

ANALYSIS AND DESIGN OF A HYPERBOLIC COOLING TOWER

by

HSUE-BIN CHEN

B. S., Taiwan Provincial College of Marine and  
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## I. INTRODUCTION AND SCOPE

The hyperboloid of revolution can be generated by rotating a hyperbola about its directrix. Shells of this type are built throughout the world as cooling towers to lower the temperature of coolants used in electricity generating plants and chemical plants. This type of shell has proven to be efficient for use in reinforced concrete natural draught cooling towers for the conservation and reuse of the coolant (water).

The purpose of this report is to present the solutions for the stress resultants for the membrane and bending analysis and the corresponding displacements for cooling towers under dead load and wind load. Numerical results comparing solutions obtained by membrane theory and bending theory are presented.

## II. REVIEW OF LITERATURE

The first hyperbolic natural draught reinforced concrete cooling tower was designed by Prof. Van Iterson of the Dutch State Mines and installed at the Emma Colliery in 1916 (2). Towers of this type were installed at Lister Drive Power Station in Liverpool in 1925 and since then have become quite common and standard practice in Europe power stations where cooling towers are required. The typical size for old towers is shown Fig. 1. This type of tower has become a familiar sight in the United States with the first tower constructed in connection with a power station in Kentucky about fifteen years ago (1960). These structure sometimes reach over 350 feet in height and have base diameter often over 200 feet.

Immense quantities of water are required for the condensers of power stations, refineries, steel plants, etc. and sites with adequate cooling water are becoming rare; thus there is a need for the development of natural draught cooling towers for cooling and reusing large quantities of water.

The one-sheet hyperboloid is a convenient geometry for cooling towers with its straight-line generators for both structural and thermal reasons:

1. It has been proven (2) that the shear and vertical stresses are reduced by over 50% due to the "hyperbolic" shape of the shell compared with a cylinder of the same height and base diameter. Also this type stiffens the shell against wind force.
2. The momentum of the air entering the shell carries it into the center to form a vena contracta whose diameter depends on the

ratio of tower diameter to height of air inlet.

The other advantages of this hyperbolic concrete tower are (3):

1. Concrete towers are permanent.
2. There are no fans or similar equipment so there is lack of vibration due to resonance of fans and the tower. Therefore, the only power consumption is needed for pumping the water to the distribution pipes.
4. The natural draft towers minimize hazards such as fire, mist, and frozen spray.

Rish and Steel (2) discussed the treatment of the hyperboloid by assuming the shell to be made up of two truncated cones with a cylinder in between. Martin and Scriven (4) and Martin, Maddock, and Scriven (5) presented numerical solutions for dead load and wind load stresses and displacements in a particular shell. Gould and Lee (6,7,8,9) presented the membrane solutions and bending solutions for the stress resultants and displacements in hyperbolic cooling towers subjected to dead load, earthquake load, and wind load. The influence of the various shell parameters on the magnitude of the stress resultants and displacements is studied by these researchers and design tables are given to facilitate the design of such structures.



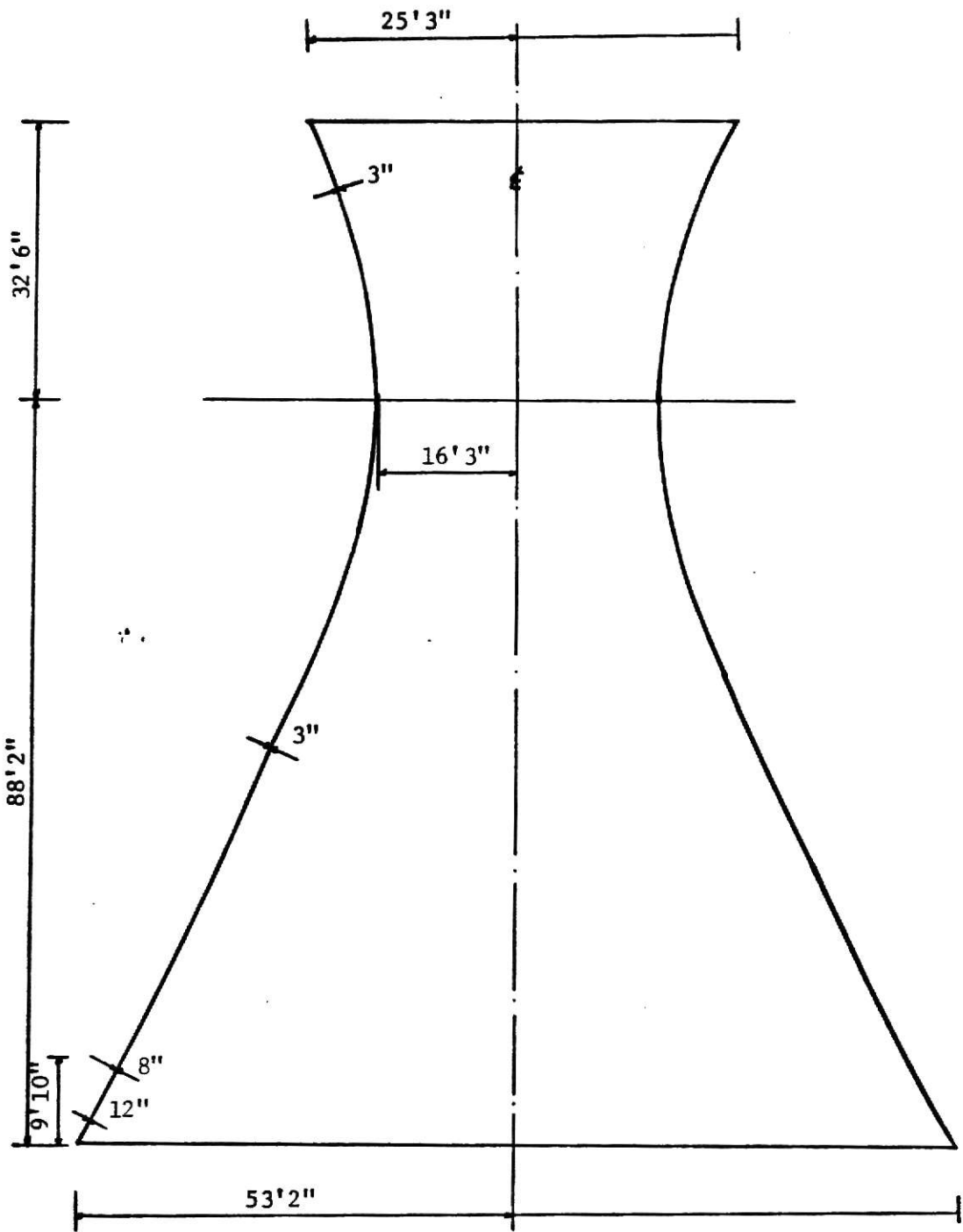


FIG. 1. Typical Old Natural Draught Cooling Tower

### III. DESIGN CONSIDERATIONS FOR HYPERBOLIC COOLING TOWERS (11)

#### 1. Size selection

Chilton (10) gave a formula to enable the size of cooling towers to be determined for a given cooling duty. This was

$$D = \frac{A_b (H)^{\frac{1}{2}}}{C (C)^{\frac{1}{2}}} \text{ --- (3.1)}$$

where

$A_b$  = the base area of the tower measured at pond sill;

$H$  = the height of the tower measured above sill level;

- $C$  is an efficiency factor known as the performance coefficient. In the past values of this have been in the region of 5.2 where water loadings were over 750 lb/hr/sq.ft, but new types of packing bring this down to give a  $C$  value of 5.0.

The Duty Coefficient  $D$  may be worked out from the formula:

$$\frac{W_L}{D} = 90.59 \frac{\Delta h}{\Delta T} (\Delta t + 0.3124 \Delta h)^{\frac{1}{2}} \text{ --- (3.2)}$$

where

$\Delta h$  = the change in total heat of the air passing through the tower;

$\Delta T$  = the change of temperature of the water passing through the tower;

$W_L$  = the water load in lb/hr.

$\Delta t$  = the change between the dry bulb air temperature and aspirated wet bulb air temperature.

