A LOCAL EXTRAPOLATION METHOD FOR HYPERBOLIC CONSERVATION LAWS: THE ENO AND GOODMAN-LEVEQUE UNDERLYING SCHEMES AND SUFFICIENT CONDITIONS FOR TVD PROPERTY

by

DONALD OMEDO ADONGO

BSc., Egerton University, Kenya, 1990
M.S., Clemson University, 2000

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
2008

Abstract

We start with linear single variable conservation laws and examine the conditions under which a local extrapolation method (LEM) with upwinding underlying scheme is total variation diminishing TVD. The results are then extended to non-linear conservation laws. For this later case, we restrict ourselves to convex flux functions f, whose derivatives are positive, that is, $f'' > 0$ and $f' > 0$. We next show that the Goodman-LeVeque flux satisfies the conditions for the LEM to be applied to it. We make heavy use of the CFL conditions, the geometric properties of convex functions apart from the martingle type properties of functions which are increasing, continuous, and differentiable.

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Dedication

I dedicate this work to my son, Abong'o Adongo, who always wondered why my school never closed in the evening.

Preface

In Chapter 1, we give a background of hyperbolic conservation laws and the problems faced in solving them both analytically and numerically. We also discuss some traditional numerical methods used.

In computing solutions to equations of the form $\mathbf{u}_t(x,t) + \mathbf{f}(\mathbf{u}(x,t))_x = 0$ numerically, many difficulties do arise. Using Godunov's method, which is first order accurate, the numerical results are very smeared in regions near the discontinuities. The first order accurate method has a large amount of "numerical viscosity" that smoothes the solution just as physical viscosity would, but to an unrealistic extent by several orders of magnitude. The numerical viscosity may be eliminated by using some standard second order methods but dispersive effects leading to large oscillations in the numerical solution are introduced. A solution to the discontinuity problem, is to use "shock tracking", whereby some explicit procedure for tracking the location of discontinuities is incorporated into the standard numerical methods.

Ideally it is preferable to have schemes that produce sharp approximations to discontinuous solutions automatically, without explicit tracking and use of jump conditions. These are the so called "shock capturing" schemes. Various approaches have been used to develop these high resolution schemes. Examples include essentially non-oscillatory schemes (ENO) due to Harten, Engquist, Osher and Chakravarthy [4]; Total Variation Diminishing schemes (TVD) due to Harten [3] , Goodman and LeVeque [2], Van Leer [7].

In Chapter 2, we discuss the ENO schemes due to Osher, *et al* and the scheme due to Goodman and LeVeque. These are methods that are at least second order accurate on smooth solutions and yet give well resolved, non oscillatory solutions.

Yang [18] developed a local extrapolation method (LEM) that increases the order of an r^{th} order scheme by one, using ENO underlying schemes. We restate the LEM and then examine the conditions under which the LEM with the upwinding underlying scheme is TVD under the more general assumptions that the problem has convex, positive flux, that is, the flux function f has $f''(u) > 0$ and $f'(u) > 0$. The main tool used is the CFL condition (named for Courant, Friedrichs and Lewy). It states that a numerical scheme can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as Δt and Δx go to zero. It is a necessary condition for stability which in turn is sufficient for convergence. The CFL condition is set up so that the flux across any cell interface does not depend on flux emanating from any neighboring cells. In addition to the general properties of continuous, differentiable and increasing functions, geometric properties of convex functions are also used.

We then show that the flux of Goodman and LeVeque's scheme, satisfies the conditions required by the LEM in [18]. Goodman-LeVeque's scheme is a second order method and TVD. The geometric nature of Goodman and LeVeque's scheme makes its implementation relatively easy, and computation-wise it is less expensive compared to the ENO schemes of the same order. The LEM is desirable because it is less expensive to implement than to use for example the ENO schemes of equivalent order.

In Chapter 3, we discuss some general stability and entropy conditions. Various forms are introduced and we then make a connections between the entropy conditions and the discussions on the Riemann problem in Chapter 1.3. We state the Lax-Wendroff theorem - if the numerical solution of a conservative scheme converges, it converges towards a weak solution - which is a useful tool towards determining whether numerical solutions obtained satisfy the entropy conditions or not.

In Chapter 4, we give some numerical examples for both the linear and non-linear conservation laws. A few spatial step lengths h are considered whereas the parameter λ for the LEM is fixed at 0.8. The ENO schemes, both second and third order, Goodman-LeVeque scheme and LEM are considered in various cases. We also compute the mean computation times for the various schemes in solving a linear conservation law.

Chapter 1

Introduction

The initial value problem of hyperbolic systems of conservation laws is given by

$$\begin{aligned} \mathbf{u}_t + \sum_{i=1}^{d} (\mathbf{f}_i(\mathbf{u}))_{x_i} &= 0 \\ \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}), \end{aligned} \tag{1.0.1}$$

where $\mathbf{u} \in \mathbb{R}^m$ is an m-dimensional vector of conserved quantities, $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{f}_i : \mathbb{R}^m \to \mathbb{R}^m$. The hyperbolicity of system means that

$$\sum_{i=1}^{d} \xi_i \left(\frac{\partial \mathbf{f}_i}{\partial \mathbf{u}} \right)$$

has m real eigenvalues and a complete set of eigenvectors for all collections of real numbers $\{\xi_i\}_{i=1}^{d}$. Conservation laws arise in fields like physics and engineering. Some examples are

- Euler equations of gas dynamics
- Aerodynamics
  - modeling of wing flutter
  - modeling flow patterns around rotating helicopter blades
  - modeling flow patterns around the blades of a turbine
- Meteorology and weather predictions: the weather fronts are just a shock waves, i.e. discontinuities in pressure and temperature.
- study of explosions and blast waves

1.1 Conservation Laws in 1-D

In one space dimension the conservation laws are of the form

$$\mathbf{u}_t(x,t) + \mathbf{f}(\mathbf{u}(x,t))_x = 0. \tag{1.1.1}$$

Note that the $m \times m$ Jacobian matrix $\mathbf{f}'(u)$ is diagonalizable - since we have a complete set of m linearly independent eigenvectors - and all its eigenvalues are real. $\mathbf{u}$ is an $m-$dimensional vector of conserved quantities so that u_j is the density function for the jth state variable (conserved quantity). The total quantity of this state variable in the interval $[x_{\kappa-1}, x_\kappa]$ at time t is given by $\int_{x_{\kappa-1}}^{x_\kappa} u_j(x,t)dx$. Conservation of the state variables means that $\int_{-\infty}^{\infty} u_j(x,t)dx$ is constant with respect to t. f is called the flux function.

Note that if $f'(\mathbf{u}) \in \mathbb{R}^{m\times m}$ then we can write

$$f'(\mathbf{u}) = \mathcal{R}\Lambda\mathcal{R}^{-1} \tag{1.1.2}$$

where R is the matrix of right eigenvectors. Letting $\tilde{\mathbf{u}} = \mathcal{R}^{-1}\mathbf{u}$ the system (1.1.1) is reduced to

$$\tilde{\mathbf{u}}_t + \Lambda\tilde{\mathbf{u}}_x = 0 \tag{1.1.3}$$

which is just a set of m decoupled advection equations. If $f'(\mathbf{u})$ is not a constant then both $\mathcal{R}$ and Λ may depend on x and/or t and (1.1.3) does not hold.

A well known fact is that the solution of (1.1.1) may develop discontinuities in itself or its derivatives in a finite time no matter how smooth the initial function $\mathbf{u}_0(x)$ is (see examples A.1, A.2 and A.3 in Appendix A). A typical solution of (1.1.1) is a piecewise smooth function whose domain consists of regions where the solution is smooth, separated by discontinuities, for example, shocks, contact discontinuities and the wave fronts of rarefaction waves.

The shock tube problem is an example that illustrates a solution of the form described above. In this case consider a tube filled with gas, initially divided by a membrane into two sections. The gas has higher density and pressure in one half of the tube than in the other half, with zero velocity everywhere. At initial time ($t = 0$), the membrane is suddenly removed and the gas allowed to flow. A net motion in the direction of lower pressure is expected and a uniform flow across the tube may be assumed.

The flow has three distinct waves separating regions in which the state variables are constant.

- A shock wave propagates into the region of lower pressure, across which the density and pressure jump to higher values and all of the state variables are discontinuous.
- A contact discontinuity follows the shock wave. Across it the density is discontinuous but the velocity and pressure are constant.
- A rarefaction wave moves in the opposite direction. All state variables are continuous and there is a smooth transition. As this wave passes through, the gas is rarefied (the density of the gas decreases).

1.2 Low Order Numerical Schemes

The hyperbolic problem (1.1.1) is discretized by space-time finite differences. The half plane $\{(x,t) : -\infty < x < \infty,\ t > 0\}$ is discretized by choosing a spatial grid size $h := \Delta x$ and a temporal step $k := \Delta t$. The grid points (x_j, t^n) are then given by

$$x_j = jh, \quad j \in \mathbb{Z}, \qquad t^n = nk, \quad n \in \mathbb{N}. \tag{1.2.1}$$

For subsequent discussions in this dissertation we set

$$\lambda := \frac{k}{h}$$

and define

$$x_{j+\frac{1}{2}} = x_j + \frac{h}{2}.$$

We look for discrete solutions U_j^n which approximate the values $u(x_j, t^n)$ of the exact solution for any j, n. Any explicit finite-difference method can be written in the form

$$U_j^{n+1} = U_j^n - \lambda \left(g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}} \right), \tag{1.2.2}$$

where $g_{j+\frac{1}{2}} = g(U_j^n, U_{j+1}^n)$ for every j and $g(\cdot,\cdot)$ is some function called the *numerical flux*. Some examples of explicit finite difference schemes are

1. Upwinding

$$U_j^{n+1} = U_j^n - \lambda \Delta_{\pm x} f(U_j^n), \tag{1.2.3}$$

   where

$$\Delta_{+x} f(U_j^n) = f(U_{j+1}^n) - f(U_j^n) \quad \text{and} \quad \Delta_{-x} f(U_j^n) = f(U_j^n) - f(U_{j-1}^n).$$

   Backward differences, that is $\Delta_{-x} f(U_j^n)$, are used if $f'(U_j^n) > 0$ while forward differences, that is $\Delta_{+x} f(U_j^n)$, are used if $f'(U_j^n) < 0$.

2. Forward Euler/centered

$$U_j^{n+1} = U_j^n - \frac{\lambda}{2}\left[f(U_{j+1}^n) - f(U_{j-1}^n)\right] \tag{1.2.4}$$

3. Lax-Friedrichs

$$U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{\lambda}{2}\left[f(U_{j+1}^n) - f(U_{j-1}^n)\right] \tag{1.2.5}$$

4. Lax-Wendroff

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{\lambda}{2}\left\{\left[1 - \lambda A_{j+\frac{1}{2}}\right]\left[f(U_{j+1}^n) - f(U_j^n)\right] + \left[1 + \lambda A_{j-\frac{1}{2}}\right]\left[f(U_j^n) - f(U_{j-1}^n)\right]\right\} \\ &= U_j^n - \frac{\lambda}{2}\left[f(U_{j+1}^n) - f(U_{j-1}^n)\right] \\ &\quad + \frac{1}{2}\lambda^2\left\{A_{j+\frac{1}{2}}\left[f(U_{j+1}^n) - f(U_j^n)\right] - A_{j-\frac{1}{2}}\left[f(U_j^n) - f(U_{j-1}^n)\right]\right\} \end{aligned} \tag{1.2.6}$$

   where

$$A_{j\pm\frac{1}{2}} = f'\left(\frac{1}{2}[U_j^n + U_{j\pm 1}^n]\right).$$

Definition 1.1. *Let u be the exact solution of the conservation law (1.1.1). The local truncation error $L(u;k,h)$ of the numerical scheme (1.2.2) is given by*

$$L(u;k,h) = \frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{k} + \frac{g(u(x_j,t^n), u(x_{j+1},t^n)) - g(u(x_{j-1},t^n), u(x_j,t^n))}{h} \tag{1.2.7}$$

We say the scheme is of order p in time and of order q in space (for suitable integers p and q), if for a sufficiently smooth solution of the exact problem, we have that

$$L(u;k,h) = O(k^p + h^q).$$

Both Upwinding and Lax-Friedrichs schemes are of first order whereas the Lax-Wendroff scheme is of second order. A drawback to using Lax-Wendroff scheme in the above form is that evaluation of the Jacobian is needed. So it is more expensive to use than the other forms that only use the flux function. Ways to avoid using the Jacobian include using two step procedures (see [8]). The Lax-Wendroff scheme may also produce oscillations in the solution. The Upwinding and Lax-Friedrichs schemes have smearing effects around the discontinuities. Hence they do not capture the true profiles of the solution around the discontinuity. Generally the low order schemes are not adapted to handling the discontinuities effectively.

Definition 1.2. *A finite difference scheme is said to be consistent if the local truncation error $L(u;k,h)$ goes to zero at k and h tend to zero independently.*

Definition 1.3. *A numerical scheme is said to be convergent if*

$$\lim_{k,h\to 0} \max_{j,n} |u(x_j,t^n) - U_j^n| = 0. \tag{1.2.8}$$

Definition 1.4. *A numerical method for a hyperbolic problem is said to be stable if, for any time T, there exists two positive constants C_T and δ, such that*

$$||\mathbf{U}^n||_\Delta \le C_T ||\mathbf{U}^0||_\Delta, \tag{1.2.9}$$

for any n such that $nk \le T$ and for any k,h such that $0 < k \le \delta$, $0 < h \le \delta$. Note that $||\cdot||_\Delta$ is a suitable discrete norm.

An example of a suitable discrete norm is

$$||\mathbf{V}||_{\Delta,p} = \left(h \sum_{j=-\infty}^{\infty} |V_j|^p \right)^{\frac{1}{p}} \qquad \text{for } p = 1,2 \tag{1.2.10}$$

or

$$\|\mathbf{V}\|_{\Delta,\infty} = \sup_j |V_j|.$$

Definition 1.5. *Let $a = f'(u) = const$. Courant, Friedrichs and Lewy showed that a necessary and sufficient condition for any explicit finite difference scheme to be stable is that*

$$|a\lambda| = \left|a\frac{k}{h}\right| \leq 1,$$

which is known as the CFL condition. The quantity $a\lambda$ is referred to as the CFL number.

In the case when $f'(u)$ is not constant the CFL condition is given by

$$k \leq \frac{h}{\sup_{x\in\mathbb{R},t>0} |f'(u(x,t))|}.$$

For the hyperbolic system where $f'(u) \in \mathbb{R}^{m\times m}$ the stability condition becomes

$$\left|\lambda_p \frac{k}{h}\right| \leq 1, \quad p = 1,\cdots,m$$

where $\{\lambda_p\}$ are the eigenvalues of $f'(u)$.

Definition 1.6. *The total variation of a solution U is defined to be*

$$TV(U) = \sum_{j=-\infty}^{\infty} |U_{j+1} - U_j|.$$

The numerical method $U^{n+1} = \mathcal{E}_n U$ is called total variation diminishing (TVD) if

$$TV(U^{n+1}) \leq TV(U^n)$$

for all grid functions U^n.

1.3 The Riemann Problem

The Riemann problem is the conservation equation together with a piecewise constant initial condition having a single jump discontinuity,

$$\begin{aligned} \mathbf{u}_t + \mathbf{f}(\mathbf{u})_x &= 0, \qquad \mathbf{u},\mathbf{f} \in \mathbb{R}^m \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x) = \begin{cases} \mathbf{u}_l & \text{if } x < 0 \\ \mathbf{u}_r & \text{if } x > 0. \end{cases} \end{aligned} \tag{1.3.1}$$

As the solution evolves the shock propagates with some speed, $s(t)$. The *Rankine-Hugoniot jump condition* is

$$s(\mathbf{u}_r - \mathbf{u}_l) = \mathbf{f}(\mathbf{u}_r) - \mathbf{f}(\mathbf{u}_l). \tag{1.3.2}$$

This can be written in the form $s[\![\mathbf{u}]\!] = [\![\mathbf{f}]\!]$ where $[\![\cdot]\!]$ represents the jump across the shock. For a scalar conservation law the shock speed is

$$s = \frac{f(u_r) - f(u_l)}{u_r - u_l} \tag{1.3.3}$$

that is, for the Riemann problem, the constant number s is the speed of the resulting shock.

For a better understanding of solutions to the Riemann problems, we shall look at some examples.

Definition 1.7. *For the scalar case ($m = 1$), the function $u(x,t)$ is a weak solution of the conservation law (1.1.1) if for all test functions $\phi(x,t) \in C_0^1(\mathbb{R},\mathbb{R})$ we have that*

$$\int_0^\infty \int_{-\infty}^\infty [u\phi_t + f(u)\phi_x] dx dt = -\int_{-\infty}^\infty u(x,0)\phi(x,0) dx. \tag{1.3.4}$$

Example 1.1. *Consider Burgers' equation*

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{1.3.5}$$

with piecewise constant initial data

$$u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0. \end{cases}$$

The solution depends on whether $u_l > u_r$ or $u_l < u_r$.

Case 1.1. $u_l > u_r$

Using (1.3.3), the shock speed is

$$s = \frac{\frac{u_r^2}{2} - \frac{u_l^2}{2}}{u_r - u_l} = \frac{1}{2}(u_r + u_l). \tag{1.3.6}$$

The unique weak solution is

$$u(x,t) = \begin{cases} u_l & \text{if } x - st < 0 \\ u_r & \text{if } x - st > 0. \end{cases}$$
$$= \begin{cases} u_l & \text{if } x < st \\ u_r & \text{if } x > st. \end{cases} \tag{1.3.7}$$

Note that the characteristic speed to the left of the shock is $f'(u) = u_l$ and to the right it is $f'(u) = u_r$. We have that $\frac{1}{u_l} < \frac{1}{u_r}$. Hence the characteristics in each of the regions where u is constant go into the shock $x = st$ as time advances.

Case 1.2. $u_l < u_r$

This case has infinitely many solutions, one of them being (1.3.7) but with characteristics now going out of the shock. This solution is not stable to perturbations.

A weak solution that is stable to perturbations is the rarefaction wave, namely,

$$u(x,t) = \begin{cases} u_l & \text{if } x < u_l t \\ \frac{x}{t} & \text{if } u_l t \le x \le u_r t \\ u_r & \text{if } x > u_r t. \end{cases} \tag{1.3.8}$$

Consider a Riemann problem where we assume the flux is convex i.e. $f'' > 0$, and $u_l < u_r$. Then the rarefaction wave is given by

$$u(x,t) = \begin{cases} u_l & \text{if } x < f'(u_l)t \\ v(\frac{x}{t}) & \text{if } f'(u_l)t \le x \le f'(u_r)t \\ u_r & \text{if } x > f'(u_r)t \end{cases} \tag{1.3.9}$$

where $v(\xi)$ is the solution to $f'(v(\xi)) = \xi$.

Lemma 1.1. *For non convex flux, the solution to the IVP for nonlinear hyperbolic systems of conservation laws*

$$u_t(x,t) + f(u)_x = 0, \qquad t > 0, \; -\infty < x < \infty$$
$$u(x,0) = u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

is

$$u(x,t) = u\left(\frac{x}{t}\right) = u(\xi), \tag{1.3.10}$$

where $u(\xi)$ *satisfies*

$$f(u(\xi)) - \xi w(\xi) = \begin{cases} \min_{u_l \le w \le u_r} [f(w) - \xi w], & u_l < u_r \\ \\ \max_{u_r \le w \le u_l} [f(w) - \xi w], & u_r < u_l. \end{cases} \tag{1.3.11}$$

Theorem 1.1.

$$u(\xi) = \begin{cases} -\frac{d}{d\xi}\left(\min_{u_l \le w \le u_r} [f(w) - \xi w]\right), & u_l < u_r \\ \\ -\frac{d}{d\xi}\left(\max_{u_r \le w \le u_l} [f(w) - \xi w]\right), & u_r < u_l. \end{cases} \tag{1.3.12}$$

For a more detailed discussion and proofs of Lemma 1.1 and Theorem 1.1 see [11].

1.4 Conservative methods

Definition 1.8. *A numerical scheme is said to be in conservative form if it can be written in the form*

$$U_j^{n+1} = U_j^n - \lambda[g(U_{j-p}^n, \cdots, U_{j+q}^n) - g(U_{j-p-1}^n, \cdots, U_{j+q-1}^n)] \tag{1.4.1}$$

for some function g *called the numerical flux function and where* $\lambda = \frac{k}{h}$ *with* k *the temporal step and* h *the spatial grid size.*

The simplest case is when $p = 0$ and $q = 1$ so that

$$U_j^{n+1} = U_j^n - \lambda[g(U_j^n, U_{j+1}^n) - g(U_{j-1}^n, U_j^n)]. \tag{1.4.2}$$

For brevity, we shall use the notation

$$g_{j+\frac{1}{2}}^n = g_{j+\frac{1}{2}}[U^n] = g(U_{j-p}^n, \cdots, U_{j+q}^n). \tag{1.4.3}$$

and so

$$g_{j-\frac{1}{2}}^n = g_{j-\frac{1}{2}}[U^n] = g(U_{j-p-1}^n, \cdots, U_{j+q-1}^n). \tag{1.4.4}$$

Example 1.2.

- *Forward Euler/Centered is conservative with numerical flux*

$$g_{j+\frac{1}{2}}^n = \frac{1}{2}[f(U_{j+1}^n) + f(U_j^n)]. \tag{1.4.5}$$

- *Lax-Friedrichs Scheme is conservative with numerical flux*

$$g^n_{j+\frac{1}{2}} = \frac{1}{2}\left[f(U^n_{j+1}) + f(U^n_j) - \frac{1}{\lambda}(U^n_{j+1} - U^n_j)\right]. \tag{1.4.6}$$

- *Lax-Wendroff is conservative with numerical flux*

$$g^n_{j+\frac{1}{2}} = \frac{1}{2}[f(U^n_{j+1}) + f(U^n_j)] - \frac{1}{2}\lambda A_{j+\frac{1}{2}}[f(U^n_{j+1}) - f(U^n_j)]. \tag{1.4.7}$$

where

$$A_{j+\frac{1}{2}} = f'\left(\frac{1}{2}[U^n_j + U^n_{j+1}]\right).$$

The method (1.4.1) simulates the exact relation

$$\overline{u}^{n+1}_j = \overline{u}^n_j - \lambda[\overline{f}^n_{j+\frac{1}{2}} - \overline{f}^n_{j-\frac{1}{2}}] \tag{1.4.8}$$

of the conservation law, where,

- $\overline{u}^n_j = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t^n)dx$ is the cell-average
- $\overline{f}^n_{j+\frac{1}{2}} = \frac{1}{k}\int_{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}},t))dt$ is the average flux

Definition 1.9. *The method (1.4.1) is consistent with the conservative law (1.1.1) if*

$$g(u,\cdots,u) = f(u) \tag{1.4.9}$$

for all u in the domain of f.

A sufficient condition for consistency is for g to be a Lipschitz continuous function in each variable (see for example [8, 9]).

Chapter 2

High Resolution TVD Schemes

We define a high resolution scheme to be one which

- achieves high order accuracy in the regions where the true solution is smooth
- produces sharp profiles for the shocks and contact discontinuities
- avoids superfluous oscillations around the discontinuities
- gets the correct positions and speeds of the discontinuities

We first describe Godunov's scheme on which many high resolution schemes are based.

- Define a piecewise constant function $w(x, t^n)$ which takes on the value U_j on $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$.
- Solve the conservation law exactly up to time t^{n+1} with initial data $w(x, t^n)$ to obtain $w(x, t^{n+1})$ [for k sufficiently small we have a sequence of Riemann problems].

  Then
  $$U_j^{n+1} = \frac{1}{h} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} w(x, t^{n+1}) dx.$$

2.1 Essentially Non-Oscillatory Schemes (ENO)

The ENO schemes were originally developed by Chakravarthy, Engquist, Harten and Osher. We define the so called sliding average as

$$\overline{u}(x_j,t^n) = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(\xi,t^n)d\xi = \int_{-\frac{1}{2}}^{\frac{1}{2}} u(x_j+\eta h,t^n)d\eta. \tag{2.1.1}$$

Now given the cell average values, $\{\overline{u}_j^n : j \in \mathbb{Z}\}$ we want an approximate $q(x)$ to $u(x,t^n)$. The process of finding $q(x)$ is called reconstruction and the most efficient way is via primitive functions. $q(x)$ should be as simple as possible, say, piecewise polynomial.

