equation with Equation (2.37) gives us what we need. ∎

2.2.5 Lemma. *Let W satisfy the following*

$$\bar{\lambda} \le L(W) \le \bar{\Lambda} \ \ \text{in } B_r \qquad \text{and } W \ge 0 \ , \tag{2.38}$$

then there exists a positive constant, C, such that

$$\sup_{\partial B_r} W \ge W(0) + Cr^2 \ . \tag{2.39}$$

Proof. Let u solve

$$L(u) = 0 \ \text{ in } B_r \qquad \text{and } u = W \ \text{ on } \partial B_r \ . \tag{2.40}$$

Then the Weak Maximum Principle gives:

$$\sup_{\partial B_r} u \geq u(0). \tag{2.41}$$

Let v solve

$$L(v) = L(W) \ \text{ in } B_r \quad \text{ and } v = 0 \ \text{ on } \ \partial B_r\ . \tag{2.42}$$

Notice that $v_0(x) := \frac{|x|^2 - r^2}{2n}$ solves

$$\Delta(v_0) = 1 \ \text{ in } B_r \quad \text{ and } v_0 = 0 \ \text{ on } \ \partial B_r\ . \tag{2.43}$$

By Lemma (2.2.4) above, there exist constants C_1, C_2, such that $C_1 v_0(x) \leq v(x) \leq C_2 v_0(x)$ in $B_{r/4}$. In particular,

$$-v(0) \geq C_2 \frac{r^2}{2n}. \tag{2.44}$$

By the definitions of u and v, we know $W = u + v$, therefore by Equations (2.41) and (2.44) we have

$$\sup_{\partial B_r} W(x) = \sup_{\partial B_r} u(x) \geq u(0) = W(0) - v(0) \geq W(0) + C_2 \frac{r^2}{2n}\ . \tag{2.45}$$

■

2.2.6 Lemma. *Take w as above, and assume that $w(0) = \gamma > 0$. Then $w > 0$ in a ball B_{δ_0} where $\delta_0 = C_0\sqrt{\gamma}$*

Proof. By Theorem (2.2.2), we know that if $w(x_0) = 0$, then

$$\gamma = |w(x_0) - w(0)| \leq C|x_0|^2, \tag{2.46}$$

which implies $|x_0| \geq C\sqrt{\gamma}$. ■

2.2.7 Lemma (Nondegenerate Increase on a Polygonal Curve)**.** *Let w be exactly as above except that we assume that everything is satisfied in B_2 instead of B_1. Suppose again that $w(0) = \gamma > 0$, but now we may require γ to be sufficiently small. Then there exists a positive constant, C, such that*

$$\sup_{B_1} w(x) \geq C + \gamma. \tag{2.47}$$

Proof. We can assume without loss of generality that there exists a $y \in B_{1/3}$ such that $w(y) = 0$. Otherwise we can apply the maximum principle along with Lemma (2.2.5) to get:

$$\sup_{B_1} w(x) \geq \sup_{B_{1/3}} w(x) \geq \gamma + C, \tag{2.48}$$

and we would already be done.

By Lemmas (2.2.5) and (2.2.6), there exist $x_1 \in \partial B_{\delta_0}$, such that

$$w(x_1) \geq w(0) + C\frac{\delta_0^2}{2n} = (1 + C_1)\gamma. \tag{2.49}$$

For this x_1 and $B_{\delta_1}(x_1)$ where $\delta_1 = C_0\sqrt{w(x_1)}$, Lemma (2.2.6) guarantees the existence of an $x_2 \in \partial B_{\delta_1}(x_1)$, such that

$$w(x_2) \geq (1 + C_1)w(x_1) \geq (1 + C_1)^2\gamma. \tag{2.50}$$

Repeating the steps we can get finite sequences $\{x_i\}$ and $\{\delta_i\}$ with $x_0 = 0$ such that

$$w(x_i) \geq (1 + C_1)^i\gamma \quad \text{and} \quad \delta_i = |x_{i+1} - x_i| = C_0\sqrt{w(x_i)}. \tag{2.51}$$

Observe that as long as $x_i \in B_{1/3}$, because of the existence of $y \in B_{1/3}$ where $w(y) = 0$ we know that $\delta_i \leq 2/3$, and so x_{i+1} is still in B_1. Pick N to be the smallest number which satisfies the following inequality:

$$\Sigma_{i=0}^{N}\delta_i = \Sigma_{i=0}^{N} C_0\sqrt{\gamma}(1+C_1)^{\frac{i}{2}} \geq \frac{1}{3}, \tag{2.52}$$

that is

$$N \geq \frac{2\ln\left[\frac{(1+C_1)^{\frac{1}{2}}-1}{3C_0\sqrt{\gamma}}+1\right]}{\ln(1+C_1)} - 1. \tag{2.53}$$

Plugging this into Equation (2.51) gives

$$\begin{aligned} w(x_N) &\geq \gamma(1+C_1)^{\frac{2\ln\left[\frac{(1+C_1)^{\frac{1}{2}}-1}{3C_0\sqrt{\gamma}}+1\right]}{\ln(1+C_1)}-1} \\ &= \frac{\gamma}{1+C_1}\left(\frac{(1+C_1)^{\frac{1}{2}}-1}{3C_0\sqrt{\gamma}}+1\right)^2 \\ &= (\tilde{C}_0+\tilde{C}_1\sqrt{\gamma})^2 \\ &\geq C_2(1+\gamma)\ , \end{aligned}$$

where the last inequality is guaranteed by the fact that we allow γ to be sufficiently small.

■

2.2.8 Lemma. *Take w as above, but assume that $0 \in \overline{\{w>0\}}$. Then*

$$\sup_{\partial B_1} w(x) \geq C. \tag{2.54}$$

Proof. By applying the maximum principle and the previous lemma this lemma is immediate. ■

2.2.9 Theorem (Nondegeneracy)**.** *With $C = C(n,\lambda,\Lambda,\bar{\lambda},\bar{\Lambda}) > 0$ exactly as in the previous* *lemma, and if* $0 \in \overline{\{w > 0\}}$, *then for any* $r \leq 1$ *we have*

$$\sup_{x \in B_r} w(x) \geq Cr^2 \,. \tag{2.55}$$

Proof. Assume there exists some $r_0 \leq 1$, such that

$$\sup_{x \in B_{r_0}} w(x) = C_1 r_0^2 < C{r_0}^2 \,. \tag{2.56}$$

Notice that for $\gamma \leq 1$,

$$u_\gamma(x) := \frac{w(\gamma x)}{\gamma^2} \tag{2.57}$$

is also a solution to the same type of problem with a new operator $\tilde{L}$ and new function $\tilde{h}$ defined in B_1, but the new operator has the same ellipticity as the old operator, and the new $\tilde{h}$ has the same bounds and properties that h had. Now in particular for $u_{r_0}(x) = \frac{w(r_0 x)}{{r_0}^2}$, we have for any $x \in B_1$

$$u_{r_0}(x) = \frac{w(r_0 x)}{{r_0}^2} \leq \frac{1}{r_0^2} \sup_{x \in B_{r_0}} w(x) = C_1 < C \,, \tag{2.58}$$

which contradicts the previous lemma. ∎

2.2.10 Corollary (Free Boundary Has Zero Measure)**.** *The Lebesgue measure of the set*

$$\partial\{w = 0\}$$

is zero.

Proof. The idea here is to use nondegeneracy together with regularity to show that contained in any ball centered on the free boundary, there has to be a proportional subball where w is strictly positive. From this fact it follows that the free boundary cannot have any Lebesgue points. Since the argument is essentially identical to the proof within Lemma 5.1 of[BT] that $\mathcal{P}$ has measure zero, we will omit it. ∎

2.2.11 Remark (Porosity)**.** In fact, more can be said from the same argument. Indeed, it shows that the free boundary is strongly porous and therefore has a Hausdorff dimension strictly less than n. (See[M] for definitions of porosity and other relevent theorems and references.)

2.2.12 Corollary (Removing the "Gap")**.** *The existence, uniqueness, regularity, and nondegeneracy theorems from this section and the previous section all hold whenever*

$$L(w) = h$$

is replaced by

$$L(w) = \chi_{\{w>0\}} f \ .$$

2.3 Equivalence of the Obstacle Problems

There are two main points to this section. First, we deal with the comparatively simple task of getting existence, uniqueness, and continuity of certain minimizers to our functionals in the relevant sets. Second, and more importantly we show that the minimizer is the solution of an obstacle problem of the type studied in the previous two sections. We start with some definitions and terminology.

We continue to assume that a^{ij} is strictly and uniformly elliptic and we keep L defined exactly as above. We let $G(x, y)$ denote the Green's function for L for all of $\mathbb{R}^n$ and observe that the existence of G is guaranteed by the work of Littman, Stampacchia, and Weinberger. (See[LSW].)

Let

$$C_{sm,r} := \min_{x\in\partial B_r} G(x,0)$$

$$C_{big,r} := \max_{x\in\partial B_r} G(x,0)$$

$$G_{sm,r}(x) := \min\{G(x,0), C_{sm,r}\}$$

and observe that $G_{sm,r} \in W^{1,2}(B_M)$ by results from[LSW] combined with the Cacciopoli Energy Estimate. We also know that there is an $\alpha \in (0,1)$ such that $G_{sm,r} \in C^{0,\alpha}(\overline{B_M})$ by the De Giorgi-Nash-Moser theorem. (See[GT] or[HL] for example.) For M large enough to guarantee that $G_{sm}(x) := G_{sm,1}(x) \equiv G(x,0)$ on ∂B_M, we define:

$$H_{M,G} := \{w \in W^{1,2}(B_M) \;:\; w - G_{sm} \in W_0^{1,2}(B_M)\;\}$$

and

$$K_{M,G} := \{\; w \in H_{M,G} \;:\; w(x) \leq G(x,0) \text{ for all } x \in B_M \;\}.$$

(The existence of such an M follows from[LSW], and henceforth any constant M will be large enough so that $G_{sm,1}(x) \equiv G(x,0)$ on ∂B_M.)

Define:

$$\Phi_\epsilon(t) := \begin{cases} 0 & \text{for } t \geq 0 \\ \\ -\epsilon^{-1}t & \text{for } t \leq 0\,, \end{cases}$$

$$J(w,\Omega) := \int_\Omega (a^{ij}D_iwD_jw - 2R^{-n}w)\;, \quad \text{and}$$

$$J_\epsilon(w,\Omega) := \int_\Omega (a^{ij}D_iwD_jw - 2R^{-n}w + 2\Phi_\epsilon(G-w))\;.$$

2.3.1 Theorem (Existence and Uniqueness)**.**

$$\textit{Let } \ell_0 := \inf_{w \in K_{M,G}} J(w, B_M) \quad \textit{and}$$

$$\textit{let } \ell_\epsilon := \inf_{w \in H_{M,G}} J_\epsilon(w, B_M) \; .$$

Then there exists a unique $w_0 \in K_{M,G}$ *such that* $J(w_0, B_M) = \ell_0$*, and there exists a unique* $w_\epsilon \in H_{M,G}$ *such that* $J_\epsilon(w_\epsilon, B_M) = \ell_\epsilon$ *.*

Proof. Both of these results follow by a straightforward application of the direct method of the Calculus of Variations. ∎

2.3.2 Remark. Notice that we cannot simply minimize either of our functionals on all of $\mathbb{R}^n$ instead of B_M as the Green's function is not integrable at infinity. Indeed, if we replace B_M with $\mathbb{R}^n$ then

$$\ell_0 = \ell_\epsilon = -\infty$$

and so there are many technical problems.

