DIVERGENCE FORM EQUATIONS ARISING IN MODELS FOR INHOMOGENEOUS MATERIALS

by

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Abstract

This paper will examine some mathematical properties and models of inhomogeneous materials. By deriving models for elastic energy and heat flow we are able to establish equations that arise in the study of divergence form uniformly elliptic partial differential equations. In the late 1950's DeGiorgi and Nash showed that weak solutions to our partial differential equation lie in the Holder class. After fixing the dimension of the space, the Holder exponent guaranteed by this work depends only on the ratio of the eigenvalues. In this paper we will look at a specific geometry and show that the Holder exponent of the actual solutions is bounded away from zero independent of the eigenvalues.

Contents

1 Introduction

We wish to study properties of inhomogeneous materials. The study of inhomogeneous materials frequently leads to the mathematical problem of studying solutions of the partial differential equation

$$\mathrm{div}(A(\vec{x})\nabla U) = 0 \;, \tag{1.1}$$

where $A(\vec{x})$ is a constant matrix in each material. When the individual materials being mixed are homogeneous, as we will assume, $A(\vec{x})$ is always equal to a scalar times the identity matrix, I in each separate material. This scalar value is a physical constant which differs for each inhomogeneity, and can represent the shear modulus, the thermal conductivity, or the electrical conductivity of the material. In all of these cases we can say in addition that this constant is positive.

There is already a large body of work devoted to the study of equations of the form $\mathrm{div}(A(\vec{x})\nabla U) = 0$ when $A(\vec{x})$ is a matrix which satisfies the following for positive constants Λ and λ.

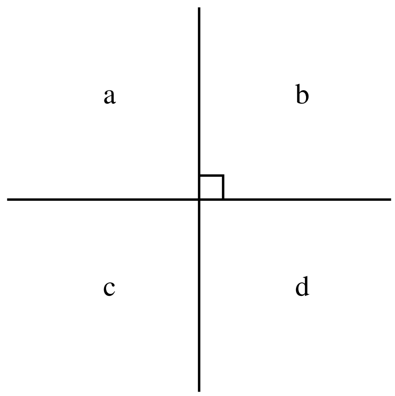
i. The boundedness property: $||A(\vec{x})\,\vec{v}|| \leq \Lambda||\vec{v}||$ for all $\vec{v} \in \mathbb{R}^n$ and $\vec{x} \in \Omega$.

ii. The positivity property: $(A(\vec{x})\vec{v}) \cdot \vec{v} \geq \lambda||\vec{v}||^2$ for all $\vec{v} \in \mathbb{R}^n$ and $\vec{x} \in \Omega$.

The positivity property (ii) is this context is called *coercivity* or $uniform$ *ellipticity*. Because of our physical model, we will always fit within this framework, which is called the theory of divergence form uniformly elliptic partial differential equations.

The first step in this theory is to make a suitable definition of solution so that uniqueness of an appropriate boundary value problem can be proven. Indeed because $A(\vec{x})$ is not assumed to be continuous, we need to define a notion of "weak solution" to get started. After defining a good notion of weak solution the first obvious question is, "How smooth are these so called solutions?"

In the much celebrated work of DeGiorgi and Nash it was shown that these weak solutions lie in the Hölder class C^γ for $\gamma > 0$. After fixing the dimension of the space, the γ guaranteed by this work depends only on the ratio $\frac{\Lambda}{\lambda}$. However, in the one sector geometry, or in the quadrant case with $ac = bd$ (See,[BT] and figure below), the Hölder exponent of the actual solutions is bounded away from zero independent of the eigenvalues, Λ and λ. In DeGiorgi's proof, the only use of the partial differential equation is to obtain the Cacciopoli estimate. We seek to improve on this result in certain cases by identifying classes of geometries for which we could potentially produce a Cacciopoli energy estimate with constants independent of the eigenvalues.



2 Heat Flow and Anti-Plane Shear

In this section we will derive the heat equation and the equation for the model of anti-plane shear from physical properties, and in each case we will end up deriving the equation $\mathrm{div}(A(\vec{x})\nabla u) = 0$. As mentioned in the introduction, solutions of this partial differential equation are typically studied in the context of inhomogeneous materials, and we will be focusing our attention on this equation for much of this paper. In this section we will also examine some physical constants and the mathematics behind them.

2.1 Heat Equation

The derivation of the heat equation is based on the following two assumptions:

i. Thermal energy (heat) is proportional to temperature (denoted by u).

ii. Heat flow is proportional to the negative of the temperature gradient.

In our first assumption it should be noted that there will be no phase change in the material. Indeed the first assumption will be false if the change in temperature causes a change in phase in this case. There will be a jump discontinuity in the energy at the temperature where the phase change occurs, and the size of the jump discontinuity will be determined by the latent heat of the material. Now, let us assume our domain is a $1 \times 1 \times 1$ brick denoted by $R \subset \mathbb{R}^3$, and consider an arbitrary open set $\Omega \subset R$. If we define $E(\Omega, t)$ to be the energy at time t in Ω, then by our first assumption

$$E(\Omega, t) = \int_{\Omega} k_s u d\vec{x},$$

where $k_s(\vec{x})$ is the specific heat capacity of the material at $\vec{x} \in \Omega$. Then it follows that

$$\frac{d}{dt}E(\Omega, t) = \int_\Omega k_s u_t d\vec{x},$$

where u_t is the partial derivative of u with respect to time. The rate of change of energy with respect to time is also equal to the flux of heat into Ω, and by our second assumption this quantity is proportional to the negative temperature gradient. Hence

$$\frac{d}{dt}E(\Omega, t) = \int_{\partial\Omega} \sigma \frac{\partial u}{\partial n} d\mathcal{H}^{n-1},$$

where $\sigma(\vec{x})$ is heat conductivity of the material at $\vec{x} \in \Omega$, and n is the outward unit normal vector. By the divergence theorem,

$$\int_{\partial\Omega} \sigma \frac{\partial u}{\partial n} d\mathcal{H}^{n-1} = \int_\Omega \operatorname{div}(\sigma \nabla u) d\vec{x},$$

and thus

$$\int_\Omega k_s u_t d\vec{x} = \int_\Omega \operatorname{div}(\sigma \nabla u) d\vec{x}.$$

This leads us to the pointwise equation

$$k_s u_t = \operatorname{div}(\sigma \nabla u),$$

by the usual arguments of the calculus of variations. If the material is homogeneous, then σ and k_s are each constant, and therefore we can pass the divergence to the inside to get the heat equation

$$u_t = c^2 \Delta u, \tag{2.1}$$

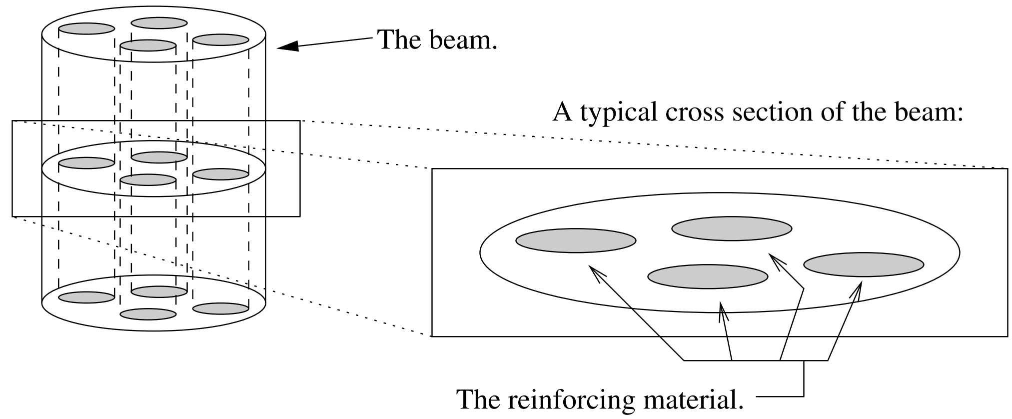
where $c^2 := \frac{\sigma}{k_s}$. If the boundary data is independent of time and we wait for thermal equilibrium, then we will get $\Delta u = 0$ in the homogeneous case. In the inhomogeneous case if we wait for thermal equilibrium we will get

$$\operatorname{div}(A(\vec{x}) \nabla u) = 0, \tag{2.2}$$

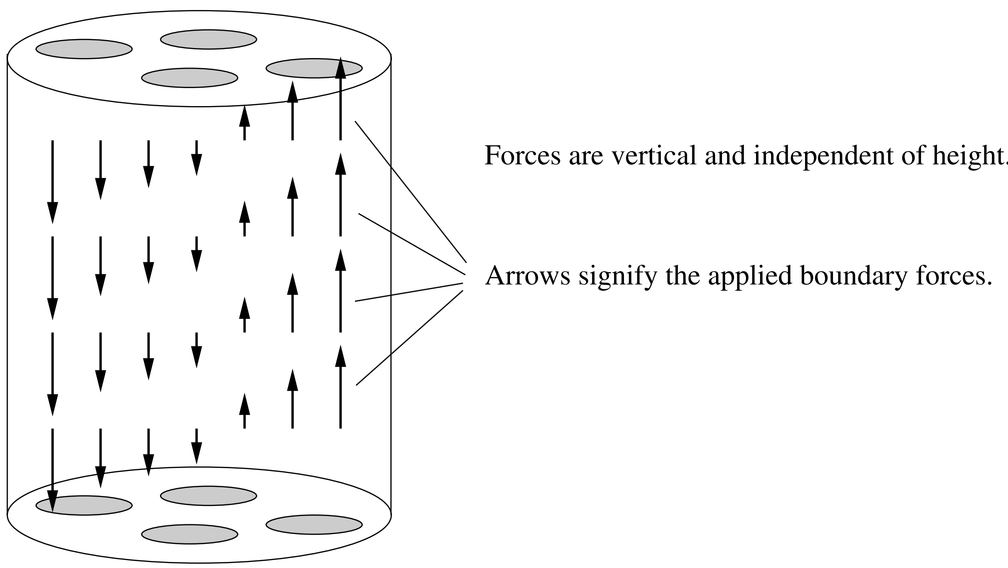
where $A(\vec{x})$ is a scalar matrix such that the scalar value is the physical constant σ.