Now fix a $j_0 \in \mathbb{Z}^+$ and consider

$$p_{j+\frac{1}{2}} = h\sum_{k=j_0}^{j}\overline{u}_k^n = h\sum_{k=j_0}^{j}\frac{1}{h}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} u(\xi,t^n)d\xi = \int_{x_{j_0-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(\xi,t^n)d\xi. \tag{2.1.2}$$

Hence

$$p_{j+\frac{1}{2}} = p(x_{j+\frac{1}{2}})$$

where

$$p(x) = \int_{x_{j_0-\frac{1}{2}}}^{x} u(\xi,t^n)d\xi$$

is an antiderivative of u, that is, $p'(x) = u(x,t^n)$. Thus, we may first obtain an approximation $P(x)$ to $p(x)$. Then

$$q(x) = P'(x). \tag{2.1.3}$$

$P(x)$ is obtained by interpolation. If we want,say, $q(x)$ to be a polynomial of degree $r-1$ on $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ then $\deg P(x) = r$ and so $r+1$ interpolation points are required. These must be chosen from among $\cdots, x_{j-\frac{5}{2}}, x_{j-\frac{3}{2}}, x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}, x_{j+\frac{3}{2}}, x_{j+\frac{5}{2}}, \cdots$. The schemes are named so, because small oscillations on the scale of the interpolation error are still possible.

For consistency we require

$$p(x_{j+\frac{1}{2}}) - p(x_{j-\frac{1}{2}}) = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(\xi,t^n)d\xi = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} q(x)dx = P(x_{j+\frac{1}{2}}) - P(x_{j-\frac{1}{2}}).$$

This can be achieved if $x_{j-\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$ are among the interpolation points, that is, $P(x_{j\pm\frac{1}{2}}) = p(x_{j\pm\frac{1}{2}})$. The choice of the collection of $r+1$ interpolation points is not unique. For example, if $r = 2$ the choice is $\{x_{j-\frac{3}{2}}, x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\}$ or $\{x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}, x_{j+\frac{3}{2}}\}$. The ENO approach is to choose the collection so that the function p is the smoothest i.e., it has the least oscillation, on the chosen stencil - namely grid points. If you want to approximate the function outside an interval containing a discontinuity then the discontinuous part should not be used to compute the approximation. For complete details and several variations see [4]. One way of computing the smoothest p is to note that

- if u is smooth

$$\begin{aligned}
\Delta U_{j-\frac{1}{2}} &= U_{j+\frac{1}{2}} - U_{j-\frac{1}{2}} = O(h) \\
\Delta^2 U_{j-\frac{1}{2}} &= O(h^2) \\
&\vdots \\
\Delta^r U_{j-\frac{1}{2}} &= O(h^r)
\end{aligned}$$

- if the difference crosses a discontinuity

$$\Delta^k U_j = O(1)$$

Therefore

$$|\Delta^k U_j|_{disc} \gg |\Delta^k U_j|_{smooth}.$$

Hence to determine the smoothest p

Procedure 2.1. *Let $I(j)$ be the starting (left most) point on the stencil for the interpolation on $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$.*

$I(j) = j - \frac{1}{2}$

Do $k = 2, \cdots, r$

if $|\Delta^k p(x_{I(j)}| > |\Delta^k p(x_{I(j)-1}|$

then $I(j) = I(j) - 1$

otherwise

I(j)=I(j)

endif

enddo.

2.2 A Local Extrapolation Method

Lemma 2.1. *The scheme (1.4.1) is accurate of order r if, for all sufficiently smooth solutions w, the numerical flux can be written as*

$$g_{j+\frac{1}{2}}[\overline{w}^n] = \overline{f}^n_{j+\frac{1}{2}} + \alpha(x_{j+\frac{1}{2}}, t^n)h^r + O(h^{r+1}) \tag{2.2.1}$$

where $\alpha(x,t)$ *is Lipschitz continuous for* x *and* t *with*

$$\overline{w}^n_j = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} w(x,t^n)dx \quad \text{and} \quad \overline{f}^n_{j+\frac{1}{2}} = \frac{1}{k}\int_{t^n}^{t^{n+1}} f(w(x_{j+\frac{1}{2}},t))dt.$$

We prove the above Lemma which is stated without proof in [18].

Proof. Let $\lambda = \frac{k}{h}$ and consider the local truncation error, $L(\overline{w}^n_j;\lambda)$, in the conservative scheme,

$$L(\overline{w}^n_j;\lambda) = \overline{w}^{n+1}_j - \overline{w}^n_j + \lambda\left(g_{j+\frac{1}{2}}[\overline{w}^n] - g_{j-\frac{1}{2}}[\overline{w}^n]\right). \tag{2.2.2}$$

On applying the assumption (2.2.1) we get

$$\begin{aligned} L(\overline{w}^n_j;\lambda) &= \overline{w}^{n+1}_j - \overline{w}^n_j + \lambda\left(\overline{f}^n_{j+\frac{1}{2}} - \overline{f}^n_{j-\frac{1}{2}}\right) + \lambda\left[\alpha(x_{j+\frac{1}{2}},t^n) - \alpha(x_{j-\frac{1}{2}},t^n)\right]h^r + O(h^{r+1}) \\ &= \lambda\left[\alpha(x_{j+\frac{1}{2}},t^n) - \alpha(x_{j-\frac{1}{2}},t^n)\right]h^r + O(h^{r+1}), \end{aligned}$$

since $\overline{w}$ satisfies the conservation law (see (1.4.8)). Now

$$\begin{aligned} |L(\overline{w}^n_j;\lambda)| &= \left|\alpha(x_{j+\frac{1}{2}},t^n) - \alpha(x_{j-\frac{1}{2}},t^n)\right| \cdot O(h^r) \\ &\leq K\left|x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}\right| \cdot O(h^r) \qquad \text{since } \alpha \text{ is Lipschitz continuous} \\ &= O(h^{r+1}). \end{aligned} \tag{2.2.3}$$

□

In Yang [18] the following LEM which raises the order of accuracy of the underlying scheme by one is introduced. The only assumption is that for all sufficiently smooth solutions w the numerical flux function of the underlying scheme satisfies

$$g_{j+\frac{1}{2}}[\overline{w}^n] = \overline{f}^n_{j+\frac{1}{2}} + \alpha(x_{j+\frac{1}{2}}, t^n)h^r + \beta(x_{j+\frac{1}{2}}, t^n)h^{r+1} + O(h^{r+2}) \tag{2.2.4}$$

where $g_{j+\frac{1}{2}}[v]$ and α have Lipschitz continuous first derivative and β is Lipschitz continuous. The LEM algorithm extrapolates the numerical fluxes of the underlying scheme between a fine grid with step sizes h and τ and a coarse grid with step sizes $\mathcal{H} = 2h$ and $\mathcal{T} = 2\tau$.

Algorithm 2.1. [18, Yang 2000] (The linear LEM for an r th order underlying scheme)

(i) Set up the initial condition on the fine grid numerically.

For $j = 0, \pm 1, \pm 2, \cdots,$

$U_j^0 = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} w^0(x)dx.$

(ii) For $n = 0, 1, 2, \cdots,$

1. *Determine λ according to the maximum wave speed and the CFL number.*
2. *Compute the numerical solution at $t^n = n\mathcal{T}$ on the coarse grid:*

   For $j = 0, \pm 1, \pm 2, \cdots,$ set

   $V_j^n = (U_{2j-1}^n + U_{2j}^n)/2$ *and* $W_j^n = (U_{2j}^n + U_{2j+1}^n)/2.$
3. *Advance one step on the fine grid with the underlying scheme:*

   For $j = 0, \pm 1, \pm 2, \cdots,$

   $U_j^{n+\frac{1}{2}} = U_j^n - \lambda(g_{j+\frac{1}{2}}[U^n] - g_{j-\frac{1}{2}}[U^n]),$

   Save $g_{j+\frac{1}{2}}[U^n]$ as well as $U_j^{n+\frac{1}{2}}$
4. *For $j = 0, \pm 1, \pm 2, \cdots,$ evaluate $g_{j+\frac{1}{2}}[U^{n+\frac{1}{2}}]$*
5. *Evaluate the numerical flux on the coarse grid:*

(a) For $j = 2i+1, i = 0, \pm 1, \pm 2, \cdots,$ *evaluate* $G^n_{j+\frac{1}{2}} = g_{j+\frac{1}{2}}[W^n]$.

(b) For $j = 2i, i = 0, \pm 1, \pm 2, \cdots,$ *evaluate* $G^n_{j+\frac{1}{2}} = g_{j+\frac{1}{2}}[V^n]$.

6. *Evaluate the flux increment for local extrapolation:*

   For $j = 0, \pm 1, \pm 2, \cdots,$

$$\tilde{g}_{j+\frac{1}{2}} = \left\{ \frac{1}{2} \left(g_{j+\frac{1}{2}}[U^n] + g_{j+\frac{1}{2}}[U^{n+\frac{1}{2}}] \right) - G^n_{j+\frac{1}{2}} \right\} \frac{1}{(2^r - 1)} \tag{2.2.5}$$

7. *Completion of one time step of the algorithm.*

   For $j = 0, \pm 1, \pm 2, \cdots,$

$$U_j^{n+1} = U_j^{n+\frac{1}{2}} - \lambda \left[\left(g_{j+\frac{1}{2}}[U^{n+\frac{1}{2}}] + 2\tilde{g}_{j+\frac{1}{2}} \right) - \left(g_{j-\frac{1}{2}}[U^{n+\frac{1}{2}}] + 2\tilde{g}_{j-\frac{1}{2}} \right) \right].$$

Definition 2.1. *The minmod function is defined to be*

$$minmod(x_1, x_2) = \frac{sign(x_1) + sign(x_2)}{2} \min(|x_1|, |x_2|) \tag{2.2.6}$$

and in general

$$minmod(x_1, x_2, \cdots, x_k) = minmod(minmod(x_1, x_2, \cdots, x_{k-1}), x_k). \tag{2.2.7}$$

To turn off spurious oscillations, the minmod function is used. The following modifications are made to Algorithm 2.1

Algorithm 2.2. [18, Yang 2000] (Componentwise LEM)

(i) The same as the step (i) in Algorithm 2.1.

(ii) For $n = 0, 1, 2, \cdots,$ *the steps 1-6 are the same as the corresponding steps in Algorithm 2.1.*

7. *For* $j = 0, \pm 1, \pm 2, \cdots,$

$$g^{n-ext}_{j+\frac{1}{2}} = minmod\left(\beta\tilde{g}_{j+\frac{3}{2}}, \tilde{g}_{j+\frac{1}{2}}, \beta\tilde{g}_{j-\frac{1}{2}} \right) \tag{2.2.8}$$

8. *For* $j = 0, \pm 1, \pm 2, \cdots$,

$$U_j^{n+1} = U_j^{n+\frac{1}{2}} - \lambda\left[\left(g_{j+\frac{1}{2}}[U^{n+\frac{1}{2}}] + 2g_{j+\frac{1}{2}}^{n-ext}\right) - \left(g_{j-\frac{1}{2}}[U^{n+\frac{1}{2}}] + 2g_{j-\frac{1}{2}}^{n-ext}\right)\right]. \qquad (2.2.9)$$

Theorem 2.1. *Consider the linear advection equation*

$$u_t + au_x = 0, \qquad a = const.$$

If the underlying scheme is the upwinding scheme, (1.2.3) and $\beta \le 2$*, then Algorithm 2.2 is TVD for* $0 < \lambda a \le 1$.

Proof. Note that $g_{j+\frac{1}{2}}[u] = au_j$ and the order of underlying scheme is $r = 1$. So

$$\begin{aligned}
\tilde{g}_{j+\frac{1}{2}}^n &= \frac{aU_j^n + aU_j^{n+\frac{1}{2}}}{2} - \frac{aU_j^n + aU_{j-1}^n}{2} \\
&= \frac{a}{2}\{U_j^n + U_j^n - \lambda[aU_j^n - aU_{j-1}^n]\} - \frac{a}{2}[U_j^n + U_{j-1}^n], \qquad \text{on using (1.2.3)} \\
&= \frac{a}{2}\{(2-\lambda a)U_j^n + \lambda a U_{j-1}^n\} - \frac{a}{2}[U_j^n + U_{j-1}^n]
\end{aligned}$$

that is

$$\begin{aligned}
\tilde{g}_{j+\frac{1}{2}}^n &= \frac{a}{2}(1-\lambda a)U_j^n + \frac{a}{2}(\lambda a - 1)U_{j-1}^n \\
&= \frac{a}{2}(1-\lambda a)[U_j^n - U_{j-1}^n].
\end{aligned}$$

Letting $\Delta U_{j-\frac{1}{2}}^n = U_j^n - U_{j-1}^n$ we get

$$\tilde{g}_{j+\frac{1}{2}}^n = \frac{a}{2}(1-\lambda a)\Delta U_{j-\frac{1}{2}}. \qquad (2.2.10)$$

The scheme becomes

$$\begin{aligned}
U_j^{n+1} &= U_j^n - \lambda[aU_j^n - aU_{j-1}^n] \\
&\quad -\lambda\left\{\left(aU_j^n - \lambda a[aU_j^n - aU_{j-1}^n] + 2g_{j+\frac{1}{2}}^{n-ext}\right)\right. \\
&\qquad \left. -\left(aU_{j-1}^n - \lambda a[aU_{j-1}^n - aU_{j-2}^n] + 2g_{j-\frac{1}{2}}^{n-ext}\right)\right\} \\
&= \{1 - \lambda a - \lambda a + (\lambda a)^2\}U_j^n + \{\lambda a - (\lambda a)^2 + \lambda a - (\lambda a)^2\}U_{j-1}^n \\
&\quad +(\lambda a)^2 U_{j-2}^n - 2\lambda\left(g_{j+\frac{1}{2}}^{n-ext} - g_{j-\frac{1}{2}}^{n-ext}\right) \\
&= (1-\lambda a)^2 U_j^n + 2\lambda a(1-\lambda a)U_{j-1}^n + (\lambda a)^2 U_{j-2}^n - 2\lambda\left(g_{j+\frac{1}{2}}^{n-ext} - g_{j-\frac{1}{2}}^{n-ext}\right).
\end{aligned}$$

Hence

$$\begin{aligned} U^{n+1}_{j+1} - U^{n+1}_{j} &= (1-\lambda a)^2[U^{n+1}_{j+1} - U^{n}_{j}] + 2\lambda a(1-\lambda a)[U^{n+1}_{j+1} - U^{n}_{j-1}] + (\lambda a)^2[U^{n+1}_{j+1} - U^{n}_{j-2}] \\ &= -2\lambda\left(g^{n-ext}_{j+\frac{3}{2}} - 2g^{n-ext}_{j+\frac{1}{2}} + g^{n-ext}_{j-\frac{1}{2}}\right) \end{aligned}$$

which can be written in the more compact form

$$\begin{aligned} \Delta U^{n+1}_{j+\frac{1}{2}} &= (1-\lambda a)^2\Delta U^{n}_{j+\frac{1}{2}} + 2\lambda a(1-\lambda a)\Delta U^{n}_{j-\frac{1}{2}} + (\lambda a)^2\Delta U^{n}_{j-\frac{3}{2}} \\ &\quad -2\lambda\left(g^{n-ext}_{j+\frac{3}{2}} - 2g^{n-ext}_{j+\frac{1}{2}} + g^{n-ext}_{j-\frac{1}{2}}\right). \end{aligned} \tag{2.2.11}$$

Assume that $\lambda a \leq \frac{1}{2}$.

Now by definition and using (2.2.10)

$$\begin{aligned} g^{n-ext}_{j-\frac{1}{2}} &= minmod\left\{\beta\tilde{g}^{n}_{j+\frac{1}{2}},\ \tilde{g}^{n}_{j-\frac{1}{2}},\ \beta\tilde{g}^{n}_{j-\frac{3}{2}}\right\} \\ &= minmod\left\{\beta\frac{a}{2}(1-\lambda a)\Delta U^{n}_{j-\frac{1}{2}},\ \frac{a}{2}(1-\lambda a)\Delta U^{n}_{j-\frac{3}{2}},\ \beta\frac{a}{2}(1-\lambda a)\Delta U^{n}_{j-\frac{5}{2}}\right\}. \end{aligned} \tag{2.2.12}$$

Using the definition of minmod function and (2.2.12), we observe that

$$\left(2\lambda g^{n-ext}_{j-\frac{1}{2}}\right)\left(2\lambda a(1-\lambda a)\Delta U^{n}_{j-\frac{1}{2}}\right) \geq 0, \tag{2.2.13}$$

that is, $g^{n-ext}_{j-\frac{1}{2}}$ and the second term on right of (2.2.11) have the same sign. We also have that

$$2\lambda\left|g^{n-ext}_{j-\frac{1}{2}}\right| \leq 2\lambda\beta\frac{a}{2}(1-\lambda a)\left|\Delta U_{j-\frac{1}{2}}\right| \leq 2\lambda a(1-\lambda a)\left|\Delta U_{j-\frac{1}{2}}\right| \quad \text{since } \beta \leq 2. \tag{2.2.14}$$

Similarly

$$\begin{aligned} g^{n-ext}_{j+\frac{3}{2}} &= minmod\left\{\beta\tilde{g}^{n}_{j+\frac{5}{2}},\ \tilde{g}^{n}_{j+\frac{3}{2}},\ \beta\tilde{g}^{n}_{j+\frac{1}{2}}\right\} \\ &= minmod\left\{\beta\frac{a}{2}(1-\lambda a)\Delta U^{n}_{j+\frac{3}{2}},\ \frac{a}{2}(1-\lambda a)\Delta U^{n}_{j+\frac{1}{2}},\ \beta\frac{a}{2}(1-\lambda a)\Delta U^{n}_{j-\frac{1}{2}}\right\} \end{aligned} \tag{2.2.15}$$

so that

$$\left(2\lambda g^{n-ext}_{j+\frac{3}{2}}\right)\left(2\lambda(1-\lambda a)^2\Delta U^{n}_{j+\frac{1}{2}}\right) \geq 0, \tag{2.2.16}$$

and

$$2\lambda\left|g^{n-ext}_{j+\frac{3}{2}}\right| \leq 2\lambda\frac{a}{2}(1-\lambda a)\left|\Delta U_{j+\frac{1}{2}}\right| \leq (1-\lambda a)^2\left|\Delta U_{j+\frac{1}{2}}\right| \qquad \text{since } \lambda a \leq \frac{1}{2}. \tag{2.2.17}$$

Lastly

$$\begin{aligned} g^{n-ext}_{j+\frac{1}{2}} &= minmod\left\{\beta\tilde{g}^n_{j+\frac{3}{2}},\ \tilde{g}^n_{j+\frac{1}{2}},\ \beta\tilde{g}^n_{j-\frac{1}{2}}\right\} \\ &= minmod\left\{\beta\frac{a}{2}(1-\lambda a)\Delta U^n_{j+\frac{1}{2}},\ \frac{a}{2}(1-\lambda a)\Delta U^n_{j-\frac{1}{2}},\ \beta\frac{a}{2}(1-\lambda a)\Delta U^n_{j-\frac{3}{2}}\right\}. \end{aligned} \tag{2.2.18}$$

This implies that either $4\lambda g^{n-ext}_{j+\frac{1}{2}} = 0$ or the first three terms on the right of (2.2.11) and $4\lambda g^{n-ext}_{j+\frac{1}{2}}$ have the same sign.

In view of the above observations, we rewrite (2.2.11) as

$$\begin{aligned} \Delta U^{n+1}_{j+\frac{1}{2}} &= \left[(1-\lambda a)^2\Delta U^n_{j+\frac{1}{2}} - 2\lambda g^{n-ext}_{j+\frac{3}{2}}\right] + \left[2\lambda a(1-\lambda a)\Delta U^n_{j-\frac{1}{2}} - 2\lambda g^{n-ext}_{j-\frac{1}{2}}\right] \\ &\quad + \left[(\lambda a)^2\Delta U^n_{j-\frac{3}{2}} + 4\lambda g^{n-ext}_{j+\frac{1}{2}}\right] \end{aligned}$$

and on taking absolute values yields

$$\begin{aligned} \left|\Delta U^{n+1}_{j+\frac{1}{2}}\right| &\leq \left|(1-\lambda a)^2\Delta U^n_{j+\frac{1}{2}} - 2\lambda g^{n-ext}_{j+\frac{3}{2}}\right| + \left|2\lambda a(1-\lambda a)\Delta U^n_{j-\frac{1}{2}} - 2\lambda g^{n-ext}_{j-\frac{1}{2}}\right| \\ &\quad + \left|(\lambda a)^2\Delta U^n_{j-\frac{3}{2}} + 4\lambda g^{n-ext}_{j+\frac{1}{2}}\right| \\ &\qquad\qquad \text{and since the paired terms have the same sign} \\ &= (1-\lambda a)^2\left|\Delta U^n_{j+\frac{1}{2}}\right| - 2\lambda\left|g^{n-ext}_{j+\frac{3}{2}}\right| + 2\lambda a(1-\lambda a)\left|\Delta U^n_{j-\frac{1}{2}}\right| - 2\lambda\left|g^{n-ext}_{j-\frac{1}{2}}\right| \\ &\quad + (\lambda a)^2\left|\Delta U^n_{j-\frac{3}{2}}\right| + 4\lambda\left|g^{n-ext}_{j+\frac{1}{2}}\right|. \end{aligned}$$

We rearrange the above to get

$$\begin{aligned} \left|\Delta U^{n+1}_{j+\frac{1}{2}}\right| &\leq (1-\lambda a)^2\left|\Delta U^n_{j+\frac{1}{2}}\right| + 2\lambda a(1-\lambda a)\left|\Delta U^n_{j-\frac{1}{2}}\right| + (\lambda a)^2\left|\Delta U^n_{j-\frac{3}{2}}\right| \\ &\quad - 2\lambda\left(\left|g^{n-ext}_{j+\frac{3}{2}}\right| - 2\left|g^{n-ext}_{j+\frac{1}{2}}\right| + \left|g^{n-ext}_{j-\frac{1}{2}}\right|\right). \end{aligned} \tag{2.2.19}$$

We now assume that $\frac{1}{2} \leq \lambda a \leq 1$.

By definition of minmod function and (2.2.15) we have

$$\left(2\lambda g^{n-ext}_{j+\frac{3}{2}}\right)\left(2\lambda a(1-\lambda a)\Delta U^n_{j-\frac{1}{2}}\right) \geq 0, \tag{2.2.20}$$

and

$$2\lambda \left|g^{n-ext}_{j+\frac{3}{2}}\right| \leq 2\lambda\beta\frac{a}{2}(1-\lambda a)\left|\Delta U_{j-\frac{1}{2}}\right| \leq 2\lambda a(1-\lambda a)\left|\Delta U_{j-\frac{1}{2}}\right| \quad \text{since } \beta \leq 2. \tag{2.2.21}$$

Similarly

$$\left(2\lambda g^{n-ext}_{j-\frac{1}{2}}\right)\left(2\lambda(\lambda a)^2\Delta U^n_{j-\frac{3}{2}}\right) \geq 0, \tag{2.2.22}$$

and

$$2\lambda \left|g^{n-ext}_{j-\frac{1}{2}}\right| \leq 2\lambda\frac{a}{2}(1-\lambda a)\left|\Delta U_{j-\frac{3}{2}}\right| \leq (\lambda a)^2\left|\Delta U_{j-\frac{3}{2}}\right| \qquad \text{since } \frac{1}{2} \leq \lambda a \Rightarrow 1-\lambda a \leq \lambda a. \tag{2.2.23}$$

Finally, as in the previous case, either $4\lambda g^{n-ext}_{j+\frac{1}{2}} = 0$ or the first three terms on the right of (2.2.11) and $4\lambda g^{n-ext}_{j+\frac{1}{2}}$ have the same sign.

