NATURAL FREQUENCIES OF ISOTROPIC CIRCULAR PLATES OF VARIABLE THICKNESS

by 6408

BABGAUNDA A. PATIL

B.E., University of Bombay, India, 1968

A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1971

Approved by:

Hugh S. Walker
Major Professor

THIS BOOK CONTAINS NUMEROUS PAGES WITH THE ORIGINAL PRINTING BEING SKEWED DIFFERENTLY FROM THE TOP OF THE PAGE TO THE BOTTOM.

THIS IS AS RECEIVED FROM THE CUSTOMER.

TABLE OF CONTENTS

LIST OF SYMBOLS

a, b	outer and inner radius of plate, respectively
a_o, b_o, c_o, d_o	undetermined constants in deflection solution
a_k, b_k	coefficients of Frobenius series for specific exponents of singularity
A_k, B_k	$= a_k/a_o$, b_k/b_o
a_{ij}	elastic constants
c	$= (a_{11}/a_{22})^{1/2}$
D	= flexural rigidity of the plate
	$= E_r h^3/12$
D_o	$= E_r h_o{}^3/12$
E_r	= material constant
	$= a_{22}/(a_{11}a_{22} - a_{12}^2)$
f	frequency, cycles/sec.
$\overline{f}$	= frequency parameter
	$= \omega a^2\sqrt{\rho h_o/D_o}$
g_k	coefficients of Frobenius series in general form
h	plate thickness
h_o, h_a	thickness of the plate at $r = o$, $r = a$ respectively
H	$= 1-(h_a/h_o)$, non-dimensional thickness coefficient
j	exponents of singularity in Frobenius series
k	index of Frobenius series
M_r, M_θ, $M_{r\theta}$	radial, circumferential and twisting moments per unit length
n	exponents of thickness expression
$p(r, t)$	applied load

r	radial coordinate
R	$= r/a$, non-dimensional radial coordinate
S	plate area
t	time
w	lateral displacement
X, Y, Z	expressions defined by Eqs. (24), (25), (26), respectively
z	coordinate normal to and measured from the median surface of the plate
δ	step function notation
ε_r, ε_θ	radial and circumferential normal strains
ν	$= -a_{12}/a_{22}$
λ	$= \rho\omega^2 h_o a^4/D_o$, a frequency parameter
μ	mass density/unit area
ρ	mass density/unit volume
σ_r, σ_θ	radial and circumferential bending stresses
$T_{r\theta}$	shearing stress
θ	circumferential coordinate
ω	circular frequency, rad./sec.
D_r, w_{rrr}, η_r	$= \partial D/\partial r$, $\partial^3 w/\partial r^3$, $\partial\eta/\partial r$, respectively
$(\ )'$	$= d(\ )/dR$

1. INTRODUCTION

The literature of recent years contains many analyses of isotropic and orthotropic circular plates of both uniform and variable thickness. The axisymmetric flexural vibrations of a cylindrically aelotropic circular plate has been analysed by Ki. S. Joung [5].* In this investigation clamped as well as simply supported plates of constant and variable thickness were considered. The present analysis is the extension of this work and concerns itself with the investigation of the natural frequencies of isotropic circular plates of variable thickness.

The derivation of the differential equation of motion and the boundary conditions of the axisymmetric flexural vibration of a cylindrically aelotropic circular plate is based upon the variational calculas. The advantage of using the variational calculas is that in addition to differential equations appropriate boundary conditions are obtained at the same time.

In what follows, the governing equation for the free vibration of such plates is derived in terms of the lateral deflections of their median surface. This equation is then solved by the method of Frobenius. The characteristic equations for the natural frequencies are derived for a clamped and a simply supported plate, and some numerical examples are considered. Finally, a brief discussion of the results is presented.

---

*Numbers in brackets designate references at the end of report.

2. DERIVATION OF GOVERNING EQUATION

First, it is assumed that the circular plates analysed in the present investigation are governed by the small deflection theory of plates. That is, the following simplifying assumptions are made:

(1) The maximum thickness of the plate is small in comparison with the radius of plate.

(2) The magnitude of lateral deflection is small in comparison with the local thickness.

(3) The rotations are very small compared to the strains and as a result the stretching of the median surface of the plate is considered negligible.

(4) An element of the plate along a normal to the median surface in the undeformed plate remains straight and normal to the deformed median surface and its extension is negligible.

The above assumptions lead to the strain displacement relations

$$\begin{aligned} \varepsilon_r &= -zw_{rr} \\ \varepsilon_\theta &= -z(r^{-1}w_r + r^{-2}w_{\theta\theta}) \\ \gamma_{r\theta} &= -2z(r^{-1}w_\theta)_r \end{aligned} \qquad (1)$$

in polar coordinates. In the above equation, ε_r and ε_θ are the radial and circumferential normal strains, $\gamma_{r\theta}$ is the shearing strain, w is the lateral displacement of the median surface of the plate, r and θ are the radial and circumferential coordinates, z is the coordinate normal to and measured from the median surface of the plate (see Fig. 1) and w_r, w_{rr}, w_θ = $\frac{\partial w}{\partial r}$, $\frac{\partial^2 w}{\partial r^2}$, $\frac{\partial w}{\partial \theta}$ respectively.

In view of the fact that the small deflection theory of the plate is

THIS BOOK CONTAINS NUMEROUS PAGES WITH DIAGRAMS THAT ARE CROOKED COMPARED TO THE REST OF THE INFORMATION ON THE PAGE.

THIS IS AS RECEIVED FROM CUSTOMER.

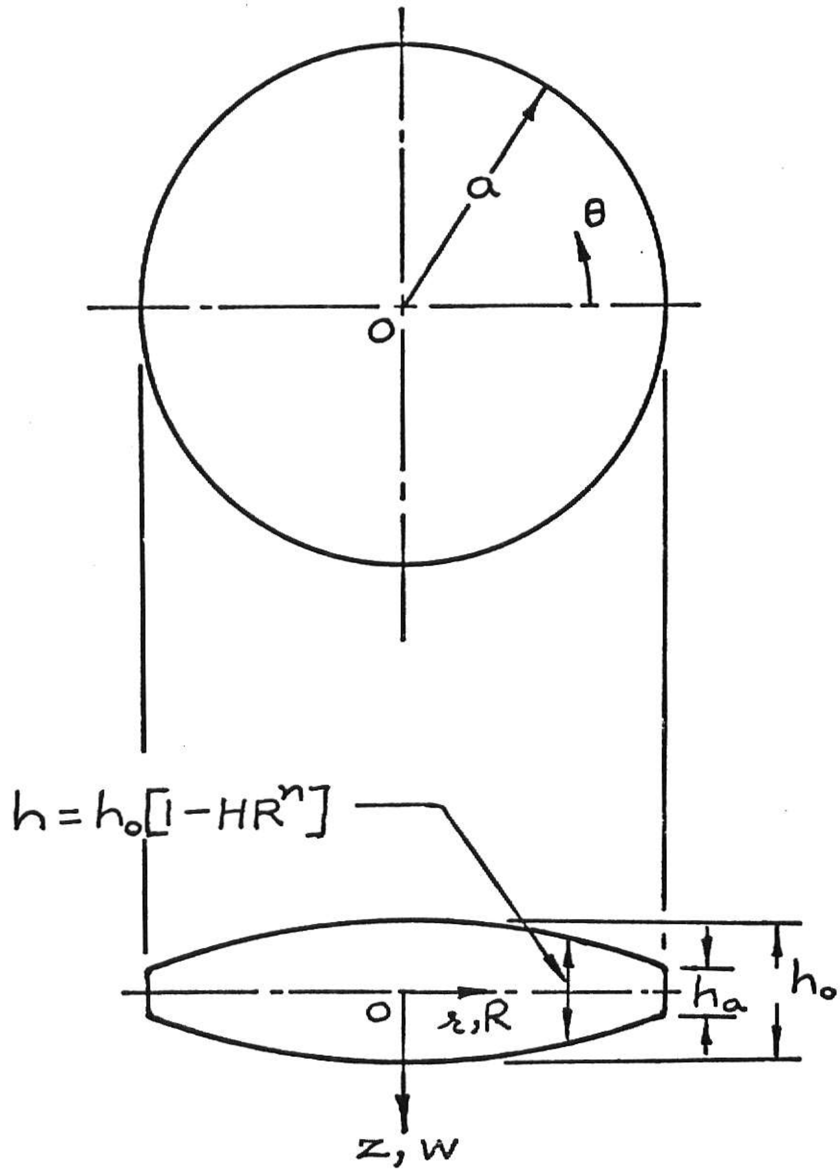

Fig. 1. Plate geometry.

