SOBOLEV SPACES

by

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Abstract

The goal for this paper is to present material from Gilbarg and Trudinger's *Elliptic Partial Differential Equations of Second Order* chapter 7 on Sobolev spaces, in a manner easily accessible to a beginning graduate student. The properties of weak derivatives and there relationship to conventional concepts from calculus are the main focus, that is when do weak and strong derivatives coincide. To enable the progression into the primary focus, the process of mollification is presented and is widely used in estimations. Imbedding theorems and compactness results are briefly covered in the final sections. Finally, we add some exercises at the end to illustrate the use of the ideas presented throughout the paper.

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Chapter 1

Sobolev Spaces

The motivation to the theory of this chapter comes from wanting a different approach to Poisson's equation. By the divergence theorem, a $C^2(\Omega)$ solution of $\Delta u = f$ satisfies the integral identity

$$\int\limits_\Omega Du \cdot D\varphi dx = -\int\limits_\Omega f\varphi dx \text{for all} \varphi \in C_0^1(\Omega) \tag{1.1}$$

The bilinear form

$$(u, \varphi) = \int\limits_\Omega Du \cdot D\varphi dx \tag{1.2}$$

is an inner product on the space $C_0^1(\Omega)$ and the completion of $C_0^1(\Omega)$ under the metric induced by (1.2) is consequently a Hilbert space, which we call $W_0^{1,2}(\Omega)$.

Furthermore, for appropriate f the linear functional F defined by $F(\varphi) = -\int_\Omega f\varphi dx$ may be extended to a bounded linear functional on $W_0^{1,2}(\Omega)$ satisfying $(u, \varphi) = F(\varphi)$ for all $\varphi \epsilon C_0^1(\Omega)$. Therefore, a generalized solution to the Dirichlet problem, $\Delta u = f$,$u = 0$ on $\partial\Omega$, is readily established. The question of classical existence is accordingly transformed into the question of regularity of generalized solutions under the appropriately smooth boundary conditions. We need to examine the class of Sobolev spaces, that is, the $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ spaces of which the space $W_0^{1,2}(\Omega)$ is a member. Some of the inequalities we treat will also be necessary for the development of the theory of quasilinear equations.

1.1 L^p Spaces

[1] Throughout this chapter Ω will denote a bounded domain in $\mathbb{R}^n$. By a measurable function on Ω we shall mean an equivalence class of measurable functions on Ω which differ only on a subset of measure zero. Any pointwise property attributed to a measurable function will thus be understood to hold in the usual sense for some function in the same equivalence class. The supremum and infimum of a measurable function will then be understood as the essential supremum and infimum. We will now introduce the main inequalities used throughout this chapter.

For $p \geq 1$, we let $\mathrm{L}^p(\Omega)$ denote the classical Banach space consisting of measurable functions on Ω that are p-integrable. The norm in $\mathrm{L}^p(\Omega)$ is defined by

$$\|u\|_{\mathrm{L}^p(\Omega)} = \left(\int_{\Omega} |u|^p dx \right)^{1/p} \tag{1.3}$$

When u is a vector or matrix function the same notation will be used, the norm $|u|$ denoting the usual Euclidean norm. For $p = \infty$, $\mathrm{L}^\infty(\Omega)$ denotes the Banach space of bounded functions on Ω with the norm

$$\|u\|_{\infty;\Omega} = \|u\|_{\mathrm{L}^\infty(\Omega)} = \sup_{\Omega} |u| \tag{1.4}$$

In the following we shall use $\|u\|_p$ for $\|u\|_{\mathrm{L}^p(\Omega)}$ when there is no ambiguity. We shall need the following inequalities in dealing with integral estimates: *Young's inequality.*

$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}; \tag{1.5}$$

this holds for positive real numbers a, b, p, q satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Proof.

$$\begin{aligned} ab &= \exp(\log(ab)) \\ &= \exp(\frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q)) \\ &\leq \frac{1}{p}\exp(\log(a^p)) + \frac{1}{q}exp(\log(b^q)) \qquad \text{by convexity of exp} \\ &= \frac{a^p}{p} + \frac{b^q}{q} \end{aligned}$$

□

The case $p = q = 2$ of inequality (1.5) is known as Cauchy's inequality. Replacing a by $\epsilon^{1/p}a$, b by $\epsilon^{-1/p}b$ for positive ϵ, we obtain a useful interpolation inequality

$$\begin{aligned} ab &\leq \frac{\epsilon a^p}{p} + \frac{\epsilon^{-q/p}b^q}{q} \\ &\leq \frac{\epsilon a^p}{p} + \frac{\epsilon a^p}{q} + \frac{\epsilon^{-q/p}b^q}{q} + \frac{\epsilon^{-q/p}b^q}{p} \\ &= (\frac{1}{p} + \frac{1}{q})(\epsilon a^p) + (\frac{1}{p} + \frac{1}{q})(\epsilon^{-q/p}b^q) \\ &= \epsilon a^p + \epsilon^{-q/p}b^q \end{aligned} \tag{1.6}$$

The following equation is commonly known as *Hölder's inequality*,

$$\int_\Omega uv dx \leq \|u\|_p \|v\|_q; \tag{1.7}$$

this holds for functions $u \in \mathrm{L}^p(\Omega), v \in \mathrm{L}^q(\Omega), 1/p + 1/q = 1$ and is a consequence of Young's inequality. When $p = q = 2$, Hölder's inequality reduces to the well known Schwarz inequality. The reason that the expression (1.3) defines a norm on $\mathrm{L}^p(\Omega)$ is a consequence of Hölder's inequality.

$$|\Omega|^{-1/p}\|u\|_p \leq |\Omega|^{-1/q}\|u\|_q \text{ for } u \in \mathrm{L}^q(\Omega), \ \ p \leq q \tag{1.8}$$

A useful application of Hölder's inequality and L^P spaces results in the following inequality:

$$\|u\|_q \leq \|u\|_p^{\lambda}\|u\|_r^{1-\lambda} \text{ for } u \in \mathrm{L}^r(\Omega), \ \text{ where } \ p \leq q \leq r \ \text{ and } \ 1/q = \lambda/p + (1-\lambda)/r \tag{1.9}$$

Proof.

$$\begin{aligned}
\|u\|_q^q &= \int_\Omega |u|^q dx \\
&= \int_\Omega |u|^{q-\alpha} \cdot |u|^\alpha dx && \text{where we will choose } \alpha \text{ later} \\
&\leq \left(\int_\Omega |u|^{\frac{(q-\alpha)p}{\lambda q}} dx\right)^{\frac{\lambda q}{p}} \left(\int_\Omega |u|^{\frac{\alpha r}{(1-\lambda)q}} dx\right)^{\frac{(1-\lambda)q}{r}} && \text{by Hölder} \\
&= \left(\int_\Omega |u|^p dx\right)^{\frac{\lambda q}{p}} \left(\int_\Omega |u|^r dx\right)^{\frac{(1-\lambda)q}{r}} && \text{letting } \alpha = (1-\lambda)q \\
&= \|u\|_p^{\lambda q}\|u\|_r^{(1-\lambda)q}
\end{aligned}$$

□

Combining inequalities (1.6) and (1.9), we obtain and interpolation inequality for L^p norms, namely,

$$\|u\|_q \leq \epsilon\|u\|_r + \epsilon^{-\mu}\|u\|_p, \ \text{ where } \ \mu = \frac{1/p - 1/q}{1/q - 1/r} \tag{1.10}$$

Proof. From (1.9) if we let $\|u\|_p^\lambda = b$ and $\|u\|_r^{1-\lambda} = a$ then from (1.6) there exists $m, l \in \mathbb{R}$ such that $ab \leq \ \epsilon a^l + \epsilon^{-m/l}b^m$. Letting $m = \frac{1}{\lambda}$ and $l = \frac{1}{1-\lambda}$ then $\mu = m/l = \frac{1-\lambda}{\lambda}$ and from (1.9) we have $\lambda = \frac{pr-pq}{qr-qp}$ and $1 - \lambda = \frac{qr-pr}{qr-pq}$ so that $\mu = \frac{qr-pr}{pr-pq} = \left(\frac{q-p}{pq}\right)\left(\frac{qr}{r-q}\right) = \frac{1/p-1/q}{1/q-1/r}$ □

We shall also have occasion to use a generalization of Hölder's inequality to m functions, $u_1, \dots, u_m$, lying respectively in spaces $L^{p_1}, \dots, L^{p_m}$, where $1/p_1 + \cdots + 1/p_m = 1$. The resulting inequality, obtainable from the case $m = 2$ by and induction argument, is then

$$\int_\Omega u_1 \cdots u_m dx \leq \|u_1\|_{p_1} \cdots \|u_m\|_{p_m} \tag{1.11}$$

It is also of interest to study the L^p norm as a function of p. Writing

$$\Phi_p(u) = \left(\frac{1}{|\Omega|} \int_\Omega |u|^p dx \right)^{1/p} \tag{1.12}$$

for $p > 0$, we see that Φ is non-decreasing in p for fixed u, by inequality (1.8), while the inequality (1.9) shows that Φ is logarithmically convex in p^{-1}. Note that $\Phi_p(u) = |\Omega|^{-1/p} \|u\|_p$ for $p \geq 1$. Although the functional Φ does not extend the L^p norm as a norm for values of p less than one, it will nevertheless be useful for later purposes.

We also note here some of the well known functional analytic properties of the L^P spaces. The space $L^p(\Omega)$ is separable for $p < \infty$, $C^0(\bar{\Omega})$ being in particular a dense subspace. The dual space of $L^p(\Omega)$ is isomorphic to $L^q(\Omega)$ provided $1/p + 1/q = 1$ and $p < \infty$. Hence $L^p(\Omega)$ is reflexive for $1 < p < \infty$. The number q, the *Hölder conjugate* of p, will often be denoted p'. Finally, $L^2(\Omega)$ is a *Hilbert space* under the scalar product

$$(u, v) = \int_\Omega uv dx \tag{1.13}$$

1.2 Regularization and Approximation by Smooth Functions

Let us define local analogues of the $L^p(\Omega)$ spaces by letting $L^p_{loc}(\Omega)$ denote the linear space of measurable functions locally p-integrable in Ω. Although they are not normed spaces are readily topologized. Namely, a sequence $\{u_m\}$ converges to u in the sense of $L^p_{loc}(\Omega)$ if $\{u_m\}$ converges to u in $L^p(\Omega')$ for each $\Omega' \subset\subset \Omega$.

