BACKWARD ITERATION IN THE UNIT BALL

by

OLENA OSTAPYUK

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
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Abstract

We consider iteration of an analytic self-map f of the unit ball in the N-dimensional complex space $\mathbb{C}^N$. Many facts were established about such maps and their dynamics in the 1-dimensional case (i.e. for self-maps of the unit disk), and we generalize some of them in higher dimensions.

In one dimension, the classical Denjoy-Wolff theorem states the convergence of forward iterates to a unique attracting fixed point, while backward iterates have much more complicated nature. However, under additional conditions (when the hyperbolic distance between two consecutive points stays bounded), backward iteration sequence converges to a point on the boundary of the unit disk, which happens to be a fixed point with multiplier greater than or equal to 1.

In this paper, we explore backward-iteration sequences in higher dimension. Our main result shows that in the case when f is hyperbolic or elliptic, such sequences with bounded hyperbolic step converge to a point on the boundary, other than the Denjoy-Wolff (attracting) point. These points are called boundary repelling fixed points (BRFPs) and possess several nice properties.

In particular, in the case when such points are isolated from other BRFPs, they are completely characterized as limits of backward iteration sequences. Similarly to classical results, it is also possible to construct a (semi) conjugation to an automorphism of the unit ball. However, unlike in the 1-dimensional case, not all BRFPs are isolated, and we present several counterexamples to show that. We conclude with some results in the parabolic case.

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In particular, in the case when such points are isolated from other BRFPs, they are completely characterized as limits of backward iteration sequences. Similarly to classical results, it is also possible to construct a (semi) conjugation to an automorphism of the unit ball. However, unlike in the 1-dimensional case, not all BRFPs are isolated, and we present several counterexamples to show that. We conclude with some results in the parabolic case.

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Acknowledgments

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Chapter 1

Introduction

1.1 1-dimensional case: forward iteration

Let f be an analytic self-map of the open unit disk $\mathbb{D}$ in the complex plane. The classical Schwarz's lemma says, that if $f(0) = 0$, then

$$|f(z)| \leq |z| \qquad \textit{and} \qquad |f'(0)| \leq 1,$$

and if equality holds for a point $z \neq 0$, then $f(z) = cz$ with $|c| = 1$ (rotation around the center). In other words, unless f is a rotation, the (Euclidean) distance between images of z and 0 is smaller then the distance between z and 0.

To have a similar statement about any two points in the unit disk, we need to replace Euclidean distance by pseudo-hyperbolic distance:

Theorem 1.1 (Point-invariant form of Schwarz's lemma)**.** *For any analytic self-map f of the unit disk and any $z, w \in \mathbb{D}$,*

$$\left| \frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}} \right| \leq \left| \frac{z - w}{1 - z\overline{w}} \right|,$$

and equality holds for some distinct pair iff f is an automorphism of $\mathbb{D}$; i.e. f is contraction in the pseudo-hyperbolic metric $d(z, w) := \left|\frac{z-w}{1-z\overline{w}}\right|$:

$$d(f(z), f(w)) \leq d(z, w).$$

Denote $f_n = f^{\circ n}$ the n^{th} iterate of f and consider the sequence of forward iterates $z_n = f_n(z_0)$, which is also called an f-orbit of z_0. By Schwarz's lemma, the sequence $d(z_n, z_{n+1})$ is decreasing; moreover, as the following theorem states, all forward iteration sequences must converge to the same point in the closed disk:

Theorem 1.2 (Denjoy-Wolff)**.** *If f is not an elliptic automorphism, then there exists a unique point $p \in \overline{\mathbb{D}}$ (called the Denjoy-Wolff point of f) such that the sequence of iterates $\{f_n\}$ converges to p uniformly on compact subsets of $\mathbb{D}$.*

Based on the dynamical behavior, we can divide self-maps of the disk into three classes:

If the Denjoy-Wolff point p is inside of the unit disk, then f is called elliptic. In this case p is a fixed point of f (i.e. $f(p) = p$) and $|f'(p)| \leq 1$. When $|f'(p)| = 1$, f is an elliptic automorphism (up to change of variables, rotation around the center).

If the Denjoy-Wolff point p is on the boundary of the unit disk and $f'(p) < 1$ (in the sense of non-tangential limits), then f is called hyperbolic. p is again a fixed point of f, now in the sense of non-tangential limits. Forward iterates tend to the Denjoy-Wolf point along non-tangential directions.

If the Denjoy-Wolff point p is on the boundary of the unit disk and $f'(p) = 1$ (in the sense of non-tangential limits), then f is called parabolic. Similarly to the hyperbolic case, p is a fixed point of f; but now forward iterates can converge tangentially to the boundary. For more details on the parabolic case, see Chapter 8.

For examples of orbits in each case, refer to Figure 1.1.

Since in every case $|f'(p)| \leq 1$, the point p can thus be called "attracting". When, in the hyperbolic and parabolic cases, $p \in \partial\mathbb{D}$, a more geometrical property (Julia's lemma for the point p) holds:

$$\forall R > 0 \quad f\left(H(p, R)\right) \subseteq H(p, cR), \tag{1.1.1}$$

where $H(p, R)$ is a horocycle at $p \in \partial\mathbb{D}$ of radius R (see Figure 1.2),

$$H(p, R) := \left\{ z \in \mathbb{D} : \frac{|p - z|^2}{1 - |z|^2} < R \right\}.$$

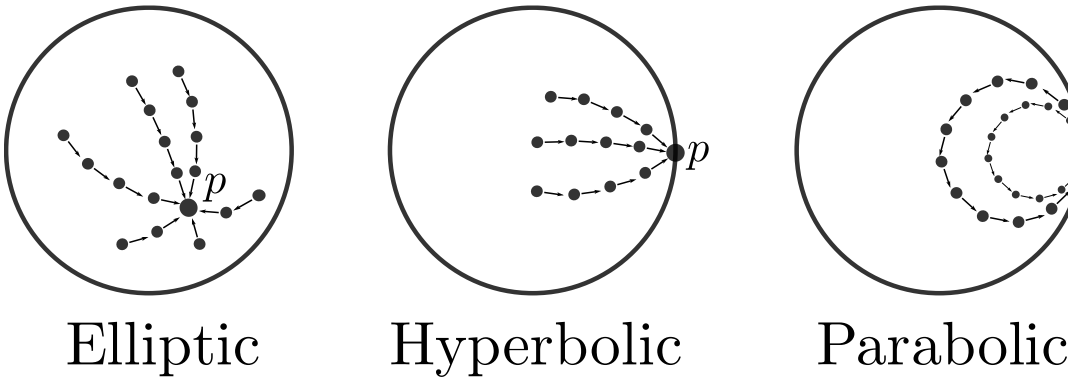

Figure 1.1: *Denjoy-Wolff point p and typical orbits in the elliptic, hyperbolic and parabolic cases.*

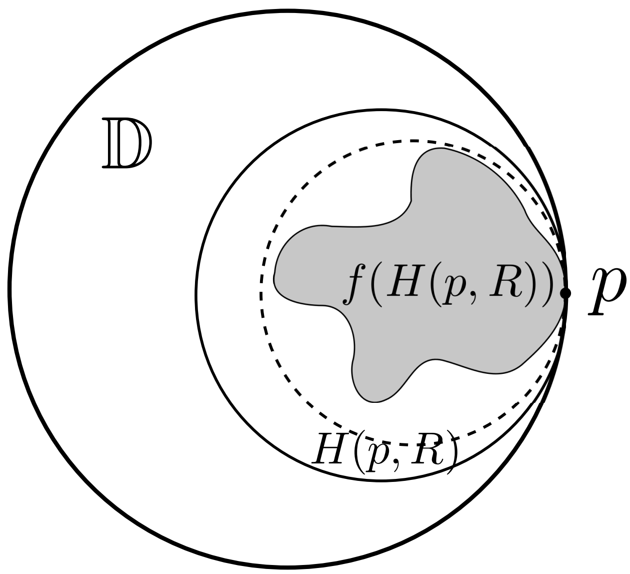

Figure 1.2: *Julia's lemma at the Denjoy-Wolff point $p \in \partial\mathbb{D}$.*

Here $c = f'(p)$ is the smallest c such that (1.1.1) holds. We will call it the multiplier or the dilatation coefficient of f at p.

A natural way to describe the dynamics of the function is to conjugate it to a simpler function with well-known behavior, usually to an automorphism. The problem can be formulated as follows: given a map f, find an intertwining map (change of variable) ψ, which solves the equation:

$$\psi \circ f = \eta_f \circ \psi,$$

where η_f is a simple map (e.g. an automorphism) of a complex manifold Ω (usually, a half-plane or a disk). In other words, the following diagram (Figure 1.3) commutes.

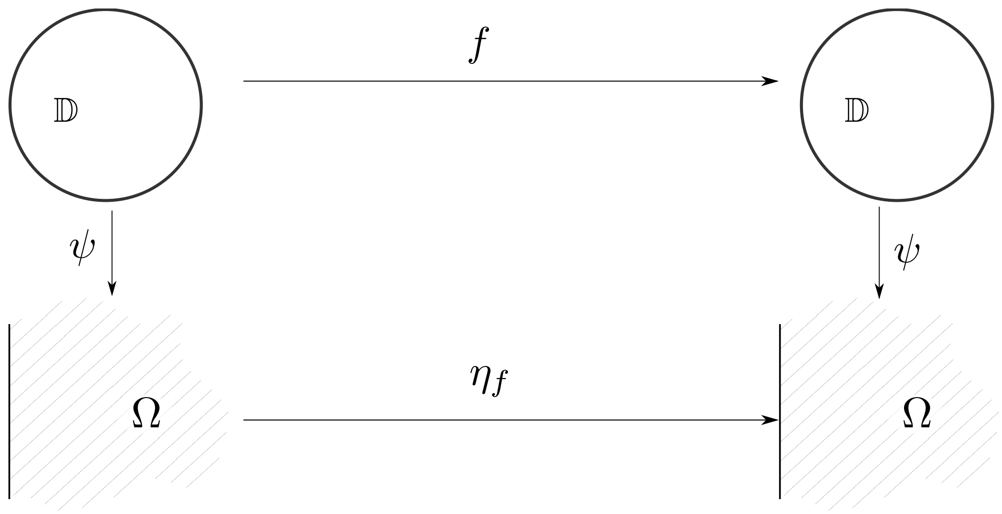

Figure 1.3: *(Semi-)conjugation for the map f.*

In the hyperbolic case, Valiron [34] showed that there is an analytic map $\psi : \mathbb{D} \to \mathbb{H}$ (where $\mathbb{H}$ is the right half-plane) with some regularity properties, which solves the Schröder equation:

$$\psi \circ f = \frac{1}{c}\psi, \tag{1.1.2}$$

and so ψ conjugates f to a multiplication in $\mathbb{H}$.

In the parabolic case, f can be conjugated to a translation in a half-plane or in the whole plane, as proved by Pommerenke [30], and Baker and Pommerenke [2].

Conjugations for elliptic maps with $f'(p) \neq 0$ were found by Koenigs [20]; such maps are conjugated to a multiplication by $f'(p)$ in the plane:

$$\psi \circ f = f'(p) \cdot \psi$$

with $\psi : \mathbb{D} \to \mathbb{C}$.

The elliptic case with $f'(p) = 0$ was considered by Böttcher in [6], and the following conjugation was found:

$$\psi \circ f = \psi^n$$

where ψ defined in a neighborhood of p, and n is the smallest integer such that $f^{(n)}(p) \neq 0$.

Conjugation to a linear-fractional transformation in all cases simultaneously was shown by Cowen in [13].

The question of uniqueness of the intertwining map has also been explored and answered; in the elliptic case in [20], in the hyperbolic case by Bracci and Poggi-Corradini in [9], in the parabolic case by Poggi-Corradini in [29] and by Contreras, Díaz-Madrigal and Pommerenke in [12].

1.2 1-dimensional case: backward iteration

Definition 1.3. We will call a sequence of points $\{z_n\}_{n=0}^{\infty}$ a backward-iteration sequence for f if $f(z_{n+1}) = z_n$ for $n = 0, 1, 2, \ldots$.

In general, such sequences may not exist. (For example, $f(z) = cz$ with $|c| < 1$ has none). Note that in the backward iteration case the sequence $d(z_n, z_{n+1})$ is increasing, so we will impose an upper bound on the pseudo-hyperbolic step:

$$d(z_n, z_{n+1}) \leq a, \quad \forall n, \tag{1.2.1}$$

for some fixed $a < 1$.

This condition is nontrivial, for an example of a map that admits a backward-iteration sequence with unbounded steps, see section 2 of [28].

A backward-iteration sequence satisfying (1.2.1) must converge to a point on the boundary of $\mathbb{D}$:

Theorem 1.4 (Poggi-Corradini, [26])**.** *Suppose f is an analytic map with $f(\mathbb{D}) \subseteq \mathbb{D}$ (and not an elliptic automorphism). Let $\{z_n\}_{n=0}^{\infty}$ be a backward-iteration sequence for the function f with bounded pseudo-hyperbolic steps $d_n = d(z_n, z_{n+1}) \uparrow a < 1$. Then the following hold:*

1. *There is a point $q \in \partial\mathbb{D}$ such that $z_n \to q$ as n tends to infinity, and q is a fixed point for f with a well-defined multiplier $f'(q) = \alpha < \infty$.*

2. *When $q \neq p$, where p is the Denjoy-Wolff point, then $\alpha > 1$, so we can call q a* boundary repelling fixed point. *If $q = p$, then f is necessarily of parabolic type.*

3. *When $q \neq p$, the sequence z_n tends to q along a non-tangential direction.*

4. *When $q = p$, z_n tends to q tangentially.*

In this case Julia's lemma holds for the point q with multiplier $\alpha > 1$:

$$\forall R > 0 \quad f\left(H(q, R)\right) \subseteq H(q, \alpha R), \tag{1.2.2}$$

where $\alpha = f'(q)$ is the smallest number such that this holds.

For backward iteration, the following conjugation result was obtained in [27]:

Theorem 1.5 (Poggi-Corradini)**.** *Suppose f is an analytic self-map of the unit disc $\mathbb{D}$ and 1 is a boundary repelling fixed point for f with multiplier $1 < \alpha < \infty$. Let $a = (\alpha - 1)/(\alpha + 1)$ and $\eta(z) = (z - a)/(1 - az)$. Then there is an analytic map ψ of $\mathbb{D}$ with $\psi(\mathbb{D}) \subseteq \mathbb{D}$, which has non-tangential limit 1 at 1, such that*

$$\psi \circ \eta(z) = f \circ \psi(z), \tag{1.2.3}$$

for all $z \in \mathbb{D}$.

Note that the order of the functions in (1.2.3) is different than in classical conjugation results.

1.3 Unit ball in $\mathbb{C}^N$.

Consider a self-map of the N-dimensional unit ball

$$\mathbb{B}^N = \left\{ Z \in \mathbb{C}^N : \|Z\| < 1 \right\},$$

In the unit ball, we have generalizations for the (pseudo-)hyperbolic distance, Schwarz's lemma, horocycles and Julia's lemma; see Chapter 2.

And a version of the Denjoy-Wolff theorem also holds:

Theorem 1.6 (Hervé [16], MacCluer [24]). *Let $f : \mathbb{B}^N \to \mathbb{B}^N$ be a holomorphic map without fixed points in $\mathbb{B}^N$. Then the sequence of iterates $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{B}^N$ to the constant map $Z \mapsto p$ for a (unique) point $p \in \partial\mathbb{B}^N$ (called the Denjoy-Wolff point of f); and the number*

$$c := \liminf_{Z \to p} \frac{1 - \|f(Z)\|}{1 - \|Z\|} \in (0, 1] \qquad (1.3.1)$$

is called the multiplier or the boundary dilatation coefficient of f at p.

The map f is called hyperbolic if $c < 1$ and parabolic if $c = 1$.

Unlike in the one-dimensional case, there may be many fixed points inside the unit ball $\mathbb{B}^N$. Even if the fixed point is unique, forward iterates need not converge to it (consider rotations). We will call a function f unitary on a slice if there exist ζ and η in $\partial\mathbb{B}^N$ with $f(\lambda\zeta) = \lambda\eta$ for all $\lambda \in \mathbb{D}$. Functions that are not unitary on any slice are precisely those for which strict inequality occurs in the multidimensional Schwarz lemma and for them forward iterates converge to the (unique) point 0 (see [14]). Note that even if the function f has more than one fixed point, the sequence of forward iterates may still converge, see [1].

Definition 1.7. We will call a self-map of the unit ball f attracting-elliptic, if it has a unique fixed point inside $\mathbb{B}^N$ and it is conjugate via an automorphism to a self-map which fixes zero and is not unitary on any slice.

In the rest of this work we will consider only self-maps of the ball that are attracting-elliptic, hyperbolic or parabolic.