So we write

$$\begin{aligned}\Delta U^{n+1}_{j+\frac{1}{2}} &= \left[(1-\lambda a)^2\Delta U^n_{j+\frac{1}{2}} + 4\lambda g^{n-ext}_{j+\frac{1}{2}}\right] + \left[2\lambda a(1-\lambda a)\Delta U^n_{j-\frac{1}{2}} - 2\lambda g^{n-ext}_{j+\frac{3}{2}}\right] \\ &\quad + \left[(\lambda a)^2\Delta U^n_{j-\frac{3}{2}} - 2\lambda g^{n-ext}_{j-\frac{1}{2}}\right]\end{aligned}$$

and on taking absolute values

$$\begin{aligned}\left|\Delta U^{n+1}_{j+\frac{1}{2}}\right| &\leq \left|(1-\lambda a)^2\Delta U^n_{j+\frac{1}{2}} + 4\lambda g^{n-ext}_{j+\frac{1}{2}}\right| + \left|2\lambda a(1-\lambda a)\Delta U^n_{j-\frac{1}{2}} - 2\lambda g^{n-ext}_{j+\frac{3}{2}}\right| \\ &\quad + \left|(\lambda a)^2\Delta U^n_{j-\frac{3}{2}} - 2\lambda g^{n-ext}_{j-\frac{1}{2}}\right| \\ &\qquad\qquad \text{and since the paired terms have the same sign} \\ &= (1-\lambda a)^2\left|\Delta U^n_{j+\frac{1}{2}}\right| + 4\lambda\left|g^{n-ext}_{j+\frac{1}{2}}\right| + 2\lambda a(1-\lambda a)\left|\Delta U^n_{j-\frac{1}{2}}\right| - 2\lambda\left|g^{n-ext}_{j+\frac{3}{2}}\right| \\ &\quad + (\lambda a)^2\left|\Delta U^n_{j-\frac{3}{2}}\right| - 2\lambda\left|g^{n-ext}_{j-\frac{1}{2}}\right|.\end{aligned}$$

which is just inequality (2.2.19).

Hence (2.2.19) holds for $0 < \lambda a \leq 1$. Now summing (2.2.19) over all j

$$\begin{aligned}\sum_j\left|\Delta U^{n+1}_{j+\frac{1}{2}}\right| &\leq \sum_j(1-\lambda a)^2\left|\Delta U^n_{j+\frac{1}{2}}\right| + \sum_j 2\lambda a(1-\lambda a)\left|\Delta U^n_{j-\frac{1}{2}}\right| + \sum_j(\lambda a)^2\left|\Delta U^n_{j-\frac{3}{2}}\right| \\ &\quad -2\lambda\left(\sum_j\left|g^{n-ext}_{j+\frac{3}{2}}\right| - 2\sum_j\left|g^{n-ext}_{j+\frac{1}{2}}\right| + \sum_j\left|g^{n-ext}_{j-\frac{1}{2}}\right|\right)\end{aligned} \tag{2.2.24}$$

that is,

$$\begin{aligned}\sum_j \left|\Delta U^{n+1}_{j+\frac{1}{2}}\right| \;\le\; & \sum_j \left|\Delta U^n_{j+\frac{1}{2}}\right| + \sum_j(-2\lambda a + a^2\lambda^2)\left|\Delta U^n_{j+\frac{1}{2}}\right| + \sum_j 2\lambda a(1-\lambda a)\left|\Delta U^n_{j-\frac{1}{2}}\right| \\ & + \sum_j (\lambda a)^2\left|\Delta U^n_{j-\frac{3}{2}}\right| - 2\lambda\left(\sum_j\left|g^{n-ext}_{j+\frac{3}{2}}\right| - 2\sum_j\left|g^{n-ext}_{j+\frac{1}{2}}\right| + \sum_j\left|g^{n-ext}_{j-\frac{1}{2}}\right|\right).\end{aligned} \tag{2.2.25}$$

Since $\Delta U^n_{j+\frac{1}{2}} = \Delta U^n_{j-\frac{1}{2}} = \Delta U^n_{j-\frac{3}{2}} = 0$ for sufficiently large j and in turn $g^{n-ext}_{j+\frac{3}{2}} = g^{n-ext}_{j+\frac{1}{2}} = g^{n-ext}_{j-\frac{1}{2}} = 0$, with the exception of the first term on the right of (2.2.25), the rest of the terms cancel out to yield

$$\sum_j \left|\Delta U^{n+1}_{j+\frac{1}{2}}\right| \le \sum_j \left|\Delta U^n_{j+\frac{1}{2}}\right|.$$

□

Theorem 2.2. *Consider the conservation law (1.1.1) under the assumption that $f' > 0$ and $f'' > 0$. If the underlying scheme is the upwinding scheme, (1.2.3), $0 < \mu \le \frac{1}{2}$ and $\beta \le 2\left(\frac{1-\mu}{2-\mu}\right)$, then Algorithm 2.2 is TVD for $0 < \lambda\|f'\|_\infty \le \mu$.*

Proof. Note that $g_{j+\frac{1}{2}}[u] = f(u_j)$ and $r = 1$. Since f is smooth, use the mean value theorem to write

$$\begin{aligned} f(U^n_j) - f(U^n_{j-1}) &= f'(\xi^n_j)[U^n_j - U^n_{j-1}] \\ &= f'(\xi^n_j)\Delta U^n_{j-\frac{1}{2}} \end{aligned} \tag{2.2.26}$$

for some ξ^n_j between U^n_{j-1} and U^n_j (see Figure 2.2). So

$$\begin{aligned} f(U^n_j - \lambda[f(U^n_j) - f(U^n_{j-1})]) &= f(U^n_j - \lambda f'(\xi^n_j)\Delta U^n_{j-\frac{1}{2}}) \\ &= f(U^n_j) - \lambda f'(\alpha^n_j)f'(\xi^n_j)\Delta U^n_{j-\frac{1}{2}} \end{aligned} \tag{2.2.27}$$

for some α^n_j between $U^n_j - \lambda f'(\xi^n_j)\Delta U^n_{j-\frac{1}{2}}$ and U^n_j, on using Taylors' theorem. Similarly

$$\begin{aligned} f(\frac{1}{2}[U^n_{j-1} + U^n_j]) &= f(U^n_j - \frac{1}{2}[U^n_j - U^n_{j-1}]) \\ &= f(U^n_j) - \frac{1}{2}f'(\eta^n_j)\Delta U^n_{j-\frac{1}{2}} \end{aligned} \tag{2.2.28}$$

for some η_j^n between $\frac{1}{2}[U_{j-1}^n + U_j^n]$ and U_j^n.

But note that $f'(\xi_j^n)$ is the slope of the line segment joining the points $(U_{j-1}^n, f(U_{j-1}^n))$ and $(U_j^n, f(U_j^n))$ where as $f'(\eta_j^n)$ is the slope of the line segment joining the points $(\frac{1}{2}[U_{j-1}^n + U_j^n], f(\frac{1}{2}[U_{j-1}^n + U_j^n]))$ and $(U_j^n, f(U_j^n))$. Convexity of f yields, (see Figure 2.1)

$$f'(\xi_j^n) < f'(\eta_j^n). \tag{2.2.29}$$

Lemma B.2 gives us that

$$f'(\eta_j^n) < 2f'(\xi_j^n). \tag{2.2.30}$$

Let γ_j^n be a point between U_{j-1}^n and $\frac{1}{2}[U_{j-1}^n + U_j^n]$ such that $f'(\gamma_j^n)$ is the slope of the line segment joining the points $(U_{j-1}^n, f(U_{j-1}^n))$ and $(\frac{1}{2}[U_{j-1}^n + U_j^n], f(\frac{1}{2}[U_{j-1}^n + U_j^n]))$ (see Figure 2.2). Then by Lemma B.3

$$f'(\xi_j^n) = \frac{f'(\gamma_j^n) + f'(\eta_j^n)}{2}. \tag{2.2.31}$$

The underlying scheme is given by

$$\begin{aligned} U_j^{n+\frac{1}{2}} &= U_j^n - \lambda[f(U_j^n) - f(U_{j-1}^n)] \\ &= U_j^n - \lambda f'(\xi_j^n)\Delta U_{j-\frac{1}{2}}^n. \end{aligned} \tag{2.2.32}$$

So the algorithm becomes

$$\begin{aligned} U_j^{n+1} &= U_j^{n+\frac{1}{2}} - \lambda\Big\{\Big(g_{j+\frac{1}{2}}[U^{n+\frac{1}{2}}] + 2g_{j+\frac{1}{2}}^{n-ext}\Big) - \Big(g_{j-\frac{1}{2}}[U^{n+\frac{1}{2}}] + 2g_{j-\frac{1}{2}}^{n-ext}\Big)\Big\} \\ &= U_j^n - \lambda f'(\xi_j^n)\Delta U_{j-\frac{1}{2}}^n - \lambda\Big\{f(U_j^n - \lambda f'(\xi_j^n)\Delta U_{j-\frac{1}{2}}^n) + 2g_{j+\frac{1}{2}}^{n-ext} \\ &\quad - \Big(f(U_{j-1}^n - \lambda f'(\xi_{j-1}^n)\Delta U_{j-\frac{3}{2}}^n) + 2g_{j-\frac{1}{2}}^{n-ext}\Big)\Big\} \end{aligned} \tag{2.2.33}$$

that is,

$$\begin{aligned} U_j^{n+1} &= U_j^n - \lambda f'(\xi_j^n)\Delta U_{j-\frac{1}{2}}^n - \lambda\Big\{f(U_j^n) - \lambda f'(\alpha_j^n)f'(\xi_j^n)\Delta U_{j-\frac{1}{2}}^n + 2g_{j+\frac{1}{2}}^{n-ext} \\ &\quad - \Big(f(U_{j-1}^n) - \lambda f'(\alpha_{j-1}^n)f'(\xi_{j-1}^n)\Delta U_{j-\frac{3}{2}}^n + 2g_{j-\frac{1}{2}}^{n-ext}\Big)\Big\} \end{aligned} \tag{2.2.34}$$

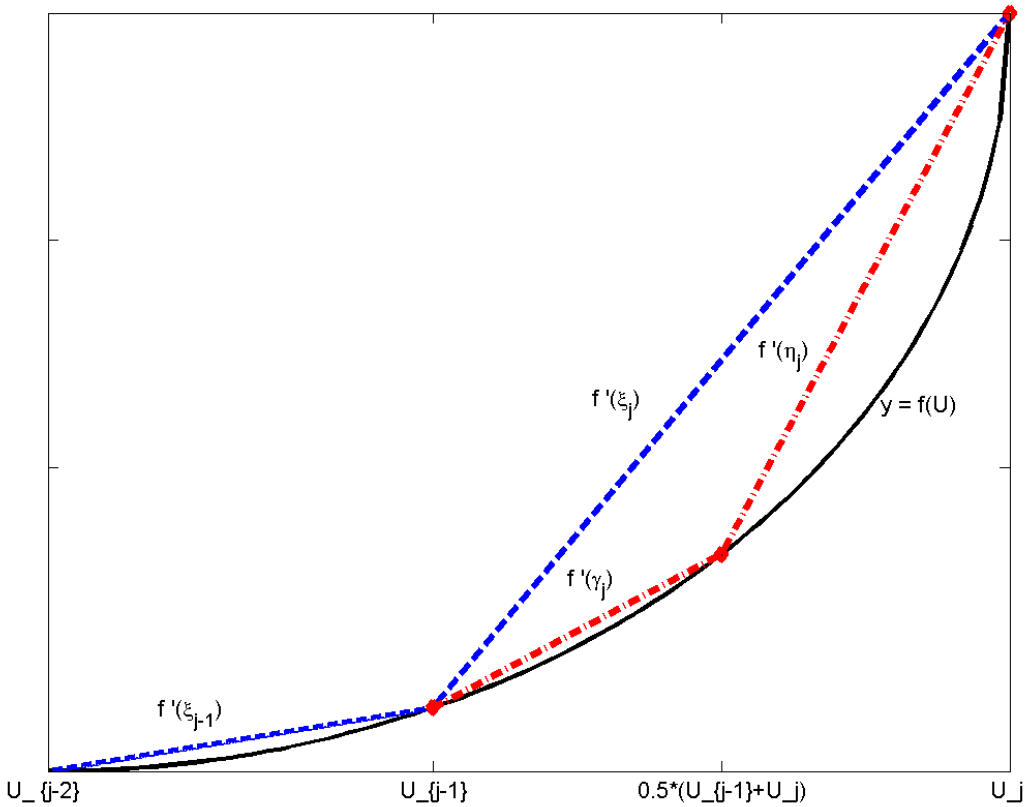

Figure 2.1: *Convexity of the flux function f and slopes of secant lines*

which simplifies to

$$\begin{aligned} U_j^{n+1} &= U_j^n - 2\lambda f'(\xi_j^n)\Delta U_{j-\frac{1}{2}}^n + \lambda^2 f'(\alpha_j^n) f'(\xi_j^n)\Delta U_{j-\frac{1}{2}}^n \\ &\quad -\lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)\Delta U_{j-\frac{3}{2}}^n - 2\lambda\left(g_{j+\frac{1}{2}}^{n-ext} - g_{j-\frac{1}{2}}^{n-ext}\right). \end{aligned} \tag{2.2.35}$$

Similarly

$$\begin{aligned} U_{j+1}^{n+1} &= U_{j+1}^n - 2\lambda f'(\xi_{j+1}^n)\Delta U_{j+\frac{1}{2}}^n + \lambda^2 f'(\alpha_{j+1}^n) f'(\xi_{j+1}^n)\Delta U_{j+\frac{1}{2}}^n \\ &\quad -\lambda^2 f'(\alpha_j^n) f'(\xi_j^n)\Delta U_{j-\frac{1}{2}}^n - 2\lambda\left(g_{j+\frac{3}{2}}^{n-ext} - g_{j+\frac{1}{2}}^{n-ext}\right). \end{aligned} \tag{2.2.36}$$

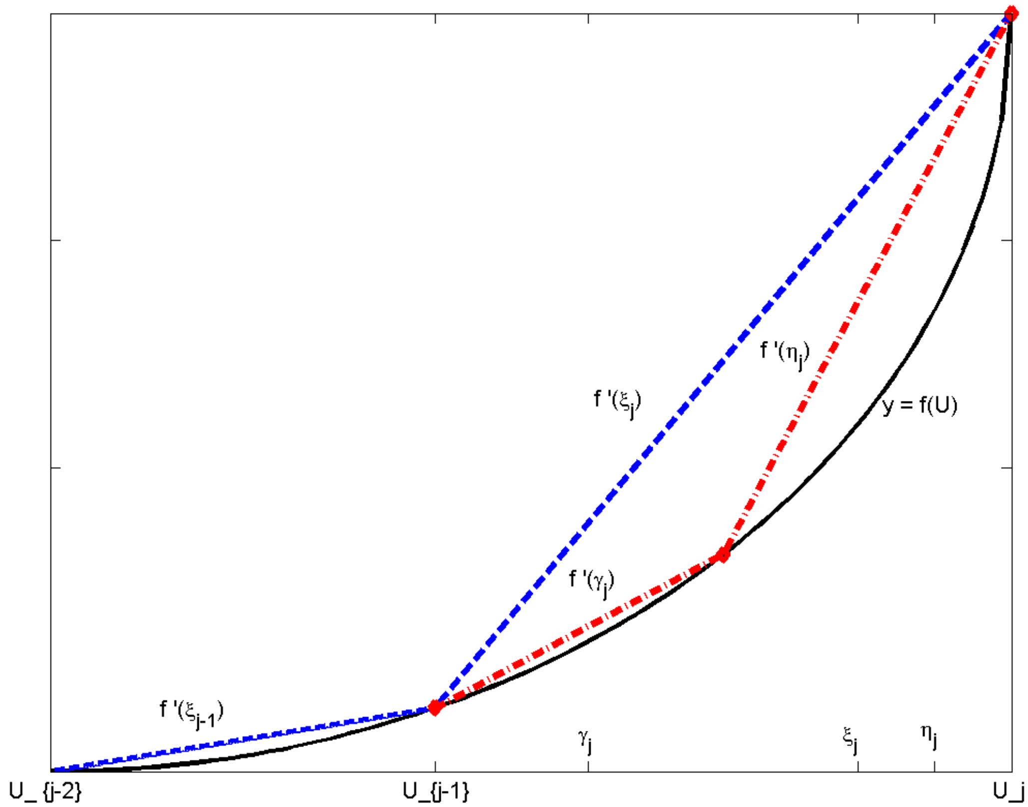

Figure 2.2: *Convexity of the flux function f and the mean value theorem*

Hence subtracting (2.2.35) from (2.2.36) yields

$$\begin{aligned}\Delta U^{n+1}_{j+\frac{1}{2}} &= \{1-2\lambda f'(\xi^n_{j+1})+\lambda^2 f'(\alpha^n_{j+1})f'(\xi^n_{j+1})\}\Delta U^n_{j+\frac{1}{2}}\\ &\quad +2\lambda f'(\xi^n_j)\{1-\lambda f'(\alpha^n_j)\}\Delta U^n_{j-\frac{1}{2}}+\lambda^2 f'(\alpha^n_{j-1})f'(\xi^n_{j-1})\Delta U^n_{j-\frac{3}{2}}\\ &\quad -2\lambda\left(g^{n-ext}_{j+\frac{3}{2}}-2g^{n-ext}_{j+\frac{1}{2}}+g^{n-ext}_{j-\frac{1}{2}}\right). \qquad (2.2.37)\end{aligned}$$

To show the TVD property we need to take absolute values in (2.2.37). At this point we do not know the signs of the terms in the equation neither do we know how they compare to each other in magnitude. We rewrite (2.2.37) as

$$\begin{aligned}\Delta U^{n+1}_{j+\frac{1}{2}} &= \Big[\{1-2\lambda f'(\xi^n_{j+1})+\lambda^2 f'(\alpha^n_{j+1})f'(\xi^n_{j+1})\}\Delta U^n_{j+\frac{1}{2}} - 2\lambda g^{n-ext}_{j+\frac{3}{2}}\Big]\\ &\quad+\Big[2\lambda f'(\xi^n_j)\{1-\lambda f'(\alpha^n_j)\}\Delta U^n_{j-\frac{1}{2}} - 2\lambda g^{n-ext}_{j-\frac{1}{2}}\Big]\\ &\quad+\Big[\lambda^2 f'(\alpha^n_{j-1})f'(\xi^n_{j-1})\Delta U^n_{j-\frac{3}{2}} + 4\lambda g^{n-ext}_{j+\frac{1}{2}}\Big].\end{aligned}\tag{2.2.38}$$

With this grouping, if we can show that the three groups have the same sign and for each group the g^{n-ext} has the same sign as the rest of the terms and less than the terms in magnitude, then we can take absolute values simplify to get an inequality.

We now determine an expression for $\tilde{g}^n_{j+\frac{1}{2}}$ by substituting (2.2.27) and (2.2.28) into (2.2.5).

$$\begin{aligned}\tilde{g}^n_{j+\frac{1}{2}} &= \frac{1}{2}\Big\{f(U^n_j)+f(U^{n+\frac{1}{2}}_j)\Big\} - G^n_{j+\frac{1}{2}}\\ &= \Big\{f(U^n_j)-\frac{1}{2}\lambda f'(\alpha^n_j)f'(\xi^n_j)\Delta U^n_{j-\frac{1}{2}}\Big\} - f(\frac{1}{2}[U^n_j+U^n_{j-1}])\\ &= \Big\{f(U^n_j)-\frac{1}{2}\lambda f'(\alpha^n_j)f'(\xi^n_j)\Delta U^n_{j-\frac{1}{2}}\Big\} - \Big\{f(U^n_j)-\frac{1}{2}f'(\eta^n_j)\Delta U^n_{j-\frac{1}{2}}\Big\}.\end{aligned}$$

Hence

$$\tilde{g}^n_{j+\frac{1}{2}} = \frac{1}{2}\{f'(\eta^n_j)-\lambda f'(\alpha^n_j)f'(\xi^n_j)\}\Delta U^n_{j-\frac{1}{2}}.\tag{2.2.39}$$

So by definition and (2.2.39)

$$\begin{aligned}g^{n-ext}_{j-\frac{1}{2}} &= minmod\Big\{\beta\tilde{g}^n_{j+\frac{1}{2}},\ \tilde{g}^n_{j-\frac{1}{2}},\ \beta\tilde{g}^n_{j-\frac{3}{2}}\Big\}\\ &= minmod\Big\{\beta\frac{1}{2}\{f'(\eta^n_j)-\lambda f'(\alpha^n_j)f'(\xi^n_j)\}\Delta U^n_{j-\frac{1}{2}},\\ &\qquad\frac{1}{2}\{f'(\eta^n_{j-1})-\lambda f'(\alpha^n_{j-1})f'(\xi^n_{j-1})\}\Delta U^n_{j-\frac{3}{2}},\\ &\qquad\beta\frac{1}{2}\{f'(\eta^n_{j-2})-\lambda f'(\alpha^n_{j-2})f'(\xi^n_{j-2})\}\Delta U^n_{j-\frac{5}{2}}\Big\}.\end{aligned}\tag{2.2.40}$$

Examining terms of (2.2.38), first we show that $g^{n-ext}_{j-\frac{1}{2}}$ has the same sign as the term that has $\Delta U^n_{j-\frac{1}{2}}$ and it is less than the same in magnitude. Now for all j

$$\begin{aligned}f'(\eta^n_j)-\lambda f'(\alpha^n_j)f'(\xi^n_j) &> f'(\xi^n_j)-\lambda f'(\alpha^n_j)f'(\xi^n_j) \quad \text{since } 0<f'(\xi^n_j)<f'(\eta^n_j)\\ &= f'(\xi^n_j)\{1-\lambda f'(\alpha^n_j)\}>0,\end{aligned}\tag{2.2.41}$$

since by assumption, we have the CFL condition $0 \le \lambda \|f'\|_\infty \le \frac{1}{2}$.