assumed to be true, the pertinant stress strain relations for such plate are [6]

$$\varepsilon_\theta = a_{11}\sigma_\theta + a_{12}\sigma_r$$
$$\varepsilon_r = a_{12}\sigma_\theta + a_{22}\sigma_r \qquad (2)$$
$$\gamma_{r\theta} = a_{66}\tau_{r\theta}$$

where σ_r and σ_θ are normal stresses in the radial and circumferential directions, $\tau_{r\theta}$ is the shearing stress, and a_{ij} are the elastic constants. These stress-strain relations together with the strain displacement relations of Eq. (1) may now be used to derive expressions for the bending moments per unit length in terms of the deflection as

$$M_r = \int_{-h/2}^{h/2} \sigma_r z dz = -D[c^2 w_{rr} + \nu(r^{-1}wr + r^{-2}w_{\theta\theta})]$$
$$M_\theta = \int_{-h/2}^{h/2} \sigma_\theta z dz = -D[\nu w_{rr} + (r^{-1}w_r + r^{-2}w_{\theta\theta})] \qquad (3)$$
$$M_{r\theta} = -\int_{-h/2}^{h/2} \tau_{r\theta} z dz = \frac{h^3}{6a_{66}} (r^{-1}w_\theta)_r$$

where, M_r, M_θ and $M_{r\theta}$ are the radial, circumferential and twisting moments per unit length, $h = h(r, \theta)$ is the local thickness of the plate,

$D = \dfrac{a_{22}h^3}{12(a_{11}a_{22} - a_{12}^2)}$, is the flexural rigidity of the plate,

$c^2 = a_{11}/a_{22}$ and $\nu = -a_{12}/a_{22}$.

For axisymmetric case above equations are reduced to

$$M_r = -D(c^2 w_{rr} + r^{-1} \nu w_r)$$
$$M_\theta = -D(\nu w_{rr} + r^{-1} w_r) \qquad (4)$$

The total internal energy due to bending can be written [8] as

$$U_b = \frac{1}{2} \int_s \int D[(cw_{rr} + r^{-1} w_r)^2 - 2(c - \nu) r^{-1} w_{rr} w_r] r dr d\theta \qquad (5)$$

According to Hamiltons principle one must require that $\delta I = 0$, where

$$I = \int_{t_1}^{t_2} (T - U_b - \Omega) dt \qquad (6)$$

In Eq. (6) t_1 and t_2 are two instants of time, Ω the potential energy of the axisymmetric time varying transverse load $p(r, t)$ given by

$$\Omega = -\int_s \int p(r, t) w r dr d\theta \qquad (7)$$

and T the total kinetic energy of the vibrating plate given by

$$T = \frac{1}{2} \int_s \int \mu(r) \dot{w}^2 r dr d\theta \qquad (8)$$

where $\mu(r)$ is the mass density per unit area.

Substituting Eqs. (5), (7), (8) into Eq. (6) we get

$$I = \int_{t_1}^{t_2} \int_s \int f(t, r, w, w_r, w_{rr}, w_t) r dr d\theta dt \qquad (9)$$

where

$$f = \frac{1}{2}\{\mu \dot{w}^2 - D[(cw_{rr} + r^{-1} w_r)^2 - 2(c - \nu) r^{-1} w_{rr} w_r] + 2p(r, t)\omega\} \qquad (10)$$

Now we apply variational technique. Let the function w receive an admissible variation $\varepsilon\eta(r, t)$, where ε is any arbitrary constant and η is an arbitrary function. Then by Taylor's theorem 1st variation δI can be written as [8]

$$\delta I = \varepsilon \int_{t_1}^{t_2} \iint_s \left[\frac{\partial f}{\partial w} - \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial w_r}\right) - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial w_t}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\partial f}{\partial w_{rr}}\right)\right] \eta\, r dr d\theta dt$$

$$+ \varepsilon \oint_c \left[\frac{\partial f}{\partial w_r} - \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial w_{rr}}\right)\right] \eta\, r dt + \varepsilon \oint_c \left(\frac{\partial f}{\partial w_{rr}}\right) \eta_r\, r dt + \varepsilon \oint_c \left(\frac{\partial f}{\partial w_t}\right) \eta\, r dr \tag{11}$$

where c denotes the bounding curve of the region s. The integral must vanish independently of the line integrals, since δI vanishes for all admissible $\eta(r, t)$. This condition yields the Euler differential equation. Hence the Euler equation is

$$\frac{\partial f}{\partial w} - \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial w_r}\right) - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial w_t}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\partial f}{\partial w_{rr}}\right) = 0 \tag{12}$$

The natural boundary conditions follow immediately from the relations

$$\eta\left[\frac{\partial f}{\partial w_r} - \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial w_{rr}}\right)\right] = 0 \quad \text{on edge} \quad r = \text{const.} \tag{13}$$

$$\eta_r \left(\frac{\partial f}{\partial w_{rr}}\right) = 0 \quad \text{on edge} \quad r = \text{const.}$$

Substituting Eq. (10) in Eqs. (12) and (13) we get the governing differential equation of motion and the boundary conditions as follows

$$D[c^2 w_{rrrr} + 2c^2 r^{-1} w_{rrr} - r^{-2} w_{rr} + r^{-3} w_r]$$

$$+ D_r[2c^2 w_{rrr} + (2c^2 + \nu) r^{-1} w_{rr} - r^{-2} w_r]$$

$$+ D_{rr}[c^2 w_{rr} + \nu r^{-1} w_r] + \mu w_{tt} = p(r, t) \tag{14}$$

$$\{[rD(c^2 w_{rrr} + c^2 r^{-1} w_{rr} - r^{-2} w_r) + rD_r(c^2 w_{rr} + \nu r^{-1} w_r)]\eta\}_b^a = 0$$

$$\{rD[c^2 w_{rr} + \nu r^{-1} w_r]\eta_r\}_b^a = 0 \tag{15}$$

where a and b are the outer and inner radii respectively.

For the case in consideration, i.e., an isotropic plate with no hole, $c^2 = 1$, $b = 0$, where $c^2 = (a_{11}/a_{22})$ and b is the inner radius. Substituting these values in Eq. (14) and (15) we get the governing differential equation as

$$D[w_{rrrr} + 2r^{-1} w_{rrr} - r^{-2} w_{rr} + r^{-3} w_r]$$
$$+ D_r[2w_{rrr} + (2 + \nu)r^{-1} w_{rr} - r^{-2} w_r] \tag{16}$$
$$+ D_{rr}[w_{rr} + \nu r^{-1} w_r] + \mu \ddot{w} = p(r, t)$$

and the boundary conditions for a clamped plate

$$w = 0 \quad \text{and} \quad w_r = 0 \tag{17a}$$

or for a simply supported plate

$$w = 0 \quad \text{and} \quad w_{rr} + \nu r^{-1} w_r = 0 \tag{17b}$$

and

$$\{D[w_{rrr} + r^{-1} w_{rr} - r^{-2} w_r] + D_r[w_{rr} + \nu r^{-1} w_r]\} = 0 \quad \text{at} \quad r = 0 \tag{18}$$

The boundary condition Eq. (18) will have to be satisfied for a clamped plate and for a simply supported plate as well.

In addition to satisfying these conditions the solution must also satisfy "regularity" conditions at r = 0, the centre of the plate. These require that at the centre displacement and slope be finite.

