THE DOUBLE OF REPRESENTATIONS OF COHOMOLOGICAL HALL ALGEBRAS

by

Xinli Xiao

B.S., Wuhan University, Wuhan, China, 2007

M.S., Nankai University, Tianjin, China, 2010

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

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Abstract

Given a quiver $\mathcal{Q}$ with/without potential, one can construct an algebra structure on the cohomology of the moduli stacks of representations of $\mathcal{Q}$. The algebra is called Cohomological Hall algebra (COHA for short). One can also add a framed structure to quiver $\mathcal{Q}$, and discuss the moduli space of the stable framed representations of $\mathcal{Q}$. Through these geometric constructions, one can construct two representations of Cohomological Hall algebra of $\mathcal{Q}$ over the cohomology of moduli spaces of stable framed representations. One would get the double of the representations of Cohomological Hall algebras by putting these two representations together. This double construction implies that there are some relations between Cohomological Hall algebras and some other algebras.

In this dissertation, we focus on the quiver without potential case. We first define Cohomological Hall algebras, and then the above construction is stated under some assumptions. We computed two examples in detail: A_1-quiver and Jordan quiver. It turns out that A_1-COHA and its double representations are related to the half infinite Clifford algebra, and Jordan-COHA and its double representations are related to the infinite Heisenberg algebra. Then by the fact that the underlying vector spaces of these two COHAs are isomorphic to each other, we get a COHA version of Boson-Fermion correspondence.

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Acknowledgments

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Chapter 1

Introduction

Cohomological Hall algebra (COHA for short) was first introduced in [38]. There are at least two motivations for Cohomological Hall algebra: one from mathematics and one from physics.

The Donaldson-Thomas invariants were introduced by R. Thomas in [68]. K. Behrend later in [1] constructed the same invariants via integrating the Behrend function with respect to a measure given by the Euler characteristic. The new point of view revealed the "motivic nature" of the invariants, and many efforts are made to construct generalized Donaldson-Thomas invariants in a more general setting.

There were several attempts made. One is due to D. Joyce and Y. Song, who developed a framework to give a rigorous definition of $\mathbb{Z}$-valued Donaldson-Thomas invariants in the case of abelian category of coherent sheaves on CY 3-fold. See e.g. [28–33]. M. Kontsevich and Y. Soibelman established two theories to produce Donaldson-Thomas invariants with values in "motives" in the case of triangulated categories. One is stating in [36], and the other is using Cohomological Hall algebra stated in [38]. Both theories produce $\mathbb{Z}$-invariants as limits of motivic Donaldson-Thomas invariants, and they are connected in the case of quivers with potential. The theory of Donaldson-Thomas invariants for quivers with potential has also been studied extensively recently due to B. Davison, S. Meinhardt and M. Reineke. See e.g. [10, 11, 44, 46]. See also [45] for a review.

It was conjectured that the generating series for Donaldson-Thomas invariants takes the form of certain infinite products with exponents $\Omega(\alpha)$ with values in $\mathbb{Z}$ or in a certain Motivic ring ([36]). This conjecture is called the Integrality Conjecture. Although it is still open in the most general case, it is proved in many speical cases. In fact, M. Kontsecvich and Y. Soibelman proved it for an arbitrary Quillen-smooth algebra with potential which covers most of 3CY categories. See e.g. [38], [36].

It would be nice if one can construct the categorification of the Donaldson-Thomas invariants. In other words, one can construct an algebraic structure on some space closely related to the Donaldson-Thomas invariants. Once we have such an algebraic structure, the Integrality conjecture is automatically true, as well as a bunch of other good properties. The algebra rising here is the Cohomological Hall algebra. Therefore it can be treated as the categorification of the Donaldson-Thomas invariants from this point of view. This direction is studied extensively for the categories coming from quiver with potentials. See e.g. [57], [13], [8]. There are other attempts to categorify Donaldson-Thomas invariants. D. Joyce with several collaborators proposed an approach to the categorification of Donaldson-Thomas invariants based on the ideas of derived algebraic geometry. See [2] and references within.

Another motivation is from Physics. In supersymmetric field theories and string theories, there are special states called BPS states. These states are the states conserved by (some) supercharges. For that reason (Hilbert) spaces of BPS states appear mathematically as cohomology spaces of moduli spaces of geometrically defined objects (which correspond to certain fields in physics). Spaces of BPS states are stable under deformations of QFTs which connect mutually dual theories. For that reason physicists often check various dualities by comparing spaces of BPS states for dual theories. See e.g. [25, 26, 47].

In the study of certain models, G. Moore and J. Harvey computed certain one-loop integrals and found two interesting facts. First, these integrals were determined purely by the spectrum of BPS states, and second that the answers involved denominator formulae for Generalized Kac-Moody algebras of the type studied previously by R. Borcherds. Therefore it was natural to think that there was an algebraic structure that one could define on the

BPS states that would be related to the denominator formulae for a Generalized Kac-Moody algebra. See e.g. [25, 26]. See also [15] for work in the same direction.

Although physicists think of BPS states as single-particle states, it is convenient mathematically to consider the space of multi-particle BPS states. It was a suggestion of M. Kontsevich and Y. Soibelman to introduce an algebra structure on the latter space and call it COHA. Single particle states correspond to a certain set of generators of COHA. From this perspective Cohomological Hall algebras can be thought of as a mathematical implementation of the idea of BPS algebra, or more precisely, algebra of closed BPS states.

However, as stated by G. Moore in [48], current methods and results have not been carefully related to the actual properties of BPS wave functions of quiver quantum mechanics. Thus, to explain in detail the relation of Cohomological Hall algebras to the BPS states of quiver quantum mechanics is a very interesting question.

Besides the closed BPS states, there are also open BPS states. Conjecturally (see e.g. [65] and [24]) this algebra acts on the "space of open BPS states" which are often described in terms of cohomology of schemes. This is related to many physics models as well as mathematical models. Hence representation theory of Cohomological Hall algebras (BPS algebras) should lead to new results as well as new connections with geometric representation theory. There are some work in this direction. see e.g. [67] and references within.

Previous work

As mentioned in the previous section, COHA coming from quivers with/without potentials is of great interests. We would like to quickly go over the work on this topic.

Fix a quiver $\mathcal{Q} = (I, H)$. One considers the moduli stack $[M_{\mathbf{d}}/G_{\mathbf{d}}]$ of all the representations of $\mathcal{Q}$ of a fixed dimension vector $\mathbf{d}$ for all dimension vectors $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{|I|}$. M. Kontsevich and Y. Soibelman defines an associative algebraic structure on $\oplus_{\mathbf{d}} H^*([M_{\mathbf{d}}/G_{\mathbf{d}}])$ and call it the Cohomological Hall algebra associated to the quiver $\mathcal{Q}$. See Chapter 2 for details. It can be generalized to the smooth algebra with potential case. See [38].

Starting from a quiver $\mathcal{Q}$, one can construct another quiver $\overline{\mathcal{Q}}$ by adding the reverse arrow for each arrow of $\mathcal{Q}$. There is an algebra called the preprojective algebra $\Pi_{\mathcal{Q}}$ associated to the new quiver $\overline{\mathcal{Q}}$. It is interesting because the moduli stack of the representations of $\Pi_{\mathcal{Q}}$ is equal to the cotangent bundle of the moduli stack of the representations of $\mathcal{Q}$. On the cohomology of the moduli stack of representations of $\Pi_{\mathcal{Q}}$, one can construct an associated algebra structure in a same manner of Cohomological Hall algebra. The resulted algebra is called the preprojective COHA in [70] or just Cohomological Hall algebra in [64]. For preprojective COHA, see also [71, 72]. Using the similar construction, but with K-Theory instead of equivariant cohomology, Schiffmann and Vasserot defined K-theoretic Hall algebras. See e.g. [63] for details.

One can add a loop to each vertex of $\overline{\mathcal{Q}}$, and get a "triple" quiver $\hat{\mathcal{Q}}$ of the original quiver $\mathcal{Q}$. There is a canonical way to define a potential W on $\tilde{\mathcal{Q}}$, and there is an algebra $J(\tilde{\mathcal{Q}}, W)$ associated to the pair $(\tilde{\mathcal{Q}}, W)$. Using the similar idea to define the multiplication, one can construct an associated algebra called the critical Cohomological Hall algebra. It is originally defined in [38]. The algebra itself as well as its relations to the preprojective COHA is studied extensively. See e.g. [8–11, 59].

Following the above construction, some extra structures can be added to the quiver as well as its representations, and some modified moduli stacks are obtained in this way. In fact, these modified stacks are usually moduli spaces. Thus we are able to construct a representations of COHA on the cohomology of these moduli spaces. Two types of the constructions are generally studied. One is the Nakajima's quiver varieties. The other one is the moduli space of stable framed representations.

M. Reineke did a lot of work on the moduli space of stable framed representations. See e.g. [14, 52, 54, 56–58]. Based on his work, H. Franzen studied the COHA module structures on some of the moduli spaces. See [17, 18].

Current work

The focus of this dissertation is quivers without potential case. By analogy with conventional Hall algebra of a quiver, which gives the "positive" part of a quantization of the corresponding

Lie algebra, one may want to define the "double" COHA, for which the one defined in [38] would be a "positive part". See [65] for detailed discussions.

The aim of this dissertation is to define and study two representations of Cohomological Hall algebras over some moduli spaces, and combine them into a single representation of some algebra. The relation of this algebra to the "full" (or "double") COHA in is discussed. We focus on the A_1-quiver case and the Jordan quiver case in this dissertation.

Contents of chapters

The dissertation is organized as below.

Chapter 2 is a brief introduction to quiver, representations and the smooth models of quiver moduli.

Chapter 3 is the introduction to Cohomological Hall algebra. Two examples are computed at the end. They are the main topics in this dissertation: A_1-COHA and Jordan-COHA.

Chapter 4 is the general construction of the increasing operators and decreasing operators of COHA. Our construction is similar to Nakajima's construction of representations of Heisenberg algebras on the homology of Hilbert schemes.

Chapter 5 is the detailed computations in the case of A_1 quiver. The main tool is Schubert calculus of Grassmannians.

Chapter 6 consists of the detailed computations in the case of Jordan quiver. The computation is based on the cellular decompositions of non-commutative Hilbert schemes.

Chapter 7 contains a version of Boson-Fermion correspondence which is a direct corollary of the previous chapters, and some discussions about future work.

There are four appendices at the back of the dissertation.

Appendix A is about Chow ring, Borel-Moore homology and Poincaré duality. The result would be used mainly in Chapter 6.

Appendix B is a review of equivariant cohomology.

Appendix C is a review of cohomology of quotient stacks. The aim of this appendix is to provide background for a pullback formula when we compute the COHA action in both

Chapter 5 and 6.

Appendix D contains some results of Hilbert schemes. The results are mainly used in Chapter 6 when we compare the non-commutative Hilbert schemes to the Hilbert schemes in the $N = 1$ case.

Chapter 2

Smooth models of quiver moduli

In the case of quivers without potential, when one studies representations of quivers, it is useful to consider varieties or other geometric objects corresponding to representations. Among all these different constructions, one can consider the stable framed moduli space of quivers. We follow the definition given by Reineke to discuss the moduli space of stable framed representations of quivers. See e.g. [14]. For stable framed objects in triangulated categories, see [37] and [65]. For comparison to other constructions, see e.g. [50].

2.1 Moduli of quiver representations

The idea of stability conditions comes from the geometric invariant theory. A. King in [35] applied the geometric invariant theory to the quiver moduli problem, and defined (semi)stability conditions purely algebraically. Our current form of stability conditions is due to A. Rudakov which is just a reformulation of that of A. King. See [60]. The stability conditions can be generalized to triangulated category. See [3] for details. In this dissertation, we would focus on the stability conditions of the category of quiver representations.

2.1.1 Quivers and the stack of representations

We follow [6] in this section. See also [55] and [35].

Definition 2.1.1. A *quiver* is a quadruple $\mathcal{Q} = (I, H, s, t)$, where I and H are finite sets (called the set of *vertices*, resp. *arrows*) and $s, t : H \to I$ are maps assigning to each arrow its *source*, resp. *target*.

Remark 2.1.2. We shall denote the vertices by $i, j, \ldots$. An arrow with source i and target j will be denoted by $\alpha : i \to j$. In the rest of this dissertation, s and t would be omitted if there are no confusions.

Definition 2.1.3. A *representation* (V, T) of a quiver $\mathcal{Q}$ over field $\mathbb{K}$ consists of a family of $\mathbb{K}$-vector spaces $\{V_i\}_{i\in I}$, together with a family of $\mathbb{K}$-linear maps $T_{\alpha:i\to j} : V_i \to V_j$ indexed by the arrows $\alpha \in H$. If there are no confusions it would be denoted by V or T or $(T_\alpha)_\alpha$ for simplicity.

Remark 2.1.4. We will focus on the complex field $\mathbb{C}$ in the dissertation.

Definition 2.1.5. A representation V of $\mathcal{Q}$ is *finite-dimensional* if all the vector spaces $\{V_i\}_{i\in I}$ are finite-dimensional. In this case, the family $\underline{\dim} V := (\dim V_i)_{i\in I}$ is called the *dimension vector* of V.

For a quiver $\mathcal{Q} = (I, H)$, denote each vertex by e_i for $i \in I$ and consider it as an edge from vertex i to i. Define product of two arrows $\alpha : i \to j$ and $\beta : k \to l$ for $\alpha, \beta \in H$ by

$$\alpha\beta = \begin{cases} \alpha\beta, & l = i, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1.1}$$

and

$$e_i^2 = e_i, \quad e_p\alpha = \delta_{p,j}\alpha, \quad \alpha e_p = \delta_{p,i}\alpha. \tag{2.1.2}$$

Then there is an algebra $\mathbb{K}\mathcal{Q}$ generated by $\{e_i\}_{i\in I}$, $\{\alpha\}_{\alpha\in H}$ using the product defined above over field $\mathbb{K}$. The algebra is called the *path algebra* of $\mathcal{Q}$. $\mathbb{K}\mathcal{Q}$ is an associative algebra.

Theorem 2.1.6. *Fix an arbitrary quiver $\mathcal{Q}$. The category of all representations of $\mathcal{Q}$ is equivalent to the category of all left modules of $\mathbb{K}\mathcal{Q}$. Furthermore, the category of all representations of $\mathcal{Q}$ is an abelian category.*

Proof. This is a classical result. See e.g. [6] and the references within for details. □

Fix a finite quiver $\mathcal{Q}$ and a dimension vector $\mathbf{d} = (d^i)_{i\in I}$. Fix the complex coordinate vector spaces $\mathbb{V}_{d^i} := \mathbb{C}^{d^i}$ for all $i \in I$. Let a_{ij} denote the number of arrows of quiver $\mathcal{Q}$ from vertex i to j. Define

$$M_{\mathbf{d}} = M_{\mathbf{d}}(\mathcal{Q}) := \bigoplus_{\alpha:i\to j} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d^i}, \mathbb{C}^{d^j}) \simeq \prod_{i,j} \mathbb{C}^{a_{ij}d^i d^j}. \tag{2.1.3}$$

There is a reductive linear algebraic group

$$G_{\mathbf{d}} := \prod_{i\in I} \mathrm{GL}(d^i, \mathbb{C}) \tag{2.1.4}$$

acting on $M_{\mathbf{d}}$ via the base change action

$$(g_i)_i \cdot (T_\alpha)_\alpha = (g_j T_\alpha g_i^{-1})_{\alpha:i\to j}. \tag{2.1.5}$$

Definition 2.1.7. $M_{\mathbf{d}}$ defined above is called the *space of representations* of $\mathcal{Q}$ of dimension $\mathbf{d}$. $G_{\mathbf{d}}$ is called the *gauge group* of $M_{\mathbf{d}}$. The quotient stack $[M_{\mathbf{d}}/G_{\mathbf{d}}]$ is the *stack of representations* of $\mathcal{Q}$ with dimension vector $\mathbf{d}$.

Example 2.1.8. The A_1*-quiver* is the quiver with one vertex $\mathbf{1}$ and no loops. Since there is only one vertex, the dimension lattice is just $\mathbb{Z}$. The path algebra is $\mathbb{K}$ since there are no nontrivial paths of A_1-quiver. Thus the category of all representations of A_1-quiver is equivalent to the category of $\mathbb{K}$-modules, which is the category of all vector spaces. When $\mathbf{d} = d \in \mathbb{Z}$, M_d of A_1-quiver is isomorphic to $\mathbf{pt}$, since the only linear map $T : \mathbb{V}_d \to \mathbb{V}_d$ is trivial. We still have the gauge group G_d action given by base change. The resulted stack is the quotient stack $[\mathbf{pt}/G_d]$.

Example 2.1.9. The *Jordan quiver* is the quiver with one vertex $\mathbf{1}$ and one loop $l : \mathbf{1} \to \mathbf{1}$. The dimension lattice is $\mathbb{Z}$. The path algebra is $\mathbb{K}[x]$, the polynomial algebra with one generator x. Thus the category of representations of Jordan quiver is the category of $\mathbb{K}[x]$-

modules. When $\mathbf{d} = d = 1 \in \mathbb{Z}$, $M_1 \simeq \mathbb{C}$. In fact, a representation of Jordan quiver of dimension 1 is a linear map from $\mathbb{C}$ to $\mathbb{C}$. If we choose a basis $v_0 \in \mathbb{C}$, we can get a 1×1 matrix representation of the map, which can be identified with a number $\lambda : v_0 \mapsto \lambda v_0$ where $\lambda \in \mathbb{C}$. $G_1 = \mathbb{C}^*$ acts on M_1 via the formula $\mu \cdot \lambda = \mu\lambda\mu^{-1} = \lambda$ for all $\mu \in \mathbb{C}^*$. Therefore the gauge group action in this case ($d = 1$) is trivial.

2.1.2 Stability conditions from Geometric invariant theory

Let V be a finite dimensional vector space with a linear action of a reductive algebraic group G. Our goal is define a "good" quotient of this action. If the action is not free, we cannot have a natural geometric quotient. That is why we need Geometric invariant theory to help us construct different versions of the quotient.

A regular function $f : V \to \mathbb{K}$ on V is called an invariant if $f(gv) = f(v)$ for all $g \in G$ and $v \in V$. We denote by $\mathbb{K}[V]$ the ring of rational functions on V, and by $\mathbb{K}[V]^G$ the subring of invariant functions of the ring $\mathbb{K}[V]$. It's obvious that $\mathrm{Spec}(\mathbb{K}[V]) = V$. $\mathbb{K}[V]$ has a natural $\mathbb{N}$-graded ring structure.

Definition 2.1.10. A *character* of G is a morphism of algebraic groups $\chi : G \to \mathbb{K}^*$. A regular function f is called *χ-semi-invariant* if $f(gv) = \chi(g)f(v)$ for all $g \in G$ and $v \in V$. We denote by $\mathbb{K}[V]^{G,\chi}$ the subspace of χ-semi-invariants, and by $\mathbb{K}[V]^G_\chi := \oplus_{n\geq 0}\mathbb{K}[V]^{G,\chi^n}$ the subring of semi-invariants for all powers of the character χ.

Remark 2.1.11. It is obvious that $\mathbb{K}[V]^G_\chi$ is a $\mathbb{N}$-graded ring. In fact it is a $\mathbb{N}$-graded subring of $\mathbb{K}[V]$ with $\mathbb{K}[V]^G$ as the subring of degree 0 elements.

Definition 2.1.12. A vector $v \in V$ is called *χ-semistable* if there exists a function $f \in \mathbb{K}[V]^{G,\chi}$ for some $n \geq 1$ such that $f(v) \neq 0$. Denote by $V^{\chi-sst}$ the subset of χ-semistable points. A vector $v \in V$ is called *χ-stable* if v is χ-semistable, its orbit Gv is closed in $V^{\chi-sst}$, and its stabilizer in G is zero-dimensional. Denote by $V^{\chi-st}$ the subset of stable points.

Definition 2.1.13. $V^{\chi-sst}//G := \mathrm{Proj}(\mathbb{K}[V]^G_\chi)$. There is a natural morphism $\pi : V^{\chi-sst} \to V^{\chi-sst}//G$.

The importance of the moduli space of stable points or semistable points lie in the following propositions.

Theorem 2.1.14 ([55]). *The variety $V^{\chi-sst}//G$ parametrizes the closed orbits of G in $V^{\chi-sst}$. The restriction of π to $V^{\chi-st}$ has as fibres precisely the G-orbits in $V^{\chi-st}$. If the G-action on $V^{\chi-st}$ is free, the morphism $V^{\chi-st} \to V^{\chi-st}/G$ is a G-principal bundle. And we have the following diagram:*

$$\begin{tikzcd} V^{\chi-st} \arrow[r, hook] \arrow[d] & V^{\chi-sst} \arrow[d, "\pi"] \\ V^{\chi-st}/G \arrow[r, hook] & V^{\chi-sst}//G \end{tikzcd} . \tag{2.1.6}$$

2.1.3 Stability conditions of quiver representations

Fix a quiver $\mathcal{Q} = (I, H)$.

Definition 2.1.15. A $\mathbb{Z}$-linear form $\Theta : \mathbb{Z}^I \to \mathbb{Z}$ over the dimension lattice is called *a stability condition*.

Definition 2.1.16. The *slope* of a dimension vector $\mathbf{d} = (d^i)_{i\in I}$ is

$$\mu^\Theta(\mathbf{d}) = \frac{\Theta(\mathbf{d})}{\sum_{i\in I} d^i}. \tag{2.1.7}$$

The *slope of a representation* V is defined to be $\mu^\Theta(V) := \mu^\Theta(\underline{\dim}(V))$.

Definition 2.1.17. A representation V is called Θ-*semistable* if $\mu^\Theta(V) \geq \mu^\Theta(V')$ for all subrepresentation $V' \subset V$. It is called Θ-*stable* if $\mu^\Theta(V) > \mu^\Theta(V')$ for all nontrivial subrepresentation $V' \subset V$.

Remark 2.1.18. We usually drop Θ from the notation μ^Θ if there is no confusion.

Lemma 2.1.19 (The seesaw property). *Let $0 \to W \to V \to U \to 0$ be a short exact sequence of representations of a quiver $\mathcal{Q}$. Fix a stability condition Θ. Then either $\mu(W) \geq \mu(V) \geq \mu(U)$, or $\mu(W) \leq \mu(V) \leq \mu(U)$. Furthermore, considering the three equalities $\mu(W) = \mu(V)$, $\mu(W) = \mu(V/W)$ and $\mu(V) = \mu(V/W)$, any one implies the other two.*

Proof. Since the representations of quiver $\mathcal{Q}$ form an abelian group, and both Θ and $\underline{\dim}$ are linear forms over the dimension lattice, for a short exact sequence $0 \to W \to V \to U \to 0$, we have $\Theta(W) + \Theta(U) = \Theta(V)$, and $\underline{\dim}(W) + \underline{\dim}(U) = \underline{\dim}(V)$. The results follow immediately from an algebraic computation. □

Remark 2.1.20. This seesaw property is the core feature of the theory the stability of abelian category. In fact, this is the definition property of A. Rudakov's stability condition. See [60] for details.

Proposition 2.1.21. *Let Θ be a stability condition. Let $\Omega = k\Theta + a$ for $a \in \mathbb{Z}$ and $k \in \mathbb{N}$ be another stability condition defined in the obvious way. A representation V is Θ-stable (resp. Θ-semistable) if and only if V is Ω-stable (resp. Ω-semistable). In this case we call Θ and Ω are equivalent.*

Proof. For an arbitrary dimension vector $\mathbf{d} = (d^i)_{i\in I}$, we have

$$\mu^{\Omega}(\mathbf{d}) = \frac{k\Theta(\mathbf{d}) + a\sum_{i\in I} d^i}{\sum_{i\in I} d^i} = k\mu^{\Theta}(\mathbf{d}) + a. \tag{2.1.8}$$

Then for any two dimension vectors $\mathbf{d}$ and $\mathbf{n}$, and $a \in \mathbb{Z}$ and $k \in \mathbb{N}$, $\mu^{\Theta}(\mathbf{d}) > \mu^{\Theta}(\mathbf{n})$ implies $\mu^{\Omega}(\mathbf{d}) > \mu^{\Omega}(\mathbf{n})$. □

The following Corollary follows immediately from Proposition 2.1.21.

Corollary 2.1.22. *If $|I| = 1$, all stability conditions are equivalent to* 0*.*

2.1.4 Moduli of quiver representations

Fix a quiver $\mathcal{Q} = (I, H)$ and a dimension vector $\mathbf{d}$. Then we have a vector space $M_{\mathbf{d}}$ with a reductive algebraic group $G_{\mathbf{d}}$-action. Recall that $M_{\mathbf{d}} = \oplus_{i,j} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{d^i}, \mathbb{C}^{d^j})$, and $G_{\mathbf{d}} = \prod_{i\in I} \mathrm{GL}(d^i, \mathbb{C})$. Now let us apply the construction from Section 2.1.2.

The characters of the general linear group are just integer powers of the determinant map. Thus for each character χ of $G_{\mathbf{d}}$ there exists a tuple of integers $\theta = (\theta^i)_{i \in I}$, such that

$$\chi((g_i)_{i\in I}) = \prod_{i\in I} \det(g_i)^{\theta^i}, \quad \forall (g_i)_{i\in I} \in G_{\mathbf{d}}. \tag{2.1.9}$$

Note that since the gauge group acts by conjugation, the diagonally embedded scalar matrices in $G_{\mathbf{d}}$ act trivially on $M_{\mathbf{d}}$. Therefore the action of the group factor through $PG_{\mathbf{d}} = G_{\mathbf{d}}/\mathbb{K}^*$. Then the character should satisfy that $\sum_{i\in I} \theta^i = 0$.

Now starting from a stability condition $\Theta : \mathbf{d} \mapsto \sum_{i\in I} d^i\theta^i$ where $\theta^i \in \mathbb{Z}$ for $i \in I$, we can construct a character

$$\chi_\Theta((g_i)_{i\in I}) := \prod_{i\in I} \det(g_i)^{\Theta(\mathbf{d}) - (\sum_{i\in I} d^i)\theta^i}. \tag{2.1.10}$$

On the other hand, if there is a character $\chi((g_i)_{i\in I}) = \prod_{i\in I} \det(g_i)^{\theta^i}$ for a tuple of integers $(\theta^i)_{i\in I}$ which satisfy $\sum_{i\in I} \theta^i = 0$, we can construct a stability condition Θ such that $\chi = \chi_\Theta$ by solving a linear system.

The key is the following theorem by A. King.

Theorem 2.1.23 ([35])**.** *A point in $M_{\mathbf{d}}$ corresponding to a representation M is χ_Θ-semistable (resp. χ_Θ-stable) in the sense of Geometric invariant theory if and only if M is Θ-semistable (resp. Θ-stable) in the sense of quiver representations.*

Then it is possible to construct the moduli space of stable representations or semistable representations of a quiver $\mathcal{Q}$. Denote by $M_{\mathbf{d}}^{sst} = M_{\mathbf{d}}^{\Theta-sst}(\mathcal{Q})$ (resp. $M_{\mathbf{d}}^{st} = M_{\mathbf{d}}^{\Theta-st}(\mathcal{Q})$) the subset of the variety $M_{\mathbf{d}}(\mathcal{Q})$ corresponding to Θ-semistable (resp. Θ-stable) representations. Applying the construction in Section 2.1.2 and we have the following definitions.

Definition 2.1.24. The moduli space $\mathcal{M}_{\mathbf{d}}^{sst}(\mathcal{Q})$ (resp. $\mathcal{M}_{\mathbf{d}}^{st}(\mathcal{Q})$) is defined to be $M_{\mathbf{d}}^{sst}//G_{\mathbf{d}}$ (resp. $M_{\mathbf{d}}^{st}/G_{\mathbf{d}}$).

Remark 2.1.25. Following Proposition 2.1.14, $\mathcal{M}_{\mathbf{d}}^{st}(\mathcal{Q})$ parametrizes isomorphism classes of Θ-stable representations of $\mathcal{Q}$ of dimension vector $\mathbf{d}$. $\mathcal{M}_{\mathbf{d}}^{sst}(\mathcal{Q})$ parametrizes isomorphism classes of μ-polystable representations of $\mathcal{Q}$ of dimension vector $\mathbf{d}$.

2.2 Stable framed representations

2.2.1 The extended quiver $\hat{\mathcal{Q}}_{\mathbf{n}}$ and the extended stability condition

We follow [14] in this section.

Definition 2.2.1. Let $\mathcal{Q} = (I, H)$ be a quiver with r vertices. Fix a dimension vector $\mathbf{d}$ and a stability condition Θ. Fix a non-trivial dimension vector $\mathbf{n} = (n^i)_{i\in I} \in \mathbb{Z}^I$ called *the framed structure*. The *extended quiver* $\hat{\mathcal{Q}}_{\mathbf{n}}$ and the *extended stability condition* $\hat{\Theta}$ are defined by the following data:

1. A quiver $\hat{\mathcal{Q}}_{\mathbf{n}}$ whose vertices are those of $\mathcal{Q}$, together with one additional vertex ∞,
2. The arrows of $\hat{\mathcal{Q}}_{\mathbf{n}}$ are those of $\mathcal{Q}$, together with n^i arrows from ∞ to i for $i \in I$,
3. An extended dimension vector $\hat{\mathbf{d}}$, with $\hat{d}^i = d^i$ and $\hat{d}^\infty = 1$,
4. A new $\hat{\Theta}$, with $\hat{\theta}^i = \theta^i\ \forall i \in I$ and $\theta^\infty = \mu(\mathbf{d}) + \epsilon$ for some sufficiently small $\epsilon \in \mathbb{Q}_{>0}$.