Let ρ be a non-negative function in $C^\infty(\mathbb{R}^n)$ vanishing outside the unit ball $B_1(0)$ and satisfying $\int \rho \, \mathrm{d}x = 1$. Such a function is often called a *mollifier*. A typical example is the function ρ given by

$$\rho(x) = \begin{cases} c \; exp\left(\frac{1}{|x|^2-1}\right) & \text{for} |x| \leq 1 \\ 0 \quad \text{for} |x| \geq 1 \end{cases}$$

where c is chosen so that $\int \rho \, \mathrm{d}x = 1$ and whose graph has the familiar bell shape.

For $u \in L^1_{loc}(\Omega)$ and $h > 0$, the *regularization* of u, denoted by u_h, is then defined by the convolution

$$u_h(x) = h^{-n} \int\limits_{\Omega} \rho\left(\frac{x-y}{h}\right) u(y) \, \mathrm{d}y \tag{1.14}$$

provided $h <$dist$(x, \partial\Omega)$. It is clear that u_h belongs to $C^\infty(\Omega')$ for any $\Omega' \subset\subset \Omega$ provided $h <$dist$(\Omega', \partial\Omega)$. Furthermore, if u belongs to $L^1(\Omega)$, Ω bounded, the u_h lies in $C^\infty_0(\mathbb{R}^n)$ for arbitrary $h > 0$. As h tends to zero, the function $y \mapsto h^{-n}\rho(x - y/h)$ tends to the Dirac delta distribution at the point x. The significant feature of regularization, which we partly explore now, is the sense in which u_h approximates u as h tends to zero. It turns out, roughly stated, that if u lies in a local space, then u_h approximates u in the natural topology of that space.

Definition For any domain $\Omega \in \mathbb{R}^n$ we say Ω' is *strongly contained* in Ω if $\bar{\Omega}' \subset \Omega$ is compact. For this we write $\Omega' \subset\subset \Omega$.

Lemma 1.2.1. *Let $u \in C^0(\Omega)$. Then u_h converges to u uniformly on any domain $\Omega' \subset\subset \Omega$*

Proof. We have

$$u_h(x) = h^{-n} \int\limits_{|x-y|\le h} \rho\left(\frac{x-y}{h}\right) u(y)\,\mathrm{d}y$$
$$= \int\limits_{|z|\le 1} \rho(z)u(x-hz)\,\mathrm{d}z \qquad\qquad \text{putting } z = \frac{x-y}{h}$$

notice that the Jacobian, when making this change of variables, is h^n hence eliminating the h^{-n}. Therefore, if $\Omega' \subset\subset \Omega$ and $2h < \mathrm{dist}(\Omega', \partial\Omega)$, we have $|u - u_h| = |\int\limits_{|z|\le 1} \rho(z)u(x)\,\mathrm{d}z - \int\limits_{|z|\le 1} \rho(z)u(x-hz)\,\mathrm{d}z| \le \int\limits_{|z|\le 1} \rho(z)|u(x) - u(x-hz)|\,\mathrm{d}z$ so that

$$\sup_{\Omega'} |u - u_h| \le \sup_{x\in\Omega'} \int\limits_{|z|\le 1} \rho(z)|u(x) - u(x-hz)|\,\mathrm{d}z$$
$$\le \sup_{x\in\Omega} \sup_{|z|\le 1} |u(x) - u(x-hz)|.$$

Since u is uniformly continuous over the set $B_h(\Omega') = \{x|\mathrm{dist}(x,\Omega') < h\}$, u_h tends to u uniformly on Ω'. $\square$

The convergence in (1.2.1) would be uniform over all of Ω if u vanished continuously on $\partial\Omega$. More generally if $u \in C^0(\overline{\Omega})$ we can define an extension $\tilde{u}$ of u such that $\tilde{u} = u$ in Ω and $\tilde{u} \in C^0(\tilde{\Omega})$ for some $\tilde{\Omega} \supset\supset \Omega$. Then $\tilde{u}_h$, the regularization of $\tilde{u}$ in $\tilde{\Omega}$, converges to u uniformly in Ω as $h \to 0$.

The process of regularization can also be used to approximate Hölder continuous functions. In particular, if $u \in C^\alpha(\Omega)$, $0 \le \alpha \le 1$, then

$$[u_h]_{\alpha;\Omega'} \le [u]_{\alpha;\Omega''} \tag{1.15}$$

where $\Omega'' = B_h(\Omega')$ and consequently u_h tends to u in the sense of $C^{\alpha'}(\Omega')$ for every $\alpha' < \alpha$ and $\Omega' \subset\subset \Omega$,as $h \to 0$.

We turn our attention now to the approximation of functions in the $L^p_{loc}(\Omega)$ spaces.

Lemma 1.2.2. *Let $u \in L^p_{loc}(\Omega)(L^p(\Omega))$, $p< \infty$. Then u_h converges to u in the sense of $L^p_{loc}(\Omega)(L^p(\Omega))$.*

Proof. Using Holder's inequality along with the change of variables from (1.2.1), we obtain

$$
\begin{aligned}
|u_h(x)|^p &\leq \left(\int\limits_{|z|\leq 1} |\rho(z)|\ |u(x-hz)|\ \mathrm{d}z \right)^p \\
&= \left(\int\limits_{|z|\leq 1} |\rho(z)|^{1/q}\ |\rho(z)|^{1/p}\ |u(x-hz)|\ \mathrm{d}z \right)^p \\
&\leq \left(\left(\int\limits_{|z|\leq 1} |\rho(z)|\ \mathrm{d}z \right)^{1/q} \left(\int\limits_{|z|\leq 1} |\rho(z)|\ |u(x-hz)|^p\ \mathrm{d}z \right)^{1/p} \right)^p \\
&= \int\limits_{|z|\leq 1} p(z)|u(x-hz)|^p\ \mathrm{d}z \qquad\qquad \text{since} \int\limits_{|z|\leq 1} |p(z)|\ \mathrm{d}z\ = 1
\end{aligned}
$$

so that if $\Omega' \subset\subset \Omega$ and $2h <$dist$(\Omega', \partial\Omega)$,

$$
\begin{aligned}
\int\limits_{\Omega'} |u_h|^p\ \mathrm{d}x &\leq \int\limits_{\Omega'} \int\limits_{|z|\leq 1} \rho(z)|u(x-hz)|^p\ \mathrm{d}z\ \mathrm{d}x \\
&= \int\limits_{|z|\leq 1} \rho(z)\ \mathrm{d}z \int\limits_{\Omega'} |u(x-hz)|^p\ \mathrm{d}x \qquad\qquad \text{by Fubini} \\
&\leq \int\limits_{B_h(\Omega')} |u(x)|^p\ \mathrm{d}x
\end{aligned}
$$

where $B_h(\Omega') = \{x|\text{dist}(x, \Omega') < h\}$. Consequently

$$\|u_h\|_{L^p(\Omega')} \leq \|u\|_{L^p(\Omega'')}, \quad \Omega'' = B_h(\Omega') \tag{1.16}$$

The proof can now be completed by approximation based on (1.2.1) Choose $\epsilon > 0$ together with a $C^0(\Omega)$ function w satisfying $\|u - w\|_{L^p(\Omega'')} \leq \epsilon$ where $\Omega'' = B_{h'}(\Omega')$ and $2h' <$dist$(\Omega', \partial\Omega)$. By virtue of (1.2.1), we have for sufficiently small h

$\|w - w_h\|_{L^p(\Omega')} \leq \epsilon$.

Applying the estimate (1.16) to the difference $u - w$, we therefore obtain

$$\begin{aligned}\|u - u_h\|_{L^p(\Omega')} &\leq \|u - w\|_{L^p(\Omega')} + \|w - w_h\|_{L^p(\Omega')} + \|u_h - w_h\|_{L^p(\Omega')} \\ &\leq 2\epsilon + \|u - w\|_{L^p(\Omega'')} \leq 3\epsilon\end{aligned}$$

for small enough $h \leq h'$. Hence u_h converges to u in $L^p_{loc}(\Omega)$. The result for $u \in L^p(\Omega)$ can then be obtained by extending u to be zero outside Ω and applying the result for $L^p_{loc}(\mathbb{R}^n)$. □

1.3 Weak Derivatives

Let u be locally integrable in Ω and α any multi-index. Then a locally integrable function v is called the α^{th} *weak derivative* of u if it satisfies

$$\int_\Omega \phi v \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi \, dx \;\; \text{for } \; all \;\; \phi \in \; C_0^{|\alpha|}(\Omega) \tag{1.17}$$

We write $v = D^\alpha u$ and note that $D^\alpha u$ is uniquely determined up to sets of measure zero. Pointwise relations involving weak derivatives will be accordingly understood to hold almost everywhere. We call a function *weakly differentiable* if all its weak derivatives of first order exist and k *times weakly differentiable* if all its weak derivatives exist for orders up to and including k. Let us denote the linear space of k times weakly differentiable functions by $W^k(\Omega)$. Clearly $C^k(\Omega) \subset W^k(\Omega)$. Therefore, the concepts of classical theory are still valid, i.e integration by parts.

Lemma 1.3.1. *Let $u \in L^1_{loc}(\Omega)$, α a multi-index, and suppose that $D^\alpha u$ exists. Then if $dist(x, \partial\Omega) > h$, we have:*

$$D^\alpha u_h(x) = (D^\alpha u)_h(x) \tag{1.18}$$

Proof. From (1.14) we have $D^\alpha u_h(x) = D^\alpha\ h^{-n} \int\limits_\Omega \rho\left(\frac{x-y}{h}\right) u(y)\ dy$
Now since $u_h \in C^\infty(\Omega)$ we may differentiate under the integral sign to obtain:

$$\begin{aligned} D^\alpha\ h^{-n} \int\limits_\Omega \rho\left(\frac{x-y}{h}\right) u(y)\ dy &= h^{-n} \int\limits_\Omega D^\alpha_x \rho\left(\frac{x-y}{h}\right) u(y)\ dy \\ &= (-1)^{|\alpha|} h^{-n} \int\limits_\Omega D^\alpha_y \rho\left(\frac{x-y}{h}\right) u(y)\ dy \quad \text{switching derivatives} \\ &= h^{-n} \int\limits_\Omega \rho\left(\frac{x-y}{h}\right) D^\alpha u(y)\ dy \quad \text{by (1.17)} \\ &= (D^\alpha u)_h(x) \end{aligned}$$

□

Theorem 1.3.2. *Let u and v be locally integrable in Ω. Then $v = D^\alpha u$ if and only if there exists a sequence of $C^\infty(\Omega)$ functions u_m converging to u in $L^1_{loc}(\Omega)$ whose derivatives $D^\alpha u_m$ converge to v in $L^1_{loc}(\Omega)$.*

Proof. If u is weakly differentiable and $v = D^\alpha u$, then from (1.3.1), we can construct such a sequence, and thus we have $(D^\alpha u)_h(x) \to\ D^\alpha u \in L^1_{loc}$. Conversely, suppose that such a sequence exists, namely $u_m \to u$ in $L^1_{loc}(\Omega)$ as $m \to \infty$ and $D^\alpha u_m \to D^\alpha u = v$ in $L^1_{loc}(\Omega)$.