Forward iteration in the unit ball of $\mathbb{C}^N$ in the hyperbolic case was studied in [7], [8] and [17]. In [8] the Schröder equation (1.1.2) was solved with ψ being holomorphic map $\psi : \mathbb{B}^N \to \mathbb{H}$ given some additional conditions. In [17], a similar result was obtained for

Schur-class maps. In [7], f was conjugated to its linear part, assuming some regularity at the Denjoy-Wolff point. Linearization results for the large class of hyperbolic and parabolic maps of $\mathbb{B}^2$ were proved in [3]. Conjugations for elliptic maps were given in [14]; and they also follow by the classical Poincaré-Dulac theory, see [33].

Chapter 2

Unit ball: preliminaries

In this chapter we will state various results about analytic functions in the unit ball that we are going to use later in this work. Many of them are generalizations of classical theorems for the unit disk.

Consider the N-dimensional unit ball

$$\mathbb{B}^N = \left\{ Z \in \mathbb{C}^N : \|Z\| < 1 \right\},$$

where the inner product and the norm are defined as

$$\langle Z, W \rangle = \sum_{j=1}^{N} Z_j \overline{W_j} \quad \text{and} \quad \|Z\|^2 = \langle Z, Z \rangle .$$

In order to describe the group of all automorphisms of $\mathbb{B}^N$ (denoted by $Aut(\mathbb{B}^N)$), we will need the following functions. Let $a \in \mathbb{B}^N$, $a \neq 0$ and define

$$P_a(Z) = \frac{\langle Z, a \rangle}{\langle a, a \rangle} a, \qquad Q_a(Z) = Z - P_a(Z),$$

i.e. P_a is a projection on the subspace generated by a, and Q_a is a projection on the orthogonal complement. Then define

$$\gamma_a(Z) = \frac{a - P_a(Z) - (1 - \|a\|^2)^{1/2} Q_a(Z)}{1 - \langle Z, a \rangle}.$$

When $N = 1$, $P_a(Z) = Z$, $Q_a(Z) = 0$, and $\gamma_a(Z)$ is clearly an automorphism of the unit disk.

γ_a is an automorphism of the ball that maps 0 to a and a to 0 and possesses several other nice properties, see [31] or [1]. It is also the main building block for $Aut(\mathbb{B}^N)$:

Claim 2.1 (Theorem 2.2.5, [31]). *Every $\gamma \in Aut(\mathbb{B}^N)$ is of the form*

$$\gamma = U \circ \gamma_a,$$

where $a = \gamma^{-1}(0)$ and U is a unitary transformation (rotation).

Next, we will need a multi-dimensional version of Schwarz's lemma:

Theorem 2.2 (Theorem(2.2.12), [1]). *Let f be an analytic self-map of $\mathbb{B}^N$ such that $f(0) = 0$. Then*

$$\forall Z \in \mathbb{B}^N \qquad \|f(Z)\| \leq \|Z\|,$$

and

$$\forall V \in \mathbb{C}^N \qquad \|df_0(V)\| \leq \|V\|,$$

where df_0 is the derivative of f at 0.

Note that, unlike in the one-dimensional case, the equality at one point does not imply linearity of f. However, the following statement holds:

Lemma 2.3 (Hervé, [16]). *Let f be an analytic self-map of $\mathbb{B}^N$ such that $f(0) = 0$, and take $Z \in \mathbb{B}^N$, $Z \neq 0$. Then*

(i) $\|f(Z)\| = \|Z\|$ iff $\|df_0(Z)\| = \|Z\|$;

(ii) $f(Z) = Z$ iff $df_0(Z) = Z$.

Corollary 2.4. *The fixed-point set of an analytic self-map of $\mathbb{B}^N$ is either empty or an affine subset of $\mathbb{B}^N$.*

And also we can get a point-invariant version of Schwarz's lemma:

Theorem 2.5 (Proposition(2.2.17), [1])**.** *Let f be an analytic self-map of $\mathbb{B}^N$. Then for every $Z, W \in \mathbb{B}^N$*

$$\frac{|1-\langle f(Z), f(W)\rangle|^2}{\left(1-\|f(Z)\|^2\right)\left(1-\|f(W)\|^2\right)} \leq \frac{|1-\langle Z, W\rangle|^2}{\left(1-\|Z\|^2\right)\left(1-\|W\|^2\right)}. \tag{2.1}$$

In particular, if f is an automorphism of $\mathbb{B}^N$, then (2.1) is always an equality.

If we define the pseudo-hyperbolic distance in $\mathbb{B}^N$ as

$$d_{\mathbb{B}^N}(Z, W) := \left(\frac{|1-\langle Z, W\rangle|^2}{\left(1-\|Z\|^2\right)\left(1-\|W\|^2\right)}\right)^{1/2}, \tag{2.2}$$

then (2.1) can be restated as

$$d_{\mathbb{B}^N}(f(Z), f(W)) \leq d_{\mathbb{B}^N}(Z, W), \tag{2.3}$$

and every automorphism of $\mathbb{B}^N$ is an isometry in the pseudo-hyperbolic metric.

We will use the pseudo-hyperbolic metric only as a matter of convenience; in fact, everything may be reformulated using the Bergmann metric $k_{\mathbb{B}^N}$, which is related to the pseudo-hyperbolic metric by

$$d_{\mathbb{B}^N}(Z, W) = \tanh(k_{\mathbb{B}^N}(Z, W)) \quad \forall Z, W \in \mathbb{B}^N.$$

The Bergmann metric (introduced by Bergmann in [4] and [5]) is the natural generalization of the Poincaré metric in the unit disk and coincides with the Poincaré metric when $N = 1$. It is also a special case of the Carathéodory ([10]) and Kobayashi metrics ([18], [19]).

The pseudo-hyperbolic ball centered at Z_0 of radius R, i.e., the set

$$B_d(Z_0, R) := \left\{Z \in \mathbb{B}^N \,|\, d_{\mathbb{B}^N}(Z, Z_0) < R\right\}$$

has the form

$$\left\{Z \in \mathbb{B}^N \,\middle|\, \frac{\|P_{Z_0}(Z) - a\|^2}{R^2\rho^2} + \frac{\|Q_{Z_0}(Z)\|^2}{R^2\rho} < 1\right\}, \tag{2.4}$$

where

$$a = \frac{1 - R^2}{1 - R^2\|Z_0\|^2} Z_0$$

and

$$\rho = \frac{1 - \|Z_0\|^2}{1 - R^2\|Z_0\|^2}.$$

Hence $B_d(Z_0, R)$ is an Euclidean ellipsoid centered at $\frac{1 - R^2}{1 - R^2\|Z_0\|^2} Z_0$ with radius $r_1 = \frac{1 - |Z_0|^2}{1 - R^2|Z_0|^2} R$ in the subspace generated by Z_0, and larger radius $r_2 = R\sqrt{\frac{1 - \|Z_0\|^2}{1 - R^2\|Z_0\|^2}}$ in the dimensions orthogonal to Z_0.

In order to formulate a multi-dimensional Julia's lemma, we will need notion of a horosphere, a natural generalization of a horocycle, introduced by Hervé in [16]. The horosphere $H(X, R)$ of center $X \in \partial\mathbb{B}^N$ and radius $R > 0$ is

$$H(X, R) := \left\{ Z \in \mathbb{B}^N \,\middle|\, \frac{|1 - \langle Z, X \rangle|^2}{1 - \|Z\|^2} < R \right\}.$$

Horospheres are also Euclidean ellipsoids with smaller radius in the subspace generated by X and larger radius in the dimensions orthogonal to X:

$$H(X, R) = \left\{ Z \in \mathbb{B}^N \,\middle|\, \frac{\|P_X(Z) - a\|^2}{r^2} + \frac{\|Q_X(Z)\|^2}{r} < 1 \right\}, \tag{2.5}$$

where $r = R/(1 + R)$ and $a = (1 - r)X$. They possess properties similar to those of horocycles, see [1], with the most important one being the following generalization of Julia's lemma:

Theorem 2.6 (Hervé [16], see also [25]). *Let $f : \mathbb{B}^N \to \mathbb{B}^N$ be a holomorphic map and suppose $X \in \partial\mathbb{B}^N$ satisfies*

$$\liminf_{Z \to X} \frac{1 - \|f(Z)\|}{1 - \|Z\|} =: \alpha < \infty. \tag{2.6}$$

Then there exists a unique $Y \in \partial\mathbb{B}^N$ such that

$$\forall R > 0 \quad f\left(H(X, R)\right) \subseteq H(Y, \alpha R).$$

In one dimension, the notion of non-tangential limit is important for the study of the boundary behavior of analytic functions in the disk. In several variables, the non-tangential limits can be replaced by a more general approach (introduced by Koranyi and Stein in [22], [21]):

Definition 2.7. The Koranyi region $K(X, M)$ of vertex $X \in \partial\mathbb{B}^N$ and amplitude $M > 1$ is the set

$$K(X, M) = \left\{ Z \in \mathbb{B}^N \,\middle|\, \frac{|1 - \langle Z, X \rangle|}{1 - \|Z\|} < M \right\}. \tag{2.7}$$

When $N = 1$, it is the usual Stolz angle in the disk; but for $N > 1$ the region is tangent to the boundary of the ball along some directions. To illustrate this, consider $K(e_1, M)$, where $e_1 = (1, 0, \ldots, 0)$. The intersection of $K(e_1, M)$ with the one-dimensional complex subspace generated by e_1 is

$$\left\{ Z_1 \in \mathbb{D} \,\middle|\, \frac{|1 - Z_1|}{1 - |Z_1|} < M \right\},$$

which is the usual Stolz region, while its intersection with the $(2n - 1)$-dimensional real subspace defined by setting $\operatorname{Im} Z_1 = 0$ contains the ball

$$\left(\operatorname{Re} Z_1 - \frac{1}{M} \right)^2 + |Z_2|^2 + \ldots + |Z_N|^2 < \left(1 - \frac{1}{M} \right)^2,$$

which is tangent to $\partial\mathbb{B}^N$ at e_1.

Similarly to the one-dimensional case, Koranyi regions can be used to define a limit at the boundary of the ball:

Definition 2.8. We will say that the function f has K-limit λ at $X \in \partial\mathbb{B}^N$ if for any $M > 1$ $f(Z) \to \lambda$ as $Z \to X$ within $K(X, M)$.

Recall that the classical Lindelöf theorem ([23]) states that, for any analytic self-map of the unit disk, a limit along a curve ending at a boundary point determines the non-tangential limit at that point. In order to obtain its multi-dimensional version, we will need a geometric notion slightly different from Koranyi regions:

Definition 2.9. For $X \in \partial\mathbb{B}^N$, a curve $\sigma : [0,1) \to \mathbb{B}^N$ such that $\sigma(t) \to X$ as $t \to 1$ is called special if

$$\lim_{t\to 1} \frac{\|\sigma(t) - \sigma_X(t)\|^2}{1 - \|\sigma_X(t)\|^2} = 0, \tag{2.8}$$

and restricted if it is special and its orthogonal projection $\sigma_X := \langle \sigma, X\rangle X$ is non-tangential.

Definition 2.10. We will say that $f : \mathbb{B}^N \to \mathbb{B}^N$ has restricted K-limit Y at $X \in \partial\mathbb{B}^N$ if $f(\sigma(t)) \to Y$ as $t \to 1$ for any restricted curve σ.

The connection between non-tangential, Koranyi and restricted approaches is described by the following

Lemma 2.11 (Lemma (2.2.24), [1])**.** *Let $\sigma : [0,1) \to \mathbb{B}^N$ be a curve such that $\sigma(t) \to X \in \partial\mathbb{B}^N$ as $t \to 1$. Then*

(i) if σ is non-tangential, then it is restricted;

(ii) assume σ is special. If σ is restricted, then it lies eventually in a Koranyi region with vertex X. Conversely, if σ lies in a Koranyi region, it is restricted.

Therefore, restricted K-limit is a weaker notion than K-limit: a function having a K-limit has a restricted K-limit, and a function having a restricted K-limit has a non-tangential limit. For more properties and examples, see [1].

In the N-dimensional case, we need a limit along a special curve to imply restricted K-limit (and, consequently, non-tangential limit):

Theorem 2.12 (Čirka, [11])**.** *Let $f : \mathbb{B}^N \to \mathbb{C}$ be a bounded holomorphic function such that there is a special curve $\sigma : [0,1) \to \mathbb{B}^N$ with $\sigma(t) \to X \in \partial\mathbb{B}^N$ as $t \to 1$ such that*

$$\lim_{t\to 1} f(\sigma(t)) = \lambda \in \mathbb{C}.$$

Then f has restricted K-limit λ at X.

The last theorem will be the generalization of Julia-Wolff-Carathéodory theorem. The main difference with the one-dimensional case is that the behavior of the radial component is quite different from the behavior of the tangential component of f.

Theorem 2.13 (Rudin, [31])**.** *Let f be an analytic self-map of $\mathbb{B}^N$ and $X \in \partial\mathbb{B}^N$ be such that*

$$\liminf_{Z \to X} \frac{1 - \|f(Z)\|}{1 - \|Z\|} = \alpha < \infty.$$

Then f has K-limit $Y \in \partial\mathbb{B}^N$, and the following functions are bounded in every Koranyi region:

(i) $\dfrac{1 - \langle f(Z), Y\rangle}{1 - \langle Z, X\rangle}$,

(ii) $\dfrac{Q_Y\left(f(Z)\right)}{\left(1 - \langle Z, X\rangle\right)^{1/2}}$.

Moreover, the function (i) has restricted K-limit α at X, and the function (ii) has K-limit 0 at X.

In some cases it will be more convenient to use the Siegel domain (or Siegel half-space, introduce by Siegel, [32]):

$$\mathbb{H}^N := \left\{(z, w) \in \mathbb{C} \times \mathbb{C}^{N-1} : \operatorname{Re} z > \|w\|^2\right\},$$

which is biholomorphic to $\mathbb{B}^N$ via the Cayley transform $\mathcal{C} : \mathbb{B}^N \to \mathbb{H}^N$:

$$\mathcal{C}(z, w) := \left(\frac{1+z}{1-z}, \frac{w}{1-z}\right) \quad \text{and} \quad \mathcal{C}^{-1}(z, w) = \left(\frac{z-1}{z+1}, \frac{2w}{z+1}\right). \tag{2.9}$$

The Siegel domain relates to the unit ball in the same way as (right) half-plane relates to the unit disk; all results can be formulated for either $\mathbb{B}^N$ or $\mathbb{H}^N$.

We will use the same notations for the points in $\mathbb{B}^N$ and their images in $\mathbb{H}^N$, when this is not likely to cause confusion. We will also denote by (z, w) an N-dimensional vector either in $\mathbb{B}^N$ or $\mathbb{H}^N$ with $z \in \mathbb{C}$ being the first component and $w \in \mathbb{C}^{N-1}$ being the last $N - 1$

components. Using (2.2) and (2.9), we can derive the pseudo-hyperbolic distance in $\mathbb{H}^N$:

$$\begin{aligned} d^2_{\mathbb{H}^N}((z,w),(\tilde{z},\tilde{w})) &:= d^2_{\mathbb{B}^N}(\mathcal{C}^{-1}(z,w),\mathcal{C}^{-1}(\tilde{z},\tilde{w})) \\ &= 1 - \frac{4(\operatorname{Re} z - \|w\|^2)(\operatorname{Re}\tilde{z} - \|\tilde{w}\|^2)}{|z+\bar{\tilde{z}} - 2\langle w,\tilde{w}\rangle|^2} \quad \forall (z,w),(\tilde{z},\tilde{w}) \in \mathbb{H}^N. \end{aligned} \tag{2.10}$$

Using Claim 2.1 and Cayley transform (2.9), it is not difficult to show that all automorphisms of $\mathbb{H}^N$ are of the form

$$\gamma = \delta_t \circ h_a \circ \mu_U,$$

where h_a ($a = (a_1, \tilde{a}) \in \partial\mathbb{H}^N$) is translation (fixes infinity):

$$h_a(z,w) := (z + a_1 + 2\langle w, \tilde{a}\rangle, w + \tilde{a});$$

δ_t ($t > 0$) is dilatation (fixes 0 and infinity):

$$\delta_t(z,w) := (t^2 z, tw);$$

and $\mu_U(z,w) := \mathcal{C}(U(\mathcal{C}^{-1}(z,w)))$ is an automorphism of $\mathbb{H}^N$ fixing $(1,0)$ (recall that U is a rotation of $\mathbb{B}^N$).

These automorphisms will be extensively used in this work in order to map a generic point on the boundary of $\mathbb{H}^N$ either to 0 or infinity and a point inside of $\mathbb{H}^N$ to $(1,0)$.

Chapter 3

Main results

The main goal of this thesis is to study backward iterates in the unit ball $\mathbb{B}^N$. The following results are generalizations of Theorem 1.4 and Theorem 1.5 to higher dimensions.