For the present axisymmetric problem the thickness can vary only in radial direction and it is assumed to be given by

$$h = h_o\left[1 - \frac{h_o - h_a}{h_o}\left(\frac{r}{a}\right)^n\right] = h_o\,(1 - HR^n) \tag{19}$$

where, h_o and h_a represent the thickness of the plate at the centre $r = 0$ and at the edge of the plate $r = a$, respectively, n is any positive integer and definitions of R and H are obvious. (For a uniform thickness plate $H = 0$). Thus by definition, the quantity

$$D = E_r h^3/12 \quad \text{becomes} \quad D = E_r h_o^3 (1 - HR^n)^3/12 ,$$

where

$$E_r = a_{22}/(a_{11}a_{22} - a_{12}^2) = 1/a_{22}\,(1 - \nu^2)$$

Since this quantity must be positive for a full plate everywhere in the interval $0 \leq r \leq a$ it is required that H be less than unity ($H < 1$). Substitution of the thickness h given by Eq. (19) into Eq. (14) and letting $p(r, t) = 0$, we get the governing differential equation for the free vibrations of an axisymmetric, isotropic plate having thickness variation $h = h_o(1 - HR^n)$ as

$$\begin{aligned}
&(1 - 2HR^n + H^2R^{2n})w_{RRRR} + 2[1 - (2 + 3n)HR^n + (1 + 3n)H^2R^{2n})\frac{w_{RRR}}{R} \\
&+ \{-1 + [2 - 3n(1 + \nu + n)]HR^n + [-1 + 3n(1 + \nu + 3n)]H^2R^{2n}\}\frac{w_{RR}}{R^2} \\
&+ \{1 - [(2 - 3n) + 3\nu n(n - 1)]HR^n + (1 - 3n)(1 - 3\nu n)H^2R^{2n}\}\frac{w_R}{R^3} \\
&+ \frac{\rho h_o a^4}{D_o} w_{tt} = 0
\end{aligned} \tag{20}$$

where $D_o = E_r h_o^3/12$ and ρ is the mass density/unit volume.

3. METHOD OF SOLUTION

In attempting to solve the governing Equation (20) non-dimensional variable $R = r/a$ has been introduced. Next, the displacement w is assumed to have the form

$$w(R, t) = W(R)e^{i\omega t} \tag{21}$$

where $W(R)$ is a function of R alone, $i = \sqrt{-1}$ and ω is the circular frequency expressed in rad/sec. When Eq. (21) is substituted into Eq. (20) the time and space variables are separated and an ordinary differential equation in terms of W is obtained as

$$\begin{aligned}&(1 - 2HR^n + H^2R^{2n})W'''' + 2[1 - (2 + 3n)HR^n + (1 + 3n)H^2R^{2n}]\frac{W'''}{R}\\&+ \{-1 + [2 - 3n(1 + \nu + n)]HR^n + [-1 + 3n(1 + \nu + 3n)]H^2R^{2n}\}\frac{W''}{R^2}\\&+ \{1 - [(2 - 3n) + 3\nu n(n - 1)]HR^n + (1 - 3n)(1 - 3\nu n)H^2R^{2n}\}\frac{W'}{R^3}\\&-\lambda W = 0\end{aligned} \tag{21}$$

where

$$\lambda = \frac{\rho\omega^2 h_o a^4}{D_o} \tag{22}$$

and a prime over a symbol denotes differentiation with respect to R.

For the solution of Eq. (14) the method of Frobenius is adopted. A series solution for W, about the regular singular point $R = 0$, is assumed in the form

$$W(R) = \sum_{k=0}^{\infty} g_k R^{j+k} \tag{23}$$

where k is the index of the summation, j denotes the exponents of singularity, the g_k represent the coefficients of the series, and g_o is, by assumption coefficient of the first term in the series. Since Eq. (21) is a fourth order differential equation, Frobenius' method will yield four exponents of singularity in the solution. Hence, the right hand side of Eq. (23) represents the sum of four series each corresponding to a particular value of j; furthermore, the g_k then represent four sets of coefficients, one for each series.

Substituting the assumed solution (23) into Eq. (21) the following equation is obtained:

$$\sum_{k=0}^{\infty} X(j+k) g_k R^{j+k-4} - \sum_{k=n}^{\infty} Y(j+k-n) H g_{k-n} R^{j+k-4} +$$

$$- \sum_{k=2n}^{\infty} Z(j+k-2n) H^2 g_{k-2n} R^{j+k-4} - \sum_{k=4}^{\infty} \lambda g_{k-4} R^{j+k-4} = 0 \qquad (23)'$$

where the indices of summations have been manipulated in such a manner that R appears to the same power in each term and the functions X, Y and Z are defined as

$$X(j+k) = (j+k)(j+k-2)[(j+k-1)^2 - 1] \qquad (24)$$

$$Y(j+k-n) = (j+k-n)\{(j+k-n-1)[2(j+k-n-2)(j+k+2n-1) + \\ -2 + 3n(1+\nu+n)] + 2 - 3n + 3\nu n(n-1)\} \qquad (25)$$

$$Z(j+k-2n) = (j+k-2n)\{(j+k-2n-1)[-(j+k+4n-1)(j+k-2n- \\ 2) + 1 - 3n(1+\nu+3n)] + (1-3n)(3\nu n - 1)\} \qquad (26)$$

The functional notations $X(j+k)$, $Y(j+k-n)$ and $Z(j+k-2n)$ have been adopted to indicate that these expressions depend not only on the material property ν and the power of the thickness variation, n, but also on the value of the exponent of singularity, j, and index of summation, k, for which they are to be evaluated. Upon introducing the step function notation

$$\delta(k - m) = \begin{cases} 1 & k - m \geq 0 \\ 0 & k - m < 0 \end{cases} \tag{27}$$

and collecting the coefficients of successive powers of R, Eq. (23) may be written in the form

$$\begin{aligned} & X(j)g_o R^{j-4} + \\ & + \{X(j + 1)g_1 - \delta(1 - n)Y(j + 1 - n)Hg_{1-n} \\ & \quad - \delta(1 - 2n)Z(j + 1 - 2n)H^2 g_{1-2n}\}R^{j-3} \\ & + \{X(j + 2)g_2 - \delta(2 - n)Y(j + 2 - n)Hg_{2-n} \\ & \quad - \delta(2 - 2n)Z(j + 2 - 2n)H^2 g_{2-2n}\}R^{j-2} \\ & + \{X(j + 3)g_3 - \delta(3 - n)Y(j + 3 - n)Hg_{3-n} \\ & \quad - \delta(3 - 2n)Z(j + 3 - 2n)H^2 g_{3-2n}\}R^{j-1} \\ & + \sum_{k=4}^{\infty} \{X(j + k)g_k - \delta(k - n)Y(j + k - n)Hg_{k-n} \\ & \quad - \delta(k - 2n)Z(j + k - 2n)H^2 g_{k-2n} + \lambda g_{k-4}\}R^{j+k-4} = 0 \end{aligned} \tag{28}$$

Now, for w(R) given by Eq. (23) to be a solution of Eq. (21) the coefficient of each term in Eq. (28) must vanish identically. Thus, equating to zero the coefficient of the term R^{j-4}, the term with the lowest power of R, yields the indicial equation

$$X(j) = j(j - 2)[(j - 1)^2 - 1] = 0 \tag{29}$$

The four roots of this equation are

$$j = 0, 0, 2, 2 \tag{30}$$

which represents two sets of equal roots, each set differing by an integer. For each value of j in Eq. (30), the vanishing of the coefficients of the terms R^{j-3}, R^{j-2} and R^{j-1} gives the following equations for determining g_1, g_2 and g_3:

$$X(j + 1)g_1 = \delta(1 - n)Y(j + 1 - n)Hg_{1-n} + \delta(1 - 2n)Z(j + 1 - 2n)H^2g_{1-2n} \tag{31}$$

$$X(j + 2)g_2 = \delta(2 - n)Y(j + 2 - n)Hg_{2-n} + \delta(2 - 2n)Z(j + 2 - 2n)H^2g_{2-2n} \tag{32}$$

$$X(j + 3)g_3 = \delta(3 - n)Y(j + 3 - n)Hg_{3-n} + \delta(3 - 2n)Z(j + 3 - 2n)H^2g_{3-2n} \tag{33}$$

Finally, for the determination of each of the coefficients g_k for $k \geq 4$, the following recurrence relation is obtained when the coefficient of the term R^{j+k-4} is equated to zero:

$$X(j + k)g_k = \delta(k - n)Y(j + k - n)Hg_{k-n} + \delta(k - 2n)Z(j + k - 2n)H^2g_{k-2n} + \lambda g_{k-4} \qquad (k \geq 4) \tag{34}$$

This relation determines each of the coefficients $g_k (k \geq 4)$ in terms of the preceding g's, and hence in terms of g_o, for each j in Eq. (30).

An inspection of Eqs. (31) through (34) indicates that it is not possible to write a simple expression for the g_k explicitly in terms of g_o for a general value of n. However, for any particular n the corresponding solution W(R), can be written [7] as

$$W(R) = a_o \sum_{k=0,4,5,\ldots}^{\infty} A_k R^k + c_o\{\frac{\partial}{\partial j}[\sum_{k=0}^{\infty} A_k R^{j+k}]\}\Big|_{j=0} +$$

$$+ b_o \sum_{k=0}^{\infty} B_k R^{k+2} + d_o\{\frac{\partial}{\partial j}[\sum_{k=0}^{\infty} B_k R^{j+k}]\}\Big|_{j=2} \tag{35}$$

In Eq. (35) a_o, b_o, c_o and d_o are undetermined constants, and $A_k = a_k/a_o$ and $B_k = b_k/b_o$ where the a_k and b_k correspond to the g_k calculated with j = 0 and 2 respectively.