Then we have

$$\int_{\Omega} u_m \varphi \, dx \to \int_{\Omega} u\varphi \, dx \quad \text{for all } \varphi \in \mathcal{C}_0(\Omega)$$

since if $K = \operatorname{supp}(\varphi) \subset\subset \Omega$

$$\begin{aligned} \left| \int_{\Omega} (u_m - u)\varphi \, dx \right| &= \left| \int_{K} (u_m - u)\varphi \, dx \right| \\ &\leq \sup_K |\varphi| \int_K |u_m - u| dx \to \ 0 \end{aligned}$$

Hence, for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$ the $L^1_{loc}(\Omega)$ convergence of u_m and $D^\alpha u_m$ gives the following:

$$\begin{aligned} \int_{\Omega} u D^\alpha \varphi \, dx &= \lim_{m\to\infty} \int_{\Omega} u_m D^\alpha \varphi \, dx \\ &= (-1)^{|\alpha|} \lim_{m\to\infty} \int_{\Omega} D^\alpha u_m \varphi \, dx \\ &= (-1)^{|\alpha|} \int_{\Omega} D^\alpha u \varphi \, dx \end{aligned}$$

therefore, u is weakly differentiable and $D^\alpha u = v$. □

The results stemming from (1.3.2), allows most of the applications from the classical theory of differential calculus to be extended to weak derivatives simply by approximation. In particular, we have the product formula:

$$D(uv) = uDv + vDu \tag{1.19}$$

which holds for all $u,\ v \in W^1(\Omega)$ such that $uv,\ uDv + vDu \in L^1_{loc}(\Omega)$.(See problem 4) Another part of classical theory is the change of variables formula, that is, if ψ maps Ω onto a domain $\tilde{\Omega} \subset \mathbb{R}^n$ with $\psi \in C^1(\Omega)$ and $\psi^{-1} \in C^1(\tilde{\Omega})$ and if $u \in W^1(\Omega),\ v = u \circ \psi^{-1}$, then

$v \in W^1(\tilde{\Omega})$ and we have the formula:

$$D_i u(x) = \frac{\partial y_j}{\partial x_i} D_{y_j} v(y) \tag{1.20}$$

which holds for almost every $x \in \Omega$, $y \in \tilde{\Omega}$, $y = \psi(x)$.

One more thing to note is that locally uniformly Lipschitz continuous functions are weakly differentiable, that is, $C^{0,1}(\Omega) \subset W^1(\Omega)$. This is justified by observing that any function in $C^{0,1}(\Omega)$ will be absolutely continuous on any line segment in Ω, and furthermore, its partial derivatives exist almost everywhere and satisfy (1.17). Therefore, they coincide almost everywhere with the weak derivatives. Using regularization, we can prove that a function is weakly differentiable if and only if it is equivalent to a function that is absolutely continuous on almost all line segments in Ω parallel to the coordinate axis and whose partial derivatives are locally integrable.(See problem 5) This gives one an alternate way of deriving the preceding and following sections properties of weak derivatives.

1.4 The Chain Rule

We will now consider a simple type of chain rule for weak derivatives.

Lemma 1.4.1. *Let $f \in C^1(\mathbb{R})$, $f' \in L^\infty(\mathbb{R})$ and $u \in W^1(\Omega)$. Then the composite function $f \circ u \in W^1(\Omega)$ and $D(f \circ u) = f'(u)\, Du$.*

Proof. Let u_m, $m = 1, 2, \ldots \in C^1(\Omega)$, such that $\{u_m\} \to u$, $\{Du_m\} \to Du$ in $L^1_{loc}(\Omega)$. Then for $\Omega' \subset\subset \Omega$, we have:

$$\begin{aligned}
\int\limits_{\Omega'} |f(u_m(x)) - f(u(x))|dx &= \int\limits_{\Omega'} \frac{|f(u_m(x)) - f(u(x))|}{|u_m(x) - u(x)|}|u_m(x) - u(x)|dx \\
&= \int\limits_{\Omega'} |f'(\xi)||u_m(x) - u(x)|dx \\
&\leq \sup|f'| \int\limits_{\Omega'} |u_m(x) - u(x)|dx \to 0 \qquad \text{since } u_m \to u \text{ as } m \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_{\Omega'} |f'(u_m(x))Du_m(x) - f'(u(x))Du(x)|dx = \\
&= \int\limits_{\Omega'} |f'(u_m(x))Du_m(x) - f'(u_m(x))Du(x) + f'(u_m(x))Du(x) - f'(u(x))Du(x)|dx \\
&\leq \int\limits_{\Omega'} |f'(u_m(x))Du_m(x) - f'(u_m(x))Du(x)|dx + \int\limits_{\Omega'} |f'(u_m(x))Du(x) - f'(u(x))Du(x)|dx \\
&\leq \sup|f'| \int\limits_{\Omega'} |Du_m(x) - Du(x)|dx + \int\limits_{\Omega'} |f'(u_m(x)) - f'(u(x))||Du(x)|dx
\end{aligned}$$

Passing to a subsequence, say $\{u_{m_k}\}$, which must converge a.e. in Ω' to u. We have, by continuity of f', that $\{f'(u_{m_k})\}$ converges to $f'(u)$ a.e. in Ω'. Therefore, the last integral tends to 0 by dominated convergence. Giving that $\{f(u_m)\} \to f(u)$ and $\{f'(u_m)Du_m\} \to f'(u)Du$ establishing the formula. $\square$

The positive and negative parts of a function are defined by

$$u^+ = \max\{u, 0\}, \quad u^- = \min\{u, 0\}$$

Moreover, we have $u = u^+ + u^-$ and $|u| = u^+ - u^-$. Using the previous lemma, we can derive a chain rule for these functions as well.

Lemma 1.4.2. *Let $u \in W^1(\Omega)$; then $u^+, u^-, |u| \in W^1(\Omega)$ and*

$$Du^+ = \begin{cases} Du & if\ u > 0 \\ 0 & if\ u \leq 0 \end{cases}$$

$$Du^- = \begin{cases} 0 & if\ u \geq 0 \\ Du & if\ u < 0 \end{cases} \tag{1.21}$$

$$D|u| = \begin{cases} Du & if\ u > 0 \\ 0 & if\ u = 0 \\ -Du & if\ u < 0 \end{cases}$$

Proof. Let $\epsilon > 0$ be given, and define

$$f_\epsilon(u) = \begin{cases} (u^2 + \epsilon^2)^{1/2} & \text{if } \ u > 0 \\ 0 & \text{if } \ u \leq 0 \end{cases}$$

so that $\lim\limits_{\epsilon \to 0} f_\epsilon(u) = u^+$. Applying (1.4.1), we have, for any $\varphi \in \mathcal{C}_0^1(\Omega)$

$$\int\limits_\Omega f_\epsilon(u) D\varphi \, dx = -\int\limits_{u>0} \varphi \frac{uDu}{(u^2 + \epsilon^2)^{1/2}} \, dx$$

taking ϵ to zero, we obtain

$$\int\limits_\Omega u^+ D\varphi \, dx = -\int\limits_{u>0} \varphi Du \, dx$$

thus establishing (1.21) for u^+. Now for u^- it follows from the previous since $u^- = -(-u)^+$, and $|u| = u^+ - u^-$. $\square$

Lemma 1.4.3. *Let $u \in W^1(\Omega)$. Then $Du = 0$ a.e. on any set where u is constant.*

Proof. This is a trivial consequence of (1.4.2), since $Du = Du^+ + Du^-$ and if u is constant on some set $Du^+ = 0$ and $Du^- = 0$ but weak derivatives are only determined up to sets of measure zero, hence $Du = 0$ a.e. on that set. $\square$

We call a function piecewise smooth if it is continuous and has piecewise continuous first derivatives. We now introduce the following generalization of the previous two lemmas.

Theorem 1.4.4. *Let f be a piecewise smooth function on $\mathbb{R}$ with $f' \in L^\infty(\mathbb{R})$. Then if $u \in W^1(\Omega)$, we have $f \circ u \in W^1(\Omega)$. Furthermore, letting L denote the set of corner points of f, we have*

$$D(f \circ u) = \begin{cases} f'(u)Du & if\ u \notin L \\ 0 & if\ u \in L \end{cases} \tag{1.22}$$

Proof. By induction, we may consider the case of only one corner point. Without loss of generality we may take it a the origin. Let $f_1, f_2 \in \mathcal{C}(\mathbb{R})$ be functions satisfying the following conditions

1. $f_1', f_2' \in L^\infty(\mathbb{R})$
2. $f_1(u) = f(u)$ for $u \geq 0$
3. $f_2(u) = f(u)$ for $u \leq 0$

Then we have $f(u) = f_1(u^+) + f_2(u^-)$ and the result follows from (1.4.1) and (1.4.2). $\square$

1.5 The $W^{k,p}$ Spaces

We will now classify the $W^{k,p}(\Omega)$ spaces and their norm. Theses spaces are Banach spaces analogous to $C^{k,\alpha}(\bar{\Omega})$ spaces in a certain sense. The changes that are necessary are the following: continuous differentiability is replaced by weak differentiability, and Hölder continuity by p-integrability.

Definition For $p \geq 1$ and k a non-negative integer, we let

$$W^{k,p}(\Omega) = \{u \in W^k(\Omega);\ D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}$$

These spaces are linear and a norm is introduced by defining

$$\|u\|_{k,p;\Omega} = \|u\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^p dx \right)^{1/p} \tag{1.23}$$

We shall use $\|u\|_{k,p}$ for $\|u\|_{k,p;\Omega}$ when there is no ambiguity. Also, and equivalent norm on $W^{k,p}(\Omega)$ would be

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_p. \tag{1.24}$$

The verification that $W^{k,p}(\Omega)$ is a Banach space under (1.23) can be seen in the next chapter, problem 9.

By taking the closure of $C_0^k(\Omega)$ in $W^{k,p}(\Omega)$, we obtain another Banach space which we call $W_0^{k,p}(\Omega)$. However, the spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ do not coincide for bounded Ω. The relation to the Hilbert spaces,$H^k(\Omega)$ and $H_0^k(\Omega)$, are a from the case $p = 2$ under the scalar product

$$(u, v)_k = \int_\Omega \sum_{|\alpha| \leq k} D^\alpha u D^\alpha v \; dx \tag{1.25}$$

Further functional analytic properties of $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ follow by considering their natural imbedding into the product of N_k copies of $L^p(\Omega)$, where N_k is the number of multi-indices α satisfying $|\alpha| \leq k$.

