Approximation of p-modulus in the plane with discrete grids

by

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B.A., College of Education in Jeddah, Saudi Arabia, 2003

M.S., King Abdulaziz University, Saudi Arabia, 2009

AN ABSTRACT OF A DISSERTATION

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Department of Mathematics
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Abstract

This thesis contains four chapters. In the first chapter, the theory of continuous p-modulus in the plane is introduced and the background p-modulus properties are provided. Modulus is a minimization problem that gives a measure of the richness of families of curves in the plane. As the main example, we compute the modulus of a 2-by-1 rectangle using complex analytic methods. We also introduce discrete modulus on a graph $G = (V, E)$ and its basic properties. We end the first chapter by providing the relationship between connecting modulus and harmonic functions. This is the fact that computing the modulus of the family of walks from a to b is equivalent to minimizing the energy over all potentials with boundary values 0 at a and 1 at b.

In the second chapter, we are interested in the connection between the continuous and the discrete modulus. We study the behavior of side-to-side modulus under some grid refinements and find an upper bound for the discrete modulus using the concept of Fulkerson duality between paths and cuts. These calculations show that the refinement will lower the discrete modulus. Since connecting modulus can also be computed by minimizing the Dirichlet energy of potential functions, we recall an argument of Jacqueline Lelong-Ferrand, that shows how refining a square grid in a "geometric" fashion, naturally decreases the 2- the energy of a potential. This monotonicity can be used to prove the convergence between continuous and discrete modulus. We first review the linear theory of discrete holomorphicity and harmonicity as provided by Skopenkov and Werness. Instead of reviewing their work in full generality, we present the outline of their arguments in the special case of square grids. Then use these results to prove the convergence between the continuous and discrete case. We believe that our method of proof generalizes to the full case of quadrangular grids that Werness studies.

In the third chapter, we show how to generalize all our proofs for 2-modulus to the case of quadrangular grids with some geometric conditions on the lengths of edges and the angles between them.

In the last chapter, a connection with potentials when $2 < p < \infty$ is discussed in the square grid case. We study the behavior of side-to-side p-modulus under the same refinements as before and we find upper bound for the p-modulus, but only when $p > 2$. The rest of the chapter is dedicated to generalizing the results from Chapter 2 to the case $2 < p < \infty$.

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To the memory of **my father**, Mousa Alrayes, who always believed in me and my ability to be successful. You are gone but your belief in me has made this journey possible. My dedication to you is a small way to say “Thank you “and I miss you very much.

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Preface

In the mathematical theory of conformal and quasiconformal mappings, the modulus of a collection of curves Γ is a way of measuring the size of Γ that is (quasi)invariant under (quasi)conformal mappings. That is, for any conformal map f the modulus of Γ is equal to the modulus of $f(\Gamma)$. One also works with the conformal extremal length which is the reciprocal of the modulus. The fact that extremal length and conformal modulus are conformal invariants makes them useful tools in the study of conformal and quasi-conformal mappings.

For a graph $G = (V, E)$ and Γ a family of paths in the graph, the originally definition of the discrete extremal length was introduced by Duffin[2]. Consider a function $\rho : E \to [0, \infty)$. The ρ -length of a path is defined as the sum of $\rho(e)$ over all edges in the path, counted with multiplicity. The "area" $A(\rho)$ is defined as $\sum_{e\in E} \rho(e)^2$. The extremal length of Γ is then defined as

$$\sup_{\rho} \frac{A(\rho)}{\ell(\Gamma)^2}.$$

If G is interpreted as a resistor network, where each edge has unit resistance, then the effective resistance between two sets of vertices is precisely the extremal length of the collection of paths with one endpoint in one set and the other endpoint in the other set.

For reasons of convenience, we will instead work with modulus, which is essentially the reciprocal of extremal length.

Definition 0.1 (Modulus on the plane)**.** If ρ is a Borel measurable, real-valued non-negative function on X, then we call ρ a density. The ρ-length of a curve, $\ell_\rho(\gamma) := \int_\gamma \rho ds$ and we say ρ is an admissible density for Γ if $\ell_\rho(\gamma) \geq 1$ for all $\gamma \in \Gamma$. The set of admissible densities for Γ defined by $\mathrm{Adm}_X(\Gamma) := \{\rho : X \to [0, \infty) \mid \ell_\rho(\gamma) \geq 1 \ \forall \gamma \in \Gamma\}$. Now, we define the modulus of a family of curves Γ as

$$\mathrm{Mod}_X(\Gamma) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \int_X \rho^2 d\mu.$$

Definition 0.2 (Modulus on the graph)**.** We will define $\mathcal{E}_X(\rho) := \int_X \rho^2 d\mu$, and call it the energy of the density ρ. The $\rho - length$ of $\gamma, \ell_\rho(\gamma)$ is defined by

$$\ell_\rho(\gamma) := \sum_{i=1}^{r} \rho(e_i).$$

the energy of a density ρ is

$$\mathcal{E}(\rho) := \sum_{e \in E} \rho(e)^2$$

The modulus of Γ is defined as

$$\mathrm{Mod}_2(\Gamma) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \mathcal{E}(\rho)$$

Theorem 0.3 (Main Theorem)**.** *Suppose Ω is a simply connected Jordan domain in the complex plane and four distinct boundary points $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \partial\Omega$ are fixed so as to create a pair of opposite sides $E = \partial\Omega(\zeta_1, \zeta_2)$ and $F = \partial\Omega(\zeta_3, \zeta_4)$. Let Γ be the family of continuous curves connecting E and F in Ω.*

Suppose also that Ω is approximated by a quadrilateral lattice domain Ω_n with underlying graph Q_n whose mesh-size tends to zero. Assume that there are corresponding sides E_n and F_n converging to E and F respectively in the Hausdorff topology. Let Γ_n be the family of paths on the graph Q_n connecting E_n to F_n. Then,

$$\mathrm{Mod}_2(\Gamma_n) \longrightarrow \mathrm{Mod}_2(\Gamma).$$

Finally, we generalize most of Werness and Skopenkov theorems to the p case when $p > 2$ but to prove their convergence in this case we need to assume that the limit function of a sequence of discrete p-harmonic functions is a p-harmonic function. It is still an open *problem to prove that this limit is p-harmonic.*

Chapter 1

Background on Modulus

1.1 Modulus in the plane

Definition 1.1. If (X, d, μ) is a metric measure space, then we say that $\gamma : [0, 1] \to X$ traces out a curve if γ is continuous. We say that γ is **rectifiable** if its **total variation**, denoted $TV(\gamma)$, is finite. We define the total variation of γ as

$$TV(\gamma) := \sup_{0=t_0<t_1<\ldots<t_N=1} \sum_{j=0}^{N-1} |\gamma(t_{j+1}) - \gamma(t_j)|.$$

Definition 1.2. We recall that a function $f : (X, \mathfrak{M}, \mu) \to (Y, \mathfrak{N}, \nu)$ is **Borel-measurable** if for every B a Borel set in $\mathfrak{N}$, we have $f^{-1}(B)$ is a Borel set in $\mathfrak{M}$. Moreover, it can be shown that it suffices to show that for every G open subset of Y, that $f^{-1}(G)$ is a Borel set in the σ-algebra $\mathfrak{M}$.

By definition, a **path family** in a domain Ω is a non-empty set Γ of countable unions of rectifiable arcs in Γ.

To define the modulus, we must discuss what 'the set of admissible densities for Γ', denoted $Adm(\Gamma)$ means. To this end, we must define a density. If ρ is a Borel measurable, real-valued non-negative function on X, then we call ρ a density. Moreover, we can define the **ρ-length** of a curve, $\ell_\rho(\gamma) := \int_\gamma \rho ds$ and we say ρ is an **admissible density for Γ**

if $\ell_\rho(\gamma) \geq 1$ for all $\gamma \in \Gamma$. Finally, we define **the set of admissible densities for** Γ by $\mathrm{Adm}_X(\Gamma) := \{\rho : X \to [0,\infty) \mid \ell_\rho(\gamma) \geq 1 \ \forall \gamma \in \Gamma\}$. Now, we define the modulus of a family of curves Γ as

$$\mathrm{Mod}_\Omega(X) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \int_X \rho^2 d\mu.$$

We will define $\mathcal{E}_X(\rho) := \int_X \rho^2 d\mu$, and call it the **energy** of the density ρ. When $E \subset \bar{\Omega}$ and $F \subset \bar{\Omega}$ the modulus $\mathrm{Mod}_\Omega(E, F)$ from E to F is defined by

$$\mathrm{Mod}_\Omega(E, F) = \mathrm{Mod}_\Omega(\Gamma)$$

where Γ is the family of connected curves in Ω that join E and F.

1.1.1 Properties of the modulus

Proposition 1.3. (Basic properties of the modulus).

1. *If* $\Gamma := \{$*all curves that are not locally rectifiable*$\}$, *then* $\mathrm{Mod}\,\Gamma = 0$.

2. *If* $\Gamma := \emptyset$ *then* $\mathrm{Mod}\,\Gamma = 0$.

3. *If there exists* $\gamma_0 \in \Gamma$ *such that* $\gamma_0(t) \equiv z_0$, *then* $\mathrm{Mod}\,\Gamma = \infty$.

Proposition 1.4. (Monotonicity). *If* $\Gamma_1 \subset \Gamma_2$ *then* $\mathrm{Mod}\,\Gamma_1 \leq \mathrm{Mod}\,\Gamma_2$.

Proposition 1.5. (Subadditivity). *If* $\{\Gamma_j\}_{j\in\mathbb{N}}$ *is a countable collection of families of curves, then* $\mathrm{Mod}\,(\cup_j \Gamma_j) \leq \sum_j \mathrm{Mod}(\Gamma_j)$.

Proposition 1.6. (Extension Rule). *If* Γ *is a family of curves in* $\Omega \subset \Omega'$ *then* $\mathrm{Mod}_\Omega(\Gamma) = \mathrm{Mod}_{\Omega'}(\Gamma)$.

It means that the modulus depend only on the path family Γ and not the domain Ω, and for this reason we will write $\mathrm{Mod}(\Gamma)$ instead of $\mathrm{Mod}_\Omega(\Gamma)$

Proof of proposition 1.6. Let $\rho \in \mathrm{Adm}_{\Omega}(\Gamma)$. Then define

$$\tilde{\rho}(z) = \begin{cases} \rho(z) & z \in \Omega \\ 0 & \Omega' \setminus \Omega. \end{cases}$$

It follows that $\tilde{\rho} \in \mathrm{Adm}_{\Omega'}(\Gamma)$ since $\forall \gamma \in \Gamma$,

$$\ell_{\tilde{\rho}}(\gamma) = \int_{\gamma} \rho(z)|dz| = \int_0^1 \tilde{\rho}(\gamma(t))|\gamma'(t)|dt = \int_0^1 \rho(\gamma(t))|\gamma'(t)|dt \geq 1,$$

since $\rho \in \mathrm{Adm}_{\Omega}(\Gamma)$ and $\gamma \subset \Omega$. Moreover,

$$\mathcal{E}_{\Omega}(\tilde{\rho}) = \iint_{\Omega'} \tilde{\rho}^2 dA = \iint_{\mathrm{supp}(\tilde{\rho})} \tilde{\rho}^2 dA = \iint_{\Omega} \tilde{\rho}^2 dA = \iint_{\Omega} \rho^2 dA = \mathcal{E}_{\Omega}(\rho), .$$

taking the infimum over $\rho \in \mathrm{Adm}_{\Omega}(\Gamma)$ results in $\mathrm{Mod}_{\Omega'}(\Gamma) \leq \mathrm{Mod}_{\Omega}(\Gamma)$.

To see the other direction, start with $\tilde{\rho} \in \mathrm{Adm}_{\Omega'}(\Gamma)$ and define $\rho = \tilde{\rho}\big|_{\Omega}$. It follows just as before that $\rho \in \mathrm{Adm}_{\Omega}(\Gamma)$ and $\mathcal{E}_{\Omega}(\rho) = \mathcal{E}_{\Omega'}(\tilde{\rho})$. This time taking the infimum over $\tilde{\rho} \in \mathrm{Adm}_{\Omega'}(\Gamma)$ yields $\mathrm{Mod}_{\Omega}(\Gamma) \leq \mathrm{Mod}_{\Omega'}(\Gamma)$, completing the proof. □

Proposition 1.7. (Serial Role) *Let Γ_1 and Γ_2 be path families contained in disjoint open sets Ω_1 and Ω_2 respectively, and let Γ be a path family contained in a domain $\Omega_1 \cup \Omega_2 \subset \Omega$. If each $\gamma \in \Gamma$ contains some $\gamma_1 \in \Gamma_1$ and some $\gamma_2 \in \Gamma_2$,then*

$$\frac{1}{\mathrm{Mod}\,\Gamma} \geq \frac{1}{\mathrm{Mod}\,\Gamma_1} + \frac{1}{\mathrm{Mod}\,\Gamma_2}.$$

Proof. Let $\rho_j \in \mathrm{Adm}(\Gamma_j)$. Define $\rho(z) = a\mathbb{1}_{\Omega_1}(z)\rho_1(z) + b\mathbb{1}_{\Omega_2}(z)\rho_2(z)$, for some $a + b = 1$ to be chosen later. Then for all $\gamma \in \Gamma$ there exists $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ so that

$$\int_{\gamma} \rho|dz| \geq \int_{\gamma_1} \rho|dz| + \int_{\gamma_2} \rho|dz| = a\int_{\gamma_1} \rho_1|dz| + b\int_{\gamma_2} \rho_2|dz| \geq a + b,$$

so that $\rho \in \mathrm{Adm}(\Gamma)$. Moreover,

$$\iint \rho^2 dA = a^2 \iint_{\Omega_1} \rho_1^2 + b^2 \iint_{\Omega_2} \rho_2^2 := a^2 x + b^2 y.$$

Since $a + b = 1$, we have

$$a^2 x + b^2 y = a^2 x + (1-a)^2 y = a^2(x+y) - 2ay + y := f(a).$$

We want to minimize f over $a \in [0,1]$. So, $f'(a) = 2a(x+y) - 2y = 0$, yields $a = \frac{y}{x+y}$ and consequently $b = \frac{x}{x+y}$. So,

$$\mathrm{Mod}\,\Gamma \leq \iint \rho^2 dA = \frac{y^2 x + x^2 y}{(x+y)^2} = \frac{xy}{x+y} = \frac{1}{\frac{1}{x} + \frac{1}{y}}.$$

Taking the infimum over ρ_1 and ρ_2 and recalling the definitions of x and y achieves the desired result.

□

Proposition 1.8. (Parallel Rule). *If $\Gamma_1 \subset \Omega_1$ and $\Gamma_2 \subset \Omega_2$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. Then for any $\Omega \supset \Omega_1 \cup \Omega_2$ and Γ a family of curves in Ω if $\forall \gamma \in \Gamma_j$ there exists $\gamma' \in \Gamma$ so that $\gamma' \subset \gamma$. Then $\mathrm{Mod}\,\Gamma \geq \mathrm{Mod}\,\Gamma_1 + \mathrm{Mod}\,\Gamma_2$.*

Proof.

Let $\rho \in \mathrm{Adm}(\Gamma)$. Define $\rho_j(z) := \rho(z)\mathbb{1}_{\Omega_j}(z)$ for $j \in \{1,2\}$. Then by assumption for any $j \in \{1,2\}$ and $\gamma_j \in \Gamma_j$, there exists γ' in Γ so that $\gamma' \subset \gamma_j$. It follows that $\rho_j \in \mathrm{Adm}(\Gamma_j)$ since,

$$\ell_{\rho_j}(\gamma_j) = \int_{\gamma_j} \rho_j(z)|dz| = \int_{\gamma_j} \rho(z)|dz| \geq \int_{\gamma'} \rho(z)|dz| \geq 1.$$

Consequently,

$$\mathrm{Mod}(\Gamma_1) + \mathrm{Mod}(\Gamma_2) \leq \iint_{\Omega_1} \rho_1^2 dA + \iint_{\Omega_2} \rho_2^2 dA \leq \iint_{\Omega} \rho^2 dA.$$

Taking the infimum over $\rho \in \mathrm{Adm}(\Gamma)$ yields the desired result.

$\square$

Proposition 1.9. (Symmetry Rule). *If $T : \Omega \to \Omega$ is an involution, i.e., $T \circ T(z) = z$, and $T(\Gamma) = \Gamma$, then*

$$\mathrm{Mod}(\Gamma) = \inf_{\substack{\rho \in \mathrm{Adm}(\Gamma) \\ \rho = \rho \circ T |\det DT|}} \iint_\Omega \rho^2 dA,$$

Proof. It is clear that

$$\mathrm{Mod}(\Gamma) \leq \inf_{\substack{\rho \in \mathrm{Adm}(\Gamma) \\ \rho = \rho \circ T |\det DT|}} \iint_\Omega \rho^2 dA,$$

Now, Suppose ρ is admissible. Define $\tilde{\rho} := (\rho \circ T)|\det DT|$. Then

$$\int_\gamma \tilde{\rho}(s) ds = \int_\gamma (\rho \circ T)|\det DT||dz| = \int_{T^{-1} \circ \gamma} \rho(w) dw = \int_{T \circ \gamma} \rho(w) dw \geq 1,$$

where the second equality follows by a change of variables and the third equality since $T \circ T = \mathrm{Id}$, and the final inequality since $T(\Gamma) = \Gamma$. Hence $\tilde{\rho} \in \mathrm{Adm}(\Gamma)$. So, we define $\rho' = \frac{\rho + \tilde{\rho}}{2}$. It is clear that ρ' is admissible since it is a convex combination of admissible densities. Moreover,

$$\rho' \circ T = \frac{\rho \circ T + \tilde{\rho} \circ T}{2} = \frac{\tilde{\rho} + \rho \circ T \circ T}{2} = \rho'.$$

Note, $(\rho')^2 = \frac{1}{4}\left(\rho^2 + \tilde{\rho}^2 + 2\rho\tilde{\rho}\right) = \frac{1}{2}\rho^2 + \frac{1}{2}\tilde{\rho}\rho$, since $\tilde{\rho}^2 = (\rho \circ T)(\rho \circ T)|\det DT|^2 = \rho^2 \circ T$. Consequently,

$$\begin{aligned} &\inf_{\substack{\sigma \in \mathrm{Adm}(\Gamma) \\ \sigma = \rho \circ T |\det DT|}} \iint_\Omega \sigma^2 dA \leq \iint_\Omega (\rho')^2 dA = \frac{1}{2} \iint_\Omega \rho^2 dA + \frac{1}{2} \iint \tilde{\rho}\rho dA \\ \leq &\frac{1}{2}\left[\iint_\Omega \tilde{\rho} + \left(\iint_\Omega \tilde{\rho}^2 dA\right)^{\frac{1}{2}} \left(\iint_\Omega \rho^2 dA\right)^{\frac{1}{2}}\right] = \iint_\Omega \rho^2 dA, \end{aligned}$$

where the second line follows from Cauchy-Schwarz inequality and again that $\tilde{\rho}^2 = \rho^2 \circ T$. Taking the infimum over $\rho \in \mathrm{Adm}(\Gamma)$, we attain the desired result. $\square$

1.1.2 Conformal invariance

Theorem 1.10. (Conformal Invariance of Modulus). *If $\varphi : \Omega \to \varphi(\Omega)$ is analytic and one-to-one, and Γ is a family of curves in Ω, then $\varphi(\Gamma) := \{\varphi \circ \gamma : [0,1] \to \varphi(\Omega)\}$ satisfies,*

$$\mathrm{Mod}_{\Omega}(\Gamma) = \mathrm{Mod}_{\varphi(\Omega)}(\varphi(\Gamma)).$$

Proof. Fix arbitrary $\rho \in \mathrm{Adm}(\varphi(\Gamma))$. Then $\int_{\varphi\circ\gamma} \rho(w)|dw| \geq 1$ for all $\gamma \in \Gamma$. Define $\widetilde{\rho}(z) := \rho(\varphi(z))|\varphi'(z)| \quad \forall z \in \Omega$. Then we observe,

$$\int_{\gamma} \widetilde{\rho}(z)|dz| = \int_{\gamma} \rho(\varphi(z))|\varphi'(z)||dz| = \int_{\varphi\circ\gamma} \rho(w)|dw|.$$

So we can conclude that $\widetilde{\rho} \in \mathrm{Adm}(\Gamma)$. Hence,

$$\mathrm{Mod}_{\Omega}(\Gamma) \leq \iint_{\Omega} (\widetilde{\rho}(z))^2 \, dA(z) = \iint_{\Omega} (\rho(z))^2 \, |\varphi'(z)|^2 dA(z) = \iint_{\varphi(\Omega)} \rho^2(w) dA(w).$$

Since $\rho \in \mathrm{Adm}(\varphi(\Gamma)$ was arbitrary, we can then take the infimum over all $\rho \in \mathrm{Adm}(\varphi(\Gamma))$ to attain, $\mathrm{Mod}_{\Omega}(\Gamma) \leq \mathrm{Mod}_{\varphi(\Omega)}(\varphi(\gamma))$. Repeating the process with φ^{-1} in the place of φ results in the opposite inequality, and we achieve the desired result. $\square$

Proposition 1.11. *If Ω is a Jordan domain and $\xi_1, \xi_2, \xi_3, \xi_4 \in \partial\Omega$ are distinct and ordered counter clockwise, then there exists a unique $h > 0$ and a unique conformal map $\varphi : \Omega \to R$ where R is normalized so that $R := \{z : 0 < \mathrm{Re}\, z < 1 \text{ and } 0 < \mathrm{Im}\, z < h\}$, satisfying $\varphi(\xi_1) = hi, \varphi(\xi_2) = 0, \varphi(\xi_3) = 1,$ and $\varphi(\xi_4) = 1 + hi$.*

Proof. The proposition claims both uniqueness and existence. First we will show existence.

(Existence)

By the Riemann Mapping theorem, there exists $\varphi_1 : \Omega \to \mathbb{H}$ where $\mathbb{H} = \{z : \mathrm{Im}\, z \geq 0\}$ and by Schwarz reflection principle and Caratheodory's theorem, φ can be extended continuously to $\bar{\Omega}$, so WLOG $x_j := \varphi(\xi_j) \in \mathbb{R}$ for $j \in \{1,2,3,4\}$. Then by a linear fractional transformation φ_2 we can send $x_1 \mapsto 0, x_2 \mapsto 1, x_3 \mapsto \lambda, x_4 \mapsto \infty$ for some $1 < \lambda < \infty$. Then, we can define a linear fractional transformation, φ_3 that takes $0 \mapsto -A, 1 \mapsto -1, \lambda \mapsto 1$, and

$\infty \mapsto A$. Since linear fractional transformations are uniquely defined by where they send three points, this is not immediately obvious. So to see this, we will explicitly define the inverse of φ_3 as

$$\varphi_3^{-1}(z) = \left(\frac{z+A}{-z+A}\right)\left(\frac{A+1}{A-1}\right).$$

Since we want $\varphi_3^{-1}(1) = \lambda$ we have to choose the correct A. So we define A so that, $\varphi_3^{-1}(1) = \left(\frac{A+1}{A-1}\right)^2 = \lambda$. This means that $A = \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}$. Since $\lambda > 1$, A is positive, also we see that $A > 1$ as needed to keep the ordering of our points correct along the real line (i.e., $\varphi_3 \circ \varphi_2 \circ \varphi_1(\xi_j) < \varphi_3 \circ \varphi_2 \circ \varphi_1(\xi_k)$ whenever $j < k$. Then the Schwarz-Christoffel integral,

$$\int_0^z \frac{1}{\sqrt{(\zeta^2 - A^2)(\zeta^2 - 1)}} d\zeta,$$

will take the points $-A, -1, 1, A$ to the corners of a rectangle of height h depending on A, and width 1 and will preserve the order as desired.