The solution for W(R) of Eq. (21) must satisfy the "regularity" and boundary conditions discussed previously. In order that the displacement be finite at the centre of the plate (R = 0) second and fourth terms in Eq. (35) are inadmissible since they become infinite at the origin; hence $c_o = d_o = 0$. This condition insures that the slope is finite at the centre. The boundary conditions to be satisfied at the centre (Eq. 18) and at the boundary (Eq. 17a and 17b) may be written in terms of nondimensional variable R as

$$D(RW''' + W'' - R^{-1}W') + RD'(W'' + \nu RW') = 0 \quad \text{at} \quad R = 0 \tag{36}$$

for a clamped plate

$$W = 0 \quad \text{and} \quad W' = 0 \quad \text{at} \quad R = 1 \tag{37a}$$

or for a simply supported plate

$$W = 0 \qquad W'' + \nu R^{-1}W' = 0 \quad \text{at} \quad R = 1 \tag{37b}$$

Eq. (36) is satisfied upon substituting Eq. (35) with $c_o = d_o = 0$, and using the definition of D′. With these results the solution becomes

$$W(R) = a_o \sum_{k=0,4,5,\ldots}^{\infty} A_k R^k + b_o \sum_{k=0}^{\infty} B_k R^{k+2} \tag{38}$$

where the remaining constants are to be determined from the boundary conditions at the edge of plate.

When boundary conditions (37a) and (37b) are enforced at the edge of the plate (R = 1), the following simultaneous equations are obtained:

$$\begin{aligned} a_o \sum_{k=0,4,5,\ldots}^{\infty} A_k + b_o \sum_{k=0}^{\infty} B_k &= 0 \\ a_o \sum_{k=4,5,\ldots}^{\infty} kA_k + b_o \sum_{k=0}^{\infty} (k+2)B_k &= 0 \end{aligned} \tag{39a}$$

and

$$\begin{aligned} a_o \sum_{k=0,4,5,\ldots}^{\infty} A_k + b_o \sum_{k=0}^{\infty} B_k &= 0 \\ a_o \sum_{k=4,5,\ldots}^{\infty} k(k+\nu-1)A_k + b_o \sum_{k=0}^{\infty} (k+2)(k+\nu+1)B_k &= 0 \end{aligned} \tag{39b}$$

where Eqs. (39a) and (39b) correspond to the clamped and simply supported plate, respectively. The solution of Eqs. (39a) and (39b) for non-trivial values of a_o and b_o requires that the determinant of the coefficients of a_o and b_o in Eqs. (39a) and (39b) vanishes. These conditions yield the following characteristic equations for determination of natural frequencies:

for a clamped plate

$$\left[\sum_{k=0,4,5,\ldots}^{\infty} A_k\right]\left[\sum_{k=0}^{\infty} (k+2)B_k\right] - \left[\sum_{k=0}^{\infty} B_k\right]\left[\sum_{k=4,5,\ldots}^{\infty} kA_k\right] = 0 \tag{40a}$$

for a simply supported plate

$$[\sum_{k=0,4,5,\ldots}^{\infty} A_k][\sum_{k=0}^{\infty} (k + 2)(k + \nu + 1)B_k]$$

$$- [\sum_{k=0}^{\infty} B_k][\sum_{k=4,5,\ldots}^{\infty} k(k + \nu + 1)A_k] = 0 \qquad (40b)$$

The roots of Eqs. (40a) and (40b) give the values of λ which, when substituted into the equation

$$f = \frac{\omega}{2\pi} = \frac{\lambda^{1/2}}{2\pi} (\frac{D_o}{\rho h_o a^4})^{1/2} \qquad (41)$$

determine the natural frequencies of the plate.

4. ILLUSTRATIVE EXAMPLE

The equations developed in the previous section were applied to several clamped and simply supported plates of various shapes. In particular, plates of uniform thickness and those with thickness variations corresponding to n = 1 (straight taper) and n = 3 (curved form) were considered. The numerical calculations were made restricting the series in Eqs. (40a) and (40b) to include terms up to twenty-second power in λ. The results for $\bar{f} = \lambda^{1/2} = \omega a^2 \sqrt{\rho h_o / D_o}$ corresponding to lowest four frequencies were computed and are presented in the following tables. Mode shapes for various cases are also presented.

Clamped Plate, $\nu = 1/3$

n	H	$\bar{f}_1$	$\bar{f}_2$	$\bar{f}_3$	$\bar{f}_4$
1	0.1	9.424	37.44	84.36	149.3
1	0.2	8.602	34.96	79.19	140.8
1	0.3	7.795	32.49	74.01	131.8
1	0.4	6.987	29.96	68.64	123.7
1	0.5	6.176	27.35	63.11	113.6
3	0.1	9.580	38.26	86.30	153.5
3	0.2	8.936	36.68	83.26	146.6
3	0.3	8.282	35.05	80.26	141.8
3	0.4	7.620	33.35	76.94	137.7
3	0.5	6.945	31.51	73.33	130.3
uniform thickness		10.21	39.77	89.11	158.2

Simply Supported Plate, $\nu = 1/3$

n	H	$\bar{f}_1$	$\bar{f}_2$	$\bar{f}_3$	$\bar{f}_4$
1	0.1	4.712	28.12	70.28	130.6
1	0.2	4.439	26.46	66.25	123.4
1	0.3	4.163	24.78	62.14	116.1
1	0.4	3.882	23.06	57.90	108.6
1	0.5	3.593	21.30	53.52	100.2
3	0.1	4.857	29.00	72.30	134.3
3	0.2	4.732	28.20	70.31	130.7
3	0.3	4.607	27.36	68.22	126.3
3	0.4	4.483	26.46	65.98	122.5
3	0.5	4.355	25.48	63.58	115.9
uniform thickness		4.983	29.75	74.19	138.2

Simply Supported Plate, $\nu = 0.3$

uniform thickness		4.935	29.72	74.15	138.3

All computations were performed on an IBM 360 computer at the University's Computer Center.

