CLUSTER AUTOMORPHISMS AND HYPERBOLIC CLUSTER ALGEBRAS

by

IBRAHIM A SALEH

B.A., Cairo University 1995
M.S., Cairo University 2002
M.S., Kansas State University 2008

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

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Department of Mathematics
College of Arts and Sciences

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Abstract

Let $\mathcal{A}_n(S)$ be a coefficient free commutative cluster algebra over a field K. A cluster automorphism is an element of $Aut._K K(t_1, \cdots, t_n)$ which leaves the set of all cluster variables, χ_S, invariant. In Chapter 2, the group of all such automorphisms is studied in terms of the orbits of the symmetric group action on the set of all seeds of the field $K(t_1, \cdots, t_n)$.

In Chapter 3, we set up for a new class of non-commutative algebras that carry a non-commutative cluster structure. This structure is related naturally to some hyperbolic algebras such as, Weyl Algebras, classical and quantized universal enveloping algebras of sl_2 and the quantum coordinate algebra of $SL(2)$. The cluster structure gives rise to some combinatorial data, called cluster strings, which are used to introduce a class of representations of Weyl algebras. Irreducible and indecomposable representations are also introduced from the same data.

The last section of Chapter 3 is devoted to introduce a class of categories that carry a hyperbolic cluster structure. Examples of these categories are the categories of representations of certain algebras such as Weyl algebras, the coordinate algebra of the Lie algebra sl_2, and the quantum coordinate algebra of $SL(2)$.

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Manhattan, Kansas

July 27, 2012

Dedication

To my Mother and my precious family

Introduction

Cluster algebras were invented by S. Fomin, and A. Zelevinsky [1, 15 ,16, 17, 34]. A cluster algebra is a commutative algebra with a distinguished set of generators called cluster variables and particular type of relations called mutations. A quantum version was introduced in [10] and [2]. The original motivation was to create an algebraic framework to study total positivity and dual canonical basis in coordinate rings of certain semi simple algebraic groups. It was inspired by the discovery of a connection between total positivity and canonical basis due to G. Lusztig, [24].

First chapter of the thesis serves as a theory preliminaries. First section contains an axiomatic definition, first examples and some basic structural properties of cluster algebras. Second section is devoted to give a brief introduction to the quantum version of cluster algebras, first introduced in [10] and [2]. Section 3 provides one of the original motivating examples, namely the cluster structure on the algebra of functions of double Bruhat Cells. Along with the example, we provide a machinery of producing totally positive basis for the double Bruhat, more details are in [1], [13]. Section 4 of Chapter 1 is where we explain a relation between the cluster algebras and the root systems which highlighting the relation between the cluster theory and the heart of the Lie theory.

The second chapter's main topic is answering the question, what does it mean for two cluster algebras to be isomorphic. Among different ways of defining the isomorphisms in this case, is to consider the algebra isomorphisms that reserve the set of cluster variables, which is considered to be the core of the algebra. Thus we define a cluster algebra isomorphism as an K-algebra isomorphism $\phi : A(S) \to A(S')$ such that $\phi(\chi_S) = \chi_{S'}$. This definition does not require that ϕ should be compatible with mutations. One could also define a cluster isomorphism as an algebra isomorphism sending clusters to clusters or require the mutation relations to be preserved. Under certain conditions, these different definitions are equivalent, (Corollary 2.2.4).

In Chapter 2, we initiate the study of the cluster automorphisms of a commutative coefficient free cluster algebra, $\mathcal{A}_n(S)$. The cluster automorphisms are the field automorphisms that leave χ_S invariant, (Definition 2.1.1). It turned out that, under certain conditions, leaving χ_S, invariant is equivalent to leaving the *cluster structure*, the set of all seeds of $\mathcal{A}_n(S)$, invariant, (Theorem 2.2.3). The group of all such automorphisms is called the cluster group of $\mathcal{A}_n(S)$. Our original motivation was to study the irreducible elements in any cluster algebra through studying the *cluster automorphisms*. Despite we introduced and studied the cluster automorphisms, we are still relatively far from this initial aim.

Also in the same chapter we study the action of the symmetric group on the set $\mathcal{S}$, of all seeds of the field $\mathcal{F} = K(t_1, t_2, \ldots, t_n)$. We show that in the simply-laced cluster algebras, the orbits of such action are subsets of the orbits of the mutations group action on $\mathcal{S}$, (Theorem 2.1.4). However, the simply-laced hypothesis is necessary, (Example 2.1.5).

Every two seeds and a permutation group element define a field automorphism, which we call an exchange automorphism. The subgroup of $Aut._K\mathcal{F}$, generated by the set of all exchange automorphisms, is called the exchange group of $\mathcal{A}_n(S)$.

The main result of the second chapter, is providing a description for the intersection of the cluster group and the exchange group for any coefficient free cluster algebra satisfying the Fomin-Zelevinsky positivity conjecture. The description is in terms of the orbits of the symmetric group action on $\mathcal{S}$ and the cluster pattern data (Theorem 2.2.3).

Hyperbolic algebras were first introduced by A. Rosenberg in [30], and his motivation was to find a ring theoretical framework to study the representation theory of some important *small algebras* such as the first Heisenberg algebra, Weyl algebra, and the universal enveloping algebra of the Lie algebra $sl(2)$. A complete list of small algebras and their representations theory using the hyperbolic algebra as the framework can be found in [30].

The relations on the hyperbolic algebras give raise to a non-commutative cluster structure, which is first introduced in Chapter 3 of this thesis. We were motivated by the rich combinatorial structure comes with any cluster structure, to be used to reclassify the rep-

resentations theory of hyperbolic algebras.

Chapter 3 works as a setting up for the cluster structure in the hyperbolic algebras, and details are given for our two running examples Weyl algebra and the quantum coordinate algebra of $SL(2,k)$.

In the last decade, the age of the cluster algebras theory, the theory has witnessed a remarkable growth due to the many links that have been discovered with a wide range of subjects. Recently, D. Hernandez and B. Leclerc in [20] and [22], and Nakajima in [28] have started using the rich cluster algebra structure to solve some classical representation theory problems. Chapter 3 has to do with this trend. The main idea is, we show that a partial relaxing of the commutativity relations of the frozen variables and cluster variables, extends the theory to include some essential objects in representation theory, such as hyperbolic algebras. We also, introduce and study non commutative seeds, that are different from the quantum seeds introduced in [10], and [2]. We show that this type of seeds exists naturally and the cluster algebras of these seeds are naturally related to some known hyperbolic algebras. The non-commutativity is controlled by a ring (the ring of coefficients) automorphism θ, and in the case of $\theta = id$ we get the Fomin and Zelevinsky cluster algebra.

Weyl algebras (algebras of differential operators with polynomial coefficients) among others are one of the most important examples of hyperbolic algebras. They are essentially relevant for the theory of infinite dimensional representations of Lie algebras. They appear as primitive quotients of universal enveloping algebras of nilpotent Lie algebras, which reduces the study of irreducible representations of nilpotent Lie algebras to study simple modules over Weyl algebras. In the case of reductive Lie algebras, Weyl algebras also appear as algebras of differential operators on (translations of) big Shubert cells. The algebra of differential operator on the big Shubert cells is used to develop methods in non-commutative geometry to reduce the study of the irreducible representations of reductive Lie algebras to the study of simple modules over Weyl algebras. More generale hyperbolic algebras play a similar role (via quantum D- modules on quantum flag varieties) for quantized enveloping

algebras.

In the last section of Chapter 3, we introduce a class of categories that carry a hyperbolic cluster structure. Examples of these categories are the categories of representations of Weyl algebras, the coordinate algebra of the Lie algebra sl_2, and the quantum coordinate algebra of $SL(2)$.

Notations: Through out the first two chapters, K is a field and $\mathcal{F} = K(\mathbb{G})(\tau_1, \ldots \tau_n)$ is the field of rational functions in n independent (commutative) variables over the filed of fractions $\mathbb{K} = K(\mathbb{G})$ of the group ring $K[\mathbb{G}]$, where $\mathbb{G}$ is a free abelian group, written multiplicatively, generated by the elements $f_1, \ldots, f_t$. We always denote (b_{ij}) for the square matrix B, (c_{ij}) for C, etc., and $[1, n] = \{1, 2, \ldots, n\}$.

Chapter 1

Cluster Algebra Basics

1.1 Preliminaries

Most of the material of this section is based on [1] [2] [15] [16] [17] and [18].

Definitions 1.1.1. 1. A seed p of rank n in $\mathcal{F}$ is a pair $(\widetilde{X}, \widetilde{B})$ where $\widetilde{X} = \{x_1, \ldots, x_n, f_{n+1}, \ldots, f_m\}$ with $m = n+t$ such that $f_{n+1}, \ldots, f_m$ generate the free abelian group $\mathbb{P}$ and the elements of $X = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$ form a transcendence basis of $\mathcal{F}$ over the field of rational functions $\mathbb{K} = K(\mathbb{P})$, and $\widetilde{B} = (\widetilde{b}_{ij})$ is an $m \times n$ integral matrix with rows labeled by the elements of $\widetilde{X}$ and columns labeled by elements of X with the following two conditions

- $\widetilde{B}$ has full rank n
- The sub matrix B of $\widetilde{B}$ formed from the first n rows is *skew-symmetrizable*, i.e., $d_i b_{ik} = -d_k b_{ki}$ for some positive integers d_i with $i, k \in [1, n]$. The matrix B is called the exchange matrix of p.

The elements of $f_{n+1}, \ldots, f_m$ are called the frozen variables, elements of X are called cluster variables and $m = t + n$ is called the size of $\mathtt{p}$.

2. The diagram of the skew-symmetrizable matrix $B = (b_{ij})$ is the weighted directed

graph, $\Gamma(B)$, with set of vertices $[1,n]$ such that there is an edge from i to j if and only if $b_{ij}>0$ and this edge is assigned the weight $|b_{ij}b_{ji}|$.

