PLATE BENDING FINITE ELEMENT ANALYSIS OF BEAMS WITH WEB OPENINGS

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INTRODUCTION

1. Introduction

In the present construction of steel buildings, openings through the webs of steel beams are frequently necessary to accommodate the passage of pipes, ducts and other utility components. Thus, the strength of the beam may be weakened to the extent that reinforcing is required in the vicinity of the openings. In the past few years, both analytical and experimental investigations have been made of the stresses around various openings with and without reinforcing. Several theoretical solutions have been verified and are available for certain cases of this problem. The purpose of this study was to determine the accuracy of an analytical solution based on a 'Finite Element Method' by comparing these results with those of an experimental program carried out at Kansas State University(1)*. The results were also compared with the results obtained using a Vierendeel Analysis (1).

An A36 W12x45 steel beam with a 6" x 9" rectangular web opening at middepth, subjected to combined bending and shear, was treated as a three dimensional plate structure in this study, The finite element method was used to investigate the stress distribution around the rectangular hole. An existing computer program (2) was made operational as a requirement of this report.

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*Numbers in parentheses refer to corresponding items in the " References ".

2. Objectives

The primary objectives of this report were:

a. To obtain a solution to the problem using the finite element method.

b. To compare the results of this study with the experimental results and with predictions based on the so-called Vierendeel Method of analysis (1).

3. Scope

The study was limited to A36 steel W12x45 shapes with a 6" x 9" rectangular web opening centered on the neutral axis of the beams, which were subjected to combined bending and shear with various values of the moment-shear ratio obtained by placing a given concentrated load at the center of the span and simultaneously varying the shear span length. Both reinforced and unreinforced openings were studied. The reinforcing consisted of horizontal bars located above and below the opening, on just one side of the web.

LITERATURE REVIEW

In 1932, Muskhelishvili introduced the application of the comformal mapping technique and complex integration to the problems of plane elasticity (3).

In 1958, Heller, Brock and Bart presented a solution by the complex variable method associated with Muskhelishvili for the stresses around a rectangular opening with rounded corners in a uniformly loaded plate (4). In 1962, they used the same method to investigate the stresses around a rectangular opening with rounded corners in a beam subjected to bending and shear (5). In both cases, they reached the conclusion that the maximum value of the boundary stress is a function of both aspect ratio (height-to-width) and corner radius.

Snell, in 1962, used the finite element method to study the effects of various reinforcing configurations for rectangular openings in plates subjected to uniaxial tension (6). After his analytical and experimental studies, he reached the following conclusions: (a) The finite element method could be used for the solution of this type of problem. (b) Reinforcing strips parallel to the direction of the applied load could effectively be used to reduce stress concentration in plates with rectangular openings. (c) The most effective length for these strips was closely approximated by placing the reinforcing in such a way that the ends were on lines drawn outward from the corners of the opening in the plate at 45° angle to the plate axes.

In 1964, Segner made a study of the reinforcing requirements around large rectangular web openings in W shape beams subjected to varying combinations of bending moment and shear (7). His theoretical approach was based on the theory that a member having such openings centered on the neutral axis acts as a Vierendeel truss and thus has a point of contraflexure at mid-length of each opening above and below the opening in the tee section. After his experimental study, he concluded that the Vierendeel theory was an appropriate analogy for this problem. Since then, the so-called 'Vierendeel Method' has frequently been used by designers to calculate the elastic stresses around rectangular holes in the web of W shape beams.

Bower developed an analytical method, in 1966, also using the complex stress function to predict the elastic stresses around elliptic and circular holes in the webs of W shape beams under a uniform load. The applicability of this analysis depends on the size and shape of the web hole and on the magnitude of the moment-shear ratio at the center of the hole (8). In the same year, he conducted tests on simply supported W shape beams with circular or rectangular web openings loaded by concentrated loads (9). He concluded that for circular and rectangular holes the elastic analysis could accurately predict the tangential stress along the hole and the bending stress on transverse cross sections in the vicinity of the hole, when the hole did not exceed half of the web depth. He also concluded that the Vierendeel analysis predicts a reasonably accurate bending stress except for local stress concentrations at the hole corners.

In 1969, Cheng experimentally analyzed the stresses around a rectangular web opening in a W shape beam using the photostress method and electrical resistance strain gage techniques (10). One of his conclusions was that simple beam theory could not be used to predict normal stress within the region on either side of the opening for a distance approximately equal to the depth of the beam. In this region the normal bending stress distribution is non-linear.

METHOD OF ANALYSIS

1. Introduction

The concept of the finite element method was originally introduced by Turner et al. in 1956 (11). O. C. Zienkiewicz and Y. K. Cheng (12) also presented the theory necessary for the analysis of a plane elastic continua. By using this method a plane elastic continua is divided into elements interconnected at a finite number of nodes. When the force-displacement relationships for the individual elements are determined, the general 'displacement method' of structural analysis procedure can be conveniently followed.