The definition of $g^{n-ext}_{j-\frac{1}{2}}$ includes $\Delta U^n_{j-\frac{1}{2}}$ whose coefficient is positive from (2.2.41). So $g^{n-ext}_{j-\frac{1}{2}}$ must have the same sign as $\Delta U^n_{j-\frac{1}{2}}$ or is zero and thus

$$\left(2\lambda g^{n-ext}_{j-\frac{1}{2}}\right)\left(2\lambda f'(\xi^n_j)\{1-\lambda f'(\alpha^n_j)\}\Delta U^n_{j-\frac{1}{2}}\right) \ge 0 \tag{2.2.42}$$

and

$$2\lambda \left|g^{n-ext}_{j-\frac{1}{2}}\right| \;\le\; 2\lambda\beta\frac{1}{2}\{f'(\eta^n_j)-\lambda f'(\alpha^n_j)f'(\xi^n_j)\}\left|\Delta U^n_{j-\frac{1}{2}}\right|. \tag{2.2.43}$$

We want

$$2\lambda \left|g^{n-ext}_{j-\frac{1}{2}}\right| \le 2\lambda f'(\xi^n_j)\{1-\lambda f'(\alpha^n_j)\}\left|\Delta U^n_{j-\frac{1}{2}}\right|. \tag{2.2.44}$$

(2.2.44) is true if

$$\beta\frac{1}{2}\{f'(\eta^n_j)-\lambda f'(\alpha^n_j)f'(\xi^n_j)\} \le f'(\xi^n_j)\{1-\lambda f'(\alpha^n_j)\}, \tag{2.2.45}$$

that is, if

$$\left(1-\frac{\beta}{2}\right)\lambda f'(\alpha^n_j)f'(\xi^n_j) \le f'(\xi^n_j) - \frac{\beta}{2}f'(\eta^n_j). \tag{2.2.46}$$

But (2.2.46) holds if

$$\left(1-\frac{\beta}{2}\right)\mu f'(\xi^n_j) \le f'(\xi^n_j) - \frac{\beta}{2}f'(\eta^n_j) \qquad \text{since } 0 < \lambda\|f'\|_\infty \le \mu. \tag{2.2.47}$$

Substituting (2.2.31) into the above and simplifying we get

$$\frac{\mu}{2}\left(1-\frac{\beta}{2}\right)(f'(\gamma^n_j)+f'(\eta^n_j)) \le \frac{1}{2}f'(\gamma^n_j) + \left(\frac{1}{2}-\frac{\beta}{2}\right)f'(\eta^n_j). \tag{2.2.48}$$

Now (2.2.48) is true if

$$\frac{\mu}{2}\left(1-\frac{\beta}{2}\right) \le \frac{1}{2}-\frac{\beta}{2} \qquad \text{and} \qquad \frac{\mu}{2}\left(1-\frac{\beta}{2}\right) \le \frac{1}{2}, \tag{2.2.49}$$

that is, if

$$0 < \beta \le 2\left(\frac{1-\mu}{2-\mu}\right). \tag{2.2.50}$$

Similarly

$$\begin{aligned} g^{n-ext}_{j+\frac{3}{2}} &= minmod\Big\{\beta\tilde{g}^n_{j+\frac{5}{2}},\ \tilde{g}^n_{j+\frac{3}{2}},\ \beta\tilde{g}^n_{j+\frac{1}{2}}\Big\} \\ &= minmod\Big\{\beta\frac{1}{2}\{f'(\eta^n_{j+2}) - \lambda f'(\alpha^n_{j+2})f'(\xi^n_{j+2})\}\Delta U^n_{j+\frac{3}{2}}, \\ &\qquad \frac{1}{2}\{f'(\eta^n_{j+1}) - \lambda f'(\alpha^n_{j+1})f'(\xi^n_{j+1})\}\Delta U^n_{j+\frac{1}{2}}, \\ &\qquad \beta\frac{1}{2}\{f'(\eta^n_j) - \lambda f'(\alpha^n_j)f'(\xi^n_j)\}\Delta U^n_{j-\frac{1}{2}}\Big\}. \end{aligned} \tag{2.2.51}$$

Using (2.2.41) and the definition (2.2.51) , we have that $g^{n-ext}_{j+\frac{3}{2}}$ has the same sign as $\Delta U^n_{j+\frac{1}{2}}$ or is zero and thus considering (2.2.37) and the CFL condition, we have

$$\Big(2\lambda g^{n-ext}_{j+\frac{3}{2}}\Big)\Big(\{1 - 2\lambda f'(\xi^n_{j+1}) + \lambda^2 f'(\alpha^n_{j+1})f'(\xi^n_{j+1})\}\Delta U^n_{j+\frac{1}{2}}\Big) \geq 0 \tag{2.2.52}$$

and

$$2\lambda g^{n-ext}_{j+\frac{3}{2}} \leq 2\lambda\frac{1}{2}\{f'(\eta^n_{j+1}) - \lambda f'(\alpha^n_{j+1})f'(\xi^n_{j+1})\}\left|\Delta U^n_{j+\frac{1}{2}}\right|.$$

We need to show that

$$2\lambda g^{n-ext}_{j+\frac{3}{2}} \leq \{1 - 2\lambda f'(\xi^n_{j+1}) + \lambda^2 f'(\alpha^n_{j+1})f'(\xi^n_{j+1})\}\left|\Delta U^n_{j+\frac{1}{2}}\right|. \tag{2.2.53}$$

Now (2.2.53) is true if

$$\lambda\{f'(\eta^n_{j+1}) - \lambda f'(\alpha^n_{j+1})f'(\xi^n_{j+1})\} \leq 1 - 2\lambda f'(\xi^n_{j+1}) + \lambda^2 f'(\alpha^n_{j+1})f'(\xi^n_{j+1})$$

which is equivalent to showing that

$$0 \leq 1 - \lambda f'(\eta^n_{j+1}) - 2\lambda f'(\xi^n_{j+1}) + 2\lambda^2 f'(\alpha^n_{j+1})f'(\xi^n_{j+1}). \tag{2.2.54}$$

Noting that

$$\begin{aligned} U^n_{j+1} - \lambda f'(\xi^n_j)\Delta U^n_{j+\frac{1}{2}} &\geq U^n_{j+1} - \frac{1}{2}\Delta U^n_{j+\frac{1}{2}} \qquad \text{since } \lambda\|f'\|_\infty \leq \frac{1}{2} \\ &= \frac{1}{2}[U^n_{j+1} + U^n_j], \end{aligned}$$

we have that the slope of the secant line joining $U^n_{j+1} - \lambda f'(\xi)\Delta U^n_{j+\frac{1}{2}}$ and U^n_j is greater than the slope of the secant line joining $\frac{1}{2}[U^n_{j+1} + U^n_j]$ and U^n_j by the convexity of f, that is,

$$f'(\alpha_{j+1}) \geq f'(\eta_{j+1}) \qquad \text{for } 0 < \lambda\|f'\|_\infty \leq \frac{1}{2}. \tag{2.2.55}$$

So if

$$0 \le 1 - \lambda f'(\alpha_{j+1}^n) - 2\lambda f'(\xi_{j+1}^n) + 2\lambda^2 f'(\alpha_{j+1}^n) f'(\xi_{j+1}^n) \qquad (2.2.56)$$

is true then (2.2.54) would be implied, that is, if subtracting a larger positive quantity yields a non negative number then certainly it must be true for a smaller positive quantity. We rewrite (2.2.56) as

$$\begin{aligned} 1 - \lambda f'(\alpha_{j+1}^n) - 2\lambda f'(\xi_{j+1}^n)\{1 - \lambda f'(\alpha_{j+1}^n)\} &= \{1 - \lambda f'(\alpha_{j+1}^n)\}\{1 - 2\lambda f'(\xi_{j+1}^n)\} \\ &\ge 0 \qquad \text{since } 0 < \lambda \|f'\|_\infty \le \frac{1}{2}. \qquad (2.2.57) \end{aligned}$$

Hence (2.2.54) holds and in turn (2.2.53) is true.

Finally note that the definition $g_{j+\frac{1}{2}}^{n-ext}$ includes $\Delta U_{j+\frac{1}{2}}^n$, $\Delta U_{j-\frac{1}{2}}^n$ and $\Delta U_{j-\frac{3}{2}}^n$. So either $4\lambda g_{j+\frac{1}{2}}^{n-ext} = 0$ or the first term in each grouping of (2.2.38) and $4\lambda g_{j+\frac{1}{2}}^{n-ext}$ have the same sign. Therefore, we can take absolute values in (2.2.38) to get

$$\begin{aligned} \left|\Delta U_{j+\frac{1}{2}}^{n+1}\right| &\le \left|\{1 - 2\lambda f'(\xi_{j+1}^n) + \lambda^2 f'(\alpha_{j+1}^n) f'(\xi_{j+1}^n)\}\Delta U_{j+\frac{1}{2}}^n - 2\lambda g_{j+\frac{3}{2}}^{n-ext}\right| \\ &\quad + \left|2\lambda f'(\xi_j^n)\{1 - \lambda f'(\alpha_j^n)\}\Delta U_{j-\frac{1}{2}}^n - 2\lambda g_{j-\frac{1}{2}}^{n-ext}\right| \\ &\quad + \left|\lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)\Delta U_{j-\frac{3}{2}}^n + 4\lambda g_{j+\frac{1}{2}}^{n-ext}\right| \\ &\qquad \text{and since the pairwise terms have the same sign and the} \\ &\qquad \text{coefficients of the } \Delta U^n\text{'s are nonnegative} \\ &= \{1 - 2\lambda f'(\xi_{j+1}^n) + \lambda^2 f'(\alpha_{j+1}^n) f'(\xi_{j+1}^n)\}\left|\Delta U_{j+\frac{1}{2}}^n\right| - 2\lambda\left|g_{j+\frac{3}{2}}^{n-ext}\right| \\ &\quad + 2\lambda f'(\xi_j^n)\{1 - \lambda f'(\alpha_j^n)\}\left|\Delta U_{j-\frac{1}{2}}^n\right| - 2\lambda\left|g_{j-\frac{1}{2}}^{n-ext}\right| \\ &\quad + \lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)\left|\Delta U_{j-\frac{3}{2}}^n\right| + 4\lambda\left|g_{j+\frac{1}{2}}^{n-ext}\right| \end{aligned}$$

that is,

$$\begin{aligned} \left|\Delta U_{j+\frac{1}{2}}^{n+1}\right| &\le \{1 - 2\lambda f'(\xi_{j+1}^n) + \lambda^2 f'(\alpha_{j+1}^n) f'(\xi_{j+1}^n)\}\left|\Delta U_{j+\frac{1}{2}}^n\right| \\ &\quad + 2\lambda f'(\xi_j^n)\{1 - \lambda f'(\alpha_j^n)\}\left|\Delta U_{j-\frac{1}{2}}^n\right| + \lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)\left|\Delta U_{j-\frac{3}{2}}^n\right| \\ &\quad - 2\lambda\left(\left|g_{j+\frac{3}{2}}^{n-ext}\right| - 2\left|g_{j+\frac{1}{2}}^{n-ext}\right| + \left|g_{j-\frac{1}{2}}^{n-ext}\right|\right). \qquad (2.2.58) \end{aligned}$$

Now summing over all integers j,

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\left|\Delta U_{j+\frac{1}{2}}^{n+1}\right| &\leq \sum_{j=-\infty}^{\infty}\{1-2\lambda f'(\xi_{j+1}^{n})+\lambda^{2}f'(\alpha_{j+1}^{n})f'(\xi_{j+1}^{n})\}\left|\Delta U_{j+\frac{1}{2}}^{n}\right| \\ &+\sum_{j=-\infty}^{\infty}2\lambda f'(\xi_{j}^{n})\{1-\lambda f'(\alpha_{j}^{n})\}\left|\Delta U_{j-\frac{1}{2}}^{n}\right|+\sum_{j=-\infty}^{\infty}\lambda^{2}f'(\alpha_{j-1}^{n})f'(\xi_{j-1}^{n})\left|\Delta U_{j-\frac{3}{2}}^{n}\right| \\ &-2\lambda\left(\sum_{j=-\infty}^{\infty}\left|g_{j+\frac{3}{2}}^{n-ext}\right|-2\sum_{j=-\infty}^{\infty}\left|g_{j+\frac{1}{2}}^{n-ext}\right|+\sum_{j=-\infty}^{\infty}\left|g_{j-\frac{1}{2}}^{n-ext}\right|\right).\end{aligned} \tag{2.2.59}$$

Keeping the first term on the right hand side of the above inequality as it is but re-indexing the rest yields

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\left|\Delta U_{j+\frac{1}{2}}^{n+1}\right| &\leq \sum_{j=-\infty}^{\infty}\left|\Delta U_{j+\frac{1}{2}}^{n}\right|+\sum_{j=-\infty}^{\infty}\{-2\lambda f'(\xi_{j+1}^{n})+\lambda^{2}f'(\alpha_{j+1}^{n})f'(\xi_{j+1}^{n})\}\left|\Delta U_{j+\frac{1}{2}}^{n}\right| \\ &+\sum_{j=-\infty}^{\infty}2\lambda f'(\xi_{j+1}^{n})\{1-\lambda f'(\alpha_{j+1}^{n})\}\left|\Delta U_{j+\frac{1}{2}}^{n}\right|+\sum_{j=-\infty}^{\infty}\lambda^{2}f'(\alpha_{j+1}^{n})f'(\xi_{j+1}^{n})\left|\Delta U_{j+\frac{1}{2}}^{n}\right| \\ &-2\lambda\left(\sum_{j=-\infty}^{\infty}\left|g_{j+\frac{1}{2}}^{n-ext}\right|-2\sum_{j=-\infty}^{\infty}\left|g_{j+\frac{1}{2}}^{n-ext}\right|+\sum_{j=-\infty}^{\infty}\left|g_{j+\frac{1}{2}}^{n-ext}\right|\right)\end{aligned} \tag{2.2.60}$$

which simplifies to the desired inequality, namely

$$\sum_{j=-\infty}^{\infty}\left|\Delta U_{j+\frac{1}{2}}^{n+1}\right| \leq \sum_{j=-\infty}^{\infty}\left|\Delta U_{j+\frac{1}{2}}^{n}\right|. \tag{2.2.61}$$

□

Theorem 2.3. *Assume that $f'>0$ and $f''>0$. Furthermore, for $1\leq c\leq 2$ let $f'(\eta_j^n)<cf'(\xi_j^n)$ where $f'(\xi_j^n)$ and $f'(\eta_j^n)$ are as defined on page 22. Then for the conservation law (1.1.1), if the underlying scheme is the upwinding scheme (1.2.3), $\frac{c}{2}\leq s\leq 1$ and $\beta\leq\frac{2(1-s)}{c-s}$, then Algorithm 2.2 is TVD for $\frac{c}{2}\leq\lambda f'\leq s$.*

The upper bound for c in the above theorem is given by Lemma B.2, that is

$$f'(\eta_j^n)<2f'(\xi_j^n). \tag{2.2.62}$$

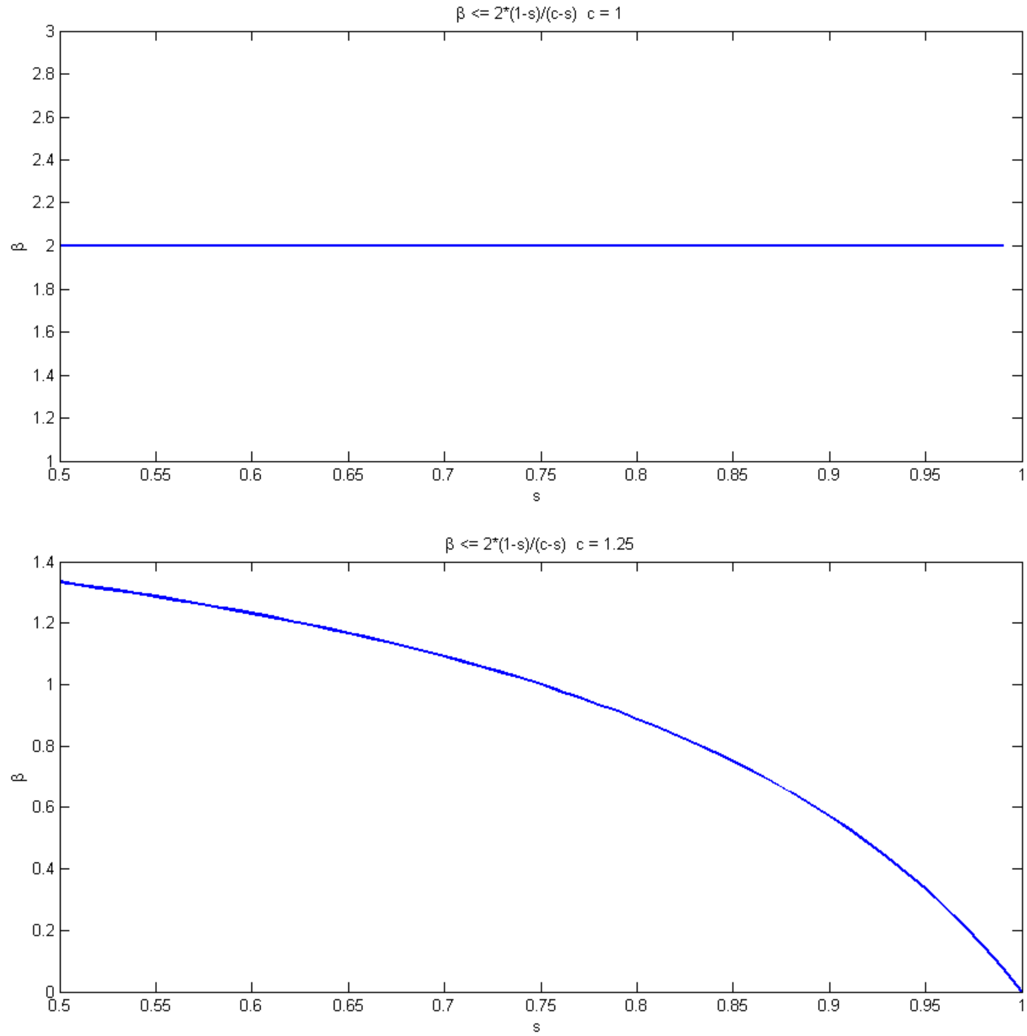

Figure 2.3: *TVD Regions for Theorem 2.3*

Proof. As in the previous theorem (2.2.37) holds. By definition we have

$$\left(2\lambda g_{j-\frac{1}{2}}^{n-ext}\right)\left(2\lambda\frac{1}{2}\{f'(\eta_{j-1}^{n}) - f'(\alpha_{j-1}^{n})f'(\xi_{j-1}^{n})\}\Delta U_{j-\frac{3}{2}}^{n}\right) \geq 0 \tag{2.2.63}$$

and

$$2\lambda\left|g_{j-\frac{1}{2}}^{n-ext}\right| \quad \leq \quad 2\lambda\frac{1}{2}\{f'(\eta_{j-1}^{n}) - \lambda f'(\alpha_{j-1}^{n})f'(\xi_{j-1}^{n})\}\left|\Delta U_{j-\frac{3}{2}}^{n}\right|. \tag{2.2.64}$$

In (2.2.37) we want

$$2\lambda\left|g_{j-\frac{1}{2}}^{n-ext}\right| \leq \lambda^{2} f'(\alpha_{j-1}^{n})f'(\xi_{j-1}^{n})\}\Delta U_{j-\frac{3}{2}}^{n}. \tag{2.2.65}$$

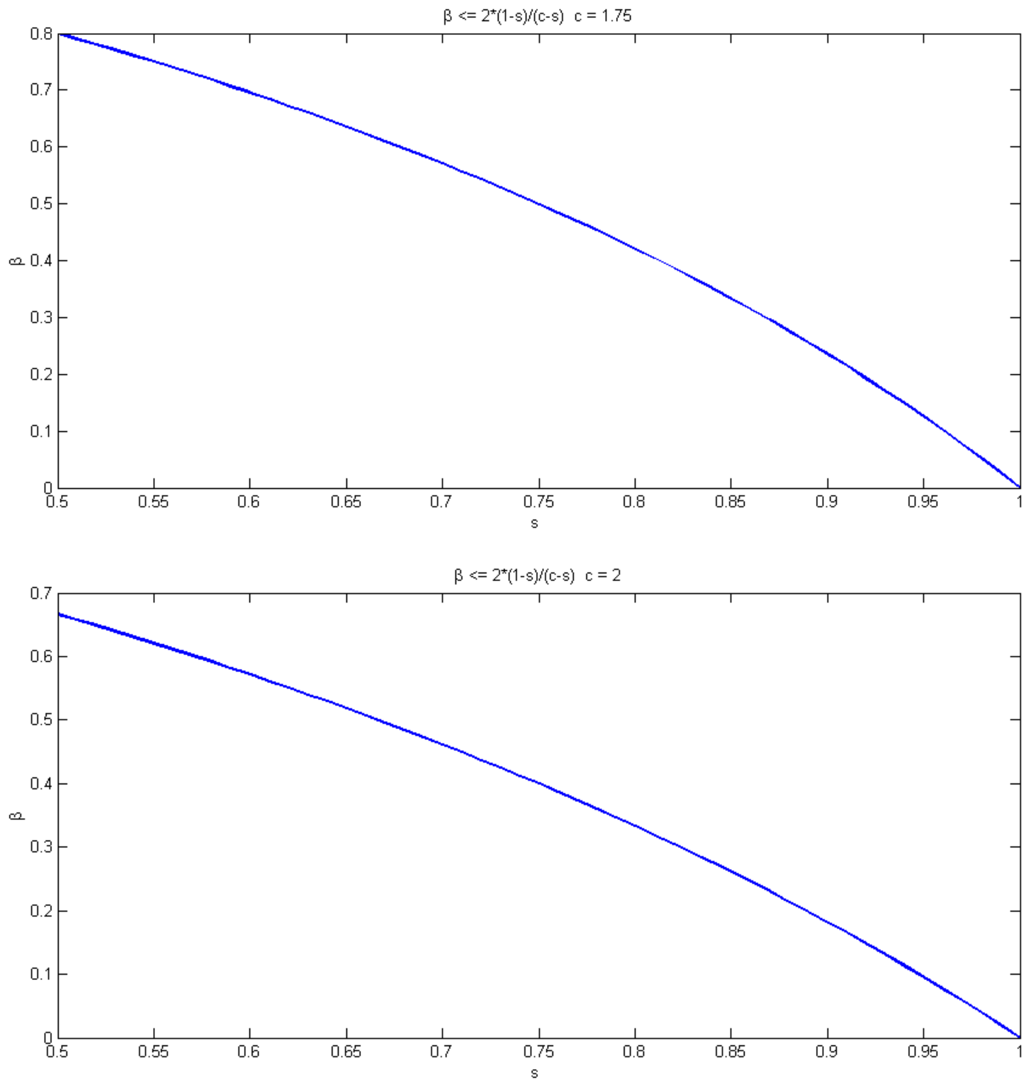

Figure 2.4: *TVD Regions for Theorem 2.3*

(2.2.65) is true if

$$f'(\eta_{j-1}^n) - \lambda f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n) \leq \lambda f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n) \tag{2.2.66}$$

that is, if

$$f'(\eta_{j-1}^n) \leq 2\lambda f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n). \tag{2.2.67}$$

On using the given assumptions, (2.2.67) is true if

$$cf'(\xi_{j-1}^n) \leq 2\lambda f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n), \tag{2.2.68}$$

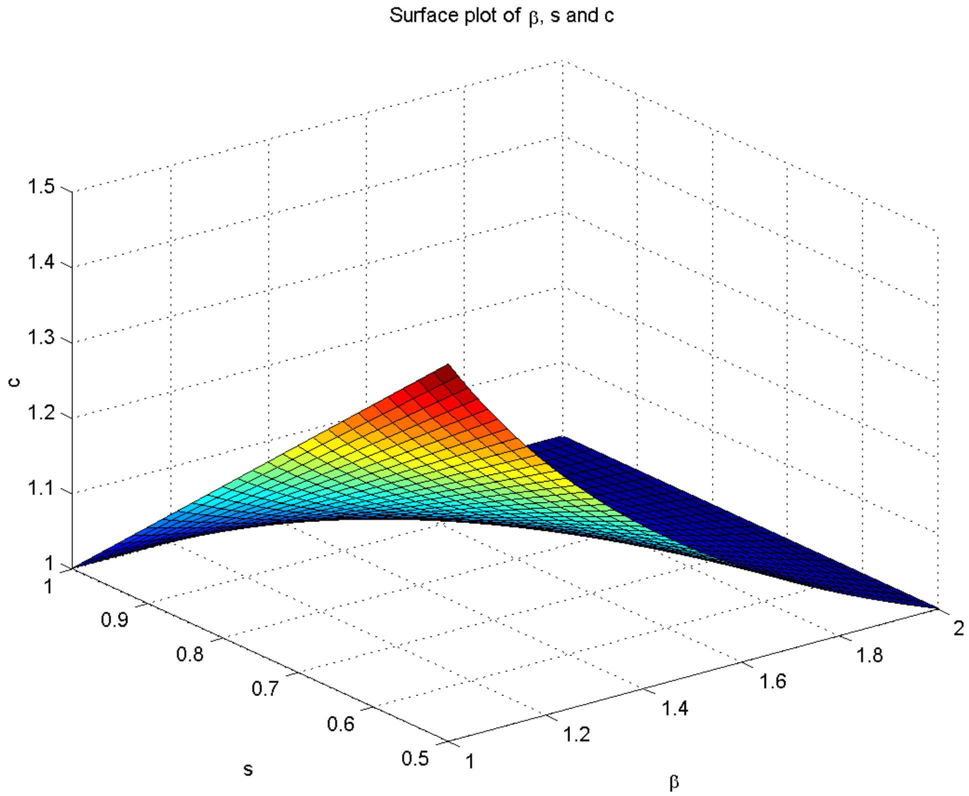

Figure 2.5: *TVD regions surface plot for Theorem 2.3*

that is, if

$$c \leq 2\lambda f'(\alpha_{j-1}^n). \tag{2.2.69}$$

This holds by the given CFL condition. Hence (2.2.65) holds.

Similarly

$$\left(2\lambda g_{j+\frac{3}{2}}^{n-ext}\right)\left(2\lambda\beta\frac{1}{2}\{f'(\eta_j^n) - \lambda f'(\alpha_j^n)f'(\xi_j^n)\}\Delta U_{j-\frac{1}{2}}^n\right) \geq 0 \tag{2.2.70}$$

and

$$2\lambda\left|g_{j+\frac{1}{2}}^{n-ext}\right| \leq 2\lambda\beta\frac{1}{2}\{f'(\eta_j^n) - \lambda f'(\alpha_j^n)f'(\xi_j^n)\}\left|\Delta U_{j-\frac{1}{2}}^n\right|. \tag{2.2.71}$$

In (2.2.37) we want

$$2\lambda\left|g_{j+\frac{3}{2}}^{n-ext}\right| \leq 2\lambda f'(\xi_j^n)\{1 - \lambda f'(\alpha_j^n)\}\left|\Delta U_{j-\frac{1}{2}}^n\right|. \tag{2.2.72}$$

(2.2.72) is true if

$$\beta\frac{1}{2}\{f'(\eta_j^n) - \lambda f'(\alpha_j^n)f'(\xi_j^n)\} \leq f'(\xi_j^n)\{1 - \lambda f'(\alpha_j^n)\}, \tag{2.2.73}$$

that is, if

$$\left(1 - \frac{\beta}{2}\right)\lambda f'(\alpha_j^n)f'(\xi_j^n) \leq f'(\xi_j^n) - \frac{\beta}{2}f'(\eta_j^n). \tag{2.2.74}$$

But (2.2.74) holds if

$$\left(1 - \frac{\beta}{2}\right)sf'(\xi_j^n) \leq f'(\xi_j^n) - \frac{\beta}{2}f'(\eta_j^n) \qquad \text{since } \frac{c}{2} < \lambda f' \leq s \tag{2.2.75}$$

which is true if

$$\left(1 - \frac{\beta}{2}\right)sf'(\xi_j^n) \leq f'(\xi_j^n) - \frac{\beta}{2}cf'(\xi_j^n) \qquad \text{since } f'(\eta_j^n) < cf'(\xi_j^n). \tag{2.2.76}$$

Now (2.2.76) is true if

$$\left(1 - \frac{\beta}{2}\right)s \leq \left(1 - \frac{\beta}{2}c\right) \tag{2.2.77}$$

that is, if

$$\beta \leq 2\left(\frac{1-s}{c-s}\right). \tag{2.2.78}$$

Hence (2.2.72) holds since the above equation is true by assumption.

