SUMMABILITY PROCEDURES APPLIED TO FOURIER SERIES

by 500

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Introduction

In this report there is a discussion of summability of series in general, followed by an introduction to summability of Fourier Series. In that the definition of "sum" of an infinite series is merely that - a definition - since it is physically impossible to add up an infinite number of terms, other more general definitions of a "sum" can hardly be faulted.

In addition, in many of the applications of series, it is not necessary that the series be convergent. Often merely being summable in some sense is sufficient and, indeed, some workers have used strictly divergent series with a fair degree of success.

It is necessary to begin by citing a few fundamental definitions and theorems. Not all theorems are proved and many important concepts and theorems have to be omitted due to lack of space.

In dealing with many of the concepts and ideas, particular cases are used. Thus, when $|a_n - L|$ is referred to, one usually thinks of real (or complex) numbers. However, in general, the definition or theorem usually is valid in other instances, where $|a_n - L|$ refers to the "distance between" two elements a_n and L. In the next few paragraphs, the elements are numbers, in the real (or complex) domain.

A sequence is defined as a function whose domain is the set of non-negative integers, or portion thereof. If the function is denoted by f, its value at n is given by f(n). The sequence itself is the set $\{(n, f(n)):n = 0, 1, 2, \ldots\}$, the set of all pairs (n, f(n)), with n a non-negative integer. Since the domain is usually the same, it is customary to shorten the notation and just write $\{f(n)\}$ instead of $\{(n, f(n))\}$. Thus the sequence $\{(n, \frac{1}{n}):n\ 1, 2, \ldots\}$ would be written simply as $\frac{1}{n}$. Also, $\{a_n\}$ is used to denote the sequence $\{(n, a_n)\}$.

A sequence $\{a_n\}$ may have different values a_n for different values of n. Suppose that as n increases the different a_n's tend to cluster around some fixed number L. If there is a number L such that $|a_n - L|$ can be made arbitrarily small for all sufficiently large n, the sequence $\{a_n\}$ is said to converge to L. Quantitatively, if, given $\epsilon > 0$, there is an N such that, for all $n > N$, $|a_n - L| < \epsilon$, then $\{a_n\}$ converges to L. This can be written as $\lim_{n\to\infty} a_n = L$. If no such limit exists, the sequence is said to diverge.

A very important way of creating a sequence $\{s_n\}$ is by addition. Suppose the elements to be added are $u_1, u_2, \ldots, u_n, \ldots$. Let

$$
\begin{aligned}
s_1 &= u_1, \\
s_2 &= u_1 + u_2, \\
&\;\vdots \\
s_n &= u_1 + u_2 + \ldots + u_n = \sum_{k=1}^{n} u_k, \\
&\;\vdots
\end{aligned}
\tag{1}
$$

Consider $\sum_{k=1}^{\infty} u_k$, that is consider the sum s_n as n increases without bound. The limit of s_n, as n increases without bound, is called an infinite series.

The sequence $\{s_n\}$ given by (1) is called the sequence of partial sums of the series. The n^{th} partial sum is s_n. If $\{s_n\}$ converges to a limit S as n increases without bound, $\lim_{n\to\infty} s_n = S$, then "the series converges to its sum S". This is written as

$$\sum_{k=1}^{\infty} u_k = \lim_{n\to\infty} \left(\sum_{k=1}^{n} u_k \right) = S.$$

Theorem 1: A necessary condition for the convergence of $\sum_{k=1}^{\infty} u_k$ is that $\lim_{k\to\infty} u_k = 0$. A series is said to be divergent if its sequence of partial sums is divergent.

Although a series may be divergent, it may still be useful. Much can be done with a divergent series if it is handled by certain methods called summability procedures. A summability process is defined as a method of assigning a "sum" to a series. This report will be restricted to those methods for which the "sum" of a convergent series is the same as the sum in the ordinary sense of convergence. Methods with this property are called "regular". In order to preserve the analogy between convergence and summability, a summability process should satisfy the following conditions:

I. If $\sum_{k=0}^{\infty} u_k = S$, then $\sum_{k=1}^{\infty} u_k = S - u_0$, and conversely.

II. If $\sum_{k=0}^{\infty} u_k = S$, $\sum_{k=0}^{\infty} v_k = T$, then $\sum_{k=0}^{\infty} (u_k + v_k) = S + T$.

III. If $\sum_{k=0}^{\infty} u_k = S$, then $\sum_{k=0}^{\infty} \alpha u_k = \alpha S$, where α is any constant.

In addition,

IV. The process must be regular.

Cesàro Summability

Consider the following series,

$$(1) \qquad \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots ,$$

and let

$$\left.\begin{aligned} s_n &= a_0 + a_1 + \cdots + a_n \\ \sigma_n &= \frac{s_o + s_1 + \cdots + s_n}{n+1} \end{aligned}\right\} \quad (n = 0, 1, 2, \ldots)$$

and

If $\lim_{n\to\infty} \sigma_n = A$ we say that the series given by (1) is Cesàro summable to A or summable (C, 1) to A. (Cesàro summability (C, α) can be defined for general α, $\alpha > -1$). As an example consider the series

$$(2) \qquad \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - \cdots .$$

This series diverges, but $s_o = 1$, $s_1 = 0$, $s_2 = 1$, $s_3 = 0$, ... so that

$$\sigma_n = \begin{cases} \frac{1}{2} & \text{if n is odd} \\ \frac{1}{2} + (2n+2)^{-1} & \text{if n is even.} \end{cases}$$

Hence the $\lim_{n\to\infty} \sigma_n = \frac{1}{2}$ and the series given by (2) is Cesàro summable to $\frac{1}{2}$.

Theorem 1: Cesàro summability is a regular method.

Proof: Suppose that a given series is convergent, with sum A and, therefore, $\lim_{n\to\infty} s_n = A$. Then for any $\varepsilon > 0$, there exists a number m such that $|s_n - A| < \frac{\varepsilon}{2}$ whenever $n \geq m$. Now consider

$$\sigma_n - A = \frac{s_o + s_1 + \cdot \cdot \cdot + s_n - (n+1)A}{n+1} = \frac{1}{n+1} \sum_{i=0}^{n} (s_i - A).$$

For $n > m$,

$$\sigma_n - A = \frac{1}{n+1} \sum_{i=0}^{m-1} (s_i - A) + \frac{1}{n+1} \sum_{i=m}^{n} (s_i - A)$$

and hence

(3) $$|\sigma_n - A| \leq \frac{1}{n+1} \sum_{i=0}^{m-1} |s_i - A| + \frac{1}{n+1} \sum_{i=m}^{n} |s_i - A|.$$

Since m is a fixed number

(4) $$\frac{1}{n+1} \sum_{i=0}^{m-1} |s_i - A| < \frac{\varepsilon}{2}$$

for all sufficiently large n, $n > M$, say. But since $|s_n - A| < \frac{\varepsilon}{2}$ for $n \geq m$

(5) $$\frac{1}{n+1} \sum_{i=m}^{n} (s_i - A| < \frac{n - m + 1}{n+1} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} \quad .$$

Combining equations (3), (4), and (5), $|\sigma_n - A| < \varepsilon$ provided $n > m + M$, which proves the theorem.

Summability (C, 1) can be generalized to summability (C, k) as follows: Given a sequence $\{a_n\} = a_0, a_1, a_2, \ldots, a_n \ldots$, let

$$s_n^{(0)} = a_0 + a_1 + \ldots + a_n,$$

$$s_n^{(1)} = s_0^{(0)} + s_1^{(0)} + \ldots + s_n^{(0)},$$