3. For a squire matrix $B=(b_{ij})$ with integer entries, the *Cartan counterpart* of B is denoted by $A(B)=(a_{ij})$ where a_{ij} are given by

$$a_{ij}=\begin{cases}2, & \text{if } i=k,\\ -|b_{ij}| & \text{if } i\neq k.\end{cases} \tag{1.1.1}$$

Definition 1.1.2 (**Seed mutation**). For each fixed $k\in\{1,\dots,n\}$, and each given seed $(\widetilde{X},\widetilde{B})$ we define a new pair $\mu_k(\widetilde{X},\widetilde{B})=(\widetilde{X}',\widetilde{B}')$ by setting $X'=(x_1',\dots,x_n')$ with

$$x_i'=\begin{cases}x_i, & \text{if } i\neq k,\\ \frac{(\prod_{b_{ji}>0}f_j^{b_{ji}})(\prod_{b_{ji}>0}x_j^{b_{ji}})+(\prod_{b_{ji}<0}f_j^{-b_{ji}})(\prod_{b_{ji}<0}x_j^{-b_{ji}}}{x_i)}, & \text{if } i=k,\end{cases} \tag{1.1.2}$$

and $B'=(b_{ij}')$ with

$$b_{ij}'=\begin{cases}-b_{ij}, & \text{if } k\in\{i,j\},\\ b_{ij}+\frac{|b_{ik}|b_{kj}+b_{ik}|b_{kj}|}{2}, & \text{otherwise.}\end{cases} \tag{1.1.3}$$

The operation μ_k is called a mutation in $k-$direction.

Remark 1.1.3. 1. One can see that $\mu_k^2=1$ for all $k\in[1,n]$, and $\{x_1,\dots,x_{i-1},x_i',x_{i+1},\cdots,x_n\}$ is always a transcendence basis of $\mathcal{F}$ over $\mathbb{K}$ for all $i\in[1,n]$, and B' is skew-symmetrizable. So $(\mu_k(\widetilde{X}),\mu_k(\widetilde{B}))$ is again a new seed.

2. The following relation is an equivalent relation on $\mathcal{S}$, the set of all seeds in $\mathcal{F}$.

$$\forall p_1,p_2\in\mathcal{S},\quad p_1\sim p_1 \text{ if and only if } p_2=\mu_{i_1}\mu_{i_2}\cdots\mu_{i_q}(p_1) \tag{1.1.4}$$

for some sequence of mutations $\mu_{i_1},\mu_{i_2},\dots,\mu_{i_q}, i_j\in[1,n], j\in[1,q]$

In this case $\mu_{i_1}\mu_{i_2}\dots\mu_{i_q}(p)$ is called mutation-equivalent to p.

Definition 1.1.4 (Distinguished seeds). A seed $p = (\widetilde{X}, \widetilde{B})$ is called a *distinguished seed* if the exchange matrix B satisfies the following two conditions

- $$b_{ij}b_{ik} \geq 0, \quad \forall \ \ i, j, k \in [1, n], \tag{1.1.5}$$
- the *Cartan counterpart* $A(B) = (a_{ij})$ is of finite type as a Cartan matrix.

Furthermore, the type of the seed p is determined by the Cartan-Killing type of $A(B)$.

Definition 1.1.5 (Geometric cluster algebra). Fix $p = (\widetilde{X}, \widetilde{B}) \in \mathcal{S}$. Let S denote the mutation equivalence class of p, and $\mathcal{X}_S$ be the set of all cluster variables in S, i.e., the union of all clusters of S. The cluster algebra $\mathcal{A}_n(S)$ of rank n, associated to the initial geometric seed $p = (\widetilde{X}, \widetilde{B})$ (of size $m = n + t$) is defined to be the $\mathbb{Z}[\mathbb{P}]-$subalgebra of $\mathcal{F}$ generated by $\mathcal{X}_S$ that is

$$\mathcal{A}_n(S) := \mathbb{Z}[\mathbb{P}][\mathcal{X}_S] \subset \mathcal{F} \tag{1.1.6}$$

Definition 1.1.6 (Cluster pattern and Cluster structure of $\mathcal{A}_n(S)$ [17]).

1. Let $\mathbb{T}_n = (V, E)$ be the n-regular tree, where V is the set of vertices and E is the edges. The cluster pattern $\mathbb{T}_n(S)$ of $\mathcal{A}_n(S)$ is define to be the triple $(\mathbb{T}_n, f, l)$ with $f : V \to S$ and $l : E \to [1, n]$ are two maps such that for $e \in E$ connecting v and v' with $l(e) = k$, we have $\mu_k(f(v)) = f(v')$.
2. The cluster structure of $\mathcal{A}_n(S)$ is the n-regular graph with set of vertices S and edges labeled by $[1, n]$ such that if $k \in [1, n]$ connecting v and v', then $\mu_k(v) = v'$.

Remark 1.1.7. One can see that, the cluster pattern and the cluster structure of $\mathcal{A}_n(S)$ can be completely determined by any seed in S.

Definition 1.1.8. A cluster algebra $\mathcal{A}_n(S)$ is called of finite type if S is a finite set. Equivalently if $\mathcal{X}_S$ is a finite set.

The details for the following two theorems are available in [16].

Theorem 1.1.9. ***(Finite type classification).*** *For a cluster algebra $\mathcal{A}_n(S)$, the following are equivalent:*

- *$\mathcal{A}_n(S)$ is of finite type*
- *for every seed $(\widetilde{X}, \widetilde{B})$ in S, the entries of the exchange matrix $B = (b_{ij})$ satisfy the inequalities $|b_{ij}b_{ji}| \leq 3$, for all $i, j \in [1, n]$*
- *S contains a distinguished seed.*

In such case, The cluster type of $\mathcal{A}_n(S)$ is the same as the Cartan-Killing type of the Cartan counter part the distinguished seed.

Theorem 1.1.10. *Every finite type Cartan matrix corresponds to one and only one, finite type cluster algebra, up to a field automorphism that is restricted to algebra isomorphism.*

Remark 1.1.11 (**[34]**). A seed $p = (\widetilde{X}, \widetilde{B}) \in S$, with $X = (x_1, x_2, \ldots, x_n)$, is said to be acyclic if there is a linear ordering of $\{1, 2, \cdots n\}$ such that $b_{ij} \geq 0$ for all $i < j$. In this case $\mathcal{A}_n(S)$ is called an *acyclic cluster algebra*, and the following is satisfied

$$\mathcal{A}_n(S) = \mathbb{Z}[\mathbb{P}][x_k, x'_k; k \in [1, n]], \tag{1.1.7}$$

and $\mathcal{A}_n(S)$ is finitely generated as an algebra over $\mathbb{Z}[\mathbb{P}]$.

Theorem 1.1.12. ***(Laurent Phenomenon).*** *The cluster algebra $\mathcal{A}_n(S)$ is contained in the integral ring of Laurent polynomials $\mathbb{Z}[\mathbb{P}][X^{\pm}]$, for any cluster X, i.e.,*

$$\mathcal{A}_n(S) \subset \mathbb{Z}[\mathbb{P}][X^{\pm}] = \mathbb{Z}[\mathbb{P}][x_1^{\pm}, x_2^{\pm}, \ldots, x_n^{\pm}]. \tag{1.1.8}$$

More precisely, every non zero element, can be uniquely written as

$$y = \frac{P(x_1, x_2, \ldots, x_n)}{x_1^{\alpha_1} \ldots x_n^{\alpha_n}}, \tag{1.1.9}$$

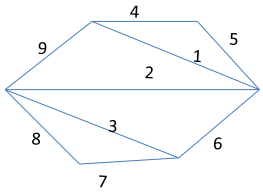
where $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$, *and* $P(x_1, x_2, \ldots, x_n)$ *is in* $\mathbb{Z}[\mathbb{P}][x_1, x_2, \ldots, x_n]$, *which is not divisible by any cluster variables* $x_1, x_2, \ldots, x_n$.

Conjecture 1.1.13. Fomin-Zelevinsky Positivity Conjecture. For any cluster algebra $\mathcal{A}_n(S)$ if y is a cluster variable, then the polynomial $P(x_1, x_2, \ldots, x_n)$ (appeared in (1.1.9)) has nonnegative integer coefficients.

The conjecture has been proved in many cases including classical type cluster algebras [16], rank two affine cluster algebras as in [33], acyclic cluster algebra [14], cluster algebras arising from spaces [27], and more.

Example 1.1.14. Cluster algebras coming from polygon triangulations. From the above definition of cluster algebra, one can see that starting with a skew symmetrizable matrix we can associate a transcendence basis to get a seed and use this seed to generate a cluster algebra using mutations. So, having a method of generating skew symmetrizable matrices is equivalent to having a method of generating cluster algebras. One way to generate skew symmetrizable matrices is to associate a matrix to each non crossing triangulations of the polygon, as we see in the following.

Let T be a fixed triangulation of the $(n+3)$-gone $\mathbb{P}_n$. Label all the diagonals with numbers starting with the internal diagonals, i.e., the labels of the internal diagonals are from the set $\{1, \ldots, n\}$ and the edges take labels from $\{n+1, \ldots, 2n+3\}$. Let $\widetilde{B}(\mathrm{T}) = (b_{ij})$ be the adjacent matrix associated to the triangulation T. We define $\widetilde{B}(\mathrm{T})(b_{ij})$ as follows; it has exactly $m = 2n + 3$ rows and n columns, the rows are associated to all diagonals (internals and edges) and the columns associates to diagonals only. So, every diagonal encodes one cluster variable and the edges correspond to the frozen variables. Consider the following triangulation of the hexagon, and the entries of $\widetilde{B}(\mathrm{T})(b_{ij})$ in this case are given by



$$\widetilde{b}_{ij} := \begin{cases} +1, & \text{if } i \text{ and } j \text{ share one vertex such that } i \text{ following } j \text{ clockwise } \angle_j^i, \\ 0, & \text{if } i \text{ and } j \text{ do not show up in some triangle ,} \\ -1, & \text{if } i \text{ and } j \text{ share one vertex such that } i \text{ following } j \text{ counterclockwise } \angle_i^j. \end{cases} \tag{1.1.10}$$

The above triangulation for the hexagon corresponds to the seed $(\widetilde{X}, B)$ where $\widetilde{X} = (x_1, x_2, x_3, f_4, f_5, f_6, f_7, f_8, f_9)$ and the matrix $\widetilde{B}$ given by

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \tag{1.1.11}$$

1.2 Quantum cluster algebra

All the material of this subsection are quoted from [2], [10]

Definition 1.2.1. Compatible pairs Let $\widetilde{B}$ and $\widetilde{X}$ be as in definition 1.1.1, and $\Lambda = (\lambda_{ij})$ be a skew symmetric $m \times m$ integral matrix with row and columns labeled by the elements of $\widetilde{X}$. The pair $(\Lambda, \widetilde{B})$ is said to be a compatible pair if

$$\sum_{k=1}^{m} b_{kj}\lambda_{ki} = \begin{cases} d_j, & \text{if } i = j, \\ 0, & \text{if } i \neq k \end{cases} \tag{1.2.1}$$

Definition 1.2.2. Based quantum tours, toric frames and quantum seeds.