The new slope $\hat{\mu}$ is defined through $\hat{\Theta}$ and $\hat{\mathbf{d}}$. The representation space of $\hat{\mathcal{Q}}_{\mathbf{n}}$ of dimension $\hat{\mathbf{d}}$, the moduli space of semistable framed representations and of stable framed representations can be defined using the extended stability condition for the extended quiver.

Notation 2.2.2. We use $((V, T), f)$ to denote a representation of $\hat{\mathcal{Q}}_{\mathbf{n}}$ of dimension $\hat{\mathbf{d}}$, where T is a representation of $\mathcal{Q}$ of dimension $\mathbf{d}$ on V, and $f = (f_{i,j} : \mathbb{C} \to V_i)_{1\le j\le n^i, i\in I}$ gives all the information about new arrows coming from the ∞ vertex.

Notation 2.2.3. For two dimension vectors $\mathbf{e} = (e^i)_{i\in I}$ and $\mathbf{d} = (d^i)_{i\in I}$, we say $\mathbf{e} \le \mathbf{d}$ (resp. $\mathbf{e} < \mathbf{d}$) if $e^i \le d^i$ (resp, $e^i < d^i$) for all $i \in I$.

Remark 2.2.4. We consider $\mathbf{d}$ as a dimension vector of the extended quiver by setting $d^\infty = 0$.

Lemma 2.2.5 ([14])**.** *For a quiver $\mathcal{Q}$,*

1. *For all $0 \ne \boldsymbol{e} \le \boldsymbol{d}$, we have $\hat{\mu}(\boldsymbol{e}) < \hat{\mu}(\hat{\boldsymbol{d}}) \Leftrightarrow \hat{\mu}(\boldsymbol{e}) \le \hat{\mu}(\hat{\boldsymbol{d}}) \Leftrightarrow \mu(\boldsymbol{e}) \le \mu(\boldsymbol{d})$.*
2. *For all $\boldsymbol{e} < \boldsymbol{d}$, we have $\hat{\mu}(\hat{\mathbf{e}}) < \hat{\mu}(\hat{\boldsymbol{d}}) \Leftrightarrow \hat{\mu}(\hat{\mathbf{e}}) \le \hat{\mu}(\hat{\boldsymbol{d}}) \Leftrightarrow \mu(\boldsymbol{e}) < \mu(\boldsymbol{d})$.*

Proof. Denote $\mu(\mathbf{d})$ by μ_d and $\mu(\mathbf{e})$ by μ_e.

$$\hat{\mu}(\hat{\mathbf{d}}) = \frac{\sum d^i\theta^i + \mu_d + \epsilon}{\sum d^i + 1} = \frac{(\sum d^i)\mu_d + \mu_d + \epsilon}{\sum d^i + 1} = \mu_d + \frac{\epsilon}{\sum d^i + 1}. \tag{2.2.1}$$

$$\hat{\mu}(\mathbf{e}) = \frac{\sum e^i\theta^i}{\sum e^i} = \mu_e. \tag{2.2.2}$$

Then

$$\hat{\mu}(\mathbf{e}) \leq \hat{\mu}(\hat{\mathbf{d}}) \Leftrightarrow \mu_e \leq \mu_d + \frac{\epsilon}{\sum d^i + 1}. \tag{2.2.3}$$

Since ϵ is a sufficiently small constant, $\mu_e \leq \mu_d + \frac{\epsilon}{\sum d^i+1}$ implies $\mu_e \leq \mu_d$. Then $\mu_e < \mu_d + \frac{\epsilon}{\sum d^i+1}$ because $\frac{\epsilon}{\sum d^i+1} > 0$. $\hat{\mu}(\mathbf{e}) < \hat{\mu}(\hat{\mathbf{d}}) \Rightarrow \hat{\mu}(\mathbf{e}) \leq \hat{\mu}(\hat{\mathbf{d}})$ is obvious. Thus the first statement is proved.

For the second statement,

$$\hat{\mu}(\hat{\mathbf{e}}) = \frac{\sum e^i\theta^i + \mu_d + \epsilon}{\sum e^i + 1} = \frac{(\sum e^i)\mu_e + \mu_e + (\mu_d - \mu_e) + \epsilon}{\sum e^i + 1} = \mu_e + \frac{\mu_d - \mu_e}{\sum e^i + 1} + \frac{\epsilon}{\sum e^i + 1}. \tag{2.2.4}$$

Then

$$\begin{aligned} \hat{\mu}(\hat{\mathbf{d}}) \geq \hat{\mu}(\hat{\mathbf{e}}) &\Leftrightarrow \mu_d + \frac{\epsilon}{\sum d^i + 1} \geq \mu_e + \frac{\mu_d - \mu_e}{\sum e^i + 1} + \frac{\epsilon}{\sum e^i + 1} \\ &\Leftrightarrow \mu_d - \mu_e \geq \epsilon(\frac{1}{\sum e^i})(\frac{\sum d^i - \sum e^i}{\sum d^i + 1}). \end{aligned} \tag{2.2.5}$$

Choose ϵ as small as possible. Since $\mathbf{e} < \mathbf{d}$, we have

$$\mu_d - \mu_e \geq \epsilon(\frac{1}{\sum e_i})(\frac{\sum d_i - \sum e_i}{\sum d_i + 1}) \Rightarrow \mu_d - \mu_e > 0. \tag{2.2.6}$$

Similarly we have $\hat{\mu}(\hat{\mathbf{d}}) > \hat{\mu}(\hat{\mathbf{e}}) \Leftrightarrow \mu_d - \mu_e > \epsilon(\frac{1}{\sum e^i})(\frac{\sum d^i - \sum e^i}{\sum d^i+1})$. If $\mu_e < \mu_d$, choose a sufficiently small $\epsilon > 0$, we have $\mu_d - \mu_e > \epsilon(\frac{1}{\sum e^i})(\frac{\sum d^i - \sum e^i}{\sum d^i+1})$. Thus $\hat{\mu}(\hat{\mathbf{d}}) > \hat{\mu}(\hat{\mathbf{e}})$. Since $\hat{\mu}(\hat{\mathbf{d}}) > \hat{\mu}(\hat{\mathbf{e}}) \Rightarrow \hat{\mu}(\hat{\mathbf{d}}) > \hat{\mu}(\hat{\mathbf{e}})$ is obvious, the second statement is proved.

□

Proposition 2.2.6 ([14])**.** *For a representation $((V,T),f)$ of $\hat{\mathcal{Q}}_{\boldsymbol{n}}$ of dimension vector $\hat{\boldsymbol{d}}$, the* *following are equivalent:*

1. *$((V,T),f)$ is $\hat{\mu}$-semistable;*

2. *$((V,T),f)$ is $\hat{\mu}$-stable;*

3. *(V,T) is a μ-semistable representation of $\mathcal{Q}$, and $\mu(U) < \mu(V)$ for all proper subrepresentations U containing the image of f.*

Proof. Let $((U,T),f)$ be a subrepresentation of $((V,T),f)$ as representations of $\hat{\mathcal{Q}}_{\mathbf{n}}$. Let $\mathbf{d} := \underline{\dim} V$. Then $\underline{\dim}((V,T),f) = \hat{\mathbf{d}}$. There are two types of $((U,T),f)$. Either U contains the 1-dimensional space associated to ∞, or not.

In the first case, denote by W the subspace of U by removing the 1-dimensional space associated to ∞. Then W is a subrepresentation of V as a representation of the original quiver $\mathcal{Q}$ who contains the image of f. Let $\mathbf{e} := \underline{\dim} W$. Then $\underline{\dim}((U,T),f) = \hat{\mathbf{e}}$. Since $\underline{\dim} W < \underline{\dim}((U,T),f)$, $\mathbf{e} < \mathbf{d}$. Therefore, by Lemma 2.2.5, $\hat{\mu}(\hat{\mathbf{d}}) \geq \hat{\mu}(\hat{\mathbf{e}}) \Leftrightarrow \hat{\mu}(\hat{\mathbf{d}}) > \hat{\mu}(\hat{\mathbf{e}}) \Leftrightarrow \mu(\mathbf{d}) > \mu(\mathbf{e})$. This implies the equivalence we need.

In the second case, U can be directly viewed as a representation of $\mathcal{Q}$. Let $\mathbf{e} := \underline{\dim}((U,T),f)$. Then $\mathbf{e} \leq \mathbf{d}$. By Lemma 2.2.5, $\hat{\mu}(\hat{\mathbf{d}}) > \hat{\mu}(\mathbf{e}) \Leftrightarrow \hat{\mu}(\hat{\mathbf{d}}) \geq \hat{\mu}(\mathbf{e}) \Leftrightarrow \mu(\mathbf{d}) > \mu(\mathbf{e})$. This also implies the equivalence we need. $\square$

Definition 2.2.7. We denote $\mathcal{M}^{st}_{\hat{\mathbf{d}}}(\hat{\mathcal{Q}}_{\mathbf{n}})$ by $\mathcal{M}^{st}_{\mathbf{d},\mathbf{n}}(\mathcal{Q})$ and call this variety a smooth model for $\mathcal{M}^{sst}_{\mathbf{d}}(\mathcal{Q})$.

2.2.2 Example: Grassmannians

Consider the quvier A_1 with one vertex and no arrows. By Corollary 2.1.22, we only need to consider the trivial stability condition. The extended quiver is as below:

$$\begin{tikzcd} \bullet & \bullet^{\infty} \arrow[l, bend right, "f"'] \arrow[l, bend left, "\cdots"] \end{tikzcd} . \tag{2.2.7}$$

Fix a dimension d and a framed structure N. A framed representation (V_d, f) is a family of maps $\{f_i : \mathbb{C} \to V_d\}_{i=1}^N$. It is equivalent to one map $f : \mathbb{C}^N \to V_d$. By abusing notations, we don't distinguish the two meanings of f.

Proposition 2.2.8. *The moduli space $\mathcal{M}_{d,N}^{st}$ of stable framed representations of A_1-quiver is a quotient space $[\operatorname{Epi}(\mathbb{C}^N, \mathbb{C}^d)/\operatorname{GL}_d(\mathbb{C})]$ where $\operatorname{Epi}(\mathbb{C}^N, \mathbb{C}^d)$ is the space of all epimorphisms from $\mathbb{C}^N$ to $\mathbb{C}^d$. This quotient space is isomorphic to the Grassmannian $\operatorname{Gr}(N-d, N)$.*

Proof. By Proposition 2.2.6, a framed representation (V, f) is stable if each proper subspace which contains $\operatorname{Im}(f)$ has a smaller slope than V. Since the stability condition is trivial, it implies that a framed representation (V, f) is stable if and only if $f : \mathbb{C}^N \to V$ is surjective. The gauge group in this case the $\operatorname{GL}_d(\mathbb{C})$ which acts on V by the natural action. □

Notation 2.2.9. In the rest of the dissertation, by abusing notations, we would use $\operatorname{Gr}(d, n)$ to denote the Grassmannian $\operatorname{Gr}(n-d, n)$. In other words, all Grassmannians in this dissertation refer to the "quotient" type Grassmannians.

2.2.3 Example: Noncommutative Hilbert schemes

Consider the quiver $\mathcal{Q}^{(m)}$ with one vertex and $m \geq 0$ loops. Fix a dimension d and a framed structure N. We have the extended quiver:

$$\begin{tikzcd} \{T_i\}_{i=1}^m \arrow[loop left, "\cdots"] \bullet & \bullet^\infty \arrow[l, bend right, "f"'] \arrow[l, bend left, "\cdots"] \end{tikzcd}. \tag{2.2.8}$$

A framed representation of $\mathcal{Q}^{(m)}$ is a pair $((V, T), f)$, where $(V, T) = (V, T_i)_{i=1}^m$ is the representation of $\mathcal{Q}^{(m)}$ and $\{f_j : \mathbb{C}^N \to V\}_{i=1}^N$ represents the framed structure. Since there is only one vertex in the original quiver $\mathcal{Q}^{(m)}$, by Corollary 2.1.22, the only stability condition is the trivial one. Then $((V, T), f)$ is stable if and only if $\operatorname{Im}(f)$ can generate the whole V under the actions of $\{T_i\}_{i=1}^m$. We denote the variety of all stable framed representations by $H_{d,N}^{(m)}$. $\operatorname{GL}(d, \mathbb{C})$ acts freely on it. The quotient gives the smooth model, which is denoted by $\mathcal{H}_{d,N}^{(m)}$.

Definition 2.2.10. The variety $\mathcal{H}_{d,N}^{(m)}$ is called the *noncommutative Hilbert schemes*.

Chapter 3

Cohomological Hall algebra

The definition of Cohomological Hall algebra is similar to the definition of conventional (constructible) Hall algebra (see e.g. [62]) or its motivic version (see e.g. [36]). The cohomology of the moduli spaces provided by representations of quivers is studied at first. It is the underlying vector space of the Cohomological Hall algebra. Then, the pullback-pushforward construction is used to define the multiplication. Two examples are computed in details: one for A_1-quiver, the other for Jordan quiver. This chapter mainly follows [38].

3.1 Cohomological Hall algebras

3.1.1 Stacks of representations and their cohomologies

Fix a quiver $\mathcal{Q} = (I, H)$ and a dimension vector $\mathbf{d} = (d^i)_{i\in I}$. We have the stack $[M_{\mathbf{d}}/G_{\mathbf{d}}]$ of representations of $\mathcal{Q}$ with dimension vector $\mathbf{d}$. We will consider the cohomology of this stack. By Appendix C.3, $H^*([M_{\mathbf{d}}/G_{\mathbf{d}}]) \simeq H^*_{G_{\mathbf{d}}}(M_{\mathbf{d}})$. Denote $\mathcal{H}_{\mathbf{d}} := H^*_{G_{\mathbf{d}}}(M_{\mathbf{d}})$.

Since $M_{\mathbf{d}}$ is a vector space, and $G_{\mathbf{d}}$-action is linear, it is equivariantly homotopicly equivalent to $\mathbf{pt}$ with $G_{\mathbf{d}}$-action trivially. Then $H^*_{G_{\mathbf{d}}}(M_{\mathbf{d}}) \simeq H^*_{G_{\mathbf{d}}}(\mathbf{pt})$. According to Example B.4.3, $H^*_{G_{\mathbf{d}}}(M_{\mathbf{d}})$ is the algebra of polynomials of $\sum_{i\in I} d^i$ variables, where each group of d^i variables are symmetric.

We introduce a $\mathbb{Z}^I_{\geq 0}$-graded abelian group

$$\mathcal{H} := \oplus_{\mathbf{d}} \mathcal{H}_{\mathbf{d}}. \tag{3.1.1}$$

This is the underlying space of Cohomological Hall algebra.

3.1.2 Multiplication

Fix any $\mathbf{d}_1$, $\mathbf{d}_2$ two dimension vectors and denote $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$. Denote by $M_{\mathbf{d}_1,\mathbf{d}}$ the space of representations of $\mathcal{Q}$ in coordinate spaces of dimensions $(d_1^i + d_2^i)_{i\in I}$ such that the standard coordinate subspaces of dimensions $(d_1^i)_{i\in I}$ form a subrepresentation. The group $G_{\mathbf{d}_1,\mathbf{d}} \subset G_{\mathbf{d}}$ consisting of transformations preserving subspaces $(\mathbb{C}^{d_1^i} \subset \mathbb{C}^{d^i})_{i\in I}$, acts on $M_{\mathbf{d}_1,\mathbf{d}}$.

Consider the following diagram:

$$[M_{\mathbf{d}_1}/G_{\mathbf{d}_1}] \times [M_{\mathbf{d}_2}/G_{\mathbf{d}_2}] \xleftarrow{p_1 \times p_2} [M_{\mathbf{d}_1,\mathbf{d}}/G_{\mathbf{d}_1,\mathbf{d}}] \xrightarrow{p} [M_{\mathbf{d}}/G_{\mathbf{d}}]. \tag{3.1.2}$$

Construct a morphism $m_{\mathbf{d}_1,\mathbf{d}_2} : \mathcal{H}_{\mathbf{d}_1} \otimes \mathcal{H}_{\mathbf{d}_2} \to \mathcal{H}_{\mathbf{d}}$ by

$$p_*(p_1^*(\alpha) \cup p_2^*(\beta)), \quad \text{for } \alpha \in H^*_{G_{\mathbf{d}_1}}(M_{\mathbf{d}_1}),\ \beta \in H^*_{G_{\mathbf{d}_2}}(M_{\mathbf{d}_2}). \tag{3.1.3}$$

Define $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ by

$$m := \sum_{\mathbf{d}_1,\mathbf{d}_2} m_{\mathbf{d}_1,\mathbf{d}_2}. \tag{3.1.4}$$

Theorem 3.1.1 ([38])**.** *The product m on $\mathcal{H}$ is associative.*

Remark 3.1.2. Note that the natural morphism of stacks $p : [M_{\mathbf{d}_1,\mathbf{d}}/G_{\mathbf{d}_1,\mathbf{d}}] \to [M_{\mathbf{d}}/G_{\mathbf{d}}]$ is a proper morphism of smooth Artin stacks. Hence it induces the pushforward map on cohomology. Therefore the above formula makes sense.

Theorem 3.1.3 ([38])**.** *The product $f_1 \cdot f_2$ of elements $f_i \in \mathcal{H}_{\boldsymbol{d}_i}$, $i = 1,2$ is given by the symmetric function $g((x_{i,a})_{i\in I, a\in\{1,\dots,d^i\}})$, where $\boldsymbol{d} := \boldsymbol{d}_1 + \boldsymbol{d}_2$, obtained from the following* *function in variables* $(x'_{i,a})_{i\in I, a\in\{1,\dots,d_1^i\}}$ *and* $(x''_{i,a})_{i\in I, a\in\{1,\dots,d_2^i\}}$*:*

$$f_1((x'_{i,a}))f_2((x''_{i,a}))\frac{\prod_{i,j\in I}\prod_{a_1=1}^{d_1^i}\prod_{a_2=1}^{d_2^j}(x''_{j,a_2}-x'_{i,a_1})^{a_{ij}}}{\prod_{i\in I}\prod_{a_1=1}^{d_1^i}\prod_{a_2=1}^{d_2^i}(x''_{i,a_2}-x'_{i,a_1})}, \tag{3.1.5}$$

by taking the sum over all shuffles for any given $i \in I$ *of the variables* $x'_{i,a}$*,* $x''_{i,a}$ *(the sum is over* $\prod_{i\in I}\binom{d^i}{d_1^i}$ *shuffles).*

Let $\mathcal{Q} = \mathcal{Q}^{(m)}$ be a quiver with just one vertex and $m \geq 0$ loops. The product formula (3.1.5) specializes to

$$(f_1 \cdot f_2)(x_1, \dots, x_{p+q}) := \sum_{\substack{i_1<\dots<i_p \\ j_1<\dots<j_q \\ \{i_1,\dots,i_p,j_1,\dots,j_q\}=\{1,\dots,p+q\}}} f_1(x_{i_1},\dots,x_{i_p})f_2(x_{j_1},\dots,x_{j_q})(\prod_{k=1}^{p}\prod_{l=1}^{q}(x_{j_l}-x_{i_k}))^{m-1} \tag{3.1.6}$$

for symmetric polynomials, where f_1 has p variables and f_2 has q variables. The product $f_1 \cdot f_2$ is a symmetric polynomial in $p+q$ variables.

3.2 Examples: A_1-quiver

If $m = 0$, the quiver $\mathcal{Q}^{(0)}$ is actually A_1-quiver. In this case, the product formula is

$$(f_1 \cdot f_2)(x_1, \dots, x_{p+q}) := \sum_{\substack{i_1<\dots<i_p \\ j_1<\dots<j_q \\ \{i_1,\dots,i_p,j_1,\dots,j_q\}=\{1,\dots,p+q\}}} \frac{f_1(x_{i_1},\dots,x_{i_p})f_2(x_{j_1},\dots,x_{j_q})}{\prod_{k=1}^{p}\prod_{l=1}^{q}(x_{j_l}-x_{i_k})}. \tag{3.2.1}$$

Notice that $\mathcal{H}_1 = H^*_{G_1}(M_1) \simeq \mathbb{Q}[x_1]$, where x_1 is the first Chern class of the tautological line bundle of the classifying space of G_1. Denote x_1^i by ϕ_i. Then $\mathcal{H}_1$ is the $\mathbb{Q}$-span of $\{\phi_0, \phi_1, \dots\}$ as a vector space.

We want to make a connection between the A_1-COHA and the Schur polynomials. We define partitions first.

Definition 3.2.1. An *partition* λ of $n \in \mathbb{N}$ is a sequence of integers $(\lambda_1, \lambda_2, \ldots)$ such that $\sum_i \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$. The maximal index k such that $\lambda_k \neq 0$ is called the length of the partition. An *d-partition* of $n \in \mathbb{N}$ is a d-tuple $(\lambda_1, \ldots, \lambda_d)$ satisfying $\lambda_1 \geq \ldots \geq \lambda_d \geq 0$ and $\sum_{i=1}^{d} \lambda_i = n$.

Notation 3.2.2. Let $k_i = \lambda_i + d - i$ for $i = 1, \ldots, d$. $\mathbf{k}(\lambda) = (k_1, \ldots, k_d)$ is an index associated to λ.

Define a polynomial related to $\mathbf{k}(\lambda)$ by

$$a_{\mathbf{k}(\lambda)}(x_1, \ldots, x_d) = \det \begin{bmatrix} x_1^{k_1} & x_2^{k_1} & \ldots & x_d^{k_1} \\ x_1^{k_2} & x_2^{k_2} & \ldots & x_d^{k_2} \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{k_d} & x_2^{k_d} & \ldots & x_d^{k_d} \end{bmatrix}. \tag{3.2.2}$$

A special case is when $\underline{\lambda_0} = (0, 0, \ldots, 0)$ and in this case the polynomial is known for *the Vandermonde determinant*:

$$a_{\mathbf{k}(\underline{\lambda_0})}(x_1, \ldots, x_d) = \det \begin{bmatrix} x_1^{d-1} & x_2^{d-1} & \ldots & x_d^{d-1} \\ x_1^{d-2} & x_2^{d-2} & \ldots & x_d^{d-2} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & 1 \end{bmatrix} = \prod_{1 \leq j < k \leq d} (x_j - x_k). \tag{3.2.3}$$

It is obvious that $a_{\mathbf{k}(\lambda)}$ is alternating. Therefore $a_{\mathbf{k}(\lambda)}$ is divisible by $a_{\mathbf{k}(\underline{\lambda_0})}$.

Definition 3.2.3. The *Schur polynomial* for λ is defined as the ratio:

$$s_\lambda(x_1, \ldots, x_d) = \frac{a_{\mathbf{k}(\lambda)}(x_1, \ldots, x_d)}{a_{\mathbf{k}(\underline{\lambda_0})}(x_1, \ldots, x_d)}. \tag{3.2.4}$$

Proposition 3.2.4. *For a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda_1 \geq \ldots \geq \lambda_d \geq 0$,*

$$s_\lambda = \phi_{k_1} \cdot \ldots \cdot \phi_{k_d}, \tag{3.2.5}$$

where s_λ denotes the Schur symmetric function of $\lambda = (k_1 - d + 1, k_2 - d + 2, \ldots, k_d)$.

Proof. Use induction on the length of λ. It is obvious when the length of the partition is 0. Now let us consider a length $d+1$ partition $\tau = (\tau_1, \tau_2, \ldots, \tau_{d+1})$. The associated index $\mathbf{k}(\tau) = (k_1, k_2, \ldots, k_{d+1})$. Let $\underline{\tau_0} = (0, \ldots, 0)$ be the trivial $(d+1)$-partition. Let λ be the length d partition which satisfies $\lambda = (\tau_2, \ldots, \tau_{d+1})$, and $\underline{\lambda_0}$ be the trivial length d partition. It is easy to see $\mathbf{k}(\lambda) = (k_2, \ldots, k_{d+1})$. Then we have

$$
\begin{aligned}
a_{\mathbf{k}(\tau)}(x_1, \ldots, x_{d+1}) = \det \begin{bmatrix} x_1^{k_1} & x_2^{k_1} & \ldots & x_{d+1}^{k_1} \\ x_1^{k_2} & x_2^{k_2} & \ldots & x_{d+1}^{k_2} \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{k_{d+1}} & x_2^{k_{d+1}} & \ldots & x_{d+1}^{k_{d+1}} \end{bmatrix} & \\
= x_1^{k_1} \det \begin{bmatrix} x_2^{k_2} & \ldots & x_{d+1}^{k_2} \\ \ldots & \ldots & \ldots \\ x_2^{k_{d+1}} & \ldots & x_{d+1}^{k_{d+1}} \end{bmatrix} - x_2^{k_1} \det \begin{bmatrix} x_1^{k_2} & x_3^{k_2} & \ldots & x_{d+1}^{k_2} \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{k_{d+1}} & x_3^{k_{d+1}} & \ldots & x_{d+1}^{k_{d+1}} \end{bmatrix} + \ldots & \\
\ldots + (-1)^d x_{d+1}^{k_1} \det \begin{bmatrix} x_1^{k_2} & \ldots & x_d^{k_2} \\ \ldots & \ldots & \ldots \\ x_1^{k_{d+1}} & \ldots & x_d^{k_{d+1}} \end{bmatrix} & \\
= x_1^{k_1} a_{\mathbf{k}(\lambda)}(x_2, \ldots, x_{d+1}) - x_2^{k_1} a_{\mathbf{k}(\lambda)}(x_1, x_3, \ldots, x_{d+1}) + \ldots & \\
\ldots + (-1)^d x_{d+1}^{k_1} a_{\mathbf{k}(\lambda)}(x_1, \ldots, x_d), &
\end{aligned}
\tag{3.2.6}
$$

and for any $s = 1, \ldots, d+1$,

$$
\begin{aligned}
a_{\mathbf{k}(\underline{\tau_0})} = \prod_{1 \le j < k \le d+1} (x_j - x_k) &= (-1)^{s-1} \Big(\prod_{\substack{1 \le j < k \le d+1 \\ j,k \ne s}} (x_j - x_k) \Big) \Big(\prod_{\substack{1 \le l \le d+1 \\ l \ne s}} (x_s - x_l) \Big) \\
&= (-1)^{s-1} a_{\mathbf{k}(\underline{\lambda_0})}(x_1, \ldots, \hat{x_s}, \ldots, x_{d+1}) \Big(\prod_{\substack{1 \le l \le d+1 \\ l \ne s}} (x_s - x_l) \Big).
\end{aligned}
\tag{3.2.7}
$$

By definition, and the product formula (3.2.1),

$$
\begin{aligned}
s_\tau(x_1,\dots,x_{d+1}) =& \frac{a_{\mathbf{k}(\tau)}(x_1,\dots,x_{d+1})}{a_{\mathbf{k}(\underline{\tau_0})}(x_1,\dots,x_{d+1})} \\
=& \frac{x_1^{k_1} a_{\mathbf{k}(\lambda)}(x_2,\dots,x_{d+1})}{a_{\mathbf{k}(\underline{\lambda_0})}(x_2,\dots,x_{d+1})(\prod_{2\le l\le d+1}(x_1-x_l))} \\
&+(-1)\frac{x_2^{k_1} a_{\mathbf{k}(\lambda)}(x_1,x_3,\dots,x_{d+1})}{(-1)a_{\mathbf{k}(\underline{\lambda_0})}(x_1,x_3,\dots,x_{d+1})(\prod_{\substack{1\le l\le d+1\\ l\neq 2}}(x_2-x_l))}+\dots \\
&\dots+(-1)^d\frac{x_{d+1}^{k_1} a_{\mathbf{k}(\lambda)}(x_1,\dots,x_d)}{(-1)^d a_{\mathbf{k}(\underline{\lambda_0})}(x_1,\dots,x_d)(\prod_{1\le l\le d}(x_{d+1}-x_l))} \\
=& \frac{x_1^{k_1} s_\lambda(x_2,\dots,x_{d+1})}{\prod_{2\le l\le d+1}(x_1-x_l)}+\frac{x_2^{k_1} s_\lambda(x_1,x_3,\dots,x_{d+1})}{\prod_{\substack{1\le l\le d+1\\ l\neq 2}}(x_2-x_l)}+\dots \\
&\dots+\frac{x_{d+1}^{k_1} s_\lambda(x_1,\dots,x_d)}{\prod_{1\le l\le d}(x_{d+1}-x_l)} \\
=& (\phi_{k_1}\cdot s_\lambda)(x_1,\dots,x_{d+1}).
\end{aligned}
\tag{3.2.8}
$$

By induction, $s_\lambda=\phi_{k_2}\cdot\dots\cdot\phi_{k_{d+1}}$. Therefore we have

$$
\begin{aligned}
s_\tau(x_1,\dots,x_{d+1}) =& (\phi_{k_1}\cdot s_\lambda)(x_1,\dots,x_{d+1}) \\
=& (\phi_{k_1}\cdot\phi_{k_2}\cdot\dots\cdot\phi_{k_{d+1}})(x_1,\dots,x_{d+1}).
\end{aligned}
\tag{3.2.9}
$$

□

Proposition 3.2.5. *$\mathcal{H}\simeq\bigwedge^*(\mathcal{H}_1)$ as algebras.*

Proof. Since $\mathcal{H}_d=H^*_{G_d}(M_d)\simeq\mathbb{Q}[x_1,\dots,x_d]^{S_d}$, $\mathcal{H}_d$ (as vector spaces) is the algebra of symmetric polynomials with d variables. Since Schur polynomials form a basis of algebra of symmetric polynomials, (see e.g. [43]), by Proposition 3.2.4, $\mathcal{H}_d\simeq\bigwedge^d(\mathcal{H}_1)$ as vector spaces. The proposition follows immediately. □

Remark 3.2.6. By the proposition, in A_1-COHA case, we would also use wedge $\wedge$ to denote the COHA multiplication.