2.3.3 Theorem (Continuity)**.** *For any* $\epsilon > 0$*, the function* w_ϵ *is continuous on* $\overline{B_M}$ *.*

See Chapter 7 of[G].

2.3.4 Lemma. *There exists* $\epsilon > 0$*,* $C < \infty$*, such that* $w_0 \leq C$ *in* B_ϵ*.*

Proof. Let $\bar{w}$ minimize $J(w, B_M)$ among functions $w \in H_{M,G}$. Then we have

$$w_0 \leq \bar{w}.$$

Set $b := C_{big,M} = \max_{\partial B_M} G(x, 0)$, and let w_b minimize $J(w, B_M)$ among $w \in W^{1,2}(B_M)$ with

$$w - b \in W^{1,2}_0(B_M).$$

Then by the weak maximum principle, we have

$$\bar{w} \leq w_b.$$

Next define $\ell(x)$ by

$$\ell(x) := b + R^{-n}\left(\frac{M^2 - |x|^2}{4n}\right) \leq b + \frac{R^{-n}M^2}{4n} < \infty. \tag{2.59}$$

With this definition, we can observe that ℓ satisfies

$$\Delta\ell = -\frac{R^{-n}}{2}, \text{ in } B_M \quad \text{and}$$
$$\ell \equiv b := \max_{\partial B_M} G \text{ on } \partial B_M.$$

Now let $\widetilde{\alpha}$ be $b + \frac{R^{-n}M^2}{4n}$. By Corollary 7.1 in[LSW] applied to $w_b - b$ and $\ell - b$, we have

$$w_b \leq b + K(\ell - b) \leq b + K\widetilde{\alpha} < \infty.$$

Chaining everything together gives us

$$w_0 \leq b + K\widetilde{\alpha} < \infty.$$

■

2.3.5 Lemma. *If* $0 < \epsilon_1 \leq \epsilon_2$, *then*

$$w_{\epsilon_1} \leq w_{\epsilon_2}.$$

Proof. Assume $0 < \epsilon_1 \leq \epsilon_2$, and assume that

$$\Omega_1 := \{w_{\epsilon_1} > w_{\epsilon_2}\}$$

is not empty. Since $w_{\epsilon_1} = w_{\epsilon_2}$ on ∂B_M, since $\Omega_1 \subset B_M$, and since w_{ϵ_1} and w_{ϵ_2} are continuous functions, we know that $w_{\epsilon_1} = w_{\epsilon_2}$ on $\partial\Omega_1$. Then it is clear that among functions with the same data on $\partial\Omega_1$, w_{ϵ_1} and w_{ϵ_2} are minimizers of $J_{\epsilon_1}(\cdot, \Omega_1)$ and $J_{\epsilon_2}(\cdot, \Omega_1)$ respectively. Since we will restrict our attention to Ω_1 for the rest of this proof, we will use $J_\epsilon(w)$ to denote $J_\epsilon(w, \Omega_1)$.

$J_{\epsilon_2}(w_{\epsilon_2}) \leq J_{\epsilon_2}(w_{\epsilon_1})$ implies

$$\begin{aligned} &\int_{\Omega_1} a^{ij} D_i w_{\epsilon_2} D_j w_{\epsilon_2} - 2R^{-n} w_{\epsilon_2} + 2\Phi_{\epsilon_2}(G - w_{\epsilon_2}) \\ \leq &\int_{\Omega_1} a^{ij} D_i w_{\epsilon_1} D_j w_{\epsilon_1} - 2R^{-n} w_{\epsilon_1} + 2\Phi_{\epsilon_2}(G - w_{\epsilon_1}) \ , \end{aligned}$$

and by rearranging this inequality we get

$$\int_{\Omega_1} (a^{ij} D_i w_{\epsilon_2} D_j w_{\epsilon_2} - 2R^{-n} w_{\epsilon_2}) - \int_{\Omega_1} (a^{ij} D_i w_{\epsilon_1} D_j w_{\epsilon_1} - 2R^{-n} w_{\epsilon_1})$$

$$\leq \int_{\Omega_1} 2\Phi_{\epsilon_2}(G - w_{\epsilon_1}) - 2\Phi_{\epsilon_2}(G - w_{\epsilon_2}) \ .$$

Therefore,

$$\begin{aligned} &J_{\epsilon_1}(w_{\epsilon_2}) - J_{\epsilon_1}(w_{\epsilon_1}) \\ = &\int_{\Omega_1} a^{ij} D_i w_{\epsilon_2} D_j w_{\epsilon_2} - 2R^{-n} w_{\epsilon_2} + 2\Phi_{\epsilon_1}(G - w_{\epsilon_2}) \\ &\quad - \int_{\Omega_1} a^{ij} D_i w_{\epsilon_1} D_j w_{\epsilon_1} - 2R^{-n} w_{\epsilon_1} + 2\Phi_{\epsilon_1}(G - w_{\epsilon_1}) \\ \leq &\, 2\int_{\Omega_1} \left[\Phi_{\epsilon_2}(G - w_{\epsilon_1}) - \Phi_{\epsilon_2}(G - w_{\epsilon_2})\right] \\ &\quad - 2\int_{\Omega_1} \left[\Phi_{\epsilon_1}(G - w_{\epsilon_1}) - \Phi_{\epsilon_1}(G - w_{\epsilon_2})\right] \\ < &\, 0 \end{aligned}$$

since $G - w_{\epsilon_1} < G - w_{\epsilon_2}$ in Ω_1 and Φ_{ϵ_1} decreases as fast or faster than Φ_{ϵ_2} decreases everywhere. This inequality contradicts the fact that w_{ϵ_1} is the minimizer of $J_{\epsilon_1}(w)$. Therefore, $w_{\epsilon_1} \le w_{\epsilon_2}$ everywhere in Ω. ■

2.3.6 Lemma. *$w_0 \le w_\epsilon$ for every $\epsilon > 0$.*

Proof. Let $S := \{w_0 > w_\epsilon\}$ be a nonempty set, let $w_1 := \min\{w_0, w_\epsilon\}$, and let $w_2 := \max\{w_0, w_\epsilon\}$. It follows that $w_1 \le G$ and both w_1 and w_2 belong to $W^{1,2}(B_M)$. Since $\Phi_\epsilon \ge 0$, we know that for any $\Omega \subset B_M$ we have

$$J(w, \Omega) \le J_\epsilon(w, \Omega) \tag{2.60}$$

for any permissible w. We also know that since $w_0 \le G$ we have:

$$J(w_0, \Omega) = J_\epsilon(w_0, \Omega)\ . \tag{2.61}$$

Now we estimate:

$$\begin{aligned}
J_\epsilon(w_1, B_M) &= J_\epsilon(w_1, S) + J_\epsilon(w_1, S^c) \\
&= J_\epsilon(w_\epsilon, S) + J_\epsilon(w_0, S^c) \\
&= J_\epsilon(w_\epsilon, B_M) - J_\epsilon(w_\epsilon, S^c) + J_\epsilon(w_0, S^c) \\
&\le J_\epsilon(w_2, B_M) - J_\epsilon(w_\epsilon, S^c) + J_\epsilon(w_0, S^c) \\
&= J_\epsilon(w_0, S) + J_\epsilon(w_\epsilon, S^c) - J_\epsilon(w_\epsilon, S^c) + J_\epsilon(w_0, S^c) \\
&= J_\epsilon(w_0, S) + J_\epsilon(w_0, S^c) \\
&= J_\epsilon(w_0, B_M)\ .
\end{aligned}$$

Now by combining this inequality with Equations (2.60) and (2.61), we get:

$$J(w_1, B_M) \leq J_\epsilon(w_1, B_M) \leq J_\epsilon(w_0, B_M) = J(w_0, B_M) \; ,$$

but if S is nonempty, then this inequality contradicts the fact that w_0 is the unique minimizer of J among functions in $K_{M,G}$. ■

Now, since w_ϵ decreases as $\epsilon \to 0$, and since the w_ϵ's are bounded from below by w_0, there exists

$$\widetilde{w} = \lim_{\epsilon \to 0} w_\epsilon$$

and $w_0 \leq \widetilde{w}$.

2.3.7 Lemma. *With the definitions as above, $\tilde{w} \leq G$ almost everywhere.*

Proof. This fact is fairly obvious, and the proof is fairly straightforward, so we supply only a sketch.

Suppose not. Then there exists an $\alpha > 0$ such that

$$\tilde{S} := \{\tilde{w} - G \geq \alpha\}$$

has positive measure. On this set we automatically have $w_\epsilon - G \geq \alpha$. We compute $J_\epsilon(w_\epsilon, B_M)$ and send ϵ to zero. We will get $J_\epsilon(w_\epsilon, B_M) \to \infty$ which gives us a contradiction. ■

2.3.8 Lemma. *$\widetilde{w} = w_0$ in $W^{1,2}(B_M)$.*

Proof. Since for any ϵ, w_ϵ is the minimizer of $J_\epsilon(w, B_M)$, we have

$$\begin{aligned} J_\epsilon(w_\epsilon, B_M) &\leq J_\epsilon(w_0, B_M) \\ &\leq \int_{B_M} a^{ij} D_i w_0 D_j w_0 - 2R^{-n} w_0 + 2\Phi_\epsilon(G - w_\epsilon), \end{aligned}$$

and after canceling the terms with Φ_ϵ we have:

$$\int_{B_M} a^{ij} D_i w_\epsilon D_j w_\epsilon - 2R^{-n} w_\epsilon \leq \int_{B_M} a^{ij} D_i w_0 D_j w_0 - 2R^{-n} w_0.$$

Letting $\epsilon \to 0$ gives us

$$J(\tilde{w}, B_M) \leq J(w_0, B_M)\ .$$

However, by Proposition (2.3.7), $\widetilde{w}$ is a permissible competitor for the problem $\inf_{w \in K_{M,G}} J(w, B_M)$, so we have

$$J(w_0, B_M) \leq J(\tilde{w}, B_M).$$

Therefore

$$J(w_0, B_M) = J(\tilde{w}, B_M),$$

and then by uniqueness, $\widetilde{w} = w_0$. ∎

Let W solve:

$$\begin{cases} L(w) = -\chi_{\{w<G\}} R^{-n} & \text{in } \ B_M \\ \\ w = G_{sm} & \text{on } \ \partial B_M\ . \end{cases} \tag{2.62}$$

The existence of such a W is guaranteed by combining Theorem (2.1.3) with Corollary (2.2.12). (Signs are reversed, so to be completely precise one must apply the theorems to the problem solved by $G - W$.)