## 2. General consideration for loading and analysis (11)

- a. The cooling tower shell should be considered to resist forces resulting from gravity loading, thermal gradient and icing, wind and earthquake, and foundation settlement. Also, temporary construction loading should be considered.
- b. When interior or exterior fill is supported on concrete, the effects of their loading should be considered in the design of the shell.
- c. Adequate stiffening of the top and the base of the shell should be provided.
- d. Cooling towers should be analyzed in accordance with recognized theories for thin elastic shell which for concrete are assumed to be uncracked, homogeneous and isotropic.
- e. The actual geometric profile, thickness variations and support conditions of the shell should be considered in the structural analysis.
- f. Equilibrium checks of internal forces and external loads should be performed regardless of the analysis method used.
- g. Results from model studies or full-scale tests may be used as a basis for the design and to check the validity of assumption involved in a mathematical analysis.

## 3. Method of analysis (11)

- a. An analysis which is based on a recognized bending theory for thin elastic shells is considered to be the most appropriate basis for the design of the tower and supporting structure. An analysis based on the membrane theory of thin shells may be satisfactory

for design provided that local bending in critical regions is accounted for by an appropriate method.

- b. Realistic boundary conditions should be considered in the analysis.
- c. Deformations which are computed from the elastic analysis should be checked to verify that they fall within the assumed limits of the applied theory.

#### 4. Stability (11)

- a. For wind load the critical shell buckling pressure may be estimated from test results. A wind buckling analysis should be made using the correct tower geometry and boundary conditions, and including the influence of dead weight. When made, the analysis should account for the influence of any anticipated shell hairline cracking.
- b. For dead load alone the critical shell buckling may be estimated by a simplified procedure which accounts for the dead load stresses in both the meridional and circumferential directions or by a dead load buckling analysis using the correct tower geometry and boundary conditions.
- c. Imperfections, which will reduce the buckling capacity are measured by deviation  $w_t$  in thickness over the arc length  $l$  where buckling capacity decreases as  $w_t/l$  increases. When imperfections larger than field tolerances occur, the engineer should make an estimate of the reduction in  $q_c$  to assure that adequate buckling capacity remains. The term  $q_c$  is the critical buckling pressure in psi along the windward meridian.

From an analysis of the wind tunnel test results reported (11), the following equation was obtained for estimating the critical shell buckling pressure.

$$q_c = C_c E (h/a)^{\alpha_c} \text{ - - - - - (3.3)}$$

in which

$C_c$  = an empirical coefficient taken to be 0.052;

$E$  = the modulus of elasticity of the concrete in psi;

$h$  = the shell thickness at the throat;

$a$  = the radius of the shell parallel circle at the throat ( the same unit as  $h$  );

$\alpha_c$  = an empirical coefficient taken to be 2.3.

The value  $q_c$  computed from this equation should be compared to the design wind pressure at the top of the tower to insure an adequate safety factor against buckling. It is intended that the design of the tower not be controlled by stability.

##### 5. Strength and serviceability requirements (11)

The cooling tower should be designed using the Strength Method according to the provisions of ACI 318-71 (16). Serviceability under working loads should be considered to insure that neither cracking nor deflections are excessive under the conditions of unfactored loading.

6. Reinforcement (11)

- a. The shell reinforcing in each direction should not be less than 0.35% of the cross sectional area of concrete.
- b. It is preferable to provide two layers of reinforcement in each direction.
- c. The maximum spacing of bars in each layers should not exceed twice the shell thickness, or no more than 18 inches.
- d. Reinforcement interrupted by openings should be replaced by not less than one and one half times the interrupted amount of reinforcement placed adjacent to the opening plus additional diagonal bars at corners of opening. For larger openings, the designer should take particular care to reinforce the opening to resist the design loads.

7. Splices in reinforcement (11)

- a. Splices in reinforcement should be designed according to the provisions of ACI 318-71 (16) except as provided in following.
- b. In case of a lap splice in tension, care should be taken to ensure the transfer of the design force without jeopardizing the integrity of the confining concrete.
- c. Splicing of reinforcing bars above a level not to exceed one-half the column spacing from the bottom should be distributed around the shell wall. No more than 1/3 of the vertical reinforcing should be spliced at one level.
- d. If splices in reinforcement are designed by a method not covered by ACI 318-71, the strength of the splices should be ensured.

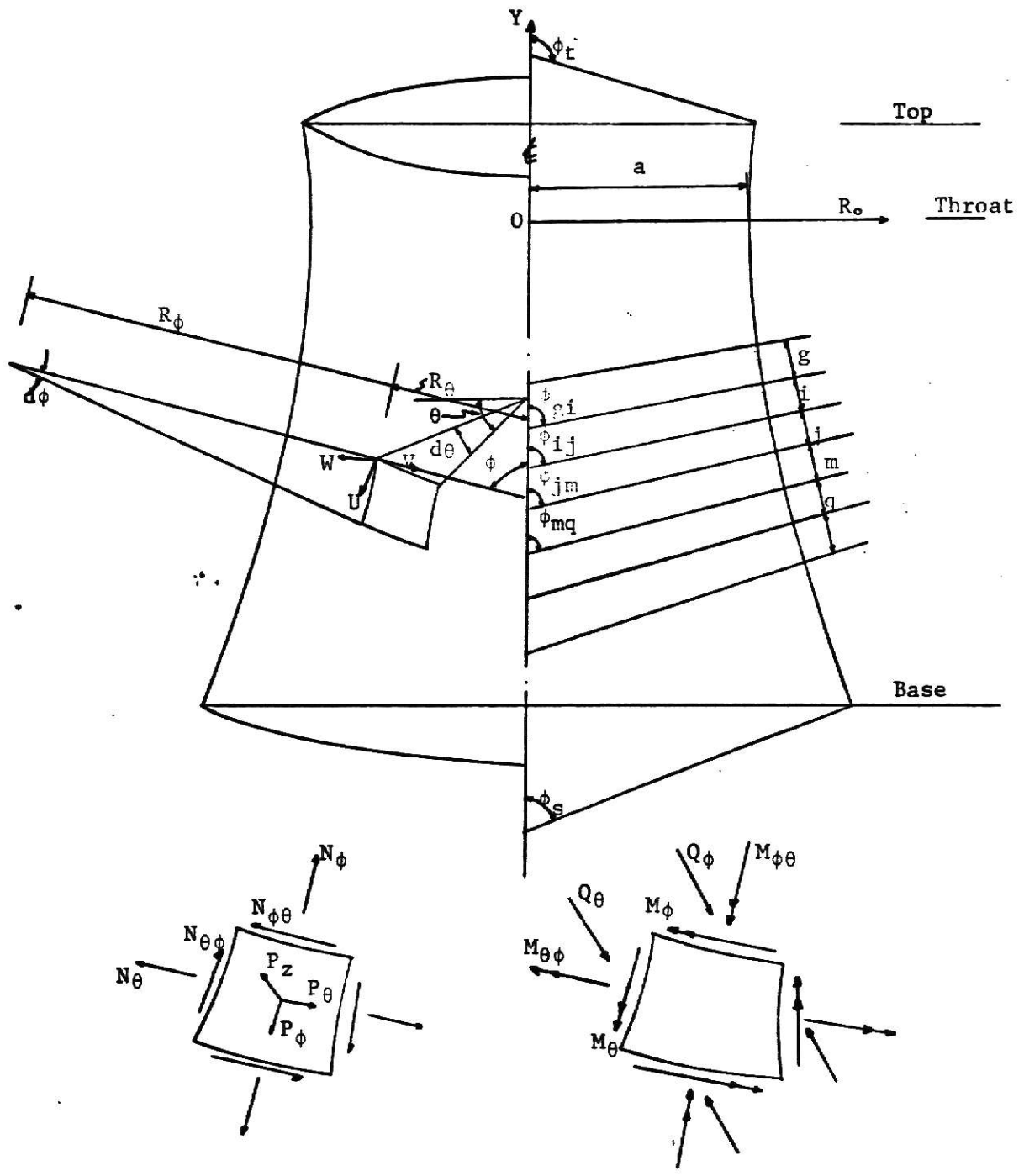


FIG. 2. Hyperboloid of Revolution

## IV. ANALYSIS OF HYPERBOLIC SHELL OF REVOLUTION

## 1. Surface geometry

The geometry of the shell surface (Fig. 2) is defined by

$$\frac{R_o^2}{a^2} - \frac{y^2}{b^2} = 1, \quad (4.1a)$$

in which  $R_o$  = the horizontal radius,  $y$  = the vertical coordinate,  
 $a$  = the throat radius and

$$b = \frac{aT}{(t^2 - a^2)^{1/2}} = \frac{bS}{(s^2 - a^2)^{1/2}}, \quad (4.1b)$$

in which  $s$  = the base radius,  $t$  = the top radius, and  $S$  and  $T$  = the  
 vertical distances from the throat to the base and the top of the shell  
 respectively.