2.2 Anti-Plane Shear

Consider the following physical situation: assume we have a very long reinforced beam positioned vertically with identical cross sections, and assume the beam is long enough so that we can approximate our calculations with an infinite beam.

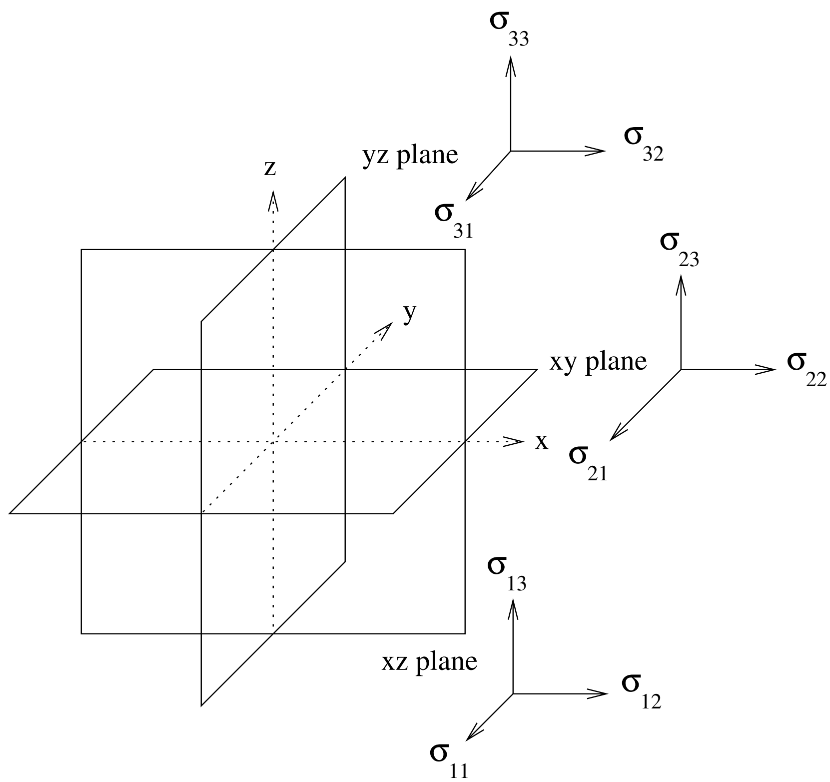


We will have a 'rubber material' pushing one side of the beam in the upward direction while the rubber material is pushing the other side of the beam in the downward direction. Then there will exist a force applied to the beam which is purely tangential in the vertical direction. We further assume that this tangential force may depend on the x and y coordinates, but it will not depend on the vertical coordinate.



It follows that we have no horizontal force applied to the boundary of the beam, and there will be no lateral stretching or compressing of the beam. In the literature, for sufficiently small forces, this physical situation is typically understood via the mathematical model known as anti-plane shear.

Before we proceed we need to make note of a few physical definitions. Stress is defined locally as the map from an infinitesimal plane containing the point in consideration to the infinitesimal force acting on that plane. By identifying the infinitesimal plane with its normal vector, the stress is then viewed as a 3×3 matrix mapping that vector to the force vector on the corresponding plane.



It can be shown by delicate physical arguments that the corresponding 3×3 matrix is symmetric. We will use σ to denote the stress. Strain is defined as the change in dimension per unit length caused by the action of a stress on a physical body. In other words, strain measures the deformation or change in shape that a body undergoes when subject to a force. We will use γ to denote the strain, which will also turn out to be a symmetric 3×3 matrix. Stress is usually expressed in psi (pounds per square inch) or Pa (Pascals, which are newtons per square meter), while strain is dimensionless and is often expressed as in./in. or cm/cm [AP]. Stress and strain are related by a response function which we denote $\mathbf{T}^D$, and in the case of linear elasticity $\mathbf{T}^D$ is approximated with a tensor, which we will call $\tilde{A}(\vec{x})$.

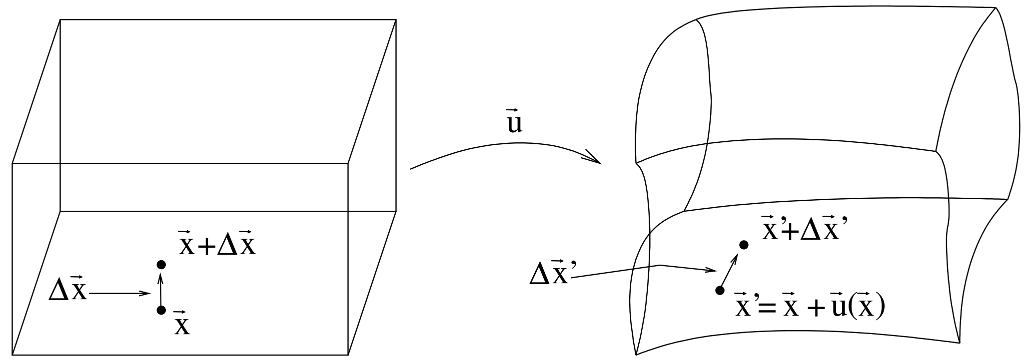
We will be considering an elastic strain which is defined as a fully recoverable strain resulting from an applied stress, or in other words a material which undergoes an elastic strain does not show any permanent deformations, i.e. it will return to the original shape once the stress is removed. In many materials sufficiently small strains are elastic, but if the force is increased until the deformation is permanent, it is labeled a plastic strain. All of this can be understood by considering the tension properties of a rubberband. If a small enough stress is applied to a rubberband, then the band will return to its original shape with no permanent deformation, that is it will retain its original tension throughout the material. This is an example of an elastic strain. On the other hand, the applied stress can be increased further, but without breaking the rubberband so that the band will lose some of its original tension. In this case there will be some internal tearing of the material, and the applied stress has caused a plastic strain. After the stress is removed the rubberband will be limp. A paper clip will also serve as a good example of elastic and plastic strains. If the stress applied to the paper clip is small enough it can still be used to hold papers effectively, again this case is an example of a elastic strain. However, if the clip gets bent too far in one direction it will no longer be able to hold the papers together. Then in this case it will not return to its original shape because the stress was too great, and this is an example of a plastic strain.

In our situation the strains and stresses we will be considering are assumed to be linearly related, which is a good model for sufficiently small stresses. In this particular physical model we will assume that each of the materials individually are homogeneous and isotropic. A point of a material is said to be isotropic if the properties are the same in all directions, i.e. a rotation of the material will have no effect. A material is said to be anisotropic if certain directions are favored over others. Examples of anisotropies include the formation of crystals in non-radially symmetric geometries, and materials that compress more easily in one direction than another.

We now consider the elastic energy associated with the model of anti-plane shear deformation. The elastic energy is the integral of the dot-product of the strain and stress tensors on the material. Because of our assumed boundary data (recall we are discussing the model of anti-plane shear), we have

$$u_1 = u_2 \equiv 0, \text{ and } \quad u_3(x_1, x_2, x_3) = u_3(x_1, x_2), \tag{2.3}$$

where $\vec{u}$ represents the displacement of the point after the forces are applied. Before we can proceed with computing the elastic energy, we need to derive the strain tensor. (We follow Symon [S].) We let the position of a point in a specific material be given by the vector $\vec{x}$. Let the point $\vec{x}$ move a small distance, under a deformation, to a new position given by $\vec{x}' = \vec{x} + \vec{u}(\vec{x})$. Let $\vec{x} + \Delta\vec{x}$ be a point that is close to $\vec{x}$, and call $\Delta\vec{x}$ the "location vector." Then after the deformation, the new position of this point will be given by $\vec{x}' + \Delta\vec{x}' = \vec{x} + \Delta\vec{x} + \vec{u}(\vec{x} + \Delta\vec{x})$.



Approximating $\vec{u}$ by the first two terms of its Taylor series expansion, we see that

$$\vec{u}(\vec{x}+\Delta\vec{x}) \approx \vec{u}(\vec{x}) + \mathbf{D}\vec{u}(\vec{x})\Delta\vec{x},$$

and therefore

$$\begin{aligned} \vec{x}' + \Delta\vec{x}' &\approx \vec{x} + \Delta\vec{x} + \vec{u}(\vec{x}) + \mathbf{D}\vec{u}(\vec{x})\Delta\vec{x} \\ &= \vec{x}' + \Delta\vec{x} + \mathbf{D}\vec{u}(\vec{x})\Delta\vec{x}. \end{aligned}$$

We conclude

$$\Delta\vec{x}' - \Delta\vec{x} \approx \mathbf{D}\vec{u}(\vec{x})\Delta\vec{x},$$

where $\mathbf{D}\vec{u}(\vec{x})$ is the corresponding change in displacement matrix.

Given a matrix M, since

$$M = \frac{1}{2}\left(M - M^T\right) + \frac{1}{2}\left(M + M^T\right),$$

we see that any matrix may be written as the sum of a skew-symmetric matrix and a symmetric matrix. A skew-symmetric matrix is any matrix A that has the property $A^T = -A$. Consider the dot product of the skew-symmetric matrix A applied to a vector $\vec{y}$ with $\vec{y}$. Then

$$(\vec{y}, A\vec{y}) = (A^T\vec{y}, \vec{y}) = (-A\vec{y}, \vec{y}) = -(A\vec{y}, \vec{y}) = -(\vec{y}, A\vec{y}),$$

since A is a real valued matrix. Thus $(\vec{y}, A\vec{y}) = 0$ for any skew-symmetric matrix, and in other words any skew-symmetric matrix applied to a vector is always perpendicular to that original vector. So skew-symmetric matrices applied to infinitesimal vectors produce infinitesimal rotations.