1. Let L be a lattice of rank m, with skew symmetric bilinear form $\Lambda : L \times L \to \mathbb{Z}$. Let q be a formal variable, and $\mathbb{Z}[q^{\pm\frac{1}{2}}] \subset \mathbb{Q}(q^{\frac{1}{2}})$ be the ring Laurent polynomials in $q^{\frac{1}{2}}$. The based quantum torus associated with L is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra $\tau = \tau(\Lambda)$ with a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis $\{X^e : e \in L\}$ and multiplication given by

$$X^e X^f = q^{\frac{\Lambda(e,f)}{2}} X^{e+f}, \quad x^0 = 1 \ \text{ and } \ (x^e)^{-1} = x^{-e}, \quad \text{for } \ e, f \in L. \tag{1.2.2}$$

Let $\mathbb{F}$ be the skew-field of fractions of τ, then one can see that $d \mapsto d \cdot 1^{-1}$ is an embedding of τ in $\mathbb{F}$.

2. A toric frame in $\mathbb{F}$ is a mapping $M : \mathbb{Z}^m \to \mathbb{F}^*$, given by $M(c) = \lambda(X^{\gamma(c)})$, for some λ an $\mathbb{F}$-automorphism, and $\gamma : \mathbb{Z}^m \to L$ is an isomorphism of lattices.

   Note that the elements $M(c)$ form a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis of an isomorphic copy $\lambda(\tau)$ of the based quantum torus τ.

3. A quantum seed is a pair $(M, \widetilde{B})$, where M is a toric frame in $\mathbb{F}$, and $\widetilde{B}$ is an $m \times n$ integral matrix with rows labeled by $[1, m]$ and columns labeled by an n-element subset $\mathbf{ex}$ of $[1, m]$, such that $(\Lambda_M, \widetilde{B})$ is a compatible pair, where Λ_M is the bilinear form obtained from Λ by transferring the form Λ from L by the lattice isomorphism γ.

Definition 1.2.3. Let $(M, \widetilde{B})$ be a quantum seed. We generate new quantum seeds by applying mutations on $(M, \widetilde{B})$ as follows. Fix $k \in \mathbf{ex}$, and $\epsilon \in \{-1, 1\}$. We define $\mu_k(M) = M' : \mathbb{Z}^m \to \mathbb{F}^*$ is given by; for the integral column vector $c = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$,

$$M'(c) = \sum_{p=0}^{c_k} \binom{c_k}{p}_{q^{\frac{d_k}{2}}} M(E_\epsilon c + \epsilon p b^k), \quad M'(-c) = M'(c)^{-1}, \tag{1.2.3}$$

where

$$\binom{r}{p}_t = \frac{(t^r - t^{-r}) \cdots (t^{r-p+1} - t^{-r+p-1})}{(t^p - t^{-p}) \cdots (t - t^{-1})},$$

and $E_\epsilon = (e_{ij})$ is an $m \times m$ matrix given by

$$e_{ij} = \begin{cases} \delta_{ij}, & \text{if } j \neq k, \\ -1, & \text{if } i = j = k\,, \\ max(0, -\epsilon b_{ij}) & \text{if } i \neq j = k \end{cases}$$

and the vector b^k is the k-th column of $\widetilde{B}$. Mutation on $\widetilde{B}$ is defined as before.

Definition 1.2.4. Let $(M, \widetilde{B})$ be a quantum seed, and let $\widetilde{X} = (x_1 \ldots, x_m)$, where $x_i = M(e_i)$, $\{e_i\}_1^n$ is a basis of $\mathbb{Z}^m$. The set of cluster variables of $(M, \widetilde{B})$ is $X = \{x_j; j \in \mathbf{ex}\}$. Let $C = \widetilde{X} - X$, and $\mathbb{P}'$ be the free group generated by elements of C, written multiplicatively. The quantum cluster algebra is defined to be the $\mathbb{Z}[q^{\pm\frac{1}{2}}][\mathbb{P}']$-subalgebra of $\mathbb{F}$ generated by all the cluster variables in every quantum seed that obtain from $(M, \widetilde{B})$ by applying some sequence of mutations.

1.3 Cluster algebras and total positivity

The classical theory of total positivity was started in the third decade of the the twentieth century by many mathematicians among whom Gantmacher, Kerin, and Schoenberg and they basically were studying the total positivity in the matrix groups. In [24] G. Lusztig introduced the totally non-negative variety $G_{\geq 0}$ and studied the structure of total non-negative elements inside the unipotent radical N of a Borel subgroup B in any reductive group G, he was motivated by his discovery of connections between total positivity and his theory of canonical basis for quantum groups. In [1], and [13] A. Zelevinsky, S. Fomin, and A. Berenstein provided the double Bruhat cells as a natural framework to study the total positivity in any reductive group using the double Bruht decomposition of G into a disjoint union of double Bruhat cells. In their study, they provided an algebraic framework for the total positivity tests which are regular functions on the double Bruhat cells, this framework is what they later called *cluster algebras*.

All results of this section are true for every reductive algebraic group. However, some statements are written for matrix algebraic groups (subgroups of the general linear group $GL_n(\mathbb{C})$). Our running example is $SL_n(\mathbb{C})$.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ with respect to a set of Chevalley generators $\{e_i\}$,$\{h_i\}$, and $\{f_i\}$, $i = 1, \ldots, r$, for $\mathfrak{n}_-$, $\mathfrak{h}$, and $\mathfrak{n}_+$ respectively. Let $\Phi \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with simple roots $\{\alpha_1, \ldots, \alpha_r\}$. The

Cartan matrix of the root system Φ is given by $A = (a_{ij}) = (\alpha_i(h_j))$. Denote the weight lattice by $\mathcal{P} = \{a \in \mathfrak{h}^* : a(h_i) \in \mathbb{Z}, i \in [1, r]\}$. The fundamental weights $\{\omega_1, \ldots, \omega_r\}$ given by $\omega_i(h_j) = \delta_{ij}$, form a $\mathbb{Z}$-basis for $\mathcal{P}$. Let $\mathtt{G}$ be the Lie group with Lie algebra $\mathfrak{g}$ and N, N_- and H be closed subgroups of $\mathtt{G}$ with Lie algebras $\mathfrak{n}, \mathfrak{n}_-$ and $\mathfrak{h}$ respectively. Every element of H can be written as $\exp(h)$ for some $h \in \mathfrak{h}$, which gives rise to a multiplicative character for the group H given by $\gamma : H \to \mathbb{C}^*$ where $\gamma(\exp(h)) = \exp(\gamma(h)), h \in H$.

Let B and B_- be two opposite Borel subgroups of $\mathtt{G}$ with unipotent radicals N and N_- respectively, that is $B = HN, B_- = HN_-$ and $H = B \cap B_-$ be the maximal torus. The Weyl group is defined by $W = \mathrm{Norm}_{\mathtt{G}}(H)/H$. W acts on H by $w(h) = w^{-1}hw$ which gives rise to an action of the weyl group W on the weight lattice $\mathcal{P}$ as follows; For $a \in H, w \in W$ and $\gamma \in \mathcal{P}$, $w(\gamma)(a) = \gamma(w(a)) = \gamma(w^{-1}aw)$. We identify W with the group of all linear transformations of $\mathfrak{h}^*$ which is a Coxeter group generated by simple reflections $s_1, \ldots, s_r$ where $s_i : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ with $s_i(\gamma) = \gamma - \gamma(h_i)\alpha_i, i = 1, \ldots, r$. A reduced word $w \in W$ is a sequence of indices $\mathrm{i} = (i_1, \ldots, i_m)$ of shortest possible length such that $w = s_{i_1} \cdots s_{i_m}$. In this case we say the length of w is m and we write $l(w) = m$. Consider the two sets of elements $\{\bar{s}_1, \ldots, \bar{s}_r\}$ and $\{\bar{\bar{s}}_1, \ldots, \bar{\bar{s}}_r\}$ from $\mathtt{G}$ given by $\bar{s}_i = \exp(-e_i)\exp(f_i)\exp(-e_i)$, $\bar{\bar{s}}_i = \exp(e_i)\exp(-f_i)\exp(e_i)$. These elements satisfy the following relations

1.
$$\bar{s}_i\bar{s}_j\bar{s}_i \cdots = \bar{s}_j\bar{s}_i\bar{s}_j \cdots \quad \text{(same equations are satisfied for } \bar{\bar{s}}_i) \tag{1.3.1}$$

2.
$$s_i = \bar{s}_i H. \tag{1.3.2}$$

Therefore for any $w \in W$ we can introduce the elements $\bar{w} \in \mathrm{Norm}_{\mathtt{G}}(H)$ with the equation $\overline{uv} = \bar{u}\bar{v}$ whenever $l(uv) = l(u) + l(v)$.

Let $\mathtt{G}_0 = N_-HN = \{x \in \mathtt{G} : x = x_-x_0x_+ \text{ for some } x_- \in N_-, x_0 \in H, x_+ \in N\}$. For a fundamental weight ω and a regular function ∇ on $\mathtt{G}$, if the restriction of ∇ on $\mathtt{G}_0$ given by $\nabla(x) = \omega(x_0)$, then we write $\nabla = \nabla^\omega$.

Definition 1.3.1. For $u, v \in W$ and ω_i a fundamental weight ***the generalized minor*** $\triangle_{u\omega_i, v\omega_i}$ is the regular function on $\mathtt{G}$ whose given on $\mathtt{G}$ by

$$\triangle_{u\omega_i, v\omega_i}(x) = \nabla^{\omega_i}(\overline{\overline{u^{-1}}} x \bar{v}). \tag{1.3.3}$$

Remark 1.3.2. The action of the generalized minors on $\mathtt{G}$ does not depend on the particular choice of the reduced word of u and v but depends on the weights $u\omega_i$ and $v\omega_i$

Important special case. Let $\mathtt{G} = SL_{r+1}$ be the special linear subgroup of GL_n. In this case, B is the subgroup of $\mathtt{G}$ of upper triangle matrices, B_- is the subgroup of of lower triangle matrices and H is the subgroup of diagonal matrices, N is the subgroup of upper triangle matrices with all diagonal entries equal one, and N_- is the subgroup of lower triangle matrices with diagonal entries are all ones.