The coordinate system is shown in Fig. 2 where the positive  
 directions of  $\phi$  and  $\theta$  as well as the load components per unit area of  
 middle surface  $P_\phi$ ,  $P_\theta$ ,  $P_z$  are indicated. The principal radii of  
 curvature  $R_\phi$  and  $R_\theta$  are given by

$$R_\phi = -a^2 b^2 \left( \frac{R_o^2}{a^4} + \frac{y^2}{b^4} \right)^{3/2}$$

$$= \frac{-a^2 b^2}{(a^2 \sin^2 \phi - b^2 \cos^2 \phi)^{3/2}}, \quad (4.1c)$$



$$R_{\theta} = a \left[ 1 + \frac{y^2}{a^2} (\bar{\alpha} + \bar{\alpha}^2) \right]^{\frac{1}{2}}$$

$$= \frac{a^2}{(a^2 \sin^2 \phi - b^2 \cos^2 \phi)^{\frac{1}{2}}} \quad (4.1d)$$

where  $\bar{\alpha} = \frac{b^2}{a^2}$ ,

and

$$R_{\phi} = \frac{a^2 \sin^2 \phi}{(a^2 \sin^2 \phi - b^2 \cos^2 \phi)^{\frac{1}{2}}} = R_{\theta} \sin \phi. \quad (4.1e)$$

## 2. Membrane theory

The differential equations of equilibrium of a shell of revolution based on membrane theory are well known (13) and given by

$$\frac{1}{R_{\phi}} \frac{\partial N_{\phi\theta}}{\partial \phi} + 2 \frac{\cot \phi}{R_{\theta}} N_{\phi\theta} + \frac{1}{R_{\theta} \sin \phi} \frac{\partial N_{\theta}}{\partial \theta} + P_{\theta} = 0, \quad (4.2a)$$

$$\frac{1}{R_{\phi}} \frac{\partial N_{\phi}}{\partial \phi} + \frac{\cot \phi}{R_{\theta}} (N_{\phi} - N_{\theta}) + \frac{1}{R_{\theta} \sin \phi} \frac{\partial N_{\phi\theta}}{\partial \theta} + P_{\phi} = 0, \quad (4.2b)$$

$$\frac{N_{\phi}}{R_{\phi}} + \frac{N_{\theta}}{R_{\theta}} - P_z = 0. \quad (4.2c)$$

The expressions relating stress resultants, strains and displacements are

$$\epsilon_{\phi} = \frac{1}{Eh} (N_{\phi} - \mu N_{\theta}), \quad (4.3a-1)$$

$$= \frac{1}{R_{\phi}} \left( \frac{\partial U}{\partial \phi} - W \right), \quad (4.3a-2)$$

$$\epsilon_{\theta} = \frac{1}{Eh} (N_{\theta} - \mu N_{\phi}), \quad (4.3b-1)$$

$$= \frac{1}{R_{\theta}} \left( \frac{1}{\sin \phi} \frac{\partial V}{\partial \theta} + U \cot \phi - W \right), \quad (4.3b-2)$$

$$\omega = \frac{2(1 + \mu)}{Eh} N_{\phi\theta}, \quad (4.3c-1)$$

$$= \frac{-1}{R_{\phi}} \frac{\partial U}{\partial \phi} + \frac{1}{R_{\theta}} U \cot \phi - \frac{1}{R_{\theta} \sin \phi} \frac{\partial V}{\partial \theta} = 0. \quad (4.3c-2)$$

#### Dead weight (12)

For shells of constant thickness, the components of the dead load are given by

$$P_{\theta} = 0, \quad P_{\phi} = g \sin \phi, \quad P_z = -g \cos \phi, \quad (4.4)$$

in which  $g$  is the dead weight per unit area of the surface. Due to symmetry of the loads,  $N_{\phi\theta} = N_{\theta\phi} = 0$ , and all terms involving derivatives with respect to  $\theta$  vanish. Upon inserting these and Eq. (4.4) into Eq. (4.2) the following equations of equilibrium will be obtained.

$$N_{\theta} R_{\phi} \cos \phi - \frac{d}{d\phi} (N_{\phi} R_{\theta}) = g \sin \phi R_{\theta} R_{\phi}, \quad (4.5a)$$

$$\frac{N_{\phi}}{R_{\phi}} + \frac{N_{\theta}}{R_{\theta}} = -g \cos \phi. \quad (4.5b)$$

Solving the differential equation (4.5a),

$$N_{\phi} = - \frac{W_d}{2\pi R_{\theta} \sin^2 \phi} \quad , \quad (4.6a)$$

in which  $W_d$  is the total vertical load above the level  $\phi$  found by integration as followed.

$$W_d = g \int 2\pi R_{\theta} R_{\phi} d\phi = 2\pi g a^4 b^2 \int_{\phi}^{\phi_t} \frac{\sin \phi d\phi}{(a^2 \sin^2 \phi - b^2 \cos^2 \phi)^2} \quad (4.6b)$$

Introducing the auxiliary variable  $\xi$ ,

$$\text{let } \cos \phi = \frac{a}{(a^2 + b^2)^{1/2}} \xi \quad (4.6c)$$

Therefore,

$$\begin{aligned} W_d &= 2\pi g a^4 b^2 \int_{\xi}^{\xi_t} \frac{a(a^2 + b^2)^{3/2}}{(a^4 - a^4 \xi^2 + a^2 b^2 - a^2 b^2 \xi^2)^2} d\xi \\ &= \frac{\pi g}{2} \frac{ab^2}{(a^2 + b^2)^{1/2}} \left[ \frac{2\xi}{1 - \xi^2} + \ln \frac{1 + \xi}{1 - \xi} \right]_{\xi}^{\xi_t} \quad (4.6d) \end{aligned}$$

$$\text{Let } f(\xi) = \frac{2\xi}{1 - \xi^2} + \ln \frac{1 + \xi}{1 - \xi} \quad (4.6e)$$

$$\text{Then } N_{\phi} = - \frac{\pi g}{2} \frac{ab^2}{(a^2 + b^2)^{1/2}} \frac{1}{2\pi R_{\theta} \sin^2 \phi} [ f(\xi) - f(\xi_t) ] \quad (4.6f)$$

Simplifying,

$$N_{\phi} = -\frac{g}{4} b^2 (a^2 + b^2)^{\frac{1}{2}} \frac{(1 - \xi^2)^{\frac{1}{2}}}{(a^2 + b^2 - a^2 \xi^2)} [f(\xi) - f(\xi_t)], \quad (4.6g)$$

and using Eq. (4.5b),

$$N_{\theta} = -\frac{ga^2}{(a^2 + b^2)^{\frac{1}{2}}} \frac{\xi}{(1 - \xi^2)^{\frac{1}{2}}} + N_{\phi} \frac{a^2}{b^2} (1 - \xi^2). \quad (4.6h)$$

The displacement components U, V and W are positive as shown in Fig. 2.