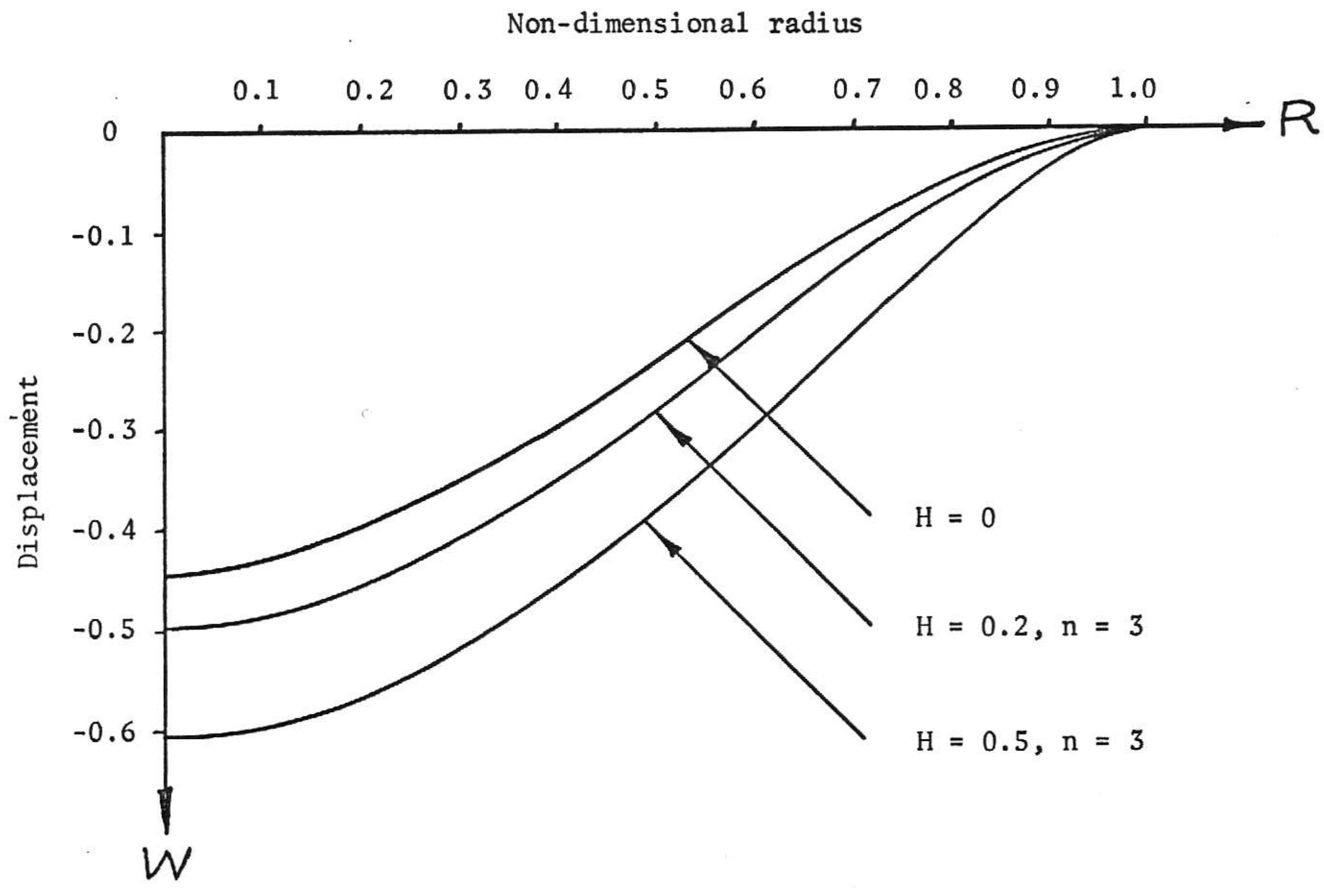

Fig. 2. Mode shapes for lowest frequency for different values of H for clamped plate ($\nu = 1/3$).

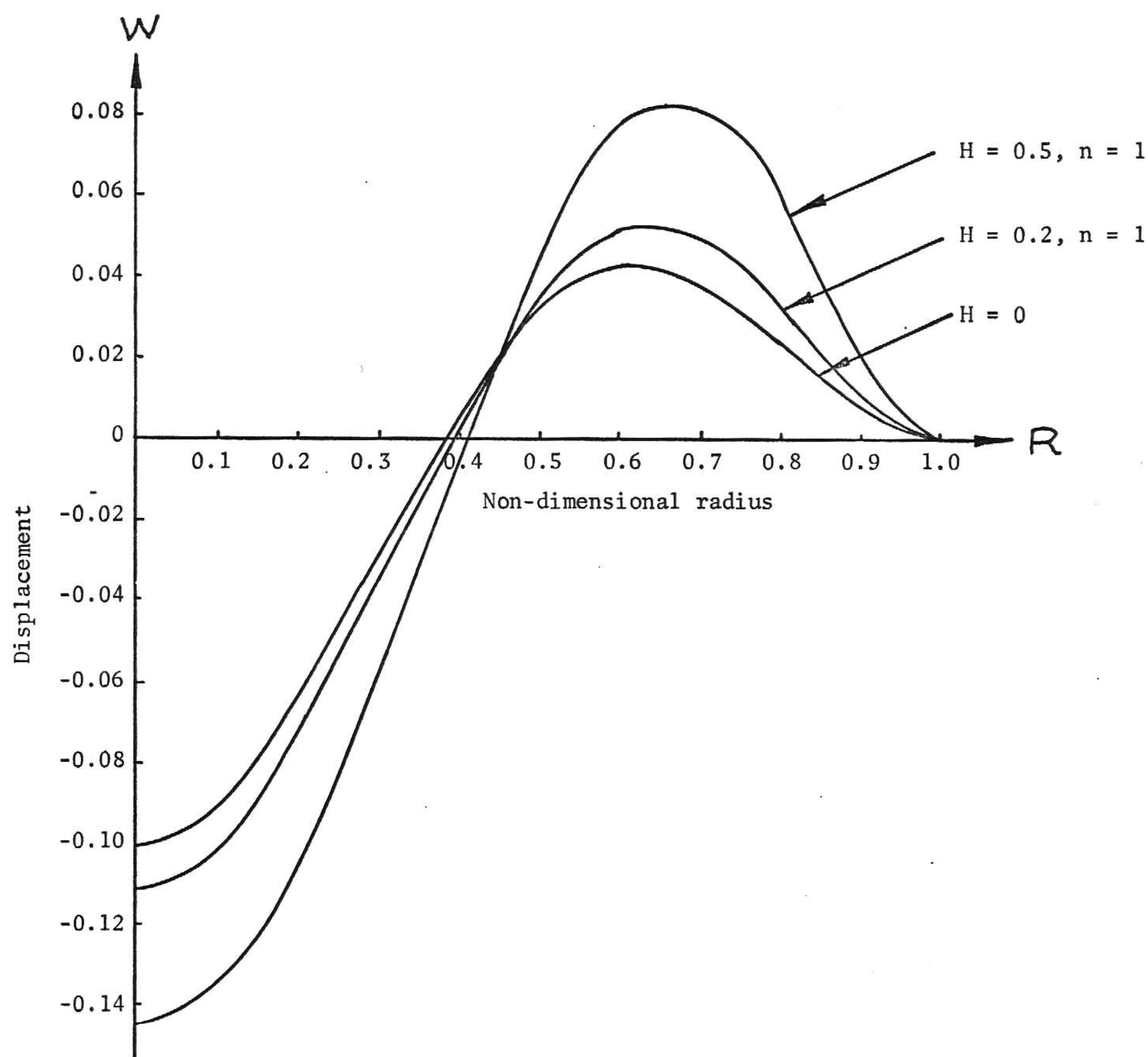

Fig. 3. Mode shapes for second lowest frequency for different values of H for clamped plate ($\nu = 1/3$).

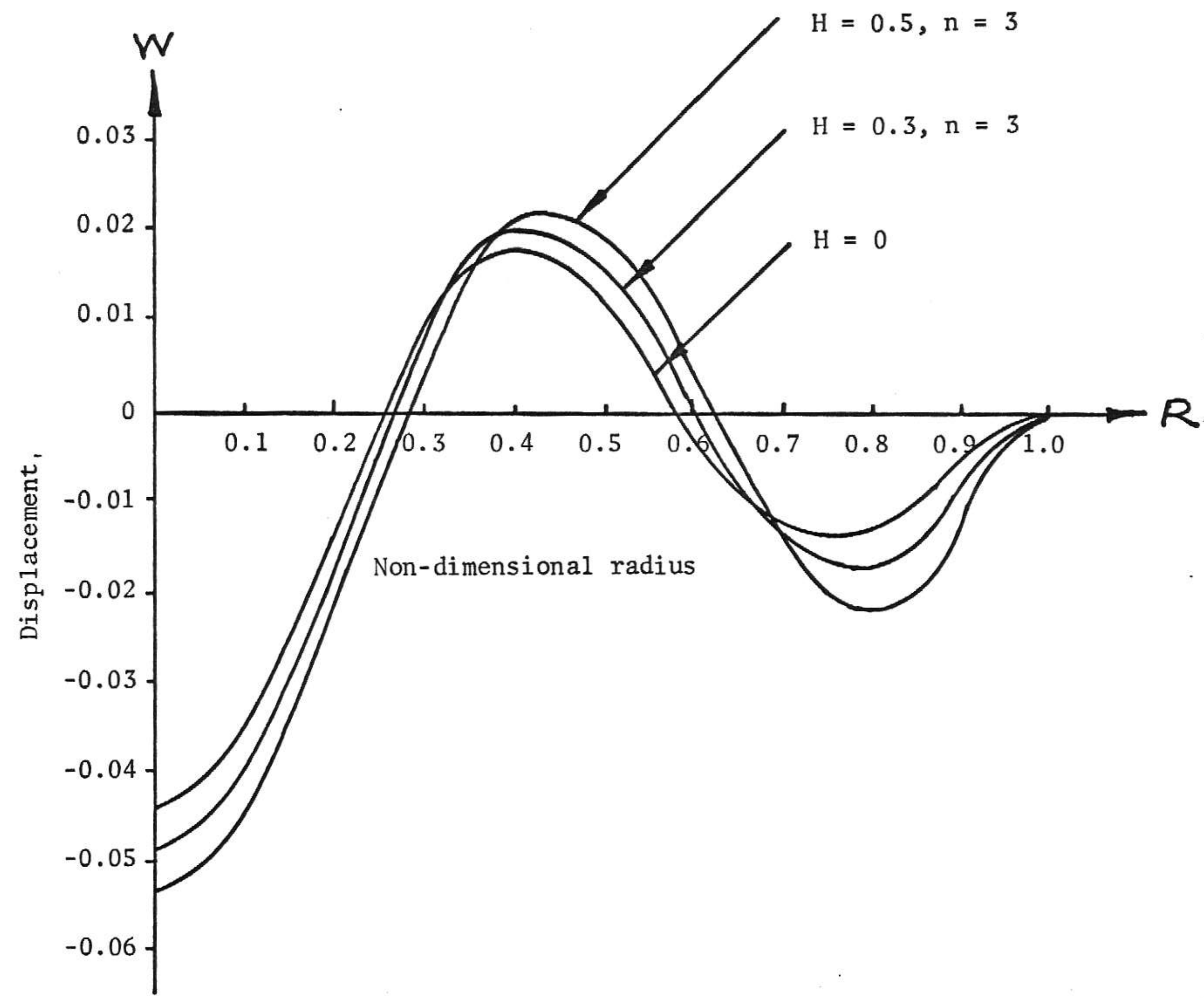

Fig. 4. Mode shapes for third lowest frequency for different values of H for clamped plate ($\nu = 1/3$).