In this report, the W shape beam is treated as a three-dimensional structure. The elements then may be subjected to both bending and 'in plane' forces. For a flat element these loadings cause independent deformations, and the stiffness matrix for plane stress and plate bending can each be determined separately..The total element stiffness matrix can then be made up by simply combining these two matrices. Flat triangular elements are used with constant strain properties for the plane stress components, and linear strain variation for bending. For bending, a non-conforming shape function is used. The matrix formulation of the finite element analysis as presented in Reference (12) is included here for the purpose of providing some insight into the method.

2. Formulation of the General Equations

The general displacement method equation is given (13) as

$$\{F\}^e = [K]^e\{\delta\}^e \tag{1}$$

in which

$\{F\}^e$ = column matrix of nodal forces for a particular element (in local coordinates),

$[K]^e$ = element stiffness matrix (in local coordinates), and

$\{\delta\}^e$ = column matrix of nodal displacements for a particular element (in local coordinates).

Internal displacements are expressed in terms of the nodal displacements by

$$\{f\} = [N]\{\delta\}^e \tag{2}$$

in which

$\{f\}$ = column matrix of internal displacements in the element, and

$[N]$ = square matrix dependent upon the element geometry, relating the internal displacements and nodal displacements.

The strains at any point within the element can be determined by

$$\{\epsilon\} = [B]\{\delta\}^e \tag{3}$$

in which

$\{\epsilon\}$ = the column matrix of total strains at any point within the element, and

$[B]$ = square matrix obatined from the appropriate strain-displacement relationships governing the element.

The relationship between stress and strain can be written as

$$\{\sigma\} = [D]\{\epsilon\} \tag{4}$$

in which

$\{\sigma\}$ = column matrix of stresses within the element, and

$[D]$ = square matrix of material constants relating internal stress to strain.

Substituting from equation (3) into equation (4) yields

$$\{\sigma\} = [D]\,[B]\{\delta\}^{e}. \tag{5}$$

The element stiffness matrix can now be derived by using an energy method (15). The total strain energy of the element is

$$U = \frac{1}{2}\int_{v}\{\epsilon\}^{T}\{\sigma\}\,dv. \tag{6}$$

Substituting from equations (3) and (5) into (6) yields

$$U = \frac{1}{2}\int_{v}(\{\delta\}^{e})^{T}\,[B]^{T}\,[D]\,[B]\{\delta\}^{e}\,dv. \tag{7}$$

The average work done by the nodal forces is

$$W = \frac{1}{2}(\{F\}^{e})^{T}\{\delta\}^{e}. \tag{8}$$

Substituting from equation (1) into equation (8) yields

$$W = \frac{1}{2}(\{\delta\}^{e})^{T}\,[K]^{e}\{\delta\}^{e}. \tag{9}$$

Since the external work W must equal the energy, U, absorbed by the element, a comparison of equations (7) and (9) reveals that

$$[K]^e = \int_v [B]^T [D] [B] \, dv. \tag{10}$$

The element stiffness matrix in local coordinates must be transformed into the global coordinate system in order to assemble the elements. Let

$$\{\delta\}^e = [T] \{\bar{\delta}\}^e \qquad \text{and} \qquad \{F\}^e = [T] \{\bar{F}\}^e \tag{11}$$

in which

$[T]$ = transformation matrix,

$\{\bar{\delta}\}^e$ = column matrix of nodal displacements for a particular element (in global coordinates), and

$\{\bar{F}\}^e$ = column matrix of nodal forces for a particular element (in global coordinates).

Substituting from equation (11) into equation (1) yields

$$\{\bar{F}\}^e = [T]^T [K]^e [T] \{\bar{\delta}\}^e. \tag{12}$$

The stiffness matrix of an element in the global coordinates then becomes

$$[\bar{K}]^e = [T]^T [K]^e [T]. \tag{13}$$

When the overall equilibrium conditions are established at the nodes of the structure, the resulting equations will contain the displacements as unknowns. Once these have been solved, the stresses can be found by using equation (5) for each element, in turn.

3. Triangular Plate Element

a. Plate in Plane Stress

Let a typical triangular plate element with nodes i, j and m noted in a counter-clockwise order be as shown in

Fig. 1. The plane displacements of a node each have two components

$$\{\delta_i\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} . \tag{14}$$

The six components of element displacements can then be listed as a column vector.

$$\{\delta\}^e = \begin{Bmatrix} \delta_i \\ \delta_j \\ \delta_m \end{Bmatrix} \tag{15}$$

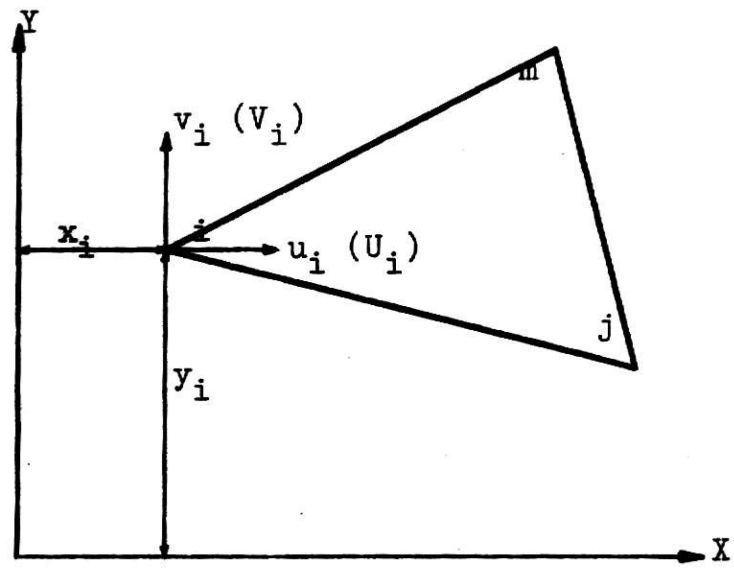

Fig. 1 Triangular Plate Plane Stress Element

The displacements at any point within an element are defined by these six values as

$$\{f\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [N]\{\delta\}^e \tag{16}$$

in which u and v are the internal displacements in the x and y directions, respectively. Two linear polynomials