2.3.9 Lemma. *$W \leq G$ in B_M.*

Proof. Let $\Omega = \{W > G\}$ and $u := W - G$. Since G is infinite at 0, and since W is bounded, and both G and W are continuous, we know there exists an $\epsilon > 0$ such that $\Omega \cap B_\epsilon = \phi$. Then if $\Omega \neq \phi$, then u has a positive maximum in the interior of Ω. However, since $L(W) = L(G) = 0$ in Ω, we would get a contradiction from the weak maximum principle. Therefore, we have $W \leq G$ in B_M. ∎

2.3.10 Lemma. *$\tilde{w} \geq W$.*

Proof. It suffices to show $w_\epsilon \geq W$, for any ϵ. Suppose for the sake of obtaining a contradiction that there exists an $\epsilon > 0$ and a point x_0 where $w_\epsilon - W$ has a negative local minimum. So $w_\epsilon(x_0) < W(x_0) \leq G(x_0)$. Let $\Omega := \{w_\epsilon < W\}$ and observe that $w_\epsilon = W$ on $\partial\Omega$. Then x_0 is an interior point of Ω and

$$L(w_\epsilon) = -R^{-n} \text{ in } \Omega.$$

However

$$L(W - w_\epsilon) \geq -R^{-n} + R^{-n} = 0 \text{ in } \Omega. \tag{2.63}$$

By the weak maximum principle, the minimum can not be attained at an interior point, and so we have a contradiction. ∎

2.3.11 Lemma. *$w_0 = \tilde{w} = W$, and so w_0 and $\tilde{w}$ are continuous.*

Proof. We already showed that $w_0 = \tilde{w}$ in lemma (2.3.8). By lemma (2.3.10), in the set where $W = G$, we have

$$W = \tilde{w} = G. \tag{2.64}$$

Let $\Omega_1 := \{W < G\}$, it suffices to show $\tilde{w} = W$ in Ω_1. By definition of W, $L(W) = -R^{-n}$ in Ω_1.

Using the fact that w_0 is the minimizer, the standard argument in the calculus of variations leads to $L(w_0) \geq -R^{-n}$. Therefore

$$L(\tilde{w} - W) = L(w_0 - W) \geq 0 \quad \text{in} \quad B_M. \tag{2.65}$$

Notice that on $\partial\Omega_1$, $W = \tilde{w} = G$. By weak maximum principle, we have

$$\tilde{w} = W \quad \text{in} \quad \Omega_1. \tag{2.66}$$

■

Using the last lemma along with our definition of W (see Equation (2.62)) we can now state the following theorem.

2.3.12 Theorem (The PDE satisfied by w_0)**.** *The minimizing function w_0 satisfies the following boundary value problem:*

$$\begin{cases} L(w_0) = -\chi_{\{w_0 < G\}} R^{-n} & \textit{in} \quad B_M \\ \\ w_0 = G_{sm} & \textit{on} \quad \partial B_M \ . \end{cases} \tag{2.67}$$

2.4 Minimizers Become Independent of M

At this point we are no longer interested in the functions from the last section, with the exception of w_0. On the other hand, we now care about the dependence of w_0 on the radius of the ball on which it is a minimizer. Accordingly, we reintroduce the dependence of w_0 on M, and so we will let w_M be the minimizer of $J(w, B_M)$ within $K(M, G)$, and consider the behavior as $M \to \infty$. As we observed in Remark (2.3.2), it is not possible to start by minimizing our functional on all of $\mathbb{R}^n$, so we have to get the key function, "V_R," mentioned by Caffarelli on page 9 of[C5] by taking a limit over increasing sets. Note that by Theorem (2.3.12) we know that w_M satisfies

$$\begin{cases} L(w_M) = -\chi_{\{G>w_M\}} R^{-n} & \text{in } \ B_M \\ \\ w_M = G_{sm} & \text{on } \ \partial B_M \ . \end{cases} \tag{2.68}$$

The theorem that we wish to prove in this section is the following:

2.4.1 Theorem (Independence from M)**.** *There exists $M \in \mathbb{N}$ such that if $M_j > M$ for $j = 1, 2,$ then*

$$w_{M_1} \equiv w_{M_2} \quad \textit{within} \quad B_M$$

and

$$w_{M_1} \equiv w_{M_2} \equiv G \quad \textit{within} \quad B_{M+1} \setminus B_M \ .$$

Furthermore, we can choose M such that $M < C(n, \lambda, \Lambda) \cdot R$.

This Theorem is an immediate consequence of the following Theorem:

2.4.2 Theorem (Boundedness of the Noncontact Set)**.** *There exists a constant $C = C(n, \lambda, \Lambda)$ such that for any $M \in \mathbb{R}$*

$$\{w_M \neq G\} \subset B_{CR} \ . \tag{2.69}$$

Proof. First of all, if $M \leq CR$, then there is nothing to prove. For all $M > 1$ the function $W := G - w_M$ will satisfy:

$$L(W) = R^{-n} \chi_{\{W>0\}}, \text{ and } 0 \leq W \leq G \text{ in } B_1^c. \tag{2.70}$$

If the conclusion to the theorem is false, then there exists a large M and a large C such that

$$x_0 \in FB(W) \cap \{B_{M/2} \setminus B_{CR}\} \ .$$

Let $K := |x_0|/3$. By Theorem (2.2.9), we can then say that

$$\sup_{B_K(x_0)} W(x) \geq CR^{-n}K^2 > CK^{2-n} \geq \sup_{B_K(x_0)} G(x) \tag{2.71}$$

which gives us a contradiction since $W \leq G$ everywhere. Now note that in order to avoid the contradiction, we must have

$$CR^{-n}K^2 \leq CK^{2-n} \ ,$$

and this leads to

$$K \leq CR$$

which means that $|x_0|$ must be less than CR. In other words, $FB(W) \subset B_{CR}$. ∎

At this point, we already know that when M is sufficiently large, the set $\{G > w_M\}$ is contained in B_{CR}. Then by uniqueness, the set will stay the same for any bigger M. Therefore, it makes sense to define w_R to be the solution of

$$Lw = -R^{-n}\chi_{\{w<G\}} \quad \text{in } \mathbb{R}^n \tag{2.72}$$

among functions $w \leq G$ with $w = G$ at infinity. Note that we can now obtain the function, "V_R," that Caffarelli uses on page 9 of[C5]. The relationship is simply:

$$V_R = w_R - G \ . \tag{2.73}$$

2.5 The Mean Value Theorem

Finally, we can turn to the Mean Value Theorem.

2.5.1 Lemma (Ordering of Sets)**.** *For any $R < S$, we have*

$$\{w_R < G\} \subset \{w_S < G\}. \tag{2.74}$$

Proof. Let B_M be a ball that contains both $\{w_R < G\}$ and $\{w_S < G\}$. Then by the discussion in Section 2, we know w_R minimizes

$$\int_{B_M} a^{ij} D_i w D_j w - 2wR^{-n}$$

and w_S minimizes

$$\int_{B_M} a^{ij} D_i w D_j w - 2wS^{-n}.$$

Let $\Omega_1 \subset\subset B_M$ be the set $\{w_S > w_R\}$. Then it follows that

$$\int_{\Omega_1} a^{ij} D_i w_S D_j w_S - 2w_S S^{-n} \leq \int_{\Omega_1} a^{ij} D_i w_R D_j w_R - 2w_R S^{-n}, \tag{2.75}$$

which implies

$$\begin{aligned} \int_{\Omega_1} a^{ij} D_i w_S D_j w_S &\leq \int_{\Omega_1} a^{ij} D_i w_R D_j w_R + 2S^{-n} \int_{\Omega_1} (w_S - w_R) \\ &< \int_{\Omega_1} a^{ij} D_i w_R D_j w_R + 2R^{-n} \int_{\Omega_1} (w_S - w_R). \end{aligned}$$

Therefore, since $w_S \equiv w_R$ on $\partial\Omega_1$, and

$$\int_{\Omega_1} a^{ij} D_i w_S D_j w_S - 2w_S R^{-n} < \int_{\Omega_1} a^{ij} D_i w_R D_j w_R - 2w_R R^{-n}, \tag{2.76}$$

we contradict the fact that w_R is the minimizer of $\int a^{ij} D_i w D_j w - 2wR^{-n}$. ■

2.5.2 Lemma. *There exists a constant* $c = c(n, \lambda, \Lambda)$ *such that*

$$B_{cR} \subset \{G > w_R\}.$$

Proof. By Lemma (2.3.4) we already know that there exists a constant

$$C = C(n, \lambda, \Lambda)$$

such that $w_1(0) \leq C$. Then it is not hard to show that

$$\|w_1\|_{L^\infty(B_{1/2})} \leq \tilde{C}. \tag{2.77}$$

By[LSW] for any elliptic operator L with given λ and Λ, we have

$$\frac{c_1}{|x|^{n-2}} \leq G(x) \leq \frac{c_2}{|x|^{n-2}}. \tag{2.78}$$

By combining the last two equations it follows that there exists a constant $c = c(n, \lambda, \Lambda)$ such that

$$B_c \subset \{G > w_1\}.$$

It remains to show that this inclusion scales correctly.

Let $v_R := G - w_R$ (so $v_R = -V_R$). Then v_R satisfies

$$Lv_R = \delta - R^{-n}\chi_{\{v_R>0\}} \quad \text{in } \mathbb{R}^n\ . \tag{2.79}$$

Now observe that by scaling our operator L appropriately, we get an operator $\tilde{L}$ with the same ellipticity constants as L, such that

$$\tilde{L}\left(R^{n-2}v_R(Rx)\right) = \delta - \chi_{\{v_R(Rx)>0\}} \,. \tag{2.80}$$

So we have

$$B_c \subset \left\{x \,\middle|\, v_R(Rx) > 0\right\},$$

which implies

$$B_{cR} \subset \left\{v_R(x) > 0\right\}. \tag{2.81}$$

■

Suppose v is a supersolution to

$$Lv = 0,$$

i.e. $Lv \leq 0$. Then for any $\phi \geq 0$, we have

$$\int_\Omega vL\phi \leq 0. \tag{2.82}$$

If $R < S$, then we know that $w_R \geq w_S$, and so the function $\phi = w_R - w_S$ is a permissible test function. We also know:

$$L\phi = R^{-n}\chi_{\{G>w_R\}} - S^{-n}\chi_{\{G>w_S\}}. \tag{2.83}$$

By observing that $v \equiv 1$ is both a supersolution and a subsolution and by plugging in our ϕ, we arrive at

$$R^{-n}|\{G > w_R\}| = S^{-n}|\{G > w_S\}|, \tag{2.84}$$

and this implies

$$L\phi = C\left[\frac{1}{|\{G > w_R\}|}\chi_{\{G>w_R\}} - \frac{1}{|\{G > w_S\}|}\chi_{\{G>w_S\}}\right]. \tag{2.85}$$