(Uniqueness). Using the conformal invariance theorem of the modulus, suppose that there are two maps φ_1, φ_2 that map Ω to the rectangles $R_{h_j} := \{z : 0 < \operatorname{Re} z < 1 \text{ and } 0 < \operatorname{Im} z < h_j\}$ for $j \in \{1, 2\}$ and $h_1 \neq h_2$. Then, consider the conformal map $\varphi_2 \circ \varphi_1^{-1}$ from R_{h_1} to R_{h_2},fixing the orders of the corners. Then, consider the family of curves $\Gamma := \{\gamma : [0,1] \to \overline{R_{h_1}} | \operatorname{Re}\gamma(0) = 0, \operatorname{Re}\gamma(1) = 1\}$. Since $\operatorname{Mod}_{R_{h_1}}(\Gamma) = \operatorname{Mod}_{R_{h_2}}(\varphi_2 \circ \varphi_1^{-1}(\Gamma)$, by the basic example below we will have that $h_1 = h_2$, a contradiction. □

1.1.3 Examples

Example 1) The Basic Example For this example we make the following claim:

Let $\mathcal{R} := \{z = x + iy \in \mathbb{C} : 0 < x < \ell, 0 < y < h\}$ denote the rectangle of height h and length ℓ, also let $E := \{z \in \overline{\mathcal{R}} : \operatorname{Re} z = 0\}$ and $F := \{z \in \overline{\mathcal{R}} : \operatorname{Re} z = \ell\}$. If $\Gamma = \Gamma_{\mathcal{R}}(E, F)$ then, $\operatorname{Mod}(\Gamma) = \frac{h}{\ell}$.

Proof. For all $0 < y < h$ define $\gamma_y(t) := t + iy$. Then if $\rho \in \operatorname{Adm}(\Gamma)$, in particular,

$\ell_\rho(\gamma_y(t)) \geq 1$, so $\ell_\rho(\gamma_y(t)) = \int_0^\ell \rho(t,y)dy \geq 1$. Using Cauchy-Sechwartz we obtain,

$$1 \leq \left[\int_0^\ell \rho(t,y)dt\right]^2 \leq \left(\int_0^\ell \rho^2(t,y)dt\right)\left(\int_0^\ell dt\right) = \ell \int_0^\ell \rho^2(t,y)dt.$$

In particular, $\frac{1}{\ell} \leq \int_0^\ell \rho^2(t,y)dt$. Integrating over y, we get

$$\frac{h}{\ell} = \int_0^h \frac{1}{\ell}dy \leq \int_{\mathcal{R}} \rho^2 dA,$$

so that $\operatorname{Mod}(\Gamma) \geq \frac{h}{\ell}$.

For the other direction, we define $\rho_0(z) = \frac{1}{\ell}\mathbb{1}_{\mathcal{R}}(z)$ and observe that $\int_{\mathcal{R}} \rho^2 = \frac{h\ell}{\ell^2} = \frac{h}{\ell}$. Hence, if $\rho_0 \in \operatorname{Adm}(\Gamma)$, then $\operatorname{Mod}(\Gamma) \leq \frac{h}{\ell}$. Indeed,

$$\ell_\rho(\gamma) = \int_0^\ell \frac{1}{\ell}|\dot\gamma(t)|dt \geq \frac{1}{\ell}\int_0^\ell |\operatorname{Re}\dot\gamma(t)|dt \geq \frac{1}{\ell}\left(\operatorname{Re}\gamma(1) - \operatorname{Re}\gamma(0)\right) \geq 1 \quad \forall \gamma \in \Gamma,$$

where the penultimate inequality follows from

$$\int_0^\ell |\operatorname{Re}\dot\gamma(t)|dt = \sup_{0=t_0<\ldots<t_N=1} \sum_{j=0}^{N-1} |\operatorname{Re}\gamma(t_{j+1}) - \operatorname{Re}\gamma(t_j)| \geq \gamma(1) - \gamma(0).$$

Thus we conclude ρ_0 is admissible, which completes the proof. □

Annulus Example Let $A := \{z : r < |z| < R\}$. Then define the family of curves $\Gamma_A := \{\gamma : [0,1] \to \mathbb{C} \big| |\gamma(0)| = r, |\gamma(1)| = R,$ and $\gamma(t) \in \overline{A}, \forall t \in [0,1].\}$. We will show that $\operatorname{Mod}(\Gamma_A) = \frac{2\pi}{\ln(R)-\ln(r)}$.

Proof. First, let $\widetilde{\Gamma}_A := \{\gamma \in \Gamma_A \big| \gamma(t) = e^{i\theta_\gamma}r(t)$ for some $\theta_\gamma \in (-\pi,\pi)\}$, that is $\widetilde{\Gamma}_A$ is the set of radial curves from Γ_A that do not intersect the negative real axis. Then clearly, $\widetilde{\Gamma}_A \subset \Gamma_A$, so that by monotonicity $\operatorname{Mod}(\widetilde{\Gamma}_A) \leq \operatorname{Mod}(\Gamma_A)$. Then, $\forall \gamma \in \widetilde{\Gamma}_A$ we have that $\gamma \subset A_s := \{z \in \mathbb{C} \setminus (-\infty, 0] \big| r < |z| < R\}$, the slit annulus. The (analytic) complex logarithm defined on the slit plane then takes A_s to $\{z \in \mathbb{C} \big| \ln(r) < \operatorname{Re} z < \ln(R)$ and $-i\pi < \operatorname{Im} z < i\pi\}$. Thus,

$\varphi_1(z) := \frac{z-(\ln(r)-\pi i)}{\ln(R)-\ln(r)}$, is analytic and one-to-one satisfying,

$$\varphi_1(\log(A_s)) = \left\{ z \in \mathbb{C} \middle| 0 < \operatorname{Re} z < 1 \text{ and } 0 < \operatorname{Im} z < h = \frac{2\pi}{\log(R) - \log(r)} \right\} := R_h.$$

Finally, we observe that the curves $\gamma \in \widetilde{\Gamma}_A$ are mapped to the horizontal curves connecting $A := \partial R \cap \{z \big| \operatorname{Re} z = 0\}$ to $B := \partial R \cap \{z \big| \operatorname{Re} z = 1\}$, so that by the basic example and the conformal invariance of the modulus we know $\operatorname{Mod}(\widetilde{\Gamma}_A) = \operatorname{Mod}(\Gamma_{R_h}(C, D)) = h = \frac{2\pi}{\ln(R)-\ln(r)}$. Hence,

$$\frac{2\pi}{\ln(R) - \ln(r)} \leq \operatorname{Mod}(\Gamma_A). \tag{1.1}$$

To see the other direction, we consider the density, $\rho_0(z) := \frac{C}{|z|}$, with C to be determined later. Choose $\gamma \in \Gamma_A$ and consider the polar form of the parametrization of γ, i.e., $\gamma(t) = r(t)e^{i\theta(t)}$. We observe,

$$\begin{aligned} \ell_{\rho_0}(\gamma) = \int_\gamma \rho_0(s)ds = \int_0^1 \rho_0(r(s)e^{i\theta(s)}) \left| r'(s)e^{i\theta(s)} + i\theta'(s)r(s)e^{i\theta(s)} \right| ds \geq \\ \int_0^1 \frac{C}{|r(s)|} |r'(s)| \, ds \geq C \int_I \frac{r'(s)}{r(s)} ds \geq C \ln(|\gamma|) \Big|_{|\gamma|=r}^{|\gamma|=R} = C \left[\ln(R) - \ln(r) \right], \end{aligned} \tag{A.1}$$

where the top row follows from a simple change of variables, and the product rule. The set $I \subset [0, 1]$ is chosen to be the set of $s \in [0, 1]$ such that $r'(s) > 0$, and the final inequality follows since the integral $\int_I \frac{r'(s)}{r(s)} ds$, is the total variation of the natural logarithm of r over the set I, which is at least as big as the evaluation of the natural logarithm at exteme values of I. In light of (A.1) and desiring to make ρ_0 be admissible, we choose $C = \frac{1}{\ln(R)-\ln(r)}$. Since $\gamma \in \Gamma_A$ was arbitrary, we then have that ρ_0 is indeed admissible. Hence,

$$\operatorname{Mod}(\Gamma_A) \leq \iint_A \rho_0^2(s) dA(s) = \frac{1}{(\ln(R) - \ln(r))^2} \int_0^{2\pi} \int_r^R \frac{1}{s^2} (s ds d\theta) = \frac{2\pi}{\ln(R) - \ln(r)}. \tag{1.2}$$

from(4.1) and(4.2) we have $\frac{2\pi}{\ln(R)-\ln(r)} \leq \operatorname{Mod}(\Gamma_A) \leq \frac{2\pi}{\ln(R)-\ln(r)}$. $\square$

1.2 Modulus of 2 by 1 rectangle using complex analysis

In the following a conformal map f from a region D to another region G is assumed to be analytic in the domain D and one-to-one and onto G. Conformal maps f satisfy $f'(z) \neq 0$, for all $z \in D$, which means that infinitesimally the map stretches uniformly in all directions. Also, the inverse function f^{-1} is a conformal map of G to D.

If we suppose that f is an analytic function which is defined in the upper half-disk $\mathbb{D}^+ := \{|z|^2 < 1, \ \mathrm{Im}(z) > 0\}$, and we further suppose that f extends continuously to the real axis, so that it takes on real values on the real axis, then f can be extended to an analytic function on the whole unit disk $\mathbb{D}$ by the formula

$$f(\bar{z}) = \overline{f(z)} \quad \forall z \in \mathbb{D}.$$

This formula states that the values taken by f at points that are symetric with respect to $\mathbb{R}$ are symmetric with respect to $\mathbb{R}$ in $f(\mathbb{D})$. This is known as the **Schwarz Reflection Principle**. There are many applications for the reflection principle. For instance, it is used to derive an explict formula for the conformal map that maps a half-plane onto the interior of a polygon. This is known as the **Schwarz-Chritoffel mapping function**, see (1.3).

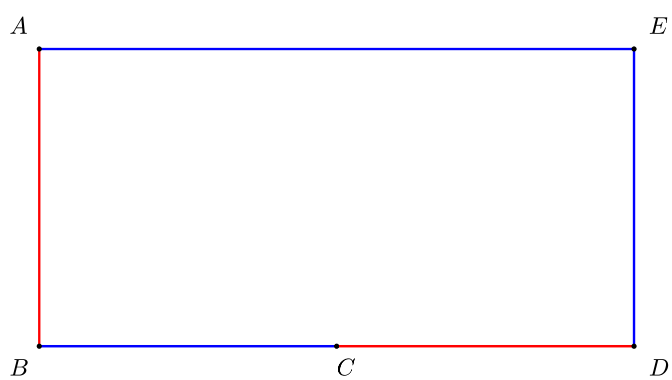

Figure 1.1: *2 by 1 rectangle*

Now, we want to compute the following connecting modulus in the 2 by 1 rectangle

$R = [-1, 1] \times [0, 1]$. using complex analysis. Namely, we want the modulus of the family of curves connecting the side $AB = \{-1\} \times [0, 1]$ to the segment $CD = [0, 1] \times \{0\}$. See Figure 1.1

By symmetry, there is a unique conformal map F that maps the unit disk $\mathbb{D}$ to the unit square such that

$$F(\bar{z}) = \bar{F}(z), \quad F(-\bar{z}) = -\bar{F}(z) \quad \forall z \in \mathbb{D}$$

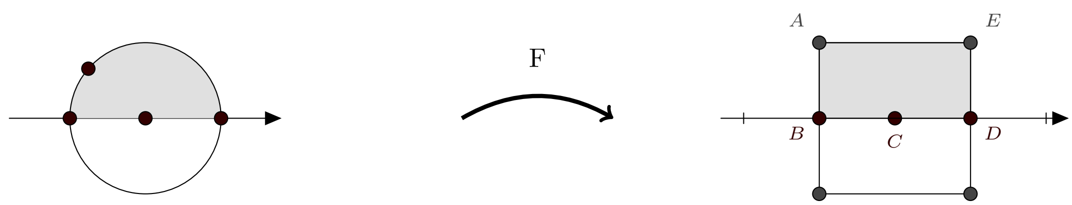

Figure 1.2: *Mapping the semidisk into 2 by 1 rectangle*

So, the quadrilateral $(R; A, B, C, D)$ is conformally equivalent to $(\mathbb{D}^+; e^{i\frac{3\pi}{4}}, -1, 0, 1)$, where $\mathbb{D}^+ = \{z \in \mathbb{C} : |z| \leq 1, \quad Im(z) > 0\}$. This means that F is a conformal map from the semidisk into a 2 by 1 rectangle see Figure 1.2. Now, we map the semidisk to the upper half plane. First, map the upper half disk to the upper half plane with semidisk removed with $f(z) = -i\frac{z+i}{z-i}$. See Figure 1.3. We have, $f(e^{i\frac{3\pi}{4}}) = -(\sqrt{2} + 1)$

Figure 1.3: *Mapping the semidisk into the upper half plane with semidisk removed*

Next, using the Joukowsky transform, $g(z) = \frac{1}{2}(z + \frac{1}{z})$, map the upper half plane with semidisk removed to the upper half plane. See Figure 1.4. We have $g(-(1 + \sqrt{2})) = -\sqrt{2}$

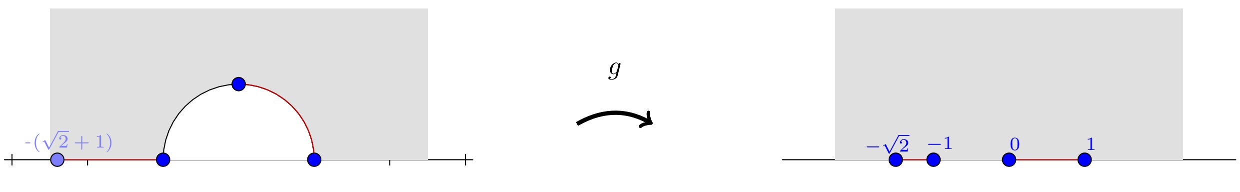

Figure 1.4: *Map the upper half plane with semidisk removed into the upper half plane*

Then, there is a unique Mobuis transformation

$$\tau(z) = \frac{az+b}{cz+d}$$

which maps $0 \to 1, \quad -1 \to -1, \quad 1 \to \lambda, \quad -\sqrt{2} \to -\lambda$. See Figure 1.5. Solving the equations we get,

$$\tau(z) = \frac{(3\lambda-1)z+(\lambda+1)}{(3-\lambda)z+(\lambda+1)}$$

and

$$\lambda = 7 - 4\sqrt{2} + (4-2\sqrt{2})\sqrt{2-\sqrt{2}}$$

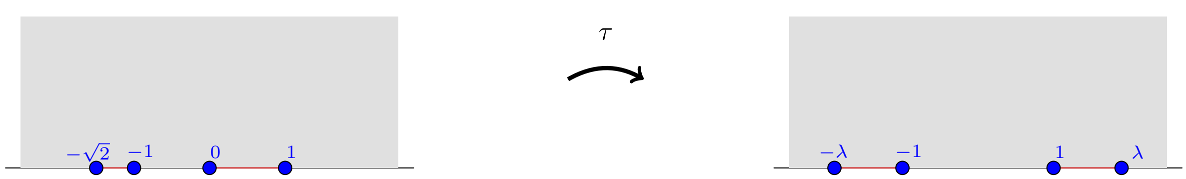

Figure 1.5: *Mapping the upper half plane into the upper half plane with equaly spaced vertices*

Using Schwarz-christoffel map see Fiigure 1.6, we can map the upperhalf plane with prevertices $-1, 1, \lambda, -\lambda$ to a rectangle by :

$$\phi(w) = C\int_0^w \frac{dz}{\sqrt{(1-z^2)(1-\kappa^2 z^2)}}dz, \tag{1.3}$$

where $\kappa := \frac{1}{\lambda}$, so $0 < \kappa < 1$ and $C > 0$ is a constant to be determined so that $\phi(1) = 1/2$.

Below p_2 depends on λ.

Let the corners of the rectangle be at

$$\frac{1}{2}, \qquad \frac{1}{2} + ip_2, \qquad -\frac{1}{2} + ip_2, \qquad -\frac{1}{2}$$

where $p_2 > 0$

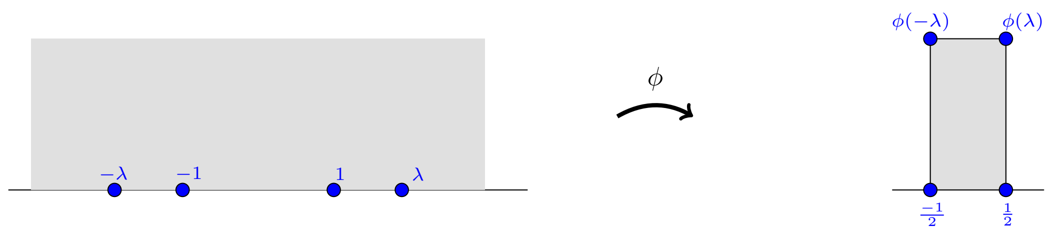

Figure 1.6: *Schwarz-christoffel map*

To evaluate C, set

$$\phi(1) = \frac{1}{2}$$

$$C \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-\kappa^2 z^2)}} = \frac{1}{2}$$

$$C = \frac{1}{2\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-\kappa^2 z^2)}}}$$

$$2C \ K(\kappa) = 1$$

Where

$$K(\kappa) := \int_0^1 [(1-z^2)(1-\kappa^2 z^2)]^{-\frac{1}{2}} dz$$

is known as a **complete elliptic integral of the first kind**[3].

By setting $z = \sin\phi$, we obtain

$$K(\kappa) = \int_0^{\frac{\pi}{2}} [1 - \kappa^2 \sin^2 \phi]^{-\frac{1}{2}} d\phi$$

Next we have,

$$ip_2 = C \int_1^{\frac{1}{\kappa}} [(1 - z^2)(1 - \kappa^2 z^2)]^{-\frac{1}{2}} dz$$

Replacing z by $\frac{1}{z}$, we get :

$$ip_2 = C \int_\kappa^1 [(z^2 - 1)(z^2 - \kappa^2)]^{-\frac{1}{2}} dz$$

and

$$p_2 = C \int_\kappa^1 [(1 - z^2)(z^2 - \kappa^2)]^{-\frac{1}{2}} dz$$

The integeral for p_2 can also be written as a complete elliptic integeral of the first kind. Namely,

$$p_2 = K(\kappa')$$

where $\kappa' = \sqrt{1 - \kappa^2}$.

Then,

$$\text{Mod}(AB, CD) = p_2 = C\ K(\kappa') = \frac{K(\kappa')}{2\ K(\kappa)} = 0.68063417306$$

1.3 Modulus on Graphs

Let G be a graph with vertex set V and edges set E. The sets of vertices and edges are assumed to be finite, with $n = |V|$ vertices and $m = |E|$ edges. The graph may be directed or undirected. We will define the weight function to be $\sigma : E \to (0, \infty)$. A graph is called unweighted if $\sigma \equiv 1$. Here, we will consider only unweighted graph.

A walk is a finite sequence of vertices $v_1, v_2, ..., v_r$ in V, with the property that $(v_i, v_{i-1}) \in E$ or a finite sequence of edges $\gamma = e_1, e_2, ..., e_r$ where $e_i = (v_i, v_{i-1})$ for $i = \{1, 2, ..., r\}$ For each walk there is a graph length $\ell(\gamma) = r$. In general, given any edge density $\rho : E \to \mathbb{R}$

the $\rho - length$ of $\gamma, \ell_\rho(\gamma)$ is defined by

$$\ell_\rho(\gamma) := \sum_{i=1}^{r} \rho(e_i).$$

This notation depends on the walk, so instead for every walk define the following *edge-usage* vector:

$$\mathcal{N}(\gamma, e) := \begin{cases} 1 & \text{if } e \in \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Then the ρ-length of γ can be written as

$$\ell_\rho(\gamma) = \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e).$$

Let Γ be a family of walks and let $\rho : E \to \mathbb{R}$ be an edge density, the graph length and $\rho - length$ of Γ are defined as

$$\ell(\Gamma) := \inf_{\gamma \in \Gamma} \ell(\gamma) \qquad and \qquad \ell_\rho(\Gamma) := \inf_{\gamma \in \Gamma} \ell_\rho(\gamma)$$

The family Γ is associated to a set of densities called the admissible set $\text{Adm}(\Gamma)$ defined as follows:

$$\text{Adm}(\Gamma) := \{\rho : E \to \mathbb{R}, \ell_\rho(\Gamma) \geq 1\}$$

1.3.1 Energy and modulus

Given a real parameter $p \geq 1$ or $p = \infty$, the $p - energy$ of a density ρ is

$$\mathcal{E}_p(\rho) := \begin{cases} \sum_{e \in E} |\rho(e)|^p & if \quad 1 \leq p < \infty \\ \max_{e \in E} |\rho(e)| & if \quad p = \infty. \end{cases}$$

For $1 \leq p \leq \infty$ the $p-modulus$ of Γ is defined as

$$\mathrm{Mod}_p(\Gamma) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \mathcal{E}_p(\rho)$$

Moreover, an edge density is called extremal for a given family Γ and a given p if

$$\mathrm{Mod}_p(\Gamma) = \mathcal{E}(\rho)$$

1.3.2 Properties of discrete p-modulus

The continuous and discrete modulus share many of the same properties although the proof can be different.

Remark 1.12.

- The modulus of an empty family is defined to be zero $\forall \rho$
- If Γ contains a constant walk then $\mathrm{Adm}(\Gamma) = \emptyset$ and $\mathrm{Mod}_p(\Gamma) = \infty$

Proposition 1.13. *For $1 \leq p < \infty$, let $G = (V, E)$ be a finite graph, and let $\Gamma_1, \Gamma_2, \Gamma$ be families of walks.*

- *(Monotonicity) If $\Gamma_1 \subset \Gamma_2$ then $\mathrm{Mod}_p \Gamma_1 \leq \mathrm{Mod}_p \Gamma_2$.*

- *(Subadditivity) If $\{\Gamma_j\}_{j \in \mathbb{N}}$ is a countable collection of families of curves, then $\mathrm{Mod}_p(\cup_j \Gamma_j) \leq \sum_j \mathrm{Mod}_p(\Gamma_j)$.*

Proof.

(Monotonicity)

If $\Gamma_1 \subset \Gamma_2$ then $\rho \in \mathrm{Adm}(\Gamma_2)$ implies that $\rho \in \mathrm{Adm}(\Gamma_1)$ Thus, $\mathrm{Adm}(\Gamma_2) \subset \mathrm{Adm}(\Gamma_1)$ and $\mathrm{Mod}_p(\Gamma_1) = \inf_{\rho \in \mathrm{Adm}(\Gamma_1)} \mathcal{E}_p(\rho) \leq \inf_{\rho \in \mathrm{Adm}(\Gamma_2)} \mathcal{E}_p(\rho) = \mathrm{Mod}_p(\Gamma_2)$

(Subadditivity)

Fix $\epsilon > 0$. For each j choose $\rho_j \in \mathrm{Adm}(\Gamma_j)$ so that $\mathcal{E}_p(\rho_j) \le \mathrm{Mod}_p(\Gamma_j) + \frac{\epsilon}{2^j}$.

Then define $\rho := (\sum\limits_{j\in\mathbb{N}} \rho_j^p)^{\frac{1}{p}}$. We will show that $\rho \in \mathrm{Adm}(\cup\Gamma_j)$. To this end let $\gamma \in \cup_j \gamma_j$. then there is k such that $\gamma \in \Gamma_k$. Since $\rho \ge \rho_k$, so that $\ell_\rho(\gamma) \ge \ell_{\rho_k}(\gamma) \ge 1$. Moreover,

$$\mathrm{Mod}_p(\Gamma) \le \mathcal{E}_p(\rho) = \sum_{e\in E} \rho(e)^p = \sum_{e\in E}\sum_{j=1}^{\infty} \rho_j(e)^p = \sum_{j=1}^{\infty}\sum_{e\in E} \rho_j(e)^p = \sum_{j=1}^{\infty} \mathcal{E}_p(\rho_j) \le \epsilon + \sum_{j=1}^{\infty} \mathrm{Mod}_p\,\Gamma_j$$

Take $\epsilon \to 0$ we will get the desired resuilt. □

Proposition 1.14. *(Parallel Rule)*

Let $G = (V, E)$. Let $\Gamma_1(s,t), \Gamma_2(s,t)$ two sets of families of walks from $s, t \in V$ such that $\Gamma_1(s,t) \cap \Gamma_2(s,t) = \emptyset$, that is , $E(\Gamma_1) \cap E(\Gamma_2) = \emptyset$ and $\Gamma_1(s,t) \cup \Gamma_2(s,t) = \Gamma(s,t)$. Then

$$\mathrm{Mod}_p(\Gamma) = \mathrm{Mod}_p(\Gamma_1) + \mathrm{Mod}_p(\Gamma_2)$$

Proof. Let $\rho \in \mathrm{Adm}(\Gamma)$. Define $\rho_j = \rho \mathbb{1}_{\Gamma_j}$ where $j = 1, 2$. For any $\gamma_i \in \Gamma_j$ there is $\gamma\prime \subset \Gamma$such that $\gamma\prime \subset \gamma_j$ for $j = 1, 2$. This implies that $\ell_{\rho_j}(\gamma_j) \ge \ell_\rho(\gamma\prime) \ge 1$. So that $\rho_j \in \mathrm{Adm}(\Gamma_j)$.Then

$$\begin{aligned}\mathcal{E}_p(\rho) = \sum_{e\in E} \rho(e)^p \\ &= \sum_{e\in E(\Gamma_1)} \rho(e)^p + \sum_{e\in E(\Gamma_2)} \rho(e)^p \\ &= \sum_{e\in E(\Gamma_1)} \rho_1(e)^p + \sum_{e\in E(\Gamma_1)} \rho_2(e)^p \\ &= \mathcal{E}_p(\rho_1) + \mathcal{E}_p(\rho_2)\end{aligned}$$

Take the infimum to both side we get the desired result.