Applying the preceding linear algebra to our case we get

$$\Delta\vec{x}' - \Delta\vec{x} \approx \frac{1}{2}\left(\mathbf{D}\vec{u}(\vec{x}) - \mathbf{D}\vec{u}(\vec{x})^T\right)\Delta\vec{x} + \frac{1}{2}\left(\mathbf{D}\vec{u}(\vec{x}) + \mathbf{D}\vec{u}(\vec{x})^T\right)\Delta\vec{x}.$$

Note that

$$\mathbf{D}\vec{u}(\vec{x}) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} =: \frac{\partial u_i}{\partial x_j}, \tag{2.4}$$

and similarly $\mathbf{D}\vec{u}(\vec{x})^T =: \frac{\partial u_j}{\partial x_i}$. Then

$$\Delta\vec{x}' - \Delta\vec{x} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\Delta\vec{x} + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\Delta\vec{x}. \tag{2.5}$$

Since the first term of Equation (2.5) is a skew-symmetric matrix it represents a rotation, as discussed above. Since a rotation is a rigid motion, it does not represent a deformation of the material. Again since we are concerned only with the deformation of the material, it is the second part of Equation (2.5) (the symmetric matrix) that we are interested in. This equation can now be read as saying that the change in the location vector after the motion, is a rigid motion plus a linear map (the strain) applied to the orginal location vector. Therefore the strain tensor is given by

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.6}$$

Recalling Equation (2.3) in this model of anti-plane shear we have the strain tensor given in matrix form

$$\gamma := \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_1} \\ 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_2} \\ \frac{1}{2}\frac{\partial u_3}{\partial x_1} & \frac{1}{2}\frac{\partial u_3}{\partial x_2} & 0 \end{bmatrix}. \tag{2.7}$$

The stress tensor associated with this elastic model is the output of the response function given by $\mathbf{T}^D(\vec{x}, \nabla\vec{u}(\vec{x})) = \mathbf{T}^D(\vec{x}, \nabla u_3)$, where $\mathbf{T}^D$ is a mapping

$$\mathbf{T}^D : (\vec{x}, F) \in \overline{\Omega} \times \mathrm{Sym}(3) \to \mathbf{T}^D(\vec{x}, F) \in \mathrm{Sym}(3),$$

where Sym(3) denotes the set of all symmetric matrices of order three [CI]. In other words the domain of the response function is the set of strains, and the range or output is the set of stresses. By linearizing $\mathbf{T}^D$ at a point $\vec{x} \in \overline{\Omega}$ we have,

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \tilde{A}(\vec{x}) \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{12} \\ 2\gamma_{23} \\ 2\gamma_{31} \end{bmatrix} \tag{2.8}$$

where $\tilde{A}(\vec{x})$ is a 6×6 matrix called the elasticity matrix. Making use of all of the additional symmetries that we enjoy at an isotropic point along with our additional assumption on $\vec{u}$, allows us to express Equation (2.8) in the simplified form,

$$\mathbf{T}^D(\vec{x}, \nabla\vec{u}_3) = A(\vec{x}) \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_1} \\ 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_2} \\ \frac{1}{2}\frac{\partial u_3}{\partial x_1} & \frac{1}{2}\frac{\partial u_3}{\partial x_2} & 0 \end{bmatrix}, \tag{2.9}$$

where $A(\vec{x})$ is a scalar. We will give some of the details in a moment. Now the elastic energy, denoted by $E(u)$, is given by

$$\begin{aligned}
& \int_\Omega \sum_{i,j} \gamma_{ij} \mathbf{T}_{ij}^D d\vec{x} \\
= & \int_\Omega A(\vec{x}) \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_1} \\ 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_2} \\ \frac{1}{2}\frac{\partial u_3}{\partial x_1} & \frac{1}{2}\frac{\partial u_3}{\partial x_2} & 0 \end{bmatrix} \bullet \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_1} \\ 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_2} \\ \frac{1}{2}\frac{\partial u_3}{\partial x_1} & \frac{1}{2}\frac{\partial u_3}{\partial x_2} & 0 \end{bmatrix} d\vec{x} \\
= & \int_\Omega A(\vec{x}) \left(\frac{1}{2} \left(\frac{\partial u_3}{\partial x_1} \right)^2 + \frac{1}{2} \left(\frac{\partial u_3}{\partial x_2} \right)^2 \right) d\vec{x} \\
= & \frac{1}{2} \int_\Omega A(\vec{x}) |\nabla u_3|^2 d\vec{x}.
\end{aligned}$$

Letting $A(\vec{x})$ absorb the 1/2 we now have the following expression of elastic energy,

$$E(u) := \int_\Omega A(\vec{x}) |\nabla u|^2 d\vec{x}. \tag{2.10}$$

When we minimize this quantity, as we will show in the next section, we will get the Euler-Lagrange equation

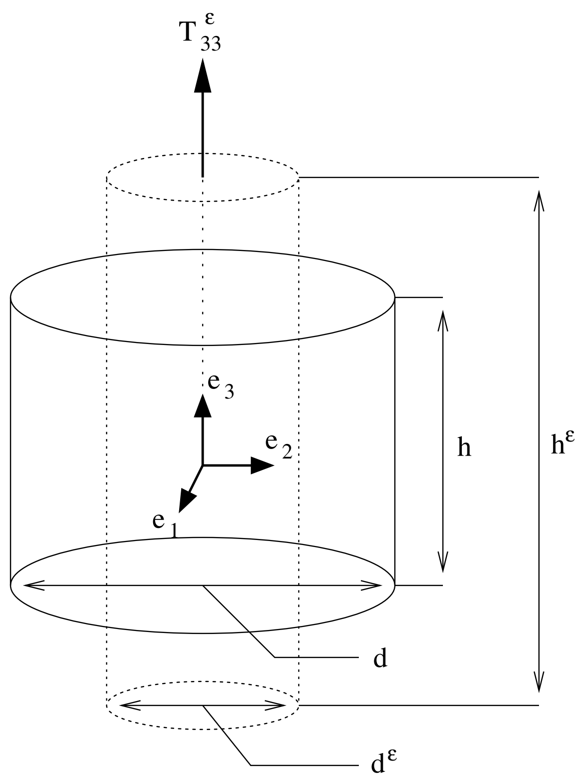
$$\operatorname{div}(A(\vec{x})\nabla u) = 0.$$

Because $A(\vec{x})$ is typically discontinuos, we also need to introduce a suitable notion of weak solution, and this we will also discuss in the next section.

In the introduction we mentioned that $A(\vec{x})$ is a physical constant which differs for each material and can represent the shear modulus, the thermal conductivity, or the electrical conductivity of the material. Before we show how $A(\vec{x})$ is calculated in applied practices we need to make a few definitions. In the isotropic case that we are considering it can be shown that the elasticity matrix $\tilde{A}(\vec{x})$ determined by two constants, λ and μ known as the Lamé constants, has the structure given by the following equation

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{12} \\ 2\gamma_{23} \\ 2\gamma_{31} \end{bmatrix} \tag{2.11}$$

(see Chung [CH]). These constants are determined completely by the Young modulus of elasticity, and the Poisson ratio (E and ν respectively.) E and ν are constituitive properties of the material under consideration, and are typically calculated by engineers via a single experiment. By considering the "uniform traction" of a circular cylinder, the Poisson ratio of the material is computed by the ratio of the relative decrease in diameter of the cylinder (lateral deformation), and the relative longitudinal increase caused by applied surface forces. The Young modulus measures the ratio of the tensile stress and the relative increase in length of the cylinder. That is, it will be given by the slope of the tensile-strain curve in the linear case.



Then by definition this experiment gives the Poisson ratio by

$$\nu = \frac{\left(\dfrac{d - d^{\epsilon}}{d}\right)}{\left(\dfrac{h^{\epsilon} - h}{h}\right)}, \tag{2.12}$$

and the Young modulus by

$$E = \frac{T^{\epsilon}_{33}}{\left(\dfrac{h^{\epsilon} - h}{h}\right)}. \tag{2.13}$$

Once E and ν are computed, it can be shown that λ and μ are given by the equations,

$$\mu = \frac{E}{2(1+\nu)}, \tag{2.14}$$

and

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}. \tag{2.15}$$

Equations (2.14) and (2.15) allow mathematicians to determine μ and λ from the Young modulus and Poisson ratio. Alternatively, a simple computation can show how E and ν can be expressed in terms of μ and λ by the following equations,

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \tag{2.16}$$

and

$$\nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{2.17}$$

From these equations an engineer can look at the elasticity matrix (or the Lamé constants) and see the constitutive properties of the isotropic material under consideration.

Using Equations (2.11), (2.14), and (2.15) leads to the following expression for an isotropic material,

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = a \begin{bmatrix} 1 & b & b & 0 & 0 & 0 \\ b & 1 & b & 0 & 0 & 0 \\ b & b & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{12} \\ 2\gamma_{23} \\ 2\gamma_{31} \end{bmatrix} \tag{2.18}$$

where

$$a = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)},$$
$$b = \frac{\nu}{1-\nu}, \quad \text{and}$$
$$c = \frac{(1-2\nu)}{2(1-\nu)}.$$

Recalling Equation (2.3) and Equation (2.6), gives us

$$\gamma_{11} = \gamma_{22} = \gamma_{33} = \gamma_{12} = 0. \tag{2.19}$$

Therefore only γ_{23} and γ_{31} are nonzero. Then the only components of the stress tensor that are nonzero are σ_{23} and σ_{31}. With our knowledge of the nonzero components of the stress tensor, and expanding Equation (2.18) we have

$$\sigma_{23} = ac\frac{\partial u_3}{\partial x_2}, \text{ and } \quad \sigma_{31} = ac\frac{\partial u_3}{\partial x_1}.$$

After a trivial calculation we see that

$$ac = \frac{E}{2(1+\nu)} = \mu. \tag{2.20}$$

For our specific model of anti-plane shear the constant matrix $A(\vec{x}) := \mu I$ where I is the order two identity matrix. Then in this model we have

$$\text{div}\,(\mu(x,y)I\nabla u_3) = 0.$$

3 Weak Solutions and Compatibility Conditions

As mentioned earlier we are concerned with the notion of a weak solution. It is important to understand where the nature of this solution comes from. We begin by trying to minimize the following expression

$$J(u) = \int_\Omega (A(\vec{x})\nabla u)\cdot\nabla u\; d\vec{x}, \tag{3.1}$$

among functions u which are equal to a given function Ψ on $\partial\Omega$. It is a consequence of the direct methods of the calculus of variations, that if Ψ and $\partial\Omega$ are sufficiently smooth, and $A(\vec{x})$ satisfies the boundedness property, and the positivity property (i and ii above), then there exists a unique minimizer of J with given boundary values. Let u_0 be this minimizer. Next, let φ be any smooth function that vanishes at the boundary. Then observe that $u_0+t\varphi = \Psi$ on $\partial\Omega$ for all t, so that $J(u_0+t\varphi) \geq J(u_0)$. Then if

$$H(t) := J(u_0+t\varphi) = \int_\Omega (A(\vec{x})(\nabla u_0 + t\nabla\varphi))\cdot(\nabla u_0 + t\nabla\varphi)d\vec{x}, \tag{3.2}$$

then $H^{'}(0) = 0$ which means

$$\int_\Omega [(A(\vec{x})\nabla u_0)\cdot\nabla\varphi + (A(\vec{x})\nabla\varphi)\cdot\nabla u_0]d\vec{x} = 0. \tag{3.3}$$

Since our matrix $A(\vec{x})$ is always a scalar, we can commute with the dot product inside the integral and after mulitplying through by $1/2$ we see that

$$\int_\Omega \nabla\varphi(A(\vec{x})\nabla u)d\vec{x} = 0. \tag{3.4}$$

Now we say that u is a *weak solution* to the partial differential Equation (1.1) if Equation (3.4) is satisfied for all $\varphi \in C_0^\infty(\overline{\Omega})$.