Now, Equation (2.82) implies

$$0 \geq \int_\Omega vL\phi = C\left[\frac{1}{|\{G > w_R\}|}\int_{\{G>w_R\}} v - \frac{1}{|\{G > w_S\}|}\int_{\{G>w_S\}} v\right]. \tag{2.86}$$

Therefore, we have established the following theorem:

2.5.3 Theorem (Mean Value Theorem for Divergence Form Elliptic PDE)**.** *Let L be any divergence form elliptic operator with ellipticity λ, Λ. For any $x_0 \in \Omega$, there exists an increasing family $D_R(x_0)$ which satisfies the following:*

1. *$B_{cR}(x_0) \subset D_R(x_0) \subset B_{CR}(x_0)$, with c, C depending only on n, λ and Λ.*
2. *For any v satisfying $Lv \geq 0$ and $R < S$, we have*

$$v(x_0) \leq \frac{1}{|D_R(x_0)|}\int_{|D_R(x_0)|} v \leq \frac{1}{|D_S(x_0)|}\int_{D_S(x_0)} v. \tag{2.87}$$

As on pages 9 and 10 of[C5], (and as Littman, Stampacchia, and Weinberger already observed using their own mean value theorem,) we have the following corollary:

2.5.4 Corollary (Semicontinuous Representative)**.** *Any supersolution v, has a unique pointwise defined representative as*

$$v(x_0) := \lim_{R\downarrow 0}\frac{1}{|D_R(x_0)|}\int_{|D_R(x_0)|} v(x)dx\ . \tag{2.88}$$

This representative is lower semicontinuous:

$$v(x_0) \leq \lim_{x\to x_0} v(x) \tag{2.89}$$

for any x_0 in the domain.

We can also show the following analogue of G.C. Evans' Theorem:

2.5.5 Corollary (Analogue of Evans' Theorem)**.** *Let v be a supersolution to $Lv = 0$, and suppose that v restricted to the support of Lv is continuous. Then the representative of v given by Equation (2.89) is continuous.*

Proof. This proof is almost identical to the proof given on pages 10 and 11 of[C5] for $L = \Delta$. ■

Chapter 3

Reifenberg Flatness of Free Boundaries

3.1 Preliminaries and Basic Results

Define the functionals:

$$D(u,\Omega) := \int_\Omega (a^{ij} D_i u D_j u) \ , \quad \text{and}$$

$$\tilde{D}(w,\Omega) := \int_\Omega (a^{ij} D_i w D_j w + 2w) \ .$$

For any bounded set $\Omega \subset \mathbb{R}^n$ we will minimize these functionals in the following sets, respectively:

$$S_{\Omega,\varphi} := \{u \in W_0^{1,2}(\Omega) \ : u \geq \varphi \ \} \ ,$$

$$\tilde{H}_{\Omega,\psi} := \{w \in W^{1,2}(\Omega) \ : \ w - \psi \in W_0^{1,2}(\Omega) \ \} \ , \quad \text{and}$$

$$\tilde{K}_{\Omega,\psi} := \{ \ w \in H_{\Omega,\psi} \ : \ w(x) \geq 0 \text{ for all } x \in \Omega \ \}.$$

When it is clear on which set we are working, we will simply write "$D(u)$" in place of "$D(u,\Omega)$" and "S_φ" in place of "$S_{\Omega,\varphi}$" and so on.

Probably the most classic version of the obstacle problem involves minimizing $D(u, B_1)$ within $S_{B_1,\varphi}$ in the case where $a^{ij} = \delta^{ij}$. (Here we use δ^{ij} to denote the usual Kronecker delta function so that $D(u)$ simplifies to the usual Dirichlet integral. See[C1],[C2],[C4], and[C5] for an analysis of this problem.) Indeed, following the same arguments given at the beginning of[C5], but for the more general a^{ij} considered here, we can establish the following theorem:

3.1.1 Theorem (Basic Results)**.** *Given an obstacle $\varphi \in W^{1,2}(B_1)$ which has a trace on ∂B_1 which is negative almost everywhere, there is a unique $u \in S_{B_1,\varphi}$ which minimizes $D(u, B_1)$. Furthermore, u is a bounded supersolution to the problem $L(u) = 0$. Finally, if φ is continuous, then u is almost everywhere equal to a function which is continuous on all of $\overline{B}_1$.*

Proof. For the proof, just follow the beginning of[C5]. (Here we need the mean value formula that we proved as Theorem (2.5.3)) ■

Turning to the regularity questions, we find it convenient to work with the height function w which is the minimizer of $\tilde{D}$ within $\tilde{K}_{B_1,\psi}$. On the other hand, one can ask if this is really the same problem as before. In the original problem with the Laplacian (in other words, with $a^{ij} = \delta^{ij}$), if the obstacle is twice differentiable, then it makes sense to take its Laplacian. In the current situation, it is not as simple to characterize the functions φ, where $L\varphi$ makes sense. The obvious route, however, is to simply assume that $L\varphi = -f$ for a function f with specified properties. If we assume that $L\varphi = -f$, and that $f \in L^\infty(B_1)$, then the two problems are completely equivalent.

We are most interested in the obstacle problem where we minimize $\tilde{D}$ within $\tilde{K}_{B_1,\psi}$. Besides requiring existence and regularity, we need to know that the minimizer, w, satisfies $w \geq 0$ and

$$\begin{aligned} L(w) &= \chi_{\{w>0\}} f \quad \text{in} \quad B_1 \\ w &= \psi \quad \text{on} \quad \partial B_1 \,. \end{aligned} \tag{3.1}$$

The proof of this fact and many of the related facts follows Chapter 2 very closely, and so we will only mention that the proof is carried out with a penalization argument. The details can be found with only very minor adjustments in Chapter 2. To summarize the relevant facts we can state the following result:

3.1.2 Theorem (Problem Equivalencies)**.** *Let φ be an obstacle which satisfies the following:*

1. $\psi := -\varphi > 0$ *on all of* ∂B_1.

2. $f := -L\varphi \in L^\infty(B_1)$.

Finally assume that $w = u - \varphi$. Then the following are equivalent:

1. *w satisfies Equation (3.1).*

2. *w minimizes $\tilde{D}$ in $\tilde{K}_{B_1,\psi}$.*

3. *$u \in W_0^{1,2}(B_1)$ satisfies $Lu = -\chi_{\{u=\varphi\}} f$.*

4. *u minimizes D in $S_{B_1,\psi}$.*

Now in order to get to the regularity of the free boundary we need two more basic facts from Chapter 2. At this point, having proven our theorem about the equivalencies between the problems, it is worth gathering a collection of assumptions that we will have for the rest of this work. We will always assume:

$$\begin{aligned}
&L(w) = \chi_{\{w>0\}} f \quad \text{in } B_1 \ , \\
&a^{ij}(x) \equiv a^{ji}(x) \ , \\
&0 < \lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for all } \xi \neq 0 \ , \\
&0 < \lambda^* \leq f \leq \Lambda^* \ , \quad \text{and} \\
&w \geq 0
\end{aligned} \tag{3.2}$$

and we will frequently assume

$$0 \in \partial\{w > 0\}. \tag{3.3}$$

Recall the optimal regularity (see Theorem (2.2.2)) and nondegeneracy (see Theorem (2.2.9)) which will give us compactness of quadratic rescalings.

3.2 Measure Stability

Now we begin a measure theoretic study of regularity which will culminate in a measure theoretic version of the theorem proven by Caffarelli in 1977. (See[C1].)

3.2.1 Lemma (Compactness I). *Let $\{a_k^{ij}\}$, $\{f_k\}$, and $\{w_k\}$ satisfy*

1. $0 < \lambda I \leq a_k^{ij} \leq \Lambda I$,
2. $0 < \lambda^* \leq f_k \leq \Lambda^*$,
3. $w_k \geq 0$, $D_i a_k^{ij} D_j w_k = \chi_{\{w_k>0\}} f_k$ *in* B_2, *and* $0 \in \partial\{w_k > 0\}$,
4. $||w_k||_{W^{1,2}(B_2)} \leq \gamma < \infty$ *and*
5. *there exists an* f *(with* $0 < \lambda^* \leq f \leq \Lambda^*$*), such that* f_k *converges to* f *strongly in* L^1.

Then there exists a $w \in W^{1,2}(B_1)$ *and an* $f \in L^\infty(B_1)$ *and a subsequence of* $\{w_k\}$ *such that along this subsequence (which we still label with "k"), we have*

A. *uniform convergence of* w_k *to* w*, and weak convergence in* $W^{1,2}$,

B. *for any* $\phi \in W_0^{1,2}(B_1)$
$$\int_{B_1} \chi_{\{w_k>0\}} f_k \phi \to \int_{B_1} \chi_{\{w>0\}} f \phi. \tag{3.4}$$

Proof. Item A follows by using standard functional analysis combined with De Giorgi-Nash-Moser theory. Since we can take a subsequence, we can assume without loss of generality that f_k converges to f pointwise almost everywhere. In the interior of both $\{w > 0\}$ and $\{w = 0\}$ it is not hard to show that $\chi_{\{w_k>0\}} f_k$ converges pointwise almost everywhere to $\chi_{\{w>0\}} f$ (for the interior of $\{w = 0\}$ one needs to use the nondegeneracy statement), so by Lebesgue's dominated convergence theorem it suffices to prove that $\partial\{w = 0\}$ has no Lebesgue points. The proof of this fact is very similar to the proof of Lemma 5.1 of[BT], but we include it here for the convenience of the reader.

Let $x_0 \in \partial\{w = 0\} \cap B_1$, and choose $r > 0$ such that

$$B_r(x_0) \subset B_1.$$

Define $W(x) := r^{-2}w(x_0 + rx)$ and $W_k(x) := r^{-2}w_k(x_0 + rx)$. After this change of coordinates, we have $0 \in \partial\{W = 0\}$, and so there exists $\{x_k\} \to 0$ such that

$$W(x_k) > 0, \text{ for all } k.$$

Now fix k so $x_k \in B_{1/8}$, take J large enough such that $i, j \geq J$ implies

$$||W_j - W||_{L^\infty(B_1)} \leq \frac{W(x_k)}{2}, \tag{3.5}$$

and

$$||W_i - W_j||_{L^\infty(B_1)} \leq \frac{\tilde{C}}{10} \tag{3.6}$$

where $\tilde{C} = \frac{C}{10}$ which is the constant from the nondegeneracy statement.

Since $W_j \to W$ in C^α, $W_J(x_k) > 0$ and nondegeneracy imply the existence of $\tilde{x} \in B_{1/2}$ such that

$$W_J(\tilde{x}) \geq C\left(\frac{1}{2} - \frac{1}{8}\right)^2 = \frac{9}{64}C > \tilde{C}. \tag{3.7}$$

Now $i \geq J$ implies $W_i(\tilde{x}) \geq \frac{9\tilde{C}}{10}$. Since W_i satisfies a uniform C^α estimate, there exists an $\tilde{r} > 0$ such that $W_i(y) \geq \frac{\tilde{C}}{2}$ for all $y \in B_{\tilde{r}}(\tilde{x})$ once $i \geq J$. From this we can conclude $B_{\tilde{r}}(\tilde{x}) \subset \{W_\infty > 0\}$.