□

Proposition 1.15. *(Serial Rule)*

Let C be a cut for $\Gamma := \Gamma(s,t)$. Define $\Gamma_1 := \Gamma(s, C)$, and $\Gamma_2 := \Gamma(t, C)$. Then for $1 \le p < \infty$

and $\frac{1}{p}+\frac{1}{q}=1$, *we have*

$$\frac{1}{\operatorname{Mod}_p\Gamma} \geq \left\{\frac{(\operatorname{Mod}_p\Gamma_1)^{\frac{q}{p}}+(\operatorname{Mod}_p\Gamma_2)^{\frac{q}{p}}}{[\operatorname{Mod}_p\Gamma_1\operatorname{Mod}_p\Gamma_2]^{\frac{q}{p}}}\right\}^{\frac{q}{p}} = \left\{\left(\frac{1}{\operatorname{Mod}_p\Gamma_1}\right)^{\frac{q}{p}}+\left(\frac{1}{\operatorname{Mod}_p\Gamma_2}\right)^{\frac{p}{q}}\right\}^{\frac{p}{q}}$$

Proof. WLOG, assume $G=(V,E)$ is connected. For $j=1,2$, assume $\Gamma_j\neq\emptyset$, and let $E_j=\cup_{\gamma\in\Gamma_j}E(\gamma)$ and for $\tilde{\rho}_j\in Adm(\Gamma_j)$ define $\rho_j=\tilde{\rho}_j\mathbb{1}_{E_j}$. Then $\rho_j\in Adm(\Gamma_j)$.

Let $\rho=a\rho_1+b\rho_2$ for some $a+b=1$ to be chosen later. Then for any $\gamma\in\Gamma$ there exists $\gamma_j\in\Gamma_j$ such that $\gamma_j\preceq\gamma$ for $j=1,2$. So that, $1=a+b\leq a\ell_{\rho_1}(\gamma_1)+b\ell_{\rho_2}(\gamma_2)=\ell_\rho(\gamma)$ and $\rho\in Adm(\Gamma)$.

Moreover,

$$\mathcal{E}_p(\rho)=\sum_{e\in E}\rho(e)^p=\sum_{e\in E}(a\rho_1(e)+b\rho_2(e))^p=a^p\ell_p(\rho_1)+b^p\ell_p(\rho_2):=a^px+b^py:=f(a,b)$$

we will minimize $f(a,b)$ subject to $a+b-1=0$.

Hence,

$$f(a)=a^px+(1-a)^py$$

and

$$\frac{\partial f}{\partial a}=pa^{p-1}x-p(1-a)^{p-1}y$$

and therefor,

$$\frac{\partial f}{\partial a}=0\Longrightarrow a=(1-a)\left(\frac{y}{x}\right)^{\frac{1}{p-1}}$$

The last equation follows since p and a are nonzero so we can solve for a.

So we have that

$$a\left(1+\left(\frac{y}{x}\right)^{\frac{1}{p-1}}\right)=\left(\frac{y}{x}\right)^{\frac{1}{p-1}}$$

Thus,

$$a=\frac{y^{\frac{1}{p-1}}}{y^{\frac{1}{p-1}}+x^{\frac{1}{p-1}}}$$

and

$$b = \frac{y^{\frac{1}{p-1}}}{x^{\frac{1}{p-1}} + x^{\frac{1}{p-1}}}$$

With the choices of a, b and the fact that that $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\mathrm{Mod}_p\,\Gamma \leq \mathcal{E}_p(\rho) = \frac{xy}{\left(x^{\frac{q}{p}} + y^{\frac{q}{p}}\right)^{p-1}}$$

By back substituting for x and y , and taking the infimum over all $\rho_j \in \mathrm{Adm}(\Gamma_j)$ we get the desired result. □

1.3.3 Modulus of connecting families

We will be interested in computing the modulus of families $\Gamma(A, B)$ of all walks connecting two sets of nodes A and B in a graph G.

Example 1.16 (Basic Example)**.** Let G be a graph consisting of k simple paths in parallel, each path taking ℓ hops to connect a given vertex s to a given vertex t, see Figure 1.7. Let

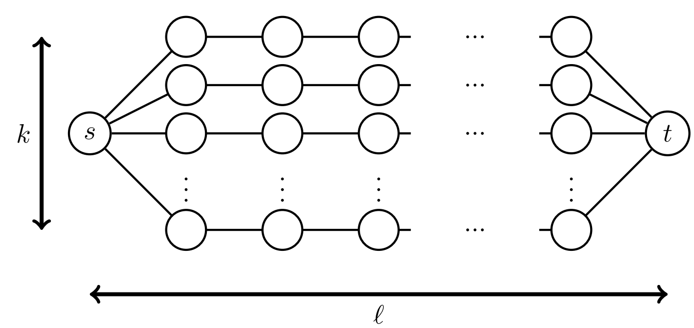

Figure 1.7: *k parallel paths with ℓ hops*

Γ be the family consisting of the k simple paths from s to t. Then $\ell(\Gamma) = \ell$ and the size of the minimal cut is k. A straightforward computation shows that

$$\mathrm{Mod}_p(\Gamma) = \frac{k}{\ell^{p-1}} \quad \text{for } 1 \leq p < \infty,$$

In particular, $\text{Mod}_p(\Gamma)$ is continuous in p, and

$$\text{Mod}_1(\Gamma) = k, \qquad \text{Mod}_2(\Gamma) = \frac{k}{\ell}, \qquad \lim_{p\to\infty} \text{Mod}_p(\Gamma)^{1/p} = \text{Mod}_{\infty,1}(\Gamma) = \frac{1}{\ell}.$$

Intuitively, when $p \approx 1$, $\text{Mod}_p(\Gamma)$ is more sensitive to the number of parallel paths, while for $p \gg 1$, $\text{Mod}_p(\Gamma)$ is more sensitive to short walks.

The case $p = 2$, tries to strike a balance between shortness of paths and number of different pathways.

1.4 Families of cuts and Fulkerson duality for p-modulus

Let $G = (V, E)$ be a graph. Fix two subsets $A, B \subset V$, such that $A \cap B = \emptyset$. Let Γ be the connecting family of all simple paths between A and B. And let $\hat{\Gamma}$ be the family of all the minimal cuts between the two sets A, B. A cut is a subset $C \subset E$ so that when C is removed from E, the remaining connected components never contain simultaneously a node $a \in A$ and a node $b \in B$. Another way to say this is that every walk from A to B must necessarily contain an edge from C. A cut C is minimal, if removing an edge from C, makes C not be a cut anymore.

A cut $\hat{\gamma} \in \hat{\Gamma}$ also has a usage vector $\hat{\mathcal{N}}(\hat{\gamma}, e)$ given by the indicator function on the set $\hat{\gamma} \subset E$. Also, a density $\eta \in \text{Adm}(\hat{\Gamma})$ is admissible for $\hat{\Gamma}$ if the usual condition

$$\hat{\mathcal{N}}\eta \geq 1,$$

is satisfied.

The two families Γ and $\hat{\Gamma}$ are related in interesting ways, and we say that $\hat{\Gamma}$ is the Fulkerson blocker of Γ. In particular, $\hat{\Gamma}$ can be seen to be the set of all extreme points of the convex set $\text{Adm}(\Gamma)$.

Given $p \in (1, \infty)$, we let $q \in (1, \infty)$ be the Hölder conjugate exponent of p so that $pq = p + q$. The p-modulus of Γ and the q-modulus of $\hat{\Gamma}$ are basically reciprocal of each others.

Then the q-modulus of the family $\hat{\Gamma}$ of cuts is equal the infimum of the q-energy of all the densities $\eta \in \mathrm{Adm}(\hat{\Gamma})$. It is a fact that

$$\mathrm{Adm}(\hat{\Gamma}) = \{\eta : \eta^T \rho \geq 1 \quad \forall \quad \rho \in \mathrm{Adm}(\Gamma)\}.$$

Theorem 1.17 ([4]). *With the notations above, we have*

$$\mathrm{Mod}_p(\Gamma)^{\frac{1}{p}} \, \mathrm{Mod}_q(\hat{\Gamma})^{\frac{1}{q}} = 1$$

When $p = q = 2$, we have the special case:

$$\mathrm{Mod}_2(\Gamma) \, \mathrm{Mod}_2(\hat{\Gamma}) = 1.$$

In the plane, there is a similar classical result that sometimes goes under the name of "conjugate modulus".

Theorem 1.18. [5] *Let Ω be a Jordan domain and let E and F be finite unions of closed subarcs of $\partial\Omega$. Assume $E \cap F \neq \emptyset$. Then there is a rectangle R having sides parallel to the axes and a conformal map ϕ of Ω onto the rectangle R with a finite number of horizontal line segments removed suvh that $\varphi \in C(\overline{\Omega})$ and $\varphi(E)$ and $\varphi(F)$ are the vertical sides of the rectangle if and only if there is an arc $\sigma \subset \partial\Omega$ such that*

$$E \subset \sigma \quad \textit{and} \quad F \cap \sigma = \emptyset$$

In this case, the modulus from E to F is the ratio of the height to the length of this rectangle. Moreover, if $\hat{\Gamma}$ is the family of curves in Ω separating E from F, then

$$\mathrm{Mod}_2(\hat{\Gamma}) = \frac{1}{\mathrm{Mod}_2(\Gamma)}.$$

1.5 Connecting modulus and harmonic functions

Let $G = (V, E)$. Let a, b be two vertices in V. Let Γ be the family of walks between a and b. The modulus of Γ can by found be minimizing the Dirichlet energy

$$\mathcal{E}_2(\phi) := \sum_{e=\{x,y\}\in E} (\phi(x) - \phi(y))^2$$

over all the *potentials* $\phi : V \to \mathbb{R}$ satisfying $\phi(a) = 0$ and $\phi(b) = 1$. The function that attains the minimum is called the *capacitary function* for a and b. Also, the minimal energy is called the *capacity* of the pair (a, b), and we write it as $\mathrm{Cap}(a, b)$.

Every such potential ϕ defines density,

$$\rho_\phi(e) := |\phi(x) - \phi(y)| \qquad \forall e = \{x, y\} \in E$$

which can be thought as the gradient of ϕ.

Also

$$\mathcal{E}_2(\rho_\phi) = \sum_{e\in E} \rho_\phi(e)^2 = \mathcal{E}_2(\phi).$$

It is a known fact, originally due to Duffin[6], that $\mathrm{Cap}(a, b) = \mathrm{Mod}_2(a, b)$.

Moreover, the capacitary function ϕ is **harmonic** at every $x \neq a, b$, meaning that

$$(L\phi)(x) := \sum_{y\sim x}(\phi(x) - \phi(y)) = 0, \tag{1.4}$$

where $L = \mathrm{diag}(A\mathbf{1}) - A$ is the **Combinatorial Laplacian** and $A(x, y) = \mathbb{1}_{x\sim y}$ is the **adjacency matrix**.

The advantage of the $p = 2$ case is that the connecting 2-modulus $\mathrm{Mod}_2(a, b)$ can be computed by solving the following Laplacian system

$$Lh = \delta_b - \delta_a,$$

where δ_x is the indicator function of the node x.

For $1 < p < \infty$, the extremal density for the modulus of a connecting family of walks can be related to a generalized voltage potential. And finding the p-modulus of Γ is equivalent[7] to minimizing:

$$\sum_{e=\{x,y\}\in E} |\phi(x) - \phi(y)|^p$$

subject to

$$\phi(a) = 0 \quad \text{and} \quad \phi(b) = 1.$$

In this case the capacitary function solves

$$(L_p\phi)(x) := \sum_{y\sim x} |\phi(x) - \phi(y)|^{p-2}(\phi(x) - \phi(y)) = 0, \qquad \forall x \neq a, b.$$

Chapter 2

Approximating a domain with square grids and study the convergence of 2-modulus

We are interested in the connection between continuous and discrete modulus. We will focus our attention on the plane, and curve families where the curves are connecting two sides of a domain. First, we will focus on comparing continuous modulus to discrete modulus on grids. Our main goal is to find an upper bound, and, if possible, establish convergence of the discrete modulus as the mesh of the grid tends to zero.

To begin, we look square grids of rectangular domains and consider the family of all curves connecting the two vertical sides.

2.1 Behavior of side-to-side modulus under grid refinements

Let R_n be a rectangular domain with a $\frac{1}{n}$-grid. Namely, R_n is a graph with nodes,

$$V_n = \{(\frac{i}{n}, \frac{j}{n}) : i_0 \leq i \leq i_1, j_0 \leq j \leq j_1\}$$

and edges

$$E_n = \{e = \{x, y\} : \quad for \quad x, y \in V_n \quad and \quad ||x - y||_\infty = 1\}$$

Let Γ_n be the family of walks in R_n from $\{\operatorname{Re} z = \frac{i_0}{n}\}$ to $\{\operatorname{Re} z = \frac{i_1}{n}\}$. Pick a cell (a square) and refine it by subdividing each side into k equal length intervals. Call the resulting graph $R_{n,k}$. Let $\Gamma_{n,k}$ be the family of walks in $R_{n,k}$, from $\{\operatorname{Re} z = \frac{i_0}{n}\}$ to $\{\operatorname{Re} z = \frac{i_1}{n}\}$, with $i_1 - i_0 = n$. Pick $\rho = \frac{1}{n}$ on the horizontal edges of the original grid and $\rho = \frac{1}{nk}$ on the new horizontal smaller edges. As discussed in Example 1.16, 2-modulus strikes a balance between short paths and number of different paths. Therefore, it is not clear what will happen to $\operatorname{Mod}_2(\Gamma_{n,k})$ relative to $\operatorname{Mod}_2(\Gamma_n)$, because, although we added longer walks, there are now more ways to go from side to side. In fact, as we will see, we have

$$\operatorname{Mod}_2(\Gamma_{n,k}) \le \operatorname{Mod}_2(\Gamma_n).$$

(For simplicity, In what follows we drop the subscript 2, that is $\operatorname{Mod}_2(\Gamma) = \operatorname{Mod}(\Gamma)$).

We have that

$$\rho(e) = \begin{cases} \frac{1}{n} & \text{if e is an old edge} \\ \frac{1}{nk} & \text{if e is a new edge} \end{cases}$$

Such a ρ is admissible for $\Gamma_{n,k}$. To see this, consider a simple path $\gamma \in \Gamma_{n,k}$ and assume that γ contains M old edges for some $M \le r := \ell(\gamma)$. If γ uses at least one new edge, then γ it must use at least k new edges and at least $n-1$ old edges. Thus, in this case,

$$\ell_\rho(\gamma) = \sum_{i=1}^{r} \rho(e_i) = M\left(\frac{1}{n}\right) + (r - M)\left(\frac{1}{nk}\right) \ge (n-1)\left(\frac{1}{n}\right) + k\left(\frac{1}{nk}\right) = 1$$

On the other hand, if γ does not use any new edges, then it must use $M \ge n$ of the old ones, and

$$\ell_\rho(\gamma) = \frac{M}{n} \ge 1.$$

$$\begin{aligned}
\mathrm{Mod}(\Gamma_{n,k}) &\leq \mathrm{Mod}(\Gamma_n) - 2\left(\frac{1}{n}\right)^2 + k(k+1)\left(\frac{1}{kn}\right)^2 \\
&= \mathrm{Mod}(\Gamma_n) - \frac{1}{n^2} + \frac{1}{kn^2} \\
&= \mathrm{Mod}(\Gamma_n) - \frac{1}{n^2}\left(1 - \frac{1}{k}\right) \qquad (2.1)
\end{aligned}$$

Now we compute the original modulus $\mathrm{Mod}(\Gamma_n)$. Again if $\rho = 1/n$ on all the horizontal edges, then ρ is admissible for Γ_n. So

$$\begin{aligned}
\mathrm{Mod}(\Gamma_n) &\leq \mathcal{E}(\rho) \\
&= n(n+1)\left(\frac{1}{n}\right)^2 \\
&= 1 + \frac{1}{n} \qquad (2.2)
\end{aligned}$$

Now let $\hat{\Gamma}_n$ be the family of all the cuts for Γ_n. In this example a cut is obtained by choosing at least one horizontal edge for each one of the $n+1$ levels of the grid. Let $\eta := \frac{1}{n+1}$ on all horizontal edges of R_n. For $\hat{\gamma} \in \hat{\Gamma}_n$, we have

$$\ell_\eta(\hat{\gamma}) = \sum_{e \in \hat{\gamma}} \eta(e) \geq (n+1)\frac{1}{n+1} = 1$$

Then, η is admissible, and we get

$$\begin{aligned}
\mathrm{Mod}(\hat{\Gamma}_n) &\leq \mathcal{E}(\eta) \\
&= n(n+1)\left(\frac{1}{n+1}\right)^2 \\
&= \frac{n}{n+1}.
\end{aligned}$$

By Fulkerson duality,

$$\mathrm{Mod}(\hat{\Gamma}_n)\mathrm{Mod}(\Gamma_n) = 1$$

So

$$\begin{aligned}\operatorname{Mod}(\Gamma_n) &= \frac{1}{\operatorname{Mod}(\hat{\Gamma}_n)} \\ &\geq \frac{n+1}{n} \\ &= 1 + \frac{1}{n} \end{aligned} \tag{2.3}$$

Hence, (2.2) and (2.3) show that

$$\operatorname{Mod}(\Gamma_n) = 1 + \frac{1}{n}. \tag{2.4}$$

Applying (2.4) to (2.1), we get

$$\operatorname{Mod}(\Gamma_{n,k}) \leq 1 + \frac{1}{n} - \frac{1}{n^2}\left(1 - \frac{1}{k}\right) = \operatorname{Mod}\Gamma_n \left[1 - \frac{k-1}{n(n+1)}\frac{1}{k}\right]. \tag{2.5}$$

Equation(2.5) gives an upper bound for modulus after one cell refinement.

2.1.1 Square grid domains

The previous calculation shows that the refinement will lower the discrete modulus. To continue, one would have to either refine repeatedly, or refine all cells simultaneously. This is where the type of modulus family seems to be of importance. In whatfollows, we will focus on general square grid domains

Definition 2.1. Let Ω be a simply connected domain in $\mathbb{C}$. We say Ω is a *square grid domain*, if Ω is tiled by a square grid Q_0.

Let E and F be disjoint arcs of the boundary of Ω that are unions of edges in Q_0, we write $\Gamma_{Q_0}(E,F)$ for the family of walks on the edges of Q_0 that connect E and F. If Q_n are grid refinements of Q_0, the goal is to show that as the mesh of Q_n tends to zero, the corresponding discrete modulus of $\Gamma_{Q_n}(E,F)$ decreases and tends to the continuous modulus $\operatorname{Mod}_\Omega(E,F)$.

As an example, we now discuss the 2-by-1 rectangle case of Section 1.2. We saw that if $E = [A, B]$, the short side, and $F = [C, D]$, half of the long side, then

$$\mathrm{Mod}_\Omega(E, F) \asymp 0.68.$$

Here we describe our numerical example that we obtained using 61 by 31 grid to approximate the 2 by 1 rectangle.

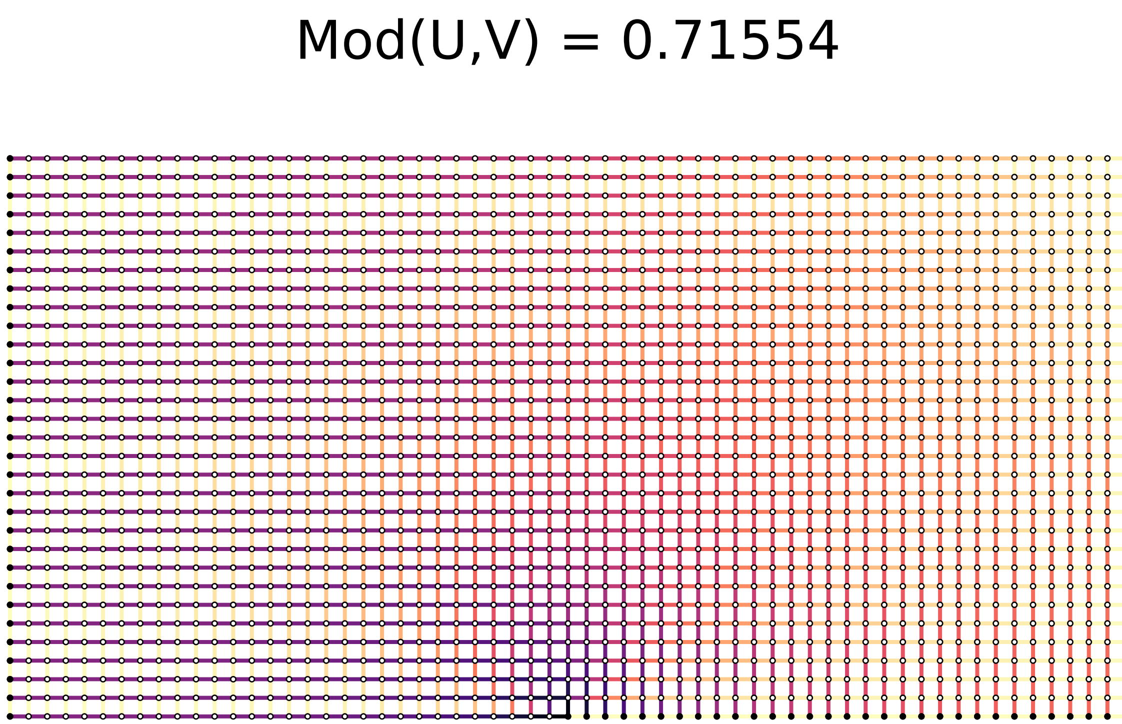

Figure 2.1: $\mathrm{Mod}(E, F) \asymp 0.715$

In the case of connecting families $\Gamma(E, F)$ the convergence of the discrete modulus to the continuous one can be proved using known results about the convergence of harmonic functions, see Theorem 2.2 below. The general case is open. For instance, little is known about the modulus of unions of connecting families, even numerical evidence.