Eliminating W between Eq. (4.3a-2) and Eq. (4.3b-2) and dropping the term involving V because of symmetry one obtains

$$\frac{\partial U}{\partial \phi} - U \cot \phi - R_{\phi} \epsilon_{\phi} + R_{\theta} \epsilon_{\theta} = 0. \quad (4.7a)$$

Substituting Eqs. (4.3a-1) and (4.3b-1) into (4.7a) and from Eq. (4.2c)  $N_{\theta}$  can be written in terms of  $N_{\phi}$  and Eq. (4.7a) becomes

$$\begin{aligned} \frac{\partial U}{\partial \phi} - U \cot \phi - \frac{N_{\phi}}{Eh} (R_{\phi} + \mu R_{\theta} + \frac{R_{\theta}^2}{R_{\phi}} + \mu R_{\theta}) + \frac{1}{Eh} (R_{\theta}^2 + \mu R_{\phi} R_{\theta}) g \cos \phi \\ = 0. \end{aligned} \quad (4.7b)$$

By applying the boundary condition,  $U = 0$  at  $\phi = \phi_s$ , the differential equation (4.7b) can be solved and

$$U = \frac{\sin \phi}{Eh} \int_{\phi_s}^{\phi} \left[ \frac{(R_{\phi}^2 + R_{\theta}^2 + 2\mu R_{\phi} R_{\theta})}{R_{\phi} \sin \phi} N_{\phi} - g R_{\theta} (R_{\theta} + \mu R_{\phi}) \cot \phi \right] d\phi. \quad (4.7c)$$

$$\begin{aligned}
W &= U \cot \phi - R_{\theta} \varepsilon_{\theta} \\
&= U \cot \phi - \frac{R_{\theta}}{Eh} (N_{\theta} - \mu N_{\phi}) \\
&= U \cot \phi + \frac{R_{\theta} N_{\phi}}{R_{\phi} Eh} (R_{\theta} + \mu R_{\phi}) - \frac{R_{\theta}^2 g \cos \phi}{Eh}. \quad (4.7d)
\end{aligned}$$

### Wind load

The equilibrium equations of the membrane theory, in the case of a hyperboloid of revolution can be reduced to the form (7),

$$\frac{r_{\theta} \sin \phi}{r_{\phi}} \frac{\partial \psi}{\partial \phi} + \frac{\partial \eta}{\partial \theta} = (p_z \cos \phi - p_{\phi} \sin \phi) r_{\theta} \sin^2 \phi, \quad (4.8a)$$

$$\frac{\partial \eta}{\partial \phi} - \frac{r_{\theta}}{\sin \phi} \frac{\partial \psi}{\partial \theta} = - (p_{\theta} \sin \phi + \frac{\partial p_z}{\partial \theta}) r_{\phi} r_{\theta}^2 \sin \phi, \quad (4.8b)$$

$$\frac{n_{\phi}}{r_{\phi}} + \frac{n_{\theta}}{r_{\theta}} - p_z = 0, \quad (4.8c)$$

in which

$$\psi = n_{\phi} r_{\theta} \sin^2 \phi, \quad \eta = n_{\phi \theta} r_{\theta}^2 \sin \phi, \quad (4.9)$$

$$\text{and } r_{\phi} = \frac{R_{\phi}}{a}, \quad r_{\theta} = \frac{R_{\theta}}{a},$$

$$n_{\phi} = \frac{N_{\phi}}{Pa}, \quad n_{\theta} = \frac{N_{\theta}}{Pa}, \quad n_{\phi \theta} = \frac{N_{\phi \theta}}{Pa}, \quad (4.10)$$

$$p_{\phi} = \frac{P_{\phi}}{Pa}, \quad p_{\theta} = \frac{P_{\theta}}{Pa}, \quad p_z = \frac{P_z}{Pa}.$$

In the above,

$P$  = a constant reference load intensity per unit area of the middle surface.

The load components may be expressed in the following form with  $n > 1$ :

$$p_\phi = p_{\phi n}(\phi) \cos n\theta, \quad p_\theta = p_{\theta n}(\phi) \sin n\theta, \quad p_z = p_{zn}(\phi) \cos n\theta, \quad (4.11a)$$

$$\text{and } \psi = \psi'(\phi) \cos n\theta. \quad (4.11b)$$

Eliminating  $n$  between Eqs. (4.8a) and (4.8b) yields

$$\frac{d^2\psi'}{d\phi^2} + \chi_1(\phi) \frac{d\psi'}{d\phi} + \chi_2(\phi)\psi' = \chi_3(\phi) \quad (4.12)$$

in which

$$\chi_1(\phi) = \left(2 \frac{r_\phi}{r_\theta} - 1\right) \cot\phi - \frac{1}{r_\phi} \frac{dr_\phi}{d\phi}, \quad (4.13a)$$

$$\chi_2(\phi) = \frac{-n^2 r_\phi}{r_\theta \sin^2\phi}, \quad (4.13b)$$

and

$$\begin{aligned} \chi_3(\phi) = & \frac{r_\phi}{r_\theta^2 \sin^2\phi} \frac{d}{d\phi} [(p_{zn} \cos\phi - p_{\phi n} \sin\phi) r_\theta^3 \sin^2\phi] - r_\phi^2 n (n p_{zn} - \\ & - p_{\theta n} \sin\phi). \end{aligned} \quad (4.13c)$$

The solution to the homogeneous part of Eq. (4.12) is (7)

$$\psi'_{jh}(\phi) = N(\phi) [C_j \cos(\rho_j \beta_j) + D_j \sin(\rho_j \beta_j)] \quad (4.14)$$

in which  $i, j$  denote the segment designation (see Fig. 2).

Then,

$$N(\phi) = \left( \frac{-r\phi}{H' r_0^2 \sin\phi} \right)^{\frac{1}{2}}, \quad (4.15a)$$

$$H' = \sqrt{-G(\phi)}, \quad (4.15b)$$

$$G(\phi) = \frac{1}{2} \frac{dX_1}{d\phi} + \frac{1}{4} X_1^2 - X_2, \quad (4.15c)$$

$$\rho(\phi) = \sqrt{1 + \Delta(\phi)}, \quad (4.15d)$$

and

$$\Delta(\phi) = \frac{1}{2G} \left[ \frac{1}{2G} \frac{d^2G}{d\phi^2} - \frac{5}{8G^2} \left( \frac{dG}{d\phi} \right)^2 \right]. \quad (4.15e)$$

$\rho(\phi)$  will be regarded as a constant  $\rho_j$  for segment  $j$  of the shell corresponding to the  $n$ th harmonic. The value of  $\rho_j$  will be taken as the average value for the segment under consideration and

$$\beta_j = \int_{\phi_{1j}}^{\phi} H'(\phi) d\phi \quad (4.15f)$$

with  $C_j$  and  $D_j$  being the constants of integration.

The particular solution to Eq. (4.12) by means of the method of variation of parameter is given by (7)

$$\psi'_{jp} = N(\phi) [A_j(\phi) \cos(\rho_j \beta_j) + B_j(\phi) \sin(\rho_j \beta_j)], \quad (4.16)$$

in which

$$A_j(\phi) = \frac{-1}{\rho_j} \int_{\phi_{1j}}^{\phi} \frac{\chi_3(\phi) \sin(\rho_j \beta_j) d\phi}{H'(\phi) N(\phi)}, \quad (4.17a)$$

and

$$B_j(\phi) = \frac{1}{\rho_j} \int_{\phi_{1j}}^{\phi} \frac{\chi_3(\phi) \cos(\rho_j \beta_j) d\phi}{H'(\phi) N(\phi)}. \quad (4.17b)$$

The integrals in Eqs. (4.17a) and (4.17b) generally must be evaluated numerically due to the complexity of the integrand. Stress resultants for  $n > 1$  with  $\psi'_j$  are determined as the sum of Eqs. (4.14) and (4.16)

$$\psi'_j = \psi'_{jh} + \psi'_{jp}. \quad (4.18)$$

Substituting back into Eqs. (4.9) and (4.11b) yields

$$n_{\phi n} = \frac{\psi'(\phi) \cos n\theta}{r_\theta \sin^2 \phi}. \quad (4.19)$$

Integrating Eq. (4.8a) in view of Eq. (4.9) yields  $n_{\theta\phi n}$ ; and considering Eq. (4.8c) gives the stress resultant  $n_{\theta n}$ . The expressions for the stress resultants in segment  $j$  for the  $n$ th harmonic are

$$n_{\phi n} = n'_{\phi n}(\phi) \cos n\theta, \quad (4.20a)$$

$$n_{\theta\phi n} = n'_{\theta\phi n}(\phi) \sin n\theta, \quad (4.20b)$$

and

$$n_{\theta n} = n'_{\theta n}(\phi) \cos n\theta. \quad (4.20c)$$



In these equations

$$n'_{\phi n}(\phi) = \frac{N}{r_{\theta} \sin^2 \phi} \{ [(C_j + A_j(\phi)) \cos(\rho_j \beta_j) + [D_j + B_j(\phi)] \sin(\rho_j \beta_j)] \}, \quad (4.21a)$$

$$n'_{\theta \phi n}(\phi) = \frac{1}{nr_{\theta}^2 \sin^2 \phi} [(p_{zn} \cos \phi - p_{\phi n} \sin \phi) r_{\theta}^3 \sin^2 \phi - \frac{r_{\theta}^2 \sin \phi}{r_{\phi}} \frac{d\psi'_j}{d\phi}], \quad (4.21b)$$

and

$$n'_{\theta n} = r_{\theta} (p_{zn} - \frac{n'_{\phi n}}{r_{\phi}}). \quad (4.21c)$$