The chain rule of (1.4.4) also extends to the spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$. In fact, a consequence of (1.4.4) and the definition of these two spaces, we have that $W^1(\Omega)$ can be replaced by $W^{1,p}(\Omega)$, and by $W_0^{1,p}(\Omega)$ if we add the stipulation that $f(0) = 0$.

We define local spaces, $W_{loc}^{k,p}(\Omega)$, to consist of functions belonging to $W^{k,p}(\Omega')$ for all $\Omega' \subset\subset \Omega$. Also,(1.3.2) shows that functions in $W_{loc}^{k,p}(\Omega)$ with compact support will in fact belong to $W_0^{k,p}(\Omega)$.

1.6 Density Theorems

Definition[2] Let U be any subset of $\mathbb{R}^n$ and let G be a collection of open sets in $\mathbb{R}^n$ which cover U, that is, $\mathrm{U} \subset \cup_{\mathrm{G_j} \in \mathrm{G}} \mathrm{G_j}$. A collection Ψ of functions $\psi \in C_0^\infty(\mathbb{R}^n)$ having the following properties:

(i) For every $\psi \in \Psi$ and every $x \in \mathbb{R}^n$, $0 \leq \psi(x) \leq 1$.

(ii) If $K \subset\subset \mathrm{U}$, all but finitely many $\psi \in \Psi$ vanish identically on K.

(iii) For every $\psi \in \Psi$ there exists $\mathrm{G_j} \in \mathrm{G}$ such that $\mathrm{supp}(\psi) \subset \mathrm{G_j}$.

(iv) For every $x \in \mathrm{U}$ we have $\sum\limits_{\psi \in \Psi} \psi(x) = 1$.

is called a C^∞-partition of unity for U subordinate to G.

From (1.2.2) and (1.3.1), we have that if u lies in $W^{k,p}(\Omega)$, then $D^\alpha u_h$ tends to $D^\alpha u$ in the sense of $L_{loc}^p(\Omega)$ as h goes to zero and for all multi-indices α satisfying $|\alpha| \leq k$. Using this, we can obtain a global approximation result.

Theorem 1.6.1. *The subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Recall that Ω is bounded in $\mathbb{R}^n$, therefore if Ω is closed then it is compact and any cover will work. So suppose Ω is open. Then let Ω_j $j = 1, 2, \ldots$ be subsets of Ω such that $\Omega_j \subset\subset \Omega_{j+1}$ for all $j = 1, 2, \ldots$ and $\cup_j \Omega_j = \Omega$. Let ψ_j, $j = 0, 1, 2, \ldots$ be a partition of unity subordinate to the covering $\{\Omega_{j+1} - \Omega_{j-1}\}$, where Ω_0 and Ω_{-1} are defined to be empty sets. That is $supp(\psi_j) \subseteq \{\Omega_{j+1} - \Omega_{j-1}\}$ for all $j = 0, 1, \ldots$. Now with this partition of unity, we have, for any arbitrary $u \in W^{k,p}(\Omega)$ and $\epsilon > 0$ we can choose h_j, $j = 1, 2, \ldots$ satisfying

$$\begin{cases} h_j \leq dist(\Omega_j, \partial\Omega_{j+1}), \quad j \geq 1 \\ ||(\psi_j u)_{h_j} - \psi_j u||_{W^{k,p}(\Omega)} \leq \frac{\epsilon}{2^j} \end{cases} \tag{1.26}$$

Writing $v_j = (\psi_j u)_{h_j}$, from (1.26) we have that only a finite number of v_j are non-vanishing on any given $\Omega' \subset\subset \Omega$. As a consequence, the function

$$v = \sum_{j=1}^{\infty} v_j \quad \text{belongs to } \mathcal{C}^\infty(\Omega).$$

Furthermore,

$$\|u - v\|_{W^{k,p}(\Omega)} \leq \sum \|v_j - \psi_j u\|_{W^{k,p}(\Omega)} \leq \epsilon.$$

This completes the proof. $\square$

Theorem (1.6.1) shows that $W^{k,p}(\Omega)$ could have been characterized as the completion of $\mathcal{C}^\infty(\Omega)$ under the norm (1.23), and in many cases this is a convenient definition.

1.7 Imbedding Theorems

This and the following section are concerned with the connection between pointwise and integrability properties of weakly differentiable functions and the integrability properties of their derivatives. One of the most simple but amazing results, is that weakly differentiable functions of one variable must be absolutely continuous. This section is aimed at proving the well known *Sobolev inequalities* for functions in $W_0^{k,p}(\Omega)$.

Before getting to the main focus in this section, we introduce the following lemma that will be used:

Lemma 1.7.1. [3] *Let $N \geq 2$ and let $f_1, f_2, \ldots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. For $x \in \mathbb{R}^N$ and $1 \leq i \leq N$ set*

$$\tilde{x_i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbb{R}^{N-1},$$

Then the function

$$f(x) = f_1(\tilde{x_1}) f_2(\tilde{x_2}) \cdots f_N(\tilde{x_N}), \qquad x \in \mathbb{R}^N$$

belongs to $L^1(\mathbb{R}^N)$ and

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^{N} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}$$

Proof. The case $N = 2$ is straight forward. Consider the case $N = 3$ then we have,

$$\begin{aligned}
\int_{\mathbb{R}^3} |f(x)| dx &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |f_1(x_2, x_3)||f_2(x_1, x_3)||f_3(x_1, x_2)| dx_1 dx_2 dx_3 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |f_1(x_2, x_3)| \left[\int_{\mathbb{R}} |f_2(x_1, x_3)||f_3(x_1, x_2)| dx_1 \right] dx_2 dx_3 \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f_1(x_2, x_3)| \left[\int_{\mathbb{R}} |f_2(x_1, x_3)|^2 dx_1 \right]^{1/2} \left[\int_{\mathbb{R}} |f_3(x_1, x_2)|^2 dx_1 \right]^{1/2} dx_2 dx_3 \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f_2(x_1, x_3)|^2 dx_1 \right]^{1/2} \left[\int_{\mathbb{R}} |f_1(x_2, x_3)| \left[\int_{\mathbb{R}} |f_3(x_1, x_2)|^2 dx_1 \right]^{1/2} dx_2 \right] dx_3
\end{aligned}$$

$$\leq \int\limits_{\mathbb{R}} \left[\int\limits_{\mathbb{R}} |f_2(x_1,x_3)|^2 dx_1\right]^{1/2} \left[\int\limits_{\mathbb{R}} |f_1(x_2,x_3)|^2 dx_2\right]^{1/2} \left[\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |f_3(x_1,x_2)|^2 dx_1 dx_2\right]^{1/2} dx_3$$

$$\leq \left[\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |f_3(x_1,x_2)|^2 dx_1 dx_2\right]^{1/2} \left[\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |f_2(x_1,x_3)|^2 dx_1 dx_3\right]^{1/2} \left[\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |f_1(x_2,x_3)|^2 dx_2 dx_3\right]^{1/2}$$

$$= \|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}\|f_3\|_{L^2(\mathbb{R}^2)}$$

The general case follows by induction, that is, assume it is true for N. Fix $x_{N+1} \in \mathbb{R}$, then by applying Hölder's inequality with $p = N$ and $q = \frac{N}{N-1}$ we obtain

$$\int\limits_{\mathbb{R}^N} |f(x)| dx_1 dx_2 \cdots dx_N \leq \|f_{N+1}\|_{L^N(\mathbb{R}^N)} \left[\int\limits_{\mathbb{R}^N} |f_1 f_2 \cdots f_N|^q dx_1 \cdots dx_N\right]^{1/q}$$

Now applying the induction assumption, we have

$$\left[\int\limits_{\mathbb{R}^N} |f_1|^q |f_2|^q \cdots |f_N|^q dx_1 \cdots dx_N\right]^{1/q} \leq \left[\prod_{i=1}^{N} \|f_i\|^q_{L^N(\mathbb{R}^{N-1})}\right]^{1/q}$$

so that

$$\int\limits_{\mathbb{R}^N} |f(x)| dx_1 \cdots dx_N \leq \|f_{N+1}\|_{L^N(\mathbb{R}^N)} \prod_{i=1}^{N} \|f_i\|_{L^N(\mathbb{R}^{N-1})}$$

Now by integrating with respect to x_{N+1} we have

$$\begin{aligned}\int\limits_{\mathbb{R}^{N+1}} |f(x)| dx_1 dx_2 \cdots dx_N dx_{N+1} &\leq \|f_{N+1}\|_{L^N(\mathbb{R}^N)} \int\limits_{\mathbb{R}} \left[\prod_{i=1}^{N} \|f_i\|_{L^N(\mathbb{R}^{N-1})} dx_{N+1}\right] \\ &\leq \prod_{i=1}^{N+1} \|f_i\|_{L^N(\mathbb{R}^N)}\end{aligned}$$

$\square$

Theorem 1.7.2.

$$W_0^{1,p}(\Omega) \subset \begin{cases} L^{np/(n-p)}(\Omega) & for\ p < n \\ C^0(\bar{\Omega}) & for\ p > n \end{cases}$$

Furthermore, there exists a constant C depending only on n and p, such that for any $u \in W_0^{1,p}(\Omega)$,

$$\begin{aligned} \|u\|_{np/(n-p)} &\leq C\|Du\|_p && for\ p < n, \\ \sup_\Omega |u| &\leq C|\Omega|^{1/n-1/p}\|Du\|_p && for\ p > n. \end{aligned} \tag{1.27}$$

Proof. We start by proving it for the case $u \in \mathcal{C}_0^1(\Omega)$ and $p = 1$. For any $u \in \mathcal{C}_0^1(\Omega)$ and i, $1 \leq i \leq n$ we have

$$\begin{aligned} |u(x)| &= |\int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|dt \\ &\leq \int_{-\infty}^{\infty} |\frac{\partial u}{\partial x_i}(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|dt \end{aligned}$$

Define these integrals as follows

$$u_i(\tilde{x_i}) \equiv \int_{-\infty}^{\infty} |\frac{\partial u}{\partial x_i}(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|dt$$

Then

$$|u(x)|^n \leq \prod_{i=1}^n u_i(\tilde{x_i})$$

So that when we apply (1.7.1) to $\int_\Omega |u(x)|^{n/(n-1)}dx$ and obtain

$$\int_\Omega |u(x)|^{n/(n-1)}dx \leq \prod_{i=1}^n \left(\int_\Omega |D_i u| dx \right)^{1/(n-1)}$$

Now using the fact that the geometric mean is dominated by the arithmetic mean, that is for any set of positive numbers say a_j

$$\left(\prod_{j=1}^{n} a_j\right)^{1/n} \leq \frac{1}{n}\sum_{j=1}^{n} a_j$$

We can now obtain the result for $p = 1$ by raising each side to the $\frac{n-1}{n}$ power, giving

$$\begin{aligned}
\|u\|_{n/(n-1)} &\leq \prod_{i=1}^{n}\left(\int_{\Omega} |D_i u| dx\right)^{1/n} \\
&\leq \frac{1}{n}\int_{\Omega}\sum_{i=1}^{n} |D_i u| dx \\
&\leq \frac{\sqrt{n}}{n}\|Du\|_1
\end{aligned}$$