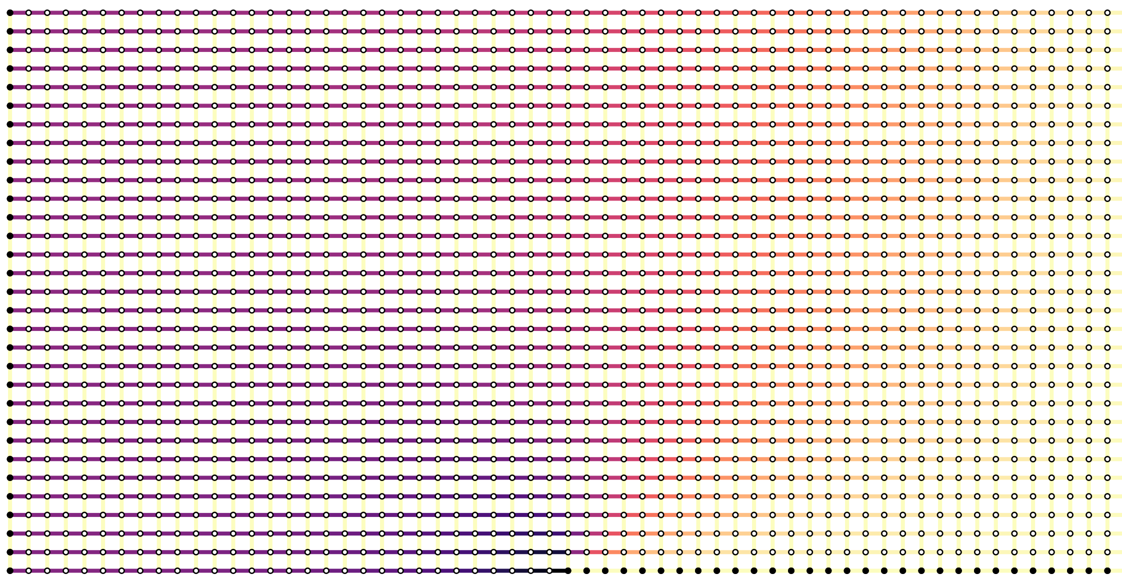

Figure 2.2: *The 2 by 1 rectangle with 61 by 31 grid – Horizontal edges*

Theorem 2.2 (Main Theorem)**.** *Let Ω be a square grid domain with initial grid Q_0, and let $E, F \subset \partial\Omega$ be two disjoint continua consisting of unions of edges of Q_0. If Q_n is a sequence of square grid refinements of Q_0 with mesh tending to zero, then*

$$\mathrm{Mod}_{Q_n}(E, F) \to \mathrm{Mod}_{\Omega}(E, F) \qquad \textit{as } n \to \infty.$$

2.2 Energy decreasing[1]

As we saw in Section 1.5, connecting modulus can also be computed by minimizing the Dirichlet energy of potential functions. In this section we recall an argument of Jacqueline Lelong-Ferrand[1] (see p. 163), that shows how refining a square grid in a "geometric" fashion,

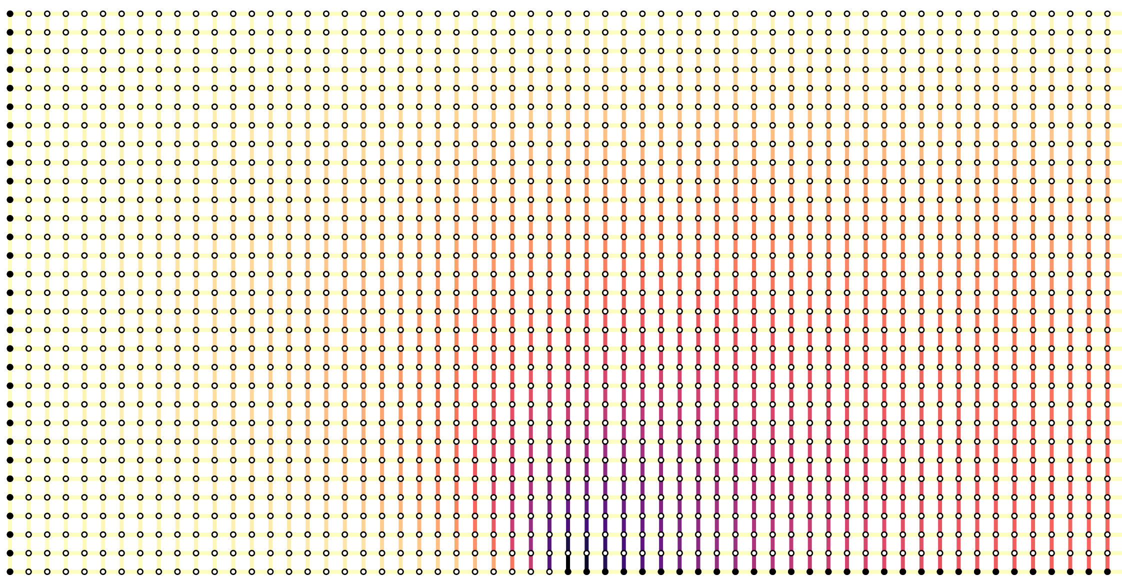

Figure 2.3: *The 2 by 1 rectangle with 61 by 31 grid – Vertical edges*

naturally decreases the 2-energy of a potential.

Namely, refine a square grid by adding a node on each edge, that we also connect to a new node in each face (see Figure 2.4). After n refinements, there exists a unique harmonic function U_n on the nodes of $Q_n \setminus (E \cup F)$ satisfying:

$$\begin{cases} U_n &= 0 \quad \text{on } E \\ U_n &= 1 \quad \text{on } F \end{cases} \tag{2.6}$$

In particular, U_n minimizes the energy

$$\mathcal{E}_2(U_n) = \sum_{e=\{x,y\} \in E(Q_n)} (U_n(x) - U_n(y))^2$$

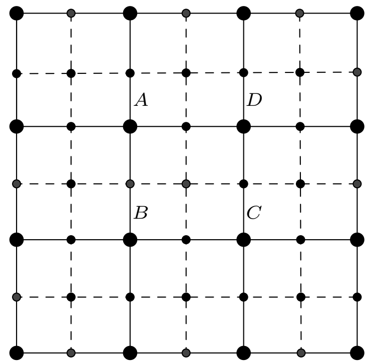

Figure 2.4: *Refining a square grid using Jacqueline Lwlong-Ferrand argument where a, b, c and d are the values of* $\bar{\phi}_n$ *at the original nodes in the positive direction and* $\frac{a+b}{2}, \frac{c+b}{2}, \frac{d+c}{2}, \frac{a+d}{2}$ *for each new node on the old edges and* $\frac{a+b+c+d}{4}$ *for the center point of the square*

over all functions on $V(Q_n)$ with the boundary values given in (4.6).

Assume that the value of U_n at the nodes of an arbitrary square, labeled in the positive direction, are a, b, c, d. Refine each square, and extend U_n to $\bar{U}_n$. The values of $\bar{U}_n$ on the old nodes are the same as U_n, but for the new nodes we set $\bar{U}_n$ equal to $\frac{a+b}{2}, \frac{c+b}{2}, \frac{d+c}{2}, \frac{a+d}{2}$ for each new node on the old edges, and we set $\bar{U}_n$ equal to $\frac{a+b+c+d}{4}$ on the new node in the middle of the old face. Now we compare the old energy to the new energy:

$$
\begin{aligned}
\mathcal{E}(U_n) - \mathcal{E}(\bar{U}_n) &= \sum_{e=\{x,y\}} (U_n(x) - U_n(y))^2 - \sum_{e=\{x,y\}} \left(\bar{U}_n(x) - \bar{U}_n(y)\right)^2 \\
&= \left[\frac{(a-b)^2}{2} + \frac{(b-c)^2}{2} + \frac{(c-d)^2}{2} + \frac{(d-a)^2}{2}\right] - \\
&\quad \left[\frac{(a-b)^2}{4} + \frac{(b-c)^2}{4} + \frac{(c-d)^2}{4} + \frac{(d-a)^2}{4}\right. \\
&\quad + \frac{(a+b-c-d)^2}{16} + \frac{(b+c-a-d)^2}{16} + \\
&\quad \left.\frac{(d+c-a-b)^2}{16} + \frac{(a+d-b-c)^2}{16}\right]
\end{aligned}
$$

Then, after some simplifications, we get:

$$\mathcal{E}(U) - \mathcal{E}(\bar{U}_n) = \frac{1}{8}\left[(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2\right] - \frac{1}{8}\left[(d-a)(b-c) + (a-b)(c-d) + (b-c)(d-a) + (a-b)(c-d)\right]$$

Finally,

$$\mathcal{E}(U_n) - \mathcal{E}(\bar{U}_n) = \frac{1}{8}\left\{[(a-b)-(c-d)]^2 + [(d-a)-(b-c)]^2\right\} \geq 0$$

This shows that

$$\mathcal{E}(U_{n+1}) \leq \mathcal{E}(U_n). \tag{2.7}$$

This monotonicity can be used to prove Theorem 2.2. Namely, in the book[1], Lelong-Ferrand develops a notion of discrete analytic function, and shows that harmonic functions are real parts of analytic functions. The energy monotonicity (2.7) allows her to extract a convergent subsequence and then Morera's Theorem is used to show that the limiting function is analytic.

Now we will prove the montoncity in general for any admissible ρ in γ_n. Consider a square grid network R_n of mesh size $1/n$, covering a simply connected polygonal domain Ω in the plane. Given two polygonal arcs E and F on $\partial\Omega$, consider a family Γ_n of walks on R_n that connect nodes in $V(R_n) \cap E$ to nodes in $V(R_n) \cap F$. A *refinement* consists in taking a square Q in R_n, choosing an integer $k = 2, 3, ...$, and replacing the four edges in $E(Q)$ by another square grid Q_k of mesh $\frac{1}{nk}$.

For simplicity, we first assume that Q has the property that $Q \cap \partial\Omega = \emptyset$. Also, we write $R_{n,k}$ for the new square grid approximation of Ω, and $\Gamma_{n,k}$ for the family of walks connecting $V(R_n) \cap E$ to $V(R_n) \cap F$ in $R_{n,k}$.

Note that $E(R_{n,k})$ can be split into old (longer) edges and new (shorter) ones. Namely

$$E(R_{n,k}) = E(R_n \setminus Q) \bigcup E(Q_k).$$

The four removed edges of Q consist of a pair of vertical edges, which we call $E_v(Q)$, and a pair of horizontal edges, called $E_h(Q)$. Likewise, the edge-set of Q_k splits into $E_v(Q_k)$ and $E_h(Q_k)$.

Assume that $\rho \in \mathbb{R}_{\geq 0}^{E(R_n)}$ is admissible for Γ_n. Define a density $\rho' \in \mathbb{R}_{\geq 0}^{E(R_{n,k})}$ as follows:

$$\rho'(e) := \begin{cases} \rho(e) & \text{for } e \in E(R_n \setminus Q) \\ \frac{1}{k} \max\limits_{E_v(Q)} \rho & \text{for } e \in E_v(Q_k) \\ \frac{1}{k} \max\limits_{E_h(Q)} \rho & \text{for } e \in E_h(Q_k) \end{cases}$$

Theorem 2.3. *ρ' is admissible for $\Gamma_{n,k}$.*

Proof. Let $\gamma \in \Gamma_{n,k}$. We want to decompose it into excursions inside Q and excursions outside of Q. To do this, we first write $V(Q)$ for the four original nodes of Q and

$$V^{\circ}(Q_k) = V(Q_k) \setminus V(Q)$$

for the new nodes in the grid $R_{n,k}$. Then, writing $\gamma = x_0\ x_1\ \cdots\ x_m$, in terms of the nodes $x_j \in V(R_{n,k})$ visited, we let

$$T_1^{\circ} := \inf\{j \geq 0 : x_j \in V^{\circ}(Q_k)\}.$$

Namely T_1 is the first time γ visits a "new" node. Then

$$S_1 := \inf\{j > T_1^{\circ} : x_j \in V(Q)\}.$$

Likewise, we define

$$T_i^{\circ} := \inf\{j > S_{i-1} : x_j \in V^{\circ}(Q_k)\},$$

and

$$S_i := \inf\{j > T_i^{\circ} : x_j \in V(Q)\}.$$

Then

$$\gamma_1 = x_0 \ \cdots \ x_{T_1^\circ - 1}$$

only uses "old" edges, and

$$\gamma_1^\circ = x_{T_1^\circ - 1} \ \cdots \ x_{S_1}$$

only uses "new" edges. Therefore, repeating this process we can write

$$\gamma = \gamma_1 \ \gamma_1^\circ \ \gamma_2 \ \gamma_2^\circ \ \cdots$$

Each subwalk γ_j that uses only old edges will have

$$\ell_{\rho'}(\gamma_j) = \ell_\rho(\gamma_j)$$

For the subwalks $\gamma_j^\circ = x_{T_j^\circ - 1} \ \cdots \ x_{S_j}$, using only new edges, there are two cases:

(a) $x_{T_j^\circ - 1}$ and x_{S_j} are neighbors in Q.

(b) $x_{T_j^\circ - 1}$ and x_{S_j} are diagonally opposite in Q.

Using these two cases, we replace each γ_j° by either one old edge of Q or two consecutive edges of Q so as to get a walk $\tilde{\gamma}$ that only visits old edges. We claim that

$$\ell_{\rho'}(\gamma) \geq \ell_\rho(\tilde{\gamma}) \geq 1.$$

The second inequality follows from the admissibility of ρ. To check the first inequality, look at case (a) first. Without loss of generality, $e' := \{x_{T_j^\circ - 1}, x_{S_j}\}$ is a horizontal edge of Q. Then, γ_j° must traverse at least k horizontal edges in $E_h(Q_k)$, hence

$$\ell_{\rho'}(\gamma_j^\circ) \geq k \left(\frac{1}{k} \max_{E_h(Q)} \rho \right) = \max_{E_h(Q)} \rho \geq \rho(e'). \tag{2.8}$$

Case (b) is analogous, except that now γ_j° will have to traverse at least k horizontal edges and at least k vertical edges. □

Theorem 2.4. *Let a and b be the old ρ on the top horizontal edge and the bottom one respectively. Let c and d be the old ρ on the right vertical edge and the left one respectively. If $a > b$ and $d > c$, for $k > \max\{\frac{a^2}{b^2}, \frac{d^2}{c^2}\}$. We have*

$$\mathcal{E}(\rho') \leq \mathcal{E}(\rho).$$

Proof. WLOG , Assume that $k > \frac{a^2}{b^2}$

$$\begin{aligned}
\mathcal{E}(\rho) - \mathcal{E}(\rho') &= \left(\sum_{e \in E_h(Q)} \rho(e)^2 - \sum_{e \in E_h(Q_k)} \rho'(e)^2 \right) + \left(\sum_{e \in E_v(Q)} \rho(e)^2 - \sum_{e \in E_v(Q_k)} \rho'(e)^2 \right) \\
&= a^2 + b^2 - k(k+1) \left(\frac{1}{k} a \right)^2 + c^2 + d^2 - k(k+1) \left(\frac{1}{k} d \right)^2 \\
&= b^2 - \frac{1}{k} a^2 + c^2 - \frac{1}{k} d^2 \\
&> b^2 - \frac{b^2}{a^2} a^2 + c^2 - \frac{b^2}{a^2} d^2 \\
&> 0 + d^2 \left[\frac{c^2}{d^2} - \frac{b^2}{a^2} \right] > 0
\end{aligned}$$

The assumpsion on k means that when the gradient is big we need to refine the grid more to get the energy decreasing.

□

The theory of discrete analytic functions has advanced since the work of Lelong-Ferrand. In particular, they can now be defined on much more general "grids". In the next section, we review the recent papers of Skopenkov and Werness, that were written for the case of grids whose "squares" are quadrilaterals with orthogonal diagonals and some uniform bound on the eccentricity.

Instead of reviewing their work in full generality, we will present the outline of their argument in the special case of square grids. The point of this being that in Chapter 4 we will extend this to the general $1 < p < \infty$ case.

2.3 Skopenkov and Werness work

2.3.1 Discrete holomorphicity

We will review the linear theory of discrete holomorphicity and harmonicity as provided by Skopenkov. The main definition of discrete holomorphicity on quadrilateral lattices is provided by a discrete form of the Cauchy-Riemann equations. The following theorems and definitions work for orthogonal lattices, whose vertices are identified with a set $V(Q) \subset \mathbb{C}$, which are collections of quadrilaterals where the diagonals of each face are orthogonal to each other. Since every bounded face of Q is a quadrilateral, our graph admits a 2-coloring of the vertices into black and white. The black vertices will be denoted by $Q^\bullet$ and the white vertices by Q°. Thus, the set of all vertices of Q may written as $V(Q) = Q^\bullet \cup Q^\circ$. Assume

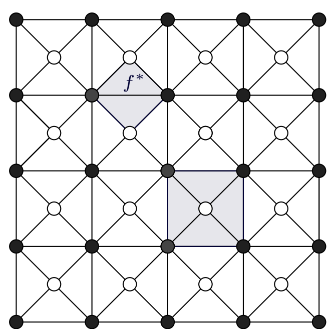

Figure 2.5: *Our original square grid can be thought as the black vertices of a* 2*-coloring grid*

we have a square grid with mesh size M. If f is a square face then $\text{Diam}(f) = \sqrt{2}M$ where M is the side-length of the square.

$$\text{Area}(f) = M^2 = \frac{1}{2}\text{Diam}(f)^2 \tag{2.9}$$

Now, For each face f assign a center point, which is the intersection of the orthogonal diagonals of the face. Then, trace a line between each vertex (the original vertex) and the center point of the face. The resulting graph is a 2-coloring graph with black and white vertices, which is unique up to interchanging the black and white vertices. The black vertices are the original one and the white are the center points of the original faces. Thus each square grid domain corresponding to a 2-coloring graph.

The resulting graph has faces f^* (see Figure 2.5) with mesh size $\frac{\sqrt{2}}{2}M$ for the new grid and $\mathrm{Diam}(f^*) = M$. I will call the set of all the center points of the square faces Q° and the set of all our original vertices (the black vertices) $Q^\bullet$.

Definition 2.5. A function $g : V(Q) \to \mathbb{C}$ is **discrete holomorphic** if for every face $f = [z_1 z_2 z_3 z_4]$ we have

$$\frac{g(z_1) - g(z_3)}{z_1 - z_3} = \frac{g(z_2) - g(z_4)}{z_2 - z_4}$$

A function $h : V(Q) \to \mathbb{R}$ is **discrete harmonic** if it is the real part of a discrete holomorphic function.

In the continuum, harmonicity may be described via the minimization of the **Dirichlet energy** :

$$\mathcal{E}_\Omega(u) := \int_\Omega |\nabla u|^2 dxdy$$

In the discrete case such a definition may be given as well. For a face f of Q we write $f \in Q$ and we will let the discrete gradient of a function $u : V(Q) \to \mathbb{R}$ on that face be the unique complex number $\nabla_Q u(f)$ such that

$$\nabla_Q u(f) \odot (z_3 - z_1) = u(z_3) - u(z_1)$$

and

$$\nabla_Q u(f) \odot (z_4 - z_2) = u(z_4) - u(z_2)$$

where $\odot$ means the dot product between two complex numbers. In particular,

$$|\nabla_Q u(f)|^2 = \frac{(u(z_3) - u(z_1))^2}{|z_3 - z_1|^2} + \frac{(u(z_4) - u(z_2))^2}{|z_4 - z_2|^2}.$$

We may now define the **Dirichlet energy**.

Definition 2.6. The discrete **Dirichlet energy** is

$$\begin{aligned}\mathcal{E}_\Omega(u) &:= \sum_{f\in Q} |\nabla_Q u(f)|^2 \cdot \mathrm{Area}(f)\\ &= \frac{1}{2}\sum_{f=[z_1z_2z_3z_4]\in Q}\left[\frac{|z_2-z_4|}{|z_1-z_3|}(u(z_3)-u(z_1))^2 + \frac{|z_1-z_3|}{|z_2-z_4|}(u(z_2)-u(z_4))^2\right]\end{aligned}$$

where the latter equality follows from the orthogonality of the lattice. In particular, for our grid we will have

$$\mathcal{E}_\Omega(u) = \frac{1}{2}\sum_{f=[z_1z_2z_3z_4]\in Q}\left[(u(z_3)-u(z_1))^2 + (u(z_2)-u(z_4))^2\right]$$

Definition 2.7. The discrete **Laplacian** of $u : V(Q) \to \mathbb{R}$ is

$$\triangle_Q u(z) = -\frac{\partial E_Q(u)}{\partial u(z)} = \sum_{f=[z_1z_2z_3z_4]\in Q}\frac{|z_2-z_4|}{|z_1-z_3|}(u(z_3)-u(z_1)).$$

Note that $\triangle_Q u$ is a weighted version of the combinatorial Laplacian L defined in (1.4), when thinking of the graph obtained from Q by keeping only the black vertices $Q^\bullet$ and connecting them across the corresponding diagonals. In particular, for our purposes we will assume that u is only defined on $Q^\bullet$.

Lemma 2.8. *For each $z \in V(Q)$, we have*

$$\triangle_Q u(z) = \sum_{\substack{f=[z_1z_2z_3z_4]\in Q\\ z_1=z}} i \,\nabla_Q\, u(f) \odot (z_2 - z_4)$$

With these definitions, many of the familiar identities from the continuous theory may be translated to the discrete theory. We will state the Green's identity and the maximum principle.

Lemma 2.9 (Maximum principle, Lemma 2.5[8])**.** *Let Q be an orthogonal lattice and let*

$U : V(Q) \to \mathbb{R}$ be discrete harmonic. Then

$$\max_{z \in Q^\bullet} U(z) = \max_{z \in Q^\bullet \cap \partial Q} U(z).$$

Lemma 2.10 (Lemma 2.4[8]). *Let Q be an orthogonal lattice and $u, v : Q^\bullet \to \mathbb{C}$ be arbitrary functions. Then*

$$\sum_{z \in Q^\bullet} [u \triangle_Q v - v \triangle_Q u] = 0$$

Note that there are no boundary terms in the Green's identity because of the symetric of the laplacian matrix and the inner product of u and v. In particular, $u^T L v = v^T L u$ and the weight is the same in our case.

2.3.2 Werness and Skopenkov convergence step by step

Here we state the main result of Werness and Skopenkov.

Theorem 2.11. *Let Ω be a simply connected "square grid" domain, as in Definition 2.1, meaning that Ω is initially tiled by a square grid Q_0. Let Q_n be the n-th refinement of Q_0 as we described in section 2.2 and let $g : \mathbb{C} \to \mathbb{R}$ be a given smooth function that will be used as boundary data for the Dirichlet problem.*

Then, the unique discrete harmonic functions U_n on Q_n with boundary values $g\big|_{\partial Q_n}$ converge uniformly to the unique continuous harmonic u on Ω with boundary values $g\big|_{\partial\Omega}$.

Moreover,

$$\mathcal{E}_2(U_n) \longrightarrow \int_\Omega |\nabla u|^2 dA, \qquad \text{as } n \to \infty.$$

In the next few sections, we present their proof in the simplified setting of square grids. The key is that these methods can be generalized to more flexible quadrangulations of the plane, as we will explain later in Chapter 3.

To guide the reader, here is a list of the main steps in their proof.

- First, they show how to approximate any given rectifiable curve with a union of square faces so that the sum of the diameters of the faces is not bigger than a constant times the length of the given curve.

- Then they provide control on the oscillation of a function on the lattice in terms of the discrete Dirichlet energy along a path.
- Also, they obtain an estimate of the oscillation between two pairs of points in terms of the semi-energy over the boundary of a ball containing the two points.
- Finally, they integrate over radii r and obtain a modulus of continuity in terms of the Dirichlet energy.
- Next, it is shown how the discrete gradients approximate the continuous ones and how the Laplacian operator approximates the continuous one.
- Then, they show how the uniform limit of a sequence of discrete harmonic functions is necessarily harmonic.
- In conclusion, the Theorem of Ascoli-Arzela is used to extract a convergent subsequence, due to the equicontinuity of the discrete energy minimizers.

Another reason for listing these steps, is to determine which steps might be easily generalized to the $p \neq 2$ case, and which ones require more work.

2.3.3 Estimate the energy and eqcontinuity

Lemma 2.12. *Let Q be a square grid and $\gamma : [0,1] \to \mathbb{C}$ be rectifiable closed loop with $\mathrm{Diam}(\gamma) \geq 4M$, where M is the side-length of squares in Q. Then*

$$\sum_{\substack{f \in Q \\ f \cap \gamma \neq \emptyset}} \mathrm{Diam}(f) \leq \frac{9}{\sqrt{2}} \ell(\gamma)$$

Proof. Fix a face f such that $f \cap \gamma \neq \emptyset$. Let $\hat{f}$ be the subgrid of Q consisting of the face f and all faces f' which share a vertex with f. We will call $\hat{f}$ the neighborhood of f.

Since $\mathrm{Diam}(\gamma) \geq 4M = 2\sqrt{2}\,\mathrm{Diam}(f)$, then $\gamma \setminus \hat{f} \neq \emptyset$. Therefore,

$$\mathrm{Diam}\left(\bigcup_{f' \in \hat{f}} \gamma \cap f'\right) \geq 2M,$$

because γ must cross the annulus $\hat{f} \setminus f$.