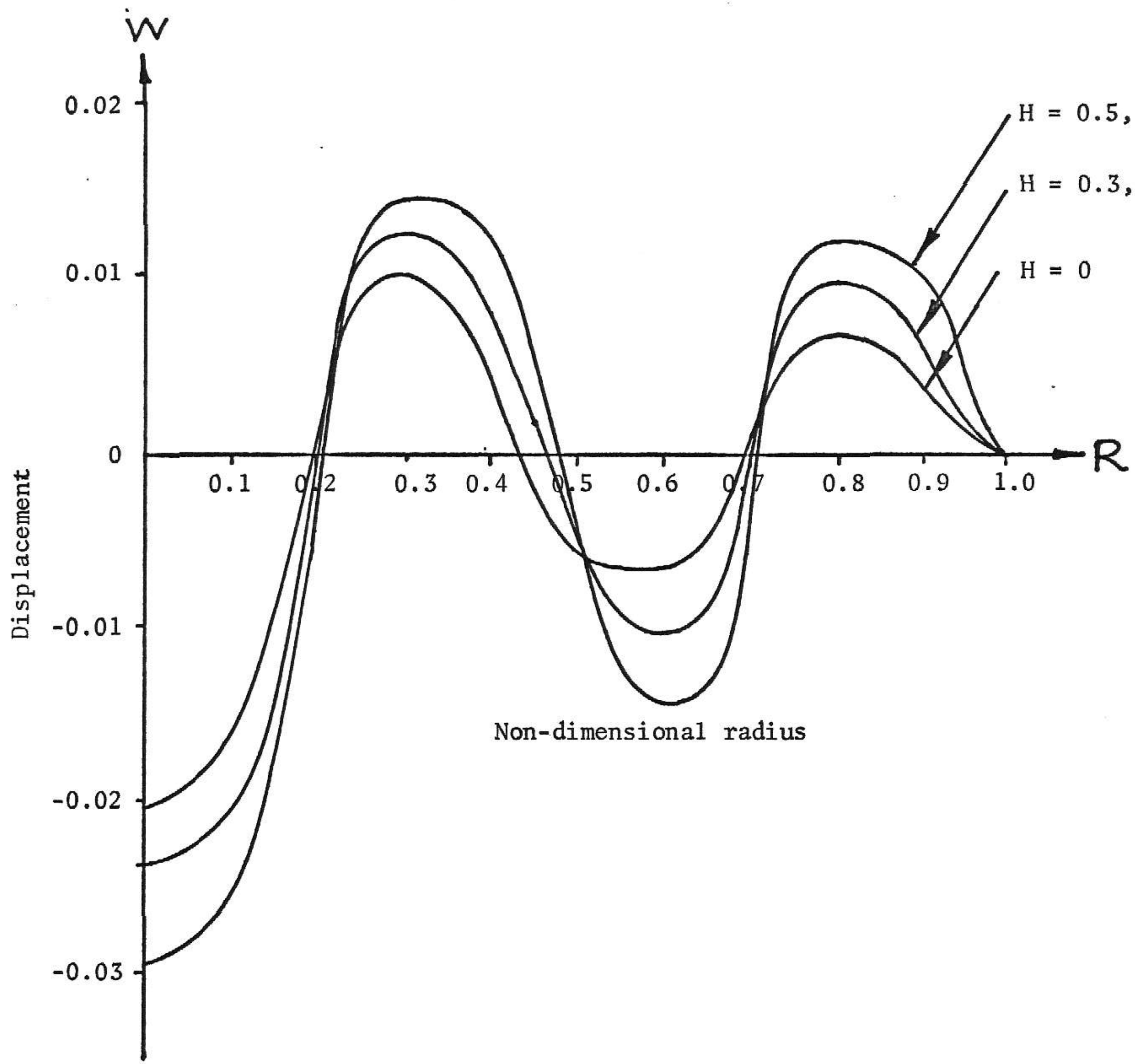

Fig. 5. Mode shapes for fourth lowest frequency for different values of H for clamped plate ($\nu = 1/3$).

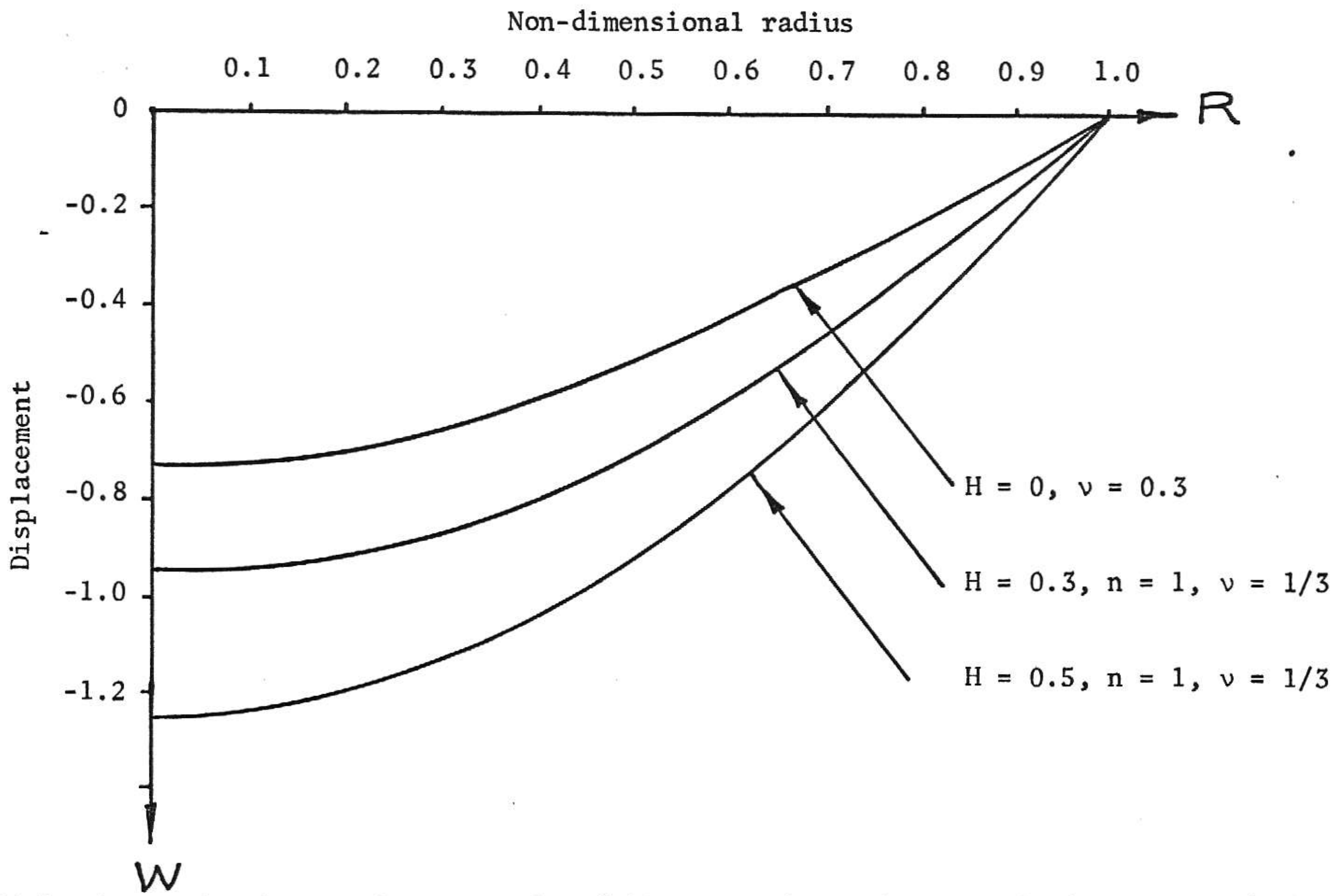

Fig. 6. Mode shapes for lowest frequency for different values of H for simply supported plate.