3.3 Examples: Jordan-quiver

If $m = 1$, the quiver $\mathcal{Q}^{(1)}$ is called the *Jordan quiver*. In this case, the product formula is

$$(f_1 \cdot f_2)(x_1, \ldots, x_{p+q}) := \sum_{\substack{i_1<\ldots<i_p \\ j_1<\ldots<j_q \\ \{i_1,\ldots,i_p,j_1,\ldots,j_q\}=\{1,\ldots,p+q\}}} f_1(x_{i_1}, \ldots, x_{i_p}) f_2(x_{j_1}, \ldots, x_{j_q}). \tag{3.3.1}$$

Using the notations introduced in the previous example. For a partition $\lambda = (k_1, \ldots, k_d)$, $k_1 \geq \ldots \geq k_d \geq 0$, define $\tilde{m}_\lambda$ to be the polynomial:

$$\tilde{m}_\lambda(x_1, \ldots, x_d) = \sum_{\sigma \in S_d} x_1^{\lambda_{\sigma(1)}} x_2^{\lambda_{\sigma(2)}} \ldots x_d^{\lambda_{\sigma(d)}}. \tag{3.3.2}$$

Remark 3.3.1. Let m_λ denote the monomial symmetric function of λ. Then $\tilde{m}_\lambda = c_\lambda m_\lambda$ for some positive integer c_λ. The number comes from the duplicate terms when permuting the powers in the definition of $\tilde{m}$ in (3.3.2). See e.g. [43] for details.

Proposition 3.3.2. *For a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda_1 \geq \ldots \geq \lambda_d \geq 0$,*

$$\tilde{m}_\lambda = \phi_{\lambda_1} \cdot \ldots \cdot \phi_{\lambda_d}. \tag{3.3.3}$$

Proof. Use inductions on the length of λ. Length 0 case is obvious. Now let $\tau = (\tau_1, \ldots, \tau_{d+1})$ be a length $d+1$ partition, and $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a length d partition where $\lambda_i = \tau_i$ for

$i = 1, \ldots, d$. Then

$$
\begin{aligned}
\tilde{m}_\tau(x_1, \ldots, x_{d+1}) = & \sum_{\sigma \in S_{d+1}} x_1^{\lambda_{\sigma(1)}} x_2^{\lambda_{\sigma(2)}} \ldots x_{d+1}^{\lambda_{\sigma(d+1)}} \\
= & \sum_{\substack{\sigma \in S_{d+1} \\ \sigma(1)=d+1}} x_1^{\lambda_{d+1}} x_2^{\lambda_{\sigma(2)}} \ldots x_{d+1}^{\lambda_{\sigma(d+1)}} + \sum_{\substack{\sigma \in S_{d+1} \\ \sigma(2)=d+1}} x_1^{\lambda_{\sigma(1)}} x_2^{\lambda_{d+1}} \ldots x_{d+1}^{\lambda_{\sigma(d+1)}} + \ldots \\
& \ldots + \sum_{\substack{\sigma \in S_{d+1} \\ \sigma(d+1)=d+1}} x_1^{\lambda_{d+1}} x_2^{\lambda_{\sigma(2)}} \ldots x_{d+1}^{\lambda_{d+1}} \\
= & (\sum_{\sigma \in S_d} x_2^{\lambda_{\sigma(1)}} \ldots x_{d+1}^{\lambda_{\sigma(d)}}) x_1^{\lambda_{d+1}} + (\sum_{\sigma \in S_d} x_1^{\lambda_{\sigma(1)}} x_3^{\lambda_{\sigma(2)}} \ldots x_{d+1}^{\lambda_{\sigma(d)}}) x_2^{\lambda_{d+1}} + \ldots \\
& \ldots + (\sum_{\sigma \in S_d} x_1^{\lambda_{\sigma(1)}} \ldots x_d^{\lambda_{\sigma(d)}}) x_{d+1}^{\lambda_{d+1}} \\
= & \tilde{m}_\lambda(x_2, \ldots, x_{d+1}) \phi_{\lambda_{d+1}}(x_1) + \tilde{m}_\lambda(x_1, x_3, \ldots, x_{d+1}) \phi_{\lambda_{d+1}}(x_2) + \ldots \\
& \ldots + \tilde{m}_\lambda(x_1, \ldots, x_d) \phi_{\lambda_{d+1}}(x_{d+1}) \\
= & (\tilde{m}_\lambda \cdot \phi_{\lambda_{d+1}})(x_1, \ldots, x_{d+1}).
\end{aligned}
\tag{3.3.4}
$$

By induction, $\tilde{m}_\lambda = \phi_{\lambda_1} \cdot \ldots \cdot \phi_{\lambda_d}$. Therefore $\tilde{m}_\tau = \phi_{\tau_1} \cdot \ldots \cdot \phi_{\tau_d} \cdot \phi_{\tau_{d+1}}$. $\square$

Proposition 3.3.3. *$\mathcal{H} \simeq \mathrm{Sym}^*(\mathcal{H}_1)$ as algebras.*

Proof. Similar to Proposition 3.2.5, the result follows immediately from the fact that the monomial symmetric polynomials form a basis of the algebra of symmetric polynomials (see e.g. [43]). $\square$

Chapter 4

A COHA module structure over the smooth model of quivers

4.1 Correspondences

4.1.1 Lusztig's model

Lusztig gives a model to describe the correspondence (3.1.2) of quiver representations. We would give a quick review about this model in this subsection. For details, see e.g. [40–42].

For $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$, let $\mathbb{V}_{\mathbf{d}} = \oplus_{i \in I} \mathbb{V}_{d^i}$, $\mathbb{V}_{\mathbf{d}_1} = \oplus_{i \in I} \mathbb{V}_{d_1^i}$ and $\mathbb{V}_{\mathbf{d}_2} = \oplus_{i \in I} \mathbb{V}_{d_2^i}$ be the standard coordinate spaces which serve as the standard underlying vector spaces of representations of quiver $\mathcal{Q}$. Let M_2 be the variety of all pairs $(W, (\mathbb{V}_{\mathbf{d}}, T))$ where $(\mathbb{V}_{\mathbf{d}}, T) \in M_{\mathbf{d}}$ and $W \subset \mathbb{V}_{\mathbf{d}}$ is a I-graded subspace of dimension $\mathbf{d}_1$ which is invariant under the action of T. Let M_1 be the variety of the quadruples $(W, (\mathbb{V}_{\mathbf{d}}, T), f_1, f_2)$ where $(W, (\mathbb{V}_{\mathbf{d}}, T)) \in M_2$, f_1 is an isomorphism $f_1 : \mathbb{V}_{\mathbf{d}_1} \simeq W$, and f_2 is an isomorphism $f_2 : \mathbb{V}_{\mathbf{d}_2} \simeq \mathbb{V}_{\mathbf{d}}/W$.

For each $(W, (\mathbb{V}_{\mathbf{d}}, T), f_1, f_2)$, let $T_1 := f_1^{-1} T|_W f_1$ and $T_2 := f_2^{-1} \bar{T} f_2$, where $T|_W$ is the map T restricted on W and $\bar{T}$ is the endomorphism on $\mathbb{V}_{\mathbf{d}}/W$ induced from $T : \mathbb{V}_{\mathbf{d}} \to \mathbb{V}_{\mathbf{d}}$. Then $T_1 : \mathbb{V}_{\mathbf{d}_1} \to \mathbb{V}_{\mathbf{d}_1}$ (resp. $T_2 : \mathbb{V}_{\mathbf{d}_2} \to \mathbb{V}_{\mathbf{d}_2}$) is a point in $M_{\mathbf{d}_1}$ (resp. $M_{\mathbf{d}_2}$).

Both M_1 and M_2 carries $G_{\mathbf{d}} \times G_{\mathbf{d}_1} \times G_{\mathbf{d}_2}$-action by

$$(g, g_1, g_2) \cdot (W, (\mathbb{V}_{\mathbf{d}}, T), f_1, f_2) = (gW, (\mathbb{V}_{\mathbf{d}}, gTg^{-1}), gf_1g_1^{-1}, \bar{g}f_2g_2^{-1}), \tag{4.1.1}$$

and

$$(g, g_1, g_2) \cdot (W, (\mathbb{V}_{\mathbf{d}}, T)) = (gW, (\mathbb{V}_{\mathbf{d}}, gTg^{-1})). \tag{4.1.2}$$

Here $\bar{g}$ is the action on $\mathbb{V}_{\mathbf{d}}/W$ induced by $g \in G_{\mathbf{d}}$ on $\mathbb{V}_{\mathbf{d}}$. Equip $M_{\mathbf{d}}$ and $M_{\mathbf{d}_1} \times M_{\mathbf{d}_2}$ with $G_{\mathbf{d}} \times G_{\mathbf{d}_1} \times G_{\mathbf{d}_2}$-action structure by setting the other groups act on non-correspondent pieces trivially.

Define $q_1 : M_1 \to M_{\mathbf{d}_1} \times M_{\mathbf{d}_2}$ by $q_1(W, (\mathbb{V}_{\mathbf{d}}, T), f_1, f_2) = ((\mathbb{V}_{\mathbf{d}_1}, T_1), (\mathbb{V}_{\mathbf{d}_2}, T_2))$. Define $q_2 : M_1 \to M_2$ by $q_2(W, (\mathbb{V}_{\mathbf{d}}, T), f_1, f_2) = (W, (\mathbb{V}_{\mathbf{d}}, T))$. Define $q_3 : M_2 \to M_{\mathbf{d}}$ by $q_3(W, (\mathbb{V}_{\mathbf{d}}, T)) = (\mathbb{V}_{\mathbf{d}}, T)$. Then we have a $G_{\mathbf{d}} \times G_{\mathbf{d}_1} \times G_{\mathbf{d}_2}$-equivariant diagram:

$$M_{\mathbf{d}_1} \times M_{\mathbf{d}_2} \xleftarrow{q_1} M_1 \xrightarrow{q_2} M_2 \xrightarrow{q_3} M_{\mathbf{d}}. \tag{4.1.3}$$

Theorem 4.1.1 ([40])**.** *q_1 is a locally trivial fibration with smooth connected fibres, q_2 is a $G_{\mathbf{d}_1} \times G_{\mathbf{d}_2}$-principal bundle and q_3 is proper.*

There is another way to talk about these spaces. Consider the space indicated by the first $\mathbf{d}_1$ coordinates of the standard coordinate space $\mathbb{V}_{\mathbf{d}}$. We denote it by $\mathbb{V}_{\mathbf{d}_1}$. Let $G_{\mathbf{d}_1,\mathbf{d}}$ be the stabilizer of $\mathbb{V}_{\mathbf{d}_1} \subset \mathbb{V}_{\mathbf{d}}$ in $G_{\mathbf{d}}$. Denote by $U_{\mathbf{d}_1,\mathbf{d}}$ the unipotent radical of $G_{\mathbf{d}_1,\mathbf{d}}$. We have canonically $G_{\mathbf{d}_1,\mathbf{d}}/U_{\mathbf{d}_1,\mathbf{d}} = G_{\mathbf{d}_1} \times G_{\mathbf{d}_2}$. Let $M_{\mathbf{d}_1,\mathbf{d}}$ be the closed subvariety of $M_{\mathbf{d}}$ consisting of all $(\mathbb{V}_{\mathbf{d}}, T)$ such that the standard coordinate subspace $\mathbb{V}_{\mathbf{d}_1} \subset \mathbb{V}_{\mathbf{d}}$ is invariant under T. Denote the natural embedding $M_{\mathbf{d}_1,\mathbf{d}} \hookrightarrow M_{\mathbf{d}}$ by $\imath$. Consider the $G_{\mathbf{d}_1,\mathbf{d}}$-equivariant map $\kappa : M_{\mathbf{d}_1,\mathbf{d}} \to M_{\mathbf{d}_1} \times M_{\mathbf{d}_2}$. We have the diagram:

$$M_{\mathbf{d}_1} \times M_{\mathbf{d}_2} \xleftarrow{q_1} G_{\mathbf{d}} \times_{U_{\mathbf{d}_1,\mathbf{d}}} M_{\mathbf{d}_1,\mathbf{d}} \xrightarrow{q_2} G_{\mathbf{d}} \times_{G_{\mathbf{d}_1,\mathbf{d}}} M_{\mathbf{d}_1,\mathbf{d}} \xrightarrow{q_3} M_{\mathbf{d}}, \tag{4.1.4}$$

where $q_1(g, (\mathbb{V}_{\mathbf{d}}, T)) = \kappa(\mathbb{V}_{\mathbf{d}}, T)$, $q_2(g, (\mathbb{V}_{\mathbf{d}}, T)) = (g, (\mathbb{V}_{\mathbf{d}}, T))$, and $q_3(g, (\mathbb{V}_{\mathbf{d}}, T)) = g(\imath((\mathbb{V}_{\mathbf{d}}, T)))$.

Theorem 4.1.2 ([42]). *$k : M_{\boldsymbol{d}_1,\boldsymbol{d}} \to M_{\boldsymbol{d}_1} \times M_{\boldsymbol{d}_2}$ is a vector bundle, q_2 is a $G_{\boldsymbol{d}_1} \times G_{\boldsymbol{d}_2}$-principal bundle and q_3 is proper.*

This two descriptions are equivalent to each other by the following proposition.

Proposition 4.1.3. *$G_{\boldsymbol{d}} \times_{U_{\boldsymbol{d}_1,\boldsymbol{d}}} M_{\boldsymbol{d}_1,\boldsymbol{d}}$ is $G_{\boldsymbol{d}} \times G_{\boldsymbol{d}_1} \times G_{\boldsymbol{d}_2}$-equivariant isomorphic to M_1. $G_{\boldsymbol{d}} \times_{G_{\boldsymbol{d}_1,\boldsymbol{d}}} M_{\boldsymbol{d}_1,\boldsymbol{d}}$ is $G_{\boldsymbol{d}}$-equivariant isomorphic to M_2.*

Proof. Recall that $M_{\mathbf{d}_1,\mathbf{d}}$ is the variety of all representations of $\mathcal{Q}$ on $\mathbb{V}_{\mathbf{d}}$ which keep the standard coordinate subspace $\mathbb{W}_{\mathbf{d}_1} \subset \mathbb{V}_{\mathbf{d}}$ invariant. Let $i : \mathbb{W}_{\mathbf{d}_1} \to \mathbb{V}_{\mathbf{d}}$ be the canonical embedding. Let $k : \mathbb{V}_{\mathbf{d}_2} \to \mathbb{V}_{\mathbf{d}}/\mathbb{W}_{\mathbf{d}_1}$ be the canonical isomorphism. Define $\phi : G_{\mathbf{d}} \times M_{\mathbf{d}_1,\mathbf{d}} \to M_1$ by

$$\phi : (g, T) \mapsto (g\mathbb{W}_{\mathbf{d}_1}, gTg^{-1}, gi, \bar{g}k), \tag{4.1.5}$$

where $\bar{g}$ is the action on $\mathbb{V}_{\mathbf{d}}/\mathbb{W}_{\mathbf{d}_1}$ induced by g on $\mathbb{V}_{\mathbf{d}}$. This is a $G_{\mathbf{d}} \times G_{\mathbf{d}_1} \times G_{\mathbf{d}_2}$-map, and the kernel is $U_{\mathbf{d}_1,\mathbf{d}}$. Thus we get the isomorphism we want. The other follows from the similar argument. $\square$

Proposition 4.1.4. *$[G_{\boldsymbol{d}} \times_{P_{\boldsymbol{d}_1,\boldsymbol{d}}} M_{\boldsymbol{d}_1,\boldsymbol{d}}/G_{\boldsymbol{d}}]$ is isomorphic to $[M_{\boldsymbol{d}_1,\boldsymbol{d}}/P_{\boldsymbol{d}_1,\boldsymbol{d}}]$ as CFGs.*

This proposition let us to use $M_{\mathbf{d}_1,\mathbf{d}}$ acted by $P_{\mathbf{d}_1,\mathbf{d}}$ as models to talk about correspondences.

4.1.2 Correspondences for stable framed representations

Fix a quiver $\mathcal{Q} = (I, H)$. Fix a framed structure $\mathbf{n} = (n^i)_{i\in I}$ and a stability condition Θ. Following Section 2.2, let $M^{\Theta-st}_{\mathbf{d},\mathbf{n}}$ be the variety of all stable framed representations of dimension $\mathbf{d}$ on the standard coordinate space $\mathbb{V}_{\mathbf{d}}$ with the framed structure $\mathbf{n}$ under the stability condition Θ. We use $M^{st}_{\mathbf{d},\mathbf{n}}$ for short. $G_{\mathbf{d}}$ acts on $M^{st}_{\mathbf{d},\mathbf{n}}$ naturally. Then there exists a moduli space $\mathcal{M}^{st}_{\mathbf{d},\mathbf{n}} = M^{st}_{\mathbf{d},\mathbf{n}}/G_{\mathbf{d}}$.

Let $((V, T), f)$ be a stable framed representation of dimension $\mathbf{d} = (d^i)_{i\in I}$. Assume that we have a $\mathbf{d}_1$-dimensional subrepresentation $W \subset V$ of $\mathcal{Q}$. That is, $W = (W_i)_{i\in I}$, $W_i \subset V_i$, $\underline{\dim} W = \mathbf{d}_1$ and $\oplus_{i\in I} W_i$ is invariant under the action of T. Then there is a natural framed representation structure on V/W. We denote it by $((V/W, \overline{T}), \overline{f})$.

Theorem 4.1.5. *Assume the slopes of V, W and V/W are the same. If $((V,T),f)$ is a stable framed representation, so is $((V/W,\overline{T}),\overline{f})$.*

Proof. Take any subspace $U/W \subset V/W$ which contains $\mathrm{Im}(\overline{f})$. Denote the preimage of U/W in V under the natural projection by U. Then $U \subset V$ contains $\mathrm{Im}(f)$, and we have the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] & V \arrow[r] & V/W \arrow[r] & 0 . \\ 0 \arrow[r] & W \arrow[r] \arrow[u, "="'] & U \arrow[r] \arrow[u, hook] & U/W \arrow[r] \arrow[u, hook] & 0 \end{tikzcd} \tag{4.1.6}$$

By Proposition 2.2.6, since $((V,T),f)$ is stable, $\mu(U) < \mu(V)$. Due to the fact that $\mu(V) = \mu(W)$, by Lemma 2.1.19, $\mu(W) > \mu(U) > \mu(U/W)$. Thus by $\mu(W) = \mu(V/W)$, we have $\mu(U/W) < \mu(V/W)$. □

This Theorem enables us to define the correspondence for smooth models in the following way. Fix a slope $\mu \in \mathbb{Q}$. Fix two dimension vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ such that $\mu^{\Theta}(\mathbf{d}_1) = \mu^{\Theta}(\mathbf{d}_2) = \mu$ and set $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$. Then $\mu^{\Theta}(\mathbf{d}) = \mu$. We apply the Lusztig's model to describe the correspondences here. Recall that $\mathbb{V}_{\mathbf{d}}$ is the standard coordinate space, and $\mathbb{V}_{\mathbf{d}_1} \subset \mathbb{V}_{\mathbf{d}}$ is the subspace consisting of the first $\mathbf{d}_1$ coordinates. Let $M^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}}$ be the subvariety of $M^{st}_{\mathbf{d},\mathbf{n}}$ which keeps $\mathbb{V}_{\mathbf{d}_1} \subset \mathbb{V}_{\mathbf{d}}$ invariant, and the subgroup of the automorphism group of $\mathbb{V}_{\mathbf{d}}$ which preserves $\mathbb{V}_{\mathbf{d}_1}$ is denoted by $P_{\mathbf{d}_1,\mathbf{d}}$. By the construction similar to diagram (4.1.4), we have the following diagram

$$M_{\mathbf{d}_1} \times M^{st}_{\mathbf{d}_2,\mathbf{n}} \xleftarrow{q_1} G_{\mathbf{d}} \times_{U_{\mathbf{d}_1,\mathbf{d}}} M^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}} \xrightarrow{q_2} G_{\mathbf{d}} \times_{P_{\mathbf{d}_1,\mathbf{d}}} M^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}} \xrightarrow{q_3} M^{st}_{\mathbf{d},\mathbf{n}}. \tag{4.1.7}$$

After taking quotient by $G_{\mathbf{d}} \times G_{\mathbf{d}_1} \times G_{\mathbf{d}_2}$, we have

$$[M_{\mathbf{d}_1}/G_{\mathbf{d}_1}] \times \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}} \xleftarrow{q_1} [M^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}}/P_{\mathbf{d}_1,\mathbf{d}}] \xrightarrow{q_2} [M^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}}/P_{\mathbf{d}_1,\mathbf{d}}] \xrightarrow{q_3} \mathcal{M}^{st}_{\mathbf{d},\mathbf{n}}. \tag{4.1.8}$$

Proposition 4.1.6. *$[M^{st}_{\boldsymbol{d}_1,\boldsymbol{d},\boldsymbol{n}}/P_{\boldsymbol{d}_1,\boldsymbol{d}}]$ can be represented by a scheme. The scheme is denoted by $\mathcal{M}^{st}_{\boldsymbol{d}_1,\boldsymbol{d},\boldsymbol{n}}$.*

Proof. Let $(W,((\mathbb{V}_{\mathbf{d}},T),f))$ be a point in $M^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}}$. Let $g\in P_{\mathbf{d}_1,\mathbf{d}}$. The action is

$$g\cdot(W,((\mathbb{V}_{\mathbf{d}},T),f))=(gW,((\mathbb{V}_{\mathbf{d}},gTg^{-1}),gf)). \tag{4.1.9}$$

Now let $g\cdot(W,((\mathbb{V}_{\mathbf{d}},T),f))=(W,((\mathbb{V}_d,T),f))$. That is,

$$(gW,((\mathbb{V}_{\mathbf{d}},gTg^{-1}),gf))=(W,((\mathbb{V}_d,T),f)). \tag{4.1.10}$$

Since W is invariant under the action of $P_{\mathbf{d}_1,\mathbf{d}}$, we have

$$gT=Tg,\ \ gf_i=f_i,\ \ i=1,\dots,N. \tag{4.1.11}$$

Fix a basis vector $\mathbf{v}\in\mathbb{W}$, and let $v_i=f_i(\mathbf{v})$. Consider the subspace U generated by v_i, $i=1,\dots N$ under the action of T. It is possible to choose a basis $v_j^k:=T^kv_j$ of U for some $k\in\mathbb{Z}_{\geq 0}$ and $j=1,\dots,N$. Then we have

$$gv_j^k=gT^kv_j=T^kgf_j(\mathbf{v})=T^kf_j(\mathbf{v})=v_j^k. \tag{4.1.12}$$

Therefore $g=\mathrm{Id}$ when restricting on U.

Assume g is not trivial. Then there is a one parameter subgroup $g_\lambda\subset G_{\mathbf{d}}$ such that g_λ fix $(W,((\mathbb{V}_{\mathbf{d}},T),f))$. Then we can decompose $\mathbb{V}_{\mathbf{d}}$ into the direct sum of weight space $\mathbb{V}_{\mathbf{d}}=\oplus_{n\geq 0}V(n)$ where $t\in g_\lambda$ acts on $V(n)$ as t^n. It is obvious that $U\subset V(0)$. Thus $V(0)$ is not trivial.

Furthermore, since $gT=Tg$ for any $g\in g_\lambda$, $V(i)$ is invariant under the action of T for any $i\in\mathbb{Z}_{\geq 0}$. Therefore, each $V(i)$ is a submodule of the original quiver $\mathcal{Q}$. Since $\mathrm{Im}(f)\subset U\subset V(0)$, $\mathrm{Im}(f)$ is not in $V(i)$ for $i>0$. Let $V_{>0}:=\oplus_{i>0}V(i)$. $V_{>0}$ is a submodule of the original quiver which doesn't contain the image of f. Then we have

$$\mathbb{V}_{\mathbf{d}}=V(0)\oplus V_{>0}. \tag{4.1.13}$$

By Proposition 2.2.6, $\mu(V_{>0}) \leq \mu(\mathbb{V}_{\mathbf{d}})$, and $\mu(V(0)) < \mu(\mathbb{V}_{\mathbf{d}})$. By Lemma 2.1.19, $\mu(V(0)) < \mu(\mathbb{V}_{\mathbf{d}}) < \mu(V_{>0})$. This a contradiction. Then $g = \mathrm{Id}$ on the whole $\mathbb{V}_{\mathbf{d}}$. This shows that $P_{\mathbf{d}_1,\mathbf{d}}$ acts on $M_{\mathbf{d}_1,\mathbf{d},\mathbf{n}}$ freely.

□

Then we build up the following correspondence diagram:

$$\begin{tikzcd} \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}} & \mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}} \arrow[l, "p_2"'] \arrow[r, "p"] \arrow[d, "p_1"] & \mathcal{M}^{st}_{\mathbf{d},\mathbf{n}} . \\ & {[M_{\mathbf{d}_1}/G_{\mathbf{d}_1}]} & \end{tikzcd} \tag{4.1.14}$$

Here

$$p(W,((\mathbb{V}_{\mathbf{d}},T),f)) = ((\mathbb{V}_{\mathbf{d}},T),f),$$

$$p_1(W,((\mathbb{V}_{\mathbf{d}},T),f)) = (W,T|_W),$$

$$p_2(W,((\mathbb{V}_{\mathbf{d}},T),f)) = ((\mathbb{V}_{\mathbf{d}}/W,\bar{T}),\bar{f}).$$

4.2 Increasing operators

The map

$$\phi^+(f) := p_*(p_1^*(\phi) \cup p_2^*(f)), \quad \text{for } \phi \in \mathcal{H}_{\mathbf{d}_1}, f \in H^*(\mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}}) \tag{4.2.1}$$

defines a morphism from $H^*(\mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}})$ to $H^*(\mathcal{M}^{st}_{\mathbf{d},\mathbf{n}})$. Since p is proper, the operator is well-defined.