Theorem 3.1. *Let f be a holomorphic self-map of $\mathbb{B}^N$ of hyperbolic or attracting-elliptic type with Denjoy-Wolff point p. Let $\{Z_n\}$ be a backward-iteration sequence for f with bounded pseudo-hyperbolic step $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a < 1$. Then:*

1. *There exists a point $q \in \partial\mathbb{B}^N$, $q \neq p$, such that $Z_n \to q$ as n tends to infinity,*
2. *$\{Z_n\}$ stays in a Koranyi region with vertex q,*
3. *Julia's lemma (1.2.2) holds for q with a finite multiplier $\alpha \geq \frac{1}{c}$, where $c < 1$ is a constant that depends on f.*

Remark 3.2. In the hyperbolic case, c is the multiplier at p; see (1.3.1).

Because of the last statement of Theorem (3.1), the multiplier $\alpha > 1$, and thus we can introduce the following

Definition 3.3. The point $q \in \partial\mathbb{B}^N$ is called a boundary repelling fixed point (BRFP) for f, if (1.2.2) holds for some $\alpha > 1$.

Remark 3.4. It follows from Julia's lemma (Theorem 2.6) that the above definition of multiplier is equivalent to (2.6).

Remark 3.5. It follows from (1.2.2) that q is also a boundary fixed point with respect to K-limits and, consequently, non-tangential limits (see the proof of Theorem 2.13 in [1] p.176).

Theorem 3.6. *Suppose f is an analytic function on $\mathbb{H}^N$ with $f(\mathbb{H}^N) \subseteq \mathbb{H}^N$ and 0 is a boundary repelling fixed point for f with multiplier $1 < \alpha < \infty$, isolated from other boundary repelling fixed points with multipliers less or equal to α. Consider the automorphism of $\mathbb{H}^N$: $\eta(z, w) = (\alpha z, \sqrt{\alpha} w)$. Then there is an analytic map ψ of $\mathbb{H}^N$ with $\psi(\mathbb{H}^N) \subseteq \mathbb{H}^N$ and $\psi(z, w) = \psi(z, 0)$, which has restricted K-limit 0 at 0, such that*

$$\psi \circ \eta(Z) = f \circ \psi(Z), \tag{3.1}$$

for every $Z \in \mathbb{H}^N$.

It follows from the proof of Theorem 3.6 (see Lemma 5.1), that every isolated boundary repelling fixed point is a limit of some backward-iteration sequence with bounded hyperbolic step. Thus in the hyperbolic and attracting-elliptic cases we have the following characterization of BRFP in terms of backward-iteration sequences: Every backward-iteration sequence with bounded hyperbolic step converges to a BRFP; and if a BRFP is isolated, then we can construct a backward-iteration sequence with bounded hyperbolic step that converges to it.

The intertwining map ψ in Theorem 3.6 satisfies $\psi(z, w) = \psi(z, 0)$ and essentially is a map from a one-dimensional subspace of $\mathbb{H}^N$ to $\mathbb{H}^N$; therefore that conjugation does not provide information about the behavior of f outside of the one-dimensional image of ψ. It then is natural to identify situations in which we can find a conjugation where the image of the intertwining map ψ has larger dimension.

Theorem 3.7. *Let $f : \mathbb{H}^N \to \mathbb{H}^N$ be analytic and expandable at 0 (see Definition 7.1) and 0 be a boundary repelling fixed point with multiplier $1 < \alpha < \infty$. Assume further that the matrix A in the definition of expandable is diagonal, and without loss of generality let its eigenvalues be $a_{j,j} = \sqrt{\alpha} e^{i\theta_j}$ for $j = 1, \ldots, L$ (L is an integer, $0 \leq L \leq N-1$) and $|a_{j,j}|^2 < \alpha$ for $j = L+1, \ldots, N-1$. Define Ω as a diagonal matrix with $\Omega_{j,j} = e^{i\theta_j}$ for*

$j = 1, \dots, L$ and $\Omega_{j,j} = 1$ for $j = L+1, \dots, N-1$. Then the conjugation (3.1) holds for $\eta(z,w) = \left(\alpha z, \Omega\alpha^{1/2}w\right)$ and intertwining map ψ such that $\psi(z,w) = \psi(p_{L+1}(z,w))$, where p_{L+1} is a projection on the first $L+1$ dimensions.

In the last chapter we will provide some new examples, in particular, functions in the two-dimensional Siegel domain that have non-isolated BRFPs, a phenomenon that never occurs in one dimension. In Example 9.3, we will show that the quadratic function $f(z,w) := (2z + w^2, w)$ is of hyperbolic type with the Denjoy-Wolff point infinity and has a curve $\{(r^2, ir) | r \in \mathbb{R}\}$ of boundary repelling fixed points, all of them having the same multiplier $\alpha = 2$.

In Example 9.5 we will describe a non-trivial way to construct a function f of the two-dimensional Siegel domain based on a function ϕ of a one-dimensional half-plane. f will behave very similarly to ϕ and will inherit many properties, however, it may have non-isolated BRFPs.

Chapter 4

Convergence of backward-iteration sequences

Proof of Theorem 3.1 (hyperbolic case). We will move to the Siegel domain $\mathbb{H}^N$. Without loss of generality we can assume that the Denjoy-Wolff point is infinity. Also denote the backward-iteration sequence as $Z_n = (z_n, w_n) \in \mathbb{C} \times \mathbb{C}^{N-1}$ and define $t_n = \operatorname{Re} z_n - \|w_n\|^2$. The image of the horosphere centered at $(1, 0)$ of radius R under the Cayley transform will be

$$\left\{ (z,w) \in \mathbb{H}^N \,\middle|\, \frac{\left|1 - \langle \mathcal{C}^{-1}(z,w), (1,0)\rangle\right|^2}{1 - \left\|\mathcal{C}^{-1}(z,w)\right\|^2} < R \right\} =$$

$$\left\{ (z,w) \in \mathbb{H}^N \,\middle|\, \frac{\left|1 - \frac{z-1}{z+1}\right|^2}{1 - \left|\frac{z-1}{z+1}\right|^2 - \frac{\|2w\|^2}{|z+1|^2}} < R \right\},$$

and after some computations this is seen to equal

$$\left\{ (z,w) \in \mathbb{H}^N \,\middle|\, \operatorname{Re} z - \|w\|^2 > \frac{1}{R} \right\};$$

i.e., any horosphere centered at the Denjoy-Wolff point ∞ will have the form

$$H(t) = \left\{ (z,w) \in \mathbb{H}^N \mid \operatorname{Re} z - \|w\|^2 > t \right\},$$

for some $t > 0$, and the Siegel domain version of the multi-dimensional Julia's lemma (Theorem 2.6) at infinity will be

$$\forall R > 0 \qquad f\left(H\left(\frac{1}{R}\right)\right) \subset H\left(\frac{1}{cR}\right)$$

or

$$\forall t > 0 \qquad f\left(H(ct)\right) \subset H(t). \tag{4.1}$$

Since $f(Z_{n+1}) = Z_n \notin H(t_n)$, by (4.1) $Z_{n+1} \notin H(ct_n)$, and, by induction, $Z_{n+k} \notin H(c^k t_n)$, $k = 1, 2, \ldots$. Thus we have

$$\operatorname{Re} z_{n+k} - \|w_{n+k}\|^2 = t_{n+k} \leq c^k t_n, \qquad k = 1, 2, \ldots \tag{4.2}$$

Since the dilatation coefficient at the Denjoy-Wolff point $c < 1$, the sequence Z_n must tend to the boundary of the Siegel domain as n tends to infinity. All we need to show now is that the limiting set on the boundary is just one point.

Define a Euclidean projection on the boundary of the Siegel domain as

$$pr(z, w) := (i \operatorname{Im} z + \|w\|^2, w).$$

It will be enough to show that $pr(Z_n)$ has a limit.

Lemma 4.1. *The Euclidean distance between projections of consecutive points of the backward-iteration sequence is bounded by*

$$\|pr(Z_n) - pr(Z_{n+1})\| \leq \tilde{C}\sqrt{t_n},$$

for some finite constant $\tilde{C}$ independent of n.

Assuming the lemma and using (4.2), we have

$$\begin{aligned} \|pr(Z_n) - pr(Z_{n+k})\| &\leq \sum_{j=0}^{k-1} \|pr(Z_{n+j}) - pr(Z_{n+j+1})\| \leq \tilde{C} \sum_{j=0}^{k-1} \sqrt{t_{n+j}} \\ &\leq \tilde{C} \sum_{j=0}^{k-1} \sqrt{c^j t_n} \leq \tilde{C}\sqrt{t_n} \sum_{j=0}^{\infty} \sqrt{c^j} = \frac{\tilde{C}\sqrt{t_n}}{1 - \sqrt{c}} \xrightarrow[n \to \infty]{} 0. \end{aligned} \tag{4.3}$$

Thus $\{pr(Z_n)\}$ is a Cauchy sequence and must have a limit $q \in \partial\mathbb{H}^N$, which is also the limit for $\{Z_n\}$. Since q is within a finite Euclidean distance from $pr(Z_1)$, it is finite and cannot coincide with the Denjoy-Wolff point infinity.

Proof of Lemma 4.1. Consider the images of Z_n and Z_{n+1} under the automorphism in $\mathbb{H}^N$ defined by

$$h_n(z, w) := (z - i \operatorname{Im} z_n + \|w_n\|^2 - 2\langle w, w_n\rangle, w - w_n),$$

which maps Z_n to $(t_n, 0)$. Denote $h_n(Z_{n+1}) = \tilde{Z}_n = (\tilde{z}_n, \tilde{w}_n) = (\tilde{x}_n + i\tilde{y}_n, \tilde{w}_n)$. h_n are called translations and they do not change the horoshperes $H(t)$ centered at infinity. We check this for the reader's convenience:

$$\begin{aligned}
&\operatorname{Re}(z - i \operatorname{Im} z_n + \|w_n\|^2 - 2\langle w, w_n\rangle) - \|w - w_n\|^2 \\
=& \operatorname{Re} z + \|w_n\|^2 - 2 \operatorname{Re}\langle w, w_n\rangle - \|w - w_n\|^2 \\
=& \operatorname{Re} z + \|w_n\|^2 - 2 \operatorname{Re}\langle w, w_n\rangle - \|w\|^2 + 2 \operatorname{Re}\langle w, w_n\rangle - \|w_n\|^2 \\
=& \operatorname{Re} z - \|w\|^2.
\end{aligned}$$

The point $(\tilde{z}_n, \tilde{w}_n)$ must satisfy two conditions (see Figure 4.1). First, the pseudo-hyperbolic step is bounded: $d_{\mathbb{H}^N}(Z_n, Z_{n+1}) \leq a$, which will take form

$$\left|\frac{\tilde{z}_n - t_n}{\tilde{z}_n + t_n}\right|^2 + \frac{4t_n\|\tilde{w}_n\|^2}{|\tilde{z}_n + t_n|^2} \leq a^2. \tag{4.4}$$

Second, by Julia's lemma (4.1)

$$t_{n+1} = \operatorname{Re} \tilde{z}_n - \|\tilde{w}_n\|^2 \leq ct_n. \tag{4.5}$$

Using (4.4) and (4.5) we obtain

$$\begin{aligned}
|\tilde{z}_n - t_n|^2 + 4t_n \operatorname{Re} \tilde{z}_n &\leq a^2|\tilde{z}_n + t_n|^2 - 4t_n\|\tilde{w}_n\|^2 + 4t_n(ct_n + \|\tilde{w}_n\|^2), \\
|\tilde{z}_n - t_n|^2 + 4t_n \operatorname{Re} \tilde{z}_n &\leq a^2|\tilde{z}_n + t_n|^2 + 4ct_n^2, \\
|\tilde{z}_n + t_n|^2 &\leq a^2|\tilde{z}_n + t_n|^2 + 4ct_n^2, \\
|\tilde{z}_n + t_n|^2 &\leq \frac{4ct_n^2}{1 - a^2}, \\
|\tilde{x}_n + t_n|^2 + |\tilde{y}_n|^2 &\leq \frac{4ct_n^2}{1 - a^2}.
\end{aligned}$$

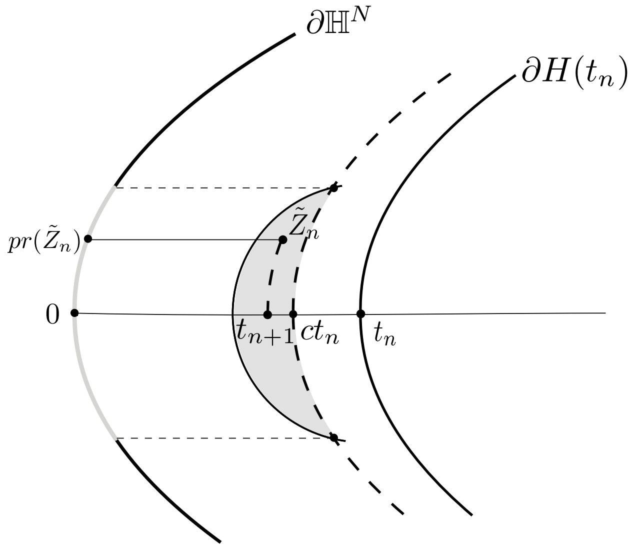

Figure 4.1: *The restriction on the point $\tilde{Z}_n = h_n(Z_{n+1})$ and its projection on the boundary of the Siegel domain. The shaded area represents the intersection of the solutions of (4.4) and (4.5).*

Thus

$$\tilde{x}_n \leq \frac{2t_n\sqrt{c}}{\sqrt{1-a^2}} - t_n = C_1 t_n, \tag{4.6}$$

$$|\tilde{y}_n| \leq \frac{2t_n\sqrt{c}}{\sqrt{1-a^2}} = C_2 t_n, \tag{4.7}$$

$$\|\tilde{w}_n\|^2 < \tilde{x}_n \leq C_1 t_n, \tag{4.8}$$

with C_1 and C_2 independent of n. Note that we must have $d_{\mathbb{H}^N}(ct_n, t_n) \leq d_{\mathbb{H}^N}(\tilde{Z}_n, (t_n, 0)) \leq a$, otherwise the backward-iteration sequence will not exist. It follows that $4c > 1 - a^2$ and $C_1 > 0$.

Now

$$pr(Z_n) = (i\,\mathrm{Im}\,z_n + \|w_n\|^2, w_n)$$

and

$$\begin{aligned} pr(Z_{n+1}) &= pr(h_n^{-1}(\tilde{z}_n, \tilde{w}_n)) \\ &= \left(i\,\mathrm{Im}(\tilde{z}_n + z_n) + 2\,\mathrm{Im}\,\langle \tilde{w}_n, w_n\rangle + \|\tilde{w}_n + w_n\|^2, \tilde{w}_n + w_n\right). \end{aligned}$$

$$
\begin{aligned}
pr(Z_{n+1}) - pr(Z_n) &= \left(i\operatorname{Im}\tilde{z}_n + 2\operatorname{Im}\langle\tilde{w}_n, w_n\rangle + \|\tilde{w}_n + w_n\|^2 - \|w_n\|^2, \tilde{w}_n\right) \\
&= \left(i\operatorname{Im}\tilde{z}_n + 2\langle\tilde{w}_n, w_n\rangle + \|\tilde{w}_n\|^2, \tilde{w}_n\right). \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
\|pr(Z_{n+1}) - pr(Z_n)\|^2 &= \left|i\operatorname{Im}\tilde{z}_n + 2\langle\tilde{w}_n, w_n\rangle + \|\tilde{w}_n\|^2\right|^2 + \|\tilde{w}_n\|^2 \\
&\leq \left(|\tilde{y}_n| + 2\|\tilde{w}_n\|\|w_n\| + \|\tilde{w}_n\|^2\right)^2 + \|\tilde{w}_n\|^2 \\
&\leq (C_2 t_n + 2C_1 t_n\|w_n\| + C_1 t_n)^2 + C_1 t_n \leq \tilde{C}^2 t_n,
\end{aligned}
$$

using (4.7), (4.8) and the fact that $t_n \to 0$, and assuming that $\|w_n\|$ is bounded.