A generalized minor on $\mathtt{G}$ is a regular function defined by sending each element x in $\mathtt{G}$ to some determinant of a sub matrix of x, we denote the minors by $\triangle_{I,J}$ with I, J being subsets of $[1, n]$ such that I and J have same cardinality, where I refers to rows and J to columns. So, for example the minor $\triangle_{124,234}$ is the determinant of the sub matrix with rows $1, 2, 4$ and columns $2, 3, 4$.

The Weyl group of the the special linear group SL_{r+1} is identified with the symmetric group $\mathfrak{S}_{r+1}$. For a fixed element $\sigma \in \mathfrak{S}_{r+1}$ the pair $(i, j) \in [1, r+1] \times [1, r+1]$ is said to be an inversion of σ if $i < j$ and $\sigma(i) > \sigma(j)$. The length of σ is defined to be the number of its inversions, and is denoted by $l(\sigma)$.

The symmetric group $\mathfrak{S}_n$ acts on the general linear group GL_n as follows; a permutation element σ acts on a matrix x by permuting the rows and the columns of x simultaneously.

Definition 1.3.3. Let G be a reductive group, and let H be the maximal torus of G. We set $H_{>0} := \{a \in H : \gamma(a) \text{ is a positive real number}, \forall \text{ weight } \gamma \in \mathcal{P}\}$. An element x of G is

said to be totally non-negative if and only if x is an element of the multiplicative semigroup $G_{\geq 0}$ generated by $H_{>0} \cup \{\exp(te_i), \exp(tf_i) : t \text{ is any positive real number}, i \in [1, r]\}$

Definition 1.3.4. A matrix of real entries B is totally positive (resp., totally nonnegative) if all its minors are positive (resp., nonnegative). We use the term TP for totally positive and TNN for totally non-negative.

Remark 1.3.5. In [1] the above two definitions of total non-negativity coincides in the case of $G = SL_n(\mathbb{C})$.

Definition 1.3.6. Let $\mathtt{G}$ be a subgroup of G. A subset $\Delta(\mathtt{G})$ of the algebra of regular functions of $\mathtt{G}$ is said to be a *total positivity test* (resp., *total non-non negativity test*) for $\mathtt{G}$ if and only if

$$x \in \mathtt{G} \text{ is TP (resp., TNN) if and only if } f(x) > 0 \text{ (resp., } f(x) \geq 0) \ \forall f \in \Delta(\mathtt{G}) \quad (1.3.4)$$

Question 1.3.7. For a given subgroup (or subset) of group G, is there a machinery to produce total positivity tests and what could be an algebraic frame work to study these tests?

In fact, the answer for this question is positive in many cases even for some non matrix groups, and it has to do with introducing the cluster algebras theory. In the following we will provide such machinery for double Bruhat cells, as an example.

Question 1.3.8. Why should we care about the positivity in double Bruhat cells?

An answer for this question in the case of G_n, the general linear group, comes from the following definition and theorem

Definition 1.3.9. For a Weyl group element u, the subsets BuB and B_-uB_- of G are called the Bruhat cells with respect to B and B_- respectively. For two elements of the Wyel group u and v, the *double Bruhat cell* with respect to B and B_- in G are given by

$$G^{u,v} = BuB \cap B_-vB_-. \quad (1.3.5)$$

In the case of GL_n, the elements u and v would be permutations in $r+1$ letters (r is the dimension of $\mathfrak{h}$).

Theorem 1.3.10. ***Double Bruhat Cells description and Bruhat decompositions.***

1. *A matrix $x \in BuB$ if and only if the following two conditions are satisfied*

   - $\triangle_{u([1,i]),[1,i]} \neq 0$ *for* $i \in [1, n-1]$*;*
   - $\triangle_{u([1,i-1]\cup\{j\}),[1,i]} = 0$ *for all* (i,j) *such that* $1 \leq i < j \leq n$ *and* $u(i) < u(j)$*.*

2. *The group G is the disjoint union of all double Bruhat cells, that is it has the following Bruhat decompositions, with respect to B and B_-:*

$$G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_- v B_-. \tag{1.3.6}$$

Remark 1.3.11. The transpose map $(-)^T : GL_n \longrightarrow GL_n$ given by $x \mapsto x^T$ provides us with a similar description for the other Bruhat cell B_-vB_- as the one provided above for BwB in the above theorem. Noticing that the transpose map sends BwB into $Bw^{-1}B$ and each minor $\triangle_{I,J}$ to $\triangle_{J,I}$.

Example 1.3.12. Let w_0 be the longest element in W. Then G^{w_0,w_0} is the open double Bruhat cell given by:

$$G^{w_0,w_0} = \{x \in G; \triangle_{w_i,w_o w_i}(x) \neq 0, \triangle_{w_o w_i,w_i}(x) \neq 0, \text{for all } \ i \in [1,r]\}. \tag{1.3.7}$$

For instance, consider the following two examples

1. Let $G = SL_2$. So, $r = 1$ and w_0 is the transpose (12). The double Bruhat cell in this case is given by

$$SL_2^{(12),(12)} = \{x \in SL_2 : \triangle_{1,2}(x) \neq 0, \triangle_{2,1}(x) \neq 0\}.$$

2. Let $G = SL_3$. Here, $r = 2$ and $w_0 = (13)$ the permutation that fixes 2 and permutes 1 and 3. Then, we have

$$SL_2^{(13),(13)} = \{x \in SL_3 : \triangle_{1,3}(x) \neq 0, \triangle_{12,32}(x) \neq 0, \triangle_{3,1}(x) \neq 0, \triangle_{23,12}(x) \neq 0\}$$

Theorem 1.3.13. *Double Bruhat cell $G^{u,v}$ is isomorphic (as an algebraic variety) to a Zarisky open subset of an affine space $\mathbb{C}^{l(u)+l(v)+r}$, where r is the rank of G.*

Definition 1.3.14. We define the totally positive part of $G^{u,v}$ by setting

$$G^{u,v}_{>0} = G^{u,v} \cap G_{\geq 0}. \tag{1.3.8}$$

Theorem 1.3.15. *The totally positive part of the open double Bruhat cell is the totally positive variety:*

$$G_{>0} = G^{w_0,w_0}_{>0}. \tag{1.3.9}$$

So, studying the total positivity of G is reduced to studding the total positivity of the double Bruhat cell at the longest element of W.

Definition 1.3.16. A Totally positive basis for $G^{u,v}$ is a collection of regular functions $F = \{f_1, \ldots, f_m\} \subset \mathbb{C}[G^{u,v}]$ with the following properties:

1. The functions $f_1, \ldots, f_m$ are algebraically independent and generate the field of rational functions $\mathbb{C}(G^{u,v})$; in particular, $m = r + l(u) + l(v)$.

2. The map $(f_1, \ldots, f_m) : G^{u,v} \longrightarrow \mathbb{C}^m$ restricts to a biregular isomorphism $U(F) \longrightarrow (\mathbb{C}_{\neq 0})^m$, where

$$U(F) = \{x \in G^{u,v} : f_k(x) \neq 0 \ \text{ for all } \ k \in [1, m]\}. \tag{1.3.10}$$

3. The map $(f_1, \ldots, f_m) : G^{u,v} \longrightarrow \mathbb{C}^m$ restricts to an isomorphism $G^{u,v}_{>0} \longrightarrow \mathbb{R}^m_{>0}$

Corollary 1.3.17. *Property three in the previous theorem implies that every totally positive basis of $G^{u,v}$ is a totally positive test.*

Theorem 1.3.18. *Each reduced word $\mathfrak{i}$ for $(u,v) \in W \times W$ gives rise to a totally positive basis $F_{\mathfrak{i}}$ consisting of generalized minors for $G^{u,v}$ given as follows:*

$$F_{\mathfrak{i}} = \{\triangle_{\gamma_k,\delta_k} : k \in [1,m]\}; \tag{1.3.11}$$

here $m = r + l(u) + l(v)$, and $\gamma_k, \delta_k \in \mathcal{P}$ are defined as follows; $\mathfrak{i}$ can be represented as a sequence of indices $(i_1, \ldots, i_m)$ from the set $\{-1, \ldots, -r, 1, \ldots, r\}$ such that $i_j = j$ for $j \in [1,r]$, and $s_{-i_{r+1}} \cdots s_{-i_m} = u$, and $s_{i_{r+1}} \cdots s_{i_m} = v$, with the convention $s_{-i} = 1$ for $i \in [1,r]$. Now we have the assignments

$$\gamma_k = s_{-i_1} \cdots s_{i_k} \omega_{|i_k|}, \qquad \delta_k = s_{i_m} \cdots s_{i_{k+1}} \omega_{|i_k|} \tag{1.3.12}$$

Example 1.3.19. Let $G = SL_3(\mathbb{C})$, and let $u = v = w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ be the order-reversing permutation (the element of maximal length in the symmetric group $W = \mathfrak{S}_3$). Take $\mathfrak{i} = (1,2,1,2,1,-1,-2,-1)$.

Then $F_{\mathfrak{i}} = \{\triangle_{1,3}, \triangle_{12,23}, \triangle_{1,2}, \triangle_{12,12}, \triangle_{1,1}, \triangle_{2,1}, \triangle_{23,12}, \triangle_{3,1}\}$. (The minors on the right-hand side are listed in the natural order, i.e., $f_1 = \triangle_{1,3}, \ldots, f_8 = \triangle_{3,1}$). One can see $\{f_1, \ldots, f_8\}$ provides a total positivity test in SL_3.

What are the relations between them and how they related to the structure of $\mathbb{C}[G^{u,v}]$?. This what we will discuss in the rest of this subsection.