Theorem 4.2.1. *$\phi_1^+(\phi_2^+(f)) = (\phi_1 \cdot \phi_2)^+(f)$ for $\phi_1 \in \mathcal{H}_{\boldsymbol{d}_1}$, $\phi_2 \in \mathcal{H}_{\boldsymbol{d}_2}$ and $f \in \mathcal{M}^{st}_{\boldsymbol{d}_3,\boldsymbol{n}}$.*

Proof. Consider the correspondence C_1 consisting of diagrams

$$\begin{tikzcd}[column sep=small] & & F_{\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3} \arrow[dr, two heads] & & \\ & & & F_{\mathbf{d}_1+\mathbf{d}_3} \arrow[dr, two heads] & \\ E_{\mathbf{d}_2} \arrow[uurr, hook] & & E_{\mathbf{d}_1} \arrow[ur, hook] & & F_{\mathbf{d}_3} \end{tikzcd} \tag{4.2.2}$$

C_2 consisting of diagrams

$$\begin{tikzcd}
 & & F_{\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3} \arrow[drr, ->>] & & \\
 & E_{\mathbf{d}_1+\mathbf{d}_2} \arrow[ur, hook] \arrow[dr, ->>] & & & \\
E_{\mathbf{d}_2} \arrow[ur, hook] & & E_{\mathbf{d}_1} & & F_{\mathbf{d}_3}
\end{tikzcd} \tag{4.2.3}$$

and C consisting of all the pieces $\{(E_1, E_{\mathbf{d}_2}, E_{\mathbf{d}_1+\mathbf{d}_2}, F_{\mathbf{d}_3}, F_{\mathbf{d}_1+\mathbf{d}_3}, F_{\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3})\}$. Here $E_{\mathbf{d}} \in M_{\mathbf{d}}$ and $F_{\mathbf{d}} \in \mathcal{M}^{st}_{\mathbf{d},\mathbf{n}}$ for any dimension vector $\mathbf{d}$. The natural projections to each components give the following diagram which is commutative.

$$\begin{tikzcd}
C \arrow[r, "q_1"] & C_1 \arrow[r, "p^1_{2,123}"] \arrow[d, "p^1_{1,13}"] & \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3,\mathbf{n}} \arrow[r, "p^{2,123}_{123}"] \arrow[d, "p^{2,123}_{13}"] \arrow[dr, "p^{2,123}_{2}"] & \mathcal{M}^{st}_{\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3,\mathbf{n}} \\
 & \mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{d}_1+\mathbf{d}_3,\mathbf{n}} \arrow[r, "p^{1,13}_{13}"] \arrow[d, "p^{1,13}_{3}"] \arrow[dr, "p^{1,13}_{1}"] & \mathcal{M}^{st}_{\mathbf{d}_1+\mathbf{d}_3,\mathbf{n}} & M_{\mathbf{d}_2} \\
 & \mathcal{M}^{st}_{\mathbf{d}_3,\mathbf{n}} & M_{\mathbf{d}_1} &
\end{tikzcd} \tag{4.2.4}$$

In the diagram, $p^{1,13}_{13}$ and $p^1_{2,123}$ are proper, and $p^1_{1,13}$ and $p^{2,123}_{13}$ are flat. Then by Proposition A.1.8, we have

$$\begin{aligned}
\phi_2^+(\phi_1^+(f)) &= {p^{2,123}_{123}}_*(p^{2,123*}_2(\phi_2) \cup p^{2,123*}_{13}({p^{1,13}_{13}}_*(p^{1,13*}_1(\phi_1) \cup p^{1,13*}_3(f)))) \\
&= p_*(p_2^*(\phi_2) \cup p_1^*(\phi_1) \cup p_3^*(f)),
\end{aligned} \tag{4.2.5}$$

where $p_1 = p^{1,13}_1 \circ p^1_{1,13} \circ q_1$, $p_2 = p^{2,123}_2 \circ p^1_{2,123} \circ q_1$, $p_3 = p^{1,13}_3 \circ p^1_{1,13} \circ q_1$ and $p = p^{2,123}_{123} \circ p^1_{2,123} \circ q_1$. Similarly, for C_2 and related correspondences and moduli spaces, we have the following diagram

$$
\begin{tikzcd}
C \arrow[r, "q_2"] & C_2 \arrow[r, "p^2_{3,123}"] \arrow[d, "p^2_{1,12}"] & \mathcal{M}^{st}_{\mathbf{d}_3,\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3,\mathbf{n}} \arrow[r, "p^{3,123}_{123}"] \arrow[d, "p^{3,123}_{12}"] \arrow[dr, "p^{3,123}_3"] & \mathcal{M}^{st}_{\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3,\mathbf{n}} \, . \\
 & M_{\mathbf{d}_1,\mathbf{d}_1+\mathbf{d}_2} \arrow[r, "p^{1,12}_{12}"] \arrow[d, "p^{1,12}_1"] \arrow[dr, "p^{1,12}_2"] & M_{\mathbf{d}_1+\mathbf{d}_2} & \mathcal{M}^{st}_{\mathbf{d}_3,\mathbf{n}} \\
 & M_{\mathbf{d}_1} & M_{\mathbf{d}_2} &
\end{tikzcd}
\tag{4.2.6}
$$

and

$$
(\phi_2 \cdot \phi_1)^+(f)) = p'_*(p'^*_2(\phi_2) \cup p'^*_1(\phi_1) \cup p'^*_3(f)), \tag{4.2.7}
$$

where p'_1, p'_2, p'_3 and p' are the natural projections from C through C_2 to $M_{\mathbf{d}_1}$, $M_{\mathbf{d}_1}$, $\mathcal{M}^{st}_{\mathbf{d}_3,\mathbf{n}}$ and $\mathcal{M}^{st}_{\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3,\mathbf{n}}$. Since in C_1 and C_2, by the universal properties of embeddings and projections, $C \xrightarrow{q_1} C_1$ and $C \xrightarrow{q_2} C_2$ are isomorphisms. Therefore we arrive at the desired result.

$\square$

The Theorem shows that Formula (4.2.1) defines a representation of $\mathcal{H} = \bigoplus_{\mathbf{d}} \mathcal{H}_{\mathbf{d}}$ on $\bigoplus_{\mathbf{d}} H^*(\mathcal{M}^{st}_{\mathbf{d},\mathbf{n}})$. It is called *the increasing representation* of COHA of the quiver Q. The operators related to the generators of the COHA are called the *increasing operators*.

4.3 Decreasing operators

Recall the diagram of correspondence

$$
\begin{tikzcd}
\mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}} & {[M^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}}/P_{\mathbf{d}_1,\mathbf{d},\mathbf{n}}]} \arrow[l, "p_2"'] \arrow[r, "p"] \arrow[d, "p_1"] & \mathcal{M}^{st}_{\mathbf{d},\mathbf{n}} \, . \\
 & {[M_{\mathbf{d}_1}/G_{\mathbf{d}_1}]} &
\end{tikzcd}
\tag{4.1.14}
$$

Choose an approximation $U_n/G_{\mathbf{d}_1}$ of $[M_{\mathbf{d}_1}/G_{\mathbf{d}_1}]$. We have the diagram

$$
\mathcal{M}^{st}_{\mathbf{d},\mathbf{n}} \xleftarrow{p} \mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}} \xrightarrow{p_{1,n} \times p_2} M_{\mathbf{d}_1} \times_{G_{\mathbf{d}_1}} U_n \times \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}}. \tag{4.3.1}
$$

To construct decreasing operators, we introduce the following conditions.

Condition A. $p_1 \times p_2 : \mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}} \to M_{\mathbf{d}_1} \times \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}}$ is a fibre bundle with fibers being projective schemes. The fiber can be embedded into the classifying space $BG_{\mathbf{d}_1}$. Furthermore all pieces has a cellular decomposition.

Assume the condition holds. Since all pieces has a cellular decomposition, the cohomology of $\mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}}$ is generated by the basis of the cohomology of the fibres as $H^*(M_{\mathbf{d}_1} \times \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}})$-modules. Since the fibre can be embedded into $BG_{\mathbf{d}_1}$, there is a projection from $H^*(BG_{\mathbf{d}_1})$ to the cohomology of the fibre. This implies that we can use COHA generators from $H^*([M_{\mathbf{d}_1}/G_{\mathbf{d}_1}])$ to denote the elements in the cohomology of fibres under the projection.

Choose the $H^*(M_{\mathbf{d}_1} \times \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}})$-basis of $H^*(\mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}})$ which are of forms $p_1^*(\phi_i)$ where $\phi_i \in H^*_{G_{\mathbf{d}_1}}(M_{\mathbf{d}_1})$ are COHA generators. We can define a $H^*(M_{\mathbf{d}_1} \times \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}})$-pairing

$$\langle \alpha, \beta \rangle := (p_1 \times p_2)_*(\alpha \cup \beta) \tag{4.3.2}$$

for $\alpha, \beta \in H^*(\mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}})$. Define the dual $D(p_1^*(\phi_i))$ of $p_1^*(\phi_i)$ by the formula

$$\langle D(p_1^*(\phi_i)), p_1^*(\phi_j) \rangle = \delta_{i,j}. \tag{4.3.3}$$

By Pioncaré duality, and the fact that the fibre is projective, the pairing is non-degenerate. Then the dual exists for any $p_1^*(\phi_i) \in H^*(\mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}})$. Again, since the basis elements in $H^*(\mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{d},\mathbf{n}})$ are pullbacks from $H^*([M_{\mathbf{d}_1}/G_{\mathbf{d}_1}])$, there exists $\phi_i^D \in H^*([M_{\mathbf{d}_1}/G_{\mathbf{d}_1}])$ for $\phi_i \in H^*([M_{\mathbf{d}_1}/G_{\mathbf{d}_1}])$ such that $D(p_1^*(\phi_i)) = p_1^*(\phi_i^D)$

The next piece we need is the pullback $\pi_2^* : H^*(\mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}}) \to H^*(M_{\mathbf{d}_1} \times \mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{n}})$. It is an isomorphism due to the fact that the natural projection to the second component $\pi_2 : M_{\mathbf{d}_1} \times \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}} \to \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}}$ is a vector bundle of rank $\dim M_{\mathbf{d}_1}$.

We need another condition to proceed.

Condition B. Let $\phi_i \in H^*([M_{\mathbf{d}_1}/G_{\mathbf{d}_1}])$, $\phi_2 \in H^*([M_{\mathbf{d}_2}/G_{\mathbf{d}_2}])$. Then $\phi_i^D \phi_j^D = (\phi_j \phi_i)^D$, where the multiplication is the COHA multiplication.

Finally we come to the definition of *decreasing operator.* For $f \in H^*(\mathcal{M}^{st}_{\mathbf{d},\mathbf{n}})$, $\phi_i \in$

$H^*([M_{\mathbf{d}_1}/G_{\mathbf{d}_1}])$,

$$\phi_i^-(f) := (\pi_2^*)^{-1}(p_1 \times p_2)_*(p_1^*(\phi_i^D) \cup p^*(f)) \in H^*(\mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{n}}). \tag{4.3.4}$$

The isomorphism $(\pi_2^*)^{-1}$ is omitted from the formula for simplicity when there is no confusion.

Remark 4.3.1. Note that the decreasing operator constructed above depends on the choices of basis. However, in the cases of A_1-quiver and Jordan quiver and $\mathbf{d}_1 = 1$, the cohomology of each degree is of dimension 1. Therefore there are no choices of basis in this dissertation.

Theorem 4.3.2. *If both Condition A and Condition B hold, $\phi_1^-(\phi_2^-(f)) = (\phi_1 \cdot \phi_2)^-(f)$, for any $f \in H^*(\mathcal{M}^{st}_{\boldsymbol{d}_3,\boldsymbol{n}})$, $\phi_1 \in H^*([M_{\boldsymbol{d}_1}/G_{\boldsymbol{d}_1}])$ and $\phi_2 \in H^*([M_{\boldsymbol{d}_2}/G_{\boldsymbol{d}_2}])$.*

Proof. Consider the same correspondence C_1, C_2 and C in Theorem 4.2.1. We have the following diagram.

$$\begin{tikzcd}
C \arrow[r, "q_1"] & C_1 \arrow[r, "p^1_{2,123}"] \arrow[d, "p^1_{1,13}"] & \mathcal{M}^{st}_{\mathbf{d}_2,\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3,\mathbf{n}} \arrow[r, "p^{2,123}_{123}"] \arrow[d, "p^{2,123}_{13}"] \arrow[dr, "p_2^{2,123}"] & \mathcal{M}^{st}_{\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3,\mathbf{n}} \\
 & \mathcal{M}^{st}_{\mathbf{d}_1,\mathbf{d}_1+\mathbf{d}_3,\mathbf{n}} \arrow[r, "p^{1,13}_{13}"] \arrow[d, "p_3^{1,13}"] \arrow[dr, "p_1^{1,13}"] & \mathcal{M}^{st}_{\mathbf{d}_1+\mathbf{d}_3,\mathbf{n}} & M_{\mathbf{d}_2} \\
 & \mathcal{M}^{st}_{\mathbf{d}_3,\mathbf{n}} & M_{\mathbf{d}_1} &
\end{tikzcd} \tag{4.3.5}$$

By the diagram and the projection formula, we have

$$\begin{aligned}
\phi_1^-(\phi_2^-(f)) &= {p_3^{1,13}}_*(p_1^{1,13*}(\phi_1^D) \cup p_{13}^{1,13*}({p_{13}^{2,123}}_*(p_2^{2,123*}(\phi_2^D) \cup p_{123}^{2,123*}(f)))) \\
&= p_{3*}(p_1^*(\phi_1^D) \cup p_2^*(\phi_2^D) \cup p^*(f)),
\end{aligned} \tag{4.3.6}$$

where $p_1 = p_1^{1,13} \circ p^1_{1,13} \circ q_1$, $p_2 = p_2^{2,123} \circ p^1_{2,123} \circ q_1$, $p_3 = p_3^{1,13} \circ p^1_{1,13} \circ q_1$ and $p = p_{123}^{2,123} \circ p^1_{2,123} \circ q_1$, $\phi_1 \in H^*([M_{\mathbf{d}_1}/G_{\mathbf{d}_1}])$, $\phi_2 \in H^*([M_{\mathbf{d}_2}/G_{\mathbf{d}_2}])$, $f \in H^*(\mathcal{M}^{st}_{\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3,\mathbf{n}})$. Similarly, for C_2 and related correspondences and moduli spaces, we have the following diagram

$$\begin{tikzcd} C \arrow[r, "q_2"] & C_2 \arrow[r, "p^2_{3,123}"] \arrow[d, "p^2_{1,12}"] & \mathcal{M}^{st}_{\mathbf{d}_3,\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3,\mathbf{n}} \arrow[r, "p^{3,123}_{123}"] \arrow[d, "p^{3,123}_{12}"] \arrow[drr, "p^{3,123}_3"] & \mathcal{M}^{st}_{\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3,\mathbf{n}} \, . & \\ & M_{\mathbf{d}_1,\mathbf{d}_1+\mathbf{d}_2} \arrow[r, "p^{1,12}_{12}"] \arrow[d, "p^{1,12}_1"] \arrow[dr, "p^{1,12}_2"] & M_{\mathbf{d}_1+\mathbf{d}_2} & & \mathcal{M}^{st}_{\mathbf{d}_3,\mathbf{n}} \\ & M_{\mathbf{d}_1} & M_{\mathbf{d}_2} & & \end{tikzcd} \tag{4.3.7}$$

and

$$(\phi_1 \cdot \phi_2)^-(f)) = p'_{3*}({p'}^*_{12}((\phi_1\phi_2)^D) \cup {p'}^*(f)), \tag{4.3.8}$$

where $p'_{12} = p^{1,12}_{12} \circ p^2_{1,12} \circ q_2$, $p'_3 = p^{3,123}_3 \circ p^2_{3,123} \circ q_2$ and $p' = p^{3,123}_{123} \circ p^2_{3,123} \circ q_2$. Since Condition B holds, we have

$$(\phi_1 \cdot \phi_2)^-(f)) = p'_{3*}({p'}^*_1(\phi_1^D) \cup {p'}^*_2(\phi_2^D) \cup {p'}^*(f)), \tag{4.3.9}$$

where $p'_1 = p^{1,12}_1 \circ p^2_{1,12} \circ q_2$ and $p'_2 = p^{1,12}_2 \circ p^2_{1,12} \circ q_2$. Then by the same argument as that in the case of increasing operators, we prove the associativity. $\square$

Chapter 5

A_1-quiver case

In Section 2.2.2, the smooth models of A_1-quiver is proved to be Grassmannians. Therefore if we apply the constructions from Chapter 4, we would arrive at an A_1-COHA module structure on the cohomology of Grassmannians.

In this chapter, we are going to take these computations in details. We start from a review on Grassmannians with an emphasis on the cohomology of Grassmannians. Then we are going to show how A_1-COHA acts, with both increasing operators and decreasing operators. Finally we will show that these two operators can be combined to give a finite Clifford algebra module structure.

5.1 Grassmannians

In this section, we mainly follow [21] and [5].

Definition 5.1.1. The *Grassmannian* $\mathrm{Gr}(d, n)$ is the variety of d-dimensional linear subspaces of $\mathbb{C}^n$.

Definition 5.1.2. Given a sequence $(d_1, \ldots, d_m)$ of positive integers with sum n, a *flag of type* $(d_1, \ldots, d_m)$ *in* $\mathbb{C}^n$ is an increasing sequence of linear subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_m = \mathbb{C}^n \tag{5.1.1}$$

such that $\dim(V_j/V_{j-1}) = d_j$ for $j = 1, \ldots, m$. The variety of all flags of type $(d_1, \ldots, d_m)$ is called the *partial flag variety* of type $(d_1, \ldots, d_m)$. The partial flag variety of type $(1, 1, \ldots, 1)$ is called the *full flag variety* (or flag variety for short), and is denoted by $\mathrm{Fl}(n)$.

Remark 5.1.3. From the definition, it is not hard to see that Grassmannian $\mathrm{Gr}(d, n)$ is the partial flag variety of type $(d, n-d)$.

Theorem 5.1.4 ([21], p.161)**.** *The cohomology of full flag variety* $\mathrm{Fl}(n)$ *is isomorphic to* $R(n) = \mathbb{Q}[x_1, \ldots, x_n]/(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n))$*, where* $e_i(x_1, \ldots, x_n)$ *represents the* i*-th elementary symmetric polynomial.*

There is a natural projection $\pi : \mathrm{Fl}(n) \to \mathrm{Gr}(d, n)$. By abuse of notations, we use the same symbol x_i to denote the classes in $\mathrm{Gr}(d, n)$ whose pullback $\pi^*(x_i)$ is $x_i \in H^*(Fl(n))$. Recall the definition of partitions and Schur polynomials from Section 3.2.

Corollary 5.1.5 ([21])**.** *The cohomology of Grassmannian* $\mathrm{Gr}(d, n)$ *is a subalgebra of* $R(n)$ *which is generated by Schur polynomials in variables* $x_1, \ldots, x_d$ *indexed by* d*-partitions. Therefore we can use* $s_\lambda(x_1, \ldots, x_d)$ *to represent classes in* $H^*(\mathrm{Gr}(d, n))$*.*

Remark 5.1.6. The partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ representing a class in $H^*(\mathrm{Gr}(d, n))$ has a restriction: $\lambda_1 \leq n - d$. See e.g. [5] for details.

Definition 5.1.7. For a d-partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ of n with $\lambda_1 \leq n - d$, the *transpose* of λ is a $(n-d)$-partition λ' defined by $\lambda'_j = \#\{\lambda_i \geq n - d + 1 - j\}$ for $1 \leq j \leq n - d$.

Let h_r (resp. e_r) stand for the r-th complete symmetric polynomial (resp. elementary symmetric polynomial). Let λ and λ' are two partitions which are transpose to each other.

Theorem 5.1.8 (Jacobi-Trudi identity. See e.g. [43])**.**

$$s_\lambda = \det(h_{\lambda_i - i + j}) = \det(e_{\lambda'_i - i + j}).$$

Based on Jacobi-Trudi identity, classes in $H^*(\mathrm{Gr}(d, n))$ have an alternative presentation.

Lemma 5.1.9. *In* $H^*(\mathrm{Gr}(d,n))$*,* $s_\lambda(x_1,\ldots,x_d) = (-1)^{|\lambda|} s_{\lambda'}(x_{d+1},\ldots,x_n)$*, where* λ' *is the transpose partition of* λ*.*

Proof. We start from the identity in the ring $R(n)[t]$. (see e.g. [21], p.163.)

$$\prod_{i=1}^{d} \frac{1}{1-x_i t} = \prod_{i=d+1}^{n} (1-x_i t). \tag{5.1.2}$$

Since

$$\prod_{i=1}^{d} \frac{1}{1-x_i t} = \sum_{r\geq 0} h_r(x_1,\ldots,x_d) t^r \tag{5.1.3}$$

and

$$\prod_{i=d+1}^{n} (1-x_i t) = \sum_{r\geq 0} e_r(x_{d+1},\ldots,x_n)(-t)^r \tag{5.1.4}$$

where h_r (resp. e_r) stands for the r-th complete symmetric polynomial (resp. elementary symmetric polynomial), we have

$$h_r(x_1,\ldots,x_d) = (-1)^r e_r(x_{d+1},\ldots,x_n), \quad r\geq 0. \tag{5.1.5}$$

By Jacobi-Trudi identity,

$$\begin{aligned} s_{\lambda'}(x_1,\ldots,x_d) &= \det(e_{\lambda_i-i+j}(x_1,\ldots,x_d)) = \det((-1)^{\lambda_i-i+j} h_{\lambda_i-i+j}(x_{d+1},\ldots,x_n)) \\ &= (-1)^{|\lambda|}\det(h_{\lambda_i-i+j}(x_{d+1},\ldots,x_n)) = (-1)^{|\lambda|} s_\lambda(x_{d+1},\ldots,x_n). \end{aligned}$$

The third identity comes from the fact that

$$\begin{aligned} \det(h_{\lambda_i-i+j} t^{\lambda_i-i+j}) &= \sum_{\omega}\sum_{i=1}^{n} (-1)^{\omega} h_{\lambda_i-i+\omega(i)} t^{\lambda_i-i+\omega(i)} \\ &= \sum_{\omega} t^{\sum_{i=1}^{n}\lambda_i-i+\omega(i)} \sum_{i=1}^{n} (-1)^{\omega} h_{\lambda_i-i+\omega(i)} \\ &= \sum_{\omega} t^{|\lambda|} \sum_{i=1}^{n} (-1)^{\omega} h_{\lambda_i-i+\omega(i)} \\ &= t^{|\lambda|}\det(h_{\lambda_i-i+j}). \end{aligned}$$

□

In this section, we would also meet Grassmann bundle.

Definition 5.1.10. Let E be a vector bundle of rank n on a scheme X, and let d be a positive integer less than n. The *Grassmann bundle of d-plane in E* is a scheme over X

$$p : \mathrm{Gr}_d(E) \to X \tag{5.1.6}$$

such that the fiber $p^{-1}(x) = \mathrm{Gr}(d, E_x)$ is the Grassmannian of the d-dimensional subspaces of E_x. p^*E is a trivial bundle of rank n on $\mathrm{Gr}(d, E)$. There would be a tautological vector bundle S on $\mathrm{Gr}(d, E)$. It is a subbundle of p^*E. Denote by Q the quotient bundle p^*E/S on $\mathrm{Gr}(d, E)$.

Theorem 5.1.11 ([20], Example 14.6.6)**.** *If X is non-singular, E a vector bundle of rank n on X, then $A^*(Gr(d, E))$ is the algebra over A^*X generated by elements a_1, …, a_d, b_1, …, b_{n-d}, modulo the relations*

$$\sum_{i=0}^{k} a_i b_{k-i} = c_k(E) \tag{5.1.7}$$

for $k = 1, \ldots, n$. (Take $a_i = c_i(S)$, $b_j = c_j(Q)$.)

When $d = 1$, $\mathrm{Gr}(1, E)$ is called the projective bundle of E, and is denoted by $P(E)$. Let the first Chern class of the tautological line bundle over $P(E)$ be denoted by ξ. In this case, the above theorem is called the *Projective bundle theorem.*

Corollary 5.1.12 (Projective bundle theorem)**.** *$H^*(P(E))$ is generated by 1, ξ, …, ξ^{n-1} as $H^*(X)$-modules.*

Then the cohomology of Grassmann Bundle can be described by Schur polynomials, in which the generators are the Chern roots, and the coefficients are in $H^*(X)$. The following theorem is cited from [20, Proposition 14.6.3].

Proposition 5.1.13 (Duality theorem)**.** *Let s_λ and s_μ be two Schur polynomials with respect to partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$, then*

$$p_*(s_\lambda \cup s_\mu) = \begin{cases} 1, & \textit{if } \lambda_i + \mu_{d-i+1} = n - d \textit{ for } 1 \leq i \leq d, \\ 0, & \textit{otherwise.} \end{cases} \tag{5.1.8}$$

5.2 Correspondence

In Section 4.1.1 we use Lusztig's model to describe correspondences. In the case of stable framed representations of A_1-quiver, the model is stated in the following way.

Recall from Section 2.2.2, a stable framed representation $((\mathbb{V}_d, T), f)$ of A_1-quiver with framed structure n satisfies $T = \mathrm{Id}_{\mathbb{V}_d}$ and $f \in \mathrm{Epi}(\mathbb{C}^n, \mathbb{C}^d)$. Since both $\mathbb{V}_d$ and T are fixed, we could write the representation f for short.

From linear algebra, if we fix an epimorphism $\tau : \mathbb{C}^n \to \mathbb{C}^d$, it is known that any epimorphism can be written in terms of τg^{-1} where $g \in \mathrm{GL}(n, \mathbb{C})$. Two epimorphisms τg_1^{-1} and τg_2^{-1} are the same if and only if $g_1 g_2^{-1}$ acts on $\ker \tau$ trivially, or equivalently $g_1 g_2^{-1} \in U_{n-d,n} \rtimes G_{d_2}$. Define a map from G_n to $\mathrm{Epi}(\mathbb{C}^n, \mathbb{C}^d)$ by $\psi : g \mapsto \tau g^{-1}$.

Lemma 5.2.1. *$\psi : G_n \times_{P_{n-d,n}} G_d \to \mathrm{Epi}(\mathbb{C}^n, \mathbb{C}^d)$ is a G_d-equivariant isomorphism.*

Let $M_{d,n}^{st}$ be the variety of all d-dimensional stable framed representations of A_1-quiver with n-framed structure. Then $G_d = \mathrm{GL}(d, \mathbb{C})$ acts on $M_{d,n}^{st}$ freely. The quotient is the smooth model $\mathcal{M}_{d,n}^{st} = M_{d,n}^{st}/G_d$. It is obviously that $\mathcal{M}_{d,n}^{st}$ is $\mathrm{Gr}(d, n)$.

Fix three dimensions $d_1 + d_2 = d$. Consider the standard coordinate space $\mathbb{V}_d$ and the standard coordinate subspace $\mathbb{V}_{d_1} \subset \mathbb{V}_d$ consisting of the first d_1 coordinates. Let $P_{d_1,d}$ be the parabolic subgroup of G_d which acts on $\mathbb{V}_{d_1} \subset \mathbb{V}_d$. Let $U_{d_1,d}$ be the subgroup of $P_{d_1,d}$ whose action on $\mathbb{V}_{d_1}$ and induced action on $\mathbb{V}_d/\mathbb{V}_{d_1}$ are both trivial. It is not hard to see that $U_{d_1,d}$ is the unipotent radical of $P_{d_1,d}$. We have $P_{d_1,d} \simeq U_{d_1,d} \rtimes (G_{d_1} \times G_{d_2})$.

Now consider M_2 who is the variety of all pairs (W, f) where $f \in M_{d,n}^{st}$ and $W \subset \mathbb{V}_d$ is a d_1-dimensional subspace who is invariant under the action of T. Since $T = \mathrm{Id}$, W can be any subspace. Using the above notation, f can be written in terms of τg^{-1} for some $g \in \mathrm{GL}(n, \mathbb{C})$.

$W \subset \mathbb{V}_d$ is one-to-one correspondent to $\mathbb{V}_d \twoheadrightarrow \mathbb{V}_d/W$ by letting $W = \ker(\mathbb{V}_d \twoheadrightarrow \mathbb{V}_{d_2})$. Then by Lemma 5.2.1 the variety of subspaces $W \subset \mathbb{V}_d$ is $G_d \times_{P_{d_1,d}} G_{d_2}$.

Lemma 5.2.2. *The variety M_2 is $G_{d_1} \times G_{d_2} \times G_d$-equivariantly isomorphic to*

$$G_n \times_{P_{n-d,n}} G_d \times_{P_{d_1,d}} G_{d_2},$$

where G_d acts naturally and $G_1 \times G_2$ acts trivially. Furthermore, $[M_2/G_{d_2}]$ is isomorphic to the variety $F_{d_2,d,n}$ of two-step flags $\mathbb{V}_n \twoheadrightarrow \mathbb{V}_d \twoheadrightarrow \mathbb{V}_{d_2}$.

Similarly, recall that M_1 is the variety of all tuples (T, V, f_1, f_2) where all notations are defined in Section 4.1.1.

Lemma 5.2.3. *The variety M_1 is $G_{d_1} \times G_{d_2} \times G_d$-equivariantly isomorphic to*

$$G_n \times_{P_{n-d,n}} G_d \times_{U_{d_1,d}} (G_{d_1} \times G_{d_2}).$$

To sum up, we have the following description of the correspondence as well as the relations on the cohomology. Recall the diagram of correspondences in A_1-quiver case:

$$\begin{tikzcd} \mathcal{M}^{st}_{d_2,n} & {[M^{st}_{d_1,d,n}/P_{d_1,d}]} \arrow[l, "p_2"'] \arrow[r, "p"] \arrow[d, "p_1"] & \mathcal{M}^{st}_{d,n} . \\ & {[M_{d_1}/G_{d_1}]} & \end{tikzcd} \tag{4.1.14}$$

Proposition 5.2.4. *The scheme $\mathcal{M}^{st}_{d_2,d,n} = [M^{st}_{d_2,d,n}/P_{d_2,d}]$ in A_1-quiver case is isomorphic as varieties to the two-step flag variety $F_{d_2,d,n}$, which is variety of the flags $\{\mathbb{C}^n \twoheadrightarrow \mathbb{C}^d \twoheadrightarrow \mathbb{C}^{d_2}\}$. p is the obvious projection from $F_{d_2,d,n}$ to $\mathrm{Gr}(d,n)$ and p_2 is the obvious projection from $F_{d_2,d,n}$ to $\mathrm{Gr}(d_2,n)$. p_1 as a morphism of quotient stacks is described as following: For any $P_{d_1,d}$-bundle $E \to B$ and $f : E \to M^{st}_{d_1,d,n}$ a $P_{d_1,d}$-equivariant map, $p_1(B \leftarrow E \xrightarrow{f} M^{st}_{d_1,d,n})$ is $B \leftarrow E \xrightarrow{f'} M_{d_1}$, where $E \to B$ is treated as a G_{d_1}-bundle by the embedding $G_{d_1} \subset P_{d_1,d}$ and $f' = \pi_1 \circ q_1 \circ f$, where $q_1 : G_d \times_{U_{d_1,d}} M^{st}_{d_1,d,n} \to M_{d_1} \times M^{st}_{d_2,n}$ is defined in the diagram (4.1.7), and π_1 is the natural projection.*

$p_1^* : H^*([M_{d_1}/G_{d_1}]) \to H^*(\mathcal{M}^{st}_{d_1,d,n})$ is defined in the following way. Choose a class $x \in H^*([M_{d_1}/G_{d_1}])$. There is a correspondent Line bundle $\mathcal{L}_x$ whose first Chern class is x. G_{d_1} acts on $M^{st}_{d_1,d,n}$ defined by the embedding $G_{d_1} \subset P_{d_1,d}$. Then we have the following lemma.