Before we procede, the following observations about the minimization of expression (3.1) for anywhere that $A(\vec{x})$ is constant should be noted. Let $\Omega' \subset \Omega$ where $A(\vec{x}) = b$ and let $\partial\Omega' = \sigma_1 \cup \sigma_2$ such that $\sigma_1 \cap \sigma_2 = \emptyset$. Suppose u is prescribed only on σ_1 and minimizes

$$I(u) := \int_{\Omega'} b|\nabla u|^2 d\vec{x},$$

then $I(u+\varphi) \geq I(u)$ as long as $\varphi|_{\sigma_1} \equiv 0$. Then for any such φ we have Equation (3.4) where Ω is replaced by Ω'. If we assume in addition that φ vanishes on all $\partial\Omega'$ then we have the following by integration by parts,

$$-\int_{\Omega'} \varphi(b\Delta u)d\vec{x} = 0.$$

Since φ is arbitrary inside Ω', we conclude $\Delta u = 0$. If we now consider the set of φ's that do not vanish on σ_2, and integrate Equation (3.4) (with Ω replaced by Ω') by parts then

$$-\int_{\Omega'} \varphi(b\Delta u)d\vec{x} + \int_{\sigma_2} \varphi b\frac{\partial u}{\partial n}d\vec{x} = 0.$$

By the the previous observation the first integral vanishes and since $\varphi|_{\sigma_2}$ is arbitrary, $\frac{\partial u}{\partial n} = 0$ on σ_2. Both of these observations will prove useful in the subsequent sections.

We now wish to establish the following compatiblity condition when $A(\vec{x})$ has a jump discontinuity across a line. In this section we will follow the exposition of Section 4.4 from Blank and Tsutsui (See [BT].)

3.1 Proposition (Sufficient pointwise compatibility conditions). *Let*

$$A(\vec{x}) = \begin{cases} a & y \geq 0 \\ b & y < 0 \end{cases}$$

Assume U is a continuous piecewise differentiable function such that $\Delta U = 0$ in B_1^+, $\Delta U = 0$ in B_1^-, and $aU_y^+(x,0) = bU_y^-(x,0)$ for all $x \in (-1,1)$. Then U is a weak solution of Equation ((1.1)).

Proof. We let $\varphi \in C_0^1(B_1)$ and use Green's identities and our assumptions to compute

$$
\begin{aligned}
&\int_{B_1} \nabla\varphi A(x)\nabla U d\vec{x} \\
&= a\int_{B_1^+} \nabla\varphi\cdot\nabla U d\vec{x} + b\int_{B_1^-} \nabla\varphi\cdot\nabla U d\vec{x} \\
&= \ -a\int_{B_1^+} \varphi\Delta U d\vec{x} + a\int_{\partial B_1^+} \varphi\frac{\partial U}{\partial n} d\mathcal{H}^1 - b\int_{B_1^-} \varphi\Delta U d\vec{x} + b\int_{\partial B_1^-} \varphi\frac{\partial U}{\partial n} d\mathcal{H}^1 \\
&= a\int_{\{y=0\}\cap\partial B_1^+} \varphi\frac{\partial U}{\partial n} dx + b\int_{\{y=0\}\cap\partial B_1^-} \varphi\frac{\partial U}{\partial n} dx \\
&= \ \int_{\{y=0\}\cap\partial B_1^+} \varphi(-aU_y^+(x,0))dx + \int_{\{y=0\}\cap\partial B_1^-} \varphi(bU_y^-(x,0))dx \\
&= \ -\int_{\{y=0\}} \varphi(aU_y^+)dx + \int_{\{y=0\}} \varphi(aU_y^+)dx \\
&= 0.
\end{aligned}
$$

Q.E.D.
The converse is also true.

3.2 Theorem (Necessary Pointwise Compatibility Condition). *Take $A(\vec{x})$ as above and if $U(x,y)$ is a continuous weak solution to (1.1), then*

$$aU_y^+(x,0) = bU_y^-(x,0)\ . \tag{3.5}$$

Before we can prove the compatibility condition above, we must first quote and prove some results. The following elliptic regularity result can be found in a paper by Li and Vogelius. (See [LV].)

3.3 Theorem. *If $U(x,y) \in H^1(B_1)$ denotes weak a solution to (1.1) and $U^\pm(x,y) = U(x,y)$ for $x \in \overline{B_1^\pm}$, then $U^\pm \in C^\infty(\overline{B_1^\pm})$.*

The techniques they employ will generalize to the case where C^∞ assumptions are replaced by real analytic, and we get the following.

3.4 Corollary. *If $U(x,y)$ is a weak solution of Equation (1.1) in B_1, then U is continuous in B_1, is real analytic on $\overline{B_1^+}$, and is real analytic on $\overline{B_1^-}$.*

3.5 Proposition. *If*

$$A(\vec{x}) = \begin{cases} a & y \geq 0 \\ b & y < 0 \end{cases}$$

then,

(a) $U(x,y) = c_0 + c_1 x$ is a weak solution of Equation (1.1) for any c_0 and $c_1 \in \mathbb{R}$

(b) If $U(x,y)$ is a weak solution of (1.1) with $U(0,0)=0$, then $U_\epsilon(x,y) = \epsilon^{-1}U(\epsilon x, \epsilon y)$ is also a weak solution of (1.1) for all ϵ, $0<\epsilon<1$.

(c) If U_n is a sequence of weak solutions of Equation (1.1) in B_1 and U_n converges to U uniformly in B_1 and ∇U_n converges to ∇U uniformly in B_1, then U is a weak solution of Equation (1.1)

Proof.

(a) Since $U_x = c_1$ and $U_y = 0$ where c_0 and $c_1 \in \mathbb{R}$, then equation (3.4) can be written as

$$I := \int_{B_1} (\varphi_x, \varphi_y) \left(A(x) \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \right) d\vec{x} = 0.$$

Then the following computation shows u satisfies our definition of a weak solution.

$$\begin{aligned} I &= \iint_{B_1^+} (\varphi_x, \varphi_y) \cdot (ac_1, 0) dxdy + \iint_{B_1^-} (\varphi_x, \varphi_y) \cdot (bc_1, 0) dxdy \\ &= ac_1 \iint_{B_1^+} \varphi_x dxdy + bc_1 \iint_{B_1^-} \varphi_x dxdy \\ &= ac_1 \int_0^1 (0-0)dy + bc_1 \int_{-1}^0 (0-0)dy = 0, \end{aligned}$$

since φ is vanishes on the boundary of B_1. This proves $U(x,y) = c_0 + c_1 x$ is a weak solution to (1.1).

(b) For the second item, let $\vec{x}' := \epsilon \vec{x}$ so that $x' = \epsilon x$, $y' = \epsilon y$, and $dA(\vec{x}') = \epsilon^2 dA(\vec{x})$. Then it follows that $\nabla_x V = \epsilon \nabla_{x'} V$. If we let $U_\epsilon(x,y) := \epsilon^{-1}U(\epsilon x, \epsilon y) = \epsilon^{-1}U(x', y')$, then we show $U_\epsilon(x,y)$ is a weak solution to (1.1) whenever $U(x,y)$ is. Suppose $\varphi \in C_0^\infty(\overline{B}_1)$, then

$$\begin{aligned} I &= \int_{B_1} \nabla\varphi(\vec{x}) A(\vec{x}) \nabla U_\epsilon(x,y) dA(\vec{x}) \\ &= \epsilon^{-1} \int_{\vec{x} \in B_1} \nabla_{\vec{x}}\varphi(\vec{x}) A(\vec{x}) \nabla_{\vec{x}} U(\epsilon x, \epsilon y) dA(\vec{x}) \\ &= \epsilon^{-1} \int_{\vec{x}' \in B_\epsilon} \nabla_{\vec{x}}\varphi(\epsilon^{-1}\vec{x}') A(\epsilon^{-1}\vec{x}') \nabla_{\vec{x}} U(x', y') \epsilon^2 dA(\vec{x}'). \end{aligned} \tag{3.6}$$

Now restricting $\varphi \in C_0^\infty(B_\epsilon) \subset C_0^\infty(B_1)$ we let $\psi(\vec{x}') := \varphi(\epsilon^{-1}\vec{x}')$. Furthermore we note that $A(\delta\vec{x}) = A(\vec{x})$ for any $\delta > 0$, then

$$\begin{aligned} I &= \epsilon \int_{\vec{x}' \in B_\epsilon} \nabla_{\vec{x}} \psi(\vec{x}') A(\vec{x}') \nabla_{\vec{x}} U(x', y') dA(\vec{x}') \\ &= \epsilon^3 \int_{\vec{x}' \in B_\epsilon} \nabla_{\vec{x}'} \psi(\vec{x}') A(\vec{x}') \nabla_{\vec{x}'} U(\vec{x}') dA(\vec{x}'). \end{aligned} \tag{3.7}$$

Since we have defined φ to vanish outside B_ϵ, and using the assumption that $U(x, y)$ is a weak solution to (1.1) in B_1 we obtain the following,

$$I = \epsilon^3 \int_{\vec{x}' \in B_1} \nabla_{\vec{x}'} \psi(\vec{x}') A(\vec{x}') \nabla_{x'} U(\vec{x}') dA(\vec{x}') = 0.$$

Therefore $U_\epsilon(x, y)$ is a weak solution of Equation (1.1) in B_1 as well.