Theorem 1.3.20. *For every reduced word $\mathfrak{i}$ for $(u,v) \in W \times W$, we associate an $m \times n$ integral matrix $\tilde{B}(\mathfrak{i}) = (b_{ij})$, such that the following are true*

1. *The pair $(F_{\mathfrak{i}}, \tilde{B}(\mathfrak{i}))$ is a seed in the field of fractions $\mathbb{C}(G^{u,v})$*
2. *For every k an exchangeable index in $[1,m]$, the mutation in the k-direction on the totally positive test $F_{\mathfrak{i}}$ given by $\mu_k(F_{\mathfrak{i}}) = F_{\mathfrak{i}} - \{f_k\} \cap \{f_k'\} \subset \mathbb{C}[G^{u,v}]$, where*

$$f'_k = \frac{\prod_{b_{ik}>0} f_i^{b_{ik}} + \prod_{b_{ik}<0} f_i^{-b_{ik}}}{f_k} \tag{1.3.13}$$

is again a totally positive basis for $G^{u,v}$.

Definition 1.3.21. For each reduced word $\mathfrak{i}$ for $(u, v) \in W \times W$, here is how we build the matrix $\tilde{B}(\mathfrak{i})$ in general.

First step: the directed graph $\tilde{\Gamma}(\mathfrak{i})$. Vertices of $\tilde{\Gamma}(\mathfrak{i})$ is the set $I_0 = \{-1, \ldots, -r, 1, \ldots, l(u)+l(v)\}$. Arrows are defined based on the following rules; For $k \in [1, r] \cup [1, l(u) + l(v)]$, we denote by k^+ is the smallest index l such that $k < l$ and $|i_l| = |i_k|$; if $|i_k| \neq |i_l|$ for $k < l$, then we set $k^+ = (u) + l(v) + 1$. Two vertices k and l, with $k < l$ are connected by an edge if and only if either k or l (or both) are exchangeable plus one of the following three conditions is satisfied

1. $l = k^+$.
2. $l < k^+ < l^+, a_{|i_k|,|i_l|} < 0$, and $\varepsilon(i_l) = \varepsilon(i_{k^+})$.
3. $l < l^+ < k^+, a_{|i_k|,|i_l|} < 0$, and $\varepsilon(i_l) = -\varepsilon(i_{k^+})$.

The edges coming from condition (1) are called horizontal and those from conditions (2) and (3) are called inclined. To determine the direction of any edge we follow the following rule; A horizontal (resp., to inclined) edge between k and l is directed from k to l if and only if $\varepsilon(i_k) = +1$) (with respect to $\varepsilon(i_l) = -1$). The directed graph $\tilde{\Gamma}(\mathfrak{i})$ provides us with the signs of $B(\mathfrak{i})$.

Second step: The entries of the matrix $\tilde{B}(\mathfrak{i})$. The rows of $\tilde{B}(\mathfrak{i})$ are labeled by the set of indices I_0 of $\tilde{\Gamma}(\mathfrak{i})$ and the columns are labeled by the set of $\mathfrak{i}$-exchangeable indices. An entry b_{kl} is determined by the following rules

1. $b_{kl} \neq 0$ if and only if there is an edge of $\tilde{\Gamma}(\mathfrak{i})$ connecting k and l; $b_{kl} > 0$ (respectively to $b_{kl} < 0$) if this edge is directed from k to l (respectively to from l to k);

2. If k and l are connected by an edge of $\tilde{\Gamma}(\mathfrak{i})$, then

$$|b_{kl}| = \begin{cases} 1, & \text{if } |i_k| = |i_l| \ \ (k \text{ and } l \text{ are connected by a horizontal edge}), \\ -a_{|i_k|,|i_l|}, & \text{if } |i_k| \neq |i_l| \ \ (k \text{ and } l \text{ are connected by a inclined edge}). \end{cases} \tag{1.3.14}$$

Proposition 1.3.22. *The matrix $\tilde{B} = \tilde{B}(\mathfrak{i})$ has a full rank n. Its principal part $B = B(\mathfrak{i})$ is skew-symmetrizable.*

Remark 1.3.23. The matrix $\tilde{B}$ can be recovered from the reduced $\mathfrak{i}$ as follows.

Let $k \in -[1,r] \cup [1, l(u)+l(v)]$ and l be an exchange vertex. Let $p = \max(k,l)$ and $q = \min(k^+, l^+)$. Then

$$b_{kl} = \begin{cases} -sgn(k-l)\varepsilon(i_p), & \text{if } p = q, \\ -sgn(k-l)\varepsilon(i_p)a_{|i_k|,|i_l|}, & \text{if } p < q \ \text{ and } \ \varepsilon(i_p)\varepsilon(i_q)(k-l)(k^{=} - l^{+}) > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{1.3.15}$$

Example 1.3.24. With the same data of example 1.3.19 we will introduce the matrix $\tilde{B}(\mathfrak{i})$ for some choice of $\mathfrak{i}$. We have $r = 2$, $u = v = w_0$. Then $m = l(u) + l(v) + 2 = 8$, and $n = 4$. Let $w_0 = s_1 s_2 s_1$, where $s_i, i = 1, 2, 3$ are the simple reflections. For simplicity we write s_j as just j, so $w_0 = 121$. The graph $\tilde{\Gamma}(\mathfrak{i})$ has exactly 8 vertices corresponding to $\{-2, -1, 1, 2, 3, 4, 5, 6\}$ which has indices $(i_{-2}, i_{-1}, i_{-1}, i_{-2}, i_{-1}, i_1, i_2, i_1)$ in order, i.e., $i_{-2} = -2, i_{-1} = -1$, etc. To find the set of exchangeable vertices, we need the following

$$-2^+ = 2, -1^+ = 1, 1^+ = 3, 2^+ = 5, 3^+ = 4, 4^+ = 6, 5^+ = 7, \ \text{ and } \ 6^+ = 7.$$

Then by definition the set of $\mathfrak{i}-$exchangeable vertices is $\{1, 2, 3, 4\}$, so $n = 4$. Based on the rules of edges we must have two strips of horizontal edges one for the vertices $\{-2, 2, 5\}$

and the other one is for the vertices $\{-1, 1, 3, 4, 6\}$. The inclined edges are between the two stripes as follows; each of the following pairs of vertices are connected by an inclined edge $\{-2, 1\}, \{2, 1\}, \{2, 3\}, \{2, 4\}$, and $\{5, 4\}$. The directions of the edges are determined as follows form k to l if $k < l$. So, The graph $\tilde{\Gamma}(\mathfrak{i})$ for $\mathfrak{i} = (1, 2, 1, -1, -2, -1)$ is as follows

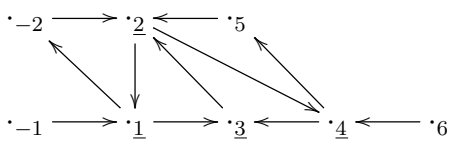
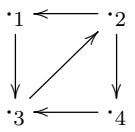


Figure 1.1. $\tilde{\Gamma}(\mathfrak{i})$.

The subgraph connecting the underlined vertices of $\tilde{\Gamma}(\mathfrak{i})$, the $\mathfrak{i}$-exchangeable vertices, corresponding to the principle part $B = B(\mathfrak{i})$. One can see B is sign-skew-symmetric.

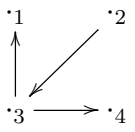
$$\tilde{B}(\mathfrak{i}) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.3.16}$$

So, the picture is completed and we have the second half of the geometric seed $\Sigma(\mathfrak{i}) = (F(\mathfrak{i}), \tilde{B}(\mathfrak{i}))$. Following the same order the elements given in example 1.2.2 for $F(\mathfrak{i})$, we can, for simplicity, we write $F(\mathfrak{i}) = \{x_{-2}, x_{-1}, x_1, x_2, x_3, x_4, x_5, x_6\}$. The set of frozen variables $Fr = \{x_{-2}, x_{-1}, x_5, x_6\}$, hence the set of cluster variables is $F(\mathfrak{i}) - Fr = \{x_1, x_2, x_3, x_4\}$. On can see the sub matrix corresponding to the $\mathfrak{i}$-exchangeable vertices can be represented by the quiver



(1.3.17)

Applying mutation at vertex 3 we get;



(1.3.18)

which is an acyclic quiver. Then the cluster algebra $\mathcal{A}_{\mathfrak{i}}$ associated with the seed $\Sigma(\mathfrak{i})$ is acyclic cluster algebra and hence it is finitely generated with set of generators

$$F(\mathfrak{i}) \cup \{x'_1, x'_2, x'_3, x'_4\},$$

where x'_1, x'_2, x'_3, x'_4 are elements of the field of fractions of the double Bruhat cell G^{w_0,w_0}. The quiver (1.1.17) can be used to write the relations that determine the new variables $x'_i, i = 1, 2, 3, 4$, as follows

$$\begin{aligned} x_1x'_1 &= x_{-1}x_2 + x_{-2}x_3 \\ x_2x'_2 &= x_{-2}x_3x_5 + x_1x_4 \\ x_3x'_3 &= x_1x_4 + x_2 \\ x_4x'_4 &= x_2x_6 + x_3x_5. \end{aligned}$$

These relations coincide with the mutations of $\Sigma(\mathfrak{i})$ at the directions $1, 2, 3$ and 4 respectively.

In the following for each double reduced word $\mathfrak{i}$ of any two elements u and v of the Weyl group W, we introduce a cluster of generalized minors that form with the matrix $B(\mathfrak{i})$ a seed in the field $\mathbb{C}(G)$ which generates a cluster algebra that is isomorphic the coordinate ring of the double Bruhat cell $\mathbb{C}[G^{u,v}]$.

Consider the following notation introduced by A. Bernstein, S. Fomin and A. Zelevinsky in [1]. For $k \in [1, l(u)+l(v)]$ we denote

$$u_{\leq k} = u_{\leq k}(\mathfrak{i}) = \prod_{l=1,\ldots,k;\varepsilon(i_l)=-1} s_{|i_l|}, \tag{1.3.19}$$

$$v_{>k} = u_{>k}(\mathfrak{i}) = \prod_{l=l(u)+l(v),\ldots,k+1;\varepsilon(i_l)=+1} s_{|i_l|}. \tag{1.3.20}$$

Notice that; the index l in the first equation above is increasing and decreasing in the second one. For $k \in -[1,r]$ we have $u_{\leq k} = e$ and $v_{>k} = v^{-1}$. For $k \in -[1,r] \cup [1, l(u)+l(v)]$ we set

$$\triangle(k, \mathfrak{i}) = \triangle_{u_{\leq k}\omega_{|i_k|}, v_{>k}\omega_{|i_k|}}. \tag{1.3.21}$$

The cluster is $F(\mathfrak{i}) = \{\triangle_{u_{\leq k}\omega_{|i_k|}, v_{>k}\omega_{|i_k|}} : k \in -[1,r] \cup [1, l(u)+l(v)]\}$.