Corollary 5.2.5. *$p_1^*(\mathcal{L}_x) \simeq M^{st}_{d_1,d,n} \times_{G_{d_1}} \mathcal{L}_x$, and thus $p_1^*(x) = c_1(M^{st}_{d_1,d,n} \times_{G_{d_1}} \mathcal{L}_x)$.*

Theorem 5.2.6. *In A_1-quiver case, Condition A holds.*

Proof. It is obvious from the above descriptions about the correspondences. □

Theorem 5.2.7. *For fixed framed structure N, $\phi_i = x^i \in H^*([M_1/G_1])$, then $\phi_i^D = \phi_{N-1-i}$ for $i = 0, \ldots, N-1$. In addition, Condition B holds.*

Proof. For general d_1, $\mathcal{M}^{st}_{d_1,d,n} \to M_1 \times \mathcal{M}^{st}_{d_2,n}$ is a Grassmann bundle over the Grassmannian $\mathrm{Gr}(d_2, n)$, and the fibre is Grassmannian $\mathrm{Gr}(d_1, N)$. By the Duality theorem of Grassmannian bundles 5.1.13, the pairing (4.3.2) is computed. If $d_1 = 1$, then $s_\lambda = \phi_\lambda$, $s_\mu = \phi_\mu$ for λ and μ are two intergers. The pairing in this case is

$$\langle \Phi_\lambda, \Phi_\mu \rangle = \delta_{\lambda+\mu=N-1}. \tag{5.2.1}$$

Thus $\phi_i^D = \phi_{N-i-1}$. If $d_1 = 2$, $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$. The associated index $\mathbf{k}(\lambda) = (k_1, k_2) = (\lambda_1 + 1, \lambda_2)$ and $\mathbf{l}(\mu) = (l_1, l_2) = (\mu_1 + 1, \mu_2)$. The pairing in this case is

$$\langle \Phi_{\mathbf{k}(\lambda)}, \Phi_{\mathbf{l}(\mu)} \rangle = \langle \phi_{k_1} \wedge \phi_{k_2}, \phi_{l_1} \wedge \phi_{l_2} \rangle = \delta_{k_1+l_2=N-1}\delta_{k_2+l_1=N-1}. \tag{5.2.2}$$

Then $(\phi_i \wedge \phi_j)^D = \phi_{N-j-1} \wedge \phi_{N-i-1} = \phi_j^D \wedge \phi_i^D$. □

Note that both p and p_2 are proper morphisms of stacks (which are in fact schemes). Then the increasing representation introduced in Section 4.2 is well-defined, and the definition of the decreasing representations introduced in Section 4.3 can be simplified.

Corollary 5.2.8. *Since p_2 is proper, and M_1 is this case is just a point,*

$$\phi_i^-(f) := p_{2*}((p_1^*(\phi_i^D)) \cup p^*(f)), \quad \textit{for } \phi_i \in \mathcal{H}_{d_1}, f \in H^*(\mathrm{Gr}(d,n)). \tag{5.2.3}$$

5.3 Increasing and decreasing operators

5.3.1 Two representations of A_1-COHA

Now we want to compute the increasing representation by the formula $p_*(p_1^*(\phi_i) \cup p_2^*(s_\lambda))$. Note that in this case, $d_1 = 1$. Recall that ϕ_i represents the polynomial $\phi_i(x) = x^i$. Using the geometric interpretation, x^i is treated as the first Chern class of the tautological line bundle over the classifying space of G_1. By Corollary 5.2.5, the line bundle will be pulled back through p_1 to the line bundle over $F_{d_2,d,n}$ associated to the corresponding character of G_{d_1} when treating G_{d_1} as a subquotient of $P_{d_2,d,n}$. Hence $p_1^*(\phi_i)$ will be the first Chern class of the line bundle described above, which is $\phi_i(x_{d_2+1}) = x^i_{d_2+1}$.

By the description in the last section, as homogeneous spaces, we have the following isomorphisms: $\mathrm{Gr}(d,n) \approx \mathrm{GL}_n(\mathbb{C})/P_{d,n}$, $\mathrm{Gr}(d_2,n) \approx \mathrm{GL}_n(\mathbb{C})/P_{d_2,n}$ and $F(d_2,d,n) \approx \mathrm{GL}_n(\mathbb{C})/P_{d_2,d,n}$. We use the formula in [4] to compute the pushforward.

Theorem 5.3.1 ([4]). *Let G be a connected reductive algebraic group over $\mathbb{C}$ and B a Borel subgroup. Choose a maximal torus $T \subset B$ with Weyl group W. The set of all positive roots of the root system of (G,T) is denoted by Δ^+. Let $P \supset B$ be a parabolic subgroup of G, with the set of positive roots $\Delta^+(P)$ and Weyl group W_P. Let L_α be the complex line bundle over G/B which is associated to the root α. The Gysin homomorphism $f_* : H^*(G/B) \to H^*(G/P)$ is given by*

$$f_*(p) = \sum_{w \in W/W_P} w \cdot \frac{p}{\prod_{\alpha \in \Delta^+ \setminus \Delta^+(P)} c_1(L_\alpha)}. \tag{5.3.1}$$

Applying Theorem 5.3.1, for $s_\lambda \in H^*(\mathrm{Gr}(d_2,n))$,

$$(\phi_i^+ \cdot s_\lambda)(x_1, \ldots, x_{d_2+1}) = \sum_{i_1 < \ldots < i_{d_2}} \frac{s_\lambda(x_{i_1}, \ldots, x_{i_{d_2}}) p_1^*(\phi_i(x_{i_{d_2+1}}))}{\prod_{j=1}^{d_2} (x_{i_j} - x_{i_{d_2+1}})}. \tag{5.3.2}$$

Similarly, the formula of the decreasing actions is

$$
\begin{aligned}
(\phi_i^- \cdot s_\lambda)(x_1,\ldots,x_{d_2-1}) &= \sum_{i_1<\ldots<i_{d_2}} \frac{s_\lambda(x_{i_1},\ldots,x_{i_{d_2}})p_1^*(\phi_i^D(x_{i_{d_2}}))}{\prod_{j=d_2+1}^n (x_{i_{d_2}}-x_j)} \\
&= \sum_{i_1<\ldots<i_{d_2}} \frac{s_\lambda(x_{i_1},\ldots,x_{i_{d_2}})p_1^*(\phi_{n-1-i}(x_{i_{d_2}}))}{\prod_{j=d_2+1}^n (x_{i_{d_2}}-x_j)}.
\end{aligned}
\tag{5.3.3}
$$

Remark 5.3.2. In Formula (5.3.3), variables x_i for $i > d_2 - 1$ appear on the right side, which do not belong to the variables on the left side. This is not a contradiction because of the formula $s_\lambda(x_1,\ldots,x_d) = (-1)^{|\lambda|}s_{\lambda'}(x_{d+1},\ldots,x_n)$ by Lemma 5.1.9. More details will be discussed in the following section.

Remark 5.3.3. The construction above actually only defines an incrasing operator $\phi_{i,d}^+$ from $H^*(\mathrm{Gr}(d,n))$ to $H^*(\mathrm{Gr}(d+1,n))$ and an decreasing operator $\phi_{i,d}^-$ from $H^*(\mathrm{Gr}(d,n))$ to $H^*(\mathrm{Gr}(d-1,n))$. The increasing operator we need is $\phi_i^+ = \sum_{d=0}^n \phi_{i,d}^+$. The decreasing operator we need is $\phi_i^- = \sum_{d=0}^n (-1)^{d-1}\phi_{i,d}^-$.

5.3.2 Increasing operators

The key result is a generalization of a result from [17].

Proposition 5.3.4. *The increasing representation structure is induced by the open embedding $j : M_{d,n}^{st} \to M_d \times \mathbb{C}^n$. The induced map $j^* : \mathcal{H} \to R_n^+$ is $\mathcal{H}$-linear and surjective. The kernel of j^* equals $\sum_{p\geq 0, q>0} \mathcal{H}_p \wedge (e_q^n \cup \mathcal{H}_q)$, where $e_q = \prod_{i=1}^d x_i$.*

Proof. In [17], the similar result for $n = 1$ is proved. It can be easily generalized to $n > 1$ case for A_1-quiver. □

The next lemma follows immediately from the definition of Schur polynomials.

Lemma 5.3.5. *$s_{(\lambda_d+1,\lambda_{d-1}+1,\ldots,\lambda_1+1)} = e_d s_\lambda$ for $s_\lambda \in \mathbb{Q}[x_1,\ldots,x_d]^{S_d}$ and $e_d = \prod_{i=1}^d x_i$. Thus $e_d^n \cup \Phi_{\mathbf{k}} = \Phi_{\mathbf{k}+\mathbf{n}}$ for $\Phi_{\mathbf{k}} \in \mathcal{H}_d$, and $\mathbf{n} = (n,n,\ldots,n)$.*

Finally, we come to the result, whose proof is straightforward.

Proposition 5.3.6. *The increasing representation R_n^+ is a quotient of $\mathcal{H} = \bigwedge^*(\mathcal{H}_1)$ whose kernel is the submodule generated by $\{\phi_i\}_{i\geq n}$. Thus R_n^+ is isomorphic to $\bigwedge^*(V(n))$ where $V(n)$ is the linear space spanned by $\phi_0, \ldots, \phi_{n-1}$ and the action is given by wedge product from left. Then $\{\phi_{k_1} \wedge \ldots \wedge \phi_{k_d}\}_{k_1 > \ldots > k_d}, 0 \leq d \leq n-1$ form a basis of R_n^+.*

5.3.3 Two presentations of classes in the cohomology of Grassmannian

Proposition 5.3.6 implies that we can use the notations introduced in section 5.1 to represent cohomology classes of Grassmannians, as well as those in COHA, since they share the same product structure. Thus in $H^*(\mathrm{Gr}(d,n))$, $\Phi_{\mathbf{k}} = \phi_{k_1} \wedge \ldots \wedge \phi_{k_d}(x_1, \ldots, x_d)$ with index $\mathbf{k} = (k_1, \ldots, k_d)$ can represent the Schur polynomial $s_{\lambda(\mathbf{k})}(x_{1,d}, \ldots, x_{d,d})$, where $0 \leq k_d < \ldots < k_1 \leq n-1$ and $\lambda = (\lambda_1, \ldots, \lambda_d) = (k_1 - d + 1, k_2 - d + 2, \ldots, k_d)$ is a d-partition.

Let λ' be the transpose partition of λ, and $\mathbf{k}' = \mathbf{k}(\lambda')$. By Lemma 5.1.9, $\Phi_{\mathbf{k}}(x_1, \ldots, x_d) = (-1)^{|\lambda|}\Phi_{\mathbf{k}'}(x_{d+1}, \ldots, x_n)$. $\Phi_{\mathbf{k}}$ is called *the ordinary presentation* of the correspondent class s_λ, and $(-1)^{|\lambda|}\Phi_{\mathbf{k}'}$ is called *the transpose presentation*.

5.3.4 Decreasing operators

Our goal is to understand the decreasing representation using the basis $\{\Phi_{\mathbf{k}}\}_{\mathbf{k}}$ of R_n^+. From Section 5.3.3, the equation (5.3.3) can be rewritten as

$$\begin{aligned}
(\phi_i^- \cdot \Phi_{\mathbf{k}})(x_1, \ldots, x_{d_2-1}) &= \sum_{i_1 < \ldots < i_{d_2}} \frac{\Phi_{\mathbf{k}}(x_{i_1}, \ldots, x_{i_{d_2}})\phi_{n-1-i}(x_{i_{d_2}})}{\prod_{j=d_2+1}^n (x_{i_{d_2}} - x_j)} \\
&= (-1)^{|\lambda(\mathbf{k})|} \sum_{i_{d_2+1} < \ldots < i_n} \frac{\Phi_{\mathbf{k}'}(x_{i_{d_2+1}}, \ldots, x_{i_n})\phi_{n-1-i}(x_{i_{d_2}})}{\prod_{j=d_2+1}^n (x_{i_{d_2}} - x_{i_j})} \qquad (5.3.4) \\
&= (-1)^{|\lambda|+n-d_2} (\phi_{n-1-i}^+ \cdot \Phi_{\mathbf{k}'})(x_{d_2}, \ldots, x_n).
\end{aligned}$$

This formula suggests an algorithm. Start from an ordinary presentation of a class $\Phi_{\mathbf{k}} = \phi_{k_1} \wedge \ldots \wedge \phi_{k_d}$ in $H^*(\mathrm{Gr}(d,n))$, where $\mathbf{k} = (k_1, \ldots, k_d)$, and $0 \leq k_d < \ldots < k_1 \leq n-1$. First we change $\Phi_{\mathbf{k}}(x_1, \ldots, x_d)$ to $(-1)^{|\lambda(\mathbf{k})|}\Phi_{\mathbf{k}'}(x_{d+1}, \ldots, x_n)$ by Lemma 5.1.9. Then apply

ϕ_i^- to $\Phi_{\mathbf{k}'}$ using formula (5.3.4) and Proposition 5.3.6. Finally change the result back to the ordinary presentation.

Definition 5.3.7. The *partial derivative operator* $\partial_i : \bigwedge^*(V(n)) \to \bigwedge^*(V(n))$ is defined by

$$\partial_i(\Phi_{\mathbf{k}}) = \begin{cases} (-1)^{s-1}\phi_{k_1} \wedge \ldots \wedge \hat{\phi}_i \wedge \ldots \wedge \phi_{k_d}, & \text{if } \phi_i \text{ appears in } \Phi_{\mathbf{k}} \text{ such that } i = k_s, \\ 0, & \text{if } \phi_i \text{ does not appear in } \Phi_{\mathbf{k}}. \end{cases} \tag{5.3.5}$$

We need the following lemma to help us to do these transformations.

Lemma 5.3.8. *If ϕ_r appears in $\Phi_{\mathbf{k}'}$, ϕ_{n-r-1} will not appear in $\Phi_{\mathbf{k}}$. On the other hand, if ϕ_r doesn't appear in $\Phi_{\mathbf{k}'}$, ϕ_{n-r-1} will appear in $\Phi_{\mathbf{k}}$.*

Proof. From Section 5.3.6, $\lambda = (\lambda_1, \ldots, \lambda_d)$ is a d-partition. The transpose partition is defined by $\lambda'_j = \#\{\lambda_i \geq j\}$ for $1 \leq j \leq n-d$. Thus we have

$$\lambda_i = \begin{cases} n-d, & \text{if } 1 \leq i \leq \lambda'_{n-d}, \\ n-d-j, & \text{if } \lambda'_{n-d-j+1} + 1 \leq i \leq \lambda'_{n-d-j} \text{ for } 1 \leq j \leq n-d-1, \\ 0, & \text{if } \lambda'_1 + 1 \leq i \leq d. \end{cases} \tag{5.3.6}$$

From $\lambda_i = k_i - d + i$ and $\lambda'_i = k'_i - (n-d) + i$, it immediately implies

$$k_i = \begin{cases} n-i, & \text{if } 1 \leq i \leq \lambda'_{n-d}, \\ n-i-j, & \text{if } \lambda'_{n-d-j+1} + 1 \leq i \leq \lambda'_{n-d-j} \text{ for } 1 \leq j \leq n-d-1, \\ d-i, & \text{if } \lambda'_1 + 1 \leq i \leq d. \end{cases} \tag{5.3.7}$$

If $1 \leq j \leq n-d-1$, $k_i = n-i-j$ for $\lambda'_{n-d-j+1} + 1 \leq i \leq \lambda'_{n-d-j}$. Thus

$$n - j - \lambda'_{n-d-j} \leq n - j - i = k_i \leq n - j - \lambda'_{n-d-j+1} - 1, \tag{5.3.8}$$

and we immediately have

$$n - k'_{n-d-j} \leq k_i \leq n - k'_{n-d-j+1} - 2, \text{ for } \lambda'_{n-d-j+1} + 1 \leq i \leq \lambda'_{n-d-j}. \tag{5.3.9}$$

Similarly,

$$n - k'_{n-d} \leq k_i \leq n - 1, \text{ if } 1 \leq i \leq \lambda'_{n-d}, \tag{5.3.10}$$

and

$$0 \leq k_i \leq n - k'_1 - 2, \text{ if } \lambda'_1 + 1 \leq i \leq d. \tag{5.3.11}$$

Therefore k_i would run over all integers between 0 and $n - k'_1 - 2$, or between $n - k'_{n-d-j}$ and $n - k'_{n-d-j+1} - 2$, or between $n - k'_{n-d}$ and $n - 1$. In other words, the only indexes appear in $\mathbf{k}(\lambda)$ would be in these ranges.

If ϕ_r doesn't appear in $\Phi_{\mathbf{k}'(\lambda)}$, there are three cases. Assume $k'_{s+1} < r < k'_s$ for $1 \leq s \leq n - d - 1$. Then $n - k'_s - 1 < n - r - 1 < n - k'_{s+1} - 1$, and it implies

$$n - k'_s \leq n - r - 1 \leq n - k'_{s+1} - 2. \tag{5.3.12}$$

Similarly, we have

$$0 \leq n - r - 1 \leq n - k'_1 - 2, \text{ if } k'_1 < r \leq n - 1, \tag{5.3.13}$$

and

$$n - k'_{n-d} \leq n - r - 1 \leq n - 1, \text{ if } 0 \leq r < k'_{n-d}. \tag{5.3.14}$$

In either case there exists some $1 \leq i \leq d$ such that $k_i = n - r - 1$.

On the other hand, assume ϕ_r appear in $\Phi_{\mathbf{k}'(\lambda)} = \phi_{k'_1} \wedge \ldots \wedge \phi_{k'_{n-d}}$. Let $r = k'_s$. Using the range introduced above, there is always a gap between $n - k'_{n-d-j+1} - 2$ and $n - k'_{n-d-j+1}$, which is $n - k'_{n-d-j+1} - 1$. This means that $n - r - 1 = n - k'_s - 1$ would never appear in $\mathbf{k}(\lambda)$.

□

Proposition 5.3.9. *The decreasing operators are the partial derivative operators:*

$$\phi_r^- \cdot \Phi_{\mathbf{k}} = \partial_r(\Phi_{\mathbf{k}}). \tag{5.3.15}$$

Proof. What we want is to compute $\phi_r^- \cdot \Phi_{\mathbf{k}}$. Based on formula (5.3.4), we have

$$\begin{aligned}(\phi_{r,d}^- \cdot \Phi_{\mathbf{k}})(x_1, \ldots, x_{d-1}) &= (-1)^{|\lambda|+n-d}(\phi_{n-1-r,d}^+ \cdot \Phi_{\mathbf{k}'})(x_d, \ldots, x_n) \\ &= (-1)^{|\lambda|+n-d}(\phi_{n-1-r,d} \wedge \phi_{k_1'} \wedge \ldots \wedge \phi_{k_{n-d}'})(x_d, \ldots, x_n).\end{aligned} \tag{5.3.16}$$

To simplify the computation, we now first compute $(\phi_r \wedge \phi_{k_1'} \wedge \ldots \wedge \phi_{k_{n-d}'})(x_d, \ldots, x_n)$.

By Lemma 5.3.8, if ϕ_{n-1-r} is not in the $\Phi_{\mathbf{k}}$, ϕ_r will appear in $\Phi_{\mathbf{k}'}$. Thus

$$(\phi_r \wedge \phi_{k_1'} \wedge \ldots \wedge \phi_r \wedge \ldots \wedge \phi_{k_{n-d}'})(x_d, \ldots, x_n) = 0. \tag{5.3.17}$$

If ϕ_{n-1-r} appears in $\Phi_{\mathbf{k}} = \phi_{k_1} \wedge \ldots \wedge \phi_{k_d}$, ϕ_r won't be in $\Phi_{\mathbf{k}'} = \phi_{k_1'} \wedge \ldots \wedge \phi_{k_{n-d}'}$. Assume $k_{s+1}' < r < k_s'$. We have

$$\phi_r \wedge \phi_{k_1'} \wedge \ldots \wedge \phi_{k_{n-d}'} = (-1)^s \phi_{k_1'} \wedge \ldots \wedge \phi_{k_s'} \wedge \phi_r \wedge \phi_{k_{s+1}'} \wedge \ldots \wedge \phi_{k_{n-d}'}. \tag{5.3.18}$$

We have to change this back to the ordinary presentation. First, let's find the partition associated to this polynomial. The index $\mathbf{l}' = (l_1', \ldots, l_{n-d+1}')$ is given by

$$l_i' = \begin{cases} k_{i-1}', & s+2 \le i \le n-d+1, \\ r, & i = s+1, \\ k_i', & 1 \le i \le s. \end{cases} \tag{5.3.19}$$

Then the new partition $\mu' = (\mu'_1, \dots, \mu'_{n-d+1})$ is given by

$$\mu'_i = \begin{cases} \lambda'_{i-1}, & s+2 \leq i \leq n-d+1, \\ r+s-n+d, & i = s+1, \\ \lambda'_i - 1, & 1 \leq i \leq s. \end{cases} \tag{5.3.20}$$

Next step is to recover the partition μ from its transpose μ'. From the definition of transpose partition, $\mu'_j = \#\{\mu_i \geq j\}$ for $1 \leq j \leq n-d+1$. Then

$$\mu_i = \begin{cases} n-d+1, & \text{if } 1 \leq i \leq \lambda'_{n-d}, \\ n-d+1-j, & \text{if } \lambda'_{n-d-j+1}+1 \leq i \leq \lambda'_{n-d-j} \text{ and } 1 \leq j \leq n-d-s-1, \\ s+1, & \text{if } \lambda'_{s+1}+1 \leq i \leq r+s-n+d, \\ s, & \text{if } r+s-n+d+1 \leq i \leq \lambda'_s - 1, \\ n-d+1-j, & \text{if } \lambda'_{n-d+2-j} \leq i \leq \lambda'_{n-d+1-j} - 1 \text{ and } n-d-s+2 \leq j \leq n-d, \\ 0, & \text{if } \lambda'_1 \leq i \leq d-1. \end{cases} \tag{5.3.21}$$

By comparing it with

$$\lambda_i = \begin{cases} n-d, & \text{if } 1 \leq i \leq \lambda'_{n-d}, \\ n-d-j, & \text{if } \lambda'_{n-d-j+1}+1 \leq i \leq \lambda'_{n-d-j} \text{ for } 1 \leq j \leq n-d-1, \\ 0, & \text{if } \lambda'_1 + 1 \leq i \leq d. \end{cases} \tag{5.3.22}$$

we notice that $\mu_i = \lambda_i + 1$ for $1 \leq i \leq r+s-n+d$ and $\mu_i = \lambda_{i+1}$ for $r+s-n+d+1 \leq i \leq d-1$. Therefore, since $l_i = \mu_i + (d-1) - i$ for $1 \leq i \leq d-1$ and $k_j = \lambda_j + d - j$ for $1 \leq j \leq d$, it is easy to see that $l_i = k_{i+1}$ for $r+s-n+d+1 \leq i \leq d-1$ and $l_i = k_i$ for $1 \leq i \leq r+s-n+d$.

Thus the resulted presentation is

$$(-1)^{n-d+s+|\lambda|+|\mu|}\phi_{k_1}\wedge\ldots\wedge\hat{\phi}_{n-r-1}\wedge\ldots\wedge\phi_{k_d}=(-1)^{r+s}\phi_{k_1}\wedge\ldots\wedge\hat{\phi}_{n-r-1}\wedge\ldots\wedge\phi_{k_d},\quad(5.3.23)$$

which is $\Phi_{\mathbf{k}}$ applied by the partial derivative of ϕ_{n-r-1} from the right hand side. If $r > k'_1$ or $r < k'_{n-d}$, the similar process will lead to the same result.

Then go back to the operator $\phi^-_{r,d}$, it is obvious that it acts on $\Phi_{\mathbf{k}}$ by taking the partial derivative of ϕ_r from the right hand side.

Finally, by definition $\phi^-_r = \sum_{d=0}^{n}(-1)^{d-1}\phi^-_{i,d}$. It can be realized as the partial derivative operators ∂_r. □

5.4 The double of representations

Use the notations from the previous subsection. Let $\{\phi^+_i\}_{i=0}^{n-1}$ be the creation operators and $\{\phi^-_i\}_{i=0}^{n-1}$ be the annihilation operators. These two set of operators satisfy the following relations.

Theorem 5.4.1. *Operators $\{\phi^+_i\}_{i=0}^{n-1}$ and $\{\phi^-_i\}_{i=0}^{n-1}$ satisfy the following relations:*

1. $\phi^+_i\phi^+_j + \phi^+_j\phi^+_i = 0,$
2. $\phi^-_i\phi^-_j + \phi^-_j\phi^-_i = 0,$
3. $\phi^+_i\phi^-_j + \phi^-_j\phi^+_i = \delta_{i,j}.$

Proof. The proof is straightforward by applying the formula in the definitions of the operators to the basis vectors of $\bigwedge^*(V(n))$. □

Definition 5.4.2. The finite *Clifford algebra* denoted by $\mathbf{Cl}_n$ is the $\mathbb{C}$-algebra with generators $\{\phi^+_i, \phi^-_i\}_{i=0}^{n-1}$ with a central element c modulo relations for all $i,\ j = 0,\ldots,n-1$:

$$\phi^+_i\phi^-_j + \phi^-_j\phi^+_i = \delta_{ij}c,\quad \phi^+_i\phi^+_j + \phi^+_j\phi^+_i = 0,\quad \phi^-_i\phi^-_j + \phi^-_j\phi^-_i = 0.\quad(5.4.1)$$

From Theorem 5.4.1, it is clear that $\bigwedge^*(V(n))$ carries a $\mathbf{Cl}_n$-module structure.

Lemma 5.4.3. *There is a canonical projection from $\bigwedge^*(V(n+1))$ to $\bigwedge^*(V(n))$ as $\boldsymbol{Cl}_r$-modules for $r = 1, \ldots, n$.*

Proof. Recall that $\bigwedge^*(V(n)) = \oplus_{d=0}^{n} H^*(\mathrm{Gr}(d,n))$ as A_1-COHA modules. There is a canonical morphism from $\mathrm{Gr}(d,n)$ to $\mathrm{Gr}(d,n+1)$ by embedding into the subspace of the first d coordinates. Then there is a canonical projection $h_n : H^*(\mathrm{Gr}(d,n+1)) \to H^*(\mathrm{Gr}(d,n))$. Furthermore, the diagram commutes:

$$\begin{tikzcd} H^*(\mathrm{Gr}(d,n+1)) \arrow[r, "h_n"] \arrow[d, "\phi_i^{\pm}"'] & H^*(\mathrm{Gr}(d,n)) \arrow[d, "\phi_i^{\pm}"] \\ H^*(\mathrm{Gr}(d\pm 1,n+1)) \arrow[r, "h_n"'] & H^*(\mathrm{Gr}(d\pm 1,n)) \end{tikzcd} \quad . \tag{5.4.2}$$

Thus the canonical projection $h = \sum h_n$ is a COHA module morphism. $\square$

The lemma enables us to consider $\varprojlim_n \bigwedge^*(V(n))$ as $\mathbf{Cl}_n$-modules. Denote by $\bigwedge$ the subspace of $\varprojlim_n \bigwedge^*(V(n))$ which consists of finite sum of wedge monomials. Recall that $\mathcal{H}$ denotes the underlying vector space of A_1-COHA.

Lemma 5.4.4. *$\mathcal{H}$ is isomorphic to $\bigwedge$ as vector spaces.*

Proof. By Proposition 5.3.6, $\varprojlim_n \bigwedge^*(V(n)) \simeq \varprojlim_n \oplus_{d=0}^{n} H^*(\mathrm{Gr}(d,n)) = \coprod_{d=0}^{\infty} H^*(\mathrm{Gr}(d,\infty))$. Thus $\bigwedge \simeq \oplus_{d=0}^{\infty} H^*(\mathrm{Gr}(d,\infty))$. Then the theorem follows from the fact that the classifying space of $\mathrm{GL}(d,\mathbb{C})$ is $\mathrm{Gr}(d,\infty)$. $\square$

Definition 5.4.5. The *half infinite Clifford algebra* denoted by $\mathbf{Cl}_{\frac{\infty}{2}}$ is the $\mathbb{C}$-algebra with generators $\{\phi_i^+, \phi_i^-\}_{i\in\mathbb{Z}_{\geq 0}}$ and a central element c modulo relations for all i, $j \in \mathbb{Z}_{\geq 0}$:

$$\phi_i^+\phi_j^- + \phi_j^-\phi_i^+ = \delta_{ij}c, \quad \phi_i^+\phi_j^+ + \phi_j^+\phi_i^+ = 0, \quad \phi_i^-\phi_j^- + \phi_j^-\phi_i^- = 0. \tag{5.4.3}$$

Theorem 5.4.6. *$\mathcal{H}$ carries a $\boldsymbol{Cl}_{\frac{\infty}{2}}$-module structure.*

Proof. By Lemma 5.4.3 and Lemma 5.4.4, $\mathcal{H}$ carries a $\mathbf{Cl}_n$-module structure for any n. The natural embedding $\mathbf{Cl}_n \to \mathbf{Cl}_{n+1}$ makes $\{\mathbf{Cl}_n\}_{n\in\mathbb{Z}_{\geq 1}}$ a direct system. The limit is $\mathbf{Cl}_{\frac{\infty}{2}}$. Then the limit induces a $\mathbf{Cl}_{\frac{\infty}{2}}$-module structure on $\mathcal{H}$. □

Remark 5.4.7. This corollary nearly shows that there is a Clifford module structure on $\mathcal{H}$. However, we still miss half of the operators since in the definition of infinite Clifford algebra, we need operators ϕ_i^+ and ϕ_i^- for all $i \in \mathbb{Z}$ while we only construct operators ϕ_i^+ and ϕ_i^- for $i \in \mathbb{Z}_{\geq 0}$. See Section 7.3 for more discussions.