Thus it is enough to show now that $\|w_n\| \leq C_3$, for a finite C_3 independent of n. Note that $w_{n+1} = w_n + \tilde{w}_n\ \forall n$ and thus

$$
\begin{aligned}
\|w_n\| &\leq \|\tilde{w}_{n-1}\| + \|\tilde{w}_{n-2}\| + \ldots + \|\tilde{w}_0\| + \|w_0\| \\
&\leq \sqrt{C_1}\left(\sqrt{t_{n-1}} + \sqrt{t_{n-2}} + \ldots + \sqrt{t_0}\right) + \|w_0\| \\
&\leq \sqrt{C_1}\sqrt{t_0}\left(\sqrt{c^{n-1}} + \sqrt{c^{n-2}} + \ldots + 1\right) + \|w_0\| \\
&\leq \frac{\sqrt{C_1}\sqrt{t_0}}{1-\sqrt{c}} + \|w_0\| =: C_3.
\end{aligned}
$$

□

Now we want to show that $\{Z_n\}$ stays in the Koranyi region with vertex q. Without loss of generality, take $q = 0$. A Koranyi region with vertex 0 in $\mathbb{H}^N$ must be the image under the Cayley transform of a Koranyi region with vertex $(-1, 0)$ in $\mathbb{B}^N$, i.e., a set of the form

$$
\left\{(z, w) \in \mathbb{H}^N \,\middle|\, \frac{|1 - \langle\mathcal{C}^{-1}(z, w), (-1, 0)\rangle|}{1 - \|\mathcal{C}^{-1}(z, w)\|} < M\right\}.
$$

Since $1 < 1 + \|\mathcal{C}^{-1}(z, w)\| < 2$, membership in this set is implied by

$$
\frac{|1 - \langle\mathcal{C}^{-1}(z, w), (-1, 0)\rangle|}{1 - \|\mathcal{C}^{-1}(z, w)\|^2} < \frac{M}{2}.
$$

The left-hand side is

$$
\frac{\left|1 + \frac{z-1}{z+1}\right|}{1 - \left|\frac{z-1}{z+1}\right|^2 - \frac{4\|w\|^2}{|z+1|^2}} = \frac{|z+1+z-1||z+1|}{|z+1|^2 - |z-1|^2 - 4\|w\|^2} = \frac{2|z||z+1|}{4\operatorname{Re} z - 4\|w\|^2};
$$

thus we need to find M independent of n such that for $Z_n = (z_n, w_n) \in \mathbb{H}^N$

$$\frac{|z_n||z_n+1|}{(\operatorname{Re} z_n - \|w_n\|^2)} < M.$$

Since $|z_n+1|$ is greater than 1 and is bounded near 0, and $\operatorname{Re} z_n - \|w_n\|^2 = t_n$, it is sufficient to show that $|z_n| \leq Ct_n$ for some constant C independent of n. Using Lemma 4.1, similarly to (4.3) we have

$$\begin{aligned}\|pr(Z_n)\| = \|pr(Z_n) - q\| &= \lim_{k\to\infty} \|pr(Z_n) - pr(Z_{n+k})\| \\ &\leq \sum_{j=0}^{\infty} \|pr(Z_{n+j}) - pr(Z_{n+j+1})\| \leq \tilde{C} \sum_{j=0}^{\infty} \sqrt{t_{n+j}} \leq \frac{\tilde{C}\sqrt{t_n}}{1-\sqrt{c}},\end{aligned}$$

so $\|pr(Z_n)\|^2 = |\operatorname{Im} z_n + \|w_n\|^2|^2 + \|w_n\|^2 \leq (\frac{\tilde{C}}{1-\sqrt{c}})^2 t_n = C_4 t_n$. It follows that $\|w_n\|^2 \leq C_4 t_n$. If there is a bound

$$\left|\operatorname{Im} z_n + \|w_n\|^2\right| = |z_n - t_n| \leq C_5 t_n, \tag{4.10}$$

then

$$|z_n| \leq |z_n - t_n| + t_n \leq (C_5+1)t_n,$$

and Z_n must stay in the appropriate Koranyi region. It is therefore enough to show (4.10).

Denote $pr_1(Z_n) = \operatorname{Im} z_n + \|w_n\|^2$, which is the first component of $pr(Z_n)$. As in (4.9)

$$pr_1(Z_{n+1}) - pr_1(Z_n) = i\tilde{y}_n + \|\tilde{w}_n\|^2 + 2\langle \tilde{w}_n, w_n\rangle$$

and thus

$$\begin{aligned}|pr_1(Z_{n+1}) - pr_1(Z_n)| &\leq |\tilde{y}_n| + \|\tilde{w}_n\|^2 + 2\|\tilde{w}_n\|\|w_n\| \\ &\leq C_2 t_n + C_1 t_n + 2\sqrt{C_1 t_n}\sqrt{C_4 t_n} = C_6 t_n.\end{aligned}$$

$$\begin{aligned}|pr_1(Z_n) - 0| &= \lim_{k\to\infty} |pr_1(Z_n) - pr_1(Z_{n+k})| \leq \sum_{k=0}^{\infty} |pr_1(Z_{n+k}) - pr_1(Z_{n+k+1})| \\ &\leq C_6 \sum_{k=0}^{\infty} t_{n+k} \leq C_6 \sum_{k=0}^{\infty} c^k t_n \leq C_5 t_n,\end{aligned}$$

which proves (4.10).

Now we will show that Julia's lemma (Theorem 2.6) is applicable to the point q. Once again, assume that $q = (-1, 0)$ in $\mathbb{B}^N$ or $q = 0$ in $\mathbb{H}^N$.

$$\liminf_{Z\to(-1,0)} \frac{1 - \|f(Z)\|}{1 - \|Z\|} \leq \liminf_{n\to\infty} \frac{1 - \|Z_n\|^2}{1 - \|Z_{n+1}\|^2}.$$

The latter liminf in $\mathbb{H}^N$ will take the form

$$\liminf_{n\to\infty} \frac{\operatorname{Re} z_n - \|w_n\|^2}{\operatorname{Re} z_{n+1} - \|w_{n+1}\|^2} \frac{|z_{n+1} + 1|^2}{|z_n + 1|^2} = \liminf_{n\to\infty} \frac{t_n}{t_{n+1}}.$$

It is enough to show that $t_{n+1} \geq K t_n$ for some constant K. Since $d(Z_n, Z_{n+1}) \leq a$, $H(t_{n+1})$ must intersect the pseudo-hyperbolic sphere (4.4), and thus

$$\frac{t_n - t_{n+1}}{t_n + t_{n+1}} \leq a,$$

and it follows that

$$t_{n+1} \geq \frac{1-a}{1+a} t_n,$$

so (1.2.2) holds with finite multiplier

$$\alpha \leq \frac{1+a}{1-a}. \tag{4.11}$$

Remark 4.2. There is another way to show that q is a BRFP with finite multiplier $\alpha \leq \frac{1+a}{1-a}$: the boundary dilatation coefficient α at $q \in \mathbb{B}^N$ can be written as

$$\begin{aligned}\frac{1}{2}\log\alpha &= \liminf_{Z\to q} \left[k_{\mathbb{B}^N}(0, Z) - k_{\mathbb{B}^N}(0, f(Z))\right] \\ &\leq \liminf_{n\to\infty} \left[k_{\mathbb{B}^N}(0, Z_{n+1}) - k_{\mathbb{B}^N}(0, Z_n)\right],\end{aligned}$$

and

$$\left[k_{\mathbb{B}^N}(0, Z_{n+1}) - k_{\mathbb{B}^N}(0, Z_n)\right] \leq k_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a',$$

where $a' = \frac{1}{2}\log\frac{1+a}{1-a}$, and (4.11) follows.

Now we will show that there is also a lower bound on α:

$$\alpha \geq \frac{1}{c}, \tag{4.12}$$

where $c < 1$.

Consider the image of 0 in $\mathbb{B}^N$ and denote $f(0) = (z_0, w_0)$. Since $0 \in \partial H((1,0),1)$ (here $H((1,0),1)$ is a horosphere centered at the Denjoy-Wolff point $(1,0)$ of radius 1), by Julia's lemma applied to $(1,0)$, $f(0) \in \overline{H}((1,0),c)$, where $c < 1$. This horosphere is a Euclidean ellipsoid, centered at $(\frac{1}{1+c}, 0)$, whose intersection with the 1-dimensional subspace, generated by $e_1 = (1,0)$ is a disk of radius $\frac{c}{1+c}$ (see (2.5)). Thus

$$\operatorname{Re} z_0 \geq \frac{1-c}{1+c}.$$

In a similar way, by Julia's lemma applied to $q = (-1,0)$, $f(0) \in \overline{H}((-1,0),\alpha)$ and

$$\operatorname{Re} z_0 \leq \frac{\alpha - 1}{\alpha + 1},$$

so we have

$$\frac{\alpha - 1}{\alpha + 1} \geq \frac{1-c}{1+c},$$

which is equivalent to $c\alpha \geq 1$ and (4.12) follows. □

Proof of Theorem 3.1 (attracting-elliptic case). Without loss of generality assume 0 is the Denjoy-Wolff point. We will need the following result on the growth of function f near the boundary of the ball:

Lemma 4.3. *Let f be a self-map of the unit ball $\mathbb{B}^N$ fixing 0 and not unitary on any slice. Fix $r_0 > 0$ and define $M(r) := \max \|f(r\mathbb{B}^N)\|$, $r \in [r_0, 1)$. Then there exists $c = c(r_0) < 1$ such that*

$$\frac{1-r}{1-M(r)} \leq c \quad \forall r \in [r_0, 1) \tag{4.13}$$

Proof. Assume the opposite: $\forall c < 1\ \exists z = z(c)$ with $\|z\| \geq r_0$ such that

$$\frac{1 - \|z\|}{1 - \|f(z)\|} > c$$

Construct the sequence $z_n := z(\frac{n-1}{n})$. Let z_0 be a partial limit of $\{z_n\}$. If $z_0 \in \mathbb{B}^N$, then $f(z_0) \in \mathbb{B}^N$ and

$$\frac{1 - \|z_0\|}{1 - \|f(z_0)\|} \geq 1 \quad \Leftrightarrow \quad 1 - \|z_0\| \geq 1 - \|f(z_0)\| \quad \Leftrightarrow \quad \|f(z_0)\| \geq \|z_0\|,$$

which is a contradiction, since $\|z_0\| \geq r_0 > 0$ by construction. Thus $z_0 \in \partial\mathbb{B}^N$ and we pick a subsequence $z_{n_k} \to z_0$. Then

$$\limsup_{k\to\infty} \frac{1 - \|z_{n_k}\|}{1 - \|f(z_{n_k})\|} \geq 1 \quad \Leftrightarrow \quad \liminf_{k\to\infty} \frac{1 - \|f(z_{n_k})\|}{1 - \|z_{n_k}\|} \leq 1$$

Applying Julia's lemma to the point $z_0 \in \partial\mathbb{B}^N$, we obtain that $\exists w_0 \in \partial\mathbb{B}^N$ such that $\forall R > 0$ $f(H(z_0, R)) \subseteq H(w_0, R)$, where $H(z, R)$ is a horosphere centered at z of radius R.

Pick R small enough such that $0 \notin \overline{H(z_0, R)}$. Let ξ be a point in $\overline{H(z_0, R)}$, closest to the origin. Since $f(\xi) \in \overline{H(w_0, R)}$, we have $\|f(\xi)\| \geq \|\xi\|$ (the horospheres have the same radius). Contradiction. □

Denote the distance to the boundary by $t_n := 1 - \|Z_n\|$. By Lemma (4.3) we have

$$t_{n+k} \leq c^k t_n \quad \forall n, k \geq 0, \tag{4.14}$$

where $c := c(\|Z_0\|)$ is as in Lemma 4.3.

Thus $t_n \leq c^n t_0 \to 0$ as n tends to infinity and the sequence $\{Z_n\}_{n=0}^{\infty}$ must tend to the boundary of the ball. Now denote by ϕ_n the angle between Z_n and Z_{n+1} seen from the origin (which is also the arc-length between radial projections of Z_n and Z_{n+1} on the boundary of the ball - see Figure 4.2).

Because $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a$, by (2.4) Z_{n+1} must be inside of the pseudo-hyperbolic ball of radius a centered at Z_n, which is the Euclidean ellipsoid centered at $\frac{1-a^2}{1-a^2\|Z_n\|^2} Z_n$ and largest semiaxis $a\sqrt{\frac{1-\|Z_n\|^2}{1-a^2\|Z_n\|^2}}$, so as Z_n tends to the boundary, for some finite constant C_7,

$$\phi_n \leq C_7(1 - \|Z_n\|)^{1/2} = C_7\sqrt{t_n}. \tag{4.15}$$

Then the arc-length between $\frac{Z_n}{\|Z_n\|}$ and $\frac{Z_{n+k}}{\|Z_{n+k}\|}$ does not exceed

$$\sum_{j=0}^{k} \phi_{n+j} \leq C_7 \sum_{j=0}^{k} \sqrt{t_{n+j}} \leq C_7\sqrt{t_n} \sum_{j=0}^{k} c^{k/2} \leq C_7 \frac{1}{1-\sqrt{c}} \sqrt{t_n},$$

which tends to 0 when n tends to infinity, so the sequence of projections must converge to some point on the boundary; denote it q. Thus the sequence Z_n must tend to q.

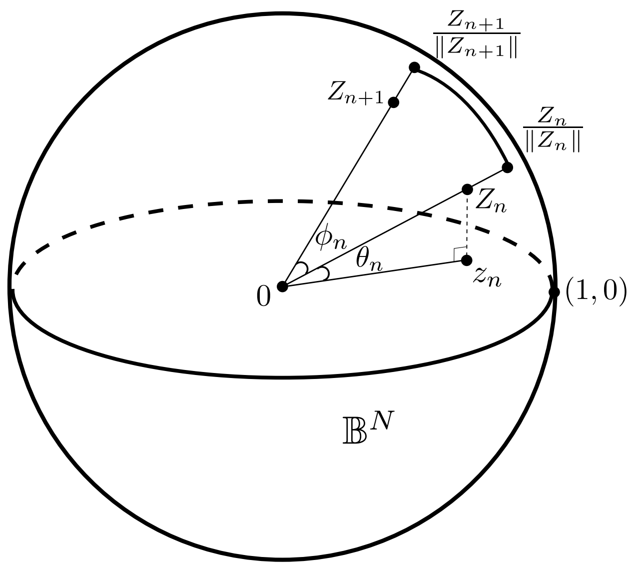

Figure 4.2: *Two consecutive points Z_n and Z_{n+1} and their radial projections on the boundary of the ball.*

The next step is to show that Z_n stays in a Koranyi region centered at q. Without loss of generality assume $q = (1, 0)$ and denote $Z_n = (z_n, w_n) \in \mathbb{C} \times \mathbb{C}^{N-1}$. We need to show that

$$\frac{|1 - z_n|}{1 - \|Z_n\|} < M \tag{4.16}$$

for some $M > 1$. By (4.14) and (4.15), The arc-length between $(1, 0)$ and the projection of Z_n on the boundary is bounded by

$$\sum_{j=n}^{\infty} \phi_j \leq C_7 \sum_{j=n}^{\infty} \sqrt{t_j} \leq C_8 \sqrt{t_n}. \tag{4.17}$$

Let θ_n be the angle between Z_n and z_n (i.e., the angle between Z_n and the plane spanned by $(1, 0)$). By (4.17), $\theta_n \leq C_8 \sqrt{t_n}$. Then

$$\begin{aligned} 1 - |z_n| = 1 - \|Z_n\| \cos\theta_n &= 1 - \cos\theta_n + \cos\theta_n - \|Z_n\| \cos\theta_n \\ &\leq 1 - \cos\theta_n + 1 - \|Z_n\| \leq C_9 t_n, \end{aligned}$$

since $1 - \cos\theta_n = \frac{\theta_n^2}{2} + o(\theta_n^3)$ as $n \to \infty$.

Since $d_{\mathbb{D}}(z_n, z_{n+1}) \le d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \le a$ and the pseudo-hyperbolic disk centered at z_n of radius a is a Euclidean disk with center $w = \frac{1-a^2}{1-a^2|z_n|^2} z_n$ and radius $r = \frac{1-|z_n|^2}{1-a^2|z_n|^2} a$,

$$|\operatorname{Arg} z_n - \operatorname{Arg} z_{n+1}| \approx \sin |\operatorname{Arg} z_n - \operatorname{Arg} z_{n+1}| \le \frac{r}{|w|} = \frac{a}{|z_n|} \frac{1-|z_n|^2}{1-a^2} \le C_{10} t_n$$

Now

$$|\operatorname{Arg} z_n| = |\operatorname{Arg} z_n - \operatorname{Arg} 1| \le \sum_{k=n}^{\infty} |\operatorname{Arg} z_k - \operatorname{Arg} z_{k+1}| \le \sum_{k=n}^{\infty} C_{10} t_k \le C_{11} t_n$$

and

$$|1 - z_n|^2 = (\operatorname{Im} z_n)^2 + (1 - \operatorname{Re} z_n)^2 = |z_n|^2 \sin^2 \operatorname{Arg} z_n + (1 - |z_n| \cos \operatorname{Arg} z_n)^2 \le$$

$$\sin^2 \operatorname{Arg} z_n + (1 - \cos \operatorname{Arg} z_n + 1 - |z_n|)^2 \le C_{12} t_n^2,$$

and (4.16) follows.