Remark 1.3.25. The above technique of obtaining the cluster $F(\mathfrak{i})$ is a deferent way to obtain the TP-basis provided from theorem 1.1.33.

Example 1.3.26. We continue with our running example SL_3 showing how to calculate the cluster variables of the cluster $F(\mathfrak{i})$ using the above technique.

Here $\mathfrak{i} = (-2,-1,1,2,1,-1,-2,-1)$. So, by the above definition of $u_{\leq k}$ and $v_{>k}$ we have the following table:

k	i_k	u	$v_{>k}$	$u_{\leq k}\omega_{\lvert i_k\rvert}$	$v_{>k}\omega_{\lvert i_k\rvert}$
-2	-2	e	w_0	$[12]$	$[23]$
-1	-1	e	w_0	$[1]$	$[3]$
1	1	e	$s_1s_2 = (231)$	$[1]$	$[2]$
2	2	e	$s_1 = (12)$	$[12]$	$[12]$
3	1	e	e	$[1]$	$[1]$
4	-1	$s_1 = (12)$	e	$[2]$	$[1]$
5	-2	$s_1s_2 = (231)$	e	$[23]$	$[12]$
6	-1	$w_0 = (13)$	e	$[3]$	$[1]$

(1.3.22)

Using the above table the cluster variables are $\triangle_{12,23}, \triangle_{1,3}, \triangle_{1,2}, \triangle_{12,12}, \triangle_{1,1}, \triangle_{2,1}, \triangle_{23,12}$ and $\triangle_{3,1}$ with the frozen cluster variables $\{\triangle_{12,23}, \triangle_{1,3}, \triangle_{23,12}, \triangle_{3,1}\}$.

The details of the following theorem can be found in [1].

Theorem 1.3.27. *Fix a reduced word* $\mathfrak{i}$ *of a pair* $(u, v) \in W \times W$. *Consider the field* $\mathfrak{F}_{\mathbb{C}} = \mathfrak{F} \otimes \mathbb{C}$. *Let* $\mathcal{A}_{\mathfrak{i}\mathbb{C}} = \mathcal{A}_{\mathfrak{i}} \otimes \mathbb{C}$ *be the complexification of the cluster algebra associated to* $\mathfrak{i}$. *The isomorphism of fields* $\varphi : \mathfrak{F}_{\mathbb{C}} \longrightarrow \mathbb{C}(G^{u,v})$ *given by*

$$\varphi(x_k) = \triangle(k, \mathfrak{i}) \quad \textit{for all} \quad k \in -[1, r] \cup [1, l(u) + l(v)] \tag{1.3.23}$$

restricts to an isomorphism of algebras $\mathcal{A}_{\mathfrak{i}\mathbb{C}} \longrightarrow \mathbb{C}[G^{u,v}]$.

1.4 Cluster algebras and root systems

Definitions 1.4.1.

1. ***Cluster monomials and full cluster monomials.*** The monomial $m = z_1^{\beta_1} \cdots z_n^{\beta_n}, \beta_i \in \mathbb{Z}_{\geq 0}, i \in [1, n]$ is a cluster monomial if and only if $(z_1, \ldots, z_n)$ is a cluster in some seed in the cluster pattern. In the case of $\beta_i \in \mathbb{Z}_{>0}, \forall i \in [1, n]$, the monomial m is called a *full cluster monomial*.

2. **Positive elements.** An element y of the cluster algebra $\mathcal{A}_n$ is said to be a positive element if it satisfies
$$y \in \bigcap_{X \in S_C} \mathbb{Z}_{\geq 0}[X^{\pm 1}].$$
Where S_C is the set of all clusters of $\mathcal{A}_n$ and $\mathbb{Z}_{\geq 0}[X^{\pm 1}]$ is the set of all Laurent polynomials in the cluster variables from the cluster X, with positive integral coefficients.

3. **indecomposable element.** An indecomposable element in $\mathcal{A}_n$ is any element that can not be written as a sum of two positive elements. (In some literatures they are called atomic elements)

4. **Denominator vector with respect to a given cluster.** From Laurent phenomenon we have that every cluster variable y can be uniquely written as

$$y = \frac{P(x_1, \ldots, x_n)}{x^{d_1} \cdots x^{d_n}} \tag{1.4.1}$$

where $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ which is not divisible by any cluster variable $x_1 \ldots x_n$. We denote $\delta(y) = (d_1, \ldots, d_n)$, and call the integer vector $\delta(y)$ *the denominator vector* of y with respect to the cluster $X = (x_1, \ldots, x_n)$. For instance, the elements of X have ***denominator vectors*** $\delta_X(x_j) = -e_j$, $\delta_X(x'_j) = e_j, j \in [1, n]$, where $e_1, ..., e_n$ are the standard basis vectors in $\mathbb{Z}^n$. One can see that the map $y \mapsto \delta(y)$ has the following valuation property $\delta_X(yz) = \delta_X(y) + \delta_X(z)$.

Let $p = (X, B)$ be a distinguished seed with Cartan matrix $A = A(B)$ and Φ be the root system associated to A , and let Q be the root lattice generated by Φ. We identify Q with $\mathbb{Z}^n$ using the basis $\Lambda = \{\alpha_1, ..., \alpha_n\}$ of simple roots in Φ. Let $\Phi_{>0}$ be the set of all positive roots associated to Λ.

Theorem 1.4.2. [16, 34].

1. *Let $p = (X, B)$ be a distinguished seed. Then the denominator vector δ_X, provides a bijection between the set of the cluster variables χ and the set*

$$\Phi_{\geq -1} = \Phi_{>0} \cup (-\Lambda) \tag{1.4.2}$$

of almost positive roots.

This parametrization also gives a bijection between the set of all cluster monomials and the root lattice $Q = \mathbb{Z}^n$.

2. *For any $\alpha = c_1\alpha_1 + \ldots + c_n\alpha_n \in \Phi_{\geq -1}$, there is a unique cluster variable $x[\alpha]$ such that*

$$x[\alpha] = \frac{P_\alpha(x_1, \ldots, x_n)}{x_1^{c_1} \cdots x_n^{c_n}}. \tag{1.4.3}$$

Where P_α is a polynomial in $x_1, ..., x_n$ with nonzero constant term; furthermore, any cluster variable is of this form.

Remark 1.4.3. Fomin-Zelevinsky positivity conjecture is equivalent to say "every cluster variable is a positive element".

Chapter 2

Cluster Automorphisms

In this chapter we introduce and study the notion of cluster automorphisms in coefficient free cluster algebras. We study them in terms of the orbits of the symmetric group action on the set of all seeds.

2.1 Cluster groups

Definition 2.1.1. Let $Aut_K(\mathcal{F})$ be the automorphism group of $\mathcal{F}$ over K. An automorphism $\varphi \in Aut_K(\mathcal{F})$ is called a cluster isomorphism of cluster algebras $\mathcal{A}_n(S)$ and $\mathcal{B}_n(S')$ over $\mathcal{F}$, if φ sends every cluster variable in $\mathcal{A}_n(S)$ to a cluster variable in $\mathcal{B}_n(S')$.

In particular, $\phi \in Aut_K(\mathcal{F})$ is called a cluster automorphism of $\mathcal{A}_n(S)$, if it leaves $\mathcal{X}_S$ invariant.

The subgroup of $Aut_K(\mathcal{F})$ of all cluster automorphisms of $\mathcal{A}_n(S)$ is called *the cluster group* of $\mathcal{A}_n(S)$ and is denoted by $C_n(S)$.

Remarks 2.1.2. 1. One can see that any cluster automorphism of a cluster algebra $\mathcal{A}_n(S)$ is an algebra automorphism of $\mathcal{A}_n(S)$ over K, i.e. $C_n(S)$ is a subgroup of $Aut_K\mathcal{A}_n(S)$, where $Aut_K\mathcal{A}_n(S)$ denotes the automorphism group of the algebra $\mathcal{A}_n(S)$ over K.

2. If $\psi : \mathcal{A}_n(S) \to \mathcal{B}_n(S')$ is a cluster isomorphism, then ψ induces a group isomorphism

between $C_n(S)$, and $C_n(S')$. To see this fact, we define the group isomorphism by $\pi : C_n(S) \to C_n(S')$, given by $\phi \mapsto \varphi$, where $\varphi(y) = \psi(\phi(\psi^{-1}(y)))$ for y a cluster variable in $\mathcal{B}_n(S')$. A routine check shows that π is a group isomorphism.

Let $\sigma \in \mathfrak{S}_n$ (a symmetric group element), and $T = (t_1, t_2, \ldots, t_n) \in \mathcal{F}$ such that $\{t_1, t_2, \ldots, t_n\}$ is a transcendence basis of $\mathcal{F}$ over K. Let σ_T be a linear automorphism of $\mathcal{F}$ over K given as follows; For $f = f(t_1, t_2, \ldots t_n) \in \mathcal{F}$

$$\sigma_T(f) := f(t_{\sigma(1)}, t_{\sigma(2)}, \ldots t_{\sigma(n)}). \tag{2.1.1}$$

Using σ_T, we will introduce an action of the symmetric group $\mathfrak{S}_n$ on the set $\mathcal{S}$ of all seeds of $\mathcal{F}$.

Definition 2.1.3. Let $X = (x_1, x_2, \ldots x_n)$ be a fixed cluster, and let $\sigma \in \Sigma_n$. For any seed $p = (Y, B) \in \mathcal{S}$, where $Y = (y_1, y_2 \ldots y_n)$, and $B = (b_{ij})$. The Laurent Phenomenon (Theorem 1.1.12) guarantees that $y_i = y_i(x_1, x_2, \ldots x_n) \in \mathbb{Z}[x_1^{\pm}, x_2^{\pm}, \ldots x_n^{\pm}]$ (i.e. y_i is a Laurent polynomial in $\{x_1, x_2, \ldots x_n\}$, for each $i \in [1, n]$). We define $\sigma_X(p)$, as follows;

$$\sigma_X(p) := (\sigma_X(Y), \sigma(B)), \tag{2.1.2}$$

where $\sigma_X(Y) = (\sigma_X(y_1), \sigma_X(y_2), \ldots, \sigma_X(y_n))$, $\sigma(B) := (b_{\sigma(i)\sigma(j)})$ and $\sigma_X(y_i)$, for $i \in [1, n]$, is as defined in (2.1.1). We write $\sigma(p)$ instead of $\sigma_X(p)$ if there is no chance of confusion.