The integration constants  $C_j$ ,  $D_j$  are determined from the boundary conditions, applying the boundary conditions  $n_{\phi} = n_{\theta \phi} = 0$  at  $\phi = \phi_t$  for the uppermost segment. The calculation of stress resultants proceeds down the shell with decreasing  $\phi$ , and the constants  $C_j$ ,  $D_j$  for any segment are obtained by equating the values of  $n'_{\phi n}$  and  $n'_{\theta \phi n}$  at the upper edge of segment  $j$  with the values of the corresponding stress resultants at the lower edge of segment  $i$  as shown in Fig. 2.

From Eqs. (4.21b) and (4.21c) and assuming  $p_{\phi n}$  and  $p_{zn}$  are constant with respect to  $\phi$  one obtains

$$C_j = [ \frac{A_i(\phi_{ij})}{\rho_i} + C_i ] \cos[\rho_i \beta_i(\phi_{ij})] + [ \frac{B_i(\phi_{ij})}{\rho_i} + D_i ] \sin[\rho_i \beta_i(\phi_{ij})], \quad (4.22a)$$

and

$$D_j = \frac{1}{Y''_j(\phi_{ij})} \{ [ \frac{A_i(\phi_{ij})}{\rho_i} + C_i ] Y'_i(\phi_{ij}) + [ \frac{B_i(\phi_{ij})}{\rho_i} + D_i ] + Y''_i(\phi_{ij}) - C_j Y'_j(\phi_{ij}) \}. \quad (4.22b)$$

in which

$$Y'_j(\phi) = \frac{dN}{d\phi} \cos(\rho_j \beta_j) - \rho_j NH' \sin(\rho_j \beta_j), \quad (4.22c)$$

$$Y''_j(\phi) = \frac{dN}{d\phi} \sin(\rho_j \beta_j) + \rho_j NH' \cos(\rho_j \beta_j). \quad (4.22d)$$

The constants for the uppermost segment t simplify to

$$C_t = 0, \quad (4.22e)$$

$$D_t = \frac{1}{\rho_t} \left( \frac{r_{\theta} r_{\phi}}{H' N} \right)_{\phi=\phi_t} \sin \phi_t (p_{zn} \cos \phi_t - p_{\phi n} \sin \phi_t). \quad (4.22f)$$

Take the load component in the form

$$p_{\phi n} = 0, \quad p_{\theta n} = 0, \quad p_{zn} = -\alpha_n, \quad (4.23a)$$

where  $\alpha_n$  is the Fourier coefficient for a particular pressure function.

$$\text{Then, } p_z = \sum_{n=0}^{\infty} p_{zn} \cos n\theta. \quad (4.23b)$$

The wind pressure distribution on a hyperbolic cooling tower is obtained by R. F. Rish and T. F. Steel from wind tests (2). The recommended formulas are summarized as follows.

$$P_z = - 1.524 \cos(1.89\theta), \quad 0^\circ < \theta < 47.6^\circ, \quad (4.23c)$$

$$P_z = + 0.69 \sin[3.61(\theta - 47.6)], \quad 47.6^\circ < \theta < 100^\circ, \quad (4.23d)$$

$$P_z = - 0.21, \quad 100^\circ < \theta < 180^\circ. \quad (4.23e)$$

The wind pressure is assumed to be constant throughout the height of the shell.

The solutions for  $n = 0$  and  $n = 1$  are presented for a wind pressure distribution of the form.

$$P_{\phi n} = 0, \quad P_{\theta n} = 0, \quad P_{zn} = - 1. \quad (23.f)$$

For  $n = 0$  the solutions is similar to that of dead load, and for  $n = 1$  to that of seismic load (6).

$$n_\phi = \alpha_0 n_{\phi 0} + \sum_{n=1}^8 \alpha_n n'_{\phi n} \cos n\theta, \quad (4.24a)$$

$$n_\theta = \alpha_0 n_{\theta 0} + \sum_{n=1}^8 \alpha_n n'_{\theta n} \cos n\theta, \quad (4.24b)$$

$$n_{\theta\phi} = \sum_{n=1}^8 \alpha_n n'_{\theta\phi n} \sin n. \quad (4.24c)$$

The stress resultant - displacement relationship can be given by  
(1, 7)

$$\frac{\partial \zeta}{\partial \phi} - \frac{r_{\theta}}{\sin \phi} \frac{\partial \gamma}{\partial \theta} = \frac{1}{\sin \phi} \left[ \frac{r_{\phi}^2 + r_{\theta}^2 + 2\mu r_{\phi} r_{\theta}}{r_{\phi}} n_{\phi} - r_{\theta} (r_{\theta} + \mu r_{\phi}) p_z \right], \quad (4.25a)$$

$$\frac{r_{\theta}^2}{r_{\phi}} \sin \phi \frac{\partial \gamma}{\partial \phi} + \frac{\partial \zeta}{\partial \theta} = 2(1 + \mu) r_{\theta} n_{\theta \phi}, \quad (4.25b)$$

$$\frac{1}{\sin \phi} \frac{\partial v}{\partial \theta} + u \cot \phi + w = r_{\theta} (n_{\theta} - \mu n_{\phi}), \quad (4.25c)$$

in which

$$\zeta = \frac{u}{\sin \phi}, \quad \gamma = \frac{v}{r_{\theta} \sin \phi}, \quad (4.26a)$$

$$u = \frac{UEh}{Pa^2}, \quad v = \frac{VEh}{Pa^2}, \quad w = \frac{WEh}{Pa^2}. \quad (4.26b)$$

In these

$E$  = Young's modulus,

$h$  = shell thickness,

and

$\mu$  = Poisson's ratio.

Taking the variable  $\gamma$  in the form

$$\gamma = \gamma'(\phi) \sin n\theta \quad (4.27)$$

and eliminating  $\zeta$  between Eqs. (4.25a) and (4.25b) leads to

$$\frac{d^2 \gamma'}{d\phi^2} + \chi_1(\phi) \frac{d\gamma'}{d\phi} + \chi_2(\phi) \gamma' = \chi_4(\phi), \quad (4.28a)$$

where

$$\begin{aligned} \chi_4(\phi) = & \frac{r_\phi}{r_\theta^2 \sin\phi} \left[ 2(1 + \mu) \frac{d}{d\phi} (r_\theta n'_{\theta\phi n} + \frac{n(r_\phi^2 + r_\theta^2 + 2\mu r_\phi r_\theta)}{r_\phi \sin\phi} n'_{\phi n} \right. \\ & \left. - n p_{zn} \frac{r_\theta(r_\theta + \mu r_\phi)}{\sin\phi} \right]. \end{aligned} \quad (4.28b)$$

Displacements for  $n = 0$  and  $n = 1$  are similar to that under dead load and seismic load respectively (6). To obtain the displacements for  $n > 1$ , the same method of variation of parameter on stress analysis is applied yielding the solution (7).