Scaling back to the original functions, we conclude x_0 is not Lebesgue point. Since x_0 was an arbitrary point of the free boundary there are no Lebesgue points in $\partial\{w > 0\}$. ■

3.2.2 Lemma (Compactness II). *If we assume everything we did in the previous lemma, and we assume in addition that $A = (A^{ij})$ is a symmetric, constant matrix with*

$$0 < \lambda I \leq A \leq \Lambda I,$$

and such that

$$||a_k^{ij} - A^{ij}||_{L^1(B_1)} \to 0,$$

then the limiting functions w and f given in the last lemma satisfy:

$$D_i A^{ij} D_j w = \chi_{\{w>0\}} f \tag{3.8}$$

in B_1. Furthermore, $0 \in \partial\{w > 0\}$.

Proof. Since $a_k^{ij} \to A^{ij}$, and there is a uniform L^∞ bound on all of a_k^{ij} and A^{ij}, we have

$$a_k^{ij} \to A^{ij} \text{ in } L^q(B_1) \tag{3.9}$$

for any $q < \infty$, in particular $a_k^{ij} \to A^{ij}$ in L^2. We have for any $\phi \in W_0^{1,2}(B_1)$,

$$\begin{aligned}\int_{B_1} a_k^{ij} D_i w_k D_j \phi = &\int_{B_1} (a_k^{ij} - A^{ij})(D_i w_k - D_i w) D_j \phi \\ &+ \int_{B_1} a_k^{ij} D_i w D_j \phi + \int_{B_1} A^{ij}(D_i w_k - D_i w) D_j \phi\end{aligned}$$

Since $a_k^{ij} \to A^{ij}$ in L^2 and $D_i w_k \rightharpoonup D_i w$, we have

$$\int_{B_1} (a_k^{ij} - A^{ij})(D_i w_k - D_i w) D_j \phi \to 0, \tag{3.10}$$

$$\int_{B_1} a_k^{ij} D_i w D_j \phi \to \int_{B_1} A^{ij} D_i w D_j \phi \tag{3.11}$$

and

$$\int_{B_1} A^{ij}(D_i w_k - D_i w) D_j \phi \to 0. \tag{3.12}$$

Therefore,

$$\int_{B_1} a_k^{ij} D_i w_k D_j \phi \to \int_{B_1} A^{ij} D_i w D_j \phi. \tag{3.13}$$

Together with Equation (3.4), we proved

$$D_i A^{ij} D_j w = \chi_{\{w>0\}} f.$$

Now in order to show that $0 \in \partial\{w > 0\}$ we observe first that $0 \in \partial\{w_k > 0\}$ implies

$$0 \in \{w = 0\}.$$

Next we suppose there exists r_0, such that $B_{2r_0} \subset \{w = 0\}$. For any k, we have

$$\sup_{x \in B_{r_0}} w_k(x) \geq C(r_0)^2 \ . \tag{3.14}$$

By picking a convergent subsequence we get a contradiction to $w = 0$ in B_{2r_0}. Therefore, we have $0 \in \partial\{w > 0\}$. ■

3.2.3 Theorem (Measure Stability)**.** *Fix positive constants $\gamma, \lambda, \Lambda, \lambda^*$, and Λ^*, and suppose w satisfies Equation (3.2), and for some constant $\mu \in [\lambda^*, \Lambda^*]$, assume that u satisfies*

$$\Delta u = \chi_{\{u>0\}} \mu \quad in\ B_1 \tag{3.15}$$

with

$$w = u, \quad on\ \partial B_1,$$

where we assume in addition that w satisfies

$$||w||_{W^{1,2}(B_1)} \leq \gamma, \quad and \quad ||w||_{C^{\alpha}(\overline{B_1})} \leq \gamma.$$

Then there exists a modulus of continuity $\sigma(\epsilon)$, such that if

$$||a^{ij} - \delta^{ij}||_{L^2(B_1)} < \sigma(\epsilon), \quad and \quad ||f - \mu||_{L^1(B_1)} < \sigma(\epsilon) \tag{3.16}$$

then

$$|\{w = 0\}\Delta\{u = 0\}| < \epsilon. \tag{3.17}$$

(We are abusing notation slightly by using μ to denote the function which is everywhere equal to μ in B_1.)

Proof. The proof of Theorem 5.4 of[BT] can be adapted to the current setting without too much difficulty, but we include it for the convenience of the reader. Suppose not. Then there exists a_k^{ij}, w_k, f_k and u_k such that,

1. $D_i a_k^{ij} D_j w_k = \chi_{\{w_k>0\}} f_k$ in B_1,
2. $a_k^{ij} \to \delta^{ij}$ in $L^2(B_1)$,
3. $f_k \to \mu$ in $L^1(B_1)$,
4. $\begin{cases} \Delta u_k = \chi_{\{u_k>0\}}\mu & \text{in } B_1 \\ u_k = w_k & \text{on } \partial B_1, \quad \text{and} \end{cases}$
5. $||w_k||_{W^{1,2}(B_1)} \leq \gamma$, and $||w_k||_{C^{\alpha}(\overline{B_1})} \leq \gamma$.

but $|\{w_k = 0\}\Delta\{u_k = 0\}| \geq \epsilon_0$ for some ϵ_0 fixed.

By applying the previous compactness lemmas to an arbitrary subsequence, there exists a w_∞ and a sub-subsequence such that

$$w_k \rightharpoonup w_\infty, \quad \text{in} \quad W^{1,2}(B_1)$$

and

$$w_k \to w_\infty \quad \text{in} \quad C^0(\overline{B}_1)$$

which implies $w_k \to w_\infty$ in $L^2(B_1)$. (We will still use "w_k" for the sub-subsequence.) Equation (3.4) is also satisfied with the constant function μ in place of f.

By standard comparison results for the obstacle problem (see for example Theorem 2.7a of[B]), there exists u such that

$$u_k \to u \text{ in } L^\infty(B_1). \tag{3.18}$$

We have for any $\phi \in W_0^{1,2}(B_1)$,

$$\begin{aligned}\int_{B_1} a_k^{ij} D_i w_k D_j \phi = &\int_{B_1} (a_k^{ij} - \delta^{ij})(D_i w_k - D_i w_\infty) D_j \phi \\ &+ \int_{B_1} a_k^{ij} D_i w_\infty D_j \phi + \int_{B_1} \delta^{ij} (D_i w_k - D_i w_\infty) D_j \phi.\end{aligned}$$

Since $a_k^{ij} \to \delta^{ij}$ in L^2 and $D_i w_k \rightharpoonup D_i w_\infty$, we have

$$\int_{B_1} (a_k^{ij} - \delta^{ij})(D_i w_k - D_i w_\infty) D_j \phi \to 0, \tag{3.19}$$

$$\int_{B_1} a_k^{ij} D_i w_\infty D_j \phi \to \int_{B_1} \delta^{ij} D_i w_\infty D_j \phi \tag{3.20}$$

and

$$\int_{B_1} \delta^{ij} (D_i w_k - D_i w_\infty) D_j \phi \to 0. \tag{3.21}$$

Therefore,

$$\int_{B_1} a_k^{ij} D_i w_k D_j \phi \to \int_{B_1} \delta^{ij} D_i w_\infty D_j \phi. \tag{3.22}$$

By Equation (3.4) with μ in place of f, we have

$$\int_{B_1} \chi_{\{w_k>0\}} f_k \phi \to \int_{B_1} \chi_{\{w_\infty>0\}} \mu\phi, \tag{3.23}$$

so w_∞ satisfies

$$\Delta w_\infty = \chi_{\{w_\infty>0\}}\mu \ \text{ in } B_1. \tag{3.24}$$

We notice that by assumption,

$$\begin{aligned}
0 < \epsilon_0 &\leq |\{w_k = 0\}\Delta\{u_k = 0\}| \\
&= ||\chi_{\{u_k>0\}} - \chi_{\{w_k>0\}}||_{L^1(B_1)} \\
&\leq ||\chi_{\{u_k>0\}} - \chi_{\{w_\infty>0\}}||_{L^1(B_1)} + ||\chi_{\{w_\infty>0\}} - \chi_{\{w_k>0\}}||_{L^1(B_1)} \\
&= I + II \ .
\end{aligned}$$

For I, since

$$\begin{cases} \Delta u_k = \chi_{\{u_k>0\}}\mu & \text{in } \ B_1 \\ \\ u_k = w_k & \text{on } \ \partial B_1 \ . \end{cases} \tag{3.25}$$

and

$$\begin{cases} \Delta w_\infty = \chi_{\{w_\infty>0\}}\mu & \text{in } \ B_1 \\ \\ w_\infty = u & \text{on } \ \partial B_1 \ . \end{cases} \tag{3.26}$$

By Theorem 2.7a of[B], we have

$$||u_k - w_\infty||_{L^\infty(B_1)} \leq ||u_k - u||_{L^\infty(\partial B_1)} \ , \tag{3.27}$$

and since $u_k \to u$ in L^∞, we have

$$||\chi_{\{u_k>0\}} - \chi_{\{w_\infty>0\}}||_{L^1(B_1)} \to 0, \tag{3.28}$$

by Corollary 4 of[C2].

For II, we know that inside $\{w_\infty > 0\}$, w_k will eventually be positive by the uniform convergence, so $\chi_{\{u_k>0\}} = \chi_{\{w_\infty>0\}}$ there. In the interior of $\{w_\infty = 0\}$, w_k will eventually be 0, since otherwise we will violate the nondegeneracy property, and so $\chi_{\{u_k>0\}} = \chi_{\{w_\infty>0\}}$ there. Finally, since $\partial\{w_\infty = 0\}$ has finite $(n-1)$-dimensional Hausdorff measure (see[C2],[C3], or[C5]), we must have $|\partial\{w_\infty = 0\}| = 0$, and therefore $II \to 0$. This convergence to 0 gives us a contradiction, since $0 < \epsilon_0 \le I + II$. ■

3.3 Weak Regularity of the Free Boundary

In this section we establish the existence of blow up limits, and use this result to show a measure-theoretic version of Caffarelli's free boundary regularity theorem. We will show the existence of blowup limits in the case where the a^{ij} and the f belong to VMO. We define VMO to be the subspace of BMO such that if $g \in BMO$ and

$$\eta_g(r) := \sup_{\rho \le r,\, y \in \mathbb{R}^n} \frac{1}{|B_\rho|} \int_{B_\rho(y)} |g(x) - g_{B_\rho(y)}|\, dx\ , \tag{3.29}$$

then $\eta_g(r) \to 0$ as $r \to 0$. For any $g \in$ VMO, $\eta_g(r)$ is referred to as the VMO-modulus. For all conventions regarding VMO we follow[BT] which in turn follows[MPS].