Note that $\mathbf{Cl}_{\frac{\infty}{2}}$ does not act on $\bigwedge^*(V(n))$. The reason is that when $r \geq n$, $\phi_r^+\phi_r^- + \phi_r^-\phi_r^+ = 0$, instead of a nontrivial c. Therefore the double of A_1-COHA cannot be $\mathbf{Cl}_{\frac{\infty}{2}}$. A possible solution is to add a series of central elements $\{c_i\}_{i\in\mathbb{Z}_{\geq 0}}$, and introduce an algebra $\mathbf{Cl}^c_{\frac{\infty}{2}}$ generated by $\{\phi_i^+, \phi_i^-, c_i\}_{i\in\mathbb{Z}_{\geq 0}}$ with relations:

$$\phi_i^+\phi_j^- + \phi_j^-\phi_i^+ = \delta_{ij}c_i, \quad \phi_i^+\phi_j^+ + \phi_j^+\phi_i^+ = 0, \quad \phi_i^-\phi_j^- + \phi_j^-\phi_i^- = 0. \tag{5.4.4}$$

This algebra acts on $\bigwedge^*(V(n))$ by $c_i = 1$ for $i = 0, \ldots, n-1$ and $c_i = 0$ for $i \geq n$. Following [65], we have the following conjecture.

Conjecture 5.4.8. *$\boldsymbol{Cl}^c_{\frac{\infty}{2}}$ is the double of A_1-COHA.*

Chapter 6

Jordan quiver case

6.1 Noncommutative Hilbert schemes

Recall the definition of noncommutative Hilbert schemes from Section 2.2.3.

Definition 6.1.1. The smooth model $\mathcal{H}^{(m)}_{d,N}$ of $\mathcal{Q}^{(m)}$ with framed structure N is called the *non-commutative Hilbert scheme.*

In this section we will focus on the Jordan quiver case, which means $m = 1$. To proceed we need the concept of compositions.

Definition 6.1.2. A *composition of* d is a sequence $\pi = (d^1, d^2 \ldots)$ with finite many nonzero terms whose sum $\sum_i d^i = d$. The maximal index k such that $d^k \neq 0$ is called the *length* of the composition. An N*-composition of* d is a N-tuple $\pi = (d^1, \ldots, d^N)$ such that $\sum_{i=1}^{N} d^i = d$.

Remark 6.1.3. In the rest part of this dissertation we make a shift on the index of N-compositions and denote it by $\pi = (d^0, d^1, \ldots, d^{N-1})$. This is for the convenience of some statements.

Remark 6.1.4. In the cellular decomposition in [53], the cell of $\mathcal{H}^{(m)}_{d,N}$ is indexed by m-ary forests with N roots. In the case of Jordan quiver, $m = 1$. A 1-ary forest with N roots and d nodes is the same thing as an N-composition of d. We are going to use the language of N-compositions instead of forests for simplicity.

6.1.1 Cellular decompositions and cohomology

Reineke found a cellular decomposition of noncommutative Hilbert schemes in [53]. The extended quiver of the Jordan quiver is pictured here.

$$\begin{tikzcd} \bullet \arrow[loop left, "T"] & \bullet^{\infty} \arrow[l, bend right, "f"'] \arrow[l, bend left] \arrow[l, phantom, "\cdots"] \end{tikzcd} \tag{6.1.1}$$

Following [53], we can describe the moduli space of stable framed representations of Jordan quiver with framed structure N in the following way. Let $\mathbb{V}_d$ be the standard d-dimensional vector space $\mathbb{C}^d$ and $\mathbb{W}$ be the standard 1-dimensional vector space $\mathbb{C}$. Denote a stable framed representation of dimension d and framed structure N by $((\mathbb{V}_d, T), f)$ where $T : \mathbb{V}_d \to \mathbb{V}_d$ is the map associated to the loop of Jordan quiver and $f = (f_i)_{i=1}^N$ with $f_i : \mathbb{W} \to \mathbb{V}_d$ for $i = 1, \dots, N$ are the maps associated to the framed structure.

Fix a basis vector $\mathbf{v} \in \mathbb{W}$. Let $v_1^i = f_i(\mathbf{v}) \in \mathbb{V}_d$ for $i = 0, \dots, N-1$ and $v_j^k = T^{j-1} v_1^k$ for $j \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}_{\geq 0}$. Then we have a set of vectors $S_{T,f} = \{v_i^k\}_{i,k}$. We call $v_i^k < v_j^l$ if either $k < l$, or $k = l$ and $i < j$. Then we have an order structure on $S_{T,f}$.

Definition 6.1.5. For each N-composition $\pi = (d^0, d^1, \dots, d^{N-1})$ of d, define Z_π as the set of all points $((\mathbb{V}_d, T), f) \in \mathcal{H}_{d,N}^{(1)}$ such that $T v_{d^k}^k$ is contained in the span of the vectors v_j^l where $v_j^l < v_{d^k}^k$ for any $k = 0, \dots, N-1$.

Theorem 6.1.6 ([53])**.** *$\{Z_\pi\}$ for π running over all N-compositions of d gives a cellular decomposition of $\mathcal{H}_{d,N}^{(1)}$ that*

$$\mathcal{H}_{d,N}^{(1)} = \bigcup_{\pi} Z_\pi. \tag{6.1.2}$$

Notation 6.1.7. For a stable framed representation $((\mathbb{V}_d, T), f) \in H_{d,N}^{(1)}$ of dimension d, there exists a unique composition $\pi = (d^0, \dots, d^{N-1})$ of d such that $((\mathbb{V}_d, T), f) \in Z_\pi$. This composition π is called the *type* of $((\mathbb{V}_d, T), f)$.

6.1.2 Representations in matrices

A representation in Z_π where $\pi = (d^0, \dots, d^{N-1})$ can be written in terms of matrices in the following way. Let $v_i^k = T^{i-1} v_1^k$ for $i = 1, \dots, d^k$ for $k = 0, \dots, N-1$. Since $Tv_{d^k}^k$ is contained in the span of the vectors v_j^l where $v_j^l < v_{d^k}^k$ for any $k = 0, \dots, N-1$, we have

$$Tv_{d^k}^k = \sum_{i=0}^{k} \sum_{j=1}^{d^i} a_j^{i,k} v_j^i, \tag{6.1.3}$$

where $\{a_j^{i,k}\}$ are coefficients in $\mathbb{C}$.

Since the representation is stable, all these vectors $\{v_i^k\}_{i=1,\dots,d_k}^{k=0,\dots,N-1}$ can generate the whole space $\mathbb{V}_d$. Thus $\{v_i^k\}_{i=1,\dots,d^k}^{k=0,\dots,N-1}$ form a basis of $\mathbb{V}_d$. Under this basis, the action T can be written as a matrix. In the case of $N = 1$, the only parition is $\pi = (d)$, and the matrix is

$$\begin{bmatrix} 0 & \dots & 0 & a_1^{0,0} \\ 1 & \dots & 0 & a_2^{0,0} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & a_d^{0,0} \end{bmatrix}. \tag{6.1.4}$$

In the case of $N > 1$, the matrix is block diagonalized and the diagonal blocks should be the same as the matrix (6.1.4). The following is an example for $N = 2$.

$$\begin{bmatrix} 0 & \dots & 0 & a_1^{0,0} & 0 & \dots & 0 & a_1^{0,1} \\ 1 & \dots & 0 & a_2^{0,0} & 0 & \dots & 0 & a_2^{0,1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 1 & a_{d^0}^{0,0} & 0 & \dots & 0 & a_{d^0}^{0,1} \\ & & & & 0 & \dots & 0 & a_1^{1,1} \\ & & & & 1 & \dots & 0 & a_2^{1,1} \\ & & & & \vdots & \ddots & \vdots & \vdots \\ & & & & 0 & \dots & 1 & a_{d^1}^{1,1} \end{bmatrix}. \tag{6.1.5}$$

6.2 Correspondences

The construction in Section 4.1.2 is applied to the Jordan quiver case in this section. Recall from Corollary 2.1.22 that if there is only one vertex, all stability conditions are equivalent to the trivial stability $\mathbf{0}$. Denote by $H_{1,d,N}$ the set of $(\lambda, ((\mathbb{V}_{d+1}, T_{d+1}), f_{d+1}))$ where $((\mathbb{V}_{d+1}, T_{d+1}), f_{d+1}) \in H^{(1)}_{d+1,N}$ is a stable framed representations in the standard coordinate space $\mathbb{V}_{d+1} \simeq \mathbb{C}^{d+1}$ of dimension $d+1$ such that the first coordinate subspace $\mathbb{V}_1 \subset \mathbb{V}_{d+1}$ of dimension 1 is invariant under the action of T_{d+1} by $\lambda : v \mapsto \lambda v$ for some $\lambda \in \mathbb{C}$. Note that $H_{1,d,N}$ is the same as $M^{st}_{1,d,N}$ introduced in Section 4.1.2. By Proposition 4.1.6, $[H_{1,d,N}/P_{1,d}]$ can be represented by a scheme. Denote it by $\mathcal{H}_{1,d,N}$.

The following theorem is the rephrase of Theorem 4.1.5.

Proposition 6.2.1. *The quotient framed representation* $((\mathbb{V}_{d+1}/\mathbb{V}_1, \overline{T_{d+1}}), \overline{f_{d+1}})$ *is stable.*

Before looking into the general case, we consider $N = 1$ case first to get some basic knowledge of the correspondences.

6.2.1 Extensions in $N = 1$ case

If $N = 1$, $\mathcal{H}^{(1)}_{d,1} \simeq \mathbb{C}^d$ by Theorem 6.1.6. By the cellular decomposition introduced in Section 6.1.1, for a point $((\mathbb{V}_d, T_d), f_d) \in H^{(1)}_{d,1}$, T_d could be written in the form

$$\begin{bmatrix} 0 & \dots & 0 & a_0 \\ 1 & \dots & 0 & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & a_{d-1} \end{bmatrix}. \tag{6.2.1}$$

Proposition 6.2.2. *There is a* 1-1 *bijection of sets between the set of monic polynomials of degree d and* $H^{(1)}_{d,1}$.

Proof. The character polynomial of the matrix (6.2.1) is

$$x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \dots - a_1 x - a_0. \tag{6.2.2}$$

Since $((\mathbb{V}_d, T_d), f_d)$ has a cyclic vector, the character polynomial is minimal. Then the character polynomial can determine $((\mathbb{V}_d, T_d), f_d)$ uniquely. □

Proposition 6.2.3. *$\mathcal{H}_{1,d,1} = [H_{1,d,1}/P_{1,d}]$ is the set of pairs (λ, mp_{d+1}) where mp_{d+1} is a monic polynomials of degree $d+1$ and $(x-\lambda)$ is a factor of mp_{d+1}.*

Proof. For $(\lambda, ((\mathbb{V}_{d+1}, T_{d+1}), f_{d+1})) \in H_{1,d,1}$, the quotient framed representation is stable. Assume the character polynomial of the quotient representation is as (6.2.2). Then the character polynomial of $((\mathbb{V}_{d+1}, T_{d+1}), f_{d+1})$ should be

$$(x-\lambda)(x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \ldots - a_1 x - a_0). \tag{6.2.3}$$

Since in $\mathbb{V}_{d+1}$ there exists a cyclic vector, the character polynomial is minimal. Then the extension is unique. □

The proposition suggests the following description of $\mathcal{H}_{1,d,1} = [H_{1,d,1}/P_{1,d}]$. Recall the correspondence diagram (4.1.14) in $N=1$ case is as follows.

$$[M_1/G_1] \times \mathcal{H}^{(1)}_{d,1} \xleftarrow{p_1 \times p_2} \mathcal{H}_{1,d,1} \xrightarrow{p} \mathcal{H}^{(1)}_{d+1,1}. \tag{6.2.4}$$

It follows from Proposition 6.2.3 immediately that the maps in diagram (6.2.4) are described by

$$\begin{aligned} p :&(\lambda, mp_{d+1}) \mapsto mp_{d+1}, \\ p_1 :&(\lambda, mp_{d+1}) \mapsto \lambda, \\ p_2 :&(\lambda, mp_{d+1}) \mapsto mp_{d+1}/(x-\lambda). \end{aligned} \tag{6.2.5}$$

It shows that the map $M_1 \times \mathcal{H}^{(1)}_{d,1} \leftarrow \mathcal{H}_{1,d,1}$ is an isomorphism, and the fiber of p is generically $d+1$ isolated points since generically there are $d+1$ different factors of a $d+1$-degree monic polynomial.

6.2.2 Relations with Hilbert schemes

To compute the pushforward of the Chow ring/cohomolgy/Borel-Moore homolgy, we need some results from Hilbert schemes. We start from the following observation.

Proposition 6.2.4. *$\mathcal{H}_{d,1}^{(1)}$ is the Hilbert scheme of points on $X = \mathbb{A}^1$.*

Proof. By Theorem D.2.6 the Hilbert scheme of points on $\mathbb{A}^1$ has a description that is exact the same as the definition of $\mathcal{H}_{d,1}^{(1)}$. □

Then we apply the Hilbert-Chow morphisms and the related stratification. For the fiber of p, let us pick a composition $\nu = (\nu_1, \nu_2, \ldots)$ of $d+1$. Composing with the Hilbert-Chow morphism ρ, $p \circ \rho$ maps $\mathcal{H}_{1,d,1}$ to $\operatorname{Sym}^{d+1}\mathbb{C}$. Since the extension is determined by the characteristic polynomial, the fiber of points in $S_\nu^{d+1}\mathbb{C}$ is $l(\nu)$ isolated points. We have

$$A_*(\mathcal{H}_{d+1,1}^{(1)}) = A_*(\mathbb{C}^{d+1}) = \mathbb{Q}[\mathbb{C}^{d+1}]. \tag{6.2.6}$$

Thus p_* restricted on $\cup_{l(\nu)<d+1} S_\nu^{d+1}\mathbb{C}$ would not contribute in the pushforward $A_*(\mathcal{H}_{1,d,1}) \to A_*(\mathcal{H}_{d+1,1}^{(1)})$. The same argument applies to $(p_1 \times p_2)_*$.

6.2.3 Extensions in general case

The goal of this section is to study diagram (4.1.14) in details.

$$[M_1/G_1] \times \mathcal{H}_{d,N}^{(1)} \xleftarrow{p_1 \times p_2} \mathcal{H}_{1,d,N} \xrightarrow{p} \mathcal{H}_{d+1,N}^{(1)}. \tag{6.2.7}$$

Pick a stable framed representation $((\mathbb{V}_d, T_d), f_d) \in \mathcal{H}_{d,N}^{(1)}$ and a representation $\lambda \in M_1$ by $\lambda : v \mapsto \lambda v$. Then we can construct an extension $(\lambda, ((\mathbb{V}_{d+1}, T_{d+1}), f_{d+1})) \in \mathcal{H}_{1,d,N}$ who has a subrepresentation on $\mathbb{V}_1 \subset \mathbb{V}_{d+1}$ is λ and the associated quotient representation $((\mathbb{V}_{d+1}/\mathbb{V}_1, \overline{T_{d+1}}), \overline{f_{d+1}}) \simeq ((\mathbb{V}_d, T_d), f_d)$.

Without loss of generosity, we assume the type of $((\mathbb{V}_d, T_d), f_d)$ is $\pi = (d^0, \ldots, d^{N-1})$. Let v_1^k be the image in $\mathbb{V}_d$ of the k-th arrow. Using the method from Section 6.1.1, we get a base of $\mathbb{V}_d$ by generating under the action of T_d. Denote it by $\{v_i^k\}_{i=1,\dots,d^k}^{k=0,\dots,N-1} \subset \mathbb{V}_d$. Assume

$$\begin{aligned} T_d v_i^k &= v_{i+1}^k, \text{ for } i = 1, \dots, d^k - 1, && (6.2.8) \\ T_d v_{d^k}^k &= \sum_{j=1}^{k} \sum_{i=1}^{d^k} a_i^{j,k} v_i^j, && (6.2.9) \end{aligned}$$

for $k = 0, \dots, N-1$ and some coefficients $\{a_i^{j,k}\} \in \mathbb{C}$.

Take an identification of $\mathbb{V}_{d+1}/\mathbb{V}_1 \simeq \mathbb{V}_d \subset \mathbb{V}_{d+1}$ as vector spaces. Then all these vectors $\{v_i^k\}_{i=1,\dots,d^k}^{k=0,\dots,N-1}$ can also be treated as vectors in $\mathbb{V}_{d+1}$. Use the same method to get the set $\{\hat{v}_i^k\}$ of vectors generated by $\{\hat{v}_1^k\}^{k=0,\dots,N-1}$ under T_{d+1} who are the images of the arrows in $\mathbb{V}_{d+1}$. Choose a basis vector of $\mathbb{V}_1 \subset \mathbb{V}_{d+1}$ and denote it by v_0.

Since $\overline{T_{d+1}} = T_d$, for $k = 0, \dots, N-1$, we have

$$\begin{aligned} \hat{v}_1^k &= v_1^k + b_0^k v_0, , && (6.2.10) \\ T_{d+1} v_i^k &= v_{i+1}^k + b_i^k v_0, \text{ for } i = 1, \dots, d^k - 1, && (6.2.11) \\ T_{d+1} v_{d^k}^k &= \sum_{j=1}^{k} \sum_{i=1}^{d^k} a_i^{j,k} v_i^j + b_{d^k}^k v_0, && (6.2.12) \end{aligned}$$

for some coefficients $\{b_i^k\} \in \mathbb{C}$.

We can then write down the matrix of T_{d+1} under the base of $\mathbb{V}_d$ and v_0. The following is an example in $N=2$ case:

$$\begin{bmatrix} \lambda & b_1^0 & \dots & b_{d^0-1}^0 & b_{d^0}^0 & b_1^1 & \dots & b_{d^1-1}^1 & b_{d^1}^1 \\ 0 & 0 & \dots & 0 & a_1^{0,0} & 0 & \dots & 0 & a_1^{0,1} \\ 0 & 1 & \dots & 0 & a_2^{0,0} & 0 & \dots & 0 & a_2^{0,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{d^0}^{0,0} & 0 & \dots & 0 & a_{d^0}^{0,1} \\ & & & & & 0 & \dots & 0 & a_1^{1,1} \\ & & & & & 1 & \dots & 0 & a_2^{1,1} \\ & & & & & \vdots & \ddots & \vdots & \vdots \\ & & & & & 0 & \dots & 1 & a_{d^1}^{1,1} \end{bmatrix}. \tag{6.2.13}$$

Denote by $H_{1,\pi} \subset H_{1,d,N}$ the set of all pairs whose quotient representation is of type $\pi = (d^0, \dots, d^{N-1})$. Recall $P_{1,d}$ denote the subgroup of $\mathrm{GL}(d+1)$ which keeps $\mathbb{V}_1$ invariant. $H_{1,\pi}$ carries a $P_{1,d}$-action automatically since the gauge group action comes from basis change and it does not change the type of a stable framed representation. The action is free. Then the quotient is a scheme. We denote it by $Z_{1,\pi}$.

Proposition 6.2.5. *$\mathcal{H}_{1,d,N}$ can be decomposed into the union of $Z_{1,\pi}$, for π running through all N-compositions of d.*

To find the canonical form of the matrix under the automorphism group $P_{1,d}$ is the same thing as to find a good basis for the matrix the first basis vector of which has to be the basis of $\mathbb{W}$. Let $\hat{v}_0 = \mu v_0$. Since $\hat{v}_0$ should be a basis vector of $\mathbb{V}_1$, $\mu \in \mathbb{C}^*$.

Lemma 6.2.6. *If a 1-dimensional representation λ on $\mathbb{V}_1$ and a d-dimensional stable framed representation $((\mathbb{V}_d, T_d), f_d)$ of type $\pi = (d^0, \dots, d^{N-1})$ are fixed, the extension can be described by $[h_0, \dots, h_{N-1}] \in \mathbb{C}P^{N-1}$.*

Proof. Set $\hat{v}_i^k = T_{d+1}^{i-1} \hat{v}_1^k$. By induction, we have

$$\hat{v}_i^k = v_i^k + b_{i-1}^k(\lambda)\hat{v}_0/\mu, \tag{6.2.14}$$

where $b_i^k(\lambda) = b_i^k + b_{i-1}^k\lambda + \ldots + b_0^k\lambda^i$. Then

$$\begin{aligned} T_{d+1}\hat{v}_{d^k}^k &= \sum_{j=1}^{k}\sum_{i=1}^{d^k} a_i^{j,k} v_i^j + b_{d^k}^k \hat{v}_0/\mu + b_{d^k-1}^k(\lambda)\lambda\hat{v}_0/\mu \\ &= \sum_{j=1}^{k}\sum_{i=1}^{d^k} a_i^{j,k}\hat{v}_i^j + b_{d^k}^k(\lambda)\hat{v}_0/\mu - \sum_{j=1}^{k}\sum_{i=1}^{d^k} a_i^{j,k} b_{i-1}^j(\lambda)\hat{v}_0/\mu \\ &= \sum_{j=1}^{k}\sum_{i=1}^{d^k} a_i^{j,k}\hat{v}_i^j + h_k\hat{v}_0/\mu, \end{aligned} \tag{6.2.15}$$

where $h_k = b_{d^k}^k(\lambda) - \sum_{j=1}^k \sum_{i=1}^{d^k} a_i^{j,k} b_{i-1}^j(\lambda)$ is a number algebraically depending on $a_i^{j,k}$, b_i^k and λ, and is not affected by μ. Then the extension can be described by $(h_0, \ldots, h_{N-1}) \in \mathbb{C}^N$. There is a $\mathbb{C}^*$ action on it induced from the group action on M_1. It is defined by $\mathbb{C}^* \ni \mu \cdot (h_1, \ldots, h_N) = (h_1/\mu, \ldots, h_N/\mu)$. □

The next proposition follows immediately from Lemma 6.2.6.

Proposition 6.2.7. *$Z_{1,\pi} \to M_1 \times Z_\pi$ is the projective bundle of the trivial vector bundle $M_1 \times Z_\pi \times \mathbb{C}^N$, i.e., $Z_{1,\pi} \simeq M_1 \times Z_\pi \times \mathbb{C}P^{N-1}$.*

Corollary 6.2.8. *$Z_{1,\pi}$ has a cellular decomposition $\coprod_{k=0}^{N-1} M_1 \times Z_\pi \times X_k$, where X_k is the $N-k-1$-dimensional cell of $\mathbb{C}P^{N-1}$.*

Proof. By Lemma 6.2.6, the fiber at point $\lambda \times ((\mathbb{V}_d, T_d), f_d)$ can be described by a point $[h_0, \ldots, h_{N-1}] \in \mathbb{C}P^{N-1}$. Let $X_k = \{[h_0, \ldots, h_{N-1}],\ h_0 = \ldots = h_{k-1} = 0, h_k \neq 0\}$. Then $X_k \simeq \mathbb{A}^{N-k-1}$ and $\coprod_{k=0}^{N-1} X_k$ gives a cellular decomposition of $\mathbb{C}P^{N-1}$. □

Notation 6.2.9. Denote the cell $M_1 \times Z_\pi \times X_k$ by $F_{\pi,k}$. Denote the restriction of p_1 and p_2 to $Z_{1,\pi}$ by $p_{1,\pi}$ and $p_{2,\pi}$.

Corollary 6.2.10. *Consider the case when the subrepresentation is of dimension 2 ($d_1 = 2$). The fiber of $Z_{2,\pi} \to M_2 \times Z_\pi$ is generically $\mathbb{C}P^{N-1} \times \mathbb{C}P^{N-1}$.*

Proof. Using notations introduced in the case of $d_1 = 1$. In the case of arbitrary d_1, in the k-th block, we can apply T_{d+1} to $\hat{v}_1^k$. The sequence $\hat{v}_1^k, \ldots, \hat{v}_{d^k}^k$ are linear independent and the subspace spanned by these vectors intersects with $\mathbb{V}_{d_1}$ only at $\{0\}$. However, $T_{d+1}\hat{v}^k_{d^k}$ would have a component in $\mathbb{V}_{d_1}$ and the component is denoted by u_k. The extension is stable if and only if $\{u_k\}_{k=0}^{N-1}$ can generate $\mathbb{V}_{d_1}$ under the action of T. Note that in $d_1 = 1$ case, choosing a basis $\hat{v}_0$ in $\mathbb{V}_1$, $u_k = h_k\hat{v}_0$ for some coefficient h_k and h_k is computed in Lemma 6.2.6.

In $d_1 = 2$ case, generically T_{d+1} restricting on $\mathbb{V}_2$ has a weight decomposition $\mathbb{V}_2 = \mathbb{C}w_1 \oplus \mathbb{C}w_2$, where $T_{d+1}w_i = \alpha_i w_i$ gives the weight vector. Then we can choose w_1 and w_2 as basis of $\mathbb{V}_2$, and denote $u_k = h^1_k w_1 + h^2_k w_2$. Then an extension can be denoted by a pair of N-tuples $(\{h^1_k\}_{k=0}^{N-1}, \{h^2_k\}_{k=0}^{N-1})$. It is stable if and only if both N-tuples $\{h^1_k\}_{k=0}^{N-1}$ and $\{h^2_k\}_{k=0}^{N-1}$ are non-zero. Similar to $d_1 = 1$ case, rescaling w_1 (resp. w_2) is the same as rescaling $\{h^1_k\}_{k=0}^{N-1}$ (resp. $\{h^2_k\}_{k=0}^{N-1}$). This shows that the equivalent classes of $\{h^i_k\}_{k=0}^{N-1}$ form $\mathbb{C}P^{N-1}$ for both $i = 1$ and 2. □

Remark 6.2.11. Here we didn't consider the non-generic case. However, those parts form a subscheme of M_2 with lower dimension. Since $A_*(M_2)$ contains only dimension 2 cycles, the non-generic parts won't contribute to the pushforward.

Now we want to consider the map $Z_{1,\pi} \to H^{(1)}_{d+1,N}$. Let $e_k := (0, \ldots, 0, 1, 0, \ldots, 0)$ denote the composition of 1 whose only nontrivial component is the k-th entry for $k = 0, \ldots, N-1$. The set of all N-compositions carries a natural semigroup structure. It is obvious that the image of $p : Z_{1,\pi} \to H^{(1)}_{d+1,N}$ has to be contained in $\coprod_{i=0}^{N-1} Z_{\pi+e_i}$

Lemma 6.2.12. *The representation in the cell $F_{\pi,k}$ is mapped into $Z_{\pi+e_k}$.*

Proof. Let $(\lambda, ((\mathbb{V}_{d+1}, T_{d+1}), f_{d+1})$ be a representation in $Z_{1,\pi}$ indexed by $[h_1, \ldots, h_N])$. Assume that it is in the cell $M_1 \times Z_\pi \times X_k$. Then $h_0 = \ldots = h_{k-1} = 0$ and $h_k \neq 0$. Then $T_{d+1}\hat{v}^k_{d^k}$ is not in the span of $\{\hat{v}^1_1, \ldots, \hat{v}^k_{d^k}\}$. In addition, if we set $\hat{v}^k_{d^k+1} := \hat{T}\hat{v}^k_{d^k}$, we have $\hat{T}\hat{v}^k_{d^k+1}$ is in the span of $\{\hat{v}^1_1, \ldots, \hat{v}^k_{d^k+1}\}$. Therefore it is a representation of type $\pi + e_k$. □

Notation 6.2.13. Denote the restriction of p to $F_{\pi,k}$ by $p_{\pi,k}$.

Lemma 6.2.14. $p_{\pi,k}(p^{-1}_{1,\pi}(M_1) \cap p^{-1}_{2,\pi}(Z_\pi) \cap (F_{\pi,k})) = Z_{\pi+e_k}$.