To obtain the result in full generality, replace $|u|$ by powers of $|u|$. Hence if $q > 1$

$$\begin{aligned}
\||u|^q\|_{n/(n-1)} &\leq \frac{\sqrt{n}}{n}\int_{\Omega} |D(|u|^q)| dx \\
&\leq q\frac{\sqrt{n}}{n}\int_{\Omega} |u|^{q-1}|Du| dx \\
&\leq \frac{q\sqrt{n}}{n}\||u|^{q-1}\|_{p'}\|Du\|_p \qquad \text{by Hölder's inequality}
\end{aligned}$$

Letting $q = \frac{(n-1)p}{n-p}$ we obtain the desired result for $1 \leq p < n$. To extend this to functions $u \in W_0^{1,p}(\Omega)$, let $\{u_i\}$ be a sequence of $\mathcal{C}_0^\infty(\Omega)$ functions converging strongly to u in $W_0^{1,p}(\Omega)$. Then an application of the inequality to $u_i - u_j$ yields

$$\|u_i - u_j\|_{np/(n-p)} \leq C\|u_i - u_j\|_{1,p}$$

Whence $u_i \longrightarrow u$ in $L^{np/(n-p)}(\Omega)$, so by density of these functions and this convergence, the result follows for the case $1 \le p < n$.

For the case $p > n$, let $\tilde{u} = \frac{|u|\sqrt{n}}{\|Du\|_p}$. We will first assume $|\Omega| = 1$ (valid since Ω is bounded subset of $\mathbb{R}^n$).

$$\|\tilde{u}^\eta\|_{n/(n-1)} \le \eta \|\tilde{u}^{\eta-1}\|_{p/(p-1)}$$

so that

$$\begin{aligned} \|\tilde{u}\|_{\eta(n/(n-1))} &\le \eta^{1/\eta} \|\tilde{u}\|^{1-1/\eta}_{(p/(p-1))(\eta-1)} \\ &\le \eta^{1/\eta} \|\tilde{u}\|^{1-1/\eta}_{\eta(p/(p-1))} \qquad \text{since } |\Omega| = 1 \end{aligned}$$

Let $\delta = \frac{n(p-1)}{p(n-1)} > 1$ and substitute δ^ν for η where $\nu = 1, 2, \ldots$ to obtain

$$\|\tilde{u}\|_{(n/(n-1))\delta^\nu} \le \delta^{\nu\delta^{-\nu}} \|\tilde{u}\|^{1-\delta^{-\nu}}_{(n/(n-1))\delta^{\nu-1}} \quad \text{for } \nu = 1, 2, \ldots$$

Iterating from $\nu = 1$ and using the process from the previous case, we get for any ν

$$\|\tilde{u}\|_{\delta^\nu} \le \delta^{\sum \nu\delta^{-\nu}} \equiv \chi$$

Using problem 1 from the next chapter, we have as $\nu \longrightarrow \infty$

$$\operatorname*{ess\,sup}_{\Omega} \tilde{u} \le \chi$$

Replacing $\tilde{u}$ with $\frac{|u|\sqrt{n}}{\|Du\|_p}$ gives the desired inequality

$$\operatorname*{ess\,sup}_{\Omega} |u| \le \frac{\chi}{\sqrt{n}} \|Du\|_p$$

To eliminate the restriction $|\Omega| = 1$, we consider the mapping $y_i \mapsto |\Omega|^{1/n} x_i$ giving

$$\operatorname*{ess\,sup}_{\Omega} |u| \leq \frac{\chi}{\sqrt{n}} |\Omega|^{1/n - 1/p} \|Du\|_p$$

$\square$

Remark. The best constant C satisfying (1.27) for the case $p < n$ was calculated by Rodemich, who showed that

$$C = \frac{1}{n\sqrt{\pi}} \left(\frac{n!\Gamma(n/2)}{2\Gamma(n/p)\Gamma(n+1-n/p)} \right)^{1/n} \gamma^{1-1/p}, \text{where } \gamma = \frac{n(p-1)}{n-p}$$

When $p = 1$, this number reduces to the well known isoperimetric constant

$$n^{-1}(\omega_n)^{-1/n}$$

A Banach space $\mathfrak{B}_1$ is said to be continuously imbedded in a Banach space $\mathfrak{B}_2$, if there exists a bounded, linear, one-to-one mapping $I : \mathfrak{B}_1 \longrightarrow \mathfrak{B}_2$. Therefore Theorem 1.7.1 may be expressed as $W_0^{1,p}(\Omega) \longrightarrow L^{np/(n-p)}(\Omega)$ if $p < n$, and $\longrightarrow C^0(\bar{\Omega})$ if $p > n$. If we then iterate the result of Theorem 1.7.1 k times, we can then obtain the following Corollary for extensions of $W_0^{k,p}(\Omega)$.

Corollary 1.7.3.

$$W_0^{k,p}(\Omega) \longrightarrow \begin{cases} L^{np/(n-kp)}(\Omega) & for\ kp < n \\ C^m(\bar{\Omega}) & for\ 0 \leq m < k - \frac{n}{p} \end{cases}$$

The second case is a consequence of the first, together with the case $p > n$ in the previous theorem. The estimates (1.27) and their extension to the spaces $W_0^{k,p}(\Omega)$ also show that a norm on $W_0^{k,p}(\Omega)$ equivalent to (1.23) may be defined by

$$\|u\|_{W_0^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha|=k} |D^\alpha u|^p dx \right)^{1/p} \tag{1.28}$$

In general, $W_0^{k,p}(\Omega)$ cannot be replaced by $W^{k,p}(\Omega)$ in the previous Corollary. However, this replacement can be made for a large class of domains Ω, which includes for example domains with Lipschitz continuous boundaries. More generally, if Ω satisfies a uniform interior cone condition then there is an imbedding

$$W^{k,p}(\Omega) \longrightarrow \begin{cases} L^{np/(n-kp)}(\Omega) & \text{for } kp < n \\ \mathcal{C}_B^m(\Omega) & \text{for } 0 \le m < k - \frac{n}{p} \end{cases} \tag{1.29}$$

where $\mathcal{C}_B^m(\Omega) = \{u \in \mathcal{C}^m(\Omega) | D^\alpha u \in L^\infty(\Omega) \ \text{ for } |\alpha| \le m\}$.

1.8 Potential Estimates and Imbedding Theorems

The imbedding results of the preceding section can be alternatively derived and improved upon with the use of potential estimates. Let $\mu \in (0,1]$ and define the operator V_μ on $L^1(\Omega)$ by the Riesz potential

$$(V_\mu f)(x) = \int_\Omega |x-y|^{n(\mu-1)} f(y) dy \tag{1.30}$$

The fact that V_μ is well defined and maps $L^1(\Omega)$ into itself will appear as a result of the next lemma.

Lemma 1.8.1. *The operator V_μ maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ for any q, $1 \le q \le \infty$ satisfying*

$$0 \le \delta = \delta(p,q) = p^{-1} - q^{-1} < \mu \tag{1.31}$$

Furthermore, for any $f \in L^p(\Omega)$,

$$\|V_\mu f\|_q \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_n^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_p. \tag{1.32}$$

The case $p > n$ of the Sobolev imbedding theorem from the previous section may be fine tuned through the following lemma.

Lemma 1.8.2. *Let Ω be convex and $u \in W^{1,1}(\Omega)$. Then*

$$|u(x) - u_S| \le \frac{d^n}{n|S|} \int\limits_\Omega |x-y|^{1-n} |Du(y)| dy \quad a.e. \ in\ \Omega \tag{1.33}$$

where

$$u_S = \frac{1}{|S|}\int\limits_S u\, dx, \quad d = diam\Omega$$

and S is any measurable subset of Ω

Proof. By (1.6.1), it is enough to show (1.33) for $u \in \mathcal{C}^1(\Omega)$. We then have for $x, y \in \Omega$ by the fundamental theorem of Calculus,

$$u(x) - u(y) = -\int\limits_0^{|x-y|} D_r u(x + r\omega) dr, \quad \omega = \frac{y-x}{|y-x|}$$

Now integrate this with respect to y over the set S to obtain on the left hand side

$$\begin{aligned} &\int\limits_S (u(x) - u(y))dy = \int\limits_S u(x)dy - \int\limits_S u(y)dy \\ &= u(x)\int\limits_S dy - \int\limits_S u(y)dy = |S|u(x) - \frac{|S|}{|S|}\int\limits_S u(y)dy \\ &= |S|[u(x) - u_S] \end{aligned}$$

The RHS doesn't depend on y, therefore we have

$$|S|[u(x)-u_S] = -\int\limits_S dy \int\limits_0^{|x-y|} D_r u(x+r\omega)dr$$

Define

$$V(x) = \begin{cases} |D_r u(x)| & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

With this new function V defined on all of $\mathbb{R}^n$ we have

$$\begin{aligned} |u(x)-u_S| &\leq \frac{1}{|S|} \int\limits_{|x-y|<d} dy \int\limits_0^\infty V(x+r\omega)dr \\ &= \frac{1}{|S|} \int\limits_0^\infty \int\limits_{|\omega|=1} \int\limits_0^d V(x+r\omega)\rho^{n-1}\, d\rho\, d\omega\, dr \\ &= \frac{d^n}{n|S|} \int\limits_0^\infty \int\limits_{\omega=1} V(x+r\omega)d\omega\, dr \\ &= \frac{d^n}{n|S|} \int\limits_\Omega |x-y|^{1-n} |D_r u(y)| dy \end{aligned}$$

□

We now prove the imbedding theorem of Morrey

Theorem 1.8.3. *Let $u \in W_0^{1,p}(\Omega)$, $p > n$. Then $u \in C^\gamma(\bar{\Omega})$, where $\gamma = 1 - n/p$. Furthermore, for any ball B*

$$\underset{\Omega\cap B}{osc}\ u \leq C\mathbb{R}^\gamma \|Du\|_p \tag{1.34}$$

where $C = C(n,p)$.