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y, \text{ and}$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y \qquad (17)$$

are chosen to represent the displacement field for the element. On substituting the boundary conditions

$$u = u_i \quad \text{and} \quad v = v_i \quad \text{at} \quad (x_i, y_i),$$
$$u = u_j \quad \text{and} \quad v = v_j \quad \text{at} \quad (x_j, y_j), \qquad (18)$$
$$\text{and} \quad u = u_m \quad \text{and} \quad v = v_m \quad \text{at} \quad (x_m, y_m)$$

into equation (17), the six undetermined coefficients can be determined. For example,

$$u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i,$$
$$u_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j, \text{ and} \qquad (19)$$
$$u_m = \alpha_1 + \alpha_2 x_m + \alpha_3 y_m.$$

We can solve for α_1, α_2 and α_3 in terms of the nodal displacements u_i, u_j and u_m. In the same way α_4, α_5 and α_6 can be obtained. We finally find that

$$u = \frac{1}{2\Delta}[(a_i+b_i x+c_i y)u_i+(a_j+b_j x+c_j y)u_j+(a_m+b_m x+c_m y)u_m], \text{ and}$$
$$v = \frac{1}{2\Delta}[(a_i+b_i x+c_i y)v_i+(a_j+b_j x+c_j y)v_j+(a_m+b_m x+c_m y)v_m] \qquad (20)$$

in which

$$a_i = x_j y_m - x_m y_j,$$
$$b_i = y_j - y_m = y_{jm},$$
$$c_i = x_m - x_j = x_{mj},$$

$$
\begin{aligned}
a_j &= x_m y_i - x_i y_m, \\
b_j &= y_m - y_i = y_{mi}, \\
c_j &= x_i - x_m = x_{im}, \\
a_m &= x_i y_j - x_j y_i, \\
b_m &= y_i - y_j = y_{ij}, \text{ and} \\
c_m &= x_j - x_i = x_{ji}
\end{aligned}
\tag{21}
$$

and where

$$
2\Delta = \det \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} = 2 \text{ (area of triangle } i, j, m \text{)}. \tag{22}
$$

We can represent the relations in equation (20) in the form of equation (16)

$$
\{f\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [N]\{\delta\}^e = [\, IN_i', IN_j', IN_m' \,]\{\delta\}^e \tag{23}
$$

where I is a two by two identity matrix and

$$
\begin{aligned}
N_i' &= \frac{(a_i + b_i x + c_i y)}{2\Delta}, \\
N_j' &= \frac{(a_j + b_j x + c_j y)}{2\Delta}, \text{ and} \\
N_m' &= \frac{(a_m + b_m x + c_m y)}{2\Delta}.
\end{aligned}
\tag{24}
$$

The calculation of the coefficients can be simplified if the reference coordinates are taken as the centroid of the element. When that is done, the relationships

$$x_i+x_j+x_m = y_i+y_j+y_m$$

and

$$a_i = \frac{2\Delta}{3} = a_j = a_m \tag{25}$$

result.

The total strain at any point within the element can be defined in terms of the displacements by well-known (14) relationships

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} . \tag{26}$$

Taking the appropriate partial derivatives of equation (20), results in

$$\{\epsilon\} = \frac{1}{2\Delta} \begin{bmatrix} b_i, & 0, & b_j, & 0, & b_m, & 0 \\ 0\ , & c_i, & 0\ , & c_j, & 0\ , & c_m \\ c_i, & b_i, & c_j, & b_j, & c_m, & b_m \end{bmatrix} \{\delta\}^e \tag{27}$$

or, to correspond with equation (3)

$$\{\epsilon\} = [B]\{\delta\}^e$$

where

$$[B] = \frac{1}{2\Delta}\begin{bmatrix} b_i, & 0, & b_j, & 0, & b_m, & 0 \\ 0, & c_i, & 0, & c_j, & 0, & c_m \\ c_i, & b_i, & c_j, & b_j, & c_m, & b_m \end{bmatrix} . \tag{28}$$

The relationship between stress and strain is defined by equation (4). For the plane stress case, three components of stress correspond to the strains already defined as

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} . \tag{29}$$

For a linearly elastic, homogeneous and isotropic material, the matrix of material constants for this case is obtained from Hooke's law as

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & (1-\mu)/2 \end{bmatrix} . \tag{30}$$

For the plane strain case, a similar matrix can be formed.