Proof. By Corollary 6.2.8, $p^{-1}_{1,\pi}(M_1) \cap p^{-1}_{2,\pi}(Z_\pi) \subset \coprod_{k=0}^{N-1} F_{\pi,k}$. Then this corollary follows from Lemma 6.2.12. □

Proposition 6.2.15. *The fiber of $F_{\pi,k} \to Z_{\pi+e_k}$ is generically the set of d^k+1 isolated points.*

Proof. Choose the first k where $h_k \neq 0$. Choose an appropriate v_0 to change h_k to be 1. Now let $\hat{v}^k_{d^k+1} = T_{d+1}\hat{v}^k_{d^k}$. Use $\hat{v}^k_1, \ldots, \hat{v}^k_{d^k+1}$ as a basis. Once λ and $a_i^{k,k}$ are determined, $a_i^{k,l}$ are determined for any $l > k$. Thus the fiber is determined by the k-th block. Using the result from $N = 1$ case, the fiber is $d^k + 1$ isolated points generically. □

Remark 6.2.16. It is similar to Section 6.2.2 that it is only necessary to consider the open dense subset consisting of representations with different eigenvalues in the block considered when computing the pushforward p_* and $(p_1 \times p_2)_*$. The complement don't contribute.

The above propositions can be summarised into the following theorem.

Theorem 6.2.17. *The diagram*

$$[M_1/G_1] \times \mathcal{H}^{(1)}_{d,N} \xleftarrow{p_1 \times p_2} \mathcal{H}_{1,d,N} \xrightarrow{p} \mathcal{H}^{(1)}_{d+1,N}. \tag{6.2.7}$$

can be decomposed into the following pieces

$$M_1 \times Z_\pi \xleftarrow{p_{1,\pi} \times p_{2,\pi}} Z_{1,\pi} = \coprod_{k=0}^{N-1} F_{\pi,k} \tag{6.2.16}$$

and

$$F_{\pi,k} \xrightarrow{p_{\pi,k}} Z_{\pi+e_k} \tag{6.2.17}$$

where π runs through all N-compositions of d and $k = 0, \ldots, N-1$. Here, (6.2.16) is the projective bundle of the rank N trivial vector bundle over $M_1 \times Z_\pi$, and the fiber of (6.2.17) is is generically $d^k + 1$ isolated points.

6.3 The construction of increasing operators

Now we are going to compute the cohomology of each piece of

$$[M_1/G_1]\times\mathcal{H}^{(1)}_{d,N}\xleftarrow{p_1\times p_2}\mathcal{H}_{1,d,N}\xrightarrow{p}\mathcal{H}^{(1)}_{d+1,N}. \tag{6.2.7}$$

6.3.1 The cohomology of $[M_1/G_1]$

By Example 2.1.9, the stack of $[M_1/G_1]$ is defined by a trivial $G_1=\mathbb{C}^*$-action on $M_1=\mathbb{C}$. The cohomology of $[M_1/G_1]$ is the equivariant cohomology $H^*_{G_1}(M_1)$. Since the action is trivial,

$$H^*([M_1/G_1])\simeq H^*_{G_1}(M_1)\simeq H^*(EG_1\times_{G_1}M_1)\simeq H^*(BG_1)\otimes H^*(M_1). \tag{6.3.1}$$

Let x denote the generator of $H^*(BG_1)$ and $[M_1]$ the fundamental class of M_1. Then the generator of the cohomology (6.3.1) is $x\otimes[M_1]$. Since $H^*(M_1)\simeq\mathbb{Q}[M_1]$, we also denoted $x\otimes[M_1]$ by x for simplicity.

$H^*([M_1/G_1])$ can be considered as $\mathcal{H}_1$ of the Jordan-COHA. Using notations introduced in Section 3.3, $\phi_k=x^k$ for $k=0,1,2,\ldots$. The degree of ϕ_k is $2k$.

6.3.2 The cohomology of $\mathcal{H}^{(1)}_{d,N}$ and $\mathcal{H}^{(1)}_{d+1,N}$

The cellular decomposition of $\mathcal{H}^{(1)}_{d,N}$ and $\mathcal{H}^{(1)}_{d+1,N}$ are stated in Theorem 6.1.6. By Theorem A.1.10 we have the following theorem.

Theorem 6.3.1 ([53]). *The Borel-Moore homology $H^{BM}_*(\mathcal{H}^{(1)}_{d,N})$ is a $\mathbb{Q}$-vector space with a basis given by the classes of the closures $\overline{Z_\pi}$ for π running through N-compositions of d.*

Let $\pi=(d^0,\ldots,d^{N-1})$. Denote $[\overline{Z}_\pi]\in H^{BM}_*(\mathcal{H}^{(1)}_{d,N})$ by $\frac{\phi_0^{d^0}\phi_1^{d^1}\cdot\ldots\cdot\phi_{N-1}^{d^{N-1}}}{d^0!d^1!\ldots d^{N-1}!}$.

Proposition 6.3.2. *$H^{BM}_*(\mathcal{H}^{(1)}_{d,N})\simeq\mathbb{Q}[\phi_0,\ldots,\phi_{N-1}]_d$ as vector spaces.*

6.3.3 The cohomology of $\mathcal{H}_{1,d,N}$

By Proposition 6.2.5, $H_*^{BM}(\mathcal{H}_{1,d,N})$ is determined by classes $[\overline{Z_{1,\pi}}]$. By Corollary 6.2.8 and Theorem A.1.10, $H_*^{BM}(Z_{1,\pi})$ is generated by classes $\{[\overline{F_{\pi,k}}]\}_{k=0}^{N-1}$. On the other hand, by Proposition 6.2.7 and the Projective Bundle Theorem (Corollary 5.1.12), $H^*(Z_{1,\pi}) \simeq \mathbb{Q}[\xi_\pi]/(\xi_\pi^N)$, where ξ_π is the first Chern class of the tautological line bundle on $Z_{1,\pi} \simeq M_1 \times Z_\pi \times \mathbb{C}P^{N-1}$. The two presentations are connected by the Poincaré duality (see e.g. Theorem A.2.3): $[\overline{F_{\pi,k}}] = \xi_\pi^k \cap [\overline{Z_{1,\pi}}]$.

6.3.4 The construction of increasing operators

Recall from Section 4.1.2, the increasing operator is defined by

$$\phi_i^+(f) := p_*(p_1^*(\phi_i) \cup p_2^*(f)) \tag{6.3.2}$$

for $f \in H^*(\mathcal{H}_{d,N}^{(1)})$.

Lemma 6.3.3. $p_{1,\pi}^*(x^k) = \xi_\pi^k$.

Proof. $\mathbb{C}^* \subset P_{1,d}$ acts on $H_{1,\pi}$. The action is induced from the $\mathbb{C}^*$-action on M_1. Since $\mathbb{C}^*$ acts on M_1 trivially, and $Z_{1,\pi} = M_1 \times Z_\pi \times \mathbb{C}P^{N-1}$, the result follows immediately from Proposition C.3.8. □

Lemma 6.3.4. $\phi_k^+([\overline{Z_\pi}]) = (d^k + 1)[\overline{Z_{\pi+e_k}}]$.

Proof. It follows from the description of the correspondence in Theorem 6.2.17:

$$\begin{aligned} \phi_k^+([\overline{Z_\pi}]) &= p_*(p_{1,\pi}^*(\phi_k) \cap p_{2,\pi}^*([\overline{Z_\pi}])) = p_*(p_{1,\pi}^*(\phi_k) \cap [\overline{p_{2,\pi}^{-1}(Z_\pi)}] \cap [\overline{Z_{1,\pi}}]) \\ &= p_*(\xi_\pi^k \cap [\overline{Z_{1,\pi}}]) = p_*([\overline{F_{\pi,k}}]) = (d^k+1)[\overline{Z_{\pi+e_k}}]. \end{aligned} \tag{6.3.3}$$

□

Proposition 6.3.5. $\phi_k^+(\phi_0^{d^0} \cdot \ldots \cdot \phi_k^{d^k} \cdot \ldots \cdot \phi_{N-1}^{d^{N-1}}) = \phi_0^{d^0} \cdot \ldots \cdot \phi_k^{d^k+1} \cdot \ldots \cdot \phi_{N-1}^{d^{N-1}}$.

Proof. It is a translation of Lemma 6.3.4 to the notation introduced in Section 6.3.2. □

Theorem 6.3.6. *$\oplus_d H^*(\mathcal{H}^{(1)}_{d,N}) \simeq \mathbb{Q}[\phi_0, \phi_1, \ldots, \phi_{N-1}]$ as a Jordan-COHA left module.*

Proof. It follows immediately from Proposition 6.3.2 and Proposition 6.3.5. □

6.4 The construction of decreasing operators

The construction from Section 4.3 is applied here. First, we need to check Condition A. For the same reason as Remark D.2.5, Section 6.2.2 and Remark 6.2.16, we can only focus on the part where all points are distinct.

Proposition 6.4.1. *In Jordan-quiver case, Condition A holds.*

Proof. The base has a cell decomposition. On each piece the projection is a fibre bundle with the fibre $\mathbb{C}P^{N-1}$ due to Proposition 6.2.7. Therefore if you put all pieces together you would get a fibre bundle with fibres $\mathbb{C}P^{N-1}$. □

Recall that diagram (4.1.14) can be decomposed into pieces

$$M_1 \times Z_\pi \xleftarrow{p_{1,\pi} \times p_{2,\pi}} Z_{1,\pi} = \coprod_{k=0}^{N-1} F_{\pi,k} \tag{6.2.16}$$

and

$$F_{\pi,k} \xrightarrow{p_{\pi,k}} Z_{\pi+e_k}. \tag{6.2.17}$$

Lemma 6.4.2. *Let τ be a composition of $d+1$. Then $p^*([\overline{Z_\tau}]) = \sum_{k=0}^{N-1} [\overline{F_{\tau-e_k,k}}]$.*

Lemma 6.4.3. *$\phi_i^D = \phi_{N-1-i}$.*

Proof. It follows from the fact that the fibre is $\mathbb{C}P^{N-1}$ and Proposition 5.1.13 (Duality theorem). □

Corollary 6.4.4. *In Jordan-quiver case, Condition B holds.*

Proof. We need to study extension by 2-dimensional subrepresentations, following Corollary 6.2.10. The fibre is generically $\mathbb{C}P^{N-1} \times \mathbb{C}P^{N-1}$. The it carries a cellular decomposition based on the cellular decomposition of $\mathbb{C}P^{N-1}$. Let X_k denote the cell whose codimension is k. Then the cell of $\mathbb{C}P^{N-1} \times \mathbb{C}P^{N-1}$ is of the form $X_k \times X_l$ for $0 \leq k, l \leq N-1$. The pairing is

$$\langle [\overline{X_i}] \times [\overline{X_j}], [\overline{X_k}] \times [\overline{X_l}] \rangle = \langle [\overline{X_i}], [\overline{X_k}] \rangle \langle [\overline{X_j}], [\overline{X_l}] \rangle = \delta_{i+k=N-1} \delta_{j+l=N-1}. \tag{6.4.1}$$

Let ξ_1 (resp. ξ_2) be the first Chern class of the tautological bundle with respect to the first copy (resp. the second copy) of $\mathbb{C}P^{N-1}$ of the fibre. By Poincaré duality, $\xi_1^i \xi_2^j \cap [\mathbb{C}P^{N-1} \times \mathbb{C}P^{N-1}] = [\overline{X_i}] \times [\overline{X_j}]$. Then we have

$$\langle \xi_1^i \xi_2^j, \xi_1^k \xi_2^l \rangle = \delta_{i+k=N-1} \delta_{j+l=N-1}. \tag{6.4.2}$$

Thus $D(\xi_1^i \xi_2^j) = \xi_1^{N-i-1} \xi_2^{N-j-1}$.

Similarly to $\dim \mathbb{W} = 1$ case, the COHA generators from $H^*([M_2/G_2])$ can be pullbacked to the Chern class of the line bundle on the fibres. In $H^*([M_2/G_2])$ the class $\phi_i \phi_j$ represents the class

$$\phi_i \cdot \phi_j(x_1, x_2) = x_1^i x_2^j + x_1^j x_2^i. \tag{6.4.3}$$

It would be pullbacked to $\xi_1^i \xi_2^j + \xi_1^j \xi_2^i$. Then

$$\begin{aligned} D(p_1^*(\phi_i \cdot \phi_j)) &= D(\xi_1^i \xi_2^j + \xi_2^i \xi_1^j) = \xi_1^{N-i-1} \xi_2^{N-j-1} + \xi_1^{N-j-1} \xi_2^{N-i-1} \\ &= p_1^*(\phi_{N-i-1} \cdot \phi_{N-j-1}) = p_1^*(\phi_i^D \cdot \phi_j^D). \end{aligned} \tag{6.4.4}$$

Here all the product $\xi_1^i \xi_2^j$ is the cup product. Then $(\phi_i \cdot \phi_j)^D = \phi_i^D \cdot \phi_j^D$. □

Lemma 6.4.5. $\phi_k^-([\overline{Z_\tau}]) = [\overline{Z_{\tau - e_k}}]$.

Proof. Take a generator $x^k = \phi_k \in H^*(M_1/G_1)$. Recall that x can be represented by the first Chern class of tautological line bundle $\mathcal{L}$ over the classifying space BG_1. We can construct a line bundle $\mathcal{L}_k$ on BG_1 whose first Chern class is $x^k = \phi_k$. Then we have $p_{1,\pi}^*(x^k) = p_{1,\pi}^*(c_1(\mathcal{L}_k)) = c_1(p_{1,\pi}^*(\mathcal{L}_k))$.

Recall that $Z_{1,\pi}$ is generically a fibre bundle whose fibre is $\mathbb{C}P^{N-1}$, and the fibre can be realized as $\mathbb{C}^N\backslash\{0\}$ quotient by $G_1 = \mathbb{C}^*$. Thus $Z_{1,\pi}$ is generically isomorphic to $M_1 \times Z_\pi \times (\mathbb{C}^N\backslash\{0\})/G_1$. By the Projection bundle theorem (Corollary 5.1.12), $H^*(Z_{1,\pi})$ is generated by $1, \xi, \ldots, \xi^{N-1}$ as $H^*(M_1 \times Z_\pi)$-modules, where ξ is the first Chern class of the tautological line bundle on the fibres. Furthermore, this suggests that $p_{1,\pi}^*(\mathcal{L}_k) = M_1 \times Z_\pi \times (\mathbb{C}^N\backslash\{0\}) \times_{G_1} \mathcal{L}_k$. Thus $p_{1,\pi}^*(x^k) = \xi^k$. Since the fibre is a projective space, $D\xi^k = \xi^{N-1-k}$.

Now consider the pushforward. Since $Z_{1,\pi}$ is a fibre bundle over $M_1 \times Z_\pi$, the pushforward $(p_1 \times p_2)_*$ is integration over fibres, and only the top classes can be mapped to non-trivial classes. Thus, we have

$$(p_1 \times p_2)_*(\xi^{N-1-k} \cup [\overline{F_{\pi,j}}]) = \delta_{k,j}[\overline{M_1 \times Z_\pi}]. \tag{6.4.5}$$

It implies that

$$(\pi_2^*)^{-1}(p_1 \times p_2)_*(D\xi^k \cup \sum_{j=0}^{N-1}[\overline{F_{\tau-e_j,j}}]) = (\pi_2^*)^{-1}[\overline{M_1 \times Z_{\tau-e_k}}] = [\overline{Z_{\tau-e_k}}]. \tag{6.4.6}$$

□

Proposition 6.4.6. *$\phi_k^-(f) = \partial_k(f)$ for $f \in H^*(\mathcal{H}_{d,N}^{(1)})$.*

Proof. The formula follow from Lemma 6.4.5 and Proposition 6.4.6, and then translate it to the notations introduced in Section 6.3.2. We have

$$\phi_k^-(\phi_0^{d^0} \cdot \ldots \cdot \phi_k^{d^k} \cdot \ldots \cdot \phi_{N-1}^{d^{N-1}}) = \frac{1}{d^k}\phi_0^{d^0} \cdot \ldots \cdot \phi_k^{d^k-1} \cdot \ldots \cdot \phi_{N-1}^{d^{N-1}}. \tag{6.4.7}$$

□

6.5 The combination of the increasing and decreasing operators

Finally, we come to the main result of this chapter. This section is parallel to Section 5.4.

Definition 6.5.1. The *finite Heisenberg algebra* $\mathbf{Heis}_n$ is an associative unital algebra over $\mathbb{C}$ generated by $\{x_i^+, x_i^-\}_{i=0}^{n-1}$ with a central element c modulo relations

$$x_i^+x_j^+ = x_j^+x_i^+, \quad x_i^-x_j^- = x_j^-x_i^-, \quad x_i^-x_j^+ - x_j^+x_i^- = \delta_{i,j}c. \tag{6.5.1}$$

Definition 6.5.2. The *infinite Heisenberg algebra* **Heis** is an associative unital algebra over $\mathbb{C}$ generated by $\{x_i^+, x_i^-\}_{i\in\mathbb{Z}_{\geq 0}}$ with a central element c modulo relations

$$x_i^+x_j^+ = x_j^+x_i^+, \quad x_i^-x_j^- = x_j^-x_i^-, \quad x_i^-x_j^+ - x_j^+x_i^- = \delta_{i,j}c. \tag{6.5.2}$$

Theorem 6.5.3. *The two set of operators of the Jordan-COHA defines a representation of* $\boldsymbol{Heis}_N$ *on* $\oplus_{d\geq 0}H^*(\mathcal{H}_{d,N}^{(1)}) \simeq \mathbb{Q}[\phi_0, \ldots, \phi_{N-1}]$.

Proof. It follows immediately from Proposition 6.3.5 and Proposition 6.4.6. □

Notation 6.5.4. Let $\mathbb{H}_N := \oplus_{d\geq 0}H^*(\mathcal{H}_{d,N}^{(1)})$.

Lemma 6.5.5. *There is a canonical projection from* $\mathbb{H}_{N+1}$ *to* $\mathbb{H}_N$.

Proof. Let $((\mathbb{V}_d, T), f) \in \mathcal{H}_{d,N}^{(1)}$. Let the type of it be $\pi = (d^0, \ldots, d^{N-1})$. We can add one more framed arrow by setting $\tilde{f} = (f_i)_{i=0}^N$ where $\tilde{f}_i = f_i$ for $i = 0, \ldots, N-1$ and $\tilde{f}_N = 0$. Then we get a new representation $((\mathbb{V}_d, T), \tilde{f})$. It is obvious that $((\mathbb{V}_d, T), \tilde{f}) \in H_{d,N+1}^{(1)}$ and the type of it is $\tilde{\pi} = (d^0, \ldots, d^{N-1}, 0)$. This map induces an embedding $\mathcal{H}_{d,N}^{(1)} \hookrightarrow \mathcal{H}_{d,N}^{(1)}$, and the embedding induces a projection $H^*(\mathcal{H}_{d,N+1}^{(1)}) \to H^*(\mathcal{H}_{d,N}^{(1)})$. The projection is $\mathbb{Q}[\phi_0, \ldots, \phi_N]_d \to \mathbb{Q}[\phi_0, \ldots, \phi_{N-1}]_d$ via $\phi_i \mapsto \phi_i$ for $i = 0, \ldots, N-1$ and $\phi_N \mapsto 0$. Then after taking direct sum over d, we have $\mathbb{H}_{N+1} \to \mathbb{H}_N$. □

The lemma enables us to consider the $\varprojlim_N \mathbb{H}_N$. Denote by $\mathbb{H}$ the subspace of the limit which consists of finite sum. By the same construction as Section 5.4, we have the following theorem.

Theorem 6.5.6. *$\mathbb{H}$ is isomorphic to $\mathcal{H}$ as vector spaces. Thus $\mathcal{H}$ carries a **Heis**-module structure.*

Proof. The proof is parallel to the proof of Theorem 5.4.6. $\square$

Similar to the A_1-case, we can define the modified infinite Heisenberg algebra $\mathbf{Heis}^c$ by generators $\{x_i^+, x_i^-\}_{i\in\mathbb{Z}_{\geq 0}}$ and a series of central elements $\{c_i\}_{i\in\mathbb{Z}_{\geq 0}}$ modulo relations

$$x_i^+ x_j^+ = x_j^+ x_i^+, \quad x_i^- x_j^- = x_j^- x_i^-, \quad x_i^- x_j^+ - x_j^+ x_i^- = \delta_{i,j} c_i. \tag{6.5.3}$$

Conjecture 6.5.7. *$\mathbf{Heis}^c$ is the double of the Jordan-COHA*

See [65] for more discussions.

Chapter 7

Application and future discussion

In this chapter, we first review the simplest version of the classic Boson-Fermion correspondence. Then we states the main theorem of this dissertation which is a similar relation between our constructions of A_1-quiver and Jordan quiver. Finally we give a conjectures regarding these relations which shows some future directions of this topic.

7.1 Boson-Fermion correspondence: classic version

The origin of the Boson-Fermion correspondence can be traced back to the Jacobi triple product identiy

$$\sum_{n\in\mathbb{Z}} t^n q^{\frac{n^2}{2}} = \prod_{n\geq 1}(1-q^n)(1+tq^{n-\frac{1}{2}})(1+t^{-1}q^{n-\frac{1}{2}}), \tag{7.1.1}$$

which one should consider as an equality of two generating functions

$$\sum_{n\in\mathbb{Z},m\in\mathbb{Z}_{\geq 0}} b_{n,m}t^n q^{\frac{m}{2}} = \sum_{n\in\mathbb{Z},m\in\mathbb{Z}_{\geq 0}} f_{n,m}t^n q^{\frac{m}{2}} \tag{7.1.2}$$

with nonnegative integral coefficients. The boson-fermion correspondence can be viewed as a cateogrification of this identity. It is an isomorphism of two double graded vector spaces, bosonic Fock space and fermionic Fock space. For classical references on this topic, see [34]. See also [19, 61].

7.1.1 Fermionic Fock spaces

Definition 7.1.1. An infinite expression of the form

$$\underline{i_0} \wedge \underline{i_1} \wedge \dots$$

is called *semi-infinite monomial* if i_0, $i_1, \dots$ are integers and

$$i_0 > i_1 > \dots, \; i_n = i_{n-1} - 1 \text{ for } n \gg 0. \tag{7.1.3}$$

Let F be the complex vector space with a basis consisting of all semi-infinite monomials. This F is called *the full fermionic Fock space.*

Define the *charge* decomposition

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)} \tag{7.1.4}$$

by letting

$$|m\rangle = \underline{m} \wedge \underline{m-1} \wedge \underline{m-2} \wedge \dots \tag{7.1.5}$$

denote the *vacuum vector* of charge m and $F^{(m)}$ denote the linear span of all semi-infinite monomials of charge m. It is not hard to see that, for a monomial $\underline{i_0} \wedge \underline{i_1} \wedge \dots \in F^{(m)}$, $i_k = m - k$ for $k \gg 0$.

For $j \in \mathbb{Z}$ define the wedging and contracting operators ψ_j and ψ_j^* on F by:

$$\psi_j(\underline{i_0} \wedge \underline{i_1} \wedge \dots) = \begin{cases} 0, & \text{if } j = i_s \text{ for some } s, \\ (-1)^{s+1} \underline{i_0} \wedge \dots \wedge \underline{i_s} \wedge \underline{j} \wedge \underline{i_{s+1}} \wedge \dots, & \text{if } i_s > j > i_{s+1}. \end{cases} \tag{7.1.6}$$

$$\psi_j^*(\underline{i_0} \wedge \underline{i_1} \wedge \dots) = \begin{cases} 0, & \text{if } j \neq i_s \text{ for all } s, \\ (-1)^s \underline{i_0} \wedge \underline{i_1} \wedge \dots \wedge \underline{i_s} \wedge \underline{i_{s+1}} \wedge \dots, & \text{if } j = i_s. \end{cases} \tag{7.1.7}$$

Note that

$$\psi_j(F^{(m)}) \subset F^{(m+1)}, \quad \psi_j^*(F^{(m)}) \subset F^{(m-1)}. \tag{7.1.8}$$

These operators ψ_j and ψ_j^* are called *free fermions*. They satisfy the following relations:

$$\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}, \quad \psi_i\psi_j + \psi_j\psi_i = 0, \quad \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0. \tag{7.1.9}$$

Definition 7.1.2. The *infinite Clifford algebra* denoted by $\mathbf{Cl}_\infty$ is the $\mathbb{C}$-algebra with basis $\{\psi_i, \psi_i^*\}_{i\in\mathbb{Z}}$ and relations for all i, $j \in \mathbb{Z}$.

$$\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}, \quad \psi_i\psi_j + \psi_j\psi_i = 0, \quad \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0. \tag{7.1.10}$$

Therefore the operators ψ_j and ψ_j^* generate the infinite Clifford algebra $\mathbf{Cl}_\infty$. It is obvious that F is a $\mathbf{Cl}_\infty$-module.

7.1.2 Bosonic Fock spaces

Definition 7.1.3. Define the *full bosonic Fock space* to be the polynomial algebra on indeterminates p_1, $p_2, \ldots$ and q, q^{-1}.

$$B = \mathbb{C}[p_1, p_2, \ldots; q, q^{-1}]. \tag{7.1.11}$$

The full bosonic Fock space carries a natural representation of the oscillator algebra by the following formula

$$\begin{aligned} r^B(s_m) &= m\frac{\partial}{\partial p_m}, \quad r^B(s_{-m}) = p_m \text{ for } m > 0, \\ r^B(s_0) &= q\frac{\partial}{\partial q}, \quad r^B(K) = 1. \end{aligned}$$

B has a charge decomposition

$$B = \bigoplus_{m\in\mathbb{Z}} B^{(m)}, \tag{7.1.12}$$

where $B^{(m)} := q^m\mathbb{C}[p_1, p_2, \ldots]$.

7.1.3 Boson-Fermion correspondence

Theorem 7.1.4 ([34])**.** *There is an isomorphism of vector spaces $\sigma : F \to B$. Furthermore, the isomorphism induces isomorphisms on each charge subspaces: $\sigma_m : F^{(m)} \to B^{(m)}$ for all $m \in \mathbb{Z}$.*

7.2 Boson-Fermion correspondence: COHA version

Theorem 7.2.1. *There is an isomorphism of vector spaces $\sigma : \bigwedge \to \mathbb{H}$.*

Proof. By Theorem 5.4.6, the underlying vector space $\mathcal{H}$ of A_1-COHA carries a $\mathbf{Cl}_{\frac{\infty}{2}}$-module structure. By Theorem 6.5.6, the underlying vector space $\mathcal{H}$ of Jordan-COHA carries a **Heis**-module structure. However, by Section 3.1.1 both A_1-COHA and Jordan-COHA share the same underlying vector spaces, which is $\oplus_d H^*_{G_d}(M_d) \simeq \oplus_d H^*_{G_d}(\mathbf{pt})$. Then we can immediately get a correspondence between these two representations. □

7.3 Generalization

The A_1-COHA is related to the infinite Clifford algebra since it contains free fermions with positive energies. The Jordan-COHA is related to the infinite Heisenberg algebra, and thus it is related to the oscillator algebras. Therefore, there should be a relation between the COHA version Boson-Fermion correspondence (Theorem 7.2.1) and the classic Boson-Fermion correspondence (Theorem 7.1.4).

Conjecture 7.3.1. *There is a construction to double the double of A_1-COHA and realize it as the infinite Clifford algebra $\boldsymbol{Cl}_\infty$.*

In other words, we want to construct the Fermion Fock space geometrically. There are some other work in this direction. See e.g. [19, 61]. We would like to find a COHA construction.

Appendix A

Review of Intersection Theory

A.1 Chow ring

A.1.1 Definitions

Definition A.1.1. Let X be an algebraic scheme. A k-*cycle* on X is a finite formal sum $\sum n_i[V_i]$ where V_i are k-dimensional subvarieties of X and n_i are integers. The group of k-cycles modulo rational equivalence is denoted by $A_k(X)$. The direct sum $A_*(X) = \oplus_{k \in \mathbb{Z}_{\geq 0}} A_k(X)$ is called the *Chow group*.

Definition A.1.2. Let X be an n-dimensional non-singular variety. Set $A^p(X) := A_{n-p}(X)$, and $A^*(X) = \oplus_k A^k(X)$. There is a ring structure on $A^*(X)$ which represents the intersection of two cycles. The group with the ring structure is called the *Chow ring*.

Remark A.1.3. See [20] for more details about the definitions of the product.

A.1.2 Functorial properties

Definition A.1.4. Let $f : X \to Y$ be a proper morphism of varieties. Let V be a subvariety of X of dimension k. Set $W = f(V)$. The rational functions over V (resp. W) is denoted by $R(V)$ (resp. $R(W)$). f induce an embedding of $R(W)$ into $R(V)$ and we can compute

$[R(V) : R(W)]$ if W has the same dimension as V. Define

$$f_*[V] = \begin{cases} [R(V) : R(W)][W], & \text{if } \dim(W) = \dim(V), \\ 0, & \text{if } \dim(W) < \dim(V). \end{cases} \tag{A.1.1}$$

Proposition A.1.5 (See e.g.[20]). *The push-forward map f_* is a morphism of abelian groups $A_k X \to A_k Y$. It makes A_* a covariant functor for proper morphisms.*

Definition A.1.6. Let $f : X \to Y$ be a flat morphism of varieties. Let V be a k-dimensional subvariety of X. Define the map by

$$f^*[V] = [f^{-1}(V)]. \tag{A.1.2}$$

Proposition A.1.7 (See e.g.[20]). *The pull-back map f^* is a morphism of abelian groups $A_k Y \to A_{k+n} X$ where $n = \dim(Y) - \dim(X)$. It makes A^* a contravariant functor for flat morphisms.*

Proposition A.1.8 ([20]). *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \tag{A.1.3}$$

be a fibre square, with g flat and f proper. Then g' is flat, f' is proper, and $f'_* g'^* = g^* f_*$.