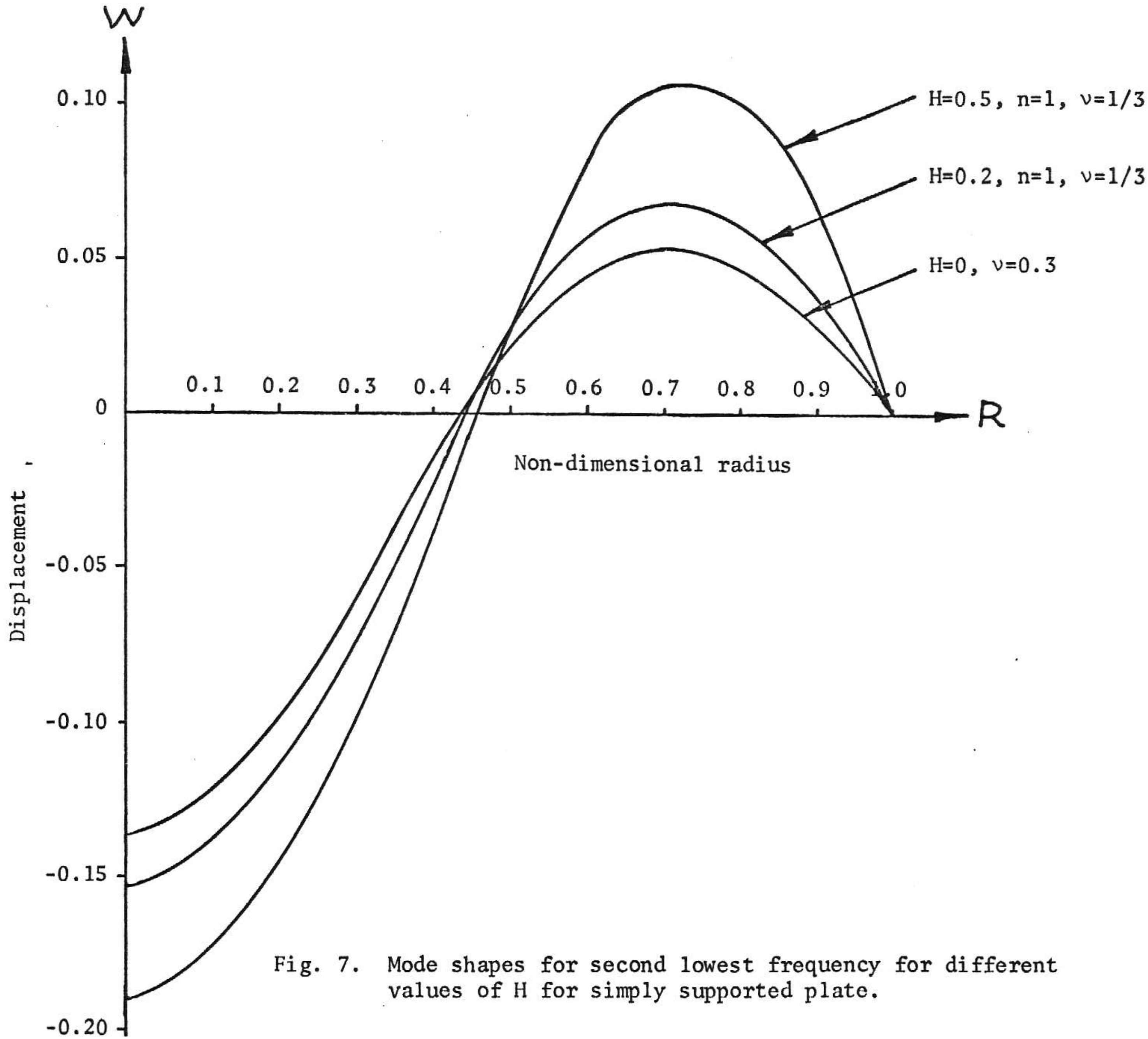

Fig. 7. Mode shapes for second lowest frequency for different values of H for simply supported plate.

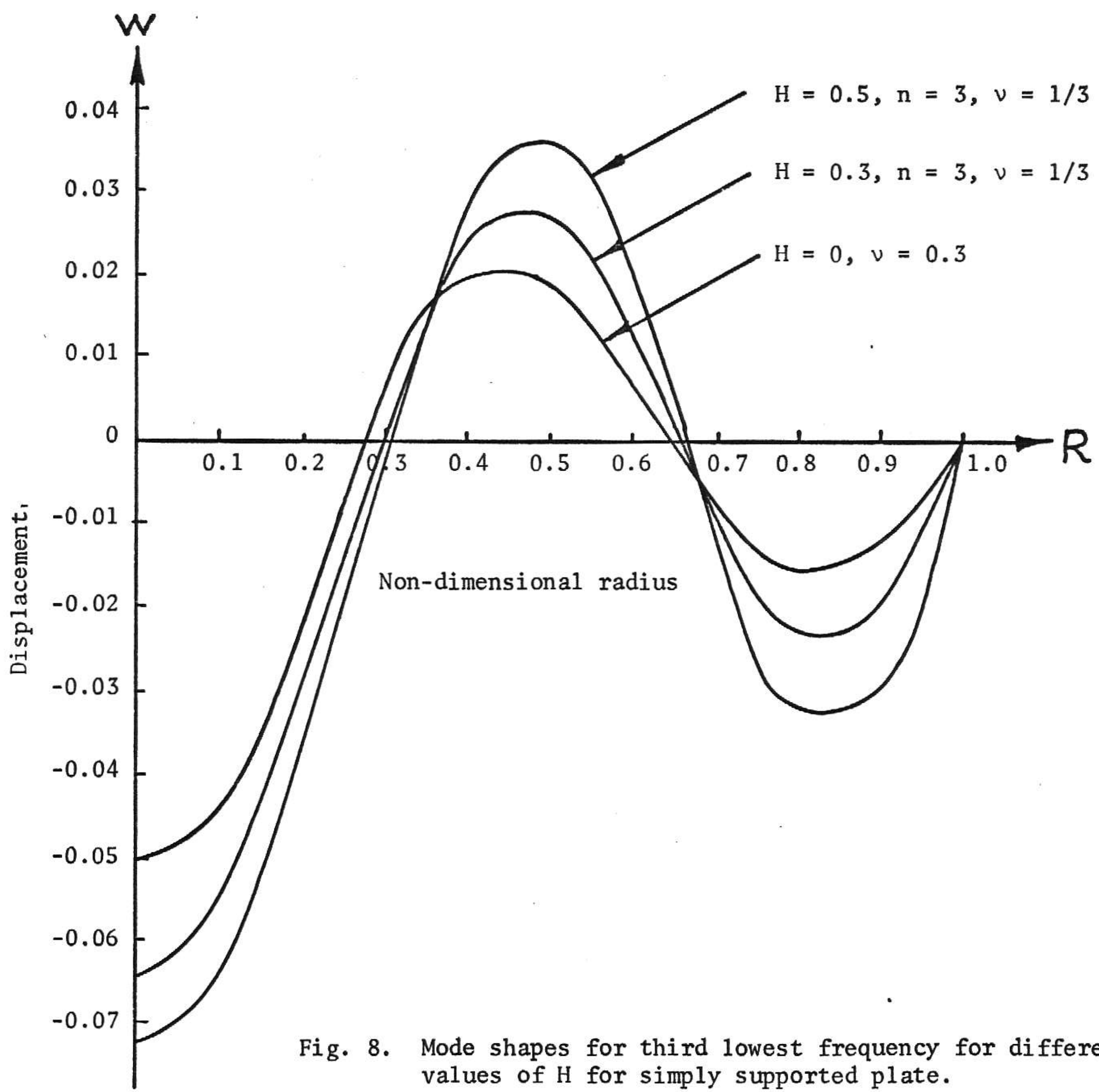

Fig. 8. Mode shapes for third lowest frequency for different values of H for simply supported plate.