3.3.1 Theorem (Existence of Blowup Limits I)**.** *Assume w satisfies Equations (3.2) and (3.3), and assume in addition that a^{ij} and f belong to VMO. Define the usual rescaling*

$$w_\epsilon(x) := \epsilon^{-2} w(\epsilon x).$$

Then for any sequence $\{\epsilon_m\} \downarrow 0$, there exists a subsequence, a real number $\mu \in [\lambda^*, \Lambda^*]$, and a symmetric matrix $A = (A^{ij})$ with

$$0 < \lambda I \leq A \leq \Lambda I$$

such that for all i, j we have

$$⨍_{B_{\epsilon_m}} a^{ij}(x)dx \rightarrow A^{ij} \tag{3.30}$$

and

$$⨍_{B_{\epsilon_m}} f(x)dx \rightarrow \mu \ , \tag{3.31}$$

and on any compact set, $w_{\epsilon_m}(x)$ converges strongly in C^α and weakly in $W^{1,2}$ to a function $w_\infty \in W^{1,2}_{loc}(\mathbb{R}^n)$, which satisfies:

$$D_i A^{ij} D_j w_\infty = \chi_{\{w_\infty > 0\}} \mu \quad \text{on } \mathbb{R}^n, \tag{3.32}$$

and has 0 in its free boundary.

Proof. This proof is so similar to the proof of Theorem 6.1 of[BT] that we leave it for an Appendix. ∎

3.3.2 Remark (Nonuniqueness of Blowup Limits)**.** Notice that the theorem does not claim that the blowup limit is unique. In fact, it is relatively easy to produce nonuniqueness even in the case with a constant right hand side, and it was done in[BT] for the nondivergence form case, but that counter-example can be copied almost exactly for the divergence form case. In the case where the coefficients of L are constant, one can use the counter-example in[B] to show nonuniqueness of blowup limits when the right hand side is only assumed to be continuous.

In fact, let $\Theta(x) = \cos(\pi \ln|\ln|x||) + 2$. In[BT] it is shown that Θ is a VMO function.

It is easy to verify that $\Theta(x) = 1$ when $|x| = \exp(-\exp(2m-1))$, and $\Theta(x) = 3$ when $|x| = \exp(-\exp(2m))$. Now define:

$$L(w) := \begin{cases} \Delta w, & \text{in } B_1/B_r \\ \Theta \Delta w, & \text{in } B_r. \end{cases}$$

Where $r < 1$ is equal to $\exp(-\exp(2m_0 - 1))$ for some large $m_0 \in \mathbb{N}$. It is not hard to show that when the scales are picked at $\epsilon = \exp(-\exp(2m-1))$, L is very "close" to Δ, and when the scales are picked at $\epsilon = \exp(-\exp(2m))$, L is very "close" to 3Δ. ("Close" of course means close in exactly the sense that we need in order to apply Theorem (3.3.1).) Therefore, as long as we can choose boundary data that gets us a regular free boundary point at the origin, we will have different blowup limits according to how we choose our sequence of ϵ's going to zero. The details of this process are carried out in[BT], but are essentially unchanged in the current setting.

3.3.3 Theorem (Caffarelli's Alternative in Measure (Weak Form)). *Assuming again Equations (3.2) and (3.3), the limit*

$$\lim_{r \downarrow 0} \frac{|\Lambda(w) \cap B_r|}{|B_r|} \tag{3.33}$$

exists and must be equal to either 0 *or* 1/2.

Proof. Here again our proof is almost identical to the proof of Theorem 6.3 of[BT], so we relegate it to an Appendix. ∎

3.3.4 Definition (Regular and Singular Free Boundary Points). A free boundary point where Λ has density equal to 0 is referred to as *singular*, and a free boundary point where the density of Λ is $1/2$ is referred to as *regular.*

The theorem above gives us the alternative, but we do not have any kind of uniformity to our convergence. Caffarelli stated his original theorem in a much more quantitative (and therefore useful) way, and so now we will state and prove a similar stronger version. We need the stronger version in order to show openness and stability under perturbation of the regular points of the free boundary.

3.3.5 Theorem (Caffarelli's Alternative in Measure (Strong Form)). *Once again assuming Equations (3.2) and (3.3), for any $\epsilon \in (0, 1/8)$, there exists an $r_0 \in (0,1)$, and a $\tau \in (0,1)$ such that*

if there exists a $t \leq r_0$ such that

$$\frac{|\Lambda(w) \cap B_t|}{|B_t|} \geq \epsilon \ , \tag{3.34}$$

then for all $r \leq \tau t$ we have

$$\frac{|\Lambda(w) \cap B_r|}{|B_r|} \geq \frac{1}{2} - \epsilon \ , \tag{3.35}$$

and in particular, 0 is a regular point according to our definition. The r_0 and the τ depend on ϵ and on the a^{ij}, but they do not depend on the function w.

3.3.6 Remark (Another version). The theorem above is equivalent to a version using a modulus of continuity. In that version there is a universal modulus of continuity σ such that

$$\frac{|\Lambda(w) \cap B_{\tilde{t}}|}{|B_{\tilde{t}}|} \geq \sigma(\tilde{t}) \tag{3.36}$$

for any $\tilde{t}$ implies a uniform convergence of the density of $\Lambda(w)$ to $1/2$ once $B_{\tilde{t}}$ is scaled to B_1. (Here we mean uniformly among all appropriate w's.)

Proof. Here again we have a proof which is almost identical to the proof of Theorem 6.5 in[BT]. On the other hand, in an effort to make things more convenient for the reader, since we use this theorem quite a bit, we will include the proof here.

We start by assuming that we have a t such that Equation (3.34) holds, and by rescaling if necessary, we can assume that $t = r_0$. Next, by arguing exactly as in the last theorem, by assuming that r_0 is sufficiently small, and by defining $s_0 := \sqrt{r_0}$, we can assume without loss of generality that

$$\fint_{B_{s_0}} \left|a^{ij}(x) - \delta^{ij}\right| \, dx \tag{3.37}$$

is as small as we like. Now we will follow the argument given for Theorem 4.5 in[B] very closely.

Applying our measure stability theorem on the ball B_{s_0} we have the existence of a function u which satisfies:

$$\begin{aligned} \Delta u = \chi_{\{u>0\}}\mu \quad &\text{in} \quad B_{s_0} \\ u \equiv w \quad &\text{on} \quad \partial B_{s_0} \ , \end{aligned} \tag{3.38}$$

and so that

$$|\{\Lambda(u)\Delta\Lambda(w)\} \cap B_{r_0}| \tag{3.39}$$

is small enough to guarantee that

$$\frac{|\Lambda(u) \cap B_{r_0}|}{|B_{r_0}|} \geq \frac{\epsilon}{2} \ , \tag{3.40}$$

and therefore

$$m.d.(\Lambda(u) \cap B_{r_0}) \geq C(n) r_0 \epsilon \ . \tag{3.41}$$

Now if r_0 is sufficiently small, then by Caffarelli's $C^{1,\alpha}$ regularity theorem for the obstacle problem (see[C4] or[C5]) we conclude that $\partial\Lambda(u)$ is $C^{1,\alpha}$ in an r_0^2 neighborhood of the origin. Furthermore, if we rotate coordinates so that $FB(u) = \{(x', x_n) \mid x_n = g(x')\}$, then we have the following bound (in $B_{r_0^2}$):

$$||g||_{C^{1,\alpha}} \leq \frac{C(n)}{r_0} \ . \tag{3.42}$$

On the other hand, because of this bound, there exists a $\gamma < 1$ such that if $\rho_0 := \gamma r_0 < r_0$, then

$$\frac{|\Lambda(u) \cap B_{\rho_0}|}{|B_{\rho_0}|} \ > \ \frac{1-\epsilon}{2} \ . \tag{3.43}$$

Now by once again requiring r_0 to be sufficiently small, we get

$$\frac{|\Lambda(w) \cap B_{\rho_0}|}{|B_{\rho_0}|} \;>\; \frac{1}{2} - \epsilon \; . \tag{3.44}$$

(So you may note that here our requirement on the size of r_0 will be much smaller than it was before; we need it small both because of the hypotheses within Caffarelli's regularity theorems and because of the need to shrink the L^p norm of $|a^{ij} - \delta^{ij}|$ and the L^1 norm of $|f - \mu|$ in order to use our measure stability theorem.)

Now since $\frac{1}{2} - \epsilon$ is strictly greater than ϵ, we can rescale B_{ρ_0} to a ball with a radius *close to* r_0, and then repeat. Since we have a little margin for error in our rescaling, after we repeat this process enough times we will have a small enough radius (which we call τr_0), to ensure that for all $r \leq \tau r_0$ we have

$$\frac{|\Lambda(w) \cap B_r|}{|B_r|} \;>\; \frac{1}{2} - \epsilon \; .$$

■

3.3.7 Corollary (The Set of Regular Points Is Open)**.** *Still assuming Equations (3.2) and (3.3), the set of regular points of $FB(w)$ is an open subset of $FB(w)$.*

Proof. The proof of this corollary is identical to the proof of Corollary 4.8 in[B] except that in place of using Theorem 4.5 of[B] we use Theorem (3.3.5) from this work:

Take r_0 and τ as in Theorem 2.4.5. By changing coordinates and rescaling, we can assume that 0 is a regular point of $FB(w)$. Since 0 is regular, there exists an $s \leq r_0$ such that

$$\frac{|\Lambda(w) \cap B_s|}{|B_s|} > \epsilon, \tag{3.45}$$

Now we know that if $r \leq \tau s$, then

$$\frac{|\Lambda(w) \cap B_r|}{|B_r|} \geq \frac{1}{2} - \epsilon. \tag{3.46}$$

Again, since $1/2 - \epsilon > \epsilon$, we have some margin for error. If $\gamma := ||x_0||$ is sufficiently small, and $x \in FB(w)$, then

$$\frac{|\Lambda(w) \cap B_{\tau s - \gamma}(x_0)|}{|B_{\tau s - \gamma}(x_0)|} > \epsilon, \tag{3.47}$$

and therefore for any $r \leq \tau(\tau s - \gamma)$ we have

$$\frac{|\Lambda(w) \cap B_r(x_0)|}{|B_r(x_0)|} \geq \frac{1}{2} - \epsilon, \tag{3.48}$$

Thus, x_0 is a regular point. ∎

3.3.8 Theorem (Existence of Blowup Limits II). *We assume Equation (3.2), and we assume a^{ij} and f belong to VMO. We let*

$$S_r := \{x \in FB(w) \cap B_r \ : \ x \textit{ is a regular point of } FB(w) \ \} \tag{3.49}$$

and we assume $S_{1/2} \neq \phi$. Let $K \subset\subset S_{1/2}$, let $\{x_m\} \subset K$, and let $\epsilon_m \downarrow 0$.