$$\gamma'_j = \gamma'_{jh} + \gamma'_{jp}. \quad (4.29)$$

In this,

$$\gamma'_{jh} = N [C'_j \cos(\rho_j \beta'_j) + D'_j \sin(\rho_j \beta'_j)], \quad (4.30a)$$

$$\gamma'_{jp} = N [A'_j(\phi) \cos(\rho_j \beta'_j) + B'_j(\phi) \sin(\rho_j \beta'_j)], \quad (4.30b)$$

$$\beta'_j = \int_{\phi_{jm}}^{\phi} H'(\phi) d\phi, \quad (4.30c)$$

$$A'_j(\phi) = \frac{1}{\rho_j} \int_{\phi_{jm}}^{\phi} \frac{-\chi_4(\phi) \sin(\rho_j \beta'_j)}{H'(\phi) N(\phi)} d\phi, \quad (4.30d)$$

$$B'_j(\phi) = \frac{1}{\rho_j} \int_{\phi_{jm}}^{\phi} \frac{\chi_4(\phi) \cos(\rho_j \beta'_j)}{H'(\phi) N(\phi)} d\phi, \quad (4.30e)$$

and  $C'_j$  and  $D'_j$  denote the constants of integration. The lower limit of integration for  $\beta'_j$ ,  $A'_j$  and  $B'_j$  is taken as the angle  $\phi_{jm}$  corresponding the boundary between segments  $j$  and  $m$  as shown in Fig. 2. By combining Eqs. (4.25) and (4.29), the displacements for segment  $j$  and the  $n$ th harmonic are given by

$$u_n = u'_n(\phi) \cos n\theta, \quad v_n = v'_n(\phi) \sin n\theta, \quad w_n = w'_n(\phi) \cos n\theta \quad (4.31)$$

in which,

$$u'_n(\phi) = \frac{-\sin\phi}{n} \left[ 2(1 + \mu) r_\theta n'_{\theta\phi n} - \frac{r_\theta^2 \sin\phi}{r_\phi} \frac{d\gamma'_j}{d\phi} \right], \quad (4.32a)$$

$$v'_n(\phi) = N(\phi) r_\theta \sin\phi \{ [C'_j + A'_j(\phi)] \cos(\rho_j \beta'_j) + [D'_j + B'_j(\phi)] \sin(\rho_j \beta'_j) \}, \quad (4.32b)$$

and

$$w'_n(\phi) = r_\theta (n'_{\theta n} - \mu n'_{\phi n}) - n v'_n \csc\phi - u'_n \cot\phi. \quad (4.32c)$$

The boundary conditions for displacements are  $u = v = 0$  at  $\phi = \phi_s$  in which  $\phi_s$  defines the base of the shell.

The integration constants  $C'_j$ ,  $D'_j$  are determined from the boundary conditions for the base of the shell. Proceed upward with increasing

$\phi$  and obtain by equating the values of  $u'_n$  and  $v'_n$  at the lower edge of segment  $j$  with the values of the corresponding displacement components at the upper edge of the segment  $m$ . Thus from Eqs. (4.30a) and (4.30b),

$$C'_j = \left[ \frac{A'_m(\phi_{jm})}{\rho_m} + C'_m \right] \cos[\rho_m \beta'_m(\phi_{jm})] + \left[ \frac{B'_m(\phi_{jm})}{\rho_m} + D'_m \right] \sin[\rho_m \beta'_m(\phi_{jm})], \quad (4.33a)$$

and

$$D'_j = \frac{1}{Z''_j(\phi_{jm})} \left\{ \left[ \frac{A'_m(\phi_{jm})}{\rho_m} + C'_m \right] Z'_m(\phi_{jm}) + \left[ \frac{B'_m(\phi_{jm})}{\rho_m} + D'_m \right] Z''_m(\phi_{jm}) - C'_j Z'_j(\phi_{jm}) \right\}, \quad (4.33b)$$

in which

$$Z'_j(\phi) = \frac{dN}{d\phi} \cos(\rho_j \beta'_j) - \rho_j N H' \sin(\rho_j \beta'_j), \quad (4.33c)$$

and

$$Z''_j(\phi) = \frac{dN}{d\phi} \sin(\rho_j \beta'_j) + \rho_j N H' \cos(\rho_j \beta'_j). \quad (4.33d)$$

The constants for the lowest segment  $s$  simplify to

$$C'_s = 0, \quad (4.33e)$$

and

$$D'_s = \frac{2(1 + \nu)}{\rho_s} \left( \frac{r_\phi n'_\theta \phi n}{r_\theta H' N} \right)_{\phi=\phi_s} \csc \phi_s. \quad (4.33f)$$

The expression for wind load displacements for the loading given by Eqs. (4.23) are given by (7)

$$u = \alpha_o u_o + \sum_{n=1}^{\infty} \alpha_n u'_n \cos n\theta, \quad (4.34a)$$

$$v = \sum_{n=1}^{\infty} \alpha_n v'_n \sin n\theta, \quad (4.34b)$$

and

$$w = \alpha_o w_o + \sum_{n=1}^{\infty} \alpha_n w'_n \cos n\theta. \quad (4.34c)$$

### 3. Bending theory

The equations of equilibrium for the bending of a shell of revolution are given by (8)

$$\frac{1}{r_\phi r_\theta \sin\phi} \left[ \frac{\partial(r_\theta \sin\phi n_\phi)}{\partial\phi} + r_\phi \frac{\partial n_{\theta\phi}}{\partial\phi} - \frac{\partial(r_\theta \sin\phi)}{\partial\phi} n_\theta \right] + \frac{q_\phi}{r_\phi} + p_\phi = 0, \quad (4.35a)$$

$$\frac{1}{r_\phi r_\theta \sin\phi} \left[ \frac{\partial(r_\theta \sin\phi n_{\theta\phi})}{\partial\phi} + r_\phi \frac{\partial n_\theta}{\partial\theta} + \frac{\partial(r_\theta \sin\phi)}{\partial\phi} n_{\theta\phi} \right] + \frac{q_\theta}{r_\theta} + p_\theta = 0, \quad (4.35b)$$

$$\frac{1}{r_\phi r_\theta \sin\phi} \left[ \frac{\partial(r_\theta \sin\phi q_\phi)}{\partial\phi} + \frac{\partial(r_\phi q_\theta)}{\partial\theta} \right] - \frac{n_\phi}{r_\phi} - \frac{n_\theta}{r_\theta} + p_z = 0, \quad (4.35c)$$

$$\frac{1}{r_\phi r_\theta \sin\phi} \left[ \frac{\partial(r_\theta \sin\phi m_{\phi\theta})}{\partial\phi} + r_\phi \frac{\partial m_{\phi\theta}}{\partial\theta} - \frac{\partial(r_\theta \sin\phi)}{\partial\phi} m_\theta \right] - q_\phi = 0, \quad (4.35d)$$



$$\frac{1}{r_\phi r_\theta \sin\phi} \left[ \frac{\partial(r_\theta \sin\phi m_{\phi\theta})}{\partial\phi} + \frac{\partial(r_\phi m_\theta)}{\partial\theta} + \frac{\partial(r_\theta \sin\phi)}{\partial\phi} m_{\theta\phi} \right] - q_\theta = 0, \quad (4.35e)$$

$$n_{\phi\theta} - n_{\theta\phi} + \frac{m_{\phi\theta}}{r_\phi} - \frac{m_{\theta\phi}}{r_\theta} = 0 \quad (4.35f)$$

in which

$$m_\phi = \frac{M_\phi}{Pa^2}, \quad m_\theta = \frac{M_\theta}{Pa^2}, \quad m_{\phi\theta} = \frac{M_{\phi\theta}}{Pa^2}, \quad m_{\theta\phi} = \frac{M_{\theta\phi}}{Pa^2}, \quad (4.36)$$

$$q_\phi = \frac{Q_\phi}{Pa}, \quad \text{and} \quad q_\theta = \frac{Q_\theta}{Pa}.$$

Equation(4.35f) is identically satisfied for  $n_{\phi\theta} = n_{\theta\phi}$ ,  $m_{\phi\theta} = m_{\theta\phi}$ .