(c) Again assume $\varphi \in C_0^\infty(\overline{B}_1)$. Then,

$$\begin{aligned} & \int_{B_1} \nabla\varphi(A(\vec{x})\nabla U) d\vec{x} \\ = & \int_{B_1} \nabla\varphi(A(\vec{x})(\nabla U - \nabla U_n + \nabla U_n)) d\vec{x} \\ = & \int_{B_1} \nabla\varphi(A(\vec{x})(\nabla U - \nabla U_n)) d\vec{x} + \int_{B_1} \nabla\varphi(A(\vec{x})\nabla U_n) d\vec{x} \\ = & \int_{B_1} \nabla\varphi(A(\vec{x})(\nabla U - \nabla U_n)) d\vec{x}. \end{aligned}$$

Now consider the following,

$$\begin{aligned} & \left| \int_{B_1} \nabla\varphi(A(\vec{x})(\nabla U - \nabla U_n)) d\vec{x} \right| \\ \leq & \int_{B_1} |\nabla\varphi||A(\vec{x})||\nabla U - \nabla U_n| d\vec{x} \\ \leq & \max_{\vec{x} \in B_1} |\nabla\varphi(\vec{x})| \cdot \max\{a, b\} \cdot \max_{\vec{x} \in B_1} |\nabla U(\vec{x}) - \nabla U_n(\vec{x})| \cdot |\pi| \\ \leq & C \cdot \max_{\vec{x} \in B_1} |\nabla U(\vec{x}) - \nabla U_n(\vec{x})|. \end{aligned} \tag{3.8}$$

Then by combining the last two computations we see that

$$\left| \int_{B_1} \nabla\varphi(A(x)\nabla U) dx \right| \leq C \cdot \max_{\vec{x} \in B_1} |\nabla U(\vec{x}) - \nabla U_n(\vec{x})|.$$

Since we assume ∇U_n converges uniformly to ∇U, and our constant C is independent of n then by taking the limit as $n \to \infty$ in the last inequality we get

$$\left| \int_{B_1} \nabla\varphi(A(x)\nabla U)dx \right| = 0.$$

This equality proves $U(\vec{x})$ is a weak solution of Equation (1.1).

Q.E.D.

We now have the sufficient tools required to prove Theorem 3.2, the necessary pointwise compatibility condition, and we begin the proof.

Proof. By Corollary 3.4, $U(x,y)$ is real analytic on $\overline{B_1^+}$ and $\overline{B_1^-}$, and therefore we express $U(x,y)$ as the following

$$U(x,y) = \begin{cases} C_0 + C_1 x + C_2^+ y + C_{11}x^2 + C_{12}^+ xy + C_{22}^+ y^2 + \cdots & y \geq 0 \\ C_0 + C_1 x + C_2^- y + C_{11}x^2 + C_{12}^- xy + C_{22}^- y^2 + \cdots & y \leq 0. \end{cases}$$

Since $U_y^+(x,0) = C_2^+$ and $U_y^-(x,0) = C_2^-$, it suffices to show $aC_2^+ = bC_2^-$. We define $\tilde{U}(x,y) := U(x,y) - C_0 - C_1 x$ and using proposition 3.5 (a) and the fact that $U(x,y)$ is linear, $\tilde{U}(x,y)$ is a solution. Then using proposition 3.5 (b) we see that $\tilde{U}_\epsilon(x,y) := \epsilon^{-1}\tilde{U}(\epsilon x, \epsilon y)$ is also a solution. After some simple computations we obtain the following

$$\tilde{U}_\epsilon(x,y) = \begin{cases} C_2^+ y + \epsilon C_{11}x^2 + \epsilon C_{12}^+ xy + \epsilon C_{22}^+ y^2 + O(\epsilon^2) & y \geq 0 \\ C_2^- y + \epsilon C_{11}x^2 + \epsilon C_{12}^- xy + \epsilon C_{22}^- y^2 + O(\epsilon^2) & y \leq 0, \end{cases}$$

Then as $\epsilon \to 0$, $\tilde{U}_\epsilon(x,y)$ converges to

$$\tilde{U}_0(x,y) = \begin{cases} C_2^+ y & y \geq 0 \\ C_2^- y & y \leq 0. \end{cases}$$

This convergence is uniform in both the function and its gradient. Then by proposition 3.5 (c), $\tilde{U}_0(x,y)$ is a weak solution of equation (1.1) given any $\varphi \in C_0^\infty(\overline{\Omega})$, and we have

$$\int_{B_1} \nabla\varphi(A(x)\nabla\tilde{U}_0)dA = 0.$$

We now compute the following,

$$
\begin{aligned}
&\int_{B_1} \nabla\varphi(A(x)\nabla\tilde{U}_0)dA \\
&= a\int_{B_1^+}(\varphi_x,\varphi_y)\cdot(0,C_2^+)dA + b\int_{B_1^-}(\varphi_x,\varphi_y)\cdot(0,C_2^-)dA \\
&= aC_2^+\int_{B_1^+}\varphi_y\ dydx + bC_2^-\int_{B_1^-}\varphi_y\ dydx \\
&= aC_2^+\left(0-\int_{-1}^{1}\varphi(x,0)\ dx\right) + bC_2^-\left(\int_{-1}^{1}\varphi(x,0)\ dx - 0\right) \\
&= \left(-aC_2^+ + bC_2^-\right)\int_{-1}^{1}\varphi(x,0)\ dx.
\end{aligned}
$$

By choosing an appropriate φ where the last integral is nonzero we have $aC_2^+ = bC_2^-$.
Q.E.D.

3.6 Theorem (Cacciopoli's Inequality). *If $\lambda I \leq A(\vec{x}) \leq \Lambda I$,*

$$\int_{B_1} \nabla\varphi A(\vec{x})\nabla u d\vec{x} = 0 \tag{3.9}$$

for all $\varphi \in C_0^1(B_1)$, and $\eta \in C_0^1(B_1)$, then

$$\int_{B_1}\eta^2|\nabla u|^2 d\vec{x} \leq C(\lambda,\Lambda)\int_{B_1}|\nabla\eta|^2 u^2 d\vec{x}. \tag{3.10}$$

Proof. We begin by letting $\varphi = \eta^2 u$ such that $\eta \in C_0^1(B_1)$. Since $\nabla\varphi = 2\eta u\nabla\eta + \eta^2\nabla u$ we have

$$\int_{B_1}\nabla\varphi A(\vec{x})\nabla u d\vec{x} = \int_{B_1} 2\eta u\nabla\eta[A(\vec{x})\nabla u]d\vec{x} + \int_{B_1}\eta^2\nabla u[A(\vec{x})\nabla u]d\vec{x},$$

Then

$$
\begin{aligned}
&\int_{B_1}\lambda\eta^2|\nabla u|^2 \\
\leq\ &\int_{B_1}\eta^2\nabla u[A(\vec{x})\nabla u]d\vec{x} \\
=\ &-\int_{B_1} 2\eta u\nabla\eta[A(\vec{x})\nabla u]d\vec{x} \\
\leq\ &\int_{B_1} 2\Lambda|\nabla u||\eta||\nabla\eta||u|d\vec{x},
\end{aligned}
$$

since $\lambda I \leq A(\vec{x}) \leq \Lambda I$. By using the fact that $0 \leq (\sqrt{\epsilon}a - \frac{b}{\sqrt{\epsilon}})^2$ is equivalent to $2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$ we have

$$\Lambda \int_{B_1} 2|\nabla u||\eta||\nabla \eta||u| d\vec{x} \leq \Lambda \int_{B_1} \left(\epsilon \eta^2 |\nabla u|^2 + \frac{1}{\epsilon} u^2 |\nabla \eta|^2 \right) d\vec{x}.$$

Let $\frac{\Lambda}{\lambda}\epsilon = 2$ so that $\epsilon = \frac{\lambda}{2\Lambda}$, then

$$\int_{B_1} \eta^2 |\nabla u|^2 \leq \frac{\Lambda}{\lambda} \int_{B_1} \left(\frac{\lambda}{2\Lambda} \eta^2 |\nabla u|^2 + \frac{2\Lambda}{\lambda} u^2 |\nabla \eta|^2 \right) d\vec{x}.$$

After a little algebra we now have,

$$\int_{B_1} \eta^2 |\nabla u|^2 d\vec{x} \leq \frac{4\Lambda^2}{\lambda^2} \int_{B_1} |\nabla \eta|^2 u^2 d\vec{x}. \tag{3.11}$$

By letting $C(\lambda, \Lambda) = \frac{4\Lambda^2}{\lambda^2}$ we have the desired result.
Q.E.D.

4 Maximum Principle

In this section we will derive the weak maximum principle and state some of the corollaries and consequences of this result for harmonic functions. Much of the material in this section can be found in the text by Gilbarg and Trudinger [GT]. For the results in this section assume that our domain $\Omega \subset R^n$ is open, bounded, and connected, and assume $U(\vec{x}) \in C^0(\overline{\Omega}) \cap C^2(\Omega)$.

4.1 Theorem (Weak Maximum Principle). *If* $\Delta U(\vec{x}) \geq 0$, *then*

$$\max_{\vec{x} \in \overline{\Omega}} U(\vec{x}) = \max_{\vec{x} \in \partial\Omega} U(\vec{x}).$$

Proof. We will arrive at a contradiction by supposing

$$\max_{\vec{x} \in \overline{\Omega}} U(\vec{x}) > \max_{\vec{x} \in \partial\Omega} U(\vec{x}).$$

Let us first prove the case for $\Delta U(\vec{x}) > 0$. Because $U(\vec{x})$ is continuous on the compact set $\overline{\Omega}$ it attains its maximum at a point $\vec{x}_0 \in \overline{\Omega}$. Then

$$U(\vec{x}_0) = \max_{\vec{x} \in \overline{\Omega}} U(\vec{x}) > \max_{\vec{x} \in \partial\Omega} U(\vec{x}).$$

Therefore $\vec{x}_0 \in \Omega$, and $\vec{x}_0$ is a local maximum. Then we know $\nabla U(\vec{x}_0) = 0$, and $U_{jj}(\vec{x}_0) \leq 0$ for $1 \leq j \leq n$. Since the preceding statement implies $\Delta U(\vec{x}_0) \leq 0$, we have our contradiction.