The stiffness matrix of the element i, j, m is defined by the relationship in equation (10) as

$$[K]^e = \int_V [B]^T [D] [B] \, t \, dx \, dy \tag{31}$$

where t is the constant thickness of the element and the integration is taken over the area of the triangular element.

Since neither of the matrices in equation (31) contains x nor y, we have

$$[K]^e = [B]^T [D] [B] t \Delta \tag{32}$$

where Δ is the area of the triangle as defined by equation (22).

b. Plate Bending Stress

Consider a triangular plate i, j, m coinciding with the x, y plane as shown in Fig. 2. At each node, displacements $\{\delta_n\}$ are introduced. These have three components: the first a deflection w_n in the Z-direction, the second a rotation $(\theta_x)_n$ about the X-axis through the node and the third a rotation $(\theta_y)_n$ about the Y-axis through the node.

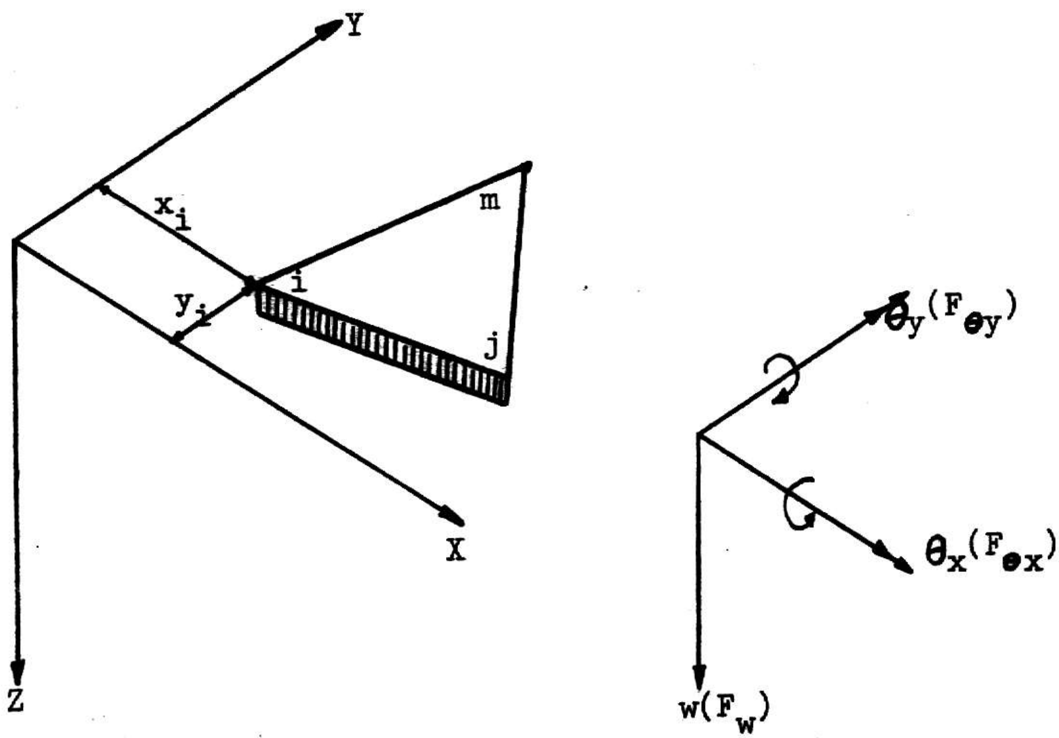

Fig. 2 Triangular Plate Element in Bending

The positive directions of the rotations are determined by the right-hand screw rule and are shown as vectors directed along the axes. The three components of nodal displacement at a node i can therefore be defined (16) as follows:

$$\left\{\delta_i\right\} = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} = \begin{Bmatrix} w_i \\ -\left(\frac{\partial w}{\partial y}\right)_i \\ \left(\frac{\partial w}{\partial x}\right)_i \end{Bmatrix} \tag{33}$$

and the corresponding force vector

$$\left\{F_i\right\} = \begin{Bmatrix} (F_w)_i \\ (F_{\theta x})_i \\ (F_{\theta y})_i \end{Bmatrix}. \tag{34}$$

The shape functions now must be definable in terms of $\left\{\delta\right\}^e$, that is in terms of nine parameters. A polynomial expression is conveniently used. We can write

$$w = f(x,y) = \alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 x^2+\alpha_5 xy+\alpha_6 y^2+\alpha_7 x^3+\alpha_8(px^2y+qxy^2) +\alpha_9 y^3 . \tag{35}$$

The undetermined coefficients α_1 to α_9 can be evaluated by writing the nine simultaneous equations linking the values of w and its slopes at the nodes when the coordinates take up their appropriate values. For instance

$$w_i = \alpha_1+\alpha_2 x_i+\alpha_3 y_i+\alpha_4 x_i^2+\alpha_5 x_i y_i+\alpha_6 y_i^2+\alpha_7 x_i^3 +\alpha_8(px_i^2 y_i+qx_i y_i^2)+\alpha_9 y_i^3,$$

$$\left(\frac{-\partial w}{\partial y}\right)_i = -\alpha_3 - \alpha_5 y_i - 2\alpha_6 y_i - \alpha_8 (p x_i^2 + 2 q x_i y_i) - 3\alpha_9 y_i^2 \text{ , and} \tag{36}$$

$$\left(\frac{\partial w}{\partial x}\right)_i = \alpha_2 + 2\alpha_4 x_i + \alpha_5 y_i + 3\alpha_7 x_i^2 + \alpha_8 (2 p x_i y_i + q y_i^2).$$