Finally either $4\lambda g_{j+\frac{1}{2}}^{n-ext} = 0$ or the first three terms on the right of (2.2.37) and $4\lambda g_{j+\frac{1}{2}}^{n-ext}$ have the same sign. Therefore

$$\begin{aligned}
\left|\Delta U_{j+\frac{1}{2}}^{n+1}\right| \leq\ & \{1 - 2\lambda f'(\xi_{j+1}^n) + \lambda^2 f'(\alpha_{j+1}^n)f'(\xi_{j+1}^n)\}\left|\Delta U_{j+\frac{1}{2}}^n\right| \\
& +2\lambda f'(\xi_j^n)\{1 - \lambda f'(\alpha_j^n)\}\left|\Delta U_{j-\frac{1}{2}}^n\right| + \lambda^2 f'(\alpha_{j-1}^n)f'(\xi_{j-1}^n)\left|\Delta U_{j-\frac{3}{2}}^n\right| \\
& -2\lambda\left(\left|g_{j+\frac{3}{2}}^{n-ext}\right| - 2\left|g_{j+\frac{1}{2}}^{n-ext}\right| + \left|g_{j-\frac{1}{2}}^{n-ext}\right|\right).
\end{aligned} \tag{2.2.79}$$

Now summing over all integers j and using similar arguments to the proof of Theorem 2.2.37, we get the desired inequality, namely

$$\sum_{j=-\infty}^{j=\infty}\left|\Delta U_{j+\frac{1}{2}}^{n+1}\right| \leq \sum_{j=-\infty}^{j=\infty}\left|\Delta U_{j+\frac{1}{2}}^n\right|. \tag{2.2.80}$$

□

2.3 Goodman-Leveque Geometric Approach

We give an overview of the geometric approach (see Goodman and Leveque [2]). To obtain second order accuracy, the solution is reconstructed from the cell averages using a piecewise linear function $\tilde{v}(x,t^n)$,

$$\tilde{v}(x,t^n) = U_j^n + s_j(x - x_j) \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \tag{2.3.1}$$

where

$$s_j = \begin{cases} 0 & \text{if } (U_{j+1} - U_j)(U_j - U_{j-1}) \le 0, \\ \\ \operatorname{sgn}(U_{j+1} - U_j) \min\left\{\left|\frac{U_{j+1}-U_j}{h}\right|, \left|\frac{U_j - U_{j-1}}{h}\right|\right\} & \text{otherwise .} \end{cases} \tag{2.3.2}$$

To obtain the numerical flux at the cell edges, the true flux f is approximated by a piecewise linear function g such that

$$g_j' = \begin{cases} \frac{f(U_j^+) - f(U_j^-)}{U_j^+ - U_j^-} & \text{if } s_j \ne 0, \\ \\ f'(U_j) & \text{if } s_j = 0, \end{cases} \tag{2.3.3}$$

where

$$U_j^{\pm} = U_j \pm \frac{h}{2} s_j. \tag{2.3.4}$$

Note that $U_j^-, U_j^+, U_{j+1}^-, U_{j+1}^+$ are monotonically ordered (by (2.3.2)) and g interpolates f at these points.

Consider the problem $u_t + g(u)_x = 0$, that is, $u_t + g'(u)u_x = 0$. Note that $g'(u(x_{j+\frac{1}{2}}, t))$ is constant for $t^n \le t \le t^{n+1}$ and so the solution here is

$$u(x_{j+\frac{1}{2}}, t) = \begin{cases} U_j^+ - (t - t^n) s_j g_j' & \text{if } f' > 0, \\ \\ U_{j+1}^- (t - t^n) s_{j+1} g_{j+1}' & \text{if } f' < 0. \end{cases}$$

Hence the numerical flux due to Goodman and LeVeque is

$$G(U;j) = \frac{1}{k}\int_{t^n}^{t^{n+1}} g(u(x_{j+\frac{1}{2}}, t))dt \tag{2.3.5}$$

$$= \begin{cases} f(U_j^+) - \frac{1}{2} k s_j (g_j')^2 & \text{if } f' > 0, \\ \\ f(U_{j+1}^-) - \frac{1}{2} k s_{j+1} (g_{j+1}')^2 & \text{if } f' < 0. \end{cases} \tag{2.3.6}$$

Theorem 2.4. *The LEM raises the order of the Goodman-LeVeque scheme by 1.*

By the theorem of Yang [18], it suffices to prove that the Goodman-LeVeque flux satisfies (2.2.4), that is, for all sufficiently smooth solutions w, the numerical flux function of the underlying scheme satisfies

$$g_{j+\frac{1}{2}}[\overline{w}^n] = \overline{f}^n_{j+\frac{1}{2}} + \alpha(x_{j+\frac{1}{2}}, t^n)h^r + \beta(x_{j+\frac{1}{2}}, t^n)h^{r+1} + O(h^{r+2})$$

where $g_{j+\frac{1}{2}}[v]$ and α have Lipschitz continuous first derivative and β is Lipschitz continuous.

We have two cases, $s_{j-1} \neq 0 \neq s_j$ and $s_j = 0$. In the second case, the Goodman-LeVeque method reduces to the first order Godunov method. W.l.o.g assume that $f' > 0$.

Case 2.1.

$$s_{j-1} \neq 0 \neq s_j. \tag{2.3.7}$$

Let w be a sufficiently smooth solution. Firstly, we note that

$$s_j = \operatorname{sgn}(\overline{w}^n_{j+1} - \overline{w}^n_j) \min\left\{\left|\frac{\overline{w}^n_{j+1} - \overline{w}^n_j}{h}\right|, \left|\frac{\overline{w}^n_j - \overline{w}^n_{j-1}}{h}\right|\right\}. \tag{2.3.8}$$

Now

$$\begin{aligned}
\overline{w}^n_j &= \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} w(x, t^n)dx \\
&= \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \Big(w(x_{j+\frac{1}{2}}, t^n) + w_x(x_{j+\frac{1}{2}}, t^n)[x - x_{j+\frac{1}{2}}] \\
&\qquad + \frac{1}{2}w_{xx}(x_{j+\frac{1}{2}}, t^n)[x - x_{j+\frac{1}{2}}]^2 + \frac{1}{6}w_{xxx}(x_{j+\frac{1}{2}}, t^n)[x - x_{j+\frac{1}{2}}]^3 + \cdots\Big)dx,
\end{aligned}$$

that is,

$$\begin{aligned}
\overline{w}^n_j &= \frac{1}{h}\Bigg[w(x_{j+\frac{1}{2}}, t^n)x + w_x(x_{j+\frac{1}{2}}, t^n)\frac{[x - x_{j+\frac{1}{2}}]^2}{2} \\
&\qquad + \frac{1}{2}w_{xx}(x_{j+\frac{1}{2}}, t^n)\frac{[x - x_{j+\frac{1}{2}}]^3}{3} + \frac{1}{6}w_{xxx}(x_{j+\frac{1}{2}}, t^n)\frac{[x - x_{j+\frac{1}{2}}]^4}{4} + \cdots\Bigg]_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \\
&= \frac{1}{h}\Big(w(x_{j+\frac{1}{2}}, t^n)h - w_x(x_{j+\frac{1}{2}}, t^n)\frac{h^2}{2} + \frac{1}{2}w_{xx}(x_{j+\frac{1}{2}}, t^n)\frac{h^3}{3} - \frac{1}{6}w_{xxx}(x_{j+\frac{1}{2}}, t^n)\frac{h^4}{4} \\
&\qquad + O(h^5)\Big)
\end{aligned}$$

which simplifies to

$$\overline{w}_j^n = w(x_{j+\frac{1}{2}},t^n) - w_x(x_{j+\frac{1}{2}},t^n)\frac{h}{2} + w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{h^2}{6} - w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{24} + O(h^4). \tag{2.3.9}$$

Similarly

$$\begin{aligned}
\overline{w}_{j+1}^n &= \frac{1}{h}\int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{3}{2}}} w(x,t^n)dx \\
&= \frac{1}{h}\Bigg[w(x_{j+\frac{1}{2}},t^n)x + w_x(x_{j+\frac{1}{2}},t^n)\frac{[x-x_{j+\frac{1}{2}}]^2}{2} \\
&\quad + \frac{1}{2}w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{[x-x_{j+\frac{1}{2}}]^3}{3} + \frac{1}{6}w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{[x-x_{j+\frac{1}{2}}]^4}{4} + \cdots\Bigg]_{x_{j+\frac{1}{2}}}^{x_{j+\frac{3}{2}}}
\end{aligned}$$

that is,

$$\begin{aligned}
\overline{w}_{j+1}^n &= \frac{1}{h}\Big(w(x_{j+\frac{1}{2}},t^n)h + w_x(x_{j+\frac{1}{2}},t^n)\frac{h^2}{2} + \frac{1}{2}w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{3} + \frac{1}{6}w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{h^4}{4} \\
&\quad + O(h^5)\Big) \\
&= w(x_{j+\frac{1}{2}},t^n) + w_x(x_{j+\frac{1}{2}},t^n)\frac{h}{2} + w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{h^2}{6} + w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{24} \\
&\quad + O(h^4).
\end{aligned} \tag{2.3.10}$$

and

$$\begin{aligned}
\overline{w}_{j-1}^n &= \frac{1}{h}\int_{x_{j-\frac{3}{2}}}^{x_{j-\frac{1}{2}}} w(x,t^n)dx \\
&= \frac{1}{h}\Bigg[w(x_{j+\frac{1}{2}},t^n)x + w_x(x_{j+\frac{1}{2}},t^n)\frac{[x-x_{j+\frac{1}{2}}]^2}{2} \\
&\quad + \frac{1}{2}w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{[x-x_{j+\frac{1}{2}}]^3}{3} + \frac{1}{6}w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{[x-x_{j+\frac{1}{2}}]^4}{4} + \cdots\Bigg]_{x_{j-\frac{3}{2}}}^{x_{j-\frac{1}{2}}}
\end{aligned}$$

that is,

$$\begin{aligned}
\overline{w}_{j-1}^n &= \frac{1}{h}\Bigg[\Bigg(w(x_{j+\frac{1}{2}},t^n)h + w_x(x_{j+\frac{1}{2}},t^n)\frac{h^2}{2} - \frac{1}{2}w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{3} + \frac{1}{6}w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{h^4}{4}\Bigg) \\
&\qquad -\Bigg(w_x(x_{j+\frac{1}{2}},t^n)\frac{(2h)^2}{2} - \frac{1}{2}w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{(2h)^3}{3} + \frac{1}{6}w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{(2h)^4}{4}\Bigg) \\
&\qquad + O(h^5)\Bigg] \\
&= w(x_{j+\frac{1}{2}},t^n) - \frac{3}{2}w_x(x_{j+\frac{1}{2}},t^n)h + \frac{7}{6}w_{xx}(x_{j+\frac{1}{2}},t^n)h^2 - \frac{5}{6}w_{xxx}(x_{j+\frac{1}{2}},t^n)h^3 \\
&\quad +O(h^4).
\end{aligned} \tag{2.3.11}$$

We examine the two cases in (2.3.8).

I. If $s_j = \text{sgn}(\overline{w}_{j+1}^n - \overline{w}_j^n)\left|(\overline{w}_{j+1}^n - \overline{w}_j^n)/h\right|$, then using (2.3.9) and (2.3.10), we have

$$\frac{\overline{w}_{j+1}^n - \overline{w}_j^n}{h} = w_x(x_{j+\frac{1}{2}},t^n) + \frac{1}{12}w_{xxx}(x_{j+\frac{1}{2}},t^n)h^2 + O(h^4) \tag{2.3.12}$$

II. If $s_j = \text{sgn}(\overline{w}_{j+1}^n - \overline{w}_j^n)\left|(\overline{w}_j^n - \overline{w}_{j-1}^n)/h\right|$, then using (2.3.9) and (2.3.11) we have

$$\frac{\overline{w}_j^n - \overline{w}_{j-1}^n}{h} = w_x(x_{j+\frac{1}{2}},t^n) - w_{xx}(x_{j+\frac{1}{2}},t^n)h + \frac{19}{24}w_{xxx}(x_{j+\frac{1}{2}},t^n)h^2 + O(h^3) \tag{2.3.13}$$

The true flux at the cell interface $x = x_{j+\frac{1}{2}}$ can be written in the form

$$\begin{aligned}
f(w(x_{j+\frac{1}{2}},t)) &= f(w(x_{j+\frac{1}{2}},t^n)) + f'(w(x_{j+\frac{1}{2}},t^n))w_t(x_{j+\frac{1}{2}},t^n)[t-t^n] \\
&\quad + \frac{1}{2}\frac{d}{dt}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n}[t-t^n]^2 \\
&\quad + \frac{1}{6}\frac{d^2}{dt^2}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n}[t-t^n]^3 \\
&\quad +O\left([t-t^n]^4\right).
\end{aligned} \tag{2.3.14}$$

So the average flux over the time interval $[t^n, t^{n+1}]$ is

$$\begin{aligned}
\overline{f}^n_{j+\frac{1}{2}} &= \frac{1}{k}\int_{t^n}^{t^{n+1}} f(w(x_{j+\frac{1}{2}},t))dt \\
&= \frac{1}{k}\Bigg\{\Bigg[f(w(x_{j+\frac{1}{2}},t^n))t + f'(w(x_{j+\frac{1}{2}},t^n))w_t(x_{j+\frac{1}{2}},t^n)\frac{[t-t^n]^2}{2} \\
&\quad + \frac{1}{2}\frac{d}{dt}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n}\frac{[t-t^n]^3}{3} \\
&\quad + \frac{1}{6}\frac{d^2}{dt^2}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n}\frac{[t-t^n]^4}{4}\Bigg]_{t=t^n}^{t^{n+1}} + O\left(k^5\right)\Bigg\} \\
&= f(w(x_{j+\frac{1}{2}},t^n)) + f'(w(x_{j+\frac{1}{2}},t^n))w_t(x_{j+\frac{1}{2}},t^n)\frac{k}{2} \\
&\quad + \frac{k^2}{6}\frac{d}{dt}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n} \\
&\quad + \frac{k^3}{24}\frac{d^2}{dt^2}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n} + O\left(k^4\right).
\end{aligned} \tag{2.3.15}$$

We now find an expression for the constant slope g'_j, in (2.3.6)

$$
\begin{aligned}
g'_j &= \frac{1}{hs_j}\left\{f\left(\overline{w}_j^n + \frac{h}{2}s_j\right) - f\left(\overline{w}_j^n - \frac{h}{2}s_j\right)\right\} \\
&= \frac{1}{hs_j}\left\{f(w(x_{j+\frac{1}{2}},t^n)) + f'(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n + \frac{h}{2}s_j - w(x_{j+\frac{1}{2}},t^n)\right]\right. \\
&\qquad +\frac{1}{2}f''(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n + \frac{h}{2}s_j - w(x_{j+\frac{1}{2}},t^n)\right]^2 \\
&\qquad +\frac{1}{6}f''(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n + \frac{h}{2}s_j - w(x_{j+\frac{1}{2}},t^n)\right]^3 \\
&\qquad +O\left(\left[\overline{w}_j^n + \frac{h}{2}s_j - w(x_{j+\frac{1}{2}},t^n)\right]^4\right) \\
&\quad -\left(f(w(x_{j+\frac{1}{2}},t^n)) + f'(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n - \frac{h}{2}s_j - w(x_{j+\frac{1}{2}},t^n)\right]\right. \\
&\qquad +\frac{1}{2}f''(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n - \frac{h}{2}s_j - w(x_{j+\frac{1}{2}},t^n)\right]^2 \\
&\qquad +\frac{1}{6}f''(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n - \frac{h}{2}s_j - w(x_{j+\frac{1}{2}},t^n)\right]^3 \\
&\qquad \left.\left. + \ O\left(\left[\overline{w}_j^n - \frac{h}{2}s_j - w(x_{j+\frac{1}{2}},t^n)\right]^4\right)\right)\right\}
\end{aligned}
$$

Let

$$
\Psi_\pm = \overline{w}_j^n \pm \frac{h}{2}s_j - w(x_{j+\frac{1}{2}},t^n)
$$

so that

$$
\begin{aligned}
g'_j &= \frac{1}{hs_j}\left\{f'(w(x_{j+\frac{1}{2}},t^n))hs_j + \left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\right]f''(w(x_{j+\frac{1}{2}},t^n))hs_j\right. \\
&\quad +\frac{1}{6}\left(3hs_j[\overline{w}_j^n]^2 - 6hs_j\overline{w}_j^n w(x_{j+\frac{1}{2}},t^n) + \frac{1}{4}h^3s_j^3 + 3hs_j w^2(x_{j+\frac{1}{2}},t^n)\right)f'''(w(x_{j+\frac{1}{2}},t^n)) \\
&\quad \left. +O(\Psi_\pm^4)\right\} \\
&= f'(w(x_{j+\frac{1}{2}},t^n)) + \left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\right]f''(w(x_{j+\frac{1}{2}},t^n)) \\
&\quad +\left(\frac{1}{2}[\overline{w}_j^n]^2 - \overline{w}_j^n w(x_{j+\frac{1}{2}},t^n) + \frac{1}{24}h^2s_j^2 + \frac{1}{2}w^2(x_{j+\frac{1}{2}},t^n)\right)f'''(w(x_{j+\frac{1}{2}},t^n)) + O(\Psi_\pm^4)
\end{aligned}
$$

Hence

$$\begin{aligned}
(g_j')^2 &= \Big[f'(w(x_{j+\frac{1}{2}},t^n)) + \Big[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\Big] f''(w(x_{j+\frac{1}{2}},t^n)) \\
&\quad + \frac{1}{2}\Big[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\Big]^2 f'''(w(x_{j+\frac{1}{2}},t^n)) + \frac{1}{24}h^2 s_j^2 f'''(w(x_{j+\frac{1}{2}},t^n))\Big]^2 + O(\Psi_\pm^4) \\
&= [f'(w(x_{j+\frac{1}{2}},t^n))]^2 + 2\Big[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\Big] f'(w(x_{j+\frac{1}{2}},t^n)) f''(w(x_{j+\frac{1}{2}},t^n)) \\
&\quad + \Big[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\Big]^2 f'(w(x_{j+\frac{1}{2}},t^n)) f'''(w(x_{j+\frac{1}{2}},t^n)) \\
&\quad + \Big[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\Big] [f''(w(x_{j+\frac{1}{2}},t^n))]^2 \\
&\quad +2\Big[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\Big]^3 f''(w(x_{j+\frac{1}{2}},t^n)) f'''(w(x_{j+\frac{1}{2}},t^n)) \\
&\quad +\frac{1}{4}\Big[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\Big]^4 [f'''(w(x_{j+\frac{1}{2}},t^n))]^2 \\
&\quad +\frac{1}{12}\Big[f'(w(x_{j+\frac{1}{2}},t^n)) + \frac{1}{2}\Big[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\Big] f''(w(x_{j+\frac{1}{2}},t^n)) \\
&\quad\ +\frac{1}{2}\Big[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\Big]^2 f'''(w(x_{j+\frac{1}{2}},t^n))\Big] f'''(w(x_{j+\frac{1}{2}},t^n)) h^2 s_j^2 \\
&\quad +\frac{1}{(24)^2} h^4 s_j^4 [f'''(w(x_{j+\frac{1}{2}},t))]^2 + O(\Psi_\pm^4). \qquad (2.3.16)
\end{aligned}$$

I). If we use (2.3.9) and (2.3.12) we get

$$\begin{aligned}
(g_j')^2 &= [f'(w(x_{j+\frac{1}{2}},t^n))]^2 \\
&\quad +2\Big[-w_x(x_{j+\frac{1}{2}},t^n)\frac{h}{2} + w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{h^2}{6} \\
&\qquad -w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{24}\Big] f'(w(x_{j+\frac{1}{2}},t^n)) f''(w(x_{j+\frac{1}{2}},t^n)) \\
&\quad +\Big[w_x^2(x_{j+\frac{1}{2}},t^n)\frac{h^2}{4} + w_x(x_{j+\frac{1}{2}},t^n) w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{6}\Big] \times \\
&\qquad\qquad f'(w(x_{j+\frac{1}{2}},t^n)) f'''(w(x_{j+\frac{1}{2}},t^n)) \\
&\quad +\Big[-w_x(x_{j+\frac{1}{2}},t^n)\frac{h}{2} + w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{h^2}{6} - w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{24}\Big] \times \\
&\qquad\qquad [f''(w(x_{j+\frac{1}{2}},t^n))]^2 \\
&\quad +2\Big[w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{8}\Big] f''(w(x_{j+\frac{1}{2}},t^n)) f'''(w(x_{j+\frac{1}{2}},t^n)) \\
&\quad +\frac{1}{12}\Big[f'(w(x_{j+\frac{1}{2}},t^n)) + \frac{1}{2}\Big[-w_x(x_{j+\frac{1}{2}},t^n)\frac{h}{2}\Big] f''(w(x_{j+\frac{1}{2}},t^n))\Big] \times \\
&\qquad\qquad f'''(w(x_{j+\frac{1}{2}},t^n)) h^2 w_x^2(x_{j+\frac{1}{2}},t^n) + O(h^4). \qquad (2.3.17)
\end{aligned}$$

that is

$$\begin{aligned}
(g_j')^2 \;=\; & [f'(w(x_{j+\frac{1}{2}},t^n))]^2 \\
& -\Big(f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n) \\
& \quad +\frac{1}{2}[f''(w(x_{j+\frac{1}{2}},t^n))]^2 w_x(x_{j+\frac{1}{2}},t^n)\Big) h \\
& +\Big(\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n) \\
& \quad +\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_x^2(x_{j+\frac{1}{2}},t^n) \\
& \quad +\frac{1}{6}\left[f''(w(x_{j+\frac{1}{2}},t^n))\right]^2\Big) h^2 \\
& +\Big(-\frac{1}{12}f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_{xxx}(x_{j+\frac{1}{2}},t^n) \\
& \quad +\frac{1}{6}f'(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)w_{xx}(x_{j+\frac{1}{2}},t^n) \\
& \quad -\frac{1}{24}[f''(w(x_{j+\frac{1}{2}},t^n))]^2 w_{xxx}(x_{j+\frac{1}{2}},t^n) \\
& \quad +\frac{1}{4}f''(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_{xxx}(x_{j+\frac{1}{2}},t^n) \\
& \quad -\frac{1}{48}f''(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_x^3(x_{j+\frac{1}{2}},t^n)\Big) h^3 \\
& +O(h^4). \qquad (2.3.18)
\end{aligned}$$

We also have that

$$\begin{aligned}
f\left(\overline{w}_j^n + \frac{h}{2}s_j\right) \;=\; & f(w(x_{j+\frac{1}{2}},t^n)) \\
& +f'(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n) + \frac{h}{2}s_j\right] \\
& +\frac{1}{2}f''(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n) + \frac{h}{2}s_j\right]^2 \\
& +\frac{1}{6}f'''(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n) + \frac{h}{2}s_j\right]^3 \\
& +O(\Psi_+^4). \qquad (2.3.19)
\end{aligned}$$

But

$$\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n) + \frac{h}{2}s_j = \frac{1}{6}w_{xx}(x_{j+\frac{1}{2}},t^n)h^2 + O(h^4)$$

so that

$$f\left(\overline{w}_j^n + \frac{h}{2}s_j\right) = f(w(x_{j+\frac{1}{2}},t^n)) + \frac{1}{6}f'(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n)h^2 + O(h^4). \tag{2.3.20}$$