Proof. Using the estimates from (1.32) and (1.33) for $S = \Omega = B$, $q = \infty$ and $\mu = n^{-1}$ we have

$$|u(x) - u_B| \leq C(n,p)\mathbb{R}^{\gamma}\|Du\|_p \quad \text{a.e. on } \Omega \cap B$$

The result now follows since

$$|u(x) - u(y)| \leq |u(x) - u_B| + |u(y) - u_B| \leq 2C(n,p)\mathbb{R}^{\gamma}\|Du\|_p \quad \text{a.e. on } \Omega \cap B$$

□

1.9 Compactness Results

Let $\mathfrak{B}_1$ be a Banach space continuously imbedded in a Banach space $\mathfrak{B}_2$. Then $\mathfrak{B}_1$ is *compactly imbedded* in $\mathfrak{B}_2$ if the imbedding operator $\mathcal{I} : \mathfrak{B}_1 \to \mathfrak{B}_2$ is compact, that is, if the images of bounded sets in $\mathfrak{B}_1$ are precompact in $\mathfrak{B}_2$. Let us now prove the *Kondrachov compactness theorem* for the spaces $W_0^{1,p}(\Omega)$.

Theorem 1.9.1. *The spaces $W_0^{1,p}(\Omega)$ are compactly imbedded (i) in the spaces $L^q(\Omega)$ for any $q < np/(n-p)$, if $p < n$, and (ii) in $C^0(\bar{\Omega})$ if $p > n$.*

Proof. We prove part (i) as part (ii) is a direct consequence of Morrey's theorem and Arzela's theorem on equicontinuous families of functions. We prove the initial case when $q = 1$. Let A be a bounded set in $W_0^{1,p}(\Omega)$, without loss of generality, we may assume that $A \in C_0^1(\Omega)$ and $\|u\|_{1,p;\Omega} \leq 1$ for all $u \in A$. For $h > 0$ define $A_h = \{u_h | u \in A\}$ where u_h is the regularization of u. We wish to show that A is precompact in $L^1(\Omega)$.

If $u \in A$ we have

$$|u_h(x)| \leq \int\limits_{|z|=1} \rho(z)|u(x - hz)|dz \leq h^{-n} \sup \rho \|u\|_1$$

and

$$|Du_h(x)| \leq h^{-1} \int\limits_{|z|\leq 1} |D\rho(z)||u(x-hz)|dz \leq h^{-n-1} \sup|D\rho|\|u\|_1$$

so that A_h is a bounded, equicontinuous subset of $C^0(\bar{\Omega})$ and hence precompact in $L^1(\Omega)$. Now estimating for $u \in A$ we have

$$\begin{aligned} |u(x) - u_h(x)| &\leq \int\limits_{|z|\leq 1} \rho(z)|u(x) - u(x-hz)|dz \\ &\leq \int\limits_{|z|\leq 1} \rho(z) \int\limits_0^{h|z|} |D_r u(x - r\omega)|dr\ dz \qquad \text{where } \omega = \frac{z}{|z|} \end{aligned}$$

now integrating over x, we obtain

$$\int\limits_\Omega |u(x) - u_h(x)|dx \leq h \int\limits_\Omega |Du|dx \leq h|\Omega|^{1-1/p}$$

consequently u_h is uniformly close to u in $L^1(\Omega)$ relative to A. Since we have shown that A_h is totally bounded in $L^1(\Omega)$ for all $h > 0$, it follows that A is also totally bounded in $L^1(\Omega)$ and hence precompact. This proves the case $q = 1$, to extend this result to all $q < np/(n-p)$ we estimate using (1.9)

$$\begin{aligned} \|u\|_q &\leq \|u\|_1^\lambda \|u\|_{np/(n-p)}^{1-\lambda} && \text{where } \lambda + (1-\lambda)\left(\frac{1}{p} - \frac{1}{n}\right) = \frac{1}{q} \\ &\leq \|u\|_1^\lambda (C\|Du\|_p)^{1-\lambda} && \text{by Theorem (1.7.2)} \end{aligned}$$

Consequently, a bounded set in $W_0^{1,p}(\Omega)$ must be precompact in $L^q(\Omega)$ for $q > 1$ $\square$

1.10 Difference Quotients

In partial differential equations, the weak or classic differentiability of functions may often be deduced through a consideration of their difference quotients. Let u be a function on a domain $\Omega \subset \mathbb{R}^n$ and denote by e_i the unit coordinate vector in the x_i direction. We define the difference quotient in the direction e_i by

$$\Delta^h u(x) = \Delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}, \quad h \neq 0 \tag{1.35}$$

The following lemmas deal with difference quotients of functions in Sobolev spaces.

Lemma 1.10.1. *Let $u \in W^{1,p}(\Omega)$. Then $\Delta^h u \in L^p(\Omega')$ for any $\Omega' \subset\subset \Omega$ satisfying $0 < h <$ dist$(\Omega', \partial\Omega)$, and we have*

$$\|\Delta^h u\|_{L^p(\Omega')} \leq \|D_i u\|_{L^p(\Omega)}.$$

Proof. We will use the fact that $C^1(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$. Therefore, suppose $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$. Then

$$\begin{aligned} \Delta^h u(x) &= \frac{u(x + he_i) - u(x)}{h} \\ &= \int_0^h \frac{1}{h} D_i u(x_1, \ldots, x_{i-1}, x_i + \xi, x_{i+1}, \ldots, x_n) d\xi \end{aligned}$$

now we apply Hölder's inequality to obtain

$$
\begin{aligned}
|\Delta^h u(x)| &= |\frac{1}{h}|\,|\int_0^h D_i u(x_1,\ldots,x_{i-1},x_i+\xi,x_{i+1},\ldots,x_n)\,d\xi| \\
&\le \frac{1}{h}\int_0^h |D_i u(x_1,\ldots,x_{i-1},x_i+\xi,x_{i+1},\ldots,x_n)|\,d\xi && \text{by} \\
&\le \frac{1}{h}\left(\int_0^h d\xi\right)^{1/q}\left(\int_0^h |D_i u|^p\,d\xi\right)^{1/p} && \text{where } 1/p+1/q=1 \\
&= \frac{h^{1/q}}{h}\left(\int_0^h |D_i u|^p\,d\xi\right)^{1/p} \\
&= h^{1/q-1}\left(\int_0^h |D_i u|^p\,d\xi\right)^{1/p}
\end{aligned}
$$

so that raising both sides to the p power, we have (using that $1/q-1=-1/p$)

$$
|\Delta^h u(x)|^p \le \frac{1}{h}\int_0^h |D_i u|^p d\xi
$$

now integrating both sides over Ω' we get

$$
\begin{aligned}
\int_{\Omega'} |\Delta^h u(x)|^p\,dx &\le \frac{1}{h}\int_{\Omega'}\int_0^h |D_i u|^p\,d\xi\,dx \\
&\le \frac{1}{h}\int_{B_h(\Omega')}\int_0^h |D_i u|^p\,d\xi\,dx \\
&= \frac{1}{h}\int_0^h\int_{B_h(\Omega')} |D_i u|^p\,dx\,d\xi && \text{by Fubini} \\
&\le \int_{\Omega} |D_i u|^p\,dx
\end{aligned}
$$

Finally, taking each side to the $1/p$ power gives the desired inequality. Generalizing this to arbitrary functions in $W^{1,p}(\Omega)$ follows by density of such functions in which we just used. $\square$

Lemma 1.10.2. *Let $u \in L^p(\Omega)$, $1 < p < \infty$, and suppose there exists a constant K such that $\Delta^h u \in L^p(\Omega')$ and $\|\Delta^h u\|_{L^p(\Omega')} \leq K$ for all $h > 0$ and $\Omega' \subset\subset \Omega$ satisfying $h <$ dist$(\Omega', \partial\Omega)$. Then the weak derivative $D_i u$ exists and satisfies $\|D_i u\|_{L^p(\Omega)} \leq K$.*

Proof. By weak compactness of bounded sets in $L^p(\Omega')$, there exists a sequence $\{h_m\}$ tending to zero and a function $v \in L^p(\Omega)$ with $\|v\|_p \leq K$ that satisfies, for all $\varphi \in C_0^1(\Omega)$,

$$\int_\Omega \varphi \Delta^{h_m} u \; dx \longrightarrow \int_\Omega \varphi v \; dx$$

Now suppose $h_m < \text{dist}(\text{supp}\varphi, \partial\Omega)$, we have

$$\int_\Omega \varphi \Delta^{h_m} u \; dx = -\int_\Omega u \Delta^{-h_m} \varphi \; dx \longrightarrow -\int_\Omega u D_i \varphi \; dx$$

Hence

$$\int_\Omega \varphi v \; dx = -\int_\Omega u D_i \varphi \; dx$$

and we have established that $v = D_i u$. $\square$

Chapter 2

Problems

1. Let Ω be a bounded domain of $\mathbb{R}^n$. If u is a measurable function on Ω such that $|u|^p \in L^1(\Omega)$ for some $p \in \mathbb{R}$, we define

$$\Phi_p(u) = \left[\frac{1}{|\Omega|}\int\limits_{\Omega} |u|^p dx\right]^{1/p}$$

Show that:

(i) $\lim\limits_{p\to\infty} \Phi_p(u) = \operatorname*{ess\,sup}\limits_{\Omega} |u|$;

(ii) $\lim\limits_{p\to-\infty} \Phi_p(u) = \operatorname*{ess\,inf}\limits_{\Omega} |u|$;

(iii) $\lim\limits_{p\to 0} \Phi_p(u) = \exp\left[\frac{1}{|\Omega|}\int\limits_{\Omega} \log|u|\ dx\right]$.

Solution.

(i) Since $|u|^p \in L^1(\Omega)$ for some $p \in \mathbb{R}$ we have $\int\limits_{\Omega} |u|^p\ dx < \infty$ for this p. Also, Ω is bounded in $\mathbb{R}^n$, hence $|\Omega| < \infty$ also. Let $\delta > 0$ be given and define the set

$$U_\delta = \{x : |u(x)| \geq \operatorname*{ess\,sup}_\Omega |u| - \delta\} \qquad \text{for} \delta < \operatorname*{ess\,sup}_\Omega |u|.$$

Then we have

$$\begin{aligned}\frac{1}{|\Omega|^{1/p}}\left(\int\limits_\Omega |u|^p \, dx\right)^{1/p} &\geq \frac{1}{|\Omega|^{1/p}}\left(\int\limits_{U_\delta} (\operatorname*{ess\,sup}_\Omega |u| - \delta)^p \, dx\right)^{1/p} \\ &= \frac{1}{|\Omega|^{1/p}}(\operatorname*{ess\,sup}_\Omega |u| - \delta)|(U_\delta)|^{1/p}\end{aligned}$$

Now since $|(U_\delta)|$ is finite and positive, we get

$$\begin{aligned}\liminf_{p\to\infty} \frac{1}{|\Omega|^{1/p}}\left(\int\limits_\Omega |u|^p \, dx\right)^{1/p} &\geq \liminf_{p\to\infty} \frac{1}{|\Omega|^{1/p}}(\operatorname*{ess\,sup}_\Omega |u| - \delta)\mu(U_\delta)^{1/p} \\ &= \operatorname*{ess\,sup}_\Omega |u| - \delta \geq \operatorname*{ess\,sup}_\Omega |u|\end{aligned}$$

Now we need to show the reverse inequality, we do this by noting that since $|u(x)| \leq \operatorname*{ess\,sup}_\Omega |u|$ for a.e. $x \in \Omega$ then for $p > q$ we have,

$$\begin{aligned}\frac{1}{|\Omega|^{1/p}}\left(\int\limits_\Omega |u|^p \, dx\right)^{1/p} &= \frac{1}{|\Omega|^{1/p}}\left(\int\limits_\Omega |u|^{p-q}|u|^q \, dx\right)^{1/p} \\ &\leq \frac{1}{|\Omega|^{1/p}}\operatorname*{ess\,sup}_\Omega |u|^{\frac{p-q}{p}}\left(\int\limits_\Omega |u|^q \, dx\right)^{1/p}\end{aligned}$$

So that taking the lim sup of both sides yields

$$\limsup_{p\to\infty} \frac{1}{|\Omega|^{1/p}}\left(\int\limits_\Omega |u|^p \, dx\right)^{1/p} \leq \operatorname*{ess\,sup}_\Omega |u|$$

Since $\lim\limits_{p\to\infty} \frac{p-q}{p} \longrightarrow 1$ and $1/p \longrightarrow 0$. The equality is now established.