Thus, since the length must sum up to at least the diameter of f

$$\sum_{f' \in \hat{f}} \ell(\gamma \cap f') \geq 2M = \sqrt{2}\,\mathrm{Diam}(f).$$

Then

$$\sum_{\substack{f \in Q \\ f \cap \gamma \neq \emptyset}} \mathrm{Diam}(f) \leq \frac{1}{\sqrt{2}} \sum_{\substack{f \in Q \\ f \cap \gamma \neq \emptyset}} \sum_{f' \in \hat{f}} \ell(\gamma \cap f')$$

Note that $\hat{f}$ is composed of at most nine faces. Thus,

$$\begin{aligned}
\sum_{\substack{f \in Q \\ f \cap \gamma \neq \emptyset}} \mathrm{Diam}(f) \leq & \frac{1}{\sqrt{2}} \sum_{f \in Q} \sum_{f' \in Q} \ell(\gamma \cap f') \mathbb{1}_{\{f \cap \gamma \neq \emptyset, f' \in \hat{f}\}} \\
& \leq \frac{1}{\sqrt{2}} \sum_{f' \in Q} \ell(\gamma \cap f') \cdot \#\{f \in Q : f \cap \gamma \neq \emptyset, f' \in \hat{f}\} \\
& \leq \frac{9}{\sqrt{2}} \sum_{f' \in Q} \ell(\gamma \cap f') \\
& \leq \frac{9}{\sqrt{2}} \ell(\gamma)
\end{aligned}$$

□

In particular, when $\gamma = \partial B(z, r)$ is inside the square grid domain and $r > 2M$, then

$$\sum_{\substack{f \in Q \\ f \cap \gamma \neq \emptyset}} \mathrm{Diam}(f) \leq Cr. \tag{2.10}$$

Definition 2.13. The **semi-energy** of a function $u : Q^{\bullet} \to \mathbb{R}$ along a path $w_0 w_1 w_2 .... w_n$

of $Q^\bullet$ is

$$\hat{\mathcal{E}}_{w_0w_1w_2........w_n}(u) = \frac{1}{M}\sum_{i=1}^{n}(u(w_i) - u(w_{i-1}))^2.$$

Lemma 2.14 (Lemma 4.1[8])**.** *Given a path $\gamma : w_0w_1w_2........w_n$,*

$$\hat{\mathcal{E}}_\gamma(u) \geq \frac{(u(w_n) - u(w_0))^2}{\ell(\gamma)}$$

where

$$\ell(\gamma) := \sum_{j=1}^{n}|w_j - w_{j-1}| = Mn$$

is the Euclidean length of the path, and M is the side-length of a square in $V(Q)$.

Proof. By Cauchy-Schwarz,

$$\begin{aligned}|u(w_n) - u(w_0)|^2 = &\leq \left[\sum_{i=1}^{n}(u(w_i) - u(w_{i-1}))^2\right] n \\ &\leq \hat{\mathcal{E}}_{w_0w_1....w_n}(u) \cdot nM.\end{aligned}$$

So the desired inequality follows. □

We will wish to use the previous lemma to obtain an estimate on the energy of a discrete harmonic function over the entire square grid in terms of the difference between the value of the function at a pair of points which implies the equicontinuity. Fix $z \in \mathbb{C}$ and $r > 0$, define the semi-energy at distance r from a point z by

$$\hat{E}_r^z(u) := \sum_{f\in Q, f\cap\partial B_r(z)\neq\emptyset} |\bigtriangledown_Q u(f)|^2 \cdot M$$

Definition 2.15. Let Q be the square grid. For any ball B_R intersect Q, we will define Q_R to be the union of all the faces that contains in $Q \cap B_R$. And for $z \in Q_R$ we will call Q_R^z the component of Q_R that contains z. Note that Q_R need not to be connected.

Lemma 2.16. *Let Q be a square grid, z, w be a pair of vertices in $Q^\bullet$, and $u : Q^\bullet \to \mathbb{R}$ be a discrete harmonic function. Take $R > |z - w| + M$ and assume that no vertices of $Q^\bullet$ lie* *on the circle of radius R about $x := \frac{(z+w)}{2}$. Let*

$$\delta_R := |u(z) - u(w)| - \max_{z',w' \in \partial Q \cap B_R} |u(z') - u(w')|$$

If $\delta_R > 0$, then there is a constant C such that

$$\hat{\mathcal{E}}_R^x(u) \geq C \frac{\delta_R^2}{R}$$

Proof. WLOG, assume $u(z) > u(w)$. Let Q_R, Q_z and Q_w be as defined in Definition(2.15). By the maximum principle, there is $z' \in \partial Q_R \cap Q^\bullet$ with $u(z') > u(z)$ and a point w' with $u(w') < u(w)$. We will consider three cases:

Case 1 : If $\partial Q_R \cap \partial Q = \emptyset$, then there are two faces that contain z' and w'. These two faces also intersect the boundary of B_R. Thus, there is a path $\gamma = w_0 w_1 .... w_r$ of black vertices that connect the two points z' and w' which is contained in faces that intersect ∂B_R.

Note that $\ell(\gamma) = rM = \sum_{i=1}^r |w_i - w_{i-1}| \leq \sum\limits_{f^* \in Q, f^* \cap \partial B_R \neq \emptyset} \text{Diam}(f^*)$, since $|w_i - w_{i-1}|$ are the diameters of the new faces f^*.

By Lemma 2.14 we have the following:

$$\begin{aligned} \hat{\mathcal{E}}_R^x(u) \geq \hat{\mathcal{E}}_{w_0 w_1 \ldots w_r} &\geq \frac{(u(z') - u(w'))^2}{rM} \\ &\geq \frac{(u(z) - u(w))^2}{rM} \geq \frac{\delta_R^2}{rM} \\ &\geq \frac{\delta_R^2}{\sum\limits_{f^* \in Q, f^* \cap \partial B_R \neq \emptyset} \text{Diam}(f^*)} \geq C \frac{\delta_R^2}{R} \end{aligned}$$

Note that the first inequality follows because the number of faces that connect z' and w' are less than or equal the number of faces that intersect the boundary of B_R. And the last inequality follows by using equation (2.10).

Case 2: If $\partial Q_R \cap \partial Q \neq \emptyset$, then if there is an arc of the circle with the same properties in Case 1 which stays inside Q, we are done. So, assume that the boundary of the circle splits into multiple components. Let $C_{z'}$ be the arc which intersect the face that contains z' and

$C_{w'}$ be the arc which intersect the face that contains w'. Let z'' be the vertex in $\partial Q_R \cap \partial Q$ at one of the end points of $C_{z'}$ and the same for w''. Then there is a path of black vertices of faces that intersect the boundary of B_R and connect z'' and z' and another path that connect w'' and w'. Also,

$$(u(z') - u(z'')) + (u(w'') - u(w')) \geq ((u(z) - u(w)) - (u(z'') - u(w'')) \geq \delta_R.$$

Thus, either $(u(z') - u(z''))$ or$(u(w'') - u(w')$ is greater than $\frac{\delta_R}{2}$. WLOG, assume that $(u(z') - u(z'')) > \frac{\delta_R}{2}$. Let $\gamma = w_0 w_1 .... w_r$ be the path that connects z'' and z' then applying the same argument as Case 1 to those points gives the desired bound.

Case 3: Assume that either z' or w' is contained in $\partial Q_R \setminus \partial Q$. WLOG, say z' is. Take z'' as in the Case 2, then there is a path of black vertices of faces intersect the boundary of B_R that connects z'' and z'.

$$u(z') - u(z'') = (u(z') - u(w')) - (u(z'') - u(w')) \geq ((u(z) - u(w)) - (u(z'') - u(w')) \geq \delta_R$$

Then the desired bound obtained as in the first case.

Case 4: If neither z' nor w' are contained in $\partial Q_R \setminus \partial Q$, then

$$0 > ((u(z) - u(w)) - (u(z') - u(w')) \geq \delta_R$$

but this contradicts our hypothesis. So, this case is impossible. □

Proposition 2.17. *Let Q be a square grid. Let $u : Q^\bullet \to \mathbb{R}$ be a discrete harmonic function. Let z, w be a pair of vertices in $Q^\bullet$. Then, there is a constant C such that for $R \geq |z-w|+M$*

$$|u(z) - u(w)| \leq C \log^{-\frac{1}{2}} \left[\frac{R}{|z-w|} \right] \mathcal{E}^{\frac{1}{2}}_{Q_R}(u) + \max_{z',w' \in \partial Q \cap B_R} |u(z') - u(w')|$$

Proof. Let $\delta_R := |u(z) - u(w)| - \max_{z',w' \in \partial Q \cap B_R} |u(z') - u(w')|$. If $\delta_R \leq 0$, then the required estimate holds automatically. Assume that $\delta_R > 0$. By Lemma 2.16 and by observing that

$\delta_r > \delta_R$ for $r < R$, we have

$$\int_{|z-w|}^{R} \hat{\mathcal{E}}_r(u)dr \geq \int_{|z-w|}^{R} C\frac{\delta_r^2}{r}dr$$
$$= C\delta_R^2 \log\left[\frac{R}{|z-w|}\right] \tag{2.11}$$

Now, by the definition of semi-energy and equation(2.9) , we get

$$\begin{aligned}
\int_{|z-w|}^{R} \hat{\mathcal{E}}_r(u)dr &\leq \int_0^R \sum_{f^*\in Q, f^*\cap\partial B_r(z)} |\bigtriangledown_Q u(f^*)|^2.\operatorname{Diam}(f^*) \\
&= \int_0^R \sum_{f^*\in Q} |\bigtriangledown_Q u(f^*)|^2.\operatorname{Diam}(f^*).\mathbb{1}_{\{f^*\cap\partial B_r\neq\phi\}} \\
&= \sum_{f^*\in Q} |\bigtriangledown_Q u(f^*)|^2.\operatorname{Diam}(f^*)\int_0^R \mathbb{1}_{\{f^*\cap\partial B_r\neq\phi\}} \\
&\leq \sum_{f^*\in Q} |\bigtriangledown_Q u(f^*)|^2.\operatorname{Diam}(f^*)\operatorname{Diam}(f^*) \\
&= \sum_{f^*\in Q} |\bigtriangledown_Q u(f^*)|^2.\operatorname{Diam}(f^*)^2 \\
&= 2\sum_{f^*\in Q} |\bigtriangledown_Q u(f^*)|^2.\operatorname{Area}(f^*) \\
&= 2E_{QR}(u)
\end{aligned} \tag{2.12}$$

Combining equations (2.11) and (2.12) and the definition of δ_R, we obtain the proposition.

□

2.3.4 Laplacian Approximation

Lemma 2.18. *(Gradient Approximation)*[8]

For any square face $f^* = [z_1z_2z_3z_4]$ we have, for any $g \in C^3(\mathbb{C})$,

$$|\bigtriangledown g - \bigtriangledown_Q g| \leq CM \max_{z\in f^*} |D^2 g(z)|$$

Lemma 2.19. *Let Q be a square grid lattice, and R be a square of side-length $r > M$ inside Q. Then for any $g \in C^3(\mathbb{C})$ we have*

$$\left| \sum_{w \in R \cap Q^\bullet} [\triangle_Q(g|_{Q^\bullet})](w) - \int_R \triangle g \quad dxdy \right| \leq C(rM \max_{z \in R} |D^2 g(z)| + r^3 \max_{z \in R} |D^3 g(z)|)$$

Proof. Take an arbitrary function $g \in C^3(\mathbb{C})$ and without loss of generality assume R is centered at 0. Expand g as

$$g(z) = a_0 + a_1 \operatorname{Re} z + a_2 \operatorname{Im} z + a_3 \operatorname{Re} z^2 + a_4 \operatorname{Im} z^2 + a_5 |z|^2 + \bar{g}(z)$$

Where $D^k \bar{g}(0) = 0$ for $k = 0, 1, 2$. We will prove Laplacian Approximation in several particular cases and then combine them together.

- The cases $g(z) = 1$, $g(z) = \operatorname{Re} z$, $g(z) = \operatorname{Im} z$ all follows immediately since all three of those functions are both harmonic and discrete harmonic.

- The case $g(z) = |z|^2$. we know that

  $$\int_R \triangle |z|^2 dxdy = \int_R 4 dxdy = 4 \operatorname{Area}(R)$$

  We wish to show that

  $$\sum_{w \in Q^\bullet \cap R} [\triangle_Q(|z|^2)](w)$$

  approximates $4 \operatorname{Area}(R)$.

  For any two points $z, w \in Q^\bullet$ we have that

  $$\operatorname{Re} \bar{w}(z - w) = \operatorname{Re}[(a - ib)(x - a + i(y - b)] = ax - a^2 - b^2 + by = ax + by - |w|^2$$

and

$$
\begin{aligned}
|z-w|^2 = |x+iy-a-ib|^2 = |(x-a)+i(y-b)|^2 = (x-a)^2+(y-b)^2 \\
= |z|^2+|w|^2-2(ax+by) = |z|^2+|w|^2-2\,\mathrm{Re}\,\bar{w}(z-w)-2|w|^2
\end{aligned}
$$

Combine the last two equalities we have that

$$
|z|^2 = |z-w|^2| + |w|^2 + 2\,\mathrm{Re}\,\bar{w}(z-w)
$$

Now, For any point $w \in Q^{\bullet}$ using the previous two identities we have that

$$
[\triangle_Q(|z|^2)](w) = [\triangle_Q(|z-w|^2 + 2\,\mathrm{Re}[\bar{w}(z-w)] + |w|^2)](w) = [\triangle_Q(|z-w|^2)](w)
$$

because the last two terms are discrete harmonic functions.

Let z' be the intersection of the diagonals of a face f. So we have the facts,

$$
\mathrm{Area}(z_1 z_2 z' z_4) = \frac{1}{2}\,\mathrm{Area}(z_1 z_2 z_3 z_4)
$$

And

$$
\mathrm{Im}[(2z'-z_2-z_4)(z_4-z_2)] = 0
$$

Then, by expanding around z' we have

$$
\begin{aligned}
[\triangle_Q(|z-w|^2)](w) = & \sum_{f=[z_1z_2z_3z_4], z_1=w} i \bigtriangledown_Q (|z-w|^2)(f)(z_4-z_2) \\
&= \sum_{f=[z_1z_2z_3z_4], z_1=w} i \bigtriangledown_Q (|z-z'|^2 + 2\operatorname{Re}[\overline{(z'-w)}(z-z')] + |z-w|^2)(f)(z_4-z_2) \\
&= \sum_{f=[z_1z_2z_3z_4], z_1=w} i \bigtriangledown_Q (2\operatorname{Re}[\overline{(z'-w)}(z-z')])(f)(z_4-z_2) \\
&= \sum_{f=[z_1z_2z_3z_4], z_1=w} 2\operatorname{Im}[\overline{(z\prime-w)}(z_4-z')] - 2\operatorname{Im}[\overline{(z'-w)}(z_2-z')] \\
&= \sum_{f=[z_1z_2z_3z_4], z_1=w} 2\operatorname{Im}[\overline{(z'-w)}\{(z_4-z')-(z_2-z'\}] \\
&= \sum_{f=[z_1z_2z_3z_4], z_1=w} 2\operatorname{Im}[\overline{(z'-w)}(z_4-z_2)] \\
&= \sum_{f=[z_1z_2z_3z_4], z_1=w} 4\operatorname{Area}(z_1z_2z'z_4) \\
&= 2\sum_{f=[z_1z_2z_3z_4], z_1=w} \operatorname{Area}(z_1z_2z_3z_4)
\end{aligned}
$$

Take the sum over all the vertex w such that $w \in R \cap Q^\bullet$ we have that

$$
\begin{aligned}
\sum_{w\in Q^\bullet \cap R} \triangle_Q(|z|^2)](w) &= \sum_{z_1,z_3\in R} 2\operatorname{Area}(f) + \sum_{z_1\in R, z_3\notin R} 2\operatorname{Area}(f) \\
&= 2\operatorname{Area}(R_Q) + \sum_{z_1\in R, z_3\notin R} 2\operatorname{Area}(f)
\end{aligned}
$$

Where R_Q is the union of all the faces contained entirely in R. The second sum is bounded by CMr where r is the side length of R and M is the mesh size of the square grid. Thus,

$$4\operatorname{Area}(R) - 2\operatorname{Area}(R_Q) \le CrM$$

- For $g(z) = \operatorname{Re} z^2$. We know that

$$\int_R \triangle \operatorname{Re}(z^2) dxdy = 0$$

As we did in the previous case we get

$$[\triangle_Q \operatorname{Re} z^2](w) = \sum_{f: z_1 = w} 2 \operatorname{Im}((z' - w)(z_2 - z_4)) = 0$$

- For $g(z) = \operatorname{Im} z^2$. This is analogous to the previous case.
- For the case when $D^k g(z) = 0$ at the center of R for $k = 0, 1, 2$. By integrating the estimate

$$| \triangle g(z)| \le Cr \max_{z \in R} |D^3 g(z)|$$

we get

$$\left| \int_R \triangle g dx dy \right| \le Cr^3 \max_{z \in R} |D^3 g(z)|.$$

Now, by the estimate $| \bigtriangledown g(z)| \le Cr^2 \max_{z \in R} |D^3 g(z)|$, $|D^2 g(z)| \le Cr \max_{z \in R} |D^3 g(z)|$, and the gradient approximation we get

$$\begin{aligned}
\left| \sum_{w \in R \cap Q^\bullet} [\triangle_Q g](w) \right| &= \left| \sum_{w \in R \cap V(Q)} \sum_{z_1 z_2 z_3 z_4 : z_1 = w} i \bigtriangledown_Q g \cdot (z_4 - z_2) \right| \\
&= \left| \sum_{z_1 \in R, z_3 \notin R} i \bigtriangledown_Q g(f) \cdot (z_4 - z_2) \right| \\
&\le \sum_{z_1 \in R, z_3 \notin R} (|\bigtriangledown_Q g(f) - \bigtriangledown g(z_1)| + |\bigtriangledown g(z_1)|) \cdot |z_2 - z_4| \\
&\le \sum_{z_1 \in R, z_3 \notin R} \left(CM \max_{z \in R} |D^2 g(z)| + Cr^2 \max_{z \in R} |D^3 g(z)| \right) |z_2 - z_4| \\
&\le \sum_{z_1 \in R, z_3 \notin R} \left(CMr \max_{z \in R} |D^3 g(z)| + Cr^2 \max_{z \in R} |D^3 g(z)| \right) |z_2 - z_4| \\
&\le Cr^2 \max_{z \in R} |D^3 g(z)| \sum_{z_1 \in R, z_3 \notin R} |z_2 - z_4| \\
&\le Cr^3 \max_{z \in R} |D^3 g(z)|
\end{aligned}$$

The last inequality follows by lemma (2.12)

- The general case when

$$g(z) = a_0 + a_1 \operatorname{Re} z + a_2 \operatorname{Im} z + a_3 \operatorname{Re} z^2 + a_4 \operatorname{Im} z^2 + a_5|z|^2 + \bar{g}(z)$$

  follows from the special cases.

□

2.3.5 Uniform limit

Lemma 2.20. *Let Q_n be a sequence of square grids approximating a domain Ω. Let $u_n : Q_n^\bullet \to \mathbb{R}$ be a sequence of discrete harmonic functions such that u_n converges uniformly to a continuous $u : \Omega \to \mathbb{R}$. Then the function u is harmonic.*

Proof. Take an arbitrary smooth function $v : \Omega \to \mathbb{R}$ that vanishes outside a compact subset $K \subset \Omega$. By Weyl's lemma it suffices to show that $\int_\Omega u \triangle v dxdy = 0$ for any such v. Consider an intermediate infinite square lattice Z_n with edge length $\sqrt{2M_n}$. For a face f of the n-th lattice Z_n denote $\tilde{u}_n(f) := \max_{z \in f \cap K} u(z)$. By continuity, $\tilde{u}_n \to u$ uniformly in the support of v and it can be extended to the whole complex plane by taking it to be zero otherwise. Thus by uniform convergence and the laplacian approximation, we have

$$\begin{aligned}
&\left| \int_\Omega u \triangle v dxdy - \sum_{z \in Q_n^\bullet} [u_n \triangle_{Q_n} (v|_{Q_n^\bullet})](z) \right| \\
&\leq \sum_{f \in Z_n : f \cap K \neq \emptyset} |\tilde{u}_n(f)| \left| \int_f \triangle v dxdy - \sum_{z \in f \cap V(Q)} [\triangle_{Q_n} v](z) \right| \\
&\leq (\#\{f \in Z_n : f \cap K \neq \emptyset\}) \max_K |u| C \left(M_n \sqrt{2M_n} \max_{z \in R} |D^2 v(z)| + \sqrt{2M_n}^3 \max_{z \in R} |D^3 v(z)| \right) \\
&\leq (\#\{f \in Z_n : f \cap K \neq \emptyset\}) (\max_K |u|) \left(\max_{z \in R} |D^2 g(z)| + \max_{z \in R} |D^3 g(z)| \right) M_n^{\frac{3}{2}} \\
&\leq \frac{\operatorname{Area}(K)}{M_n} . (\max_K |u|) . \left(\max_{z \in R} |D^2 g(z)| + \max_{z \in R} |D^3 g(z)| \right) M_n^{\frac{3}{2}} = O(M_n^{\frac{1}{2}}) \to 0 \quad as \quad n \to \infty
\end{aligned}$$

Now, By Green's identity and the harmonicity of u_n,

$$\sum_{z\in Q^\bullet}[u_n \triangle_{Q_n} v|_{Q_n^\bullet}](z) = \sum_{z\in Q^\bullet}[v|_{Q_n^\bullet} \triangle_{Q_n} u_n](z) = 0$$

Thus ,

$$\int_\Omega u \triangle v dxdy = 0$$

And we can say that u satisfies the weak Laplacian condition so, by Weyl's lemma u is harmonic.

□

We may now use these results to establish the final limit theorem.

2.3.6 The convergence

Proof of Theorem 2.11. Note that since the domain Ω is bounded, the grids Q_n are contained in some large ball B. By the Maximum principle, we know that $|U_n|$ are uniformly bounded by $\max_{z\in V(Q)} |g(z)| < \infty$.

Now, we will show that the family of functions $\{U_n\}_{n\in\mathbb{N}}$ are equicontinuous. So, we need to show that there exists some positive function $\delta(\varepsilon)$ such that for every n and for every $z, w \in Q_n^\bullet$, $|z-w| < \delta(\varepsilon)$ implies that $|U_n(z) - U_n(w)| < \varepsilon$. Suppose we are in the case $M_n < |z-w|$, let $R = (\text{Diam}(B)|z-w|)^{\frac{1}{2}}$. Then by Proposition(2.17), we have

$$\begin{aligned}
&|U_n(z) - U_n(w)| \\
&\leq C \log^{-\frac{1}{2}}\left[\frac{R}{|z-w|}\right] \mathcal{E}^{\frac{1}{2}}_{Q_{n,R}}(U_n) + \max_{z',w'\in\partial Q_n\cap B} |U_n(z') - U_n(w')| \\
&\leq C\mathcal{E}^{\frac{1}{2}}_{Q_{n,R}}(U_n) \log^{-\frac{1}{2}}[\text{Diam}(B)^{-\frac{1}{2}}|z-w|^{-\frac{1}{2}}] + (\text{Diam}(B)|z-w|)^{\frac{1}{2}}. \max_{z'\in B} |D^1 g(z')|
\end{aligned}$$

which tends to zero as $|z-w| \to 0$. Note that we have used equation (2.7) to get a uniform bound of the energy.

If we consider the case when $|z-w| < M_n$, then set $R = (\text{Diam}(B)M_n)^{\frac{1}{2}}$ just replace each $|z-w|$ with M_n in the bound above. Thus, we got

$$\begin{aligned}
&|U_n(z) - U_n(w)| \\
&\leq C \log^{\frac{-1}{2}} \left[\frac{R}{|z-w|} \right] \mathcal{E}^{\frac{1}{2}}_{Q_{n,R}}(U_n) + \max_{z',w' \in \partial Q_n \cap B} |U_n(z') - U_n(w')| \\
&\leq C \mathcal{E}^{\frac{1}{2}}_{Q_{n,R}}(U_n) \log^{-\frac{1}{2}} [\text{Diam}(B)^{-\frac{1}{2}} |z-w|^{-1} M_n{}^{\frac{1}{2}}] + (\text{Diam}(B)|z-w|)^{\frac{1}{2}} . \max_{z' \in B} |D^1 g(z')| \\
&\leq C \mathcal{E}^{\frac{1}{2}}_{Q_{n,R}}(U_n) \log^{-\frac{1}{2}} [\text{Diam}(B)^{-\frac{1}{2}} M_n{}^{-\frac{1}{2}}] + (\text{Diam}(B) M_n)^{\frac{1}{2}} . \max_{z' \in B} |D^1 g(z')|
\end{aligned}$$

If $|z-w| < \delta_0$, we can choose M_n small enough as we wish and $M_n < \epsilon_0$ for all but finitely many n. This proves the equicontinuity.

Now, by Arzela-Ascoli, we know that there exists a subsequence of the U_k converges uniformly to a u continuous on the closure of Ω. By Lemma (2.20), the limit function is harmonic in Ω. Also, for any $z \in \partial\Omega$, there exists a sequence of points $z_n \in \partial Q_n \cap Q_n^{\bullet}$ such that $z_n \to z$ as $n \to \infty$, and thus $u = g$ on $\partial\Omega$. Since this limit is unique, the entire sequence U_n converges uniformly to u as desired.

□

2.4 Proof of our Main Theorem

Here we give the proof of Theorem 2.2.

Proof. Since Ω can be thought of as a topological rectangle with a pair of opposite sides E and F, let E' and F' consist of the other pair of opposite sides.