For the conclusion of Julia's lemma to hold we need to prove that

$$\liminf_{Z \to (1,0)} \frac{1 - f(\|Z\|)}{1 - \|Z\|} < \infty.$$

Since $\{Z_n\}_{n=0}^{\infty}$ is a backward-iteration sequence tending to $(1, 0)$,

$$\liminf_{Z \to (1,0)} \frac{1 - f(\|Z\|)}{1 - \|Z\|} \le \liminf_{n \to \infty} \frac{1 - \|Z_n\|}{1 - \|Z_{n+1}\|},$$

and it is enough to show that the latter liminf is finite. Note that Z_{n+1} must be in the (Euclidean) ellipsoid centered at $\frac{1-a^2}{1-a^2\|Z_n\|^2} Z_n$ with radius $r = \frac{1-|Z_n|^2}{1-a^2|Z_n|^2} a$ in the subspace generated by Z_n, and $R = a\sqrt{\frac{1-\|Z_n\|^2}{1-a^2\|Z_n\|^2}}$ in the dimensions orthogonal to Z_n, by (2.4). Thus the point W, closest to the boundary, must have norm

$$\|W\| = \frac{1-a^2}{1-a^2\|Z_n\|^2} \|Z_n\| + \frac{1-\|Z_n\|^2}{1-a^2\|Z_n\|^2} a = \frac{(\|Z_n\| + a)(1 - a\|Z_n\|)}{1 - a^2\|Z_n\|^2}$$
$$= \frac{\|Z_n\| + a}{1 + a\|Z_n\|}$$

and

$$1 - \|Z_{n+1}\| \ge 1 - \|W\| = 1 - \frac{\|Z_n\| + a}{1 + a\|Z_n\|} = \frac{(1-a)(1-\|Z_n\|)}{1 + a\|Z_n\|}.$$

Thus

$$\frac{1 - \|Z_n\|}{1 - \|Z_{n+1}\|} \leq \frac{1 + a\|Z_n\|}{1 - a} \leq \frac{1 + a}{1 - a},$$

and Julia's lemma holds with multiplier $\alpha \leq \frac{1+a}{1-a}$. The lower bound on the multiplier $\alpha \geq \frac{1}{c}$ is a direct consequence of Lemma (4.3).

Note that the above results will hold when c is replaced with $c(\|Z_n\|)$ (as in Lemma 4.3) $\forall n \geq 0$, and since $\|Z_n\| \to 1$, for

$$c := \lim_{r_0 \to 1} c(r_0).$$

□

Chapter 5

Construction of a special backward-iteration sequence

It was shown in the previous chapter that any backward-iteration sequence with bounded hyperbolic step tends to a BRFP. Now we will show that any isolated BRFP is a limit of a special backward-iteration sequence. This special backward-iteration sequence will be a cornerstone in the construction of the conjugation near a BRFP.

We will follow an idea similar to the one in the 1-dimensional case outlined in [27]. Note that in one dimension BRFPs with multipliers bounded by a given constant are isolated, as follows from a theorem of Cowen and Pommerenke [15]. Here we will have to impose this as a hypothesis, since not all BRFPs are isolated in higher dimensions (see Example 9.3).

Lemma 5.1. *Let f be an analytic self-map of $\mathbb{B}^N$ and $(1,0)$ be a BRFP for f with multiplier $1 < \alpha < \infty$, isolated from all those BRFP's whose multipliers are less than or equal to α. Then there exists a backward-iteration sequence $\{Z_n\}_{n=0}^{\infty}$ tending to $(1,0)$ such that*

$$d(Z_n, Z_{n+1}) \leq a := \frac{\alpha - 1}{\alpha + 1}.$$

We will need the following result on the behavior of the radial and tangential components of f near the BRFP $(1,0)$:

Lemma 5.2. *Let $f : \mathbb{B}^N \to \mathbb{B}^N$ be analytic and $(1,0)$ be a fixed point for f with multiplier α (in the sense of Julia's lemma). Then the following functions are bounded in every Koranyi region:*

(1) $\dfrac{1-\pi_1(f(Z))}{1-\pi_1(Z)}$,

(2) $\dfrac{f(Z)-\pi_1(f(Z))(1,0)}{|1-\pi_1(Z)|^{1/2}}$,

where $\pi_1(Z) = \langle Z, (1,0)\rangle$. *Moreover, the function (1) has restricted* K*-limit* α *at* $(1,0)$*, and the function (2) has restricted* K*-limit* 0 *at* $(1,0)$.

Proof. Apply theorem 2.13 (i) and (ii) to the boundary fixed point $(1,0)$. □

Proof of Lemma 5.1. Let D be a small enough (Euclidean) closed ball centered at $(1,0)$ that does not contain the Denjoy-Wolff point of f or any other BRFP of f. Let $a_k = (\alpha^k-1)/(\alpha^k+1)$ and

$$H(a_k) = \left\{ Z \in \mathbb{B}^N : \frac{|1-\langle Z, e_1\rangle|^2}{1-\|Z\|^2} \leq \frac{(1-a_k)^2}{1-a_k^2} = \alpha^{-k} \right\},$$

i.e., a horosphere whose intersection with the 1-dimensional subspace generated by $e_1 = (1,0)$ is a disk with diameter with endpoints $(a_k, 0)$ and $(1,0)$. Let n_0 be the smallest integer such that $H(a_{n_0}) \subseteq D$ and $r_k = a_{n_0+k}$. (We will identify r_k with $(r_k, 0) \in \mathbb{B}^N$; that should cause no confusion). Also let $H_k = H(r_k)$, $J = \partial D \cap \mathbb{B}^N$ and γ_n be the line segment connecting r_k and $f(r_k)$.

For each k, the sequence $\{f_n(r_k)\}_n$ converges to the Denjoy-Wolff point of f, hence eventually leaves D. So there exists a smallest integer n_k such that $f_{n_k}(\gamma_k)$ intersects J. By Julia's lemma (Theorem 2.6), $f(H_{k+1}) \subseteq H_k$, so $f_j(\gamma_k)$ cannot intersect J for $j = 1, 2, \ldots, k-1$ and thus $n_k \geq k$.

Claim. $d(r_k, f(r_k)) \xrightarrow[k\to\infty]{} a$.

By Lemma 5.2,

$$\lim_{k\to\infty} \frac{1-\pi_1(f(r_k))}{1-r_k} = \alpha,$$

and by the definition of multiplier

$$\liminf_{k\to\infty} \frac{1-\|f(r_k)\|}{1-r_k} \geq \alpha. \tag{5.1}$$

By (2.2), the pseudo-hyperbolic distance d in $\mathbb{B}^N$ must satisfy the relation:

$$1-d^2(r_k,f(r_k))=\frac{(1-r_k^2)(1-\|f(r_k)\|^2)}{|1-r_k\pi_1(f(r_k))|^2}=\frac{(1+r_k)(1+\|f(r_k)\|)\dfrac{1-\|f(r_k)\|}{1-r_k}}{\left|\dfrac{1-r_k\pi_1(f(r_k))}{1-r_k}\right|^2}.$$

Now

$$\frac{1-r_k\pi_1(f(r_k))}{1-r_k}=\frac{1-r_k+r_k-r_k\pi_1(f(r_k))}{1-r_k}=1+r_k\frac{1-\pi_1(f(r_k))}{1-r_k}\longrightarrow 1+\alpha,$$

and so

$$\liminf_{k\to\infty}\left(1-d^2(r_k,f(r_k))\right)\geq\frac{4\alpha}{(1+\alpha)^2}$$

or

$$\limsup_{k\to\infty} d(r_k,f(r_k))\leq\frac{\alpha-1}{\alpha+1}=a.$$

We will need the following inequality for $d_k:=d(r_k,f(r_k))$:

$$\frac{1-\|f(r_k)\|}{1-r_k}\leq\frac{1+d_k}{1-r_kd_k}. \tag{5.2}$$

In fact, this is an instance of a more general inequality:

Claim 5.3. *For all $Z,W\in\mathbb{B}^N$ and $d:=d_{\mathbb{B}^N}(Z,W)$*

$$\frac{1-||W\|}{1-\|Z\|}\leq\frac{1+d}{1-d\|Z\|}$$

Proof. Let Δ be a closed hyperbolic ball centered at Z of (pseudo-hyperbolic) radius $d=d_{\mathbb{B}^N}(Z,W)$. By (2.4) this is a Euclidean ellipsoid, centered at $\frac{1-d^2}{1-d^2\|Z\|^2}Z$ and a disk of radius $\frac{1-\|Z\|^2}{1-d^2\|Z\|^2}d$, when intersected with the subspace generated by Z. Thus the point which is closest to the origin must be in the subspace generated by Z, and have modulus

$$\frac{1-d^2}{1-d^2\|Z\|^2}\|Z\|-\frac{1-\|Z\|^2}{1-d^2\|Z\|^2}d=\frac{(\|Z\|-d)(1+d\|Z\|)}{1-d^2\|Z\|^2}=\frac{\|Z\|-d}{1-d\|Z\|}.$$

Since $W\in\Delta$,

$$1-\|W\|\leq 1-\frac{\|Z\|-d}{1-d\|Z\|}=\frac{1+d}{1-d\|Z\|}(1-\|Z\|),$$

$$\frac{1-\|W\|}{1-\|Z\|}\leq\frac{1+d}{1-d\|Z\|}.$$

□

By taking *limsup* of both sides of (5.2),

$$\limsup_{k\to\infty} \frac{1-\|f(r_k)\|}{1-r_k} \leq \frac{1+a}{1-a} = \alpha,$$

so this together with (5.1) shows that $\lim_{k\to\infty} \frac{1-\|f(r_k)\|}{1-r_k} = \alpha$ and $\lim_{k\to\infty} d(r_k, f(r_k)) = a$.

The final steps in the construction are exactly the same as in the proof of lemma 1.4 in [27]. □

Lemma 5.4. *If $\{Z_n\}_{n=1}^{\infty}$ is a backward-iteration sequence which tends to $e_1 = (1,0)$ (BRFP with multiplier $\alpha > 1$) and $d(Z_n, Z_{n+1}) \leq a = \frac{\alpha-1}{\alpha+1}$, then its image in the Siegel domain $\mathbb{H}^N$ must possess the following properties:*

$$\lim_{n\to\infty} \frac{\operatorname{Re} z_n}{t_n} = 1, \tag{5.3}$$

$$\lim_{n\to\infty} \frac{\operatorname{Im} z_n}{t_n} = 0, \tag{5.4}$$

$$\lim_{n\to\infty} \frac{\|w_n\|^2}{t_n} = 0, \tag{5.5}$$

$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = \alpha, \tag{5.6}$$

where $t_n := \operatorname{Re} z_n - \|w_n\|^2$. In particular, the sequence $\{Z_n\}$ is special, i.e.,

$$\lim_{n\to\infty} \frac{\|Z_n - \langle Z_n, e_1\rangle e_1\|^2}{1 - \|\langle Z_n, e_1\rangle e_1\|^2} = 0.$$

Proof. By definition of multiplier

$$\liminf_{n\to\infty} \frac{1-\|Z_n\|}{1-\|Z_{n+1}\|} \geq \alpha = \frac{1+a}{1-a}.$$

Applying Claim 5.3 to Z_n, Z_{n+1} and $r_n = d(Z_n, Z_{n+1})$, we have

$$\frac{1-\|Z_n\|}{1-\|Z_{n+1}\|} \leq \frac{1+r_n}{1-r_n\|Z_{n+1}\|} \leq \frac{1+a}{1-a\|Z_{n+1}\|}.$$

Taking lim sup of both sides,

$$\frac{1-\|Z_n\|}{1-\|Z_{n+1}\|} \to \alpha$$

or, in the Siegel domain,

$$\frac{t_n}{t_{n+1}} \to \alpha,$$

so (5.6) is proved. Here we are going to use a slightly different version of the Cayley transform:

$$\mathcal{C}^{-1}(z,w) := \left(\frac{1-z}{1+z}, \frac{2w}{1+z}\right),$$

so that the BRFP $(1,0)$ will be mapped to $\mathcal{C}(1,0) = (0,0)$.

Consider the images of two consecutive points Z_n and Z_{n+1} under the automorphism $h_n : (z,w) := (z - i\operatorname{Im} z_n + \|w_n\|^2 - 2\langle w, w_n\rangle, w - w_n)$, s.t. $h_n(Z_n) = (t_n, 0)$ and denote $(\tilde{z}_n, \tilde{w}_n) := h_n(Z_{n+1})$. h_n does not change the pseudo-hyperbolic distance in $\mathbb{H}^N$, so $d((t_n,0),(\tilde{z}_n,\tilde{w}_n)) = d(Z_n, Z_{n+1}) \leq a$, which is

$$\|\tilde{z}_n - t_n\|^2 + 4t_n\|\tilde{w}_n\|^2 \leq a^2\|\tilde{z}_n + t_n\|^2,$$

$$\|\tilde{z}_n - t_n\|^2 + 4t_n(\operatorname{Re}\tilde{z}_n - t_{n+1}) \leq a^2\|\tilde{z}_n + t_n\|^2,$$

$$(1-a^2)\|\tilde{z}_n + t_n\|^2 \leq 4t_n t_{n+1},$$

$$\left|\frac{\tilde{z}_n}{t_n} + 1\right|^2 \leq \frac{4t_{n+1}}{t_n(1-a^2)}.$$

Taking limsup of both sides and using (5.6),

$$\limsup_{n\to\infty}\left|\frac{\tilde{z}_n}{t_n} + 1\right|^2 = \limsup_{n\to\infty}\left(\left|\frac{\operatorname{Re}\tilde{z}_n}{t_n} + 1\right|^2 + \left|\frac{\operatorname{Im}\tilde{z}_n}{t_n}\right|^2\right) \leq \left(1 + \frac{1}{\alpha}\right)^2.$$

Since $\operatorname{Re}\tilde{z}_n = t_{n+1} + \|\tilde{w}_n\|^2 \geq t_{n+1}$,

$$\limsup_{n\to\infty}\left(\left|\frac{t_{n+1}}{t_n} + 1\right|^2 + \left|\frac{\operatorname{Im}\tilde{z}_n}{t_n}\right|^2\right) \leq \left(1 + \frac{1}{\alpha}\right)^2,$$

$$\left(\frac{1}{\alpha} + 1\right)^2 + \limsup_{n\to\infty}\left|\frac{\operatorname{Im}\tilde{z}_n}{t_n}\right|^2 \leq \left(1 + \frac{1}{\alpha}\right)^2.$$

So,

$$\frac{\operatorname{Im} \tilde{z}_n}{t_n} \to 0, \tag{5.7}$$

which implies

$$\frac{\operatorname{Re} \tilde{z}_n}{t_n} \to \frac{1}{\alpha} \tag{5.8}$$

and

$$\frac{\|\tilde{w}_n\|^2}{t_n} = \frac{\operatorname{Re} \tilde{z}_n}{t_n} - \frac{t_{n+1}}{t_n} \to 0. \tag{5.9}$$

Now $w_{n+1} = w_n + \tilde{w}_n$, $w_{n+k} = w_n + \sum_{j=0}^{k-1} \tilde{w}_{n+j}$ $\forall k \geq 1$. Consequently,

$$\|w_{n+k}\| \geq \|w_n\| - \sum_{j=0}^{k-1} \|\tilde{w}_{n+j}\|,$$

$$0 \geq \|w_n\| - \sum_{j=0}^{\infty} \|\tilde{w}_{n+j}\|,$$

$$\|w_n\| \leq \sum_{j=0}^{\infty} \|\tilde{w}_{n+j}\|.$$

Since $\frac{t_n}{t_{n+1}} \to \alpha > 1$, we can pick ε such that $\alpha - \varepsilon > 1$; then for large enough n, $t_{n+1} \leq \frac{t_n}{\alpha-\varepsilon}$ and $t_{n+j} \leq \frac{t_n}{(\alpha-\varepsilon)^j}$.

Now by (5.9), $\forall \delta > 0$ $\exists N = N(\delta)$ s.t. $\|\tilde{w}_n\| \leq \delta\sqrt{t_n}$ for $n \geq N$. Then

$$\|w_n\| \leq \sum_{j=0}^{\infty} \delta\sqrt{t_{n+j}} \leq \delta \sum_{j=0}^{\infty} \frac{\sqrt{t_n}}{(\alpha-\varepsilon)^{j/2}} = \delta S\sqrt{t_n},$$

where S is the finite value of the infinite sum. So

$$\frac{\|w_n\|^2}{t_n} \to 0$$

and

$$\frac{\operatorname{Re} z_n}{t_n} = \frac{t_n + \|w_n\|^2}{t_n} \to 1.$$

Similarly, because $\operatorname{Im} z_{n+1} = \operatorname{Im} z_n + \operatorname{Im} \tilde{z}_n + 2 \operatorname{Im} \langle \tilde{w}_n, w_n \rangle$, we have

$$|2 \operatorname{Im} \langle \tilde{w}_n, w_n \rangle| \le 2 \|\tilde{w}_n\| \|w_n\|,$$

and using (5.7), (5.9) and (5.5),

$$\frac{\operatorname{Im} z_n}{t_n} \to 0.$$

The condition (2.8) for $(z_n, w_n) \to (1, 0)$ to be special in $\mathbb{B}^N$ is that

$$\lim_{n\to\infty} \frac{\|w_n\|^2}{1 - |z_n|^2} = 0$$

or, in $\mathbb{H}^N$ that

$$\lim_{n\to\infty} \frac{\frac{4\|w_n\|^2}{|1+z_n|^2}}{1 - \left|\frac{1-z_n}{1+z_n}\right|^2} = \lim_{n\to\infty} \frac{\|w_n\|^2}{\operatorname{Re} z_n} = 0.$$

But

$$\lim_{n\to\infty} \frac{\|w_n\|^2}{\operatorname{Re} z_n} = \lim_{n\to\infty} \frac{\frac{\|w_n\|^2}{t_n}}{\frac{\operatorname{Re} z_n}{t_n}} = 0.$$

□

Chapter 6

Conjugation at a boundary repelling fixed point

The aim of this chapter is to solve equation (3.1) in $\mathbb{B}^N$, where η is an automorphism of $\mathbb{B}^N$ with the same dilatation coefficient α at BRFP as f and $\psi : \mathbb{B}^N \to \mathbb{B}^N$ is an analytic map with some regularity at the BRFP. As in [27], the conjugating map will be obtained via the sequence of iterates f_n composed with appropriate automorphisms of $\mathbb{B}^N$. It will be convenient to build almost the entire construction in $\mathbb{H}^N$ with BRFP 0.