We can write all nine equations in matrix form,

$$\{\delta\}^e = [C]\{\alpha\} \tag{37}$$

where [C] is a nine by nine matrix depending on nodal coordinates and $\{\alpha\}$ a vector of the nine undetermined coefficients. Inverting [C] and solving for $\{\alpha\}$ yields

$$\{\alpha\} = [C]^{-1}\{\delta\}^e. \tag{38}$$

It is now possible to write the expression for the displacement within the element in the form of equation (2)

$$\{f\} = w = [N]\{\delta\}^e = [P]\,[C]^{-1}\{\delta\}^e \tag{39}$$

where

$$[P] = (1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^3, px^2y + qxy^2,\ y^3), \tag{40}$$

and

$p = q = 1$, for Tochers function (2), or

$p = 0,\ q = 1$, for Gallaghers function (2).

According to classical plate theory (16), for any point in a plate, the generalized 'strain' can be defined as

$$\{\epsilon\} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{41}$$

and the corresponding generalized 'stress' as

$$\{\sigma\} = \{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \tag{42}$$

with the notation and positive directions as shown in Fig. 3. The actual stresses are determined by such expressions as

$$\sigma_x = \frac{6M_x}{t^2}, \quad \sigma_y = \frac{6M_y}{t^2}, \text{ and } \quad \tau_{xy} = \frac{6M_{xy}}{t^2}. \tag{43}$$

Equation (41) can also be written in the form of equation (3)

$$\{\epsilon\} = [B]\{\delta\}^e. \tag{3}$$

The vector, $\{\epsilon\}$, can be obtained directly from equation (35), as

$$\{\epsilon\} = \begin{Bmatrix} (-2\alpha_4 & -6\alpha_7 x & -2p\alpha_8 y) \\ (-2\alpha_6 & -2q\alpha_8 x & -6\alpha_9 y\ ) \\ (2\alpha_5 & 4(px+qy)\alpha_8 &) \end{Bmatrix} \tag{44}$$

which can be written as

$$\{\epsilon\} = [Q]\{\alpha\} = [Q]\,[C]^{-1}\{\delta\}^e \tag{45}$$

and thus

$$[B] = [Q]\,[C]^{-1} \tag{46}$$

in which

$$[Q] = \begin{bmatrix} 0, & 0, & 0, & -2, & 0, & 0, & -6x, & -2py, & 0 \\ 0, & 0, & 0, & 0, & 0, & -2, & 0, & -2x, & -6y \\ 0, & 0, & 0, & 0, & 2, & 0, & 0, & 4(px+qy), & 0 \end{bmatrix}. \quad (47)$$

The linear relationship between stress and strain is derived as

$$\{\sigma\} = \{M\} = [D]\{\epsilon\}. \quad (48)$$

For an isotropic plate

$$[D] = \frac{Et^3}{12(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & (1-\mu)/2 \end{bmatrix}. \quad (49)$$

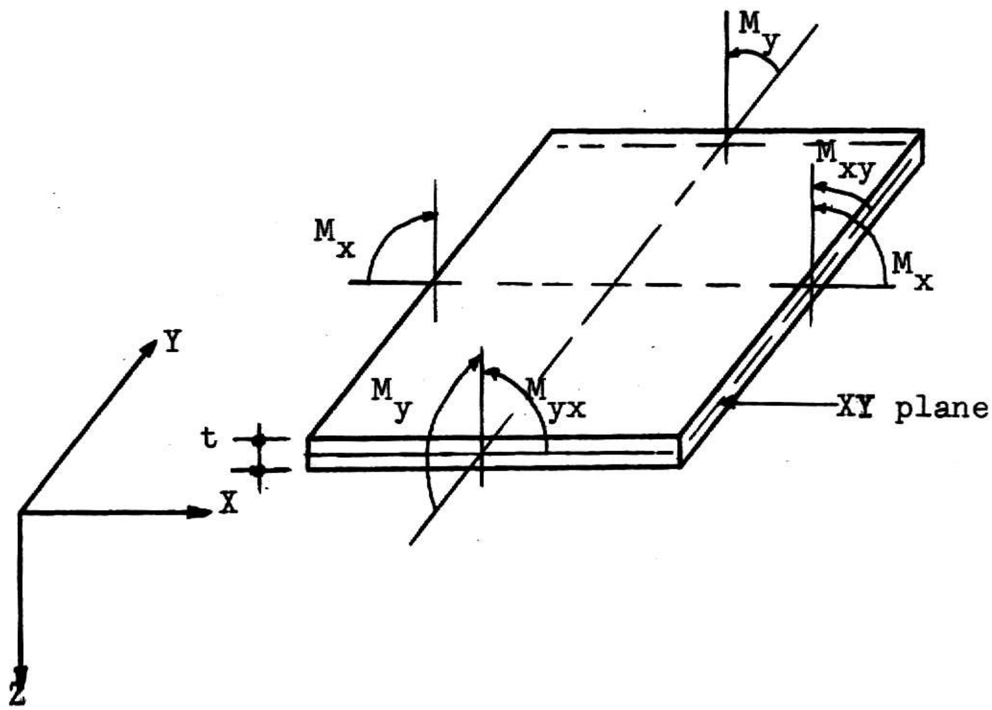

Fig. 3 Stress Resultants of 'Stresses' in Plate Bending

The stiffness matrix can now, again, be developed by an energy method as indicated in equation (10)