Using the complex formulation (8),

$$\bar{N}_\phi = n_\phi - \frac{i}{\nu} \frac{m_\theta - \mu m_\phi}{1 - \mu^2}, \quad (4.37a)$$

$$\bar{N}_\theta = n_\theta - \frac{i}{\nu} \frac{m_\phi - \mu m_\theta}{1 - \mu^2}, \quad (4.37b)$$

$$\bar{N}_{\theta\phi} = n_{\theta\phi} + \frac{i}{\nu} \frac{m_{\theta\phi}}{1 - \mu}, \quad (4.37c)$$

in which

$$\nu = \frac{h/a}{\sqrt{12(1 - \mu^2)}}, \quad (4.37d)$$

and combining Eqs. (4.35a)through (4.35e) and dropping higher order terms leads to

$$\frac{1}{r_\phi} \frac{\partial \widetilde{N}_\phi}{\partial \phi} + \frac{\cot \phi}{r_\theta} (\widetilde{N}_\phi - \widetilde{N}_\theta) + \frac{1}{r_\theta \sin \phi} \frac{\partial \widetilde{N}_{\theta\phi}}{\partial \theta} + \frac{i\nu}{r_\phi^2} \frac{\partial \widetilde{N}}{\partial \phi} = -p_\phi, \quad (4.38a)$$

$$\frac{1}{r_\phi} \frac{\partial \widetilde{N}_{\theta\phi}}{\partial \phi} + 2 \frac{\cot \phi}{r_\theta} \widetilde{N}_{\theta\phi} + \frac{1}{r_\theta \sin \phi} \frac{\partial \widetilde{N}_\theta}{\partial \theta} + \frac{i\nu}{r_\theta^2 \sin \phi} \frac{\partial \widetilde{N}}{\partial \phi} = -p_\theta, \quad (4.38b)$$

$$\frac{\widetilde{N}_\phi}{r_\phi} + \frac{\widetilde{N}_\theta}{r_\theta} - i\nu G_1(\widetilde{N}) = p_z. \quad (4.38c)$$

In the above

$$\widetilde{N} = \widetilde{N}_\phi + \widetilde{N}_\theta, \quad (4.38d)$$

and

$$G_1(\ ) = \frac{1}{r_\phi r_\theta \sin \phi} \left\{ \frac{\partial}{\partial \phi} \left[ \frac{r_\theta \sin \phi}{r_\phi} \frac{\partial}{\partial \theta} (\ ) \right] + \frac{\partial}{\partial \theta} \left[ \frac{r_\phi}{r_\theta \sin \phi} \right. \right. \\ \left. \left. \times \frac{\partial}{\partial \theta} (\ ) \right] \right\}. \quad (4.38e)$$

Consider a harmonic  $n$  and introduce

$$\begin{aligned} n_{\phi n} &= n'_{\phi n}(\phi) \cos n\theta, & n_{\theta n} &= n'_{\theta n}(\phi) \cos n\theta, & n_{\theta\phi n} &= n'_{\theta\phi n}(\phi) \sin n\theta, \\ m_{\phi n} &= m'_{\phi n}(\phi) \cos n\theta, & m_{\theta n} &= m'_{\theta n}(\phi) \cos n\theta, & m_{\theta\phi n} &= m'_{\theta\phi n}(\phi) \sin n\theta, \\ q_{\phi n} &= q'_{\phi n}(\phi) \cos n\theta, & q_{\theta n} &= q'_{\theta n}(\phi) \sin n\theta, & & \\ \widetilde{N}_\phi &= \widetilde{N}'_{\phi n}(\phi) \cos n\theta, & \widetilde{N}_\theta &= \widetilde{N}'_{\theta n}(\phi) \cos n\theta, & \widetilde{N}_{\theta\phi} &= \widetilde{N}'_{\theta\phi n}(\phi) \sin n\theta, \\ \widetilde{N} &= \widetilde{N}'_n(\phi) \cos n\theta. \end{aligned} \quad (4.39)$$

Also introduce the auxiliary variables

$$\psi = \tilde{\psi}_n(\phi) \cos n\theta, \quad (4.40a)$$

$$\tilde{\eta} = \tilde{\eta}_n(\phi) \sin n\theta, \quad (4.40b)$$

in which

$$\tilde{\psi}_n(\phi) = \tilde{N}_{\phi n} r_\theta \sin^2 \phi + i\nu \frac{r_\theta}{r_\phi} \sin \phi \cos \phi \frac{d\tilde{N}_n}{d\phi} \quad (4.40c)$$

$$\tilde{\eta}_n(\phi) = \tilde{N}_{\theta \phi n} r_\theta^2 \sin^2 \phi. \quad (4.40d)$$

In view of Eqs. (4.39) and (4.40), Eqs. (4.38a) through (4.38c) can be expressed in terms of the three dependent variables  $\tilde{\psi}_n$ ,  $\tilde{\eta}_n$  and  $\tilde{N}_n$ . Eliminating  $\tilde{\eta}_n$  leads to

$$G_{2n}(\tilde{\psi}_n) + n^2 \left(1 + \frac{i\nu k^2}{r_\theta}\right) \tilde{N}_n = \frac{1}{r_\phi r_\theta \sin \phi} \left\{ \frac{d}{d\phi} [(p'_{zn} \cos \phi - p'_{\phi n} \sin \phi) r_\theta^3 \sin^2 \phi] + np'_{\theta n} r_\phi r_\theta \sin^2 \phi \right\} \quad (4.41a)$$

and

$$G_{2n}(\tilde{N}_n) + \frac{i}{\nu} \tilde{N}_n + \frac{i}{\nu} \left( \frac{1}{r_\phi} - \frac{1}{r_\theta} \right) \frac{1}{\sin^2 \phi} \tilde{\psi}_n = \frac{i}{\nu} p'_{zn} r_\theta, \quad (4.41b)$$

in which

$$G_{2n}(\ ) = \frac{1}{r_\phi r_\theta \sin \phi} \frac{d}{d\phi} \left[ \frac{r_\theta^2 \sin \phi}{r_\phi} \frac{d(\ )}{d\phi} \right] - \frac{n^2}{r_\theta \sin^2 \phi} (\ ). \quad (4.41c)$$

Refer to the solution of the governing equation given in reference (8). The governing Eq. (4.41b) can be shown to be in the following form

$$\frac{d^2 \bar{N}_n}{d\phi^2} + \left[ \left( 2 \frac{r_\phi}{r_\theta} - 1 \right) \cot\phi - \frac{1}{r_\phi} \frac{dr_\phi}{d\phi} \right] \frac{d\bar{N}_n}{d\phi} + \left[ \frac{n^2}{\sin^2\phi} \frac{r_\phi}{r_\theta} \left( 1 - 2 \frac{r_\phi}{r_\theta} + \frac{ir_\phi^2}{r_\theta v} \right) \right] \bar{N}_n = \frac{ir_\phi^2}{r_\theta v} (n_{\phi n}^* + n_{\theta n}^*), \quad (4.42)$$

in which  $n_{\phi n}^*$  and  $n_{\theta n}^*$  are used to indicate a stress resultant derived from the membrane theory. An approximate particular integral to Eq. (4.42) is obtained by equating the coefficient of  $ir_\phi^2/r_\theta v$ , which gives

$$\bar{N}_{np} = n_{\phi n}^* + n_{\theta n}^*. \quad (4.43)$$

- Using the same technique as that of solving the homogeneous part of differential equation of membrane theory under wind load (8),

$$\bar{N}_{nh} = \frac{1}{r_\theta^{3/4} \sin^{1/2}\phi} [\bar{A} e^{-(1-i)\beta} + \bar{A}' e^{(1-i)\beta}], \quad (4.44)$$

in which

$$\beta = \int_{\phi_t}^{\phi} \frac{-r_\phi}{\sqrt{2r_\theta v}} d\phi \quad (4.45)$$

and  $\bar{A}$  and  $\bar{A}'$  are constants of integration. The complex solution is obtained from Eqs. (4.43) and (4.44) as

$$\bar{N}_n = \bar{N}_{nh} + \bar{N}_{np}. \quad (4.46)$$

Separating the complex resultants into real and imaginary parts and dropping higher order terms, yields the following explicit expressions for the stress resultants (8).