We will start with several technical statements.

Using the backward-iteration sequence $(z_n, w_n) \to 0$ as in Lemma 5.4 with $t_n = \operatorname{Re} z_n - \|w_n\|^2$, define a sequence of automorphisms τ_n of $\mathbb{H}^N$ as $\tau_n := h_n^{-1} \circ \delta_n^{-1}$, where

$$h_n(z, w) = (z + \|w_n\|^2 - iy_n - 2\langle w, w_n \rangle, w - w_n),$$

$$h_n^{-1}(z, w) = (z + \|w_n\|^2 + iy_n + 2\langle w, w_n \rangle, w + w_n),$$

$$\delta_n(z, w) = (\frac{z}{t_n}, \frac{w}{\sqrt{t_n}}),$$

$$\delta_n^{-1}(z, w) = (t_n z, \sqrt{t_n} w),$$

and, as usual, $y_n := \operatorname{Im} z_n$. Then $\tau_n(1, 0) = (z_n, w_n)$.

Lemma 6.1. *Let $\eta_k(z, w) := (\alpha^k z, \alpha^{k/2} w)$ and τ_n be defined as above. (Note that η_1 coincides with η in (3.1).) Then for each k*

(1) $\tau_{n+k}^{-1} \circ \tau_n \to \eta_k$*, uniformly on compact subsets of* $\mathbb{H}^N$*, as* n *tends to infinity, and*

(2) $\tau_{n+1}^{-1} \circ \eta^{-1} \circ \tau_n(z,w) \to (z,w)$*, uniformly on compact sets of* $\mathbb{H}^N$*, as* n *tends to infinity.*

Proof. Using the definition of τ_n and properties (5.3), (5.4), (5.5) and (5.6),

$$\tau_{n+k}^{-1} \circ \tau_n(z,w) = \delta_{n+k} \circ h_{n+k} \circ h_n^{-1} \circ \delta_n^{-1}(z,w) =$$

$$\Bigg(\frac{t_n}{t_{n+k}} z + \frac{\|w_n\|^2}{t_{n+k}} + i\frac{y_n}{t_{n+k}} + 2\frac{\sqrt{t_n}}{t_{n+k}}\langle w, w_n\rangle + \frac{\|w_{n+k}\|^2}{t_{n+k}} - i\frac{y_{n+k}}{t_{n+k}}$$

$$-\frac{2}{t_{n+k}}\left\langle \sqrt{t_n} w + w_n, w_{n+k}\right\rangle, \frac{\sqrt{t_n} w + w_n - w_{n+k}}{\sqrt{t_{n+k}}}\Bigg) \xrightarrow[n\to\infty]{} \left(\alpha^k z, \alpha^{k/2} w\right)$$
$$= \eta_k(z,w).$$

$$\tau_{n+1}^{-1} \circ \eta^{-1} \circ \tau_n(z,w) = \delta_{n+1} \circ h_{n+1} \circ \eta^{-1} \circ h_n^{-1} \circ \delta_n^{-1}(z,w) =$$
$$\Bigg(\frac{t_n}{t_{n+1}\alpha} z + \frac{\|w_n\|^2}{t_{n+1}\alpha} + i\frac{y_n}{t_{n+1}\alpha} + 2\frac{\sqrt{t_n}}{t_{n+1}\alpha}\langle w, w_n\rangle + \frac{\|w_{n+1}\|^2}{t_{n+1}} - i\frac{y_{n+1}}{t_{n+1}}$$
$$-\frac{2}{t_{n+1}}\left\langle \sqrt{t_n} w + w_n, w_{n+1}\right\rangle, \frac{\sqrt{t_n} w + w_n}{\sqrt{t_{n+1}}\sqrt{\alpha}} - \frac{w_{n+1}}{\sqrt{t_{n+1}}}\Bigg) \xrightarrow[n\to\infty]{} (z,w).$$

□

Claim 6.2. $\tau_n(z,w) \xrightarrow[n\to\infty]{} 0$ *and stays in a Koranyi region uniformly on compact ubsets of* $\mathbb{H}^N$*.*

Proof.

$$\tau_n(z,w) = \left(t_n z + \|w_n\|^2 + iy_n + 2\left\langle \sqrt{t_n} w, w_n\right\rangle, \sqrt{t_n} w + w_n\right).$$

The condition for (z,w) to be in a Koranyi region with vertex 0 in $\mathbb{H}^N$ is that a bound $M < \infty$ exist such that

$$\frac{|z|}{\operatorname{Re} z - \|w\|^2} < M.$$

For $\tau(z,w)$:

$$\frac{\left|t_n z + \|w_n\|^2 + iy_n + 2\left\langle \sqrt{t_n}w, w_n\right\rangle\right|}{t_n \operatorname{Re} z + \|w_n\|^2 + 2\sqrt{t_n}\operatorname{Re}\langle w, w_n\rangle - \|\sqrt{t_n}w + w_n\|^2}$$

$$= \frac{\left|z + \frac{\|w_n\|^2}{t_n} + \frac{iy_n}{t_n} + 2\left\langle w, \frac{w_n}{\sqrt{t_n}}\right\rangle\right|}{\operatorname{Re} z + \frac{\|w_n\|^2}{t_n} + 2\operatorname{Re}\left\langle w, \frac{w_n}{\sqrt{t_n}}\right\rangle - \|w + \frac{w_n}{\sqrt{t_n}}\|^2} \xrightarrow[n\to\infty]{} \frac{|z|}{\operatorname{Re} z - \|w\|^2}.$$

The limit is bounded on compact subsets of $\mathbb{H}^N$, so $\tau_n(z, w)$ belong to some Koranyi region. □

Claim 6.3. *Let $\phi := f \circ \eta^{-1}$ in $\mathbb{B}^N$. Then*

$$\liminf_{z\to(1,0)} \frac{1 - \|\phi(z)\|}{1 - \|z\|} = 1$$

and Lemma 5.2 is applicable.

Proof.

$$\begin{aligned}\liminf_{z\to(1,0)} \frac{1 - \|\phi(z)\|}{1 - \|z\|} &= \liminf_{z\to(1,0)} \frac{1 - \|f \circ \eta^{-1}(z)\|}{1 - \|\eta^{-1}(z)\|} \lim_{z\to(1,0)} \frac{1 - \|\eta^{-1}(z)\|}{1 - \|z\|} \\ &= \liminf_{z\to(1,0)} \frac{1 - \|f(z)\|}{1 - \|z\|} \lim_{z\to(1,0)} \frac{1 - \|\eta^{-1}(z)\|}{1 - \|z\|} = \alpha \cdot \frac{1}{\alpha} = 1,\end{aligned}$$

since η^{-1} is an automorphism that fixes $(1, 0)$ and

$$\lim_{z\to(1,0)} \frac{1 - \|\eta^{-1}(z)\|}{1 - \|z\|} = \lim_{z\to(1,0)} \frac{1 - \|\eta^{-1}(z)\|^2}{1 - \|z\|^2} = \lim_{(z,w)\to(0,0)} \frac{1 - \|\mathcal{C}^{-1}(\frac{z}{\alpha}, \frac{w}{\sqrt{\alpha}})\|^2}{1 - \|\mathcal{C}^{-1}(z, w)\|^2}$$

$$= \lim_{(z,w)\to(0,0)} \frac{1 - \left|\frac{1 - z/\alpha}{1 + z/\alpha}\right|^2 - \frac{4\|w\|^2}{\alpha|1 + z/\alpha|^2}}{1 - \left|\frac{1-z}{1+z}\right|^2 - \frac{4\|w\|^2}{|1+z|^2}} = \lim_{(z,w)\to(0,0)} \frac{\frac{\operatorname{Re} z - \|w\|^2}{\alpha}}{\operatorname{Re} z - \|w\|^2} \cdot \frac{|1 + z|^2}{\left|1 + \frac{z}{\alpha}\right|} = \frac{1}{\alpha}.$$

□

Now consider the normal family $\{f_n \circ \tau_n \circ p_1\}$, where $p_1(z, w) := (z, 0)$.

Claim 6.4. *The sequence $\tau_n \circ p_1(z, w) \to 0$ is restricted uniformly on compact subsets of $\mathbb{H}^N$.*

Proof. Note that $\tau_n \circ p_1(z,w) = (t_n z + \|w_n\|^2 + iy_n, w_n)$.

Following Definition 2.9, we need to show that $\tau_n \circ p_1(z,w)$ is special in $\mathbb{H}^N$:

$$\lim_{n\to\infty} \frac{\|w_n\|^2}{\mathrm{Re}(t_n z + \|w_n\|^2 + iy_n)} = \lim_{n\to\infty} \frac{\frac{\|w_n\|^2}{t_n}}{\mathrm{Re}\, z + \frac{\|w_n\|^2}{t_n}} = 0,$$

and that the projection on the first component is non-tangential, i.e., that

$$\frac{|t_n z + \|w_n\|^2 + iy_n|}{\mathrm{Re}(t_n z + \|w_n\|^2 + iy_n)}$$

is bounded from above. But

$$\lim_{n\to\infty} \frac{|t_n z + \|w_n\|^2 + iy_n|}{\mathrm{Re}(t_n z + \|w_n\|^2 + iy_n)} = \lim_{n\to\infty} \frac{\left|z + \frac{\|w_n\|^2}{t_n} + i\frac{y_n}{t_n}\right|}{\mathrm{Re}\, z + \frac{\|w_n\|^2}{t_n}} = \frac{|z|}{\mathrm{Re}\, z},$$

so it is bounded uniformly on compact subsets of $\mathbb{H}^N$. □

By applying Lemma 5.2 to the function $\phi = f \circ \eta^{-1}$ and the sequence $\tau_n \circ p_1(z,w)$, we obtain the following

Lemma 6.5.

$$\lim_{n\to\infty} d\left(\tau_n(p_1(z,w)), \phi(\tau_n(p_1(z,w)))\right) = 0.$$

Proof. Denote $(u_n, v_n) := \tau_n(z,0)$ and $(\tilde{u}_n, \tilde{v}_n) := \phi(\tau_n(z,0))$. Then the restricted K-limits (1) and (2) in Lemma 5.2 when translated to $\mathbb{H}^N$ become

$$\lim_{n\to\infty} \frac{\tilde{u}_n}{u_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\|\tilde{v}_n\|^2}{u_n} = 0.$$

Since $\lim_{n\to\infty} \frac{u_n}{t_n} = z$,

$$\lim_{n\to\infty} \frac{\tilde{u}_n}{t_n} = z \quad \text{and} \quad \lim_{n\to\infty} \frac{\|\tilde{v}_n\|^2}{t_n} = 0.$$

Now $d((u_n, v_n), (\tilde{u}_n, \tilde{v}_n))^2 = 1 - \dfrac{4(\mathrm{Re}\, u_n - \|v_n\|^2)(\mathrm{Re}\, \tilde{u}_n - \|\tilde{v}_n\|^2)}{|\tilde{u}_n + \bar{u}_n - 2\langle \tilde{v}_n, v_n\rangle|^2}$.

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{4(\operatorname{Re} u_n - \|v_n\|^2)(\operatorname{Re} \tilde{u}_n - \|\tilde{v}_n\|^2)}{|\tilde{u}_n + \bar{u}_n - 2\langle \tilde{v}_n, v_n\rangle|^2} \\
&= \lim_{n\to\infty} \frac{4(\operatorname{Re} \frac{u_n}{t_n} - \frac{\|v_n\|^2}{t_n})(\operatorname{Re} \frac{\tilde{u}_n}{t_n} - \frac{\|\tilde{v}_n\|^2}{t_n})}{\left|\frac{\tilde{u}_n}{t_n} + \frac{\bar{u}_n}{t_n} - 2\left\langle \frac{\tilde{v}_n}{\sqrt{t_n}}, \frac{v_n}{\sqrt{t_n}}\right\rangle\right|^2} \\
&= \frac{4(\operatorname{Re} z - 0)(\operatorname{Re} z - 0)}{|z + \bar{z} + 0|^2} = 1,
\end{aligned}
$$

that is,

$$\lim_{n\to\infty} d(\tau_n(z,0), \phi(\tau_n(z,0))) = 0.$$

□

Proof of Theorem 3.6. Consider the normal family $\{f_n \circ \tau_n \circ p_1\}$ and let ψ be one of its normal limits. Then, by Schwarz's lemma (2.3)

$$
\begin{aligned}
&d(f_n \circ \tau_n(z,0), f_{n+1} \circ \tau_{n+1}(z,0)) \le d(\tau_n(z,0), f \circ \tau_{n+1}(z,0)) \\
&\le d(\tau_n(z,0), f \circ \eta^{-1} \circ \tau_n(z,0)) + d(\eta^{-1} \circ \tau_n(z,0), \tau_{n+1}(z,0)). \qquad (6.1)
\end{aligned}
$$

The first summand in (6.1) tends to zero by lemma 6.5, and the second does by part (2) of lemma 6.1, so

$$d(f_n \circ \tau_n(z,0), f_{n+1} \circ \tau_{n+1}(z,0)) \to 0$$

as n tends to infinity. It follows that if a subsequence $\{f_{n_k} \circ \tau_{n_k} \circ p_1\}$ converges uniformly on compact subsets of $\mathbb{H}^N$ to ψ, then so does the subsequence $\{f_{n_k+1} \circ \tau_{n_k+1} \circ p_1\}$. By construction

$$f_{n_k+1} \circ \tau_{n_k+1} \circ p_1 = f \circ f_{n_k} \circ \tau_{n_k+1} \circ p_1,$$

where the left hand-side tends to ψ, and it is enough to show that $f_{n_k} \circ \tau_{n_k+1} \circ p_1 \to \psi \circ \eta^{-1}$ to prove (3.1). Note that η^{-1} and p_1 are linear functions with diagonal matrices and therefore commute, so $f_{n_k} \circ \tau_{n_k} \circ \eta^{-1} \circ p_1 \to \psi \circ \eta^{-1}$ and it is enough to show that

$$d\left(f_{n_k} \circ \tau_{n_k} \circ \eta^{-1} \circ p_1(Z), f_{n_k} \circ \tau_{n_k+1} \circ p_1(Z)\right) \to 0.$$

Applying Schwarz's lemma again,

$$d\left(f_{n_k}\circ\tau_{n_k}\circ\eta^{-1}\circ p_1(Z), f_{n_k}\circ\tau_{n_k+1}\circ p_1(Z)\right)\le d\left(\tau_{n_k}\circ\eta^{-1}(z,0),\tau_{n_k+1}(z,0)\right)$$

$$= d\left(\tau_{n_k+1}^{-1}\circ\tau_{n_k}\circ\eta^{-1}(z,0),(z,0)\right)\to 0$$

by statement (1) of Lemma 6.1, so we have

$$\psi = f\circ\psi\circ\eta^{-1},$$

which is equivalent to (3.1).

All we are left to show is that ψ fixes 0. Define $a_k := \left(\alpha^{-k},0\right)\in\mathbb{H}^N$, and note that $p_1(a_k)=a_k$. Then by definition of the sequence Z_n and τ_n and Schwarz's lemma

$$d\left(f_n\circ\tau_n(a_k), Z_k\right) = d\left(f_n\circ\tau_n(a_k), f_n(Z_{n+k})\right)\le d\left(a_k,\tau_n^{-1}\circ\tau_{n+k}(1,0)\right)$$

$$= d\left(\eta_k^{-1}(1,0),\tau_n^{-1}\circ\tau_{n+k}(1,0)\right)\to 0,$$

for any $k=1,2,\ldots$, as n tends to infinity, by (1) of Lemma 6.1. Thus we have

$$\psi(a_k)=Z_k.$$

Define the sequence

$$g_n(Z) := \tau_n^{-1}\circ\psi\circ\eta_n^{-1}(Z). \tag{6.2}$$

Then $g_n((1,0))=(1,0)$ and $g_n(a_1)=\tau_n^{-1}(\tau_{n+1}(1,0))\to\eta^{-1}(1,0)=a_1$, as n tends to infinity, by part (1) of Lemma 6.1. Hence any normal limit of g_n fixes $(1,0)$ and a_1, and, by Corollary 2.4, must fix the entire subspace, containing $(1,0)$ and a_1, i.e., the set $\left\{(z,0)\in\mathbb{H}^N\right\}$. Note that $\psi(z,w)=\psi(z,0)$ and by (6.2) $g_n(z,w)=g_n(z,0)$, so $g_n\to p_1$, since each g_n fixes a_1.