Before stating the next theorem, we need to develop some notations.
For a seed $p = (Y, B)$, *the neighborhood of a cluster variable* y_i is defined to be the subset of $\{y_1, y_2, \cdots, y_n\}$ of the cluster variables y_j, with $b_{ij} \neq 0$, and is denoted by $N_p(y_i)$. For every integral skew-symmetric matrix, $B = (b_{ij})$, we assign a quiver Q_B. We define $Q_B = (Q_{1B}, Q_{2B}, hd, tl)$, where Q_{1B} denotes the vertices, Q_{2B} denotes the arrows, and hd and tl refer to the head and tail maps respectively. We set Q_{1B} to be the set $\{1, \ldots, n\}$ and for the arrows, there is a number of arrows equals b_{ij} from i to j if and only if $b_{ij} > 0$.

The mutation operation of the matrix B can be translated to that on the associated quiver. Let $\mu_k(Q_B)$ denote the mutation at k of Q_B. First, all the arrows incident to k in Q_B

are reversed in $\mu_k(Q_B)$. Second, for each pair (a, b) of arrows in Q_B with $hd(a) = tl(b) = k$ in Q_B, add an arrow $\bar{ba}$ with $tl(\bar{ba}) = tl(a)$ and $hd(\bar{ba}) = hd(b)$. Last step, remove a number of arrows from i to j equals the number of arrows from j to i added from the second step. In other words, remove all the two cycles between i and j.

Theorem 2.1.4. *Let $p = (X, B)$ be a simply-laced seed in $\mathcal{F}$, i.e., $b_{ij} \in \{0, -1, 1\}, \forall i, j \in [1, n]$. Then for any $\sigma \in \mathfrak{S}_n$, $\sigma_X(p)$ is mutation-equivalent to p.*

Proof. Let $i, j \in \{1, 2, \ldots, n\}$ such that $x_j \in N_p(x_i)$. We will prove that $\sigma(X, B)$ is mutation-equivalent to (X, B), for every transposition $\sigma \in \mathfrak{S}_n$, that sends every $k \in \{1, 2, \ldots, n\}$ to itself except i and j. More precisely

$$\sigma(X, B) = \mu_j\mu_i\mu_j\mu_i\mu_j((X, B)) = \mu_i\mu_j\mu_i\mu_j\mu_i((X, B)). \tag{2.1.3}$$

The statement of the theorem is a direct consequence of (2.1.3), since the symmetric group is generated by transpositions.

Sketch of proof of identities (2.1.3): Note that, each simply-laced sign skew symmetric matrix must be skew symmetric. So, it is associated to a quiver, which reduces the proof of (2.1.3) to be on (X, Q_B) instead.

In the following we provide a proof for the identity (2.1.3) in some cases as examples. The proof of all other cases follow similarly:

(i) **Seeds of A_n type**

We provide a proof for A_3-type, a general A_n-type case is quite similar,

$$
\begin{aligned}
((x_1,x_2,x_3),\cdot_1\to\cdot_2\to\cdot_3) &\overset{\mu_1}{\Rightarrow} ((\frac{x_2+1}{x_1},x_2,x_3),\cdot_1\leftarrow\cdot_2\to\cdot_3)\\
&\overset{\mu_2}{\Rightarrow} ((\frac{x_2+1}{x_1},\frac{x_2x_3+x_3+x_1}{x_1x_2},x_3),\cdot_1\to\cdot_2\leftarrow\cdot_3)\\
&\overset{\mu_1}{\Rightarrow} ((\frac{x_3+x_1}{x_2},\frac{x_2x_3+x_3+x_1}{x_1x_2},x_3),\cdot_1\leftarrow\cdot_2\leftarrow\cdot_3)\\
&\overset{\mu_2}{\Rightarrow} ((\frac{x_3+x_1}{x_2},x_1,x_3),\ \begin{array}{ccc}\cdot_1 & \longleftarrow & \cdot_3\\ \downarrow & \nearrow & \\ \cdot_2 & & \end{array})\\
&\overset{\mu_1}{\Rightarrow} ((x_2,x_1,x_3),\cdot_2\to\cdot_1\to\cdot_3)
\end{aligned}
$$

(ii) **For the exchange inside the n-cycles**

We prove it for A_3-type, a general n-cycle is quite similar:

$$
\begin{aligned}
((x_1,x_2,x_3),\ \begin{array}{ccc}\cdot_1 & \longrightarrow & \cdot_3\\ \downarrow & \nearrow & \\ \cdot_2 & & \end{array}) &\overset{\mu_1}{\Rightarrow} ((\frac{x_2x_3+1}{x_1},x_2,x_3),\ \begin{array}{ccc}\cdot_1 & \longleftarrow & \cdot_3\\ \uparrow & \nearrow & \\ \cdot_2 & & \end{array})\\
&\overset{\mu_2}{\Rightarrow} ((\frac{x_2x_3+1}{x_1},\frac{x_3(x_2x_3+1)+x_1}{x_1x_2},x_3),\ \begin{array}{ccc}\cdot_2 & \longleftarrow & \cdot_1\\ \uparrow & \nearrow & \\ \cdot_3 & & \end{array})\\
&\overset{\mu_1}{\Rightarrow} ((\frac{x_3+x_1}{x_2},\frac{x_3(x_2x_3+1)+x_1}{x_1x_2},x_3),\ \begin{array}{ccc}\cdot_2 & \overset{2}{\longleftarrow} & \cdot_3\\ \downarrow & \nearrow & \\ \cdot_1 & & \end{array})\\
&\overset{\mu_2}{\Rightarrow} ((\frac{x_3+x_1}{x_2},x_1,x_3),\ \begin{array}{ccc}\cdot_2 & \overset{2}{\longrightarrow} & \cdot_3\\ \uparrow & \swarrow & \\ \cdot_1 & & \end{array})\\
&\overset{\mu_1}{\Rightarrow} ((x_2,x_1,x_3),\ \begin{array}{ccc}\cdot_2 & \longrightarrow & \cdot_3\\ \downarrow & \nearrow & \\ \cdot_1 & & \end{array}).
\end{aligned}
$$

(remark that; the number 2 written over the arrows from 3 to 2 and from 2 to 3 in third and fourth steps respectively, refers to double arrows).

(iii) **Exchange of external vertex with adjacent one which is a vertex in an n-cycle**

We provide calculations for $n = 4$ case.

$$((x_1, x_2, x_3, x_4),\ \cdot_1 \longrightarrow \cdot_2\ ) \quad \overset{\mu_1}{\Rightarrow} \quad ((\frac{x_2+1}{x_1}, x_2, x_3, x_4),\ \cdot_1 \longleftarrow \cdot_2\ )$$

$$\overset{\mu_2}{\Rightarrow} \quad ((\frac{x_2+1}{x_1}, \frac{(x_2+1)x_3x_4+x_1}{x_1x_2}, x_3, x_4),\ \cdot_1 \longrightarrow \cdot_2\ )$$

$$\overset{\mu_1}{\Rightarrow} \quad ((\frac{x_2x_4+x_1}{x_2}, \frac{(x_2+1)x_3x_4+x_1}{x_1x_2}, x_3, x_4),\ \cdot_1 \longleftarrow \cdot_2\ )$$

$$\overset{\mu_2}{\Rightarrow} \quad ((\frac{x_2x_4+x_1}{x_2}, x_1, x_3, x_4),\ \cdot_1 \longrightarrow \cdot_2\ )$$

$$\overset{\mu_1}{\Rightarrow} \quad ((x_2, x_1, x_3, x_4),\ \cdot_2 \longrightarrow \cdot_1\ ).$$

Connected cycles and different quivers shapes are similar. □

For non simply-laced type seeds, the above result is not necessarily true, we provide the following counter example.

Example 2.1.5. Consider the seed (X, B), where

$$B = (b_{ij}) = \begin{pmatrix} 0 & +2 & 0 \\ -2 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix}.$$

In the following, we show that there is no sequence of mutations $\mu_{i_1}\mu_{i_2}\dots\mu_{i_k}$, such that $\sigma_{12}(B) = \mu_{i_1}\mu_{i_2}\dots\mu_{i_k}(B)$, where

$$\sigma_{12}(B) = \begin{pmatrix} 0 & -2 & +1 \\ +2 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

If we could show that, there is no sequence of mutations that sends the entry b_{23} to zero, we will be done. We do this by showing that every sequence of mutations sends b_{23} to an odd number. First we show by induction on the length of the sequence of mutations that, any sequence of mutations sends b_{13} and b_{12} to even numbers.

For a single element sequences: one can see that, only μ_2 and μ_3 may change b_{13} and b_{12} respectively: that is μ_2 and μ_3 send b_{13} and b_{12} to 2 respectively.

Now, assume that every sequence of mutations of length k sends b_{13} and b_{12} to an even number, and let $\mu_{i_{k+1}}\mu_{i_k}\dots\mu_{i_1}$ be a sequence of length $k+1$. So, if

$$\mu_{i_k}\dots\mu_{i_1}((b_{ij})) = (b'_{ij}), \tag{2.1.4}$$

then $b'_{23} = 2d$ for some integer number d. Now we have

$$\begin{aligned} \mu_{i_{k+1}}(b'_{13}) &= b'_{13} + \frac{b'_{12}|b'_{23}| + b'_{23}|b'_{12}|}{2} \\ &= b'_{13} + d|b'_{23}| + |d|b'_{23} \\ &= b'_{13} + d \begin{cases} \pm 2b_{23}, & \text{if } b_{23}d > 0, \\ 0, & \text{if } b_{23}d < 0. \end{cases} \end{aligned}$$

Since b'_{13} is an even number then $\mu_{i_{k+1}}(b'_{13})$ must be an even too. This shows that any sequence of mutations will send b_{13} to an even number. In a similar way one can show that any sequence of mutation sends b_{12} to an even number.