Hence the numerical flux can be expressed as

$$\begin{aligned} G[\overline{w}^n;j] &= f\left(\overline{w}_j^n + \frac{h}{2}s_j\right) - \frac{1}{2}ks_j(g_j')^2 \\ &= \overline{f}_{j+\frac{1}{2}}^n + f\left(\overline{w}_j^n + \frac{h}{2}s_j\right) - \overline{f}_{j+\frac{1}{2}}^n - \frac{1}{2}ks_j(g_j')^2 \end{aligned} \tag{2.3.21}$$

which on using (2.3.12), (2.3.15), (2.3.18) and (2.3.20) yields

$$\begin{aligned} G[\overline{w}^n;j] = {} & \overline{f}_{j+\frac{1}{2}}^n \\ & + f(w(x_{j+\frac{1}{2}},t^n)) + \frac{1}{6}f'(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n)h^2 + O(h^4) \\ & - \Big[f(w(x_{j+\frac{1}{2}},t^n)) + f'(w(x_{j+\frac{1}{2}},t^n))w_t(x_{j+\frac{1}{2}},t^n)\frac{k}{2} \\ & \quad + \frac{k^2}{6}\frac{d}{dt}\left(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\right)\Big|_{t=t^n} \\ & \quad + \frac{k^3}{24}\frac{d^2}{dt^2}\left(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\right)\Big|_{t=t^n} + O\left(k^4\right)\Big] \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}k\left(w_x(x_{j+\frac{1}{2}},t^n)+\frac{1}{12}w_{xxx}(x_{j+\frac{1}{2}},t^n)h^2+O(h^4)\right)\times\\
&\Big\{[f'(w(x_{j+\frac{1}{2}},t^n))]^2\\
&-\Big(f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)\\
&\qquad+\frac{1}{2}[f''(w(x_{j+\frac{1}{2}},t^n))]^2w_x(x_{j+\frac{1}{2}},t^n)\Big)h\\
&\\
&+\Big(\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n)\\
&\qquad+\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_x^2(x_{j+\frac{1}{2}},t^n)\\
&\qquad+\frac{1}{6}\left[f''(w(x_{j+\frac{1}{2}},t^n))\right]^2\Big)h^2\\
&\\
&+\Big(-\frac{1}{12}f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_{xxx}(x_{j+\frac{1}{2}},t^n)\\
&\qquad+\frac{1}{6}f'(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)w_{xx}(x_{j+\frac{1}{2}},t^n)\\
&\qquad-\frac{1}{24}[f''(w(x_{j+\frac{1}{2}},t^n))]^2w_{xxx}(x_{j+\frac{1}{2}},t^n)\\
&\qquad+\frac{1}{4}f''(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_{xxx}(x_{j+\frac{1}{2}},t^n)\\
&\qquad-\frac{1}{48}f''(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_x^3(x_{j+\frac{1}{2}},t^n)\Big)h^3\\
&+O(h^4)\Big\}\,. \qquad (2.3.22)
\end{aligned}
$$

Simplyfying we get

$$
\begin{aligned}
G[\overline{w}^n;j] \;=\; & \overline{f}^n_{j+\frac{1}{2}} + \frac{1}{6}f'(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n)h^2 + O(h^4) \\
& -\Big[f'(w(x_{j+\frac{1}{2}},t^n))w_t(x_{j+\frac{1}{2}},t^n)\frac{k}{2} \\
& \qquad + \frac{k^2}{6}\frac{d}{dt}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n} \\
& \qquad + \frac{k^3}{24}\frac{d^2}{dt^2}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n} + O\left(k^4\right)\Big] \\
& -\frac{1}{2}k\left(w_x(x_{j+\frac{1}{2}},t^n) + \frac{1}{12}w_{xxx}(x_{j+\frac{1}{2}},t^n)h^2 + O(h^4)\right)\times \\
& \Big\{[f'(w(x_{j+\frac{1}{2}},t^n))]^2 \\
& -\Big(f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n) \\
& \qquad +\frac{1}{2}[f''(w(x_{j+\frac{1}{2}},t^n))]^2 w_x(x_{j+\frac{1}{2}},t^n)\Big)h \\
& +\Big(\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n) \\
& \qquad +\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_x^2(x_{j+\frac{1}{2}},t^n) \\
& \qquad +\frac{1}{6}\left[f''(w(x_{j+\frac{1}{2}},t^n))\right]^2\Big)h^2 + O(h^3)\Big\}.
\end{aligned}
\tag{2.3.23}
$$

Now note that

$$
\begin{aligned}
& f'(w(x_{j+\frac{1}{2}},t^n))w_t(x_{j+\frac{1}{2}},t^n)\frac{k}{2} + \frac{1}{2}kw_x(x_{j+\frac{1}{2}},t^n)[f'(w(x_{j+\frac{1}{2}},t^n))]^2 \\
&= \frac{k}{2}\Big\{f'(w(x_{j+\frac{1}{2}},t^n))[-f'(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)] \\
&\qquad + w_x(x_{j+\frac{1}{2}},t^n)[f'(w(x_{j+\frac{1}{2}},t^n))]^2\Big\} \qquad \text{using conservation law} \\
&= \frac{k}{2}[f'(w(x_{j+\frac{1}{2}},t^n))]^2\Big\{w_x(x_{j+\frac{1}{2}},t^n) - w_x(x_{j+\frac{1}{2}},t^n)\Big\} = 0.
\end{aligned}
\tag{2.3.24}
$$

In view of (2.3.24), the numerical flux (2.3.23) becomes

$$\begin{aligned}
G[\overline{w}^n;j] \;=\; & \overline{f}^n_{j+\frac{1}{2}} + \frac{1}{6}f'(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n)h^2 + O(h^4) \\
& -\Bigg[\frac{k^2}{6}\frac{d}{dt}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n} \\
& \quad + \frac{k^3}{24}\frac{d^2}{dt^2}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n} + O\left(k^4\right)\Bigg] \\
& -\frac{1}{2}k\left(\frac{1}{12}w_{xxx}(x_{j+\frac{1}{2}},t^n)h^2 + O(h^4)\right)\times \\
& \Big\{[f'(w(x_{j+\frac{1}{2}},t^n))]^2 \\
& -\Big(f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n) \\
& \quad +\frac{1}{2}[f''(w(x_{j+\frac{1}{2}},t^n))]^2 w_x(x_{j+\frac{1}{2}},t^n)\Big)h \\
& +\Big(\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n) \\
& \quad +\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_x^2(x_{j+\frac{1}{2}},t^n) \\
& \quad +\frac{1}{6}\left[f''(w(x_{j+\frac{1}{2}},t^n))\right]^2\Big)h^2 + O(h^3)\Big\}. \qquad (2.3.25)
\end{aligned}$$

The numerical flux can then be written in the form

$$G[\overline{w}^n;j] = \overline{f}^n_{j+\frac{1}{2}} + \alpha(x_{j+\frac{1}{2}},t^n)h^2 + \beta(x_{j+\frac{1}{2}},t^n)h^3 + O(h^4) \qquad (2.3.26)$$

where

$$\begin{aligned}
\alpha(x_{j+\frac{1}{2}},t^n) \;=\; & \frac{1}{6}f'(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n) \\
& -\frac{k^2}{6h^2}\frac{d}{dt}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n} \qquad (2.3.27)
\end{aligned}$$

and

$$\begin{aligned}
\beta(x_{j+\frac{1}{2}},t^n) \;=\; & -\frac{k^3}{24h^3}\frac{d^2}{dt^2}\Big(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\Big)\Big|_{t=t^n} \\
& -\frac{k}{24h}w_{xxx}(x_{j+\frac{1}{2}},t^n)[f'(w(x_{j+\frac{1}{2}},t^n))]^2. \qquad (2.3.28)
\end{aligned}$$

II For the other case (see page 37), we use (2.3.9) and (2.3.13) in (2.3.16)

$$\begin{aligned}
(g'_j)^2 = {} & [f'(w(x_{j+\frac{1}{2}},t^n))]^2 \\
& +2\left[-w_x(x_{j+\frac{1}{2}},t^n)\frac{h}{2} + w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{h^2}{6}\right. \\
& \quad \left. -w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{24}\right] f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n)) \\
& +\left[w_x^2(x_{j+\frac{1}{2}},t^n)\frac{h^2}{4} + w_x(x_{j+\frac{1}{2}},t^n)w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{6}\right]\times \\
& \qquad f'(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n)) \\
& +\left[-w_x(x_{j+\frac{1}{2}},t^n)\frac{h}{2} + w_{xx}(x_{j+\frac{1}{2}},t^n)\frac{h^2}{6} - w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{24}\right]\times \\
& \qquad [f''(w(x_{j+\frac{1}{2}},t^n))]^2 \\
& +2\left[w_{xxx}(x_{j+\frac{1}{2}},t^n)\frac{h^3}{8}\right] f''(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n)) \\
& +\frac{1}{12}\left[f'(w(x_{j+\frac{1}{2}},t^n)) + \frac{1}{2}\left[-w_x(x_{j+\frac{1}{2}},t^n)\frac{h}{2}\right] f''(w(x_{j+\frac{1}{2}},t^n))\right]\times \\
& \quad f'''(w(x_{j+\frac{1}{2}},t^n))h^2\left\{w_x^2(x_{j+\frac{1}{2}},t^n) - 2w_x(x_{j+\frac{1}{2}},t^n)w_{xx}(x_{j+\frac{1}{2}},t^n)h\right. \\
& \qquad +\left[w_{xx}^2(x_{j+\frac{1}{2}},t^n) + \frac{19}{12}w_x(x_{j+\frac{1}{2}},t^n)w_{xx}(x_{j+\frac{1}{2}},t^n)\right]h^2 \\
& \qquad \left. -\frac{19}{12}w_{xx}(x_{j+\frac{1}{2}},t^n)w_{xxx}(x_{j+\frac{1}{2}},t^n)h^3\right\} \\
& +O(h^4)
\end{aligned} \tag{2.3.29}$$

that is

$$\begin{aligned}(g_j')^2 &= [f'(w(x_{j+\frac{1}{2}},t^n))]^2\\ &-\Big(f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)\\ &\quad+\frac{1}{2}[f''(w(x_{j+\frac{1}{2}},t^n))]^2w_x(x_{j+\frac{1}{2}},t^n)\Big)h\\ &+\Big(\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n)\\ &\quad+\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_x^2(x_{j+\frac{1}{2}},t^n)\\ &\quad+\frac{1}{6}\Big[f''(w(x_{j+\frac{1}{2}},t^n))\Big]^2\Big)h^2\\ &+\Big(-\frac{1}{12}f'(w(x_{j+\frac{1}{2}},t^n))f''(w(x_{j+\frac{1}{2}},t^n))w_{xxx}(x_{j+\frac{1}{2}},t^n)\\ &\quad-\frac{1}{24}[f''(w(x_{j+\frac{1}{2}},t^n))]^2w_{xxx}(x_{j+\frac{1}{2}},t^n)\\ &\quad+\frac{1}{4}f''(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_{xxx}(x_{j+\frac{1}{2}},t^n)\\ &\quad-\frac{1}{48}f''(w(x_{j+\frac{1}{2}},t^n))f'''(w(x_{j+\frac{1}{2}},t^n))w_x^3(x_{j+\frac{1}{2}},t^n)\Big)h^3\\ &+O(h^4). \qquad (2.3.30)\end{aligned}$$

In this case

$$\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n) + \frac{h}{2}s_j = -\frac{1}{3}w_{xx}(x_{j+\frac{1}{2}},t^n)h^2 + \frac{17}{48}w_{xxx}(x_{j+\frac{1}{2}},t^n)h^3 + O(h^4)$$

so that (2.3.19) becomes

$$\begin{aligned}f\left(\overline{w}_j^n + \frac{h}{2}s_j\right) &= f(w(x_{j+\frac{1}{2}},t^n)) - \frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n)h^2\\ &+O(h^4). \qquad (2.3.31)\end{aligned}$$

Hence using (2.3.13), (2.3.15), (2.3.30) and (2.3.31) in the spirit of previous arguments yields the required result.

$$G[\overline{w}^n;j] = \overline{f}^n_{j+\frac{1}{2}} + \alpha(x_{j+\frac{1}{2}},t^n)h^2 + \beta(x_{j+\frac{1}{2}},t^n)h^3 + O(h^4) \qquad (2.3.32)$$

where

$$\alpha(x_{j+\frac{1}{2}},t^n) = -\frac{1}{3}f'(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n) - \frac{k^2}{6h^2}\frac{d}{dt}\left(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\right)\Big|_{t=t^n} \quad (2.3.33)$$

and

$$\beta(x_{j+\frac{1}{2}},t^n) = \frac{17}{48}f'(w(x_{j+\frac{1}{2}},t^n))w_{xxx}(x_{j+\frac{1}{2}},t^n) - \frac{k^3}{24h^3}\frac{d^2}{dt^2}\left(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\right)\Big|_{t=t^n} - \frac{k}{24h}w_{xxx}(x_{j+\frac{1}{2}},t^n)[f'(w(x_{j+\frac{1}{2}},t^n))]^2. \quad (2.3.34)$$

Case 2.2.

$$s_j = 0. \quad (2.3.35)$$

Recall that the method is just the first order Godunov's scheme. So rewrite

$$G[\overline{w}^n;j] = f\left(\overline{w}_j^n\right) = \overline{f}_{j+\frac{1}{2}}^n + f\left(\overline{w}_j^n\right) - \overline{f}_{j+\frac{1}{2}}^n. \quad (2.3.36)$$

But

$$f\left(\overline{w}_j^n\right) = f(w(x_{j+\frac{1}{2}},t^n)) + f'(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\right] + \frac{1}{2}f''(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\right]^2 + \frac{1}{6}f'''(w(x_{j+\frac{1}{2}},t^n))\left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\right]^3 + O\left(\left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\right]^4\right).$$

Using

$$\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n) = -\frac{1}{2}w_x(x_{j+\frac{1}{2}},t^n)h + \frac{1}{6}w_{xx}(x_{j+\frac{1}{2}},t^n)h^2 - \frac{1}{24}w_{xxx}(x_{j+\frac{1}{2}},t^n)w_{xx}(x_{j+\frac{1}{2}},t^n)h^3 + O(h^4)$$

and

$$\left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\right]^2 = \frac{1}{4}w_x(x_{j+\frac{1}{2}},t^n)h^2 - \frac{1}{6}w_x(x_{j+\frac{1}{2}},t^n)w_{xx}(x_{j+\frac{1}{2}},t^n)h^3 + O(h^4)$$

$$\left[\overline{w}_j^n - w(x_{j+\frac{1}{2}},t^n)\right]^3 = -\frac{1}{8}w_x(x_{j+\frac{1}{2}},t^n)h^3 + O(h^4)$$

we get

$$
\begin{aligned}
f\left(\overline{w}_j^n\right) &= f(w(x_{j+\frac{1}{2}},t^n)) - \frac{1}{2}f'(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)h \\
&+ \left[\frac{1}{6}f'(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n) + \frac{1}{8}f''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)\right]h^2 \\
&- \left[\frac{1}{24}f'(w(x_{j+\frac{1}{2}},t^n))w_{xxx}(x_{j+\frac{1}{2}},t^n) + \frac{1}{12}f''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)w_{xx}(x_{j+\frac{1}{2}},t^n)\right. \\
&\quad \left. + \frac{1}{48}f'''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)\right]h^3 + O\left(h^4\right).
\end{aligned} \tag{2.3.37}
$$

Hence substituting (2.3.37) and (2.3.15) into (2.3.36) gives

$$
\begin{aligned}
G[\overline{w}^n;j] &= \overline{f}_{j+\frac{1}{2}}^n + f(w(x_{j+\frac{1}{2}},t^n)) - \frac{1}{2}f'(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)h \\
&+ \left[\frac{1}{6}f'(w(x_{j+\frac{1}{2}},t^n))w_{xx}(x_{j+\frac{1}{2}},t^n) + \frac{1}{8}f''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)\right]h^2 \\
&- \left[\frac{1}{24}f'(w(x_{j+\frac{1}{2}},t^n))w_{xxx}(x_{j+\frac{1}{2}},t^n)\right. \\
&\quad + \frac{1}{12}f''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)w_{xx}(x_{j+\frac{1}{2}},t^n) \\
&\quad \left. + \frac{1}{48}f'''(w(x_{j+\frac{1}{2}},t^n))w_x(x_{j+\frac{1}{2}},t^n)\right]h^3 + O\left(h^4\right) \\
&- \left[f(w(x_{j+\frac{1}{2}},t^n)) + f'(w(x_{j+\frac{1}{2}},t^n))w_t(x_{j+\frac{1}{2}},t^n)\frac{k}{2}\right. \\
&\quad + \frac{k^2}{6}\frac{d}{dt}\left(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\right)\Big|_{t=t^n} \\
&\quad \left. + \frac{k^3}{24}\frac{d^2}{dt^2}\left(f'(w(x_{j+\frac{1}{2}},t))w_t(x_{j+\frac{1}{2}},t)\right)\Big|_{t=t^n} + O\left(k^4\right)\right]
\end{aligned}
$$

which can be simplified to

$$\begin{aligned}
G[\overline{w}^n; j] &= \overline{f}^n_{j+\frac{1}{2}} - \frac{1}{2} f'(w(x_{j+\frac{1}{2}}, t^n)) w_x(x_{j+\frac{1}{2}}, t^n) h \\
&+ \left[\frac{1}{6} f'(w(x_{j+\frac{1}{2}}, t^n)) w_{xx}(x_{j+\frac{1}{2}}, t^n) + \frac{1}{8} f''(w(x_{j+\frac{1}{2}}, t^n)) w_x(x_{j+\frac{1}{2}}, t^n) \right] h^2 \\
&- \left[\frac{1}{24} f'(w(x_{j+\frac{1}{2}}, t^n)) w_{xxx}(x_{j+\frac{1}{2}}, t^n) \right. \\
&\quad + \frac{1}{12} f''(w(x_{j+\frac{1}{2}}, t^n)) w_x(x_{j+\frac{1}{2}}, t^n) w_{xx}(x_{j+\frac{1}{2}}, t^n) \\
&\quad \left. + \frac{1}{48} f'''(w(x_{j+\frac{1}{2}}, t^n)) w_x(x_{j+\frac{1}{2}}, t^n) \right] h^3 + O\left(h^4\right) \\
&- \left[+ f'(w(x_{j+\frac{1}{2}}, t^n)) w_t(x_{j+\frac{1}{2}}, t^n) \frac{k}{2} \right. \\
&\quad + \frac{k^2}{6} \frac{d}{dt} \left(f'(w(x_{j+\frac{1}{2}}, t)) w_t(x_{j+\frac{1}{2}}, t) \right) \Big|_{t=t^n} \\
&\quad \left. + \frac{k^3}{24} \frac{d^2}{dt^2} \left(f'(w(x_{j+\frac{1}{2}}, t)) w_t(x_{j+\frac{1}{2}}, t) \right) \Big|_{t=t^n} + O\left(k^4\right) \right]. \qquad (2.3.38)
\end{aligned}$$

We can write (2.3.38) as

$$G[\overline{w}^n; j] = \overline{f}^n_{j+\frac{1}{2}} + \alpha(x_{j+\frac{1}{2}}, t^n) h + \beta(x_{j+\frac{1}{2}}, t^n) h^2 + O(h^3) \qquad (2.3.39)$$

where

$$\begin{aligned}
\alpha(x_{j+\frac{1}{2}}, t^n) &= -\frac{1}{2} f'(w(x_{j+\frac{1}{2}}, t^n)) w_x(x_{j+\frac{1}{2}}, t^n) \\
&\quad - \frac{k}{2h} f'(w(x_{j+\frac{1}{2}}, t^n)) w_t(x_{j+\frac{1}{2}}, t^n) \qquad (2.3.40)
\end{aligned}$$

and

$$\begin{aligned}
\beta(x_{j+\frac{1}{2}}, t^n) &= \frac{1}{6} f'(w(x_{j+\frac{1}{2}}, t^n)) w_{xx}(x_{j+\frac{1}{2}}, t^n) + \frac{1}{8} f''(w(x_{j+\frac{1}{2}}, t^n)) w_x(x_{j+\frac{1}{2}}, t^n) \\
&\quad - \frac{k^2}{6h^2} \frac{d}{dt} \left(f'(w(x_{j+\frac{1}{2}}, t)) w_t(x_{j+\frac{1}{2}}, t) \right) \Big|_{t=t^n} \qquad (2.3.41)
\end{aligned}$$

Chapter 3

Entropy

3.1 General

It was observed in Chapter 1.3 that weak solutions of (1.1.1) may not be unique in general. An additional condition is required to pick out the physically relevant vanishing viscosity solution. Geometrically, this requires the characteristics to propagate toward the shocks. Note that

$$\text{classical solution} \subset \text{entropy solution} \subset \text{weak solution}. \tag{3.1.1}$$

The first form is due to Oleinik. Assume $f'' > 0$.

Definition 3.1. *$u(x,t)$ is the entropy solution to (1.1.1) if*

$$\frac{u(x+a,t)-u(x,t)}{a} \le \frac{E}{t}, \qquad a>0,\ t>0,\ x\in\mathbb{R}, \tag{3.1.2}$$

where E is independent of x, t and a.

Implications of the the condition are that for fixed t and letting x range from $-\infty$ to ∞, one can only jump down, that is, in one direction across a discontinuity. Intuitively, a solution should satisfy (3.1.2) because from example A.1, we have that for smooth solutions $u_0' \ge 0$ and

$$u_x = \frac{u_0'}{1+tu_0'f''(u)}. \tag{3.1.3}$$

If $u_0' > 0$ then

$$u_x \le \frac{u_0'}{tu_0'f''(u)} = \frac{1}{tf''(u)} \le \frac{E}{t}, \tag{3.1.4}$$

where

$$E = \frac{1}{\inf f''}.$$

Recall the jump condition

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} = f'(\xi), \tag{3.1.5}$$

where ξ is between u_l and u_r. An entropy condition, also due to Oleinik, for an arbitrary scalar flux function f is given by the following definition.

Definition 3.2. *$u(x,t)$ is the entropy solution if all discontinuities have the property that*

$$\frac{f(u) - f(u_l)}{u - u_l} \geq s \geq \frac{f(u) - f(u_r)}{u - u_r} \tag{3.1.6}$$

for all u between u_l and u_r.

Hence for $f'' > 0$ we have the entropy condition

$$f'(u_l) > s > f'(u_r). \tag{3.1.7}$$

Across shock discontinuities we must have $u_l > u_r$.

3.2 Entropy/Entropy-flux pairs

Definition 3.3. *Two smooth functions $\Phi, \Psi : \mathbb{R} \to \mathbb{R}$ comprise an entropy/entropy-flux pair for the conservation law $u_t + f(u)_x = 0$ provided*

$$\Phi \text{ is convex}$$

and

$$\Phi'(y)f'(y) = \Psi'(y), \qquad y \in \mathbb{R}$$

For each entropy/entropy-flux pair Φ, Ψ the entropy condition for $u(x,t)$ is

$$\Phi(u)_t + \Psi(u)_x \leq 0 \qquad \text{on } \mathbb{R} \times (0, \infty).$$

This means that for each non-negative test function $\phi \in C_0^\infty(\mathbb{R} \times (0,\infty))$

$$\int_0^\infty \int_{-\infty}^\infty [\Phi(u)\phi_t + \Psi(u)\phi_x] dxdt \geq 0. \tag{3.2.1}$$

Definition 3.4. $u \in C([0,\infty), L^1(\mathbb{R})) \bigcap L^\infty(\mathbb{R}\times(0,\infty))$ *is called an entropy solution of*

$$\begin{aligned} u_t + f(u)_x &= 0, \qquad (x,t) \in \mathbb{R}\times(0,\infty) && (3.2.2) \\ u(x,0) &= u_0(x) && (3.2.3) \end{aligned}$$

if it satisfies (3.2.1) for each entropy/entropy-flux pair (Φ, Ψ), *and* $u(\cdot,t) \to u_0(x)$ *in* L^1 *as* $t \to 0$.

Example 3.1. *For any convex* Φ *we can find a corresponding flux function* Ψ, *namely*

$$\Psi(y) = \int_{y_0}^{y} \Phi'(w) f'(w) dw, \qquad y \in \mathbb{R}.$$

Theorem 3.1. *For the single conservation law (3.2.2), there exists - up to a set of measure zero - a unique entropy solution.*

For a proof of the above theorem and general entropy definitions for systems of conservation laws see for example [1] (proof due to Kruzkov).

We state a theorem due to Lax and Wendroff on convergence to a weak solution.

Theorem 3.2. *Consider a sequence of grids indexed by* $l = 1, 2, \cdots$, *with mesh parameters* $k_l, h_l \to 0$ *as* $l \to \infty$. *Let* $U_l(x,t)$ *denote the numerical approximation computed with a consistent (definition 1.9) and conservative (definition 1.8) method on the* l^{th} *grid. Suppose* U_l *converges to a function* u *as* $l \to \infty$, *in the sense:*

- *Over every bounded set* $\Omega = [a,b] \times [0,T]$ *in* $x - t$ *space*

$$\int_0^T \int_a^b |U_l(x,t) - u(x,t)| dx dt \to 0 \quad \text{as } l \to \infty. \tag{3.2.4}$$

- *also assume that for each* T *there is an* $R > 0$ *such that the total variation (definition 1.6)*

$$TV(U_l(\cdot,t)) < R \qquad \text{for all } 0 \le t \le T, \quad l = 1, 2, \cdots \tag{3.2.5}$$

Then $u(x,t)$ *is a weak solution of the conservative law.*

For a proof to Theorem 3.2 see [8, 9].