Consider a straight line segment connecting $(1,0)$ and $(0,0)$. Obviously it is special curve and, by Theorem 2.12, ψ will have restricted K-limit 0 at 0 if

$$\lim_{t\to 0}\psi(t,0)=0. \tag{6.3}$$

By (6.2), $\psi = \tau_n \circ g_n \circ \eta_n$. Consider a straight line segment connecting $(\alpha^{-(n+1)}, 0)$ to $(\alpha^{-n}, 0)$. It will be mapped by η_n to a segment $[(\alpha^{-1}, 0), (1, 0)]$. Pick a point $(t, 0)$ on this segment. Then

$$\|\tau_n(g_n(t,0))\| \leq \|\tau_n(g_n(t,0)) - \tau_n(t,0)\| + \|\tau_n(t,0)\| \xrightarrow[n\to\infty]{} 0,$$

since $g_n(t,0) \to (t,0)$, $\tau_n(t,0) \to 0$ uniformly in t and $\frac{\partial \tau_n}{\partial z}$ is bounded, and (6.3) follows. $\square$

Now we can show that $\{f_n \circ \tau_n \circ p_1\}$ actually converges to ψ. By Schwarz's lemma, (3.1) and (6.2)

$$\begin{aligned} &d\left(f_n \circ \tau_n \circ p_1(z,w), \psi(z,w)\right) = d\left(f_n \circ \tau_n \circ p_1(z,w), \psi \circ \eta_n \circ \eta_n^{-1}(z,w)\right) \\ =&d\left(f_n \circ \tau_n \circ p_1(z,w), f_n \circ \psi \circ \eta_n^{-1}(z,w)\right) \leq d\left(\tau_n \circ p_1(z,w), \psi \circ \eta_n^{-1}(z,w)\right) \\ =&d\left(p_1(z,w), g_n(z,w)\right) \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Chapter 7

Conjugation for expandable maps

In this chapter we will provide conjugation for the maps with some regularity at the BRFP. This class of maps was introduced in [7]:

Definition 7.1. Let $f : \mathbb{H}^N \to \mathbb{H}^N$ be holomorphic. We will call the map f *expandable* at 0 (write $f \in \mathcal{E}^1_{\mathbb{H}^N}(0)$), if f has the following expansion near 0:

$$f(z, w) = (\alpha z + o(|z|), Aw + o(|z|^{1/2})).$$

In particular, 0 is a fixed point of f.

By definition, the multiplier of f at 0 is

$$\liminf_{(z,w)\to 0} \frac{\operatorname{Re}(\alpha z + o(|z|)) - \|Aw + o(|z|^{1/2})\|^2}{\operatorname{Re} z - \|w\|^2} \leq \lim_{x_n \to 0} \frac{\alpha x_n + \operatorname{Re}\left(o(x_n)\right) - \|o(x_n^{1/2})\|^2}{x_n} = \alpha,$$

for any $(x_n, 0) \to 0$ with $x_n \in \mathbb{R}$; thus the multiplier is finite and does not exceed α. On the other hand, by part (1) of Lemma 5.2 the multiplier must be equal to

$$\lim_{n\to\infty} \frac{\alpha z_n + o(|z_n|)}{z_n},$$

for any restricted sequence $(z_n, w_n) \to 0$, i.e., the multiplier of f at 0 is indeed α.

Remark 7.2. Note that A cannot have eigenvalues $|a_{j,j}|^2 > \alpha$, because otherwise $f(\mathbb{H}^N) \not\subset \mathbb{H}^N$.

Proof of Theorem 3.7. The construction is essentially the same as in chapter 6. We modify the definition of τ_n as follows: $\tau_n := \Omega^{-n} \circ h_n^{-1} \circ \delta_n^{-1}$, where Ω is as in the statement of Theorem 3.7. The following two limits are a generalization of lemma 6.1:

$$\tau_{n+k}^{-1} \circ \tau_n(z,w) = \delta_{n+k} \circ h_{n+k} \circ \Omega^k \circ h_n^{-1} \circ \delta_n^{-1}(z,w) =$$
$$= \Bigg(\frac{t_n}{t_{n+k}} z + \frac{\|w_n\|^2}{t_{n+k}} + i\frac{y_n}{t_{n+k}} + 2\frac{\sqrt{t_n}}{t_{n+k}} \langle w, w_n \rangle + \frac{\|w_{n+k}\|^2}{t_{n+k}} - i\frac{y_{n+k}}{t_{n+k}}$$
$$- \frac{2}{t_{n+k}} \left\langle \Omega^k(\sqrt{t_n} w + w_n), w_{n+k} \right\rangle, \frac{\Omega^k(\sqrt{t_n} w + w_n) - w_{n+k}}{\sqrt{t_{n+k}}} \Bigg) \xrightarrow[n\to\infty]{} \left(\alpha^k z, \Omega^k \alpha^{k/2} w\right) =: \eta_k(z,w).$$

(Here η_k differs from previous η_k by rotation by Ω^k.)

$$\tau_{n+1}^{-1} \circ \eta^{-1} \circ \tau_n(z,w) = \delta_{n+1} \circ h_{n+1} \circ \Omega^{n+1} \circ \eta^{-1} \circ \Omega^{-n} \circ h_n^{-1} \circ \delta_n^{-1}(z,w) =$$
$$\Bigg(\frac{t_n}{t_{n+1}\alpha} z + \frac{\|w_n\|^2}{t_{n+1}\alpha} + i\frac{y_n}{t_{n+1}\alpha} + 2\frac{\sqrt{t_n}}{t_{n+1}\alpha} \langle w, w_n \rangle + \frac{\|w_{n+1}\|^2}{t_{n+1}} - i\frac{y_{n+1}}{t_{n+1}}$$
$$- \frac{2}{t_{n+1}} \left\langle \sqrt{t_n} w + w_n, w_{n+1} \right\rangle, \frac{\sqrt{t_n} w + w_n}{\sqrt{t_{n+1}}\sqrt{\alpha}} - \frac{w_{n+1}}{\sqrt{t_{n+1}}} \Bigg) \xrightarrow[n\to\infty]{} (z,w).$$

Now $\phi(z,w) := f \circ \eta^{-1}(z,w) = f(\alpha^{-1}z, \Omega^{-1}\alpha^{-1/2}w) = (z + o(|z|), \frac{\Omega^{-1}A}{\sqrt{\alpha}} w + o(|z|^{1/2}))$.

Let $p_{L+1}(z,w) = (z, w_1, \ldots, w_L, 0, \ldots, 0)$, i.e., projection on the first $1 + L$ dimensions.

Denote $(u_n, v_n) := \tau_n(p_{L+1}(z,w))$ and $(\tilde{u}_n, \tilde{v}_n) := \phi(\tau_n(p_{L+1}(z,w)))$. Then $u_n = t_n z + \|w_n\|^2 + iy_n + 2\left\langle \sqrt{t_n} p_L(w), w_n \right\rangle$ and $v_n = \Omega^{-n}(\sqrt{t_n} p_L(w) + w_n)$. Since

$$\lim_{n\to\infty} \frac{u_n}{t_n} = \lim_{n\to\infty} \frac{t_n z + \|w_n\|^2 + iy_n + 2\left\langle \sqrt{t_n} p_L(w), w_n \right\rangle}{t_n} = z,$$

$o(|u_n|) = o(t_n)$ and $o(|u_n|^{1/2}) = o(\sqrt{t_n})$, and, consequently, $\tilde{u}_n = u_n + o(t_n)$ and

$$\tilde{v}_n = \frac{\Omega^{-1}A}{\sqrt{\alpha}} v_n + o(\sqrt{t_n}) = \frac{\Omega^{-(n+1)}A\sqrt{t_n}}{\sqrt{\alpha}} p_L(w) + \frac{\Omega^{-(n+1)}A}{\sqrt{\alpha}} w_n + o\sqrt{t_n})$$
$$= \Omega^{-n}\sqrt{t_n} p_L(w) + o(\sqrt{t_n}).$$

The pseudo-hyperbolic distance in $\mathbb{H}^N$ is

$$d^2((u_n, v_n), (\tilde{u}_n, \tilde{v}_n)) = 1 - \frac{4(\operatorname{Re} u_n - \|v_n\|^2)(\operatorname{Re} \tilde{u}_n - \|\tilde{v}_n\|^2)}{|\tilde{u}_n + \bar{u}_n - 2\langle \tilde{v}_n, v_n\rangle|^2},$$

and because

$$\begin{aligned}
&\lim_{n\to\infty} \frac{4(\operatorname{Re} u_n - \|v_n\|^2)(\operatorname{Re} \tilde{u}_n - \|\tilde{v}_n\|^2)}{|\tilde{u}_n + \bar{u}_n - 2\langle \tilde{v}_n, v_n\rangle|^2} \\
=&\lim_{n\to\infty} \frac{(\operatorname{Re} \frac{u_n}{t_n} - \frac{\|v_n\|^2}{t_n})(\operatorname{Re} \frac{u_n}{t_n} + \frac{o(t_n)}{t_n} - \|\Omega^{-n} p_L(w) + \frac{o(\sqrt{t_n})}{\sqrt{t_n}}\|^2)}{\left|\operatorname{Re} \frac{u_n}{t_n} + \frac{o(t_n)}{t_n} - \left\langle \Omega^{-n} p_L(w) + \frac{o(\sqrt{t_n})}{\sqrt{t_n}}, \frac{v_n}{\sqrt{t_n}} \right\rangle\right|^2} \\
=&\frac{(\operatorname{Re} z - \|p_L(w)\|^2)(\operatorname{Re} z - \|p_L(w)\|^2)}{|\operatorname{Re} z - \langle \Omega^{-n} p_L(w), \Omega^{-n} p_L(w)\rangle|^2} = 1,
\end{aligned}$$

$d^2((u_n, v_n), (\tilde{u}_n, \tilde{v}_n)) \to 0$, i.e., a conclusion analogous to the statement of lemma 6.5 holds.

Now let ψ be any of the normal limits of $\{f_n \circ \tau_n \circ p_{L+1}\}$. The above computations show that if $f_{n_k} \circ \tau_{n_k} \circ p_{L+1}$ converges to ψ, then $f_{n_k+1} \circ \tau_{n_k+1} \circ p_{L+1}$ also converges to ψ. It is enough to show that $f_{n_k} \circ \tau_{n_k+1} \circ p_{L+1}$ converges to $\psi \circ \eta^{-1}$ uniformly on compact subsets of $\mathbb{H}^N$. Note that $\eta^{-1} \circ p_{L+1} = p_{L+1} \circ \eta^{-1}$. Because

$$\begin{aligned}
&d\left(f_{n_k} \circ \tau_{n_k} \circ \eta^{-1} \circ p_{L+1}(z, w), f_{n_k} \circ \tau_{n_k+1} \circ p_{L+1}(z, w)\right) \\
&= d\left(\tau_{n_k+1}^{-1} \circ \tau_{n_k} \circ \eta^{-1} \circ p_{L+1}(z, w), p_{L+1}(z, w)\right) \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

$$\lim_{n\to\infty} f_{n_k} \circ \tau_{n_k+1} \circ p_{L+1}(z, w) = \lim_{n\to\infty} f_{n_k} \circ \tau_{n_k} \circ \eta^{-1} \circ p_{L+1}(z, w) = \psi \circ \eta^{-1}(z, w),$$

and (3.1) holds.

By the same reasoning as in proof of Theorem (3.6), ψ fixes 0 in the sense of restricted K-limits. □

Remark 7.3. Note that in the case when the eigenvalues of A are equal to $\sqrt{\alpha}$, f will be conjugated to the same automorphism η as in Theorem 3.6, but the intertwining map ψ will be different (its image need not be one-dimensional).

Remark 7.4. Consider the hyperbolic map $f : \mathbb{H}^N \to \mathbb{H}^N$ with the Denjoy-Wolff point infinity and BRFP 0 with multiplier $1 < \alpha < \infty$: $f(z, w) = (\alpha z, 0)$. Clearly, the image of f is one-dimensional and from (3.1) we have that the image of ψ must be one-dimensional, so the result of Theorem 3.6 cannot be improved in general. For a less trivial example, one may consider $f(z, w) = (\alpha z, \beta w)$ with $0 < |\beta|^2 < \alpha$. Now the image of f has dimension N, but

$$\bigcap_{n=1}^{\infty} f_n(\mathbb{H}^N)$$

is a one-dimensional section of $\mathbb{H}^N$ and the range of the intertwining map ψ is also one-dimensional.

Chapter 8

Parabolic case

Recall that a self-map f of either the unit disk $\mathbb{D}$ or the unit ball $\mathbb{B}^N$ is called parabolic, if the Denjoy-Wolff point p is on the boundary and the multiplier at p is equal to 1.

Note that by Schwarz lemma, the pseudo-hyperbolic distance between two consecutive forward iterates $d(z_n, z_{n+1})$ is decreasing and thus has a limit d_∞. Whether this limit is positive or zero defines the behavior of the sequence and the function.

Definition 8.1. We will call a sequence $\{z_n\}$ a zero-step (respectively, non-zero-step) sequence, if $d_\infty = \lim d(z_n, z_{n+1}) = 0$ (respectively, $d_\infty > 0$).

In the one-dimensional case, as a consequence of the theorem of Pommerenke [30], zero-step and non-zero-step properties of a sequence of forward iterates do not depend on the choice of the starting point but depend on the function only, so we can call functions parabolic zero-step and parabolic non-zero step, respectively. It is still not known if the same is true in several variables.

Remark 8.2. Here we follow the terminology introduced in [26]; Pommerenke in [30] and [2] used the term "parabolic" for the parabolic non-zero-step case and "identity" for the parabolic zero-step case.

More about the parabolic non-zero-step and zero-step cases in one dimension, including backward iteration and examples, can be found in [26].

The crucial difference between parabolic non-zero-step and zero-step functions in the unit disk is that the former are conjugated to a translation in the half-plane (Pommerenke, [30]):

$$\psi \circ f(z) = \psi(z) + ib, \qquad \forall z \in \mathbb{H};$$

while the latter are conjugated to a translation in the whole plane ([2]).

In the unit disk, forward iterates of a parabolic non-zero-step function converge to the Denjoy-Wolff point tangentially (Remark 1, [30]), see Figure 8.1. In the parabolic zero-step

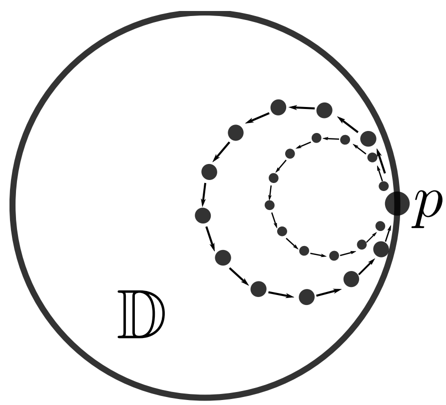

Figure 8.1: *Orbits converge tangentially to the Denjoy-Wolff point in the parabolic non-zero-step case.*

case, forward iterates may converge radially (Figure 8.2), but a complete classification of their behavior has still not been achieved.