To compute $\text{Mod}_{\Omega}(E, F)$ in the plane, when $\partial\Omega$ is smooth, it is enough to solve the mixed

Dirichlet-Neumann problem below:

$$\begin{cases} \Delta u = 0 & \text{On } \Omega \\ u = 0 & \text{On } E \\ u = 1 & \text{On } F \\ \frac{\partial u}{\partial \eta} = 0 & \text{On } E' \cup F'. \end{cases} \tag{2.13}$$

However, it turns out that the solution u for (2.13) minimizes the Dirichlet energy, over all functions with $u|_E = 0$ and $u|_F = 1$. Namely, we can minimize the energy with only the Dirichlet boundary conditions and the the Neumann conditions get satisfied automatically. This holds also in more general domains, including our case of square grid domains, as can be seen in Corollary H.2 of Garnett-Marshall[5].

Moreover, since u can also be thought as the real part of a conformal map and Ω is a Jordan domain, we can apply Carathéodory's Theorem (see Theorem I.3.1 of Garnett-Marshall[5]), and conclude that $u \in C(\overline{\Omega})$. In particular, u solves the Dirichlet Problem with its own boundary values. Therefore, we can apply the convergence result in Theorem 2.11 and conclude that there is a sequence of discrete harmonic functions U_n that are 0 on E and 1 on F whose energy converges to $\mathcal{E}(u)$.

To be precise, Theorem 2.11 requires the boundary data g to be a smooth function. So given $\epsilon > 0$, we approximate u uniformly within ϵ with a smooth function g_ϵ. Also, we may assume that $g_\epsilon = 0$ on an open set containing E and $g_\epsilon = 1$ on an open set containing F. Now, let U_ϵ and $U_{n,\epsilon}$ be the harmonic extensions of g_ϵ to Ω and Q_n respectively. Since our discrete harmonic functions U_n minimize the discrete energy among all functions on Q_n with 0 on E and 1 on F we get that

$$\mathcal{E}_{Q_n}(U_n) \leq \mathcal{E}_{Q_n}(U_{n,\epsilon}) \to \mathcal{E}_\Omega(U_\epsilon).$$

In particular,

$$\limsup_{n\to\infty} \mathcal{E}_{Q_n}(U_n) \leq \mathcal{E}_\Omega(U_\epsilon). \tag{2.14}$$

Letting $\epsilon \to 0$, by the maximum principle U_ϵ converges uniformly to u on $\overline{\Omega}$, and moreover the partial derivatives converge as well (locally on compact subsets of Ω), as can be seen by using the Poisson kernel representation on a small disk and passing the partial derivatives under the integral sign. Therefore, since the left hand-side in (2.14) does not depend on ϵ, we have

$$\limsup_{n\to\infty} \mathcal{E}_{Q_n}(U_n) \leq \mathcal{E}_\Omega(u). \tag{2.15}$$

Now we need to prove that $\liminf_{n\to\infty} \mathcal{E}_{Q_n}(u_n) \geq \mathcal{E}_\Omega(u)$. For that we need to look at the family of cuts Γ_n^{cuts}. First, note that every cut may be replaced by a walk from E' to F' along the white vertices (that are placed in the middle of each face) see Figure 2.6. Namely, every walk connecting the two components of green vertices in Figure 2.6 corresponds to a unique cut for the paths on the black vertices connecting the two components of blue vertices, and conversely, every such cut corresponds to a walk as above. Also, once the two green components are collapsed to a single node, then the usage vectors for dual paths vs. cuts will be exactly the same. Hence the modulus of the family of cuts is exactly equal to the modulus of the family of dual paths on the dual grid $\hat{Q}_n$.

So the minimal energy for the family of dual paths, $\text{Mod}_2(\Gamma_n^{\text{cuts}})$, is equal to the reciprocal of $\mathcal{E}_{Q_n}(U_n)$ (by Fulkerson duality, Theorem 1.17). On the other hand, $\hat{Q}_n$ also provides a discrete approximation of the continuous problem on Ω with 0 on E' and 1 onF'. It is known that the energy of minimizer v for the latter problem is also the reciprocal of $\mathcal{E}_\Omega(u)$. This is Theorem 1.18. So, let V_n be the potential minimizer of the discrete energy on the dual grid $\hat{Q}_n$. Applying the previous argument again, and noting that the more general form of the Skopenkov-Werness result allows for the boundary of the approximating grid to converge to $\partial\Omega$ in the Hausdorff distance, we have that

$$\limsup_{n\to\infty} \mathcal{E}_{Q_n}(V_n) \leq \mathcal{E}_\Omega(v). \tag{2.16}$$

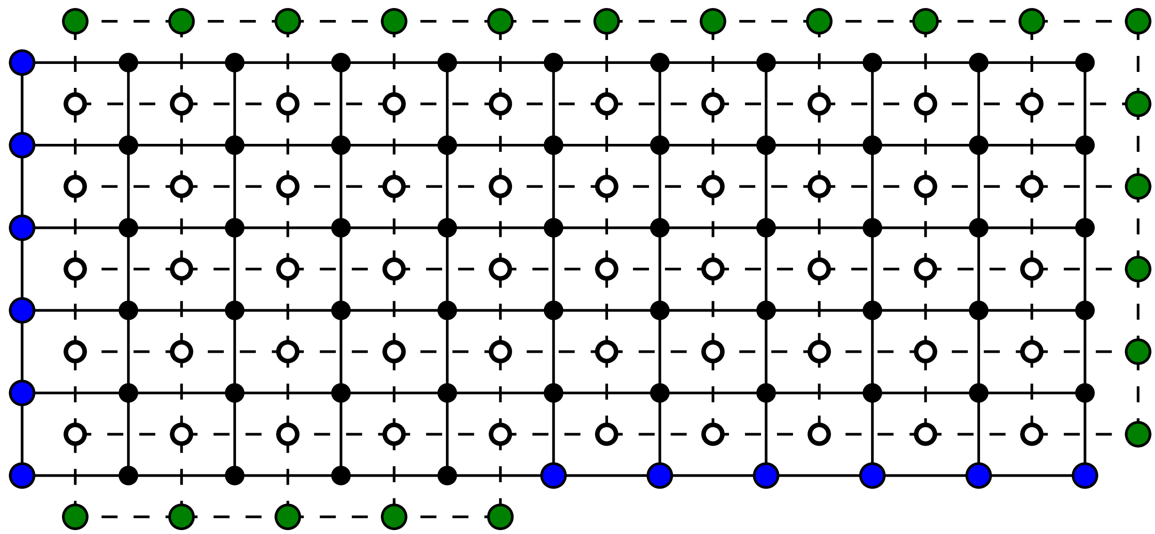

Figure 2.6: *A five-by-ten grid*

So, since $\mathcal{E}_{Q_n}(U_n) = \mathcal{E}_{Q_n}(V_n)^{-1}$, and $\mathcal{E}_\Omega(u) = \mathcal{E}_\Omega(v)^{-1}$,

$$\liminf_{n\to\infty} \mathcal{E}_{Q_n}(U_n) \geq \mathcal{E}_\Omega(u). \tag{2.17}$$

Finally, (2.15) and (2.17) complete the proof. □

Chapter 3

Generalizing to quadrangular grids with bounded geometry

We will prove the convergence of the modulus in more general lattices. We will work with a quadrilateral lattice, which is a planar graph Q with a given straight line embedding into $\mathbb{C}$ whose vertices are identified with the set $V(Q)$ such that each bounded face of Q is a quaderilateral. We will always assume that the quadrilaterals are orthogonal. See Figure 3.1

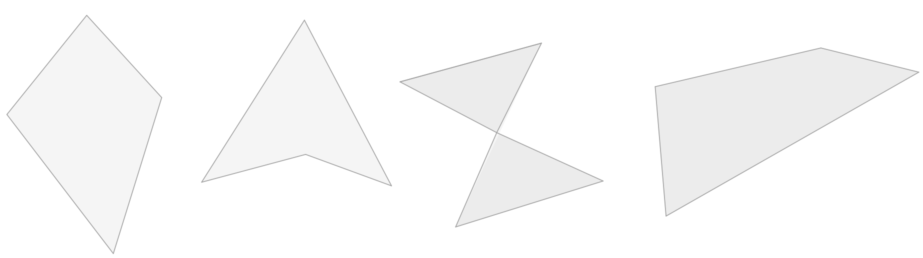

Figure 3.1: *The first two from the left are quadrilateral with orthogonal diagonals and the third is not allowed since the edges are not disjoint and the fourth is not allowed also because the diagonals are not orthogonal*

Also, we know that our graph admits a 2-coloring of the vertices as black and white, See Figure 3.2.

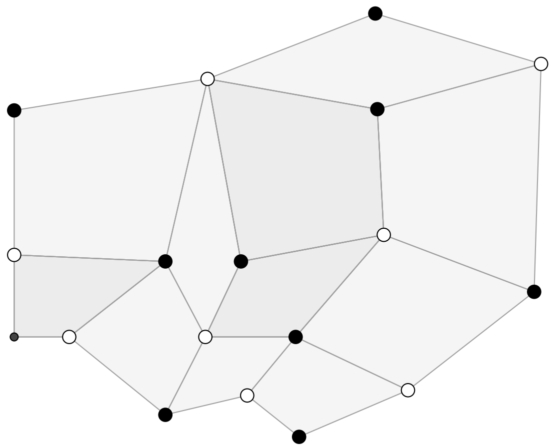

Figure 3.2: *Any quadrilateral lattice can be two-colored*

We will define the mesh size of the lattice by

$$M(Q) := \sup_{z \sim w} |z - w|$$

which is the maximal edge length of a lattice Q. Here for any simply connected domain we will approximate it by a sequence of quaderilteral lattices $\{Q_n\}$. So, we will provide the definition of the approximation as

Definition 3.1. We will say that $\{Q_n\}$ approximates a simply connected domain Ω if the following hold:

- $M_n \to 0$ as $n \to \infty$
- $d_{Haus}(\partial Q_n, \partial \Omega) \to 0$ as $n \to \infty$.

Where

$$d_{Haus}(X, Y) = \max\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\}$$

is the Hausdorff distance between the two metric spaces X and Y

Skopenkov proved the convergence between discrete and continuous Dirichlet solutions, with some uniform conditions that provide local control over the geometry of the quadrilateral lattice. Werness thinks that the uniform control is very important to prove this kind of convergence but it is a very strong condition that will excludes many reasonable lattices. Throughout this chapter we will consider Werness's K-round quadrilateral lattices.

Definition 3.2. A quadrilateral $f = [z_1 z_2 z_3 z_4]$ is K-round if all the interior angles are bounded by $\frac{2\pi}{K}$ and the ratio of the length of any pair of edges is less than K. We will call a lattice Q K-round if all faces in the lattice are K-round for a fix K. Note that always $K \geq 4$ and our square grids are 4-round lattices.

The main result for this chapter is to prove the following theorem

Theorem 3.3. *Let Ω be a simply connected domain. Let Q_n be a sequence of K-round finite orthogonal lattices that approximates a domain Ω and $g : \mathbb{C} \to \mathbb{R}$ be a given smooth function that will be used as boundary data. Then, the unique discrete harmonic functions ϕ_n on Q_n with boundary values $g\big|_{\partial Q_n}$ converge uniformly to the unique harmonic ϕ on Ω with boundary values $g\big|_{\partial\Omega}$. Moreover,*

$$\mathrm{Mod}_{Q_n}(E_n, F_n) \to \mathrm{Mod}_{\Omega}(E, F) \qquad \text{as } n \to \infty.$$

First we again recall the main steps in the Werness and Skopenkov papers. However, we skip the proofs in this more general case.

3.1 Geometric preliminaries

Lemma 3.4. [8] *If $f = [z_1 z_2 z_3 z_4]$ is K-round then the lengths of the edges are bounded below by $\frac{\mathrm{Diam}(f)}{2K}$ and the lengths of diagonals are bounded below $\frac{\mathrm{Diam}(f)}{4K^2}$.*

the next lemma is to control the area in terms of their diameters, which is generalize the equation (2.9).

Lemma 3.5. [8] *There is a constant C_K, deponding only on K such that*

$$\mathrm{Area}(f) \geq C_K \, \mathrm{Diam}(f)^2$$

The idea of the proof is to divide each quadrilateral into two triangles and then use the bound of the edges and the angle between these two edges which we know from Lemma 3.1.

Now, we will generalize Lemma 2.12 which is approximate any rectifiable curve. So, that the sum of the diameters of the faces is not bigger than a constant times the length of the given curve.

Lemma 3.6. [8] *Let Q be a K-round lattice and $\gamma : [0,1] \to \mathbb{C}$ be rectifiable closed loop with* $\mathrm{Diam}(\gamma) \geq 2M$*, where M is the side-length of squares in Q. Then there exists a constant C_K such that*

$$\sum_{\substack{f \in Q \\ f \cap \gamma \neq \emptyset}} \mathrm{Diam}(f) \leq C_K \ell(\gamma)$$

In particular, when $\gamma = \partial B(z,r)$ is inside Q and $r > 2M$, then

$$\sum_{\substack{f \in Q \\ f \cap \gamma \neq \emptyset}} \mathrm{Diam}(f) \leq C_K r. \tag{3.1}$$

For the proofs of all the geometric properties of K-round ,see[8].

3.2 Estimate the energy and equicontinuity

We will eastimate the difference between values of a discrete harmonic function at two black vertices through the distant between them. We will follow Skopenkov and Werness arguments. Through the following arguments we will be given a pair of black vertices z and w and we will also consider a ball B_R of raduis R centered at the point $x = \frac{z+w}{2}$. We will recall the definition of Q_R, Q_z and Q_w from Chapter 2 (see Definition 2.15).

Definition 3.7.

The **semi-energy** of a function $\phi : Q^\bullet \to \mathbb{R}$ along the path $w_0 w_1 w_2 .... w_n$ is

$$\hat{\mathcal{E}}_{w_0 w_1 w_2 .... w_n}(\phi) = \sum_{i=1}^{n} |\bigtriangledown \phi(f_i^*)|^2 \cdot |w_i - w_{i-1}|$$

where f_i is a quaderilteral with diagonal $w_i w_{i-1}$.

Lemma 3.8. *Let $\phi : V(Q) \to \mathbb{R}$ be any function on any quadrilateral lattice and let $w_0 w_1 .....w_n$ be a path on the black vertices of $Q^\bullet$. Then*

$$\hat{\mathcal{E}}_{p,w_0w_1w_2........w_n}(\phi) \geq \frac{|\phi(w_n) - \phi(w_0)|^2}{\ell(w_0w_1w_2........w_n)}$$

where

$$\ell(w_0w_1w_2........w_n) := \sum_{j=1}^{n} |w_j - w_{j-1}|$$

is the Euclidean length of the path.

We will apply the above lemma to estimate the amount of semi-energy at distance r from our pair of points. Fix $z \in \mathbb{C}$ and $r > 0$ then the semi-p-energy at distance r is defined to be

$$\hat{E}_r^z := \sum_{f \in Q, f \cap \partial B_r(z)} | \bigtriangledown \phi(f)|^2 \cdot \mathrm{Diam}(f)$$

Lemma 3.9. *Let Q be a K-round orthogonal lattice, z, w be a pair of vertices in $Q^\bullet$, and $\phi : Q^\bullet \to \mathbb{R}$ be a discrete harmonic function. Take $R > |z - w| + M$ restricted to those R with no vertices of Q on the circle of radius R about $\frac{(z+w)}{2}$. Let*

$$\delta_R := |\phi(z) - \phi(w)| - \max_{z', w' \in \partial Q \cap B_R} |\phi(z') - \phi(w')|$$

If $\delta_R > 0$, then there is a constant C_K depending only on K such that

$$\hat{E}_{p,R}(\phi) \geq C_K \frac{\delta_R^2}{R}$$

We now integerate this in r to obtain the desired result.

Proposition 3.10. *Let Q be a K-round orthogonal lattice. Let $\phi : Q^\bullet \to \mathbb{R}$ be a discrete harmonic function. Let z, w be a pair of black vertices. Then, there is a constant C_K depends* *only on K such that for $R \geq 2|z-w|$*

$$|u(z)-u(w)| \leq C_K \log^{-\frac{1}{2}}\left[\frac{R}{|z-w|}\right]\mathcal{E}^{\frac{1}{2}}_{Q_R}(u) + \max_{z',w' \in \partial Q \cap B_R} |u(z')-u(w')|$$

3.2.1 Laplacian Approximation

Lemma 3.11. *(Gradient Approximation)*[8]

For any K-round $f^* = [z_1 z_2 z_3 z_4]$ we have, for any $g \in C^3(\mathbb{C})$,

$$|\bigtriangledown g - \bigtriangledown_Q g| \leq C_K M \max_{z \in f^*} |D^2 g(z)|$$

Lemma 3.12. *Let Q be a K-round lattice , and R be a square of side-length $r > M$ inside Q. Then for any $g \in C^3(\mathbb{C})$ we have*

$$\left| \sum_{w \in R \cap V(Q)} [\triangle_Q(g|_{Q^\bullet})](w) - \int_R \triangle g \quad dxdy \right| \leq C_K(rM \max_{z \in R}|D^2 g(z)| + r^3 \max_{z \in R} |D^3 g(z)|)$$

We may now use these results and the results from Chapter 4 to establish the final convergence theorem.

3.3 Skopenkov and Werness convergence

Lemma 3.13. *Let Ω be a bounded simply-connected domain with smooth boundary. Let $\{Q_n\}$ be a sequence of K-round quaderilteral lattices approximating the domain. Then for any $C^2(\mathbb{C})$ smooth function $\eta : \mathbb{C} \to \mathbb{R}$, $\mathcal{E}_{Q_n}(\eta|_{Q_n}) \to \mathcal{E}_\Omega(\eta)$ as $n \to \infty$*

Theorem 3.14. *Let Ω be a simply connected domain, Ω is approximated by a quadrilateral lattice Q_0. Let Q_n be a sequence of K-round quadrilateral lattices approximating the domain and $g : \mathbb{C} \to \mathbb{R}$ be a given smooth boundary value. Then, the discrete p-harmonic functions ϕ_n on Q_n with boundary values $g_{\big|\partial Q_n}$ converge uniformly to the unique continuous p-harmonic ϕ on Ω with boundary values $g_{\big|\partial\Omega}$.*

3.4 Proof the final result

proof of main theorem 0.3. Let ϕ minimize the continuous energy with 0 on E and 1 on F. Since ϕ solves the Dirichlet Problem with its own boundary values we can apply the convergence result in Theorem 3.14 and say that there is a sequence of discrete harmonic functions ϕ'_n that are 0 on E_n and 1 on F_n whose energy converges to $\mathcal{E}(\phi)$. Since our discrete harmonic functions u_n minimize the discrete energy among all functions with 0 on E_n and 1 on F_n we get that

$$\mathcal{E}_{Q_n}(\phi_n) \leq \mathcal{E}_{Q_n}(\phi'_n) \to \mathcal{E}_\Omega(\phi)$$

Thus,

$$\limsup_{n\to\infty} \mathcal{E}_{Q_n}(\phi_n) \leq \mathcal{E}_\Omega(\phi)$$

Now we need to prove that $\liminf \mathcal{E}_{Q_n}(\phi_n) \geq \mathcal{E}_\Omega(\phi)$. For that we need to look at the family of cuts Γ_n^{cuts}. First, note that every cut may be replaced by a walk from E' to F'. Namely, every walk connecting E' and F' corresponds to a unique cut between E and F and conversely, every such cut corresponds to a walk as above. Hence the modulus of the family of cuts is exactly equal to the modulus of the family of dual paths. Also, there are E'_n and F'_n such that $d_{Haus}(E'_n, E') \to 0$ and $d_{Haus}(F'_n, F') \to 0$ as $n \to \infty$.

Lemma 3.15. *$d_{Haus}(E'_n, E') \to 0$ and $d_{Haus}(F'_n, F') \to 0$ as $n \to \infty$*

Proof. WLOG Assume that Ω tiled by a square grid domain with side-length 1. Fix a white node x in $\partial\hat{Q}_n \cup E'_n$. There is $x' \in \hat{Q}_n$ such that the edge $e = \{x, x'\}$ is perpenducular to an edge $e' = \{u_1, u_2\}$ of black vertices in ∂Q_n. Let y be the intersection point of e and e'. Then $y \in E'$ and

$$|x - y| \leq 2^{-n}$$

and

$$d_{Haus}(x, E') \leq 2^{-n}$$

which implies that

$$\sup_{x \in E'_n} d_{Haus}(x, E') \leq 2^{-n} \tag{3.2}$$

Now, fix $z \in E'$. There is an edge $\{u_1, u_2\}$ of black vertices such that $z \in \{u_1, u_2\}$. Then $|z - y| \leq 2^{-n}$. Thus by triangle inequality, $|z - x| \leq 2^{-n}$ and $d_{Haus}(z, E'_n) \leq 2^{-n}$ Since z arbitrary,

$$\sup_{z \in E'_n} d_{Haus}(z, E'_n) \leq 2^{-n} \tag{3.3}$$

Equation (3.2) and equation(3.3) implies that

$$d_{Haus}(E', E'_n) \leq 2^{-n} \to 0 \quad \text{as} \quad n \to \infty$$

□

So, let V_n be the potential minimizer of the discrete energy with 0 on E'_n and 1 on F'_n. Then Applying the previous argument again,

$$\mathcal{E}_{Q_n}(V_n) \leq \mathcal{E}_{Q_n}(V'_n) \to \mathcal{E}_{\Omega}(v)$$

. Thus,

$$\liminf_{n \to \infty} \mathcal{E}_{Q_n}(\phi_n) \geq \mathcal{E}_{\Omega}(\phi).$$

This completes the proof. □

Chapter 4

The case when $p \neq 2$, conclusion and future work

In this chapter, we will explore which results in the previous chapters extend to the $p \neq 2$ case. In particular, our goal is to modify all of our work in Chapter 2 to the case $1 < p < \infty$, in such a way that the proofs will hold in the more general case of quadrilateral lattices with orthogonal diagonals.

4.1 Behavior of side to side p-modulus under grid refinements

Let R_n be a rectangular $\frac{1}{n}$-grid. Namely, R_n is a graph with nodes,

$$V_n = \{(\frac{i}{n}, \frac{j}{n}) : i_0 \leq i \leq i_1, j_0 \leq j \leq j_1\}$$

and edges

$$E_n = \{e : e = \{x, y\} \quad for \quad x, y \in V_n \quad and \quad ||x - y||_\infty = 1\}$$

Let Γ_n be the family of walks in R_n from $\{Rez = \frac{i_0}{n}\}$ to $\{Rez = \frac{i_1}{n}\}$. Pick a cell (a square) and refine it by subdividing each side into k equal length intervals. Call the resulting graph $R_{n,k}$. Let $\Gamma_{n,k}$ be the the family of walks in $R_{n,k}$ from $\{Rez = \frac{i_0}{n}\}$ to $\{Rez = \frac{i_1}{n}\}$, with $i_1 - i_0 = n$. Pick $\rho = \frac{1}{n}$ on the horizontal edges of the original grid and $\rho = \frac{1}{nk}$ on the new horizontal smaller edges.

It is not clear what will happen to $\mathrm{Mod}_p(\Gamma_{n,k})$ rekative to $\mathrm{Mod}_p(\Gamma_n)$,because, although we added longer walks, there are now more ways to go from side to side. In fact, as we will see, we have

$$\mathrm{Mod}_p(\Gamma_{n,k}) \leq \mathrm{Mod}_p(\Gamma_n)$$

We have that

$$\rho(e) = \begin{cases} \frac{1}{n} & if \quad e \quad old \quad edge \\ \frac{1}{nk} & if \quad e \quad new \quad edge \end{cases}$$

Such a ρ is admissible. To see this, consider a simple path $\gamma \in \Gamma_{n,k}$ and assume that γ contains M old edges for some $M \leq r := \ell(\gamma)$. If γ uses at least one new edge, then it must use at least k new edges and at least $n-1$ old edges. Thus, in this case,

$$\ell_\rho(\gamma) = \sum_{i=1}^{r} \rho(e_i) = M\left(\frac{1}{n}\right) + (r-M)\left(\frac{1}{nk}\right) \geq (n-1)\left(\frac{1}{n}\right) + k\left(\frac{1}{nk}\right) = 1$$

On the other hand, if γ does not use any new edges, then it must use $M \geq n$ of the old ones, and

$$\ell_\rho(\gamma) = \frac{M}{n} \geq 1.$$

So,

$$\begin{aligned} \mathrm{Mod_p}(\Gamma_{n,k}) &\leq \mathrm{Mod_p}(\Gamma_n) - 2\left(\frac{1}{n}\right)^p + k(k+1)\left(\frac{1}{kn}\right)^p \\ &= \mathrm{Mod_p}(\Gamma_n) - \frac{2}{n^p} + \frac{k+1}{k^{p-1}n^p} \\ &= \mathrm{Mod_p}(\Gamma_n) - \frac{1}{n^p}\left(2 - \frac{k+1}{k^{p-1}}\right) \end{aligned} \tag{4.1}$$

Now, we compute the original Modulus $\mathrm{Mod}_p(\Gamma_n)$. Again if $\rho = \frac{1}{n}$ on all the horizontal edges, then ρ is admissible for Γ_n. So

$$\begin{aligned}\mathrm{Mod_p}(\Gamma_n) &\leq \mathcal{E}(\rho)\\ &= n(n+1)\left(\frac{1}{n}\right)^p\\ &= \frac{n+1}{n^{p-1}} \qquad (4.2)\end{aligned}$$

Now let $\hat{\Gamma}_n$ be the family of all the cuts for Γ_n. In this example a cut is obtained by choosing at least one horizontal edge for each one of the n+1 levels of the grid.