$$[K] = \iint [B]^T [D] [B] \, dx \, dy \, . \tag{50}$$

Substituting equation (46) and taking t as a constant within the element, yields

$$[K] = \left\{ [C]^{-1} \right\}^T \left(\iint [Q]^T [D] [Q] \, dx \, dy \right) [C]^{-1}. \tag{51}$$

c. Combining 'In Plane' And Bending Actions

From the previous derivations, we obtain the stiffness matrix and the relationships

$$\begin{Bmatrix} F_i^p \\ F_j^p \\ F_m^p \end{Bmatrix} = [K]^p \begin{Bmatrix} \delta_i^p \\ \delta_j^p \\ \delta_m^p \end{Bmatrix}$$

with $\left\{ \delta_i^p \right\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$ and $\left\{ F_i^p \right\} = \begin{Bmatrix} U_i \\ V_i \end{Bmatrix}$ (52)

for in plane (Plane stress) action,

and

$$\begin{Bmatrix} F_i^b \\ F_j^b \\ F_m^b \end{Bmatrix} = [K]^b \begin{Bmatrix} \delta_i^b \\ \delta_j^b \\ \delta_m^b \end{Bmatrix}$$

with

$$\left\{\delta_i^{\,b}\right\} = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \quad \text{and} \quad \left\{F_i^{\,b}\right\} = \begin{Bmatrix} F_{wi} \\ F_{\theta xi} \\ F_{\theta yi} \end{Bmatrix} \tag{53}$$

for plate bending action.

Before combining these stiffnesses it is important to note two facts. The first is that the displacements prescribed for 'in plane' forces do not affect the bending deformations and vice versa. The second is that rotation θ_z does not enter as a parameter into the definition of deformations in either mode. It is convenient to take this rotation into account and associate with it a fictious couple $F_{\theta z}$. It is also necessary to insert an appropriate number of zeros into the stiffness matrices. Redefining, now, the combined nodal displacements as

$$\left\{\delta_i\right\} = \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix} \tag{54}$$

and the corresponding 'forces' as

$$\left\{ F_i \right\} = \left\{ \begin{array}{c} U_i \\ V_i \\ F_{wi} \\ F_{\theta xi} \\ F_{\theta yi} \\ F_{\theta zi} \end{array} \right\} , \tag{55}$$

we can write

$$\left\{ \begin{array}{c} F_i \\ F_j \\ F_m \end{array} \right\} = [K]^e \left\{ \begin{array}{c} \delta_i \\ \delta_j \\ \delta_m \end{array} \right\} \tag{56}$$

or in the form of equation (1)

$$\left\{ F \right\}^e = [K]^e \left\{ \delta \right\}^e . \tag{1}$$

d. Transformation to Global Coordinates

The previous derivations of stiffness matrices are based on local coordinates. Transformation of coordinates to a common global system (which is denoted by $\bar{x}$, $\bar{y}$, $\bar{z}$, while the local system is denoted by x, y, z, as shown in Fig. 4) will be necessary in order to assemble the elements and write the appropriate equilibrium equations.

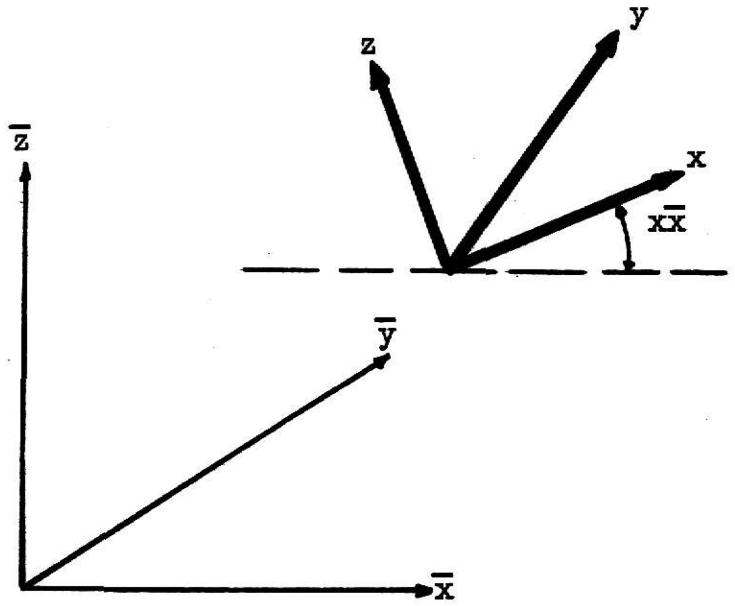

Fig. 4 Local and Global Coordinates

The forces and displacements at a node, given in the global system, are transformed to the local system by employing a matrix [L] giving