Then there exists a constant $\mu \in [\lambda^*, \Lambda^*]$, a constant symmetric matrix $A = (A^{ij})$ with $0 < \lambda \leq A \leq \Lambda$, and a strictly increasing sequence of natural numbers $\{m_j\}$ such that the sequence of functions $\{w_j\}$ defined by

$$w_j(x) := \epsilon_{m_j}^{-2} w(x_{m_j} + \epsilon_{m_j} x) \tag{3.50}$$

converges strongly in C^α (for some $\alpha > 0$) and weakly in $W^{1,2}$ on any compact set to a *function* w_∞ *which satisfies:*

$$D_i A^{ij} D_j w_\infty = \chi_{\{u_\infty > 0\}} \mu \quad on\ \mathbb{R}^n \ . \tag{3.51}$$

Furthermore 0 *is a regular point of its free boundary.*

Proof. The existence of a function $w_\infty \geq 0$ satisfying Equation (3.51) and the convergence of the w_j to w_∞ is carried out in exactly the same way as in the proof of Theorem (3.3.1) . Showing that 0 is part of the free boundary of w_∞ is also proven exactly as in Theorem (3.3.1). It remains to show that 0 is a regular point of the free boundary.

For the first part, we observe that since each x_m belongs to the regular part of the free boundary, we know that there exists an r_m such that

$$\frac{\Lambda(w) \cap B_{r_m}(x_m)}{B_{r_m}} \geq \frac{3}{8} \ . \tag{3.52}$$

There exists a small $\rho > 0$ depending only on the dimension, n, such that if $x \in B_{\rho r_m}(x_m)$, then

$$\frac{\Lambda(w) \cap B_{r_m}(x)}{B_{r_m}} \geq \frac{1}{4} \ . \tag{3.53}$$

Now the closure of the set $\{x_m\}$ is compact, and that set is covered by the open balls in the set $\{B_{\rho r_m}(x)\}$. By compactness, the set is still covered by a finite number of these balls, and their radii have a positive minimum, ρ_0. So, once $\epsilon_{m_j} < \rho_0$, we know that

$$\frac{\Lambda(w_j) \cap B_r}{B_r} \geq \frac{1}{4} \ , \tag{3.54}$$

for all r which are less than τ times ρ_0. Here τ is the constant given in the statement of Theorem (3.3.5). From this we can conclude that 0 must be a regular point of $FB(w_\infty)$. ■

3.3.9 Remark (Hausdorff Dimension)**.** Exactly as in[BT], the arguments above lead to the statement that the free boundary is strongly porous and therefore has Hausdorff dimension strictly less than n. (See[BT] and see[M] for the definition of porosity.)

3.4 Finer Regularity of the Free Boundary

In this section we show finer properties of the free boundary at regular points. Since the counter-examples in[B] and in[BT] are easily extended to the current setting, we can have regular free boundary points where the blowup limit is not unique. In spite of this fact, we show that the regular free boundary points enjoy a flatness property which is based on Reifenberg flatness. Reifenberg flatness was introduced by Reifenberg in[R], and is studied in more detail by Toro and Kenig in several papers. (See[KT1] and[KT2] for example.) For the definitions surrounding Reifenberg vanishing sets we follow the conventions in section 6 of[B], but now we must introduce a notion of sets which are "relatively Reifenberg flat."

3.4.1 Definition (Reifenberg Flatness)**.** Let $S \subset \mathbb{R}^n$ be a locally compact set, and let $\delta > 0$. Then S is $\delta-$*Reifenberg flat* if for each compact $K \subset \mathbb{R}^n$, there exists a constant $R_K > 0$ such that for every $x \in K \cap S$ and every $r \in (0, R_K]$ we have a hyperplane $L(x,r)$ containing x such that

$$D_{\mathcal{H}}(L(x,r) \cap B_r(x),\ S \cap B_r(x)) \leq 2r\delta\ . \tag{3.1}$$

Here $D_{\mathcal{H}}$ denotes the Hausdorff distance: If $A,\ B \subset \mathbb{R}^n$, then

$$D_{\mathcal{H}}(A,B) := \max\{\ \sup_{a \in A} d(a,B)\ ,\ \sup_{b \in B} d(b,A)\ \}\ . \tag{3.2}$$

We also define the following quantity, which we call the *modulus of flatness,* to get a more quantitative and uniform measure of flatness:

$$\theta_K(r) := \sup_{0<\rho\leq r} \left(\sup_{x \in S \cap K} \frac{D_{\mathcal{H}}(L(x,\rho) \cap B_\rho(x),\ S \cap B_\rho(x))}{\rho} \right)\ . \tag{3.3}$$

Finally, we will say that S is a *Reifenberg vanishing* set, if for any compact $K \subset S$

$$\lim_{r\to 0} \theta_K(r) = 0 \; . \tag{3.4}$$

3.4.2 Definition (Relatively Reifenberg Flat)**.** Let $S \subset \mathbb{R}^n$ be a locally compact set, let $K \subset\subset S$, and let $\delta > 0$. Then K is *relatively δ–Reifenberg flat with respect to S* if there exists a constant $R > 0$ such that for every $x \in K$ and every $r \in (0, R]$ we have a hyperplane $L(x, r)$ containing x such that

$$D_{\mathcal{H}}(L(x,r) \cap B_r(x),\; S \cap B_r(x)) \leq 2r\delta \; . \tag{3.5}$$

We also define the *modulus of flatness,* exactly as above, and then K is *relatively Reifenberg vanishing* if the modulus of flatness goes to zero as r approaches 0.

3.4.3 Remark. It is worth noting that the compact set K, plays a very different role in the two definitions above. In the first case, K allows us to look at bounded sets to get uniform bounds on the constant R_K which bounds the radius, while in the second case, K *is* the set that we want to show is Reifenberg vanishing, but we are allowing all of S when seeing if we are close to a plane. As a simple example, a point can never be Reifenberg flat, but viewed as a subset of a plane, it is relatively δ-Reifenberg flat.

First we need to show that our measure stability theorem can be used to show uniform closeness of our solutions to solutions of obstacle problems with constant coefficients and constant right hand side, as long as we have zoomed in far enough. In particular, we can say the following:

3.4.4 Theorem (Uniform Closeness Result)**.** *We assume Equation (3.2), and we let $u \geq 0$ satisfy:*

$$\begin{aligned} \Delta u &= \chi_{\{u>0\}}\mu \quad \textit{in} \;\; B_1 \\ u &\equiv w \quad \textit{on} \;\; \partial B_1 \; . \end{aligned} \tag{3.6}$$

We also assume that there is a fixed constant β, *and an* $\alpha \in (0,1)$ *such that* $||w||_{C^{\alpha}(\overline{B}_1)} \leq \beta$. *For any* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that if*

$$||a^{ij}(x) - \delta^{ij}||_{L^1(B_1)} < \delta \quad \textit{and} \quad ||f(x) - \mu||_{L^1(B_1)} < \delta \ , \tag{3.7}$$

then

$$||w - u||_{L^{\infty}(B_{3/4})} < \epsilon \ . \tag{3.8}$$

Proof. Some of the ideas in this proof were inspired by ideas of Li and Vogelius who in turn were following ideas of Caffarelli. (See[LV] and[C3].) Letting $A(x)$ be the matrix determined by $a^{ij}(x)$, we have in B_1 (using "divergence" notation):

$$\begin{aligned}
&\mathrm{div}\left[A(x)\left(\nabla[w(x) - u(x)]\right)\right] \\
&\quad = f(x)\chi_{\{w>0\}} - \mathrm{div}\left[(A(x) - I)\,\nabla u(x)\right] - \Delta u \\
&\quad = f(x)\chi_{\{w>0\}} - \mu\chi_{\{u>0\}} - \mathrm{div}\left[(A(x) - I)\,\nabla u(x)\right] \\
&\quad = f(x)\left(\chi_{\{w>0\}} - \chi_{\{u>0\}}\right) + \chi_{\{u>0\}}\left(f(x) - \mu\right) + \mathrm{div}\left[(I - A(x))\,\nabla u(x)\right] \\
&\quad = I + II + \mathrm{div}\left[III\right] \ .
\end{aligned}$$

After fixing $q \in (n, \infty)$, and by shrinking δ if necessary, we can use our measure stability theorem (Theorem (3.2.3)) and a simple interpolation, to ensure that the $L^{q/2}$ norm of I on B_1 is as small as we like. Using our assumptions and shrinking δ if necessary, we can make the $L^{q/2}$ norm of II on B_1 as small as we like. (The fourth line of Equation (3.2) supplies the L^{∞} bound needed for the interpolation.)

To control III we need to shrink the ball slightly. First we observe that by De Giorgi-Nash-Moser theory (see Theorem 8.29 of[GT]), there exists an $\alpha' \in (0, \alpha)$ such that

$$||u||_{C^{\alpha'}(\overline{B}_1)} \leq C(\beta, \Lambda^*) \ . \tag{3.9}$$

For any fixed $s \in (0, 1/16)$ we then have

$$||w - u||_{L^\infty(\partial B_{1-s})} \le C(\beta, \Lambda^*) s^{\alpha'} \ . \tag{3.10}$$

For any $\tilde{q} < \infty$, we can use Calderón-Zygmund Theory to show $||u||_{W^{2,\tilde{q}}(B_{1-s})} \le C$, and then by the Sobolev Imbedding Theorem we know $||u||_{W^{1,\alpha}(\overline{B_{1-s}})} \le C$, and so finally

$$||\nabla u||_{L^\infty(B_{1-s})} \le C(\beta, \Lambda, s) \ . \tag{3.11}$$

Considering the boundary value problem that $w - u$ satisfies within B_{1-s}, we have the following: By shrinking s we can make the boundary values as small as we like by Equation (3.10). We already have the $L^{q/2}$ norm of I and II as small as we like by making δ small. For III we can use Equation (3.11) to ensure that $||\nabla u||_{L^\infty(B_{1-s})}$ is under control, and then shrink δ if necessary to ensure that $||A - I||_{L^q(B_1)}$ is as small as we like. Applying Theorem 8.16 of[GT] yields the desired result. ∎

Now we have a standard corollary for obstacle type problems.

3.4.5 Corollary (Free Boundaries Are Close)**.** *Assuming Equation (3.2) again, assuming u is defined as in the previous theorem, and using $D_{\mathcal{H}}$ as the Hausdorff distance between sets defined at the beginning of this section, there exists a universal constant C such that*

$$D_{\mathcal{H}}(FB(w), FB(u)) \le C\sqrt{\epsilon} \tag{3.12}$$

where ϵ is the number given in Equation (3.8).

Proof. This result is a simple application of the nondegeneracy enjoyed by each function. Indeed, if there is a point x where one function is positive and a ball $B_r(x)$ where the other function is zero, then nondegeneracy implies that the max of the first function is Cr^2 on $\partial B_r(x)$ and this must be smaller than ϵ. ∎

Now we prove the main theorem of this chapter.