(ii) Assume $|u|^p \in L^p(\Omega)$ for some $p \in \mathbb{R}^-$. Let $\tilde{p} = -p$ and $v = \frac{1}{|u|}$, then we have $v \in L^{\tilde{p}}(\Omega)$ for $\tilde{p} \in \mathbb{R}^+$ and based on part (i) we have

$$\begin{aligned}
\lim p \to -\infty \left(\frac{1}{|\Omega|} \int\limits_{\Omega} |u|^p \, dx \right)^{1/p} &= \lim_{p\to\infty} \left(\frac{1}{|\Omega|} \int\limits_{\Omega} \left(\frac{1}{|u|} \right)^{-p} dx \right)^{-1/-p} \\
&= \lim_{\tilde{p}\to\infty} \left(\frac{1}{|\Omega|} \int\limits_{\Omega} v^{\tilde{p}} dx \right)^{-1/\tilde{p}} \\
&= (\operatorname*{ess\,sup}_{\Omega} \ v)^{-1} \qquad \text{by part (i)} \\
&= \frac{1}{1/\operatorname*{ess\,inf}_{\Omega} |u|} = \operatorname*{ess\,inf}_{\Omega} |u|
\end{aligned}$$

(iii) Assume $|u|^p \in L^1(\Omega)$ for $0 < p < 1$. Using Jensen's inequality with the convex function x^s/r with $0 < r < s < 1$ we have

$$\left(\int\limits_{\Omega} |u|^r dx \right)^{s/r} \leq \int\limits_{\Omega} |u|^s dx$$

hence $\|u\|_r \leq \|u\|_s$. Also if $0 < p < 1$, another application of Jensen's inequality will show $\int\limits_{\Omega} \log |u| \ dx \leq \log \|u\|_p$. Therefore, $\log \|u\|_{1/n}$ is decreasing and bounded below. To find the limit we use another useful inequality

$$\log(a) \leq n(a^{1/n} - 1) \quad \text{with } a = \left(\int\limits_{\Omega} |u|^{1/n} dx \right)^n$$

By assumption we have $\frac{|u|^{1/n}-1}{1/n} \leq \frac{|u|^p-1}{p}$ for the p such that $|u|^p \in L^1(\Omega)$. So taking the limit as $n \to \infty$ and dominated convergence, we obtain

$$\lim_{n\to\infty} \log \|u\|_{1/n} \le \int_\Omega \log|u|dx$$

$$\lim_{n\to\infty} \log \|u\|_{1/n} = \int_\Omega \log|u|dx \qquad \text{by the squeeze theorem}$$

Finally, since log is continuous, we have

$$\lim_{n\to\infty} \|u\|_{1/n} = \exp\left(\int_\Omega \log|u|dx\right)$$

The weighted norm will now follow as $\frac{1}{|\Omega|}$ is just a scaling of the previous argument.

2. Show that a function u is weakly differentiable in a domain Ω if and only if it is weakly differentiable in a neighborhood of every point in Ω.

Solution:

Since Ω is a domain in $\mathbb{R}^n$, it is open and bounded. Suppose a function u is weakly differentiable in Ω, then there exists a function v with the property that for any $\varphi \in \mathcal{C}_0^{|\alpha|}(\Omega)$ we have

$$\int_\Omega \varphi v\, dx = (-1)^{|\alpha|}\int_\Omega uD^\alpha\varphi\, dx \quad (by (1.17))$$

For every $x \in \Omega$ there exists an $\epsilon > 0$ so that $\mathcal{B}_\epsilon(x) \subset \Omega$. Since weak derivatives are unique up to sets of measure zero, u is weakly differentiable in a neighborhood of x. Hence u is weakly differentiable in a neighborhood of every point in Ω.

Conversely, suppose u is weakly differentiable in a neighborhood of every point $x \in \Omega$. Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be the connected components of Ω. Then without loss of generality, we may work on one connected component, say Ω_1. Fix $x_0 \in \Omega_1$, then for any $x \in \Omega_1$ there exists, by connectedness of Ω_1, a sequence $\{x_k\}_{k=1}^n$ such that the neighborhoods, say $\mathcal{B}_{\epsilon_k}(x_k)$, in which u is weakly differentiable, have the following properties: $|\mathcal{B}_{\epsilon_k}(x_k) \cap \mathcal{B}_{\epsilon_{k+1}}(x_{k+1})| \neq 0$ for $0 \leq k \leq n$ and $|\mathcal{B}_{\epsilon_n}(x_n) \cap \mathcal{B}_{\epsilon}(x)| \neq 0$. Hence, by uniqueness of weak derivatives up to sets of measure zero, $D^\alpha u$ is the same weak derivative in every $\mathcal{B}_{\epsilon_k}(x_k)$ as well as in $\mathcal{B}_\epsilon(x)$. Since x was arbitrary, this process will work for every $x \in \Omega_1$. Hence creating a cover of Ω_1 in which u is weakly differentiable. Thus showing u is weakly differentiable in Ω_1. Continuing this process on every connected component of Ω give the final result.

3. Let α, β be multi-indices and u be a locally integrable function on a domain Ω. Show that provided any two of the weak derivatives $D^{\alpha+\beta}u$, $D^\alpha(D^\beta u)$, $D^\beta(D^\alpha u)$ exist, they all exist and coincide a.e. in Ω.

Solution:

Suppose, with out loss of generality, $D^{\alpha+\beta}u$ and $D^\alpha(D^\beta u)$ exist. Then for any test function $\varphi \in \mathcal{C}_0^\infty$ we have $D^\alpha \varphi \in \mathcal{C}_0^\infty$ is also a test function, therefore

$$\begin{aligned}
\int_\Omega D^\beta u D^\alpha \varphi \, dx &= (-1)^{|\beta|} \int_\Omega u D^{\alpha+\beta} \varphi \, dx \\
&= (-1)^{|\beta|}(-1)^{|\alpha+\beta|} \int_\Omega D^{\alpha+\beta} u \varphi \, dx \\
&= (-1)^{2|\beta|+|\alpha|} \int_\Omega D^{\alpha+\beta} u \varphi \, dx \\
&= (-1)^{|\alpha|} \int_\Omega D^{\alpha+\beta} u \varphi \, dx \\
&= (-1)^{|\alpha|}(-1)^{|\beta|} \int_\Omega D^\alpha u D^\beta \varphi \, dx \\
&= (-1)^{|\alpha|} \int_\Omega D^\beta (D^\alpha u) \varphi \, dx
\end{aligned}$$

Which gives that $D^\beta(D^\alpha u)$ exists and all three are equal a.e in Ω. A similar argument will follow given the other two combinations of the weak derivatives.

4. Let $u, v \in W^1(\Omega) \cap L^\infty(\Omega)$. Prove that $uv \in W^1(\Omega) \cup L^\infty(\Omega)$ and $D(uv) = uDv + vDu$.

Sol.

Since u and v are in $W^1(\Omega)$, by Theorem 1.3.2 there exists sequences $\{u_n\}$ and $\{v_n\} \in C^\infty(\Omega)$ such that $u_n \to u$ and $v_n \to v$ in $L^1_{loc}(\Omega)$ whose derivatives also converge in $L^1_{loc}(\Omega)$. Let $\omega \subset\subset \Omega$ and $\text{supp}(\phi) \subset \bar{\omega}$. Fix $m \in \mathbb{N}$ then we have

$$\begin{aligned}
\int_\Omega u v_m D\phi \, dx = \lim_{n\to\infty} \int_{\bar{\omega}} u_n v_m D\phi \, dx &= - \lim_{n\to\infty} \int_{\bar{\omega}} (u_n D v_m + v_m D u_n)\phi \, dx \\
&= - \int_\Omega (u D v_m + v_m D u)\phi \, dx
\end{aligned}$$

now using the fact that u is bounded, and the convergence of v_m and Dv_m in $L^1_{loc}(\Omega)$

we have $\lim\limits_{m\to\infty} \int\limits_{\Omega} uDv_m\phi\ dx = \int\limits_{\Omega} uDv\phi\ dx$, and since $\|v_m\|_\infty \leq \|v\|_\infty$ by dominated convergence we have

$$\int\limits_{\Omega} uvD\phi\ dx = \int\limits_{\Omega} (uDv + vDu)\phi\ dx$$

5. Show that a function u is weakly differentiable in a domain $\Omega \subset \mathbb{R}^n$ if and only if it is equivalent to a function $\bar{u}$ that is absolutely continuous on almost all line segments in Ω parallel to the coordinate axes and whose partial derivatives are locally integrable in Ω.