$$\left\{\delta_i\right\} = [L]\left\{\overline{\delta}_i\right\} \quad \text{and} \quad \left\{F_i\right\} = [L]\left\{\overline{F}_i\right\} \tag{57}$$

in which

$$[L] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \tag{58}$$

where $[\lambda]$ is a three by three matrix of direction cosines of the angles formed between the two sets of axes. That is

$$[\lambda] = \begin{pmatrix} \lambda_{x\overline{x}} & \lambda_{x\overline{y}} & \lambda_{x\overline{z}} \\ \lambda_{y\overline{x}} & \lambda_{y\overline{y}} & \lambda_{y\overline{z}} \\ \lambda_{z\overline{x}} & \lambda_{z\overline{y}} & \lambda_{z\overline{z}} \end{pmatrix}. \tag{59}$$

In which $\lambda_{x\overline{x}}$ = cosine of the angle between the x and $\overline{x}$ axes, etc. These values can be determined from the coordinates of the nodes forming the element. For the whole set of forces acting on the nodes of an element we can then write the equation in the form of equation (11), and finally obtain the stiffness matrix in global coordinates as equation (13)

$$[\bar{K}]^e = [T]^T [K]^e [T] \qquad (13)$$

in which, matrix [T] is given by

$$[T] = \begin{pmatrix} L & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & L \end{pmatrix}. \qquad (60)$$

e. Assembly of Elements

To obtain a complete solution for the entire structure, the two conditions of displacement compatibility and equilibrium have to be satisfied throughout. When any set of nodal displacements

$$\{\delta\} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \delta_n \end{Bmatrix} \qquad (61)$$

is listed for the whole structure in which all the elements participate, the condition of displacement compatibility is automatically satisfied. The overall equilibrium of the complete assemblage is provided by establishing equilibrium at the nodes of the structure. Consider the structure to be loaded by external forces [R]

$$\{R\} = \begin{Bmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ R_n \end{Bmatrix} \qquad (62)$$

applied at the nodes.

We now have to combine all the element properties into the stiffness matrix [K] for the complete structure. It is a matter of simply superimposing the element stiffness matrices in the appropriate positions of the matrix [K]. Then the force-displacement relation for the whole structure can be written as

$$\{R\} = [K]\{\delta\}. \qquad (63)$$

The solution for the unknown displacements can be obtained once the prescribed support displacements have been substituted into equation (63). The resulting displacements are referred to the global system, and before the stresses can be computed it is necessary to transform these into the local system for each element. The element stress matrix, equation (5), for 'in plane' bending components can then be used. In the existing program used in this report, the stresses are assigned to the centroid of each element and are converted to principal stresses and their directions.

4. Solution of Equations

The element stiffness matrices are formed one after the other, and them added to the appropriate locations of the overall matrix in accordance with the nodal numbers in the problem. Since the time necessary for inversion of a matrix increases approximately as the cube of the matrix size, a partitioning scheme is used to reduce the physical size of the stiffness matrix, equation (13). The final partitioned form of the overall stiffness matrix can be shown as in Fig. 5. Physically, this corresponds to the fact that the nodal points of the structure are divided into a number of partitions connected in series as illustrated in Fig. 6.