Now assume only $\Delta U(\vec{x}) \geq 0$, and as before take $\vec{x}_0$ such that

$$U(\vec{x}_0) = \max_{\vec{x}\in\overline{\Omega}} U(\vec{x}) = \beta + \max_{\vec{x}\in\partial\Omega} U(\vec{x}) > \max_{\vec{x}\in\partial\Omega} U(\vec{x}).$$

Let

$$d = \max_{\vec{y}\in\overline{\Omega}} ||\vec{x}_0 - \vec{y}||,$$

which is finite since we are assuming Ω to be bounded. We now define $V(\vec{x}) := U(\vec{x}) + \epsilon||\vec{x} - \vec{x}_0||^2$ and let $\epsilon = \frac{\beta}{2d^2}$. Therefore

$$\max_{\vec{x}\in\partial\Omega} V(\vec{x}) \leq \max_{\vec{x}\in\partial\Omega} U(\vec{x}) + \frac{\beta}{2d^2} d^2 < \max_{\vec{x}\in\overline{\Omega}} U(\vec{x}) = U(x_0) = V(x_0)\ . \tag{4.1}$$

It follows that

$$\max_{\vec{x}\in\partial\Omega} V(\vec{x}) < \max_{\vec{x}\in\overline{\Omega}} V(\vec{x}). \tag{4.2}$$

Furthermore it easy to show

$$\Delta V(\vec{x}) = \Delta U(\vec{x}) + 2n\epsilon \geq 2n\epsilon > 0\ . \tag{4.3}$$

Armed with Equations (4.2) and (4.3) we can now invoke the first part of the proof by replacing U with V, to arrive at a contradiction.
Q.E.D.

4.2 Theorem (Mean Value Inequality). *$U(\vec{x}) \in C^2(\Omega)$, and $\Delta U \geq 0$, then for any $B := B_R(y) \subset \Omega$*

$$U(y) \leq \frac{1}{w_n R^n} \int_B U d\vec{x}, \tag{4.4}$$

where w_n is the volume of a unit ball in $\mathbb{R}^n$.

4.3 Theorem (The Hopf Lemma). *If $\Delta U \leq 0$ in Ω, where $U(\vec{x})$ is continuous, and $\vec{x}_0 \in \partial\Omega$ satisfies*

i. there exists $B_R(\vec{y}) \in \Omega$ such that $\partial B_R(\vec{y}) \cap \partial\Omega = \vec{x}_0$

ii. $U(\vec{x}) > U(\vec{x}_0)$ for all $\vec{x} \in B_R(\vec{y})$

then $\frac{\partial}{\partial r} U(\vec{x}_0) < 0$ where $r = |\vec{x} - \vec{y}|$.

Proof. Without loss of generality take $U(\vec{x}_0) = 0$ and let y be the origin. If we consider the compact subset $\partial B_{R/2}(\vec{y}) \subset B_R(\vec{y})$, U attains a positive minimum δ on this set since U is continuous, and $U(x) > 0$ for all $x \in B_R(\vec{y})$. We define $v(\vec{x}) := \Gamma(|\vec{x}|) - \Gamma(R)$, where

$$\Gamma(s) = \begin{cases} \frac{1}{n-2} s^{2-n} & n > 2 \\ -\log(s) & n = 2 \end{cases}$$

It is easy to check that $\Delta v = 0$ in $A_R(y) := B_R(\vec{y}) - B_{R/2}(\vec{y})$. From our definition of v, $v \equiv 0$ on $\partial B_R(\vec{y})$ and $v \equiv \beta$ on $\partial B_{R/2}(\vec{y})$ for some $\beta > 0$. We can now define $V := \frac{\delta}{\beta} v$. Then $\Delta V = 0$ in $A_R(\vec{y})$, and $V \leq U$ on $\partial A_R(\vec{y})$. Therefore $V \leq U$ in $A_R(\vec{y})$ by the Weak Maximum Princple. Thus

$$\frac{\partial}{\partial r} U(\vec{x}_0) \leq \frac{\partial}{\partial r} V(\vec{x}_0) = -|\vec{x}_0|^{1-n} < 0 \tag{4.5}$$

for all n.
Q.E.D.

4.4 Theorem (Strong Maximum Principle). *If $\Delta U(\vec{x}) \geq 0$, and U attains a local maximum in Ω, then U is a constant.*

Proof. We prove this theorem by assuming that U is not constant and arrive at a contradiction. Suppose there exists some $\vec{x}_0 \in \Omega$ such that $U(\vec{x}) < U(\vec{x}_0)$ for all $\vec{x} \in \Omega$ where

$$U(\vec{x}_0) = \max_{\vec{x} \in \overline{\Omega}} U(\vec{x}).$$

We consider the open set $\Omega_m := \{U(\vec{x}) < U(\vec{x}_0)\} \subset \Omega$ where $\Omega_m \neq 0$ by the assumption that U is non-constant. Let y be a point in Ω_m such that y is closer to $\partial\Omega_m$ than $\partial\Omega$. It is not difficult to show that there exists an $r > 0$ and $\vec{y} \in \Omega_m$ such that $B_r(\vec{y}) \subset \Omega_m$, $\partial B_r(\vec{y}) \cap \partial\Omega_m \ni \vec{y}_0$, and $\partial B_r(\vec{y}) \cap \partial\Omega = \emptyset$. Then we have $U(\vec{y}_0) = U(\vec{x}_0)$. Therefore $\vec{y}_0$ is a local maximum in Ω of U, and thus $\nabla U(\vec{y}_0) = 0$. On the other hand by applying the Hopf Lemma to $B_r(\vec{y})$ we see that $\nabla U(\vec{y}_0) \neq 0$. This is a contradiction, and therefore U must be a constant.
Q.E.D.

4.5 Remark. The Weak and Strong Maximum Principles hold analogously for minimums. The proof of each theorem follows as before by taking minimums and applying the appropriate negative signs.

We wish to show that in the one sector geometry the Hölder exponent is always bounded from below by 1/2. This fact follows by a delicate barrier argument. As a warm up to that proof we consider the following simpler situation. We let $S_\alpha := \{0 < \theta < \alpha\} \cap B_1$, and $\sigma_\alpha := \partial S_\alpha \cap \partial B_1$. With the preceding definition we present the following theorem.

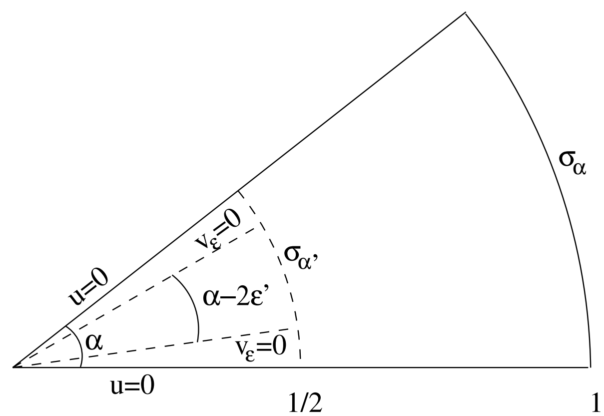
4.6 Theorem. *If $\Delta U(\vec{x}) = 0$ in S_α, $U = 0$ when $\theta = 0$ and $\theta = \alpha$, and $0 \leq \phi = U \not\equiv 0$ on ∂S_α, then the maximum γ for which it can be guaranteed that $U \in C^\gamma(\overline{S}_\alpha \cap \overline{B}_\epsilon)$ is π/α, this γ is called the sharp Hölder exponent γ of U near the origin*

Proof. We prove this theorem by assuming $\gamma \neq \pi/\alpha$ and arrive at a contradiction.

1. Assume first that $\gamma > \pi/\alpha$. In this case there exists $\epsilon > 0$ such that $\frac{\pi}{\alpha} < \frac{\pi+\alpha\epsilon}{\alpha} < \gamma$. Define $v_\epsilon := r^{\frac{\pi+\alpha\epsilon}{\alpha}} \sin\left((\frac{\pi+\alpha\epsilon}{\alpha})\theta'\right)$, where $\theta' = \theta - \frac{\alpha^2\epsilon}{2(\pi+\alpha\epsilon)} = \theta - \epsilon'$, so that $\theta' = 0$ is equivalent to $\theta = \epsilon'$ and $\theta' = \frac{\pi\alpha}{\pi+\alpha\epsilon}$ is equivalent to $\theta = \alpha - \epsilon'$. Now we let

$$S_{\alpha'} := \{\epsilon' < \theta < \alpha - \epsilon'\} \cap B_{1/2} = \left\{0 < \theta' < \frac{\pi\alpha}{\pi + \alpha\epsilon}\right\} \cap B_{1/2} \subset S_\alpha$$

and $\sigma_{\alpha'} := \partial S_{\alpha'} \cap \partial B_{1/2}$. Then we have $0 = v_\epsilon \leq U$ when $\theta' = 0$ or $\theta' = \frac{\pi\alpha}{\pi+\alpha\epsilon}$, i.e. $v_\epsilon = 0$ on the lateral boundary on our sector $S_{\alpha'}$, and $v_\epsilon > 0$ on $\sigma_{\alpha'}$. Then we have $0 = v_\epsilon \leq U$ when $\theta' = 0$ or $\theta' = \frac{\pi\alpha}{\pi+\alpha\epsilon}$, i.e. $v_\epsilon = 0$ on the lateral boundary on our sector $S_{\alpha'}$, and $v_\epsilon > 0$ on $\sigma_{\alpha'}$.



By the Strong Maximum Principle we know $U > 0$ inside S_α since $U \geq 0$ on ∂S_α and $U \not\equiv 0$. Since $\sigma_{\alpha'}$ is a compact subset of S_α, and U is a continuous function, U attains a positive minimum δ on this set. Define $M := \max(v_\epsilon) = \frac{1}{2}^{\frac{\pi+\alpha\epsilon}{\alpha}}$ on $\sigma_{\alpha'}$. We can now define $V := \frac{\delta}{M}v_\epsilon$ whose maximum is δ. Then $\Delta V = 0$ in S'_α, $0 = V \leq U$ when $\theta' = 0$ or $\theta' = \frac{\pi\alpha}{\pi+\alpha\epsilon}$, and $\max(V) = \delta = \min(U)$ on $\sigma_{\alpha'}$. Therefore $V \leq U$ in $S_{\alpha'}$ by the Weak Maximum Principle.