Now in order to show that the weak solution $u(x,t)$ obtained as a limit of our numerical solutions $U_l(x,t)$ satisfies (3.2.1), it suffices to show that a discrete entropy inequality holds, of the form

$$\Phi(U_j^{n+1}) \leq \Phi(U_j^n) - \lambda\left[\Theta(U^n;j) - \Theta(U^n;j-1)\right] \tag{3.2.6}$$

where Θ is some numerical entropy flux function that must be consistent with Ψ in the same manner that the numerical flux is required to be consistent with the true flux. So provided it can be shown that (3.2.6) holds for a suitable Θ, then following the lines of the proof of the Lax-Wendroff Theorem, one can show that the limiting weak solution $u(x,t)$ obtained as the grid is refined satisfies the entropy inequality (3.2.1).

Chapter 4

Numerical Results

4.1 Linear Conservation Laws

Example 4.1.

$$u_t + u_x = 0 \qquad x \in \mathbb{R}, \quad t \geq 0 \tag{4.1.1}$$

with initial value

$$u_0(x) = \begin{cases} \sqrt{1 - \left(\frac{x-0.5}{0.15}\right)^2} & 0.35 \leq x \leq 0.65 \\ 0 & \textit{otherwise} \end{cases} \tag{4.1.2}$$

assumed to be periodic with period one.

The problem is solved using the second and third order ENO schemes, Goodman-LeVeque scheme and lastly the LEM applied to the Goodman-LeVeque scheme. For the LEM, the parameter λ is fixed at 0.8. Numerical solutions were obtained for time $T = 0.96$.

From Figure 4.1 where the spatial step-size used was $h = 0.05$, it is not easy to see the improvements to the computed solution obtained by applying the LEM. In fact, the third order ENO scheme is much more accurate than the LEM applied to Goodman-LeVeque scheme. Decreasing the spatial step-length to $h = 0.02$ we get the results given in Figure 4.2. In this case the improvements achieved by applying the LEM to the Goodman-LeVeque

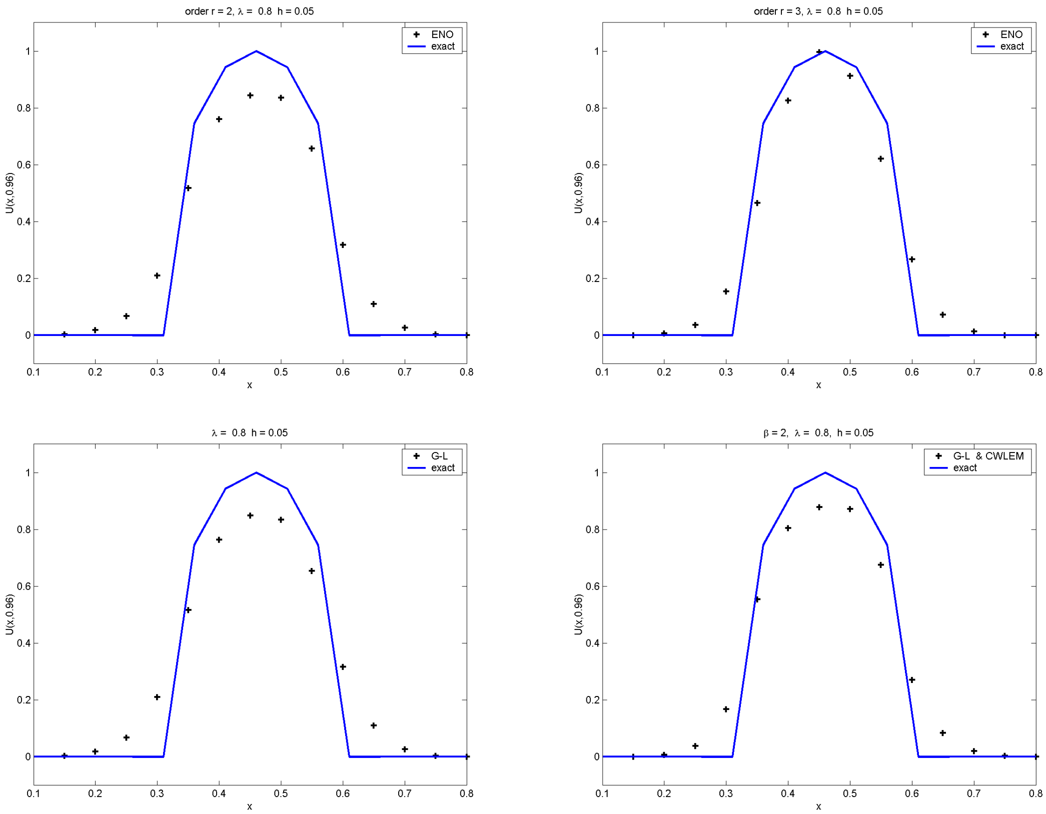

Figure 4.1: *Numerical results for example 4.1 with $h = 0.05$*

scheme are noticeable. The LEM applied to Goodman-LeVeque scheme captures the solution profile about the discontinuities much better than the second order ENO scheme or just the Goodman-LeVeque scheme. In fact it seems to be better than the third order ENO scheme in this respect. Note that the wave head seems to travel faster than the tail when the LEM is applied to the Goodman-LeVeque scheme. Numerical examples for the conservation law $u_t + u_x = 0$ show that the diffusion/compression effect on the head of the wave is different from that on the tail. The wave pattern may be distorted by this. To compensate for this, see solution in [18].

Further more, mean computation times were obtained by executing each scheme 500 times. The results are given in Table 4.1. Comparing the computational time for the Goodman-LeVeque method and the second order ENO scheme we observe that the com-

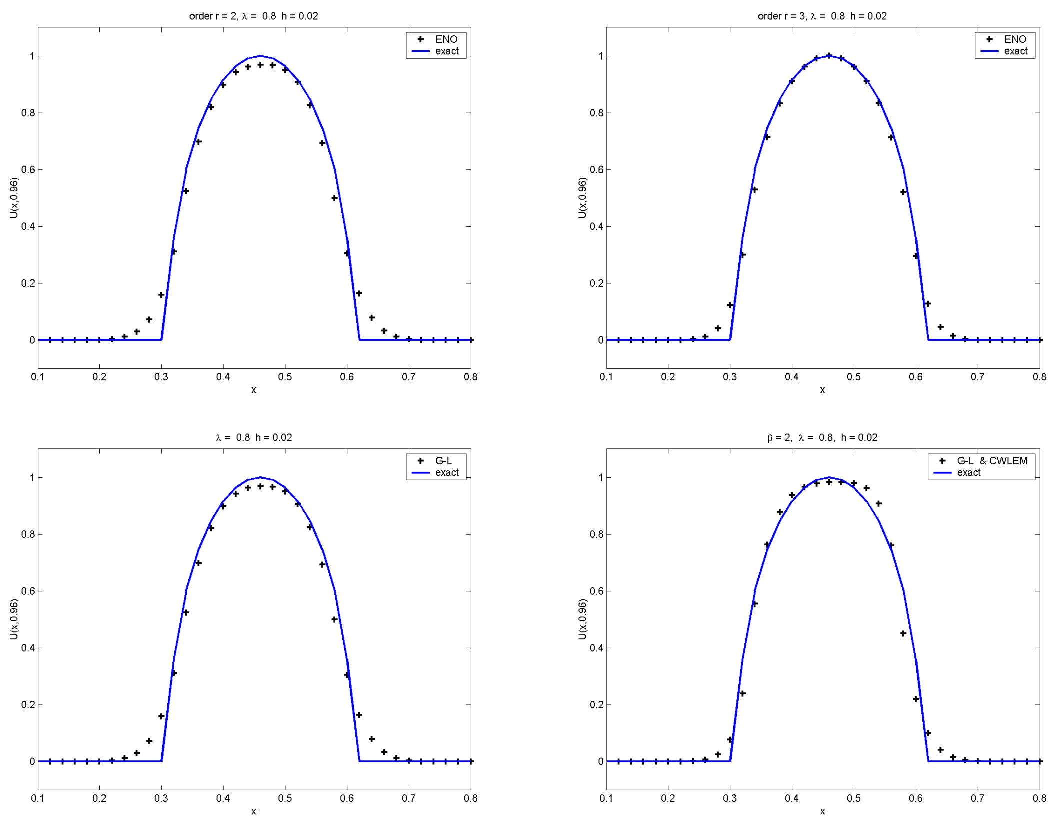

Figure 4.2: *Numerical results for example 4.1 with h = 0.02*

putation time in the former case is smaller. Given its smaller computational time and the ease with which it is implemented, the Goodman-LeVeque scheme is preferable over the ENO scheme. Now comparing third order ENO scheme and the Goodman-LeVeque scheme with LEM, we observe that the computational time of the latter is smaller. The second order ENO scheme with LEM has a better computational time but does not compare favorably to the computation time of the Goodman-LeVeque scheme with LEM. Hence the Goodman-LeVeque scheme with LEM is computation-wise less expensive compared to the ENO scheme of the same order ($r = 3$). The ENO scheme has a high computational time arising from determining the collection of interpolation points for which the interpolating polynomial is smoothest, as described on page 13.

Note that the standard deviations are small meaning that the actual computation times

scheme	λ	h	*cpu time*	*standard deviation*
ENO 2^{nd} order	0.8	0.5	0.7673	0.0460
		0.2	4.1973	0.2404
ENO 2^{nd} order + LEM	0.8	0.5	0.5556	0.0460
		0.2	2.3310	0.1126
ENO 3^{rd} order	0.8	0.5	0.8884	0.0497
		0.2	4.9126	0.2342
Goodman-LeVeque	0.8	0.5	0.1508	0.0247
		0.2	0.3809	0.0280
Goodman-LeVeque + LEM	0.8	0.5	0.1955	0.0245
		0.2	0.6641	0.0302

Table 4.1: *mean cpu times for example 4.1*

are close to the mean.

4.2 Nonlinear Conservation Laws

Example 4.2.

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad x \in \mathbb{R}, \quad t \geq 0 \tag{4.2.1}$$

with initial value

$$u_0(x) = \begin{cases} \sqrt{1 - \left(\frac{x-0.5}{0.15}\right)^2} & 0.35 \leq x \leq 0.65 \\ 0 & \textit{otherwise} \end{cases} \tag{4.2.2}$$

assumed to be periodic with period one.

In this example, we consider Burgers' equation, the initial value being a semi-ellipse. Figures 4.3 and 4.4 give us the solutions at time $T = 0.16$ when Goodman-LeVeque scheme and Goodman-LeVeque scheme with LEM respectively are used. The spatial step size in this case is $h = 0.05$. The Goodman-LeVeque scheme approximates relatively well the part of the wave that moves with positive speed (the part moving to the right).

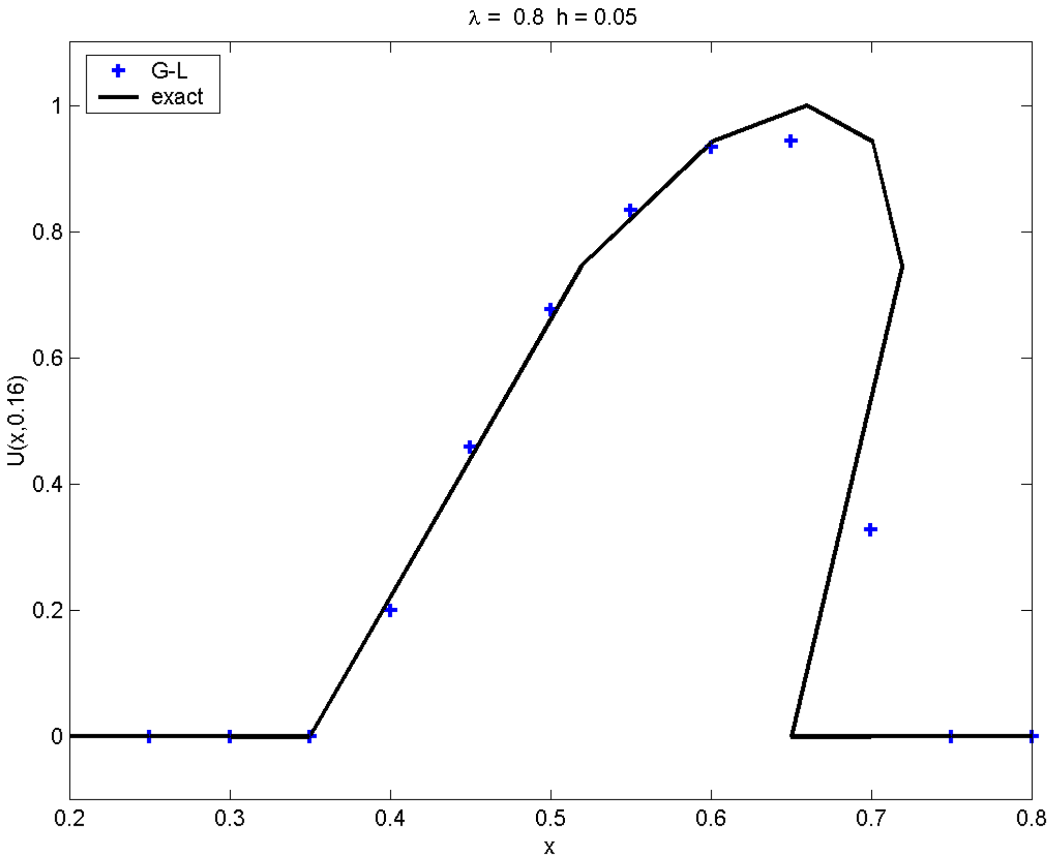

Figure 4.3: *Numerical results for example 4.2 with $h = 0.05$ - GL scheme*

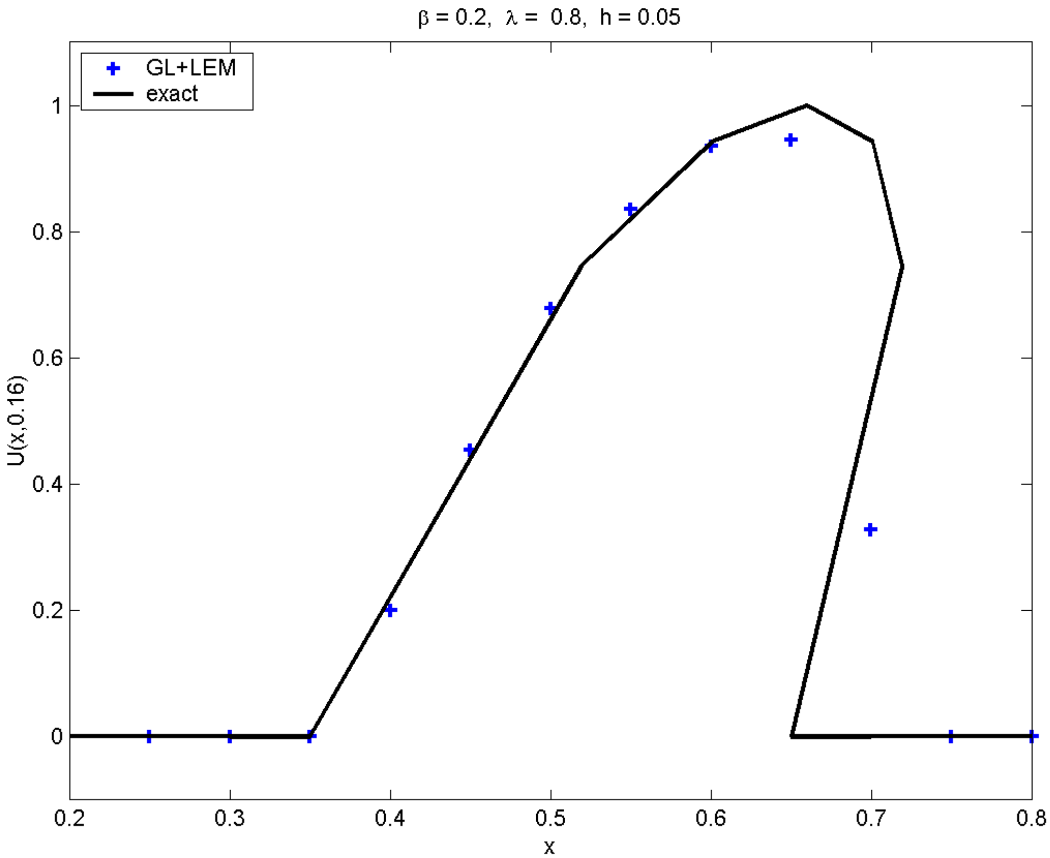

Figure 4.4: *Numerical results for example 4.2 with* $h = 0.05$, $\beta = 0.2$ *- GL + LEM scheme*

We use a smaller spatial step size, $h = 0.2$ and compute the numerical solution at time $T = 0.096$. The results are given in Figures 4.5 and 4.6. The numerical solution by the Goodman-LeVeque scheme approximates the exact solution very well as can be seen from Figure 4.5. From the plots, the improvements as a result of applying the LEM to the Goodman-LeVeque scheme are not pronounced.

We also observe that it does not take long for a discontinuity to form on the right side of the wave.

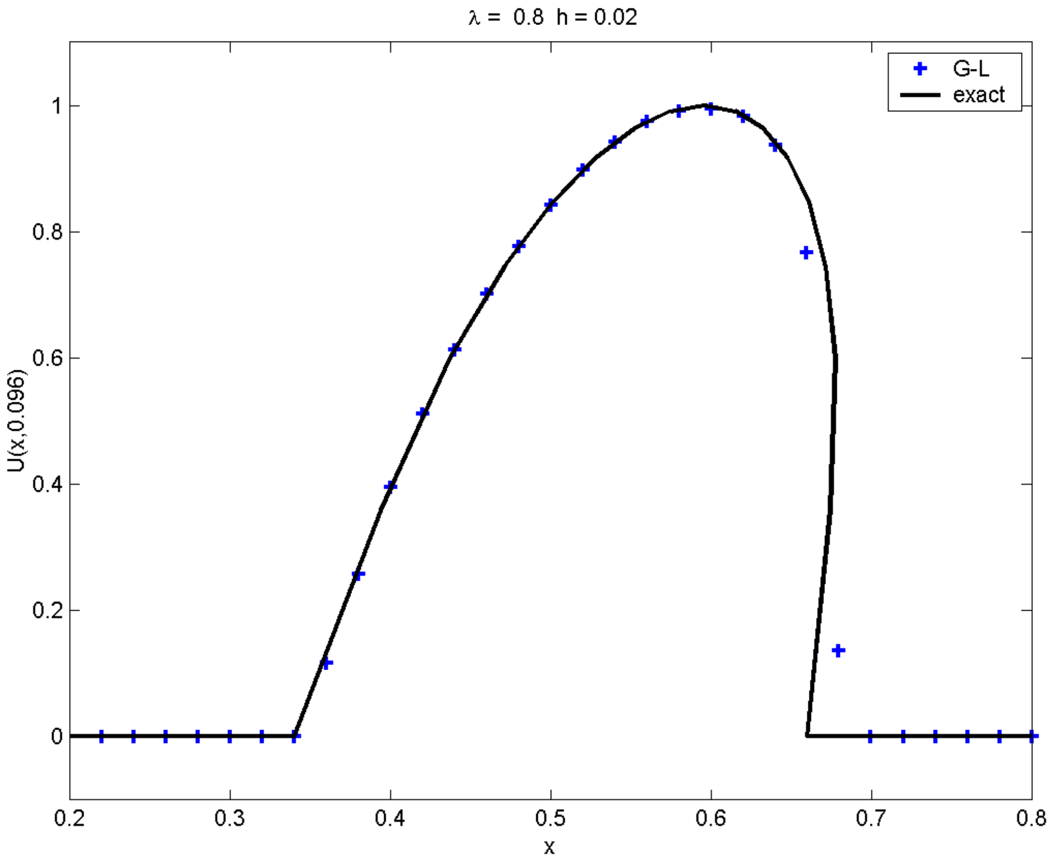

Figure 4.5: *Numerical results for example 4.2 with $h = 0.02$ - GL scheme*

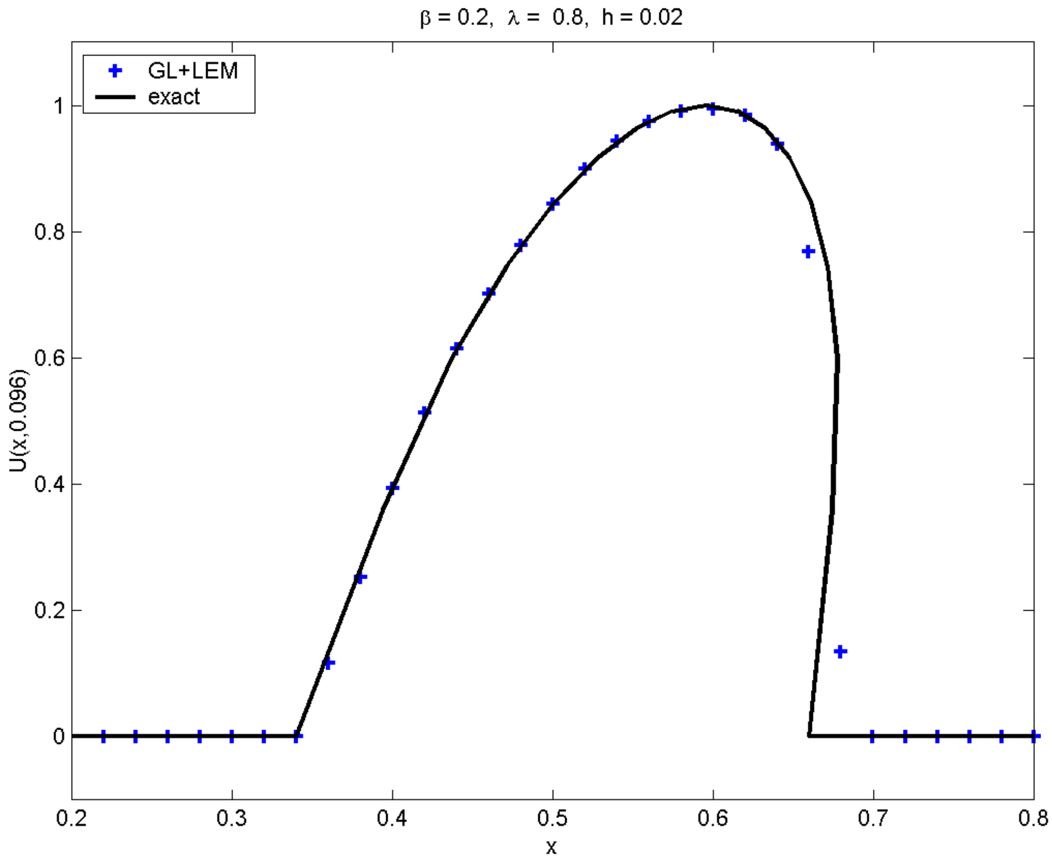

Figure 4.6: *Numerical results for example 4.2 with h = 0.02, β = 0.2 - GL + LEM scheme*

Example 4.3.

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad x \in \mathbb{R}, \quad t \geq 0 \tag{4.2.3}$$

with initial value

$$u_0(x) = \begin{cases} -0.5, & x \in (0, 0.5) \bigcup (1.5, 2) \\ 0.2 + 0.7\cos(2\pi x), & x \in (0.5, 1.5) \end{cases} \tag{4.2.4}$$

assumed periodic with period two.

In the last example, we once more consider Burgers' equation but with the initial condition being a cosine wave. First, the solution is computed at time $T = 0.24$ with a spatial step size of $h = 0.05$. Comparing Figures 4.7 and 4.8, representing the Goodman-LeVeque scheme and the Goodman-LeVeque scheme with LEM respectively, we can atleast find points of improvement when the LEM is applied to the Goodman-LeVeque scheme. This was not the case for this spatial step size when the semi-elliptical initial condition was used.