Let $\hat{\eta} = \frac{1}{n+1}$ on horizontal edges of R_n. Then η is admissible and we get

$$\begin{aligned}\mathrm{Mod_q}(\hat{\Gamma}_n) &\leq \mathcal{E}(\hat{\eta})\\ &= n(n+1)\left(\frac{1}{n+1}\right)^q\\ &= \frac{n}{(n+1)^{q-1}}\end{aligned}$$

By Fulkerson duality,

$$\mathrm{Mod_q}^{\frac{1}{q}}(\hat{\Gamma}_n)\mathrm{Mod_p}^{\frac{1}{p}}(\Gamma_n) = 1$$

So

$$\begin{aligned}\mathrm{Mod_p}^{\frac{1}{p}}(\Gamma_n) &= \frac{1}{\mathrm{Mod_q}^{\frac{1}{q}}(\hat{\Gamma}_n)}\\ &\geq \left(\frac{(n+1)^{q-1}}{n}\right)^{\frac{1}{q}}\end{aligned}$$

Thus, using that $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\begin{aligned}\mathrm{Mod}_p(\Gamma_n) &\geq \left(\frac{(n+1)^{q-1}}{n}\right)^{\frac{p}{q}} \\ &= \frac{(n+1)^{\frac{p(q-1)}{q}}}{n^{\frac{p}{q}}} \\ &= \frac{n+1}{n^{\frac{p}{q}}} \\ &= \frac{n+1}{n^{p-1}} \end{aligned} \tag{4.3}$$

Hence, combining equations (4.2) and (4.3), we have that

$$\mathrm{Mod_p}(\Gamma_n) = \frac{n+1}{n^{p-1}} \tag{4.4}$$

Apply (4.4) to (4.1), we get

$$\begin{aligned}\mathrm{Mod_p}(\Gamma_{n,k}) \leq & \frac{n+1}{n^{p-1}} - \frac{1}{n^p}\left(2-\frac{k+1}{k^{p-1}}\right) \\ =& \frac{n+1}{n^{p-1}}\left[1-\frac{1}{n(n+1)}\left(2-\frac{k+1}{k^{p-1}}\right)\right]\end{aligned} \tag{4.5}$$

Equation(4.5) gives an upper bound of p- modulus after one cell refinement.

Thus

$$\mathrm{Mod_p}(\Gamma_{n,k}) \leq \mathrm{Mod}_p(\Gamma_n)\left[1-\frac{1}{n(n+1)}\left(2-\frac{k+1}{k^{p-1}}\right)\right]$$

4.2 Decreasing of p-energy when $p \geq 2$

Here we will recall Definition 2.1 and as we saw in Section 1.5 computing connecting p-modulus is equivalent to minimizing p-energy subject to the boundary values. We will refine grids by adding a node on each edge that we also connect to a new node in each face. again we will recall an argument of Jacqueline Lelong-Ferrand that shows how refining a square grid in a geometric fashion, decreases the p-energy(see Figuer 2.4)

Namely, refine a square grid by adding a node to each edge, that we also connect to a new node in each face. After n refinements, there exist a unique p-harmonic function ϕ_n on the nodes of $Q_n \setminus (E \cup F)$ satisfying:

$$\begin{cases} \phi_n &= 0 \quad \text{on } E \\ \phi_n &= 1 \quad \text{on } F \end{cases} \tag{4.6}$$

In particular, ϕ_n minimizes the energy

$$\mathcal{E}(\phi) = \sum_{e=\{x,y\} \in E(Q_n)} |\phi(x) - \phi(y)|^p$$

over all functions on $Q_n^\bullet$ with the boundary values given in (4.6).

Assume that the value of ϕ_n at the nodes of an arbitrary square are $a, b, c, d,$ in the positive direction. Refine each square, and extend ϕ_n to $\bar{\phi}_n$. The values of $\bar{\phi}_n$ on the old nodes are the same as ϕ_n, but for the new nodes we set $\bar{\phi}_n$ equal to $\frac{a+b}{2}, \frac{c+b}{2}, \frac{d+c}{2}, \frac{a+d}{2}$ for each new node on the old edges, and we set $\bar{\phi}_n$ equal to $\frac{a+b+c+d}{4}$ on the new node in the middle of the old face.

Remark 4.1. Note that if the square that we pick it not in the boundary, each edge is sharing between two squares. So, when we compute the energy, we need to divide by 2. For example the edge between the two values a and b has energy value $|a-b|^p$. we will only consider $\frac{|a-b|^p}{2}$ when we compute thye energy of a chosen square.

Now we compare the old p-energy to the new p-energy:

$$\begin{aligned}
\mathcal{E}(\phi_n) - \mathcal{E}(\bar{\phi}_n) &= \sum_{e=\{x,y\}} |\phi_n(x) - \phi_n(y)|^p - \sum_{e=\{x,y\}} \left|\bar{\phi}_n(x) - \bar{\phi}_n(y)\right|^p \\
&= \left[\frac{|a-b|^p}{2} + \frac{|b-c|^p}{2} + \frac{|c-d|^p}{2} + \frac{|d-a|^p}{2}\right] - \\
&\left[\frac{|a - \frac{a+b}{2}|^p}{2} + \frac{|\frac{a+b}{2} - b|^p}{2} + \frac{|b - \frac{b+c}{2}|^p}{2} + \right. \\
&\frac{|\frac{b+c}{2} - c|^p}{2} + \frac{|c - \frac{c+d}{2}|^p}{2} + \frac{|\frac{c+d}{2} - d|^p}{2} + \frac{|\frac{a+d}{2} - d|^p}{2} + \frac{|d - \frac{a+d}{2}|^p}{2} \\
&+ \left|\frac{a+b-c-d}{4}\right|^p + \left|\frac{a+b-c-d}{4}\right|^p + \\
&\left.\left|\frac{a+d-b-c}{4}\right|^p + \left|\frac{a+d-b-c}{4}\right|^p\right] \\
&= \frac{2^p-2}{2^{p+1}}\left[|a-b|^p + |b-c|^p + |c-d|^p + |d-a|^p\right] \\
&- \frac{1}{4^p}\left[2|(a-d)+(b-c)|^p + 2|(a-b)+(d-c)|^p\right]
\end{aligned}$$

Put

$x = a-b, y = b-c, z = d-c, w = a-d,$

Then ,for $p \geq 2$ we get:

$$\begin{aligned}
\mathcal{E}(\phi) - \mathcal{E}(\bar{\phi}_n) &= \frac{2^p-2}{2^{p+1}}\left[|x|^p + |y|^p + |z|^p + |w|^p\right] - \frac{2}{4^p}\left[|w+y|^p + |x+z|^p\right] \\
&= \frac{2^p-2}{2^{p+1}}\left[|x|^p + |z|^p\right] - \frac{2}{4^p}|x+z|^p + \frac{2^p-2}{2^{p+1}}\left[|y|^p + |w|^p\right] - \frac{2}{4^p}|y+w|^p \\
&= \frac{1}{2^{p-1}}\left[\frac{2^p-2}{4}(|x|^p + |z|^p) - \left(\frac{|x+z|}{2}\right)^p\right] + \frac{1}{2^{p-1}}\left[\frac{2^p-2}{4}(|y|^p + |w|^p) - \left(\frac{|y+w|}{2}\right)^p\right] \\
&\geq \frac{1}{2^{p-1}}\left[\frac{1}{2}(|x|^p + |z|^p) - \left(\frac{|x|+|z|}{2}\right)^p\right] + \frac{1}{2^{p-1}}\left[\frac{1}{2}(|y|^p + |w|^p) - \left(\frac{|y|+|w|}{2}\right)^p\right]
\end{aligned}$$

The last inequality follows using $p \geq 2$.

Now, by convexity we have that,

$\frac{1}{2}(|x|^p + |z|^p) > \left(\frac{|x|+|z|}{2}\right)^p$ and $\frac{1}{2}(|y|^p + |w|^p) > \left(\frac{|y|+|w|}{2}\right)^p$

Then, we can conclude that

$$\mathcal{E}(\phi_n) - \mathcal{E}(\bar{\phi}_n) \geq 0$$

This shows that

$$\mathcal{E}(\phi_{n+1}) \leq \mathcal{E}(\phi_n). \tag{4.7}$$

This monotonicity can be used to prove the convergence between the p-modulus of a domain and the p-modulus of its grid approximation.

For now we will generalize the Skopenkov and Werness work to the case $1 \leq p \leq \infty$ but we will consider the case of grids whose faces are squares. However, our proofs should generalize to quadrangular lattices with orthogonal diagonals.

4.3 Basic facts about discrete p-harmonic functions on graphs

Let $G = (V, E)$ be a finite graph, where V is the set of all the vertices and E is the set of all the edges. For a real valued function ϕ on V and $x \in V$, $1 < p < \infty$, we will review all the definitions from the Holopainen and Soardi paper[9].

Definition 4.2. The pth power of the gradient is defined by

$$|D\phi(x)|^p = \sum_{y \sim x} |\phi(y) - \phi(x)|^p$$

Definition 4.3. The p - Dirichlet sum is defined by

$$I_p(\phi, V) = \sum_{x \in V} |D\phi(x)|^p$$

Definition 4.4. The p - Laplacian is defined by

$$\triangle_p \phi(x) = \sum_{y\sim x} \text{sign}(\phi(y)-\phi(x))|\phi(y)-\phi(x)|^{p-1} = \sum_{y\sim x}(\phi(y)-\phi(x))|\phi(y)-\phi(x)|^{p-2}$$

In the continuum, p-harmonicity may be described via the minimization of the variational integeral:

$$\int_D |\bigtriangledown h|^p dm$$

For $u \in L^p(G)$ and for every $D \subset G$ among all functions in $W^{1,p}(D)$ with same values in ∂D, where G is an oben set that is contained in a noncompact, connected and oriented manifold of class C^∞.

In the discrete such a definition may be given as well. For a face f of Q we write $f \in Q$ and we will let the discrete gradient of a function $\phi : V(Q) \to \mathbb{R}$ on that face by:

$$|D\phi(f^*)|^p = \left(|D\phi(f^*)|^2\right)^{\frac{p}{2}}$$

We may now define the discrete p-energy.

Definition 4.5. The discrete p-energy is

$$\begin{aligned}\mathcal{E}_{p,Q}(\phi) &:= \sum_{f^*\in Q} |D\phi(x)|^p \cdot \text{Area}(f^*)\\ &= \frac{1}{2}\sum_{f^*\in Q}\left[\frac{|z_2-z_4|}{|z_1-z_3|^{p-1}}|\phi(z_3)-\phi(z_1)|^p + \frac{|z_1-z_3|}{|z_2-z_4|^{p-1}}|\phi(z_2)-\phi(z_4)|^p\right]\end{aligned}$$

In the case of square lattices, which we assume throughout, the expression for the energy may be made more explicit:

$$\mathcal{E}_{p,Q}(\phi) = \frac{1}{2}\sum_{f^*\in Q}\left[\frac{1}{|z_3-z_1|^{p-2}}|\phi(z_3)-\phi(z_1)|^p + \frac{1}{|z_4-z_2|^{p-2}}|\phi(z_4)-\phi(z_2)|^p\right]$$

Definition 4.6. A function ϕ is **discrete p-harmonic in V** if $\triangle_p\phi(x) = 0$ for every $x \in V$.

Definition 4.7. Let $G = (V, E)$ be a graph. If $I_p(\phi, V) < \infty$, then ϕ is said to be **energy**

finite

4.4 Comparison principle and Maximum principle for discrete p - harmonic functions on graphs

Definition 4.8. A lower semi-continuous function $\phi : \Omega \to \mathbb{R} \cup \{+\infty\}$ that is not identically $+\infty$ is p-superharmonic function, if it is satisfies the comparison principle with respect to p-harmonic functions in every subdomain D with closure in Ω: i.e., whenever a p-harmonic function $h \in C(\overline{D})$ is such that

$$\phi(x) \geq h(x)$$

for all $x \in \partial D$ then

$$\phi(x) \geq h(x)$$

for all $x \in D$

Remark 4.9. A function h is p-harmonic if and only if h and $-h$ are super p-harmonic functions.

Theorem 4.10 (Comparison principle)**.** [9] *Let u be a p-superharmonic and v p-subharmonic functions in a finite set $S \subset V$ such that $u \geq v$ in ∂S. Then $u \geq v$ in S.*

Now, using Comparison principle we can prove a weak version of the maximum principle

Theorem 4.11 (Maximum principle (weak version))**.** *Let G be a finite graph and h be p-harmonic function on G such that $a \leq h \leq b$ on ∂G. Then $a \leq h \leq b$ on $V(G)$.*

Proof. We claim first that $h \leq b$ on $V(G)$. Set $v := h(x) - b$. By the assumption we have $v \leq 0$ on ∂G. By comparison principle, $v \leq 0$ on $V(G)$ which implies that $h \leq b$ on $V(G)$. Next, we claim $h \geq a$ on $V(G)$. Set $u := a - h(x)$. By the assumption we have $u \geq 0$ on ∂G. By comparison principle, $v \geq 0$ on $V(G)$ which implies that $h \geq a$ on $V(G)$. □

Now, we will prove the maximum principle for the discrete p-harmonic functions

Theorem 4.12. *Let $G = (V, E)$ be a finite graph. Assume that $\emptyset \neq B \subset V$ is a set of boundary points. Let $\phi : V \to \mathbb{R}$ be a function which is discrete p-harmonic on the interior points $V \setminus B$ and attains its global maximum value $m := \max_V \phi$ at an interior point. Then,*

$$m = \max_B \phi.$$

Proof. Assume ϕ has its maximum m at some interior point $x \in V(G)$. Since ϕ is discrete p-harmonic we have that

$$\sum_{y\sim x} (\phi(y) - \phi(x))|\phi(y) - \phi(x)|^{p-2} = 0$$

and $\phi(y) - \phi(x)$ is nonpositive for every y, hence we have that $\phi(y) = \phi(x)$ for all $y \sim x$, which means that the global maximum is attained at every neighbor of x as well. This can be repeated in an oil spill fashion all the way to the boundary. A similar result holds for the minimum value. □

4.5 Estimate the energy and equicontinuity

We will estimate the difference between values of a discrete p -harmonic function at two black vertices through the distant between them. we will generalize Skopenkov and Werness arguments for the case when $1 < p < \infty$ and for the given p, we will let $q \in (1, \infty)$ be the Hölder conjugate exponent of p so that $pq = p + q$.

Through the following arguments we will be given a pair of black vertices z and w and we will also consider a ball B_R of raduis R centered at the point $x = \frac{z+w}{2}$. We will recall the definition of Q_R, Q_z and Q_w from Chapter 2 (see Definition 2.15).

Definition 4.13.

The **semi-p-energy** of a function $\phi : Q^\bullet \to \mathbb{R}$ along the path $w_0w_1w_2........w_n$ is

$$\hat{E}_{p,w_0w_1w_2........w_n}(\phi) = \frac{1}{M^{p-1}} \sum_{i=1}^{n} |D\phi(f_i^*)|^p$$

Lemma 4.14.

$$\hat{E}_{w_0w_1w_2........w_n}(\phi) \geq \frac{|\phi(w_n) - \phi(w_0)|^p}{\ell^{p-1}(w_0w_1w_2........w_n)}$$

where

$$\ell(w_0w_1w_2........w_n) := \sum_{j=1}^{n} |w_j - w_{j-1}| = nM$$

is the Euclidean length of the path.

Proof.

By Hölder inequality, we see that

$$\begin{aligned}
|\phi(w_n) - \phi(w_0)|^p &= \left|\sum_{i=1}^{n}(\phi(w_i) - \phi(w_{i-1}))\right|^p \\
&\leq \left(\sum_{i=1}^{n} |\phi(w_i) - \phi(w_{i-1})|^p\right)^{\frac{p}{p}} \cdot \left(\sum_{i=1}^{n} 1\right)^{\frac{p}{q}} \\
&= \frac{1}{M^{p-1}} \sum_{i=1}^{n} |\phi(w_i) - \phi(w_{i-1})|^p.(nM)^{p-1} \\
&= \hat{E}_{p,w_0w_1w_2........w_n}(\phi).\ell^{p-1}(w_0w_1.....w_m)
\end{aligned}$$

and the result follows.

□

We will apply Lemma 4.14 to estimate the amount of semi-p-energy at distance r from our pair of points. Fix $z \in \mathbb{C}$ and $r > 0$ then the semi-p-energy at distance r is defined to be

$$\hat{E}^z_{r,p} := \sum_{f^* \in Q, f^* \cap \partial B_r(z)} |D\phi(f)|^p \cdot \text{Diam}(f)$$

Lemma 4.15. *Let Q be a square grid, z, w be a pair of vertices in $Q^\bullet$, and $\phi : Q^\bullet \to \mathbb{R}$ be a discrete p-harmonic function. Take $R > |z - w| + M$ restricted to those R with no vertices of Q on the circle of radius R about $\frac{(z+w)}{2}$. Let*

$$\delta_R := |\phi(z) - \phi(w)| - \max_{z', w' \in \partial Q \cap B_R} |\phi(z') - \phi(w')|$$

If $\delta_R > 0$, then there is a constant C_p depends only on p such that

$$\hat{E}_{p,R}(\phi) \geq C_p \frac{\delta_R^p}{R^{p-1}}$$

Proof.

WLOG, assume $\phi(z) > \phi(w)$. Let Q_R, Q_z and Q_w as defined before (see Definition 2.15). By the maximum principle, there is $z' \in \partial Q_R \cap Q^\bullet$ with $\phi(z') > \phi(z)$ and a point w' with $\phi(w') < \phi(w)$. We want to prove that the graph $Q_R \cap \partial B_R$ contains two black vertises joined by a path in $Q_R^\bullet \cap \partial B_R$. Now, consider the following cases:

Case 1 : If $\partial Q_R \cap \partial Q = \emptyset$, then there are two faces that contain z' and w'. These two faces also intersect the boundary of B_R. Thus, there is a path $\gamma = w_0 w_1 .... w_r$ of black vertices that connect the two points z' and w' which is contained in faces that intersect ∂B_R.

Note that by using equation (2.10) we have that

$$(\ell(\gamma))^{p-1} = (rM)^{p-1} = \left(\sum_{i=1}^{r} |w_i - w_{i-1}|\right)^{p-1} \leq \left(\sum_{f^* \in Q, f^* \cap \partial B_R \neq \emptyset} \mathrm{Diam}(f^*)\right)^{p-1} \leq C_p R^{p-1} \tag{4.8}$$

Since $|w_i - w_{i-1}|$ are the diameters of the new faces f^*.

Now, by using Lemma 4.15 and equation(4.8), we have the following:

$$\begin{aligned}\hat{\mathcal{E}}_{p,R}^x(\phi) \geq \hat{\mathcal{E}}_{p,w_0 w_1 \dots w_r} &\geq \frac{(\phi(z') - \phi(w'))^p}{(rM)^{p-1}} \\ &\geq \frac{(\phi(z) - \phi(w))^p}{(rM)^{p-1}} \\ &\geq \frac{\delta_R^p}{(rM)^{p-1}} \\ &\geq C_p \frac{\delta_R^p}{R^{p-1}}\end{aligned}$$

Note that the first inequality follows because the number of faces that connect z' and w' are less than or equal the number of faces that intersect the boundary of B_R.

Case 2: If $\partial Q_R \cap \partial Q \neq \emptyset$, then if there is an arc of the circle with the same properties in

Case 1 which stays inside Q, we are done. So, assume that the boundary of the circle splits into multiple components. Let $C_{z'}$ be the arc which intersect the face that contains z' and $C_{w'}$ be the arc which intersect the face that contains w'. Let z'' be the vertex in $\partial Q_R \cap \partial Q$ at one of the end points of $C_{z'}$ and the same for w''. Then there is a path of black vertices of faces that intersect the boundary of B_R and connect z'' and z' and another path that connect w'' and w'. Also,

$$(\phi(z') - \phi(z'')) + (\phi(w'') - \phi(w')) \geq ((\phi(z) - \phi(w)) - (\phi(z'') - \phi(w'')) \geq \delta_R.$$

Thus, either $(\phi(z') - \phi(z''))$ or$(\phi(w'') - \phi(w')$ is greater than $\frac{\delta_R}{2}$. WLOG, assume that $(\phi(z') - \phi(z'')) > \frac{\delta_R}{2}$. Let $\gamma = w_0 w_1 .... w_r$ be the path that connects z'' and z' then applying the same argument as Case 1 to those points gives the desired bound.

Case 3: Assume that either z' or w' is contained in $\partial Q_R \setminus \partial Q$. WLOG, say z' is. Take z'' as in the Case 2, then there is a path of black vertices of faces intersect the boundary of B_R that connects z'' and z'.

$$\phi(z') - \phi(z'') = (\phi(z') - \phi(w')) - (\phi(z'') - \phi(w')) \geq ((\phi(z) - \phi(w)) - (\phi(z'') - \phi(w')) \geq \delta_R$$

Then the desired bound obtained as in the first case.

Case 4: If neither z' nor w' are contained in $\partial Q_R \setminus \partial Q$, then

$$0 > ((\phi(z) - \phi(w)) - (\phi(z') - \phi(w')) \geq \delta_R$$

but this contradicts our hypothesis. So, this case is impossible.

□

Proposition 4.16. *Let Q be a square grid. For $p > 2$ Let $\phi : Q^\bullet \to \mathbb{R}$ be a discrete p-harmonic function. Let z, w be a pair of black vertices. Then, there is a constant C_p depends only on p such that for $R \geq 2|z - w|$*

$$|\phi(z)-\phi(w)| \le C_p \mathcal{E}_{p,Q_R}^{\frac{1}{p}}(\phi)\left(\frac{R^{p-2}|z-w|^{p-2}}{|z-w|^{p-2}-R^{p-2}}\right)^{\frac{1}{p}} + \max_{z',w'\in\partial Q\cap B_R}|\phi(z')-\phi(w')|$$

Proof.