If we now consider the half line $\theta' = \frac{\pi\alpha}{2(\pi+\alpha\epsilon)} =: \theta_h$, we have

$$U(r, \theta_h) \geq \frac{\delta}{M} r^{\frac{\pi+\alpha\epsilon}{\alpha}} \sin\left(\frac{(\pi + \alpha\epsilon)}{\alpha} \cdot \frac{\pi\alpha}{2(\pi + \alpha\epsilon)}\right) = Dr^{\frac{\pi+\alpha\epsilon}{\alpha}},$$

where $D = \frac{\delta}{M}$. By our assumption that $U \in C^\gamma$, there exists a $D' \geq 0$ such that

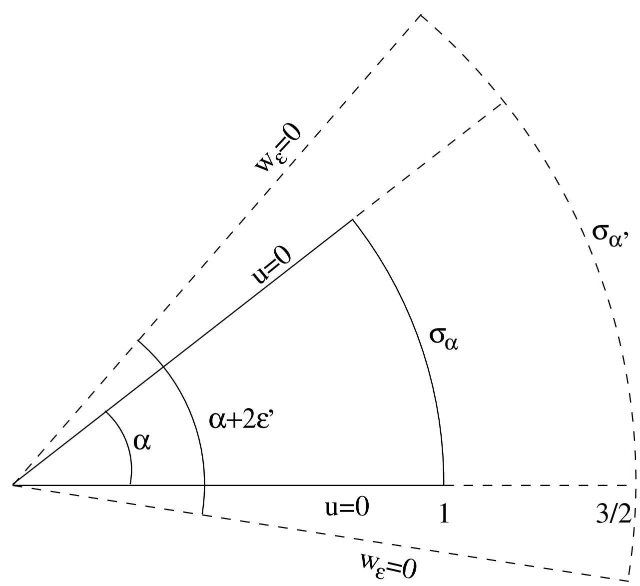
$$Dr^{\frac{\pi+\alpha\epsilon}{\alpha}} \leq D'r^\gamma$$
$$D/D' \leq r^{\gamma - \frac{\pi+\alpha\epsilon}{\alpha}} .$$

Since $\gamma > \frac{\pi+\alpha\epsilon}{\alpha}$ we get a contradiction by taking r to be sufficiently small, and therefore $\gamma \leq \pi/\alpha$.

2. Assume second that $\gamma < \pi/\alpha$: In this case there exists $\epsilon > 0$ such that $\pi/\alpha > \frac{\pi-\alpha\epsilon}{\alpha} > \gamma$. Define $w_\epsilon := r^{\frac{\pi-\alpha\epsilon}{\alpha}} \sin\left((\frac{\pi-\alpha\epsilon}{\alpha})\theta'\right)$, where $\theta' = \theta + \frac{\alpha^2\epsilon}{2(\pi-\alpha\epsilon)} = \theta + \epsilon'$, so that $\theta' = 0$ is equivalent to $\theta = -\epsilon'$ and $\theta' = \frac{\pi\alpha}{\pi-\alpha\epsilon}$ is equivalent to $\theta = \alpha + \epsilon'$. Now we let

$$S_{\alpha'} := \{-\epsilon' < \theta < \alpha + \epsilon'\} \cap B_{3/2} = \left\{0 < \theta' < \frac{\pi\alpha}{\pi - \alpha\epsilon}\right\} \cap B_{3/2} \supset S_\alpha$$

and $\sigma_{\alpha'} := \partial S_{\alpha'} \cap \partial B_{3/2}$. Then we have $0 = U < w_\epsilon$ when $\theta = 0$ or $\theta = \alpha$, and we let $M := \max(U)$ on σ_α.



By our definition of w_ϵ we see that $w_\epsilon > 0$ inside $S_{\alpha'}$. Also w_ϵ has a minimum δ on σ_α where $\delta := \min\left\{\sin\left(\frac{\pi-\alpha\epsilon}{\alpha}\theta'\right)\right\} = \sin\left(\frac{\alpha\epsilon}{2}\right) > 0$. We can now define $W := \frac{M}{\delta} w_\epsilon$ whose minimum is M. Then $\Delta W = 0$ in S_α, $0 = U < W$ when $\theta = 0$ or $\theta = \alpha$, and $\min(W) = M = \max(U)$ on σ_α. Therefore $U \leq W$ in S_α by the Weak Maximum Principle.

With our assumption that the sharp Hölder exponent of U is γ and $\gamma < \pi/\alpha$, we have $U \notin C^\beta$, where $\beta := \frac{\pi-\alpha\epsilon}{\alpha}$. By the previous statement and the definition of Hölder continuity there exists a sequence $\{\vec{x}_n\}$ such that $\{\vec{x}_n\} \to 0$, and

$$\frac{U(\vec{x}_n)}{|\vec{x}_n|^\beta} \to \infty. \tag{4.6}$$

Since $U \leq W$ in S_α and $W \in C^\beta$ we have

$$U(\vec{x}_n) \leq W(\vec{x}_n) \leq D|\vec{x}_n|^\beta,$$

and therefore

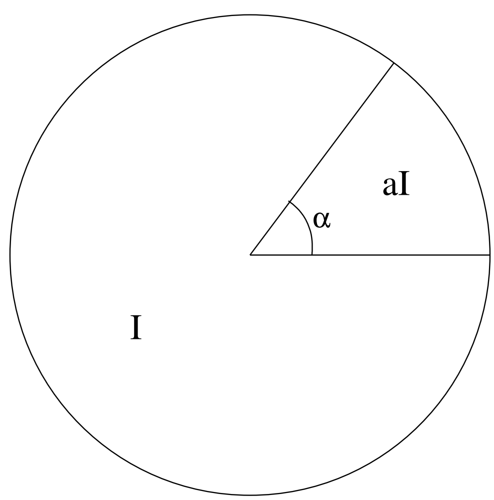
$$\frac{U(\vec{x}_n)}{|\vec{x}_n|^\beta} \leq D. \tag{4.7}$$

This is a contradiction of Equation (4.6). Therefore $\gamma \geq \pi/\alpha$.

Hence $\gamma = \pi/\alpha$ at the origin.
Q.E.D.

5 The One Sector Geometry

In this section, we will consider the one sector geometry. That is we consider a fiber-reinforced composite material where there exists a single cut in the form of a sector with angle α. In this case $A(\vec{x})$ is given by aI when $\vec{x}$ is on the inside of the sector ($0 < \theta < \alpha$), and I when $\vec{x}$ is outside of the sector ($\alpha < \theta < 2\pi$), as in the figure.



Following the procedure of separation of variables we begin by searching for solutions of the partial differential equation $\mathrm{div}(A(\vec{x})\nabla U) = 0$ such that U has the following form,

$$U(r,\theta) = \begin{cases} r^\gamma(A_{in}\cos(\gamma\theta) + B_{in}\sin(\gamma\theta)) & 0 < \theta < \alpha \\ r^\gamma(A_{out}\cos(\gamma\theta) + B_{out}\sin(\gamma\theta)) & \alpha < \theta < 2\pi. \end{cases} \tag{5.1}$$

After expressing the continuity conditions and compatibility conditions across the interfaces in a matrix form, the existence of a nontrivial separated variable solution becomes equivalent to the vanishing of the determinant of this matrix. This determinant is a function of the Hölder exponent γ, the shear modulus coefficient a, and the angle of the cut α.

From the fact that our solutions must be continuous across the sectors we have the following conditions. When $\theta = 0$,

$$A_{in} = A_{out}\cos(2\pi\gamma) + B_{out}\sin(2\pi\gamma), \tag{5.2}$$

and for $\theta = \alpha$,

$$A_{in}\cos(\gamma\alpha) + B_{in}\sin(\gamma\alpha) = A_{out}\cos(\gamma\alpha) + B_{out}\sin(\gamma\alpha). \tag{5.3}$$

By our compatibility conditions we know that $U_\theta^{out}(r,\theta) = aU_\theta^{in}(r,\theta)$ on the interfaces, where $U_\theta^{in}(r,\theta)$ is the one sided derivative on the interface that stays inside the sector $0 < \theta < \alpha$, and similarly for $U_\theta^{out}(r,\theta)$. Taking the partial derivative of (5.1) with respect to θ we see that

$$U_\theta(r,\theta) = \begin{cases} \gamma r^\gamma(B_{in}\cos(\gamma\theta) - A_{in}\sin(\gamma\theta)) & 0 < \theta < \alpha \\ \\ \gamma r^\gamma(B_{out}\cos(\gamma\theta) - A_{out}\sin(\gamma\theta)) & \alpha < \theta < 2\pi. \end{cases} \tag{5.4}$$

Therefore at $\theta = 0$ we have

$$aB_{in} = B_{out}\cos(2\pi\gamma) - A_{out}\sin(2\pi\gamma), \tag{5.5}$$

and for $\theta = \alpha$,

$$a(B_{in}\cos(\gamma\alpha) - A_{in}\sin(\gamma\alpha)) = B_{out}\cos(\gamma\alpha) - A_{out}\sin(\gamma\alpha). \tag{5.6}$$

After rewriting equations (5.2), (5.3), (5.5), and (5.6) we get the following system of equations.

$$\begin{bmatrix} 1 & 0 & -\cos(2\pi\gamma) & -\sin(2\pi\gamma) \\ 0 & a & \sin(2\pi\gamma) & -\cos(2\pi\gamma) \\ \cos(\gamma\alpha) & \sin(\gamma\alpha) & -\cos(\gamma\alpha) & -\sin(\gamma\alpha) \\ -a\sin(\gamma\alpha) & a\cos(\gamma\alpha) & \sin(\gamma\alpha) & -\cos(\gamma\alpha) \end{bmatrix} \begin{bmatrix} A_{in} \\ B_{in} \\ A_{out} \\ B_{out} \end{bmatrix} = \vec{0} \tag{5.7}$$

Here we are not interested in the trivial solution, so in order to get nontrivial solutions we require the matrix

$$\mathcal{M} := \begin{bmatrix} 1 & 0 & -\cos(2\pi\gamma) & -\sin(2\pi\gamma) \\ 0 & a & \sin(2\pi\gamma) & -\cos(2\pi\gamma) \\ \cos(\gamma\alpha) & \sin(\gamma\alpha) & -\cos(\gamma\alpha) & -\sin(\gamma\alpha) \\ -a\sin(\gamma\alpha) & a\cos(\gamma\alpha) & \sin(\gamma\alpha) & -\cos(\gamma\alpha) \end{bmatrix} \tag{5.8}$$

to be singular. Let $\Theta(\alpha, a, \gamma) :=$ determinant$(\mathcal{M})$. After a long, but elementary computation we find

$$\Theta(\alpha, a, \gamma) = a^2 A(\gamma, \alpha) + 2aB(\gamma, \alpha) + A(\gamma, \alpha) \tag{5.9}$$

where

$$\begin{aligned} A(\gamma, \alpha) &:= \sin(\gamma\alpha)\sin(\gamma(2\pi - \alpha)) \\ B(\gamma, \alpha) &:= [1 - \cos(\gamma\alpha)\cos(\gamma(2\pi - \alpha))] \ . \end{aligned} \tag{5.10}$$

6 Small Values of the Shear Modulus

In this section we will discuss various results for the first positive γ when $a = 0$ and then use the implicit function theorem to extend some of these results to the case where a is very small.