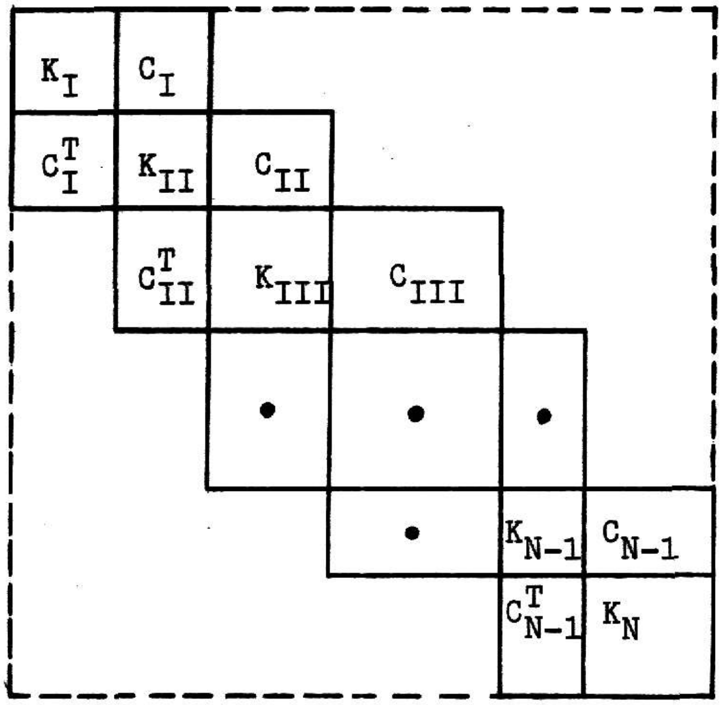

Fig. 5 Partitioned Stiffness Matrix

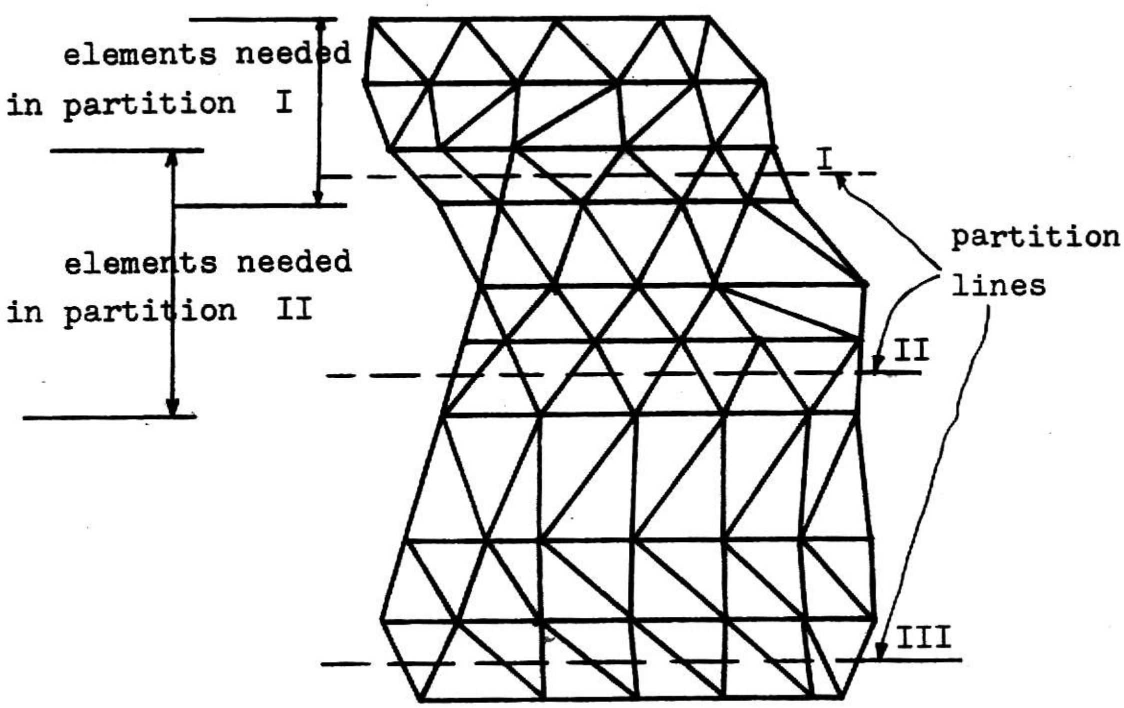

Fig. 6 Partitioning of A Structure

The final partitioned form of the overall matrix is sub-divided into convenient parts which are written in a tridiagonalized manner as follows:

$$\begin{bmatrix} K_I & C_I & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ C_I^T & K_{II} & C_{II} & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & C_{II}^T & K_{III} & C_{III} & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & K_{N-1} & C_{N-1} & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & C_{N-1}^T & K_N & \cdot \end{bmatrix} \begin{Bmatrix} \delta_I \\ \delta_{II} \\ \delta_{III} \\ \cdot \\ \cdot \\ \delta_{N-1} \\ \delta_N \end{Bmatrix} = \begin{Bmatrix} P_I \\ P_{II} \\ P_{III} \\ \cdot \\ \cdot \\ P_{N-1} \\ P_N \end{Bmatrix} . \quad (64)$$

This system of equations will be solved as follows:

The first two matrix equations can be written as

$$[K_I]\{\delta_I\} + [C_I]\{\delta_{II}\} = \{P_I\}$$

and

$$[C_I]^T\{\delta_I\} + [K_{II}]\{\delta_{II}\} + [C_{II}]\{\delta_{III}\} = \{P_{II}\}. \tag{65}$$

The first equation will yield

$$\{\delta_I\} = [K_I]^{-1}\{P_I\} - [K_I]^{-1}[C_I]\{\delta_{II}\} \tag{66}$$

and substituting into the second yields

$$([K_{II}] - [C_I]^T[K_I]^{-1}[C_I])\{\delta_{II}\} + [C_{II}]\{\delta_{III}\}$$

$$= \{P_{II}\} - [C_I]^T[K_I]^{-1}\{P_I\}. \tag{67}$$

By defining new symbols,

$$[\overline{K}_{II}] = ([K_{II}] - [C_I]^T[K_I]^{-1}[C_I]) \text{ and}$$

$$\{\overline{P}_{II}\} = \{P_{II}\} - [C_I]^T[K_I]^{-1}\{P_I\} \tag{68}$$

equation (67) may be written as

$$[\overline{K}_{II}]\{\delta_{II}\} + [C_{II}]\{\delta_{III}\} = \{\overline{P}_{II}\}, \tag{69}$$

from which $\{\delta_{II}\}$ can be obtained as $\{\delta_I\}$ is found in equation (66) and then substituted into the next row equation to give $[\overline{K}_{III}]$ and $\{\overline{P}_{III}\}$.

This process of substitution and elimination goes on until the last row is reached, that is,

$$[\overline{K}_N]\{\delta_N\} = \{\overline{P}_N\} \tag{70}$$