3.4.6 Theorem (Free Boundary Regularity)**.** *Once again we assume Equation (3.2) and we assume that a^{ij} and f belong to VMO. As in Equation (3.49) we define S_r to be the set of regular points of the free boundary within B_r. Let $K \subset\subset S_{1/2}$. Then K is relatively Reifenberg vanishing with respect to $S_{1/2}$.*

Proof. Fix$\epsilon > 0$. We will demonstrate that there is a radius $\tilde{r} > 0$ such that for any $x \in K$, and any positive $r < \tilde{r}$ there is a hyperplane $H(r, x)$ such that

$$D_{\mathcal{H}}(FB(w) \cap B_r(x), H(r, x) \cap B_r(x)) \leq r\epsilon \ . \tag{3.13}$$

We start by using the compactness of K in almost the same way as in Theorem (3.3.8). Namely, we know that for every $x \in K$ there exists an r_x such that

$$\frac{|\Lambda(w) \cap B_{r_x}(x)|}{|B_{r_x}|} \geq \frac{49}{100} \ . \tag{3.14}$$

Next, there exists a small $\rho > 0$ depending only on the dimension, n, such that if $y \in B_{\rho r_x}(x) \cap FB(w)$, then

$$\frac{|\Lambda(w) \cap B_{r_x}(y)|}{|B_{r_x}|} \geq \frac{48}{100} \ . \tag{3.15}$$

Now K is compact, and is therefore covered by the open balls in the set $\{B_{\rho r_x}(x)\}$. By compactness, the set is still covered by a finite number of these balls, and their radii have a positive minimum, ρ_0. Using Theorem (3.3.5) guarantees that for all $r < \tau\rho_0$, and for all $x \in K$, we have

$$\frac{|\Lambda(w) \cap B_r(x)|}{|B_r|} \geq \frac{48}{100} \ . \tag{3.16}$$

Here τ is the constant given in the statement of Theorem (3.3.5). Henceforth, the argument becomes completely independent of whatever point in the free boundary that we wish to consider, so we can fix $x_0 \in FB(w)$, and show flatness at that point. Also, given the VMO-modulus η, we can be sure that every quantity that we wish to control below can be shrunk in a uniform and universal way by shrinking the radius that we are considering.

We consider the situation in $B_r(x_0)$ and after a linear invertible change of coordinates with eigenvalues bounded away from 0 and ∞ in a uniform way depending only on ellipticity, we can assume that the averages of a^{ij} are δ^{ij} and the average of f is μ. Then we let u solve the boundary value problem:

$$\begin{aligned} \Delta u = \mu \chi_{\{u>0\}} \quad &\text{in} \quad B_r(x_0) \\ u = w \quad &\text{on} \quad \partial B_r(x_0)\ . \end{aligned} \tag{3.17}$$

By the L^1 closeness of a^{ij} to δ^{ij} and f to μ which are controlled by the VMO-modulus along with our measure stability theorem (Theorem (3.2.3)), we can guarantee (by assuming r_1 is sufficiently small) that

$$\frac{|\Lambda(v) \cap B_{r_1}(x_0)|}{|B_{r_1}|} \geq \frac{47}{100}\ . \tag{3.18}$$

Now it follows from Caffarelli's free boundary regularity theorem (see Theorem 7 of[C4] or[C5]) that if $r_2 \leq \tau_2 r_1$ where τ_2 is suitably small, then $FB(v) \cap B_{r_2}(x_0)$ is uniformly $C^{1,\alpha}$ in $B_{r_2}(x_0)$. We can also assume that $FB(v)$ has a free boundary point as close to x_0 as we like by using the last corollary (and shrinking r_1 again if needed). Now zooming in on a uniformly $C^{1,\alpha}$ set will flatten it in a uniform way depending only on how much one zooms, so after zooming in to $r_3 := \tau_3 r_2$, where τ_3 will only depend on estimating how uniformly $C^{1,\alpha}$ functions flatten out as you zoom in, so we can have $FB(v) \cap B_{r_3}(x_0)$ within $r_3 \cdot \epsilon/2$ of a plane. Now we invoke Corollary (3.4.5) again to guarantee that $FB(w)$ is within $r_3 \cdot \epsilon/2$ of $FB(v)$ and we are done. ■

3.4.7 Remark (Choosing r)**.** It is worth remarking that the r_j that work for all of the estimates in the last proof must be found *before* finding the function u, and then in Equation

(3.17) we can use $r = r_3$.

3.4.8 Remark (Nondivergence Form Case)**.** The Theorem above (and the next corollary) can be extended without any difficulty to the nondivergence form setting. On the other hand, in the nondivergence form setting, since the functions will have stronger convergence to their blowup limits, it is very likely that the Weiss-type Monotonicity formula can be used to give an easier proof. In the divergence form case, the presence of the Dirichlet integral within the Weiss-type monotonicity functional coupled with the weak convergence in $W^{1,2}$ to the blowup limit makes it difficult to move back and forth from the original function to its blowup limit.

3.4.9 Corollary (Blowup Classification)**.** *Any blowup found in Theorem (3.3.8) must be homogeneous of degree two, and therefore in the right coordinate system, it willl be a constant times* $(x_n^+)^2$.

Proof. By Theorem (3.4.6) any blowup found in Theorem (3.3.8) will have to be a global solution to the obstacle problem with a free boundary which is a hyperplane. Then by applying a combination of the Cauchy-Kowalevski theorem and Holmgren's uniqueness theorem we conclude (after a possible rotation and change of coordinates) that the blowup limit is $C(x_n^+)^2$. ■

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Appendix A

Proof of Theorem 3.3.1

Proof. We define the matrix

$$A_r^{ij} := \fint_{B_r} a^{ij}(x)dx, \tag{A.1}$$

then this matrix also satisfies the elliptic setting. We can take a subsequence of the radii ϵ_n such that each scalar $A_{\epsilon_n}^{ij}$ converges to a real number A^{ij}. With this subsequence, we also know:

$$\fint_{B_{\epsilon_n}} |a^{ij}(x) - A_{\epsilon_n}^{ij}|dx \leq \eta(\epsilon_n) \to 0, \tag{A.2}$$

where η is just taken to be the maximum of all of the VMO moduli for each of the a^{ij}s and by the triangle inequality this lead to

$$\fint_{B_{\epsilon_n}} |a^{ij}(x) - A^{ij}|dx \to 0. \tag{A.3}$$

Now we observe that if $a^{ij,n}(x) := a^{ij}(\epsilon_n x)$ then the rescaled function $w_n := w_{\epsilon_n}$ satifies the equation:

$$\fint_{B_1} |a^{ij,n}(x) - A^{ij}|dx \leq \eta(\epsilon_n) \to 0. \tag{A.4}$$

So Lemma 3.2.1 gives us exactly what we need. ∎

Appendix B

Proof of Theorem 3.3.3

Proof. We will suppose that

$$\limsup_{r\downarrow 0} \frac{|\Lambda(w)\cap B_r|}{|B_r|} > 0 \tag{B.1}$$

and show that in this case the limit exists and is equal to 1/2. It follows immediately from this assumption that there exists a sequence $\{\epsilon_n\} \to 0$ such that(for some $\delta > 0$) we have

$$\frac{|\Lambda(w_{\epsilon_n})\cap B_1|}{|B_1|} > \delta \tag{B.2}$$

for all n. We can extract a subsequence, and guarantee the existence of a symmetric positive definite matrix A^{ij} which satified the elliptic setting., and a $w_\infty \in W^{2,p}_{loc}(\mathbb{R}^n)$, such that if $a^{ij,n}(x) := a^{ij}(\epsilon_n, x)$, then

$$⨍_{B_{\epsilon_m}} a^{ij}(x)dx \to A^{ij}, \tag{B.3}$$

$$⨍_{B_{\epsilon_m}} f(x)dx \to \mu\ , \tag{B.4}$$

and

$$D_i A^{ij} D_j w_\infty = \chi_{\{w_\infty>0\}}\mu \ \text{ on } \mathbb{R}^n \tag{B.5}$$

and 0 is in $FB(w_\infty)$. Furthermore, we will have w_n converging to w_∞ in both $W^{2,p}$ and $C^{1,\alpha}$ on every compact set.

Now we diagonalize the matrix A^{ij}, and then we dilate the individual coordinates so that in the new coordinate system we have $A^{ij} = \delta^{ij}$.

Now we let u_n denote the solution to

$$\begin{cases} \Delta u_n = \chi_{\{u_n>0\}}\mu & \text{in } \; B_1 \\ \\ u_n = w_n & \text{on } \; \partial B_1 \; . \end{cases} \tag{B.6}$$

Applying our measure stability to u_n and w_n we can make $|\Lambda(u_n)\Delta\Lambda(w_n)|$ as small as we like for n sufficiently large. In particular, we now have:

$$\frac{|\Lambda(w_n) \cap B_1|}{|B_1|} > \frac{\delta}{2} \tag{B.7}$$

Since w_n converges uniformly to w_∞ on every compact set, it follows that u_n converges uniformly to w_∞ on ∂B_1, and now we have

$$\frac{|\Lambda(w_\infty) \cap B_1|}{|B_1|} > \frac{\delta}{2} \tag{B.8}$$

We can invoke the $C^{1,\alpha}$ at regular points to guarantee that w_∞ at the origin, and this implies that

$$\lim_{r\downarrow 0} \frac{|\Lambda(w_\infty) \cap B_r|}{|B_r|} > \frac{1}{2}. \tag{B.9}$$

Now it remains to do two things. First we need to pass this result from w_∞ back to our subsequence of radii for w, but second we will need to show that we get the same limit along any sequence of radii converging to zero. The first step is a consequence of combining our measure stability with Corollary 4 of [C2]. Indead, for any $r > 0$,

$$\lim_{n\to\infty} \left(\frac{|\Lambda(w_n) \cap B_r|}{|B_r|} - \frac{|\Lambda(w_\infty) \cap B_r|}{|B_r|} \right) = 0. \tag{B.10}$$

On the other hand, by our rescaling, this equation becomes

$$\lim_{n\to\infty} \left(\frac{|\Lambda(w) \cap B_{(r\epsilon_n)}|}{|B_{(r\epsilon_n)}|} - \frac{|\Lambda(w_\infty) \cap B_r|}{|B_r|} \right) = 0, \tag{B.11}$$

which implies that

$$\lim_{n\to\infty} \frac{|\Lambda(w) \cap B_{(r\epsilon_n)}|}{|B_{(r\epsilon_n)}|} = \frac{1}{2}. \tag{B.12}$$

Finally, we wish to be able to replace $r\epsilon_n$ with r in previous equation. Suppose that we have a different sequence of radii converging to zero(which we call s_k) such that

$$\frac{|\Lambda(w) \cap B_{s_k}|}{|B_{s_k}|} \neq \frac{1}{2} \tag{B.13}$$

At this point we are led to a contradiction in one of two ways. If the limit above does not equal zero, then we can get convergence to a global solution with properties which contradict the Caffarelli Alternative. On the other hand, if the limit does equal zero, then we use the continuity of the function:

$$g(r) := \frac{|\Lambda(w) \cap B_r|}{|B_r|} \tag{B.14}$$

to get an interlacing sequence of radii which we can call $\tilde{s_k}$ and which converge to zero such that $g(s_k) = 1/4$, and then we proceed as in the first case. ■