   Sol.[4]

   First suppose u is weakly differentiable. Consider a rectangular cell in Ω of the form

$$R \equiv [a_1, b_1]x \cdots x[a_n, b_n]$$

   whose side lengths are rational. We have seen in the previous chapter that the regularizers of u converge to u in the local norm. Define $x \in R$ as $x = (\tilde{x}, x_i)$ where $\tilde{x} \in \mathbb{R}^{n-1}$ and $x_i \in [a_i, b_i]$ for $1 \leq i \leq n$, then from Fubini's theorem there is a sequence $\{\epsilon_k\} \to 0$ such that

$$\lim_{\epsilon_k \to 0} \int\limits_{a_i}^{b_i} |u_{\epsilon_k}(\tilde{x}, x_i) - u(\tilde{x}, x_i)| + |Du_{\epsilon_k}(\tilde{x}, x_i) - Du(\tilde{x}, x_i)| dx_i = 0$$

   for almost all $\tilde{x}$. Since u_{ϵ_k} is smooth, for each such $\tilde{x}$ and for every $\eta > 0$, there is

$M > 0$ such that for $b \in [a_i, b_i]$ and $k > M$

$$\begin{aligned} |u_{\epsilon_k}(\tilde{x}, b) - u_{\epsilon_k}(\tilde{x}, a_i)| &\leq \int_{a_i}^{b_i} |Du_{\epsilon_k}(\tilde{x}, x_i) dx_i \\ &\leq \int_{a_i}^{b_i} |Du(\tilde{x}, x_i) dx_i + \eta \end{aligned}$$

If $\{u_{\epsilon_k}(\tilde{x}, a_i)\}$ converges as $\epsilon_k \to 0$ (which may be assumed without loss of generality), this shows that the sequence $\{u_{\epsilon_k}\}$ is uniformly bounded on $[a_i, b_i]$. Moreover, as a function of x_i, the u_{ϵ_k} are absolutely continuous, uniformly with respect to ϵ_k, because the L^1 convergence of Du_{ϵ_k} to Du implies that for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\int_E |Du_{\epsilon_k}(\tilde{x}, x_i)| dx_i < \epsilon \quad \text{whenever} \quad |E| < \delta$$

Thus, by Arzela-Ascoli theorem, $\{u_{\epsilon_k}\}$ converges uniformly on $[a_i, b_i]$ to an absolutely continuous function that agrees almost everywhere with u. The general case follows by the familiar diagonalization process.

Now suppose u has such a representative $\bar{u}$. Then $\bar{u}\varphi$ also possesses the absolute continuity properties of $\bar{u}$, whenever $\varphi \in C_0^\infty(\Omega)$. Hence, for $1 \leq i \leq n$ it follows that

$$\int_\Omega \bar{u} D_i\varphi \, dx = -\int_\Omega D_i\bar{u}\varphi \, dx$$

on almost every line segment in Ω whose end-points belong to $\mathbb{R}^n - \text{supp}\varphi$ and is parallel to the i^{th} coordinate axis. Applying Fubini implies that the weak derivative $D_i u$ has $D_i\bar{u}$ as a representative.

6. Let Ω be a domain in $\mathbb{R}^n$ containing the origin. Show that the function γ given by $\gamma(x) = |x|^{-\alpha}$ belongs to $W^k(\Omega)$ provided $k + \alpha < n$.

   Sol.

   First we will show $\gamma(x)$ is weakly differentiable if $\alpha + 1 < n$ the main result will follow by induction on k. Let $\phi^\epsilon(x) \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $\phi^\epsilon(x) = 1$ in $B_\epsilon(0)$, $\phi^\epsilon(x) = 0$ outside $B_{2\epsilon}(0)$ and $|\partial_i \phi^\epsilon(x)| \leq C/\epsilon$.

   Define
$$\gamma^\epsilon(x) = \frac{1 - \phi^\epsilon(x)}{|x|^\alpha} \in C^\infty$$

   where $\gamma^\epsilon = \gamma$ in $|x| \geq 2\epsilon$. Integration by parts yields

$$\int\limits_\Omega [\partial_i \gamma^\epsilon(x)]\varphi \, dx = -\int\limits_\Omega \gamma^\epsilon(x)(\partial_i \varphi) dx$$

   and
$$\partial_i \gamma^\epsilon(x) = \frac{-\alpha}{|x|^{\alpha+1}} \frac{x_i}{|x|}[1 - \phi^\epsilon(x)] - \frac{1}{|x|^\alpha}\partial_i \phi^\epsilon(x)$$

   We have $|\partial_i \phi^\epsilon| \leq C/\epsilon$ and $|\partial_i \phi^\epsilon| = 0$ if $|x| \leq \epsilon$ and $|x| \geq 2\epsilon$, therefore $|\partial_i \phi^\epsilon| \leq C/|x|$ so that $|\partial_i \gamma^\epsilon| \leq \frac{\alpha}{|x|^{\alpha+1}} - \frac{C}{|x|^{\alpha+1}} = \frac{C'}{|x|^{\alpha+1}}$. Also $\lim\limits_{\epsilon \to 0^+} \partial_i \gamma^\epsilon(x) = \frac{-\alpha}{|x|^{\alpha+1}} \frac{x_i}{|x|}$ pointwise a.e. By dominated convergence, we have

$$\partial_i \gamma(x) = \frac{-\alpha}{|x|^{\alpha+1}} \frac{x_i}{|x|} \quad \text{if} \ \ \alpha + 1 < n$$

   The main result now follows by induction on k.

7. Let Ω be a domain in $\mathbb{R}^n$. Show that a function $u \in C^{0,1}(\Omega)$ if and only if u is weakly differentiable with locally bounded weak derivatives.

   Sol.

   Assume, first that $u \in C^{0,1}(\Omega)$. Then there exists a constant $C \in \mathbb{R}$ such that for a.e.

$x, y \in \Omega$ we have $|u(x) - u(y)| \leq C|x - y|$. Using difference quotients we have for all $\phi \in C_0^1(\Omega)$

$$\int_\Omega \frac{u(x + he_i) - u(x)}{h}\phi(x)\,dx = \int_\Omega u(x)\frac{\phi(x - he_i) - \phi(x)}{h}dx$$

The right hand side converges as $h \to 0$ by dominated convergence theorem to

$$-\int_\Omega u(x)D_i\phi(x)\,dx$$

Therefore the left hand side also converges. We need to show that this is indeed a function in $L^\infty(\Omega)$. For any $\varphi \in C_0^1(\Omega)$ define the functional $T(\varphi)$ by

$$T(\phi) := \lim_{h \to 0}\int_\Omega \frac{u(x + he_i) - u(x)}{h}\varphi\,dx$$

Then T is linear (trivial consequence since derivatives and limits are linear) and $|T(\varphi)| \leq \|\varphi\|_1$ since

$$\text{for all } \; h > 0 |T(\varphi)| \leq \int_{\text{supp}\varphi} |\varphi| dx = \|\varphi\|_1$$
$$\Rightarrow |T(\varphi)| \leq \|\varphi\|_1$$

There for T is a bounded linear functional. Applying Reisz representation theorem, we have

$$\text{there exists a unique bounded } v \in L^\infty \text{ such that } T(\varphi) \equiv \int_\Omega v\varphi\,dx \text{ for all}$$
$$\varphi \in L^\infty \subset C_0^1(\Omega)$$

Therefore, for $\varphi \in C_0^1(\Omega)$ we have $-\int_\Omega uD_i\varphi\,dx = T(\varphi) = \int_\Omega v\varphi\,dx$. Hence $v = D_i u$.

Now assume u is weakly differentiable with locally bounded weak derivatives. Let u_ϵ be the usual mollification of u. Then $u_\epsilon \to u$ a.e. in Ω as $\epsilon \to 0$ and $\|Du_\epsilon\|_\infty \leq \|Du\|_\infty$. Choose any two points $x, y \in \Omega$ with $x \neq y$ then,

$$\begin{aligned} u_\epsilon(x) - u_\epsilon(y) &= \int_0^1 \frac{d}{dt} u_\epsilon(tx + (1-t)y)dt \\ &= \int_0^1 Du_\epsilon(tx + (1-t)y)dt(x-y) \end{aligned}$$

Taking absolute values gives

$$|u_\epsilon(x) - u_\epsilon(y)| \leq \|Du_\epsilon\|_\infty |x-y| \leq \|Du\|_\infty |x-y|$$

Letting $\epsilon \to 0$ we have the desired result, that is u is Lipschitz.

8. Show that the norms (1.23) and (1.24) are equivalent norms on $W^{k,p}(\Omega)$.

Sol.

Using the fact that all norms on $\mathbb{R}^n$ are equivalent, consider the following inequality

$$C_1 \sum_{j=1}^N |a_j| \leq \left(\sum_{j=1}^N |a_j|^p \right)^{1/p} \leq C_2 \sum_{j=1}^N |a_j|$$

This will hold for all $(a_1, \ldots, a_N) \in \mathbb{R}^N$ and some constants C_1, C_2. Now let N be the number of indices α such that $|\alpha| \leq k$ and applying the inequality to $\left(\int_\Omega |D^\alpha u|^p \, dx \right)^{1/p}$ we have

$$C_1 \sum_{|\alpha| \leq k} \|D^\alpha u\|_p \leq \left(\sum_{j=1}^N \int_\Omega |D^\alpha u|^p \, dx \right)^{1/p} \leq C_2 \sum_{|\alpha| \leq k} \|D^\alpha u\|_p$$

for some constants C_1 and C_2. Thus proving equivalence.

9. Prove that the space $W^{k,p}(\Omega)$ is complete under either of the norms (1.23), (1.24).

Sol.

Since in problem 8, we have established equivalence of these two norms, it is sufficient to prove this for either norm. Let $\{u_n\}$ be a Cauchy sequence in $W^{k,p}(\Omega)$, then for $|\alpha| \leq k$ we have $D^\alpha u_n$ is also a Cauchy sequence in $L^p(\Omega)$ by completeness of $L^p(\Omega)$. Let $u_n \to u$ and $D^\alpha u_n = v_n \to v$ in $L^p(\Omega)$. We wish to show that $u \in W^{k,p}(\Omega)$. Using the definition of weak derivative, we have

$$\int\limits_\Omega u_n D^\alpha \phi \, dx = (-1)^{|\alpha|} \int\limits_\Omega v_n \phi \, dx$$

for $\phi \in C_0^\infty(\Omega)$. Applying Holder to the differences we wish to calculate yields

$$\int\limits_\Omega (u_n - u) D^\alpha \phi \, dx \leq \|u_n - u\|_p \|D^\alpha \phi\|_q \to 0$$

$$\int\limits_\Omega (v_n - v) \phi \, dx \leq \|v_n - v\|_p \|\phi\|_q \to 0$$

These two equations hold since v_n, u_n converge in L^p to v, u resp. and for any smooth test functions L^q norms will be bounded. Therefore we have the following

$$\lim_{n\to\infty} \int\limits_\Omega u_n D^\alpha \phi \, dx = \int\limits_\Omega u D^\alpha \phi \, dx$$

$$\lim_{n\to\infty} \int\limits_\Omega v_n \phi \, dx = \int\limits_\Omega v \phi \, dx$$

$$\text{and} \int\limits_\Omega u D^\alpha \phi \, dx = \lim_{n\to\infty} \int\limits_\Omega u_n D^\alpha \phi \, dx = \lim_{n\to\infty} (-1)^{|\alpha|} \int\limits_\Omega v_n \phi \, dx = (-1)^\alpha \int\limits_\Omega v \phi \, dx$$

Hence $f \in W^{k,p}(\Omega)$.

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