6.1 Lemma.

$$\Theta(\alpha, a, \gamma) = a^2 \Theta(\alpha, 1/a, \gamma) \ . \tag{6.1}$$

Proof. Simply examine Equation (5.9).
Q.E.D.

6.2 Remark. An obvious consequence of this lemma is that the a values that make $\Theta(\alpha, a, \gamma) = 0$ for fixed α and γ are reciprocals. Since the a values that make $\Theta(\alpha, a, \gamma) = 0$ are reciprocals and since we can multiply the partial differential equation by $1/a$ without changing the solutions, we can now assume $0 < \alpha \le \pi$.

We are interested in our first positive γ that satisfies Equation (5.9). Let $\{\gamma_n\}$ be the set of all nonnegative zeros of $\Theta(\alpha, a, \cdot)$, let $\gamma_0 = 0$, and label the γ_n so that $\gamma_n < \gamma_{n+1}$ for all n. It is trivial to check that when $\gamma = 0$ the expression in (5.9) equals zero. So γ_1 is the first positive solution to equation (5.9).

6.3 Lemma. *If $a = 0$, then $\gamma_1 = \frac{\pi}{2\pi - \alpha}$.*

Proof. $\Theta(\alpha, 0, \gamma) = \sin(\gamma\alpha)\sin(\gamma(2\pi - \alpha))$ If $\Theta(\alpha, 0, \gamma) = 0$, then $\sin(\gamma\alpha) = 0$ or $\sin(\gamma(2\pi - \alpha)) = 0$. So $\gamma\alpha = \pi$ or $\gamma(2\pi - \alpha) = \pi$, and therefore $\gamma_1 = \min\{\frac{\pi}{\alpha}, \frac{\pi}{2\pi - \alpha}\}$. Since $\alpha \le \pi$ for all α, we have

$$\begin{aligned} 2\alpha &\le 2\pi \\ \alpha &\le 2\pi - \alpha \\ \frac{1}{2\pi - \alpha} &\le \frac{1}{\alpha} \\ \frac{\pi}{2\pi - \alpha} &\le \frac{\pi}{\alpha} \ . \end{aligned}$$

Thus $\gamma_1 = \frac{\pi}{2\pi-\alpha}$.
Q.E.D.

6.4 Remark. $\Theta(\alpha, 0, \frac{\pi}{2\pi-\alpha}) = 0$ for all $\alpha \in (0, \pi]$. This fact follows by examining Equation (5.9).

6.5 Lemma. *If $a > 0$, then $\gamma_1 > \frac{\pi}{2\pi-\alpha}$ independent of a.*

Proof. First consider the case when $\gamma = \frac{\pi}{2\pi-\alpha}$. Then $\Theta\left(\alpha, a, \frac{\pi}{2\pi-\alpha}\right)$ is equal to

$$2a\left[1-\cos\left(\frac{\alpha\pi}{2\pi-\alpha}\right)\cos(\pi)\right] = 2a\left[1+\cos\left(\frac{\alpha\pi}{2\pi-\alpha}\right)\right] > 0\,. \tag{6.2}$$

Now suppose $0 < \gamma < \frac{\pi}{2\pi-\alpha}$. We view Equation (5.9) as a quadratic function of a, and show that the zeros are always negative, and arrive at a contradiction. We start by showing that B and A are both positive. Solving equation (5.9) for a we see

$$a = \frac{-B(\gamma,\alpha) \pm \sqrt{B(\gamma,\alpha)^2 - A(\gamma,\alpha)^2}}{A(\gamma,\alpha)}\,. \tag{6.3}$$

Since $\cos(\gamma\alpha) \in (-1, 1)$, it is clear that $B(\gamma,\alpha) = [1-\cos(\gamma\alpha)\cos(\gamma(2\pi - \alpha))] > 0$. Now it must be shown that $A(\gamma,\alpha) = \sin(\gamma\alpha)\sin(\gamma(2\pi-\alpha) > 0$. We establish this inequality by showing $\gamma(2\pi-\alpha) < \pi$ and $\gamma\alpha < \pi$. The first inequality is immediate by our assumption $\gamma < \frac{\pi}{2\pi-\alpha}$. The second inequality follows by using $\frac{\pi}{2\pi-\alpha} \leq \frac{\pi}{\alpha}$, which was shown in the proof of Lemma 6.3. Thus,

$$0 > \frac{-B(\gamma,\alpha)}{A(\gamma,\alpha)} > \frac{-B(\gamma,\alpha) - \sqrt{B(\gamma,\alpha)^2 - A(\gamma,\alpha)^2}}{A(\gamma,\alpha)}\,, \tag{6.4}$$

and so one of our solutions of $\Theta(\alpha, \cdot, \gamma) = 0$ is negative. On the other hand, by Remark (6.2) a values that make $\Theta(\alpha, a, \gamma) = 0$ for fixed α and γ are reciprocals, so both solutions are negative. Since we are assuming $a > 0$, we have a contradiction. Thus $\gamma_1 > \frac{\pi}{2\pi-\alpha}$.
Q.E.D.

6.6 Theorem. *For any $\epsilon > 0$ and for a fixed $\alpha \in (0, \pi)$, we can choose $a > 0$ but small enough to guarantee $\gamma_1 \in \left(\frac{\pi}{2\pi-\alpha}, \frac{\pi}{2\pi-\alpha} + \epsilon\right)$.*

Proof. This estimate is a consequence of the Implicit Function Theorem, along with the following facts

a. $\Theta(\alpha, a, \gamma)$ is C^∞ in all its variables.

b. $\Theta\left(\alpha, 0, \frac{\pi}{2\pi-\alpha}\right) = 0$ for all a.

c. $\Theta\left(\alpha, a, \frac{\pi}{2\pi-\alpha}\right) > 0$ for all $a > 0$.

d. $\Theta_\gamma\left(\alpha, 0, \frac{\pi}{2\pi-\alpha}\right) < 0$.

Item c. follows from the proof of Lemma 6.5. The only item left that is not immediately clear is item d. Consider $\Theta_\gamma(\alpha, a, \gamma)$ which is equal to

$$\begin{aligned}&(a^2+1)[\alpha\cos(\gamma\alpha)\sin(\gamma(2\pi-\alpha)) + (2\pi-\alpha)\sin(\gamma\alpha)\cos(\gamma(2\pi-\alpha))] \\ &+ 2a[\alpha\sin(\gamma\alpha)\cos(\gamma(2\pi-\alpha)) + (2\pi-\alpha)\cos(\gamma\alpha)\sin(\gamma(2\pi-\alpha))]\ .\end{aligned} \tag{6.5}$$

Then,

$$\Theta_\gamma\left(\alpha, 0, \frac{\pi}{2\pi-\alpha}\right) = (\alpha - 2\pi)\sin\left(\frac{\pi\alpha}{2\pi-\alpha}\right) < 0\ . \tag{6.6}$$

Q.E.D.

7 Results for Rational Angles

We will discuss various results for rational multiples of π for α, and in this section we will express $r \in \mathbb{Q}$ in lowest terms as p/q, so $\alpha = \frac{p\pi}{q}$.

7.1 Lemma. *The following hold true for all a.*

a. $\Theta\left(\frac{p\pi}{q}, a, q\right) = 0$.

b. $\Theta\left(\frac{p\pi}{q}, a, 0\right) = 0$.

c. $\Theta_\gamma\left(\frac{p\pi}{q}, a, q\right) = 0$.

d. $\Theta_\gamma\left(\frac{p\pi}{q}, a, 0\right) = 0$

Proof. All of these follow by inspection of equations (5.9)and (6.5).
Q.E.D.

7.2 Lemma. $\Theta\left(\frac{p\pi}{q}, a, \gamma\right) = \Theta\left(\frac{p\pi}{q}, a, \gamma + q\right)$ *for all a and $q/p \in \mathbb{Q}$*

Proof. We begin by observing $\Theta\left(\frac{p\pi}{q}, a, \gamma + q\right)$ is equal to

$$\begin{aligned}&(a^2+1)\left[\sin\left(\frac{\gamma p\pi}{q} + p\pi\right)\sin\left(\gamma\left(2\pi - \frac{p\pi}{q}\right) - q\pi\right)\right] + \\ &2a\left[1 - \cos\left(\frac{\gamma p\pi}{q} + p\pi\right)\cos\left(\gamma\left(2\pi - \frac{p\pi}{q}\right) - q\pi\right)\right]\end{aligned} \tag{7.1}$$

We must consider the following two cases, when $q \in 2\mathbb{N}$ and $q \in 2\mathbb{N}+1$. If $q \in 2\mathbb{N}$, then it is clear that $\Theta\left(\frac{p\pi}{q}, a, \gamma\right) = \Theta\left(\frac{p\pi}{q}, a, \gamma + q\right)$. If $q \in 2\mathbb{N}+1$, then each $\sin\left(\frac{\gamma p\pi}{q} + p\pi\right) = -\sin\left(\frac{\gamma p\pi}{q}\right)$, and similarly for $\cos\left(\frac{\gamma p\pi}{q} + p\pi\right)$. This observation completes the proof.
Q.E.D.

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