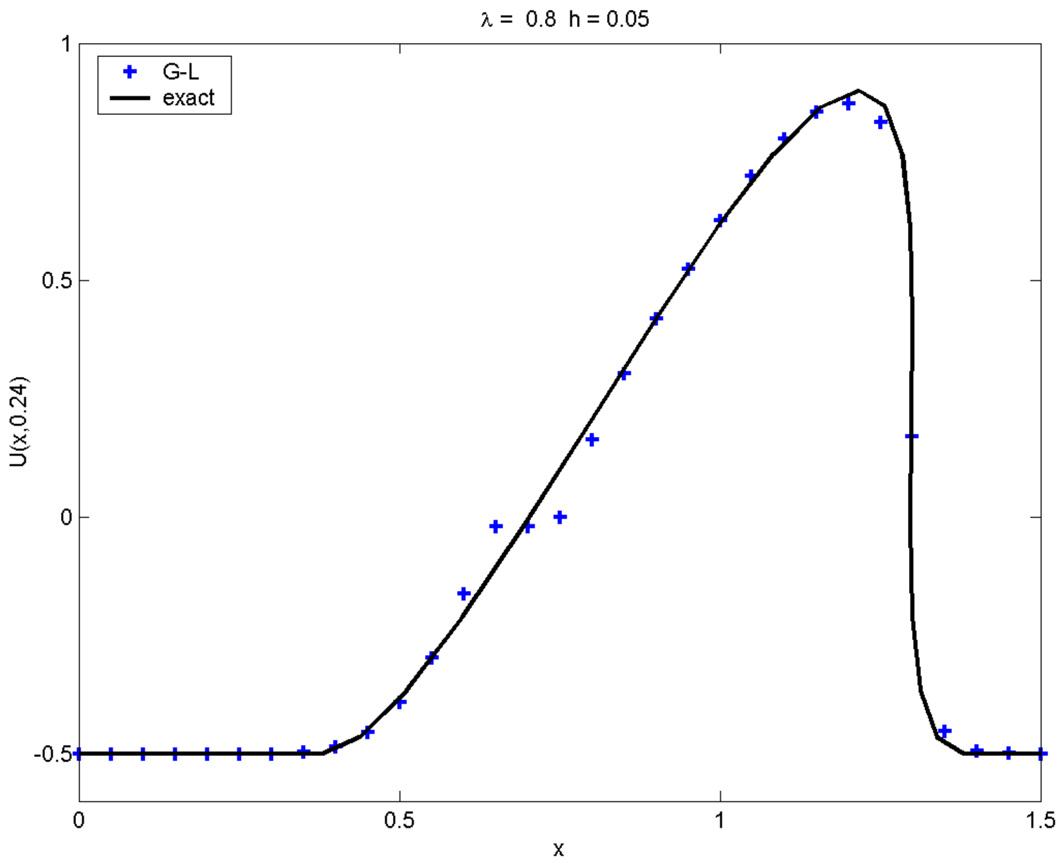

Figure 4.7: *Numerical results for example 4.3 with h = 0.05 - GL scheme*

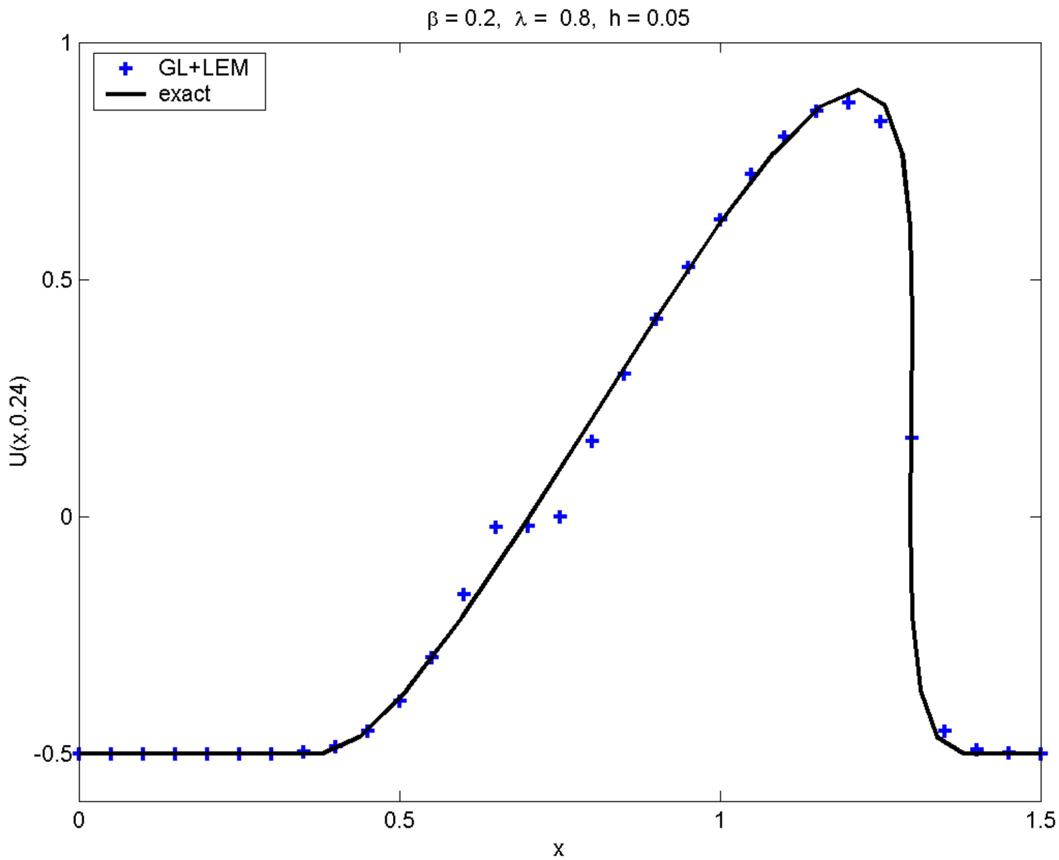

Figure 4.8: *Numerical results for example 4.3 with h = 0.05, β = 0.2 - GL + LEM scheme*

The step size is decreased to $h = 0.02$. The numerical solution using the Goodman-LeVeque scheme was computed at time $T = 0.24$ whereas the solution using the Goodman-LeVeque scheme with LEM was computed at $T = 0.224$, less than the former. The results are given in Figures 4.9 and 4.10. Applying the LEM yielded more noticeable improvements in this case. This is especially true around the peak of the wave.

A discontinuity forms on the right side of the wave at a very small value T. Of course this forms when the peak is directly opposite the part of the wave with value -0.5.

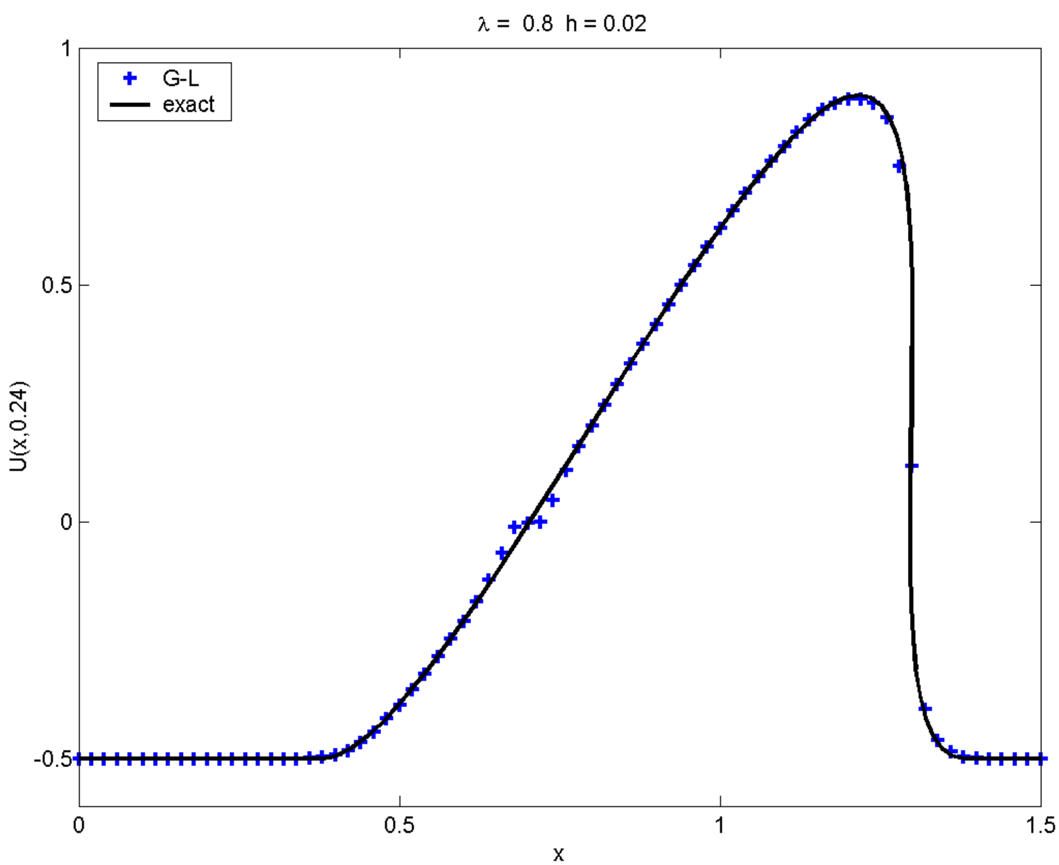

Figure 4.9: *Numerical results for example 4.3 with $h = 0.02$ - GL scheme*

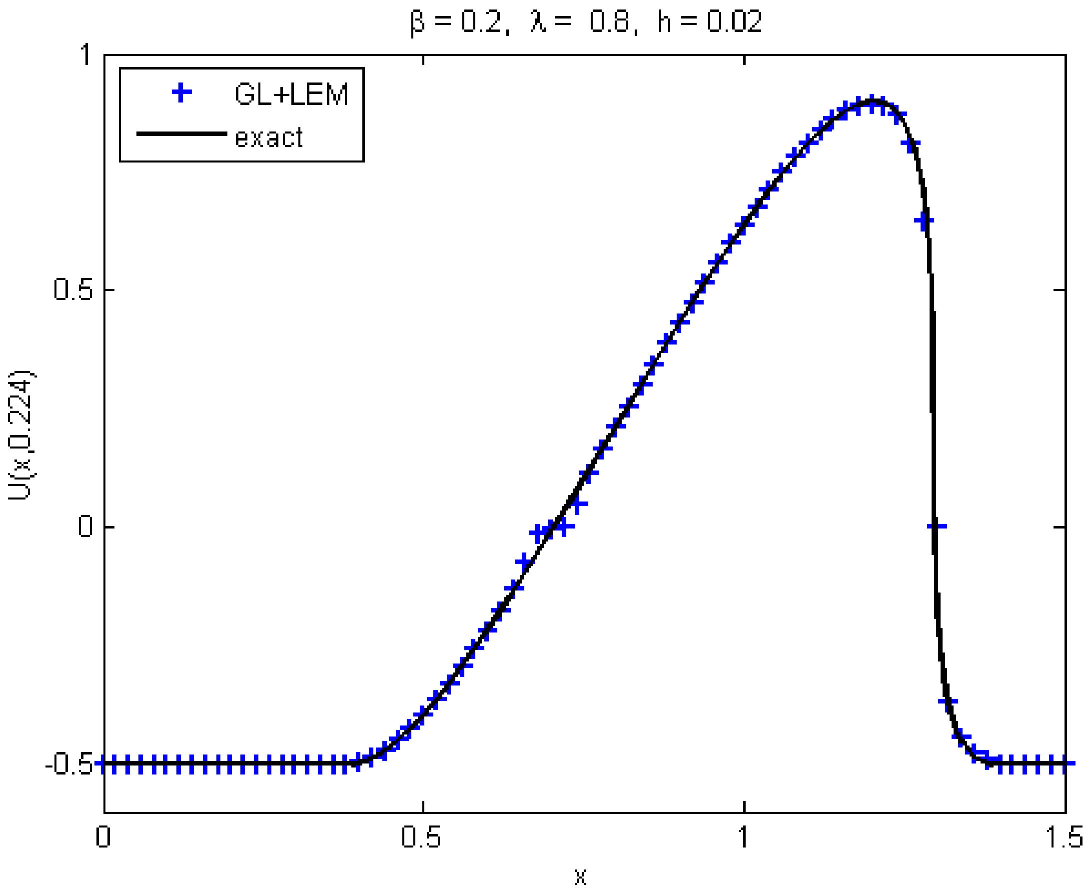

Figure 4.10: *Numerical results for example 4.3 with $h = 0.02$, $\beta = 0.2$ - GL + LEM scheme*

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Appendix A

Discontinuous Solutions of 1D Conservation Laws

Example A.1. *[13] Consider using the method of characteristics to solve (1.1.1). The characteristics are straight lines and since the solution u is constant along characteristics, it is implicitly given by*

$$u(x,t) = u_0(x - tf'(u(x,t))), \qquad t > 0. \tag{A.0.1}$$

Suppose u_0 is a differentiable function and taking t sufficiently small, then from (A.0.1) we get

$$u_t = u_0'[-f'(u) - tf''(u)u_t] \quad \textit{and} \quad u_x = u_0'[1 - tf''(u)u_x], \tag{A.0.2}$$

so that

$$u_t = -\frac{f'(u)u_0'}{1 + tu_0'f''(u)} \quad \textit{and} \quad u_x = \frac{u_0'}{1 + tu_0'f''(u)}. \tag{A.0.3}$$

Now assume that $f'' > 0$. If $u_0'(x) \geq 0$ for all x then by (A.0.3) we have ∇u is bounded for all $t > 0$ and the solution u exists for all time. If $u_0' < 0$ at some point, both u_t and u_x are unbounded for some finite value of t.

Example A.2. *Consider the Burger's equation, that is, $f(u) = \frac{1}{2}u^2$ in (1.1.1). Let the initial condition be $u_0 = \sin x, \quad 0 \leq x \leq 2\pi$. Since solution is constant along characteristics, the maxima of the sine wave travels to the right with speed 1 and the minima to the left with speed -1. The wave breaks into discontinuities at some finite time when the two fronts meet.*

Example A.3. *[12] Another example is to consider the Burgers' equation but with initial condition*

$$u(x,0) = u_0(x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ 0 & x \geq 1. \end{cases} \tag{A.0.4}$$

The characteristic line emanating from the point $(x_0, 0)$ *is given by*

$$x(t) = x_0 + tu_0(x_0) = \begin{cases} x_0 + t, & x_0 \leq 0, \\ x_0 + t(1 - x_0), & 0 \leq x_0 \leq 1, \\ x_0 & x_0 \geq 1. \end{cases} \tag{A.0.5}$$

The characteristic lines do not intersect only if $t < 1$. *The solution cannot be continuous at the intersection and in this case a classical solution does not exist.*

Appendix B

TVD Schemes

Lemma B.1. *Consider a scheme of the form*

$$U_j^{n+1} = U_j^n - C_{j-1}(U_j^n - U_{j-1}^n) - D_j(U_j^n - U_{j-2}^n). \tag{B.0.1}$$

Sufficient conditions for the scheme to be TVD are

$$\begin{aligned} 0 &\le D_j \quad \forall j \\ C_j + D_{j+1} &\le 1 \quad \forall j \\ 0 &\le C_{j-1} - D_{j+1} \quad \forall j. \end{aligned} \tag{B.0.2}$$

The proof is similar to the argument of Harten (see page 178 [8] or 116 [9]).

Proof.

$$\begin{aligned} U_{j+1}^{n+1} - U_j^{n+1} &= (1 - C_j)(U_{j+1}^n - U_j^n) + C_{j-1}(U_j^n - U_{j-1}^n) \\ &\quad - D_{j+1}(U_{j+1}^n - U_{j-1}^n) + D_j(U_j^n - U_{j-2}^n), \end{aligned} \tag{B.0.3}$$

that is

$$\begin{aligned} U_{j+1}^{n+1} - U_j^{n+1} &= (1 - C_j - D_{j+1})(U_{j+1}^n - U_j^n) + C_{j-1}(U_j^n - U_{j-1}^n) \\ &\quad - D_{j+1}(U_j^n - U_{j-1}^n) + D_j(U_j^n - U_{j-2}^n) \\ &= (1 - C_j - D_{j+1})(U_{j+1}^n - U_j^n) + (C_{j-1} - D_{j+1})(U_j^n - U_{j-1}^n) \\ &\quad + D_j(U_j^n - U_{j-2}^n). \end{aligned} \tag{B.0.4}$$

On taking absolute value of both sides and using (B.0.2) we get

$$\begin{aligned} |U_{j+1}^{n+1} - U_j^{n+1}| &\le (1 - C_j - D_{j+1})|U_{j+1}^n - U_j^n| + (C_{j-1} - D_{j+1})|U_j^n - U_{j-1}^n| \\ &\quad + D_j(|U_j^n - U_{j-1}^n| + |U_{j-1}^n - U_{j-2}^n|). \end{aligned} \tag{B.0.5}$$

Now summing over all j

$$\begin{aligned} \sum_{j=-\infty}^{\infty} |U_{j+1}^{n+1} - U_j^{n+1}| &\le \sum_{j=-\infty}^{\infty} (1 - C_j - D_{j+1})|U_{j+1}^n - U_j^n| \\ &\quad + \sum_{j=-\infty}^{\infty} (C_{j-1} - D_{j+1})|U_j^n - U_{j-1}^n| \\ &\quad + \sum_{j=-\infty}^{\infty} D_j(|U_j^n - U_{j-1}^n| + |U_{j-1}^n - U_{j-2}^n|). \end{aligned} \tag{B.0.6}$$

After re-indexing the last three summations on the right hand side of the equation above, we get

$$\begin{aligned} \sum_{j=-\infty}^{\infty} |U_{j+1}^{n+1} - U_j^{n+1}| &\le \sum_{j=-\infty}^{\infty} (1 - C_j - D_{j+1})|U_{j+1}^n - U_j^n| \\ &\quad + \sum_{j=-\infty}^{\infty} (C_j - D_{j+2})|U_{j+1}^n - U_j^n| \\ &\quad + \sum_{j=-\infty}^{\infty} (D_{j+1}|U_{j+1}^n - U_{j-1}^n| + D_{j+2}|U_{j+1}^n - U_j^n|) \end{aligned} \tag{B.0.7}$$

that is

$$\begin{aligned} \sum_{j=-\infty}^{\infty} |U_{j+1}^{n+1} - U_j^{n+1}| &\le \sum_{j=-\infty}^{\infty} (1 - C_j - D_{j+1} + C_j - D_{j+2} + D_{j+1} + D_{j+2})|U_{j+1}^n - U_j^n| \\ &= \sum_{j=-\infty}^{\infty} |U_{j+1}^n - U_j^n|. \end{aligned} \tag{B.0.8}$$

□

Note that the coefficients C_{j-1} and D_j in (B.0.1) depend on $\{U_j\}$s and a convenient form may not be easy to obtain. For example, from (2.2.35), the resulting scheme when the LEM is applied to the upwinding method is

$$\begin{aligned} U_j^{n+1} &= U_j^n - 2\lambda f'(\xi_j^n)\Delta U_{j-\frac{1}{2}}^n + \lambda^2 f'(\alpha_j^n) f'(\xi_j^n)\Delta U_{j-\frac{1}{2}}^n \\ &\quad -\lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)\Delta U_{j-\frac{3}{2}}^n - 2\lambda\left(g_{j+\frac{1}{2}}^{n-ext} - g_{j-\frac{1}{2}}^{n-ext}\right) \end{aligned} \tag{B.0.9}$$

which we can write as

$$\begin{aligned} U_j^{n+1} &= U_j^n - \left\{2\lambda f'(\xi_j^n) - \lambda^2 f'(\alpha_j^n) f'(\xi_j^n)\right\} [U_j - U_{j-1}] \\ &\quad -\lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)[U_{j-1} - U_{j-2}] - 2\lambda \left(g_{j+\frac{1}{2}}^{n-ext} - g_{j-\frac{1}{2}}^{n-ext}\right). \end{aligned} \tag{B.0.10}$$

Now note that

$$U_{j-1}^n - U_{j-2}^n = (U_{j-1}^n - U_j^n) + (U_j^n - U_{j-2}^n) = -(U_j^n - U_{j-1}^n) + (U_j^n - U_{j-2}^n).$$

So (B.0.10) becomes

$$\begin{aligned} U_j^{n+1} &= U_j^n - \left\{2\lambda f'(\xi_j^n) - \lambda^2 f'(\alpha_j^n) f'(\xi_j^n) - \lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)\right\} [U_j - U_{j-1}] \\ &\quad -\lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)[U_j - U_{j-2}] - 2\lambda \left(g_{j+\frac{1}{2}}^{n-ext} - g_{j-\frac{1}{2}}^{n-ext}\right). \end{aligned} \tag{B.0.11}$$

One may consider writing

$$\begin{aligned} U_j^{n+1} &= U_j^n - \left\{2\lambda f'(\xi_j^n) - \lambda^2 f'(\alpha_j^n) f'(\xi_j^n) - \lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)\right\} [U_j - U_{j-1}] \\ &\quad -\lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)[U_j - U_{j-2}] - \frac{2\lambda}{U_j - U_{j-2}} \left(g_{j+\frac{1}{2}}^{n-ext} - g_{j-\frac{1}{2}}^{n-ext}\right) [U_j - U_{j-2}] \end{aligned} \tag{B.0.12}$$

so that

$$\begin{aligned} U_j^{n+1} &= U_j^n - \left\{2\lambda f'(\xi_j^n) - \lambda^2 f'(\alpha_j^n) f'(\xi_j^n) - \lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)\right\} [U_j - U_{j-1}] \\ &\quad - \left\{\lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n) + \frac{2\lambda}{U_j - U_{j-2}} \left(g_{j+\frac{1}{2}}^{n-ext} - g_{j-\frac{1}{2}}^{n-ext}\right)\right\} [U_j - U_{j-2}]. \end{aligned} \tag{B.0.13}$$

Hence in this case we have that

$$C_{j-1} = 2\lambda f'(\xi_j^n) - \lambda^2 f'(\alpha_j^n) f'(\xi_j^n) - \lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n)$$

and

$$D_j = \lambda^2 f'(\alpha_{j-1}^n) f'(\xi_{j-1}^n) + \frac{2\lambda}{U_j - U_{j-2}} \left(g_{j+\frac{1}{2}}^{n-ext} - g_{j-\frac{1}{2}}^{n-ext}\right).$$

To show that the scheme (B.0.10) is TVD one would have to show that the conditions (B.0.2) are satisfied and this may not be easy.

Lemma B.2. *Let g be continuous, differentiable and increasing on an interval $[a,b]$. Let $\xi \in (a,b)$ and $\eta \in (\frac{a+b}{2}, b)$ such that $g'(\xi)$ is the slope of the secant line from the point $(a, g(a))$ to the point $(b, g(b))$ and $g'(\eta)$ is the slope of the secant from the point $(\frac{a+b}{2}, g\left(\frac{a+b}{2}\right))$ to the point $(b, g(b))$. Then*

$$g'(\eta) \le 2g'(\xi). \tag{B.0.14}$$

Proof. We note that

$$\begin{aligned} g'(\eta) &= \frac{g(b) - g\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} \\ &\le \frac{g(b) - g(a)}{b - \frac{a+b}{2}}, \end{aligned} \tag{B.0.15}$$

on using the assumption that g is increasing and therefore $g(a) < g\left(\frac{a+b}{2}\right)$. Hence

$$g'(\eta) \le 2\frac{g(b) - g(a)}{b - a} = 2g'(\xi). \tag{B.0.16}$$

□

Lemma B.3. *Let g be continuous, differentiable and increasing on an interval $[a,b]$. Let $\xi \in (a,b)$, $\gamma \in (a, \frac{a+b}{2})$ and $\eta \in (\frac{a+b}{2}, b)$ such that $g'(\xi)$ is the slope of the secant from the point $(a, g(a))$ to the point $(b, g(b))$, $g'(\gamma)$ is the slope of the secant from the point $(a, g(a))$ to the point $(\frac{a+b}{2}, g\left(\frac{a+b}{2}\right))$ and $g'(\eta)$ is the slope of the secant from the point $(\frac{a+b}{2}, g\left(\frac{a+b}{2}\right))$ to the point $(b, g(b))$. Then*

$$g'(\xi) = \frac{g'(\gamma) + g'(\eta)}{2}. \tag{B.0.17}$$

Proof. We have

$$\begin{aligned} \frac{g'(\gamma) + g'(\eta)}{2} &= \frac{1}{2}\left\{\frac{g\left(\frac{a+b}{2}\right) - g(a)}{\frac{a+b}{2} - a} + \frac{g(b) - g\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}}\right\} \\ &= \frac{g\left(\frac{a+b}{2}\right) - g(a)}{b - a} + \frac{g(b) - g\left(\frac{a+b}{2}\right)}{b - a} \\ &= \frac{g(b) - g(a)}{b - a} \\ &= g'(\xi). \end{aligned} \tag{B.0.18}$$

□