A.1.3 Cellular decomposition

Definition A.1.9. X is a scheme with a *cellular decomposition* if X has a filtration $X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$ by closed subschemes, with each $X_i - X_{i-1}$ a disjoint union of schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$.

Theorem A.1.10 ([20]). *Let X be a scheme with a cellular decomposition with notations described above. Then $A_*(X)$ is $\mathbb{Q}$-span of $\{[\overline{U_{i,j}}]\}$.*

A.2 Borel-Moore homology

Let X be a complex scheme.

Definition A.2.1. Let $\overline{X}$ be an arbitrary compactification of X such that $(\overline{X}, \overline{X}\backslash X)$ is a CW-pair. The Borel-Moore homology is defined as $H_*^{BM}(X) := H_*(\overline{X}, \overline{X}\backslash X)$, where H_* is ordinary relative homology of the pair $(\overline{X}, \overline{X}\backslash X)$.

We need Borel-Moore homology because we need *fundamental classes*.

Proposition A.2.2 ([7]). *For any complex algebraic variety X, there exists a fundamental class $[X]$. If X is irreducible of real dimension m, $[X] \in H_m^{BM}(X)$.*

The relation of Borel-Moore homology and cohomology is stated in the following theorem.

Theorem A.2.3 (Poincaré duality, see e.g. [7]). *Let M be a smooth algebraic variety of $\mathbb{R}$-dimension $2n$. Then*

$$\begin{aligned} PD : H^*(M) &\to H_{2n-*}^{BM}(M) \\ \psi &\mapsto \psi \cap [M] \end{aligned} \tag{A.2.1}$$

is an isomorphism.

Theorem A.2.4 ([20]). *If X has a cellular decomposition, then the cycle map*

$$cl : A_*(X) \to H_{2*}^{BM}(X) \tag{A.2.2}$$

is an isomorphism which doubles the degree.

Corollary A.2.5. *If X has a cellular decomposition, using the notations introduced in the previous section, $H_*^{BM}(X)$ is $\mathbb{Q}$-span of $\{[\overline{U_{i,j}}]\}$. Here by abuse of notations, we use $[\overline{U_{i,j}}]$ to denote the Borel-Moore homology class which is correspondent to the cycle $[\overline{U_{i,j}}]$.*

Remark A.2.6. We would use the notations of cohomology, Borel-Moore homology and algebraic cycles freely.

Remark A.2.7. For any complex algebraic variety X, there exists a fundamental class $1_{[X]} \in H^0(X)$ by Proposition A.2.2 and Theorem A.2.3.

Example A.2.8. $H_*^{BM}(\mathbb{R}^{2n}) \simeq \mathbb{Q}[\mathbb{R}^{2n}]$, where $[\mathbb{R}^{2n}]$ is the fundamental class in $H_{2n}^{BM}(\mathbb{R}^{2n}) \simeq H^0(\mathbb{R}^{2n})$.

Appendix B

Equivariant Cohomology

B.1 Classifying spaces

Definition B.1.1. Let G be a topological group. Let EG be a contractible space on which G acts freely. EG is called the *universal bundle* of G, and the quotient space $BG := EG/G$ is called the *classifying space* of G.

The importance of $EG \to BG$ lays in the following theorem.

Theorem B.1.2 (Classification Theorem, see e.g.[66])**.** *Let $\pi : E \to B$ be a principal G-bundle. There exists a map $f : B \to BG$ and an isomorphism of principal G-bundles $\Phi : E \simeq f^*(EG)$. Moreover, The map f and Φ are unique up to homotopy.*

Example B.1.3. Let $T = \mathbb{C}^*$ be a 1-dimensional torus. Then $BT = \mathbb{C}P^\infty$ and $ET = \mathbb{C}^\infty \backslash \{0\}$. $ET \to BT$ is the tautological bundle on $\mathbb{C}P^\infty$.

Example B.1.4. Let $T = (\mathbb{C}^*)^n$ be a n-dimensional torus. $BT = (\mathbb{C}P^\infty)^n$, and $ET = \pi_1^*\mathcal{O}(-1) \otimes \ldots \otimes \pi_n^*\mathcal{O}(-1)$, where $\pi_i : BT \to \mathbb{C}P^\infty$ is the i-th projection and $\mathcal{O}(-1)$ is the tautological bundle on $\mathbb{C}P^\infty$.

B.2 Equivariant cohomology

Definition B.2.1. Let G be a complex linear algebraic group, and let X be a complex algebraic variety with a left G-action. Find a contractible space EG with a free right G-action. Now form the quotient space

$$EG \times_G X := EG \times X/(e \cdot g, x) \sim (e, g \cdot x). \tag{B.2.1}$$

Then the *equivariant cohomology ring* $H_G^*(X)$ is defined to be the cohomology of the quotient space

$$H_G^*(X) := H^*(EG \times_G X) \tag{B.2.2}$$

where the cohomology on the right hand side is the singular cohomology with $\mathbb{Q}$ coefficients.

Remark B.2.2. Equivariant cohomology is a generalized cohomology theories. Thus for each G-space X, since each projection map $X \to \mathbf{pt}$ is G-equivariant, there is a natural map $H_G^*(\mathbf{pt}) \to H_G^*(X)$ which equips $H_G^*(X)$ a $H_G^*(\mathbf{pt})$-module structure. Denote $EG \times_G \mathbf{pt}$ by BG. Then each $H_G^*(X)$ is a $H^*(BG)$-module.

Remark B.2.3. The space EG in the Borel construction is actually the universal bundle of G due to the uniqueness of the universal bundle. Similarly, the space BG in the Borel construction is the classifying space of G.

Example B.2.4. Let $T := \mathbb{C}^*$. Then as stated in Example B.1.3, $B\mathbb{C}^* = \mathbb{C}P^\infty$. Then

$$H_{\mathbb{C}^*}^*(\mathbf{pt}) = H^*(E\mathbb{C}^* \times_{\mathbb{C}^*} \mathbf{pt}) = H^*(B\mathbb{C}^*) = \mathbb{Q}[t], \tag{B.2.3}$$

where t is the first Chern class of the tautological line bundle on $B\mathbb{C}^*$.

Example B.2.5. Let $T = (\mathbb{C}^*)^n$. As stated in Example B.1.4, $BT = (\mathbb{C}P^\infty)^n$. Then

$$H_{(\mathbb{C}^*)^n}^*(\mathbf{pt}) = H^*(E(\mathbb{C}^*)^n \times_{(\mathbb{C}^*)^n} \mathbf{pt}) = H^*(B(\mathbb{C}^*)^n) = \mathbb{Q}[t_1, \ldots, t_n], \tag{B.2.4}$$

where t_i is the first Chern class of pullback of the tautological line bundle on $\mathbb{C}P^\infty$ corresponding to the i-th projection π_i.

B.3 Approximation

The spaces EG and BG are typically infinite-dimensional, so they are not algebraic varieties. However there are finite-dimensional, non-singular algebraic varieties $E_m \to B_m$ which serve as "approximations" to $EG \to BG$. The approximation we use here was introduced by Totaro in [69].

Let G be a linear algebraic group and let n be a non negative integer. Choose a G-module V_n and a G-invariant open subset $U_n \subset V_n$ satisfying the following conditions:

1. The quotient $U_n \to U_n/G$ exists and is a principal G-bundle.
2. The codimension of $V_n \backslash U_n$ in V_n is larger than n.

$U_n \to U_n/G$ is an approximation of the universal G-bundle $EG \to BG$ in the sense of the following proposition.

Proposition B.3.1 ([12]). *Let X be a complex variety and G be an algebraic group acting on X. U_n is constructed as above. Then $H^k(X \times_G EG) \simeq H^k(X \times_G U_n)$ for $k \leq 2n$.*

Example B.3.2. Let $T := \mathbb{C}^*$ be the 1-dimensional torus. We can take $V_n := \mathbb{C}^{n+1}$ to be the natural T-module via multiplication, and $U_n := V_n \backslash \{0\}$. U_n/T is $\mathbb{C}P^n$. $\{\mathbb{C}P^n\}_n$ is an approximation of $B\mathbb{C}^*$. $H^*(\mathbb{C}P^n) = \mathbb{Q}[t]/\langle t^{n+1}\rangle$. Thus when $k \leq 2n$, $H^k(B\mathbb{C}^*) \simeq H^k(\mathbb{C}P^n)$. Furthermore, we have

$$H^*(B\mathbb{C}^*) \simeq \varprojlim_n H^*(\mathbb{C}P^n). \tag{B.3.1}$$

Using these approximation, we are able to put everything in the context of algebraic geometry.

B.4 Some bundle structure

Let G be a compact, connected Lie group, T a maximal torus, and $W := N_G(T)/T$ the Weyl group of T in G. Suppose G acts on X freely on the right such that $X \to X/G$ is a principal G-bundle. Then the natural projection $X/T \to X/G$ is a fiber bundle with fiber G/T.

Lemma B.4.1 ([27]). *The cohomology of X/G is the subspace of W-invariants of the cohomology of X/T: $H^*(X/G) \simeq H^*(X/T)^W$.*

Example B.4.2. Let $G = \mathrm{GL}(n, \mathbb{C})$ and T be the maximal torus which consists of all diagonal matrices. Then $T \simeq (\mathbb{C}^*)^n$. The Weyl group W in this case is the symmetric group of n elements. By Example B.2.5, $H^*_{(\mathbb{C}^*)^n}(\mathbf{pt}) = \mathbb{Q}[t_1, \ldots, t_n]$. The Weyl group acts on the space by permuting t_i's. Then by Lemma B.4.1, we have

$$H^*_{\mathrm{GL}(n,\mathbb{C}^*)}(\mathbf{pt}) = \mathbb{Q}[t_1, \ldots, t_n]^W, \tag{B.4.1}$$

which is the space of all symmetric polynomials with n variables.

Example B.4.3. Let $G = \prod_{i \in I} \mathrm{GL}(d^i, \mathbb{C})$. Let T be the maximal torus which consists of all diagonal matrices of each piece. Then $H^*_T(\mathbf{pt})$ is the polynomial algebra generated by $\{t_{i,j}\}_{i \in I, j=1,\ldots,d^i}$. The variables are grouped according to the index $i \in I$. The Weyl group $W = \times_{i \in I} S_{d^i}$ and it acts on $H^*_T(\mathbf{pt})$ by permuting variables in each group. Thus $H^*_G(\mathbf{pt})$ is the algebra of polynomials of $\sum_{i \in I} d^i$ variables, where each group of variables are symmetric.

Appendix C

Cohomology of Categories Fibred in Groupoids

This appendix follows [12]. See also e.g. [22].

C.1 Category fibred in groupoids

Fix a base scheme S. Let $\mathcal{S}$ be the category of S-schemes.

Definition C.1.1. A *category fibred in groupoids* (CFG) over $\mathcal{S}$ is a category $\mathcal{X}$ together with a functor $\rho : \mathcal{X} \to \mathcal{S}$ satisfying the following conditions.

1. Given an object $b \in \mathcal{X}$, let $B = \rho(b)$. If $f : B' \to B$ in $\mathcal{S}$ there exists a pullback object f^*b and a morphism $f^*b \to b$ in $\mathcal{X}$ whose image under the functor ρ is the morphism $B' \to B$. Moreover, f^*b is unique up to canonical isomorphism.

2. If $\alpha : b_1 \to b_2$ is a map in $\mathcal{X}$ such that $\rho(\alpha) = B \xrightarrow{\mathrm{Id}} B$ for some S-scheme B then α is an isomorphism.

Remark C.1.2. Given a CFG $\mathcal{X}$ and a scheme B, denote by $\mathcal{X}(B)$ the subcategory consisting of objects mapping to B and morphisms mapping to the identity.

Remark C.1.3. The category satisfying the first condition is called a *fibred category.* The second condition implies that $\mathcal{X}(B)$ is a groupoid, that is, a category whose all morphisms are isomorphisms.

Example C.1.4. Let X be a S-scheme. We can associate to its functor of points a CFG $\underline{X}$. It is the category of X-schemes viewed as fibred category over the category of S-schemes, that is, a category whose objects are S-morphisms $B \to X$ for $B \in \mathcal{S}$ and morphisms are the natural morphisms.

Definition C.1.5. A CFG $\mathcal{X}$ is *representable* if $\mathcal{X}$ is equivalent to a CFG $\underline{X}$ for some S-scheme X.

C.2 Quotient CFGs

Definition C.2.1. Let B be a scheme. A *G-torsor over B* is a smooth morphism $p : E \to B$ where G acts freely on E, p is G-invariant and there is an isomorphism of G-spaces $E \times_B E \to G \times E$.

Definition C.2.2. If X is a scheme and G is an algebraic group acting on X. Define a CFG $[X/G]$ to be the category whose objects are pairs $(E \to B, E \xrightarrow{f} X)$ where $E \to B$ is a G-torsor and $f : E \to X$ is a G-equivariant map. A morphism $(E' \to B', E' \xrightarrow{f'} X') \to (E \to B, E \xrightarrow{f} X)$ in $[X/G]$ is a Cartesian diagram of G-torsors

$$\begin{tikzcd} E' \arrow[r, "h"] \arrow[d] & E \arrow[d] \\ B' \arrow[r] & B \end{tikzcd} \tag{C.2.1}$$

such that $f' = f \circ h$.

Definition C.2.3. A CFG $\mathcal{X}$ is a *quotient CFG* if $\mathcal{X}$ is equivalent to a CFG $[X/G]$ for some scheme X with G-actions.

Example C.2.4. Let $U_n := \mathbb{C}^{n+1}\backslash\{0\}$. $\mathbb{C}^*$ acts on U_n by left multiplication. The action is free. Then we can have the geometric quotient $\mathbb{C}P^n$ as well as the quotient CFG $[U_n/\mathbb{C}^*]$. It's not hard to see $[U_n/\mathbb{C}^*] \simeq \underline{\mathbb{C}P^n}$.

Example C.2.5. Let G be a linear algebraic group over $\mathbb{C}$. Let $\mathcal{B}G$ be the CFG whose objects are G-torsors $E \to T$ and whose morphism are cartesian diagrams

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ T' \arrow[r] & T \end{tikzcd} \tag{C.2.2}$$

with the added condition that the map $E' \to E$ is G-invariant. The definition above implies that $\mathcal{B}G$ is the quotient stack $[\mathbf{pt}/G]$. When talking about objects in $[\mathbf{pt}/G]$, we usually omit the morphism from E to the point pt from the pair $(E \to B, E \to \mathbf{pt})$.

C.3 Cohomology of CFGs

Definition C.3.1. Let $\mathcal{X}$ be a CFG defined over $\mathbb{C}$. A *cohomology class* c on $\mathcal{X}$ is the data of a cohomology class $c(b) \in H^*(B)$ for every scheme B and every object b of $\mathcal{X}(B)$, with the following compatibility condition: Given schemes B' and B and objects $b' \in \mathcal{X}(B')$, $b \in \mathcal{X}(B)$ and a morphism $b' \to b$ whose image in $\mathcal{S}$ is a morphism $f : B' \to B$ then $f^*c(b) = c(b') \in H^*(B')$.

Definition C.3.2. The cup product on cohomology of spaces guarantees that the collection of all cohomology classes on $\mathcal{X}$ forms a graded skew-commutative ring. We denote this ring by $H^*(\mathcal{X})$

Definition C.3.3. If $f : \mathcal{Y} \to \mathcal{X}$ is a map of CFGs over $\mathcal{S}$ then there is a *pullback* homomorphism $f^* : H^*(\mathcal{X}) \to H^*(\mathcal{Y})$. For $\mathfrak{c} \in H^*(\mathcal{X})$, $f^*\mathfrak{c}$ is defined by $f^*\mathfrak{c}(b) = \mathfrak{c}(f(b))$ for any $b \in \mathcal{Y}(B)$.

The main results about the cohomology of quotient stacks is the relations to the equivariant cohomology stated below.

Theorem C.3.4 ([12].)**.** *Using the approximation introduced in Section B.1. Let X be a complex variety and G an algebraic group acting on X. Then the pullback map $H^k(\mathcal{X}) \to H^k(X \times_G U_n)$ is an isomorphism for $k \leq 2n$.*

Corollary C.3.5 ([12])**.** *The cohomology of quotient CFGs $H^*([X/G])$ is isomorphic to the equivariant cohomology $H^*_G(X)$.*

Let X be a scheme, and $\mathcal{X} = \underline{X}$. Fix $c \in H^*(X)$. A correspondent $\mathfrak{c} \in H^*(\mathcal{X})$ is constructed as follows. Let $\mathfrak{c}(X \xrightarrow{\mathrm{Id}} X) = c$. For each object $B \xrightarrow{b} X$, we have the diagram

$$\begin{tikzcd} B \arrow[rr, "b"] \arrow[dr, "b"'] & & X \arrow[dl, "\mathrm{Id}"] \\ & X & \end{tikzcd} . \tag{C.3.1}$$

Then $\mathfrak{c}(B \xrightarrow{b} X) = b^*(\mathfrak{c}(X \xrightarrow{\mathrm{Id}} X)) = b^*(c) \in H^*(B)$. This implies that c encodes all the data needed for $\mathfrak{c}$. On the other side, fix $\mathfrak{c} \in H^*(\mathcal{X})$, we have $H^*(X) \ni c = \mathrm{Id}^*(\mathfrak{c}(X \xrightarrow{\mathrm{Id}} X))$. This correspondence suggests that $H^*(\mathcal{X}) \simeq H^*(X)$.

Example C.3.6. Following Example C.2.4, $H^*([U_n/\mathbb{C}^*]) = H^*(\underline{\mathbb{C}P^n}) \simeq H^*(\mathbb{C}P^n)$. It is well known that $H^*(\mathbb{C}P^n) \simeq \mathbb{Z}[\xi]/\langle \xi^{n+1} \rangle$ where $\xi \in H^2(\mathbb{C}P^n)$ is the first Chern class of the tautological line bundle $\mathcal{L}$ on $\mathbb{C}P^n$. As a class of stack $[U_n/\mathbb{C}^*]$, it is correspondent to the class $b^*\xi \in H^2(B)$ of the pullback bundle $b^*\mathcal{L}$ for any diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & U_n \arrow[d] \\ B \arrow[r, "b"] & \mathbb{C}^n \end{tikzcd} . \tag{C.3.2}$$

Example C.3.7. Let $G = \mathbb{C}^*$. Consider $H^*([\mathbf{pt}/\mathbb{C}^*]) = H^*(\mathcal{B}\mathbb{C}^*) \simeq H^*(B\mathbb{C}^*)$. $H^*(B\mathbb{C}^*) \simeq \mathbb{Z}[x]$ where $x \in H^2(B\mathbb{C}^*)$ is the first Chern class of the tautological line bundle $\mathcal{L}$ on $B\mathbb{C}^*$. For each $\mathbb{C}^*$-torsor $E \to B$, by the universal property, there is a map $b : B \to B\mathbb{C}^*$ such that $E \to B$ is the pullback of $E\mathbb{C}^* \to B\mathbb{C}^*$. Then as a class of stack $\mathcal{B}\mathbb{C}^*$, x is correspondent to $c_1(b^*(\mathcal{L})) \in H^2(B)$ for $E \to B$.

Finally we want to compute the pullback in a special case. Let G acts on X freely. Then $B = X/G$ is a scheme. Let $f : [X/G] \to [\mathbf{pt}/G]$ defined by the natural projection $X \to \mathbf{pt}$. For any $t = (E \to T, E \xrightarrow{t} X)$, there exist a commutative diagram

$$\begin{tikzcd} E \arrow[r, "t"] \arrow[d] & X \arrow[r] \arrow[d] & EG \arrow[d] \\ T \arrow[r] & B \arrow[r, "b"] & BG \end{tikzcd} . \tag{C.3.3}$$

For any $\mathfrak{c} \in H^*([\mathbf{pt}/G])$, $f^*c(t) \in H^*(T)$ is gotten from the pullback following the above diagram. This implies that $f^*\mathfrak{c}$ can be gotten by b^*c where $c \in H^*(BG)$ correspondent to $\mathfrak{c}$. It is also true for the approximations of $EG \to BG$ in the following sense.

Proposition C.3.8. *Let $\{U_n/G\}_n$ be an approximation of $EG \to BG$. Let $f : [X/G] \to [\boldsymbol{pt}/G]$ be the map induced from the projection $X \to \boldsymbol{pt}$. For $c \in H^k(BG)$, $f^*c \in H^k([X/G])$ consists of data coming from the diagram*

$$\begin{tikzcd} E \arrow[r, "t"] \arrow[d] & X \arrow[r] \arrow[d] & U_n \arrow[d] \\ T \arrow[r] & B \arrow[r, "b"] & U_n/G \end{tikzcd} \tag{C.3.4}$$

for $k < 2n$.

Appendix D

Hilbert Schemes

The standard reference is [51]. We follow the presentation of [39].

D.1 General concepts

D.1.1 Definitions

Let X be a quasiprojective scheme over $\mathbb{C}$.

Definition D.1.1. Let S be a scheme. A *flat family of proper subschemes in* X *over* S is a closed subscheme $Z \subset S \times X$ such that the projection $Z \to S$ is flat and proper. If $s \in S$ is a closed point, we denote the fibre of Z over s by Z_s.

Lemma D.1.2. *Let* S *and* S' *be two schemes. Given a flat family* $Z \subset S \times X$ *of proper subschemes in* X *and a morphism* $f : S' \to S$*, the family* $Z' := (f \times Id_X)^{-1}(Z) \subset S' \times X$ *is again flat and proper over* S'*.*

Let $Z \subset X$ be a zero-dimensional subscheme. $H^0(Z, \mathcal{O}_Z)$ is an artinian $\mathbb{C}$-algebra. Define the *length* $l(Z)$ of Z by $l(Z) = \dim_{\mathbb{C}} H^0(\mathcal{O}_Z)$.

We are able to define a contravariant functor for each $n \in \mathbb{N}$ as follows:

$$\mathcal{H}ilb^n_X : S \mapsto \left\{ \begin{array}{l} Z \subset S \times X : Z \text{ is proper and flat over } S, \text{ and} \\ \\ Z_s \text{ is a zero-dimensional subscheme of length } n, \text{ for all } s \in S \end{array} \right\}. \qquad \text{(D.1.1)}$$

Theorem D.1.3 ([23])**.** *The functor $\mathcal{H}ilb^n_X$ is represented by a quasiprojective scheme $X^{[n]}$. If X is projective then $X^{[n]}$ is also projective.*

Note that the set of closed points of $X^{[n]}$ is

$$X^{[n]}(\operatorname{Spec}\mathbb{C}) = \{Z \subset X : Z \text{ is a zero-dimensional subscheme with } l(Z) = n\}. \qquad \text{(D.1.2)}$$

This is our usual understanding of Hilbert scheme of points on X.

D.1.2 Hilbert-Chow morphism

Assume X is reduced. If $x \in Z$ is a closed point, the multiplicity of x in Z is defined as $l(Z_x) := \dim_{\mathbb{C}}(\mathcal{O}_{Z,x})$.

Proposition D.1.4 ([49], Section 5.4)**.** *There exists a morphism called Hilbert-Chow morphism*

$$\rho : X^{[n]}_{red} \to S^n X \qquad \text{(D.1.3)}$$

defined by

$$\rho(Z) = \sum_{x \in X} l(Z_x)[x]. \qquad \text{(D.1.4)}$$

Let $\nu = (\nu_1, \nu_2, \ldots)$ be a composition of d. Recall that a composition is a tuple of integers $\nu_1 \geq \nu_2 \geq \ldots$ with $\sum \nu_i = d$. Then maximal number k such that $\nu_k \neq 0$ is called the *length* for the composition, and is denoted by $l(\nu)$. For each composition ν of d, we define

$$S^n_\nu X = \left\{ \sum_{i=1}^{k} \nu_i [x_i] \in S^n X \;\middle|\; x_i \neq x_j \text{ for } i \neq j \right\}. \qquad \text{(D.1.5)}$$

Then $S^n_\nu X$ has dimension $l(\nu)\dim_{\mathbb{C}} X$. Note that $S^n_{(1,1,\ldots,1)}X$ is an open dense subset. It is the nonsingular locus of $S^n(X)$.

Proposition D.1.5 ([51]). *We have the stratification of $S^n(X)$*

$$S^n X = \bigcup_\nu S^n_\nu X. \tag{D.1.6}$$

Corollary D.1.6. *We have a stratification of $X^{[n]}$*

$$X^{[n]} = \bigcup_\nu \rho^{-1}(S^n_\nu X). \tag{D.1.7}$$

The dimension of $\rho^{-1}(S^n_\nu X) = l(\nu)\dim_{\mathbb{C}} X$.

D.2 $X = \mathbb{A}^1$-case

We would go through the case $X = \mathbb{A}^1$ in this section. In this case, the set of closed points of $X^{[n]}$ is

$$\left\{\begin{array}{l} Z \subset X : Z \text{ is a zero-dimensional} \\ \text{subscheme with } l(Z) = n \end{array}\right\} = \left\{\begin{array}{l} I \subset \mathbb{C}[x] : I \text{ is an ideal of } \mathbb{C}[x] \\ \text{and } \dim \mathbb{C}[x]/I = n \end{array}\right\}. \tag{D.2.1}$$

Since $\mathbb{C}[x]$ is a prime ideal domain, each idea is generated by a polynomial in $\mathbb{C}[x]$. Then the above set is the same as

$$\{\text{monic polynomials in } \mathbb{C}[x] \text{ whose degree is } n\}. \tag{D.2.2}$$

To justify these statements, any 0-dimensional subscheme in $\mathbb{A}^1$ can be described by a monic polynomial $f(T) = T^n + a_{n-1}T^{n-1} + \ldots + a_0$ of degree n. The coefficients $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ define a point $\mathbf{a} \in \mathbb{C}^n$ and conversely any point $\mathbf{a}$ defines a polynomial and thus a subscheme.

Now consider Hilbert-Chow morphism. We start from a general statement.

Proposition D.2.1. *Let X be an irreducible, smooth, quasiprojective curve. Then for all $n \geq 0$, $X^{[n]}$ is smooth and irreducible of dimension n*

Proof. There is a version of this proposition for $\dim_{\mathbb{C}} X = 2$ case. The proof can be found in [16]. The proof of this proposition is similar. □

Lemma D.2.2. *Hilbert-Chow morphism ρ induces an isomorphism*

$$X_0^{[n]} \to S^n_{(1,1,\ldots,1)}X \tag{D.2.3}$$

and $S^n_{(1,1,\ldots,1)}X$ is dense in S^nX.

Proposition D.2.3. *Let X be a nonsingular quasiprojective curve, the Hilbert-Chow morphism $\rho : X^{[n]} \to S^nX$ is an isomorphism.*

Proposition D.2.4. $X^{[n]} = S^n(\mathbb{C}^1) \simeq \mathbb{C}^n$.

Proof. The first is by Proposition D.2.3. The second can be seen from the following formulas:

$$S^n(X) = \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n]^{S_n}) \simeq \operatorname{Spec}(\mathbb{C}[e_1, \ldots, e_n]) = \mathbb{A}^n, \tag{D.2.4}$$

where e_i is the i-th elementary polynomial for each i. □

Remark D.2.5. Note that $\dim_{\mathbb{C}} S^n_\nu X = l(\nu)$, and $l(\nu) < n$ for any ν other than $(1, 1, \ldots, 1)$. Since $A^*(\mathbb{C}^n)$ is generated by $[\mathbb{C}^n]$, it implies that the strata other than $S^d_{(1,\ldots,1)}X$ don't contribute to its Chow ring. Therefore when we do computations Chow ring of cohomology, we could only focus on the $S^d_{(1,\ldots,1)}X$ part, which is the subset on which all points are different.

D.2.1 Alternative description

There is an alternate way to talk about $X^{[n]}$.

Theorem D.2.6. *Let $X = \mathbb{A}^1$.*

$$X^{[n]} \simeq \left\{ \begin{array}{l} (B,i) : \textit{There exists no subspace } S \subsetneq \mathbb{C}^n \\ \textit{such that } B(S) \subset S \textit{ and } Im(i) \subset S \end{array} \right\} \Big/ \mathrm{GL}(n,\mathbb{C}), \tag{D.2.5}$$

where $B \in \mathrm{End}(\mathbb{C}^n)$ and $i \in \mathrm{Hom}(\mathbb{C},\mathbb{C}^n)$ with the action given by

$$g \cdot (B,i) = (gBg^{-1}, gi), \textit{ for } g \in \mathrm{GL}(d,\mathbb{C}). \tag{D.2.6}$$

By Proposition D.1.5 and its corollary, we know that $X^{[n]}$ has a stratification. The open stratum $S^n_{(1,1,\ldots,1)}X$ is the nonsingular locus of $S^n(X)$.

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