$$\begin{aligned}
 n'_{\phi n} = & \delta_4 \cot \phi \sqrt{\frac{\nu}{2r_\theta}} \{A_1 e^{-\alpha} [\cos \alpha - (\delta_2 - \delta_5) \sin \alpha] - A_2 e^{-\alpha} [\sin \alpha + (\delta_2 - \delta_5) \\
 & \cos \alpha] - A_3 e^\beta [\cos \beta + (\delta_1 + \delta_5) \sin \beta] + A_4 e^\beta [-\sin \beta + (\delta_1 + \delta_5) \cos \beta]\} \\
 & + n^*_{\phi n}, \tag{4.47a}
 \end{aligned}$$

$$\begin{aligned}
 n'_{\theta n} = & \delta_4 \{A_1 e^{-\alpha} [(1 - \delta_3) \cos \alpha + \delta_3 \sin \alpha] - A_2 e^{-\alpha} [(1 - \delta_3) \sin \alpha + \delta_3 \cos \alpha] \\
 & + A_3 e^\beta [(1 + \delta_3) \cos \beta + \delta_3 \sin \beta] + A_4 e^\beta [(1 + \delta_3) \sin \beta - \delta_3 \cos \beta]\} + n^*_{\theta n}, \\
 & \tag{4.47b}
 \end{aligned}$$

$$\begin{aligned}
 n'_{\theta \phi n} = & n \delta_4 \csc \phi \sqrt{\frac{\nu}{2r_\theta}} \{A_1 e^{-\alpha} [\cos \alpha - (\delta_2 - \delta_6) \sin \alpha] - A_2 e^{-\alpha} [\sin \alpha + (\delta_2 \\
 & - \delta_6) \cos \alpha] - A_3 e^\beta [\cos \beta + (\delta_1 + \delta_6) \sin \beta] + A_4 e^\beta [-\sin \beta + (\delta_1 + \delta_6) \\
 & \times \cos \beta]\} + n^*_{\theta \phi n}, \tag{4.47c}
 \end{aligned}$$

$$\begin{aligned}
 m'_{\phi n} = & -\nu \delta_4 \{A_1 e^{-\alpha} \{[1 - (1 - \mu) \delta_3] \sin \alpha - (1 - \mu) \delta_2 \delta_3 \cos \alpha\} + A_2 e^{-\alpha} [1 \\
 & - (1 - \mu) \delta_3] \cos \alpha + (1 - \mu) \delta_2 \delta_3 \sin \alpha\} - A_3 e^\beta [1 + (1 - \mu) \delta_3] \sin \beta
 \end{aligned}$$

$$\begin{aligned}
& + (1 - \mu)\delta_1\delta_3\cos\beta\} + A_4 e^\beta\{[1 + (1 - \mu)\delta_3]\cos\beta + (1 - \mu)\delta_1\delta_3\sin\beta\} \\
& + \delta_8 \frac{d}{d\phi} (n_{\phi n}^* + n_{\theta n}^*) - \delta_9 (n_{\phi n}^* + n_{\theta n}^*), \tag{4.47d}
\end{aligned}$$

$$\begin{aligned}
m'_{\theta n} & = -\nu\delta_4[A_1 e^{-\alpha}\{[\mu + \delta_3(1 - \mu)]\sin\alpha + [\delta_2\delta_3(1 - \mu) - \delta_7]\sin\alpha\} \\
& + A_2 e^{-\alpha}\{[\mu + \delta_3(1 - \mu)]\cos\alpha - [\delta_2\delta_3(1 - \mu) - \delta_7]\sin\alpha\} + A_3 e^\beta\{-[\mu \\
& - \delta_3(1 - \mu)]\sin\beta - [\delta_1\delta_3(1 - \mu) + \delta_7]\cos\beta\} + A_4 e^\beta\{[\mu - \delta_3(1 \\
& - \mu)]\cos\beta + [\delta_1\delta_3(1 - \mu) + \delta_7]\sin\beta\}] - \delta_8 \frac{d}{d\phi} (n_{\phi n}^* + n_{\theta n}^*) + \delta_9 (n_{\phi n}^* \\
& + n_{\theta n}^*), \tag{4.47e}
\end{aligned}$$

$$\begin{aligned}
m'_{\theta\phi n} & = \frac{\nu\delta_4(1 - \mu)}{\sin\phi} \sqrt{\frac{\nu}{2r_\theta}} \{-A_1 e^{-\alpha}[(1 + \delta_6)\sin\alpha + \delta_2\cos\alpha] + A_2 e^{-\alpha}[-(1 \\
& + \delta_6)\cos\alpha + \delta_2\sin\alpha] + A_3 e^\beta[-(1 - \delta_6)\sin\beta + \delta_1\cos\beta] + A_4 e^\beta \times \\
& [(1 - \delta_6)\cos\beta + \delta_1\sin\beta]\} + \nu \frac{r_\theta}{r_\phi} \delta_7 \sin\phi \frac{d}{d\phi} (n_{\phi n}^* + n_{\theta n}^*) - \nu\delta_7 \cos\phi \\
& \tag{4.47f}
\end{aligned}$$

$$\begin{aligned}
q'_{\phi n} & = \delta_4 \sqrt{\frac{\nu}{2r_\theta}} [A_1 e^{-\alpha}(\cos\alpha - \delta_2\sin\alpha) - A_2 e^{-\alpha}(\sin\alpha + \delta_2\cos\alpha) - A_3 e^\beta \\
& \times (\cos\beta + \delta_1\sin\beta) + A_4 e^\beta(-\sin\beta + \delta_1\cos\beta)] - \delta_{10} \frac{d}{d\phi} (n_{\phi n}^* + n_{\theta n}^*), \\
& \tag{4.47g}
\end{aligned}$$

$$q'_{\theta n} = \frac{nv}{r_{\theta} \sin \phi} [A_1 e^{-\alpha} \sin \alpha + A_2 e^{-\alpha} \cos \alpha - A_3 e^{\beta} \cos \beta + A_4 e^{\beta} \sin \beta] + \delta_{11} (n^*_{\phi n} + n^*_{\theta n}). \quad (4.47h)$$

In these equations

$$\delta_1 = 1 - \frac{1}{r_{\phi} \delta_4} \frac{d}{d\phi} (\delta_4) \sqrt{2\nu r_{\theta}}.$$

$$\delta_2 = 1 + \frac{1}{r_{\phi} \delta_4} \frac{d}{d\phi} (\delta_4) \sqrt{2\nu r_{\theta}}.$$

$$\delta_3 = \cot \phi \sqrt{\frac{\nu}{2r_{\theta}}}.$$

$$\delta_4 = \frac{1}{r_{\theta}^{3/4} \sin^{1/2} \phi}.$$

$$\delta_5 = \frac{n^2}{\sin \phi \cos \phi} \sqrt{\frac{2\nu}{r_{\theta}}}.$$

$$\delta_6 = \cot \phi \sqrt{\frac{2\nu}{r_{\theta}}}. \quad (4.48)$$

$$\delta_7 = \frac{(1 - \mu)\nu}{r_{\theta} \sin^2 \phi}.$$

$$\delta_8 = \frac{(1 - \mu)\nu^2}{r_{\phi}} \cot \phi.$$

$$\delta_9 = \frac{n^2(1 - \mu)\nu^2}{r_{\theta} \sin^2 \phi}.$$

$$\delta_{10} = (1 - \mu) \left( \frac{\nu}{r_{\theta}} \right)^2.$$

$$\delta_{11} = n \left( \frac{v}{r_\theta} \right)^2 \csc^2 \phi.$$

$$\alpha = \int_{\phi_s}^{\phi} \frac{-r_\theta}{\sqrt{2r_\theta v}} d\phi.$$

The constants of integration  $A_1$  to  $A_4$  are related to those defined in Eq. (4.44) by

$$A_1 = (\operatorname{Re} \bar{A} \cos \bar{\beta} + \operatorname{Im} \bar{A} \sin \bar{\beta}) e^{\bar{\beta}},$$

$$A_2 = (\operatorname{Im} \bar{A} \cos \bar{\beta} - \operatorname{Re} \bar{A} \sin \bar{\beta}) e^{\bar{\beta}},$$

(4.49a)

$$A_3 = \operatorname{Re} \bar{A}';$$

$$A_4 = \operatorname{Im} \bar{A}',$$

in which

$$\bar{\beta} = \int_{\phi_s}^{\phi} \frac{-r_\theta}{\sqrt{2r_\theta v}} = \alpha - \beta = \text{constant.} \quad (4.49b)$$

Only two of the four constants  $A_1$  to  $A_4$  will be present in any equation derived from the boundary conditions because for the shell  $\alpha = 0$  at the base and  $\beta = 0$  at the top.

The complete solutions for the stress resultants for dead load and earthquake load may be obtained by setting  $n = 0$  and  $n = 1$  respectively. The solutions for wind load requires a superposition of the expressions for  $n = 0$ ,  $n = 1$  and higher harmonics. Also the form of the stress