We obtained the following result for the forward iteration sequences in the unit ball:

Claim 8.3. *Let f be a parabolic self-map of the unit ball $\mathbb{B}^N$ with the Denjoy-Wolff point $(1, 0)$. If the sequence of forward iterates $\{Z_n\}_{n=1}^{\infty}$ is restricted, then it must have zero step, i.e., $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \to 0$.*

Proof. Denote $Z_n = (z_n, w_n) \in \mathbb{C} \times \mathbb{C}^{N-1}$. Since Z_n is restricted, it is special, i.e.,

$$\lim_{n\to\infty} \frac{\|w_n\|^2}{1 - |z_n|^2} = 0 \tag{8.1}$$

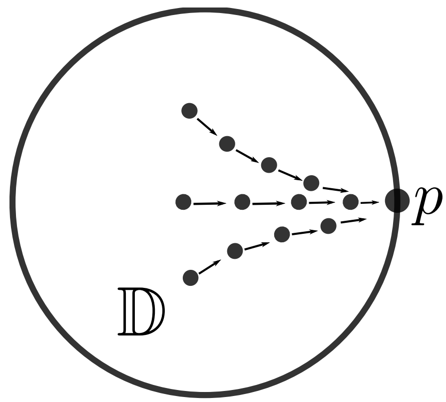

Figure 8.2: *In some cases, orbits converge radially to the Denjoy-Wolff point in the parabolic zero-step case.*

and projections on the first dimension z_n tend to 1 non-tangentially. Moreover, by Lemma 5.2

$$\lim_{n\to\infty} \frac{1-z_{n+1}}{1-z_n} = 1. \tag{8.2}$$

By (2.2), the pseudo-hyperbolic distance satisfies

$$\begin{aligned} 1 - d^2_{\mathbb{B}^N}(Z_n, Z_{n+1}) &= \frac{(1-\|Z_n\|^2)(1-\|Z_{n+1}\|^2)}{|1-\langle Z_n, Z_{n+1}\rangle|^2} \\ &= \frac{(1-|z_n|^2-\|w_n\|^2)(1-|z_{n+1}|^2-\|w_{n+1}\|^2)}{|1-z_n\overline{z_{n+1}}-\langle w_n, w_{n+1}\rangle|^2} \\ &= \frac{(1-\frac{\|w_n\|^2}{1-|z_n|^2})(1-\frac{\|w_{n+1}\|^2}{1-|z_{n+1}|^2})}{\left|\frac{1-z_n\overline{z_{n+1}}}{\sqrt{1-|z_n|^2}\sqrt{1-|z_{n+1}|^2}} - \left\langle \frac{w_n}{\sqrt{1-|z_n|^2}}, \frac{w_{n+1}}{\sqrt{1-|z_{n+1}|^2}}\right\rangle\right|^2} \end{aligned}$$

By (8.1), it is enough to show that

$$\left|\frac{1-z_n\overline{z_{n+1}}}{\sqrt{1-|z_n|^2}\sqrt{1-|z_{n+1}|^2}}\right| \to 1$$

which is equivalent to $d_{\mathbb{D}}(z_n, z_{n+1}) \to 0$.

$$d_{\mathbb{D}}(z_n, z_{n+1}) = \left|\frac{z_{n+1}-z_n}{1-\overline{z_n}z_{n+1}}\right| = \left|\frac{1-z_n-1+z_{n+1}}{1-\overline{z_n}+\overline{z_n}-\overline{z_n}z_{n+1}}\right| = \left|\frac{1-\frac{1-z_{n+1}}{1-z_n}}{\frac{1-\overline{z_n}}{1-z_n}+\overline{z_n}\frac{1-z_{n+1}}{1-z_n}}\right|$$

By (8.2), it is enough to show that the denominator is bounded away from 0, which is indeed the case when $\frac{1-\overline{z_n}}{1-z_n}$ is bounded away from -1.

$$\operatorname{Arg}\left(\frac{1-\overline{z_n}}{1-z_n}\right) = -2\operatorname{Arg}(1-z_n) \geq -\pi + \varepsilon,$$

for some $\varepsilon > 0$, because $z_n \to 1$ non-tangentially, and thus $\frac{1-\overline{z_n}}{1-z_n}$ stays away from -1. □

Remark 8.4. Since any non-tangential approach must be restricted (Lemma 2.11), it follows that every non-zero-step sequence must converge tangentially, and Claim 8.3 is a generalization of the classical one-dimensional result.

Chapter 9

Examples

In the beginning of this chapter we will describe all quadratic polynomials that map the two-dimensional Siegel domain

$$\mathbb{H}^2 := \left\{(z,w) \in \mathbb{C}^2 \,\middle|\, \operatorname{Re} z > |w|^2 \right\}$$

into itself while fixing 0, and we will completely characterize their dynamics. Some of these polynomials happen to have non-isolated BRFPs (see Example 9.3).

Claim 9.1. *A quadratic polynomial $f : \mathbb{C}^2 \to \mathbb{C}^2$ that fixes 0 maps $\mathbb{H}^2$ into $\mathbb{H}^2$ if and only if it is of the form $f(z,w) = (Az + Bw^2, Cw)$ with $A - |B| \geq |C|^2$.*

Proof. Consider the general form $f(z,w) = (f_1(z,w), f_2(z,w)) = (az + bw + cz^2 + dzw + ew^2, Az + Bw + Cz^2 + Dzw + Ew^2)$. First we will show that most coefficients must be 0.

Since $\operatorname{Re} f_1(z,w) > |f_2(z,w)|^2 \geq 0$, then $\operatorname{Re} f_1(z,0) = \operatorname{Re}(az + cz^2) > 0$ $\forall z$ such that $\operatorname{Re} z > 0$. When $z \to 0$, $az + cz^2 \sim az$, so $a > 0$. Now $\operatorname{Re} f_1(z,0) = |z|(a\cos(\operatorname{Arg} z) + |c||z|\cos(2\operatorname{Arg} z + \operatorname{Arg} c))$, we can choose $\operatorname{Arg} z$ such that $\cos(2\operatorname{Arg} z + \operatorname{Arg} c) < 0$ and $|z|$ large enough that $\operatorname{Re} f_1(z,0) < 0$, unless $|c| = 0$, when c must be 0.

Thus $f(z,0) = (az, Az + Cz^2)$, and we must have $a|z|\cos(\operatorname{Arg} z) > |z|^2|A + Cz|^2$ or $a\cos(\operatorname{Arg} z) > |z||A + Cz|^2$. The right hand side goes to ∞ as $|z| \to \infty$ unless $C = A = 0$.

Thus f must be of the form $f(z,w) = (az + bw + dzw + ew^2, Bw + Dzw + Ew^2)$. Consider the set $\{(t,1) \in \mathbb{C}^2 \mid t > 1\} \subset \mathbb{H}^2$. $f(t,1) = (at + dt + b + e, B + E + Dt)$ and $\operatorname{Re}(at + dt + b + e) < |B + E + Dt|^2$ for large enough t unless $D = 0$.

Now consider the set $\{(t^2+\varepsilon, t) \in \mathbb{C}^2 \mid t > 0\} \subset \mathbb{H}^2$. $\operatorname{Re} f_1(t^2+\varepsilon, t) \leq a(t^2+\varepsilon) + |b|t + |d|(t^2+\varepsilon)t + |e|t^2 < |Bt + Et^2|^2$ for large enough t unless $E = 0$.

To show that $d = 0$, consider $\{(z, w) \in \mathbb{C}^2 \mid z = t^{2+\varepsilon}, |w| = t, t > 1\} \subset \mathbb{H}^2$. On this set $\operatorname{Re} f_1(z, w) \leq at^{2+\varepsilon} + |b|t + |e|t^2 + |d|t^{3+\varepsilon} \cos(\operatorname{Arg} d + \operatorname{Arg} w)$. We can choose $\operatorname{Arg} w$ such that $\cos(\operatorname{Arg} d + \operatorname{Arg} w) < 0$ and choose t large enough to make $\operatorname{Re} f_1(z, w) < 0$, unless $d = 0$.

The last step is to show that $b = 0$. Consider

$$\left\{(z, w) \in \mathbb{C}^2 \mid z = t^{2-\varepsilon}, |w| = t, 1 > t > 0\right\} \subset \mathbb{H}^2.$$

On this set $\operatorname{Re} f_1(z, w) \leq at^{2-\varepsilon} + |b|t \cos(\operatorname{Arg} b + \operatorname{Arg} w) + |e|t^2$. We can choose $\operatorname{Arg} w$ such that $\cos(\operatorname{Arg} b + \operatorname{Arg} w) < 0$ and choose t close enough to 0 that $\operatorname{Re} f_1(z, w) < 0$, unless $b = 0$.

Thus f has only three nonzero terms, and (by changing notations) the function must have the form $f(z, w) = (Az + Bw^2, Cw)$. If $A - |B| \geq |C|^2$, then $\operatorname{Re}(Az + Bw^2) \geq A \operatorname{Re} z - |B||w|^2 > (A - |B|)|w|^2 \geq |C|^2|w|^2$ on $\mathbb{H}^2$ and hence $f(\mathbb{H}^2) \subseteq \mathbb{H}^2$. If $A - |B| < |C|^2$, we can choose $\operatorname{Arg} w$ such that $\cos(\operatorname{Arg} B + 2\operatorname{Arg} w) = -1$ and $\operatorname{Re} z = |w|^2 + \frac{\varepsilon}{A}|w|^2$, where $\varepsilon = |C|^2 - A + |B| > 0$, and then $\operatorname{Re}(Az + Bw^2) = A \operatorname{Re} z + |B||w|^2 \cos(\operatorname{Arg} B + 2\operatorname{Arg} w) = A \operatorname{Re} z - |B||w|^2 = (A - |B| + \varepsilon)|w|^2 = |C|^2|w|^2$; thus $f(\mathbb{H}^2) \not\subseteq \mathbb{H}^2$. □

Claim 9.2. *1. Aside from the trivial cases $A = 0$ (f must then be the zero map, because $A = 0$ forces $B = C = 0$) and $C = 0$ (f is then a one-dimensional projection) $f(z, w) = (Az + Bw^2, Cw)$ has a well-defined inverse on $\mathbb{H}^2$*

$$f^{-1}(z, w) = \left(\frac{z}{A} - \frac{B}{AC^2}w^2, \frac{w}{C}\right)$$

(though part of its image may be outside of the Siegel domain).

2. The n^{th} iterate of f has the form

$$f^{\circ n}(z, w) = \left(A^n z + \frac{A^n - C^{2n}}{A - C^2} Bw^2, C^n w\right).$$

Proof. (1) is obvious. (2) can be shown by induction. □

Now we will find the fixed points and classify the dynamical behavior of polynomials based on them.

Cases $C = 0$ (f a projection on the first dimension) and $B = 0$ (f a linear map) are trivial. So assume $B \neq 0$ and $C \neq 0$. To find the set of finite fixed points (either interior or on the boundary) we need to solve

$$\begin{cases} Az + Bw^2 = z \\ Cw = w \end{cases}$$

If $C = 1$, we can assume $A > 1$ (otherwise $B = 0$ and f is the identity). Then there are solutions $\left(-\dfrac{Bw^2}{A-1}, w\right)$. Since

$$\operatorname{Re}\left(-\frac{Bw^2}{A-1}\right) - |w|^2 \leq \frac{|B||w|^2}{A-1} - |w|^2 = \frac{|B|+1-A}{A-1}|w|^2 \leq 0,$$

any solution must be on the boundary of $\mathbb{H}^2$ and nonzero solutions exist iff $A = |B| + 1$. In this case, there are infinitely many fixed points on the boundary (see Example (9.3) below).

If $C \neq 1$, then nonzero solutions exist iff $A = 1$ and they have form $(z, 0)$. Thus we have interior fixed points.

If $C \neq 1$ and $A \neq 1$, then there are no fixed points inside of the domain and only two fixed points on the boundary (zero and infinity). One of them must be the Denjoy-Wolff point and the other a BRFP.

The dilatation coefficient at $(0, 0)$ is

$$\begin{aligned} c &= \liminf_{(z,w)\to(0,0)} \frac{\operatorname{Re}(Az + Bw^2) - |C|^2|w|^2}{\operatorname{Re} z - |w|^2} \\ &\geq \liminf_{(z,w)\to(0,0)} \frac{A\operatorname{Re} z - |B||w|^2 - |C|^2|w|^2}{\operatorname{Re} z - |w|^2} \\ &\geq \liminf_{(z,w)\to(0,0)} \frac{A\operatorname{Re} z - A|w|^2}{\operatorname{Re} z - |w|^2} = A \end{aligned}$$

and value A is attained for $z = t \to 0$ and $w = 0$. Consequently, $c = A$.

Thus if $A < 1$, then zero is the Denjoy-Wolff point of f and this is the hyperbolic case $c = A < 1$. If $A > 1$, then $(0, 0)$ is the BRFP with dilatation coefficient $A > 1$ and infinity must be the Denjoy-Wolff point. The dilatation coefficient at infinity is

$$\begin{aligned} c &= \liminf_{(z,w)\to\infty} \frac{\operatorname{Re}(Az+Bw^2)-|C|^2|w|^2}{\operatorname{Re} z-|w|^2} \frac{|z+1|^2}{|Az+Bw^2+1|^2} \\ &\le \lim_{t\to\infty} \frac{A\operatorname{Re} t}{t} \frac{|t+1|^2}{|At+1|^2} = \frac{1}{A}; \end{aligned}$$

thus $c \le \frac{1}{A} < 1$ and this is also the hyperbolic case.

Example 9.3 (Example of a quadratic function with non-isolated BRFP)**.** Consider the function $f(z,w) := (2z+w^2, w)$. Then $f^{\circ n}(z,w) = (2^n z + (2^n-1)w^2, w)$, the Denjoy-Wolff point is infinity and this is the hyperbolic case. The curve $\{(r^2, ir)|r \in \mathbb{R}\}$ is clearly the set of fixed points on the boundary. Any of those points can be mapped to $(0,0)$ by translation

$$h_r(z,w) := (z + r^2 + 2irw, w - ir)$$

with

$$h_r^{-1}(z,w) = (z + r^2 - 2irw, w + ir)$$

Then

$$\begin{aligned} h_r \circ f \circ h_r^{-1}(z,w) &= h_r \circ f(z + r^2 - 2irw, w + ir) \\ &= h_r(2z + r^2 - 2irw + w^2, w + ir) = (2z + w^2, w), \end{aligned}$$

i.e., the behavior of the function in any of those points is the same as in $(0,0)$.

The dilatation coefficient at zero is

$$c = \liminf_{(z,w)\to(0,0)} \frac{\operatorname{Re}(2z+w^2)-|w|^2}{\operatorname{Re} z-|w|^2} = 1 + \liminf_{(z,w)\to(0,0)} \frac{\operatorname{Re} z + \operatorname{Re}(w^2)}{\operatorname{Re} z - |w|^2} = 2.$$

Thus we have a set of BRFP's on the boundary with the same dilatation coefficient, none of them is isolated.

Remark 9.4. Though $(0,0)$ is a non-isolated BRFP for $f(z,w) := (2z+w^2, w)$, the statement of Lemma (5.1) still holds in this case. $Z_n = (\frac{1}{2^n}, 0)$ is clearly an example of a backward-iteration sequence with step $d = \frac{1}{3}$. Consequently, it is still possible to construct a conjugation as in Theorem 3.6.

Now we will describe another class of self-maps of $\mathbb{H}^2$; the construction of these will be based on a function on the one-dimensional half-plane $\mathbb{H}$.

Example 9.5. Let $\phi : \mathbb{H} \to \mathbb{H}$ be a holomorphic function of the right half-plane, of hyperbolic or parabolic type, with the Denjoy-Wolff point at infinity. Define a function f on $\mathbb{H}^2$ as $f(z,w) := (\phi(z-w^2)+w^2, w)$. This function is well-defined since $\forall (z,w) \in \mathbb{H}^2$ $\operatorname{Re}(z-w^2) \geq \operatorname{Re} z - |w|^2 > 0$. Moreover, by Julia's lemma in $\mathbb{H}$, $\operatorname{Re}\phi(z-w^2) \geq \operatorname{Re}(z-w^2)$ and thus $\operatorname{Re}(\phi(z-w^2)+w^2) \geq \operatorname{Re} z > |w|^2$, and the function f maps $\mathbb{H}^2$ into itself.

Claim 9.6. *Infinity is the Denjoy-Wolff point for f and f has the same type and same multiplier at infinity as ϕ. Moreover, if ϕ has a BRFP $y_0 i \in \partial\mathbb{H}$, then f has a 1-dimensional real submanifold $\{(y_0 i + t^2, t) | t \in \mathbb{R}\}$ of BRFPs.*

Proof. Iterates of f have a form $f^{\circ n}(z,w) = (\phi^{\circ n}(z-w^2)+w^2, w)$ and clearly the Denjoy-Wolff point is infinity. Assume ϕ has multiplier $c_1 \leq 1$ at infinity; then f has multiplier

$$\begin{aligned} c = &\liminf_{(z,w)\to\infty} \frac{\operatorname{Re}(\phi(z-w^2)+w^2) - |w|^2}{\operatorname{Re} z - |w|^2} \left| \frac{z+1}{\phi(z-w^2)+w^2+1} \right|^2 \\ \leq &\liminf_{z\to\infty} \frac{\operatorname{Re}\phi(z)}{\operatorname{Re} z} \left| \frac{z+1}{\phi(z)+1} \right|^2 = c_1. \end{aligned}$$

Using Julia's lemma (4.1) and the fact that $f(z,0) = (\phi(z), 0)$, we have

$$\operatorname{Re}\phi(z) \geq \frac{1}{c}, \operatorname{Re} z \qquad \textit{thatis}, \qquad \frac{\operatorname{Re}\phi(z)}{\operatorname{Re} z} \geq \frac{1}{c} \qquad \forall z \in \mathbb{H},$$

and, taking limits of both sides,

$$\frac{1}{c_1} \geq \frac{1}{c}.$$

Thus $c_1 = c$, the multipliers coincide and therefore the functions f and ϕ are of the same type (either both hyperbolic or both parabolic).

Now $f(y_0 i + t^2, t) = (\phi(y_0 i) + t^2, t) = (y_0 i + t^2, t)$ and $(y_0 i + t^2, t)$ is a BRFP for f $\forall t \in \mathbb{R}$. □

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