Let $\delta_R := |\phi(z)-\phi(w)| - \max_{z',w'\in\partial Q\cap B_R}|\phi(z\prime)-\phi(w\prime)|$. If $\delta_R \le 0$, then the required estimate holds automatically. Assume that $\delta_R > 0$. By using Lemma 4.15 and by observing that $\delta_r > \delta_R$ for $r < R$. So, we have

$$\begin{aligned}\int_{|z-w|}^{R} \hat{\mathcal{E}}_{p,r}(\phi)dr &\ge \int_{|z-w|}^{R} C_p \frac{\delta_R^p}{r^{\frac{p}{q}}}dr \\ &= C_p\delta_R^p(R^{2-p}-|z-w|^{2-p})\end{aligned}$$

for easy computation:

$$\int_{|z-w|}^{R} \hat{\mathcal{E}}_{p,r}(\phi)dr \ge C_p\delta_R^p\left(\frac{|z-w|^{p-2}-R^{p-2}}{R^{p-2}|z-w|^{p-2}}\right)$$

Now, by the definition of semi-p-energy , we get

$$\begin{aligned}
\int_{|z-w|}^{R} \hat{\mathcal{E}}_{p,r}(\phi)dr &\leq \int_0^R \sum_{f^*\in Q, f^*\cap\partial B_r(z)} |D\phi(f)|^p \cdot \text{Diam}(f^*) \\
&= \int_0^R \sum_{f\in Q} |D\phi(f^*)|^p \cdot \text{Diam}(f^*).\mathbb{1}_{\{f^*\cap\partial B_r\neq\emptyset\}} \\
&= \sum_{f\in Q} |D\phi(f^*)|^p \cdot \text{Diam}(f^*) \int_0^R \mathbb{1}_{\{f^*\cap\partial B_r\neq\emptyset\}} \\
&\leq \sum_{f^*\in Q} |D\phi(f^*)|^p \cdot \text{Diam}(f^*)\,\text{Diam}(f^*) \\
&= \sum_{f^*\in Q} |D\phi(f^*)|^p \cdot \text{Diam}(f^*)^2 \\
&= 2\sum_{f^*\in Q} |D\phi(f^*)|^p \cdot \text{Area}(f^*) \\
&= 2\mathcal{E}_{p,Q_R}(\phi)
\end{aligned}$$

By Combining the last two inequalities we have

$$C_p\delta_R^p\left(\frac{|z-w|^{p-2}-R^{p-2}}{R^{p-2}|z-w|^{p-2}}\right) \leq 2\mathcal{E}_{p,Q_R}(\phi)$$

By substituting δ_R, we have

$$C_p(|\phi(z)-\phi(w)| - \max_{z',w'\in\partial Q\cap B_R}|\phi(z\prime)-\phi(w')|)^p \leq 2\mathcal{E}_{p,Q_R}(\phi)\left(\frac{|z-w|^{p-2}-R^{p-2}}{R^{p-2}|z-w|^{p-2}}\right)^{-1}$$

By taking both sides to the power $\frac{1}{p}$

$$|\phi(z)-\phi(w)| - \max_{z',w'\in\partial Q\cap B_R}|\phi(z')-\phi(w')| \leq C_p\mathcal{E}_{p,Q_R}^{\frac{1}{p}}(\phi)\left(\frac{|z-w|^{p-2}-R^{p-2}}{R^{p-2}|z-w|^{p-2}}\right)^{-\frac{1}{p}}$$

$$|\phi(z)-\phi(w)| \leq C_p\mathcal{E}_{p,Q_R}^{\frac{1}{p}}(\phi)\left(\frac{R^{p-2}|z-w|^{p-2}}{|z-w|^{p-2}-R^{p-2}}\right)^{\frac{1}{p}} + \max_{z',w'\in\partial Q\cap B_R}|\phi(z')-\phi(w')|$$

□

4.6 p-Laplacian Approximation

Lemma 4.17. *Let Q be a square grid lattice, and R be a square inside Q, of side-length r geater than the mesh-size M. Fix $1 < p < \infty$. Then for any $g \in C^4(\mathbb{C})$ we have*

$$\left| \sum_{w \in R \cap V(Q)} [\triangle_{p,Q}(g|_{Q^\bullet})](w) - \int_R \triangle_p g \quad dxdy \right| \leq C_p \left[(r^{2(p-1)} \max_{z \in R} |D^2 g(z)| + r^{2p-1} (\max_{z \in R} |D^3 g(z)|)^{p-1} \right]$$

Proof. Take an arbitrary function $g \in C^4(\mathbb{C})$ and without loss of generality assume R is centered at 0. Expand g as

$$g(z) = a_0 + a_1 \operatorname{Re} z + a_2 \operatorname{Im} z + a_3 \operatorname{Re} z^2 + a_4 \operatorname{Im} z^2 + a_5 |z|^2 + \bar{g}(z)$$

Where $D^k \bar{g}(0) = 0$ for $k = 0, 1, 2$. We will prove Laplacian Approximation in several particular cases and then combine them together.

- The cases $g(z) = a_0$ follows immediately because it is both discrete and continuous p-Laplacian are zero.

- In the case $g(z) = \operatorname{Re} z$, we consider z a complex number such that $z = x + iy$. So, $g(z) = x$, hence $\triangle_p x = 0$, and

$$\int_R \triangle_p \operatorname{Re} z dxdy = 0.$$

  We wish to show that the sum over all the points $v \in V(Q) \cap R$ of discrete p-laplacian is zero, that is,

$$\sum_{v \in V(Q) \cap R} [\triangle_{p,Q}(\operatorname{Re} z)](v) = 0$$

  Note that for any verterx v in the graph, it has at most 4 neighbors namely v_1, v_2, v_3, v_4.

Then,

$$
\begin{aligned}
[\triangle_{p,Q}(\operatorname{Re} z)](v) &= \sum_{w\sim v} (g(w)-g(v))\,|g(w)-g(v)|^{p-2} \\
&= \sum_{w\sim v} (\operatorname{Re} w - \operatorname{Re} v)|\operatorname{Re} w - \operatorname{Re} v|^{p-2} \\
&= \sum_{w\sim v} \operatorname{Re}(w-v)|\operatorname{Re}(w-v)|^{p-2}
\end{aligned}
$$

Remark 4.18. Since we are working orthogonal diagonals, so for each face we can rotate them so they are parallel to the x and y axis. As a result we will have that v_1 and v_2 has the same real parts and v_3 and v_4 has the same real parts, v_2 and v_3 has the same imaginary parts and v_1 and v_4 has the same imaginary parts.

Thus, we will use the fact that $x = \frac{x_1+x_3}{2}$ and $x = \frac{x_2+x_4}{2}$

$$
\begin{aligned}
[\triangle_{p,Q}(\operatorname{Re} z)](v) =& \operatorname{Re}(v_1-v)|\operatorname{Re}(v_1-v)|^{p-2} + \operatorname{Re}(v_2-v)|\operatorname{Re}(v_2-v)|^{p-2} \\
&+ \operatorname{Re}(v_3-v)|\operatorname{Re}(v_3-v)|^{p-2} + \operatorname{Re}(v_4-v)|\operatorname{Re}(v_4-v)|^{p-2} \\
&= \frac{1}{2}\operatorname{Re}(v_2-v_4)|\operatorname{Re}(v_2-v_4)|^{p-2} + \frac{1}{2}\operatorname{Re}(v_4-v_2)|\operatorname{Re}(v_4-v_2)|^{p-2} \\
&+ \frac{1}{2}\operatorname{Re}(v_1-v_3)|\operatorname{Re}(v_1-v_3)|^{p-2} + \frac{1}{2}\operatorname{Re}(v_3-v_1)|\operatorname{Re}(v_1-v_3)|^{p-2} \\
&= \frac{1}{2}\operatorname{Re}(v_1-v_3+v_3-v_1)|\operatorname{Re}(v_1-v_3)|^{p-2} \\
&+ \frac{1}{2}\operatorname{Re}(v_2-v_4+v_4-v_2)|\operatorname{Re}(v_2-v_4)|^{p-2} \\
=& \, 0
\end{aligned}
$$

Take the sum over all the vertex w such that $w \in R \cap V(Q)$ we have that

$$
\sum_{w\in V(Q)\cap R} \triangle_{p,Q}(\operatorname{Re} z)(w) = 0
$$

- For $g(z) = \operatorname{Im} z$. This is analogous to the previous case.

- For $g(z) = \operatorname{Re} z^2$.

  The p- laplacian is defined by $\triangle_p g = \bigtriangledown \cdot (|\bigtriangledown g|^{p-2} \bigtriangledown g)$ and we have

$$|\bigtriangledown g|^{p-2} = (|\bigtriangledown g|^2)^{\frac{p-2}{2}}$$

  Which implies that

$$|\bigtriangledown g|^{p-2} = 2^{p-2}(x^2+y^2)^{\frac{p-2}{2}}$$

  Thus, the p-Laplacian is approximated by the following:

$$\begin{aligned}
|\triangle_p g| =& |2^{p-1}(p-2)(x^2+y^2)^{\frac{p-4}{2}}(x^2-y^2)| \\
&= |2^{p-1}(p-2)|z|^{p-4} \operatorname{Re} z^2| \\
&\leq 2^{p-1}(p-2)\left(\frac{r}{\sqrt{2}}\right)^{p-4} \frac{r}{\sqrt{2}} \cdot |\cos 2\theta| \\
&\leq 2^{p-1}(p-2)\left(\frac{r}{\sqrt{2}}\right)^{p-3} \\
&\leq C_p r^{p-3}
\end{aligned}$$

  Next, we take the integral over R,

$$\int_R |\triangle_p g| \leq C_p r^{p-3} \operatorname{Area}(R) \leq C_p r^{p-1} \leq C_p r^{2(p-1)}$$

  Now, we want to approximate the sum over all over the points $w \in V(Q) \cap R$ of the discrete p-Laplacian of $\operatorname{Re} z$. So,

$$\begin{aligned}
[\triangle_{p,Q}(\operatorname{Re} z^2)](v) &= \sum_{w\sim v} (g(w)-g(v))\,|g(w)-g(v)|^{p-2} \\
&= \sum_{w\sim v} (\operatorname{Re} w^2 - \operatorname{Re} v^2)|\operatorname{Re} w^2 - \operatorname{Re} v^2|^{p-2} \\
&= \sum_{w\sim v} \operatorname{Re}(w^2 - v^2)|\operatorname{Re}(w^2 - v^2)|^{p-2} \\
&\leq \sum_{w\sim v} |w^2 - v^2||w^2 - v^2|^{p-2} \\
&\leq \sum_{w\sim v} |w-v|^{p-1}(|w|+|v|)^{p-1} \\
&\leq \sum_{w\sim v} \left(\frac{\sqrt{2}}{2}M\right)^{p-1}\left(2\frac{r}{\sqrt{2}}\right)^{p-1}
\end{aligned}$$

Thus,

$$[\triangle_{p,Q}(\operatorname{Re} z^2)](v) \leq C_p M^{p-1} r^{p-1}$$

Now, take the sum over $x \in R \cap V(Q)$ and note that the number of vertices in R are at most $C \cdot \left(\frac{r}{M}\right)^2$

$$\sum_{x\in R} [\triangle_{p,Q}(\operatorname{Re} z^2)](x) \leq C_p r^{p-1} M^{p-1} \left(\frac{r}{M}\right)^2 \leq C_p r^p$$

Hence,

$$\sum_{x\in R} [\triangle_p(\operatorname{Re} z^2)] \leq C_p r^{2(p-1)}$$

- For $g(z) = \operatorname{Im} z^2$. This is analogous to the previous case.

- For $g(z) = |z|^2$. The p- laplacian is defined by $\triangle_p g = \triangledown \cdot (|\triangledown g|^{p-2} \triangledown g)$ and we have

$$|\triangledown g|^{p-2} = (|\triangledown g|^2)^{\frac{p-2}{2}}$$

Which implies that

$$|\bigtriangledown g|^{p-2} = 2^{p-2}(x^2+y^2)^{\frac{p-2}{2}}$$

Thus, for $p \geq 2$ the p- laplacian is

$$\begin{aligned}
|\triangle_p g| &= \left|2^{p-1}\left[(x^2+y^2)^{\frac{p-2}{2}} + (p-2)(x^2+y^2)^{\frac{p-2}{2}}\right]\right| \\
&= \left|2^{p-1}(x^2+y^2)^{\frac{p-2}{2}}(p-1)\right| \\
&\leq 2^{p-1}(p-1)|z|^{p-2} \\
&\leq C_p r^{p-2}
\end{aligned}$$

Then, take the integral over R, we have that

$$\int_R |\triangle_p g| \leq C_p r^p$$

$$\begin{aligned}
[\triangle_{p,Q}(|z|^2)](v) &= \sum_{w\sim v}(g(w)-g(v))\,|g(w)-g(v)|^{p-2} \\
&= \sum_{w\sim v}(|w|^2-|v|^2)||w|^2-|v|^2|^{p-2} \\
&= \sum_{w\sim v}(|w|-|v|)(|w|+|v|)\,|(|w|-|v|)(|w|+|v|)|^{p-2} \\
&= \sum_{w\sim v}(|w|-|v|)^{p-1}(|w|+|v|)^{p-1} \\
&\leq \sum_{w\sim v}\left(\frac{\sqrt{2}}{2}M\right)^{p-1}\left(2\frac{r}{\sqrt{2}}\right)^{p-1} \\
&\leq C_p r^{(p-1)}M^{p-1}
\end{aligned}$$

Take the sum over all $v \in R \cap V(Q)$, we have

$$\sum_{v\in R\cap V(Q)}[\triangle_{p,Q}(|z|^2)](v) \leq C_p r^{p-1}M^{p-1}\left(\frac{r}{M}\right)^2 \leq C_p r^{2(p-1)}$$

- For the case$\tilde{g}(z)$ when $D^k\tilde{g}(z) = 0$ at the center of R for $k = 0, 1, 2$. by some simplification we get

$$\triangle_p g = |\bigtriangledown g|^{p-4}\left[|\bigtriangledown g|^2 \triangle g + (p-2)\sum_{i,j=1,2}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}\frac{\partial^2 g}{\partial x_i \partial x_j}\right]$$

We will use the following estimates:

$$|\triangle g(z)| \leq Cr \max_{z\in R}|D^3 g(z)|$$

$$|\bigtriangledown g(z)| \leq Cr^2 \max_{z\in R}|D^3 g(z)|$$

$$|D^2 g(z)| \leq Cr \max_{z\in R}|D^3 g(z)|$$

Thus,

$$\begin{aligned}
|\triangle_p g| =& |\bigtriangledown g|^{p-4}\left||\bigtriangledown g|^2 \triangle g + (p-2)\sum_{i,j=1,2}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}\frac{\partial^2 g}{\partial x_i \partial x_j}\right| \\
&\leq \left(Cr^2 \max_{z\in R}|D^3 g(z)|\right)^{p-2}\left(Cr \max_{z\in R}|D^3 g(z)|\right) \\
&+ (p-2)r \max_{z\in R}|D^3 g(z)|\left(r^2 \max_{z\in R}|D^3 g(z)|\right)^{p-4} \\
&\leq C_p r^{2p-3}\left(\max_{z\in R}|D^3 g(z)|\right)^{p-1} + C_p r^{2p-7}\left(\max_{z\in R}|D^3 g(z)|\right)^{p-3}
\end{aligned}$$

By integrate the last inequality over R, we get

$$\int_R |\triangle_p g| \leq C_p r^{2p-1}\left(\max_{z\in R}|D^3 g(z)|\right)^{p-1} + C_p r^{2p-5}\left(\max_{z\in R}|D^3 g(z)|\right)^{p-3}$$

Now, by the estimate $|\bigtriangledown g(z)| \leq Cr^2 \max_{z\in R}|D^3 g(z)|$, $|D^2 g(z)| \leq Cr \max_{z\in R}|D^3 g(z)|$, and the gradient approximation we get

$$
\begin{aligned}
\left|\sum_{x\in R\cap V(Q)}[\triangle_Q g](x)\right| &= \left|\sum_{x\in R\cap V(Q)}\sum_{y\sim x}(g(y)-g(x))\,|g(y)-g(x)|^{p-2}\right| \\
&\leq \sum_{x\in R\cap V(Q)}\sum_{y\sim x}|g(y)-g(x)|^{p-1} \\
&= \sum_{x\in R\cap V(Q)}|Dg(x)|^{p-1} \\
&= \sum_{f\in R}|Dg(f)|^{p-1}\cdot|z_3-z_1| \\
&= \sum_{x\in R\cap V(Q)}(|Dg(x)-\bigtriangledown g(y)|+|\bigtriangledown g(y)|)^{p-1}\cdot|z_3-z_1| \\
&\leq \sum_{x\in R\cap V(Q}\left(CM\max_{x\in R}|D^2g(x)|+Cr^2\max_{x\in R}|D^3g(z)|\right)^{p-1}\cdot|z_3-z_1| \\
&\leq \sum_{x\in R\cap V(Q)}\left(CMr\max_{z\in R}|D^3g(z)|+Cr^2\max_{z\in R}|D^3g(z)|\right)^{p-1}\cdot|z_3-z_1| \\
&\leq C_p r^{2(p-1)}\left(\max_{x\in R}|D^3g(z)|\right)^{p-1}\sum_{x\in R\cap V(Q)}|z_3-z_1| \\
&\leq Cr^{2p-1}\left(\max_{z\in R}|D^3g(z)|\right)^{p-1}
\end{aligned}
$$

The last inequality follows by lemma (2.12)

- The general case when

$$g(z)=a_0+a_1\operatorname{Re}z+a_2\operatorname{Im}z+a_3\operatorname{Re}z^2+a_4\operatorname{Im}z^2+a_5|z|^2+\bar{g}(z)$$

follows from the special cases.

□

4.7 Gradient Approximation

Lemma 4.19. *For any square face $f = [z_1 z_2 z_3 z_4]$ and a function $g \in C^2(\Omega)$ we have*

$$||Dg(f)|^p - |\bigtriangledown g|^p| \leq C_p(\text{Diam}(f))^2 p(\max_{z\in f}|D^3 g|)^p$$

Proof. using Lemma 5.3**?** , we have

$$|Dg(f)| - |\bigtriangledown g| \leq |Dg(f) - \bigtriangledown g| \leq C\,\text{Diam}(f)\max_{z\in f}|D^2 g|$$

Using the estimate

$$|\bigtriangledown g| \leq C\,\text{Diam}^2(f)\max_{z\in R}|D^3 g(z)|$$

and

$$|D^2 g(z)| \leq C\,\text{Diam}(f)\max_{z\in R}|D^3 g(z)|$$

We have that

$$\begin{aligned}
|Dg|^p &\leq \left(C\,\text{Diam}(f)\max_{z\in f}|D^2 g| + |\bigtriangledown g|\right)^p \\
&= \left(C(\text{Diam}(f))^2\max_{z\in f}|D^3 g|\right)^p \\
&= C_p(\text{Diam}(f))^{2p}(\max_{z\in f}|D^3 g|)^p \\
&\leq C_p(\text{Diam}(f))^{2p}(\max_{z\in f}|D^3 g|)^p
\end{aligned}$$

Thus,

$$||Dg|^p - |\bigtriangledown g|^p| \leq C_p(\text{Diam}(f))^{2p}(\max_{z\in f}|D^3 g|)^p$$

□

4.8 The uniform limit

We need to show that if we are given a collection of discrete p-harmonic functions defined on grids whose meshe tends to zero and which do converge, then the limit must itself be p-harmonic in the usual sense.

Proposition 4.20. *Let Q_n be a sequence of square grids approximating a domain Ω. Let $\phi_n : Q_n^\bullet \to \mathbb{R}$ be a sequence of discrete p-harmonic functions such that ϕ_n converges uniformly to a continuous ϕ, $\phi : \Omega \to \mathbb{R}$. Then the function ϕ is p-harmonic in Ω.*

This proposition have not been proved yet in general case. Since the laplacian operator when $p = 2$ is rely different from the p-laplacian operator when $p > 2$. So, if we assume that the limit function is p-harmonic we will get the following convergence results.

We may now use these results to establish the final convergence theorem.

4.9 The convergence Results

Lemma 4.21. *Let Ω be a bounded simply-connected domain with smooth boundary. Let $\{Q_n\}$ be a sequence of square lattices approximating the domain. Then for any $C^2(\mathbb{C})$ smooth function $\eta : \mathbb{C} \to \mathbb{R}$, $\mathcal{E}_{p,Q_n}(\eta|_{Q_n}) \to \mathcal{E}_{p,\Omega}(\eta)$ as $n \to \infty$*

Proof. Let $\tilde{Q}_n$ be the union of all the faces of Q_n and all the interior edges. Since Q_n approximating the domain Ω, we have that ∂Q_n is contained within ϵ_n neighborhood of $\partial\Omega$ and $\partial\Omega$ is contained within ϵ_n neighborhood of ∂Q_n for some $\epsilon_n \to 0$ as $n \to 0$. Thus, as $n \to \infty$,

$$\text{Area}(\Omega \setminus \tilde{Q}_n) \to 0,$$

and

$$\text{Area}(\tilde{Q}_n \setminus \Omega) \to 0,$$

Moreover, there is a compact neighborhood Ω' of Ω which contains all $\tilde{Q}_n$. Thus,

$$\mathcal{E}_\Omega(\eta) - \mathcal{E}_{\tilde{Q}_n}(\eta) = \mathcal{E}_{\Omega\setminus\tilde{Q}_n}(\eta) - \mathcal{E}_{\tilde{Q}_n\setminus\Omega}(\eta) \to 0$$

By Gradient Approximation Lemma 4.19, we get $\mathcal{E}_{\tilde{Q}_n}(\eta) - \mathcal{E}_{Q_n}(\eta) \to 0$ as $n \to \infty$ and thus the lemma follows. ☐

Theorem 4.22. *Let Ω be a simply connected "square grid" domain, meaning that Ω is initially tiled by a square grid Q_0. Let Q_n be the n-th refinements of Q_0 as we describe it in section 2.2 and $g : \mathbb{C} \to \mathbb{R}$ be a given smooth function that will be used as boundary data. Then, the sequence of solutions discrete p-harmonic functions ϕ_n on Q_n with boundary values $g\big|_{\partial Q_n}$ converge uniformly to the unique p- harmonic function ϕ with boundary values $g\big|_{\partial\Omega}$. Moreover,*

$$\mathcal{E}_{p,Q_n}(\phi_n) \to \mathcal{E}_{p,\Omega}(\phi)$$

Proof.

Note that since the domain Ω is bounded, the grids Q_n are contained in some large ball B. By the Maximum principle, we know that $|\phi_n|$ are uniformly bounded by $\max_{z\in B}|g(z)| < \infty$.

We now show that the family of functions $\{\phi_n\}_{n\in\mathbb{N}}$ are equicontinuous, which is to say that there exists some positive function $\delta(\varepsilon)$ such that for every n we have that for every $z, w \in Q_n^{\bullet}$ we have that $|z - w| < \delta(\varepsilon)$ implies that $|\phi(z) - \phi(w)| < \varepsilon$.

Suppose we are in the case $M_n < |z - w|$, let $R = 3\,\mathrm{Diam}(B)|z - w|$ Then we have

$$\begin{aligned}|\phi(z) - \phi(w)| &\le C_p \mathcal{E}_{p,Q_R}^{\frac{1}{p}}(\phi)\left(\frac{R^{p-2}|z-w|^{p-2}}{|z-w|^{p-2} - R^{p-2}}\right)^{\frac{1}{p}} + \max_{z',w' \in \partial Q \cap B_R} |\phi(z') - \phi(w')| \\ &\le C_p E_{p,B}^{\frac{1}{p}}(\phi_n)\left(\frac{\mathrm{Diam}(B)^{p-2}|z-w|^{p-2}}{1 - 3\,\mathrm{Diam}(B)^{p-2}}\right)^{\frac{1}{p}} + 3\,\mathrm{Diam}(B)|z-w| \max_{z'\in B}|D^1 g(z')|\end{aligned}$$

By using equation (4.7), we know that the energy is decreasing and there is a uniform bound. So, we get a uniform bound for the energy and $|\phi_n(z) - \phi_n(w)| \to 0$ as $|z - w| \to 0$.

If we consider the case when $|z-w| < M_n$, then set $R = 3(M_n|z-w|)$. We have that

$$|\phi(z)-\phi(w)| \leq C_p E_{p,B}^{\frac{1}{p}}(\phi_n)\left(\frac{M_n^{2(p-2)}}{1-3M_n^{p-2}}\right)^{\frac{1}{p}} + 3M_n|z-w|\max_{z'\in B}|D^1 g(z')|$$

We can choose M_n small enough. This proves the eqcontinuty. Now, by Arzela-Ascoli, we know that there exists a subsequence of the ϕ_n converges uniformly to a ϕ continuous on the closure of Ω. By the assumption, the limit function is p-harmonic in Ω. Also, for any $z \in \partial\Omega$, there exists a sequence of points $z_n \in \partial Q_n \cap Q_n^{\bullet}$ such that $z_n \to z$ as $n \to \infty$, and thus $\phi = g$ on $\partial\Omega$. Since this limit is unique , the entire sequence ϕ_n converges uniformly to ϕ as desired. Moreover, there is a subsequence ϕ'_n that is 0 on E and 1 on F such that as

$$\mathcal{E}_{p,Q_n}(\phi_n) \leq \mathcal{E}_{p,Q_n}(\phi'_n) \to \mathcal{E}_{p,\Omega}(\phi)$$

Which implies that

$$\limsup \mathcal{E}_{p,Q_n}(\phi_n) \leq \mathcal{E}_{p,\Omega}(\phi)$$

Applying the theorem to the family of cuts and replace each cut with a walk from E' to F' implies that

$$\limsup \mathcal{E}_{p,Q_n}(v_n) \leq \mathcal{E}_{p,\Omega}(v)$$

Thus,

$$\liminf \mathcal{E}_{p,Q_n}(\phi_n) \geq \mathcal{E}_{p,\Omega}(\phi)$$

Thus, the final result follows. □

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