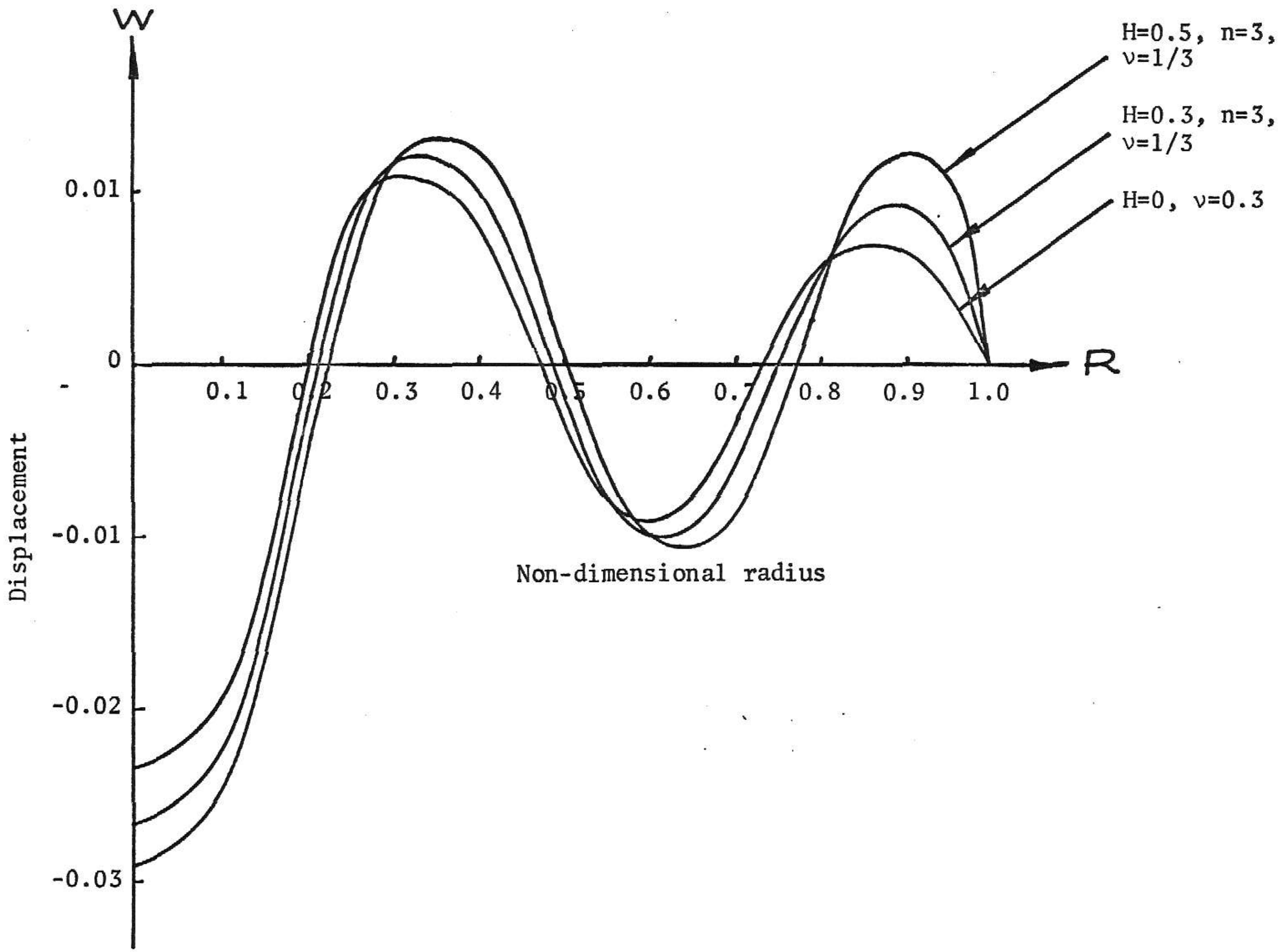

Fig. 9. Mode shapes for fourth lowest frequency for different values of H for simply supported plate.

5. DISCUSSION OF RESULTS

A method for determining the natural frequencies of clamped and simply supported isotropic plates of radially varying thickness has been presented.

The exact values of frequency parameter $\bar{f} = \omega a^2\sqrt{\rho h_o/D_o}$ and radii of nodal circles for uniform thickness plates are given in reference [9]. These values are quoted as follows, for clamped plate ($\nu = 1/3$): $\bar{f}_1 = 10.21$, $\bar{f}_2 = 39.77$, $\bar{f}_3 = 89.10$ and $\bar{f}_4 = 158.2$. The radii of nodal circles are: for $\bar{f}_1$ 1.0, for $\bar{f}_2$ 1.0 and 0.379, for $\bar{f}_3$ = 1.0, 0.583 and 0.255, and for $\bar{f}_4$ 1.0, 0.688, 0.439 and 0.191. For simply supported plate ($\nu = 0.3$): $\bar{f}_1 = 4.977$, $\bar{f}_2 = 29.76$, $\bar{f}_3 = 74.20$ and $\bar{f}_4 = 138.3$. The radii of nodal circles are: for $\bar{f}_1$ 1.0, for $\bar{f}_2$ 1.0 and 0.441, for $\bar{f}_3$ 1.0, 0.644 and 0.279, for $\bar{f}_4$ 1.0, 0.736, 0.469 and 0.204. These values are in agreement with the ones calculated in this thesis.

From an examination of numerical results it may be concluded that the natural frequencies for clamped and simply supported plates tend to decrease as the ratio of the edge thickness of the plate to its thickness at the centre is decreased, and tend to increase as the thickness exponent n increases. From the various mode shapes it may be concluded that displacement and radii of nodal circles tend to increase as the ratio of the edge thickness of the plate to its thickness at the centre is decreased.

Knowing the amplitudes from modal shapes the stress values at different points along the nodal diameter can be calculated.

REFERENCES

1. Carrier, G. F., "The Bending of the Cylindrically Aelotropic Plate," Journal of Applied Mechanics, Vol. 11, 1944, pp. A129-A134.

2. Conway, H. D., "Note on the Bending of Circular Plates of Variable Thickness," Journal of Applied Mechanics, Vol. 71, 1949, pp. 209-210.

3. Minkarah, I. A. and Hoppmann, W. H., II, "Flexural Vibration of Cylindrically Aelotropic Circular Plates," The Journal of the Accoustical Society of America, Vol. 36, 1964, pp. 470-475.

4. Salzman, A. P. and Patel, S. A., "Natural Frequencies of Orthotropic Circular Plates of Variable Thickness," Task NR 064-427, April 1968, Polytechnic Institute of Brooklyn, Department of Aerospace Engineering and Applied Mechanics.

5. Joung, K. S., "Derivation of Axisymmetric Flexural Vibration Equations of a Cylindrically Aelotropic Circular Plate," AIAA Journal, Vol. 7, No. 7, 1969, pp. 1386-1388.

6. Lekhnitskii, S. G., "Theory of Elasticity of an Anisotropic Elastic Body," translated by P. Fern, Holden-Day, San Francisco, 1963, p. 19.

7. J. Irving and N. Mullineux, "Mathematics in Physics and Engineering," by Academic Press Inc. (London) Ltd., 1959, p. 80.

8. Langhaar, H. L., "Energy Methods in Applied Mechanics," by John Wiley and Sons, Inc., New York, London, 1962.

9. Leissa, A. W., "Vibration of Plates," by National Aeronautics and Space Administration, Washington, D. C., 1969, p. 8-10.

ACKNOWLEDGEMENT

The author wishes to extend his sincere gratitude to his major advisor, Dr. H. S. Walker of Mechanical Engineering Department and Dr. C. L. Huang of Applied Mechanics Department for their valuable guidance and constant encouragement throughout the course of this work. Thanks are also due to Mrs. Karen Area for her excellent typing.

NATURAL FREQUENCIES OF ISOTROPIC CIRCULAR
PLATES OF VARIABLE THICKNESS

by

BABGAUNDA A. PATIL

B.E., University of Bombay, India, 1968

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1971

ABSTRACT

The present investigation is concerned with the natural frequencies of isotropic circular plates of variable thickness. In particular, a thickness variation of the form $h = h_o(1-HR^n)$ has been selected. The derivation of the governing differential equation for axisymmetric case is based on the method of variational calculas. The solution of this equation is obtained by an application of the method of Frobenius. Characteristic equations for the natural frequencies of clamped and simply supported plates are derived and numerical results are presented for several plates of various shapes.