Secondly, we show that every sequence of mutations sends $|b_{23}|$ to an odd number. We show this by induction on the number of occurrences of μ_1 in the sequence. Note that any sequence not containing μ_1 will not change $|b_{23}|$.

Sequences contains only one copy of μ_1: Without loss of generality, let $\mu_{i_1}\mu_{i_2}\dots\mu_{i_k}$ be a sequence of mutations such that $\mu_{i_k} = \mu_1$, and $\mu_{i_j} \neq \mu_1, \quad \forall j \in [1,k]$. This becomes clear by considering that the possible change in $|b_{23}|$ appears only after applying μ_1, and there is no change in $|b_{23}|$ due to μ_2 or μ_3. Then using same notation as in (2.1.4), we have

$$b'_{23} = \pm 1 + \frac{b'_{21}|b'_{13}| + |b'_{21}|b'_{13}}{2}. \tag{2.1.5}$$

However b'_{21} and b'_{13} are both even numbers, so $\frac{b'_{21}|b'_{13}|+|b'_{21}|b'_{13}}{2}$ must be even, and b'_{23} is an odd number.

Sequences contains more than one copy of μ_1: Assume that any sequence of mutations, with μ_1 repeated $k-$ times sends b_{23} to an odd number.

Let $\mu_{i_t}, \mu_{i_2}\dots\mu_{i_1}$ be a sequence of mutations containing μ_1, $k+1-$times, then we can assume that $\mu_{i_t} = \mu_1$. Let

$$\mu_{i_t}\dots\mu_{i_1}((b_{ij})) = (b''_{ij}), \quad \text{and} \quad \mu_{i_{t-1}}\dots\mu_{i_1}((b_{ij})) = (b'_{ij}), \tag{2.1.6}$$

then one can see that b'_{23} is an odd number and b'_{12} and b'_{13} are both even numbers. Then,

$$b''_{23} = b'_{23} + \frac{b'_{21}|b'_{13}| + b'_{13}|b'_{21}|}{2} \tag{2.1.7}$$

is a sum of an odd and even numbers, b''_{23} is an odd number. □

Definition 2.1.3 gives rise to an equivalence relation on $\mathcal{S}$, as we will see in the following definition.

Definition 2.1.6. Let $B = (b_{ij})$ and $B' = (b'_{ij})$ be any two sign skew symmetric integral matrices, and σ be an element of $\mathfrak{S}_n$. Then we say that B and B' are σ-similar if $b'_{ij} = b_{\sigma(i),\sigma(j)}$, where $\epsilon \in \{-1,+1\}$.

Now we define an equivalence relation $\sim$ on $\mathcal{S}$. Let $p = (X, B)$ and $p' = (Y, B')$ be two seeds. Then

$$p \sim p' \text{ if and only if } B \text{ and } B' \text{ are } \sigma - \text{similar for some permutation } \sigma. \tag{2.1.8}$$

This yields an equivalence relation on $\mathcal{S}$ with the equivalence class of p denoted by $[p]$ and the equivalence class of B denoted by $\langle B \rangle$. (Note, $\sim$ defines an equivalence relation on the set of all sign skew-symmetric integral $n \times n$ matrices, for all $n \in \mathbb{Z}_{\geq 0}$)

Lemma 2.1.7. *Let $p = (X, B), p' = (Y, B') \in S$ and $\sigma \in \mathfrak{S}_n$. Let $T_{pp',\sigma} \in Aut._K(F)$ be induced by $x_i \mapsto y_{\sigma(i)}$. Then,*

$$B \text{ and } B' \text{ are } \sigma - \text{similar if and only if } T_{pp',\sigma}(\mu_i(x_i)) = \mu_{\sigma(i)}(y_{\sigma(i)}), \quad \forall i \in [1, n]. \tag{2.1.9}$$

In particular, $p \sim p'$ if and only if for some permutation σ, $T_{pp',\sigma}$ sends $\mu_i(X)$ to $\mu_{\sigma(i)}(Y)$.

Proof. $\Rightarrow$) Assume that B and B' are two σ- similar seeds. Then $B' = \epsilon(\sigma(B))$. So, if $B = (b_{ij})$ and $B' = (b'_{ij})$, then $b'_{ij} = \epsilon b_{\sigma(i)\sigma(j)}$, where $\epsilon \in \{+1, -1\}$. Then we have

$$\begin{aligned}
T_{pp',\sigma}(\mu_i(x_i)) &= T_{pp',\sigma}\left(\frac{\prod_{b_{ji}>0} x_j^{b_{ji}} + \prod_{b_{ji}<0} x_j^{-b_{ji}}}{x_i}\right) \\
&= \frac{\prod_{b_{ji}>0} y_{\sigma(j)}^{b_{ji}} + \prod_{b_{ji}<0} y_{\sigma(j)}^{-b_{ji}}}{y_{\sigma(i)}} \\
&= \frac{\prod_{b_{ji}>0} y_{\sigma(j)}^{\epsilon b'_{\sigma(j)\sigma(i)}} + \prod_{b_{ji}<0} y_{\sigma(j)}^{-\epsilon b'_{\sigma(j)\sigma(i)}}}{y_{\sigma(i)}} \\
&= \begin{cases} \frac{\prod_{b'_{\sigma(j)\sigma(i)}>0} y_{\sigma(j)}^{b'_{\sigma(j)\sigma(i)}} + \prod_{b'_{\sigma(j)\sigma(i)}<0} y_{\sigma(j)}^{-b'_{\sigma(j)\sigma(i)}}}{y_{\sigma(i)}}, & \text{if } \epsilon = 1, \\ \frac{\prod_{b'_{\sigma(j)\sigma(i)}<0} y_{\sigma(j)}^{-b'_{\sigma(j)\sigma(i)}} + \prod_{b'_{\sigma(j)\sigma(i)}>0} y_{\sigma(i)}^{b'_{\sigma(j)\sigma(i)}}}{y_{\sigma(i)}}, & \text{if } \epsilon = -1. \end{cases} \\
&= \mu_{\sigma(i)}(y_{\sigma(i)})
\end{aligned}$$

$\Leftarrow$) Suppose that p and p' are not σ-similar, then $B' \neq \pm\sigma(B)$, i.e. $(b'_{ij}) \neq \pm(b_{\sigma(i)\sigma(j)})$. Then there is $i1, n]$ such that $b'_{ij_0} \neq \pm b_{\sigma(i)\sigma(j_0)}$, for some $j_0 \in [1, n]$. Now, we have

$$
\begin{aligned}
T_{pp',\sigma}(\mu_i(x_i)) &= T_{pp',\sigma}\left(\frac{\prod_{b_{ij}>0} x_j^{b_{ij}} + \prod_{b_{ik}<0} x_k^{-b_{ik}}}{x_i}\right) \\
&= T_{pp',\sigma}\left(\frac{\prod_{b_{ij}>0, j\neq j_0} x_j^{b_{ij}} \cdot x_{j_0}^{b_{ij_0}} + \prod_{b_{ik}<0} x_k^{-b_{ik}}}{x_i}\right) \\
&= \frac{\prod_{b_{ij}>0, j\neq j_0} y_{\sigma(j)}^{b_{ij}} \cdot y_{\sigma(j_0)}^{b_{ij_0}} + \prod_{b_{ik}<0} y_{\sigma(k)}^{-b_{ik}}}{y_{\sigma(i)}} \\
&= \begin{cases} \frac{\prod_{b_{ij}>0, j\neq j_0} y_{\sigma(j)}^{b_{ij}} \cdot y_{\sigma(j_0)}^{b_{ij_0}} + \prod_{b_{ik}<0} y_{\sigma(k)}^{-b_{ik}}}{y_{\sigma(i)}}, & \text{if } b_{ij_0} > 0, \\ \frac{\prod_{b_{ij}>0,} y_{\sigma(j)}^{b_{ij}} + \prod_{b_{ik}<0, j\neq j_0} y_{\sigma(k)}^{-b_{ik}} \cdot y_{\sigma(j_0)}^{b_{ij_0}}}{y_{\sigma(i)}}, & \text{if } b_{ij_0} < 0 \end{cases} \\
&\neq \mu_{\sigma(i)}(y_{\sigma(i)}).
\end{aligned}
$$

Last line is due to the fact that, $y_{\sigma(j_0)}^{b_{ij_0}}$ appears in $\mu_{\sigma(i)}(y_{\sigma(i)})$ with a different exponent. The last part of the statement is straightforward. □

Theorem 2.1.8. *Let $p = (X, B)$ and $p' = (Y, B')$ be any two σ-similar seeds. Then for any sequence of mutations $\mu_{i_k}, \mu_{i_{k-1}}, \ldots, \mu_{i_1}$, the following statements are true:*

(1) *$\mu_{i_k}\mu_{i_{k-1}} \ldots \mu_{i_1}(X, B)$ and $\mu_{\sigma(i_k)}\mu_{\sigma(i_{k-1})} \ldots \mu_{\sigma(i_1)}(Y, B')$ are σ- similar,*

(2) *$T_{pp',\sigma}(\mu_{i_k}\mu_{i_{k-1}} \ldots \mu_{i_1}(X)) = \mu_{\sigma(i_k)}\mu_{\sigma(i_{k-1})} \ldots \mu_{\sigma(i_1)}(Y)$, where $T_{pp',\sigma}$ is as defined in Lemma 2.1.7.*

Proof. Part (1) follows as a simple corollary of the identity

$$
\sigma(\mu_k(B)) = \mu_{\sigma(k)}(\sigma(B)), \quad \forall k \in [1, n]. \tag{2.1.10}
$$

One can see that; the $(\sigma(i), \sigma(j))$ entry of $\sigma(\mu_k(B))$ is b'_{ij}. Now, the entry $(\sigma(i), \sigma(j))$ of $\sigma(B)$ is b_{ij}. So, applying mutation in the direction $\sigma(k)$ on $\sigma(B)$ will result that the entry $(\sigma(i), \sigma(j))$ will change to

$$
\begin{cases} -b_{ij}, & \sigma(k) \in \{\sigma(i), \sigma(j)\} \\ b_{\sigma(i)\sigma(j)} + \frac{b_{\sigma(i)\sigma(k)}|b_{\sigma(k)\sigma(j)}| + |b_{\sigma(i)\sigma(k)}|b_{\sigma(k)\sigma(j)}}{2}, & \text{otherwise.} \end{cases}
$$

Noticing that; $k \in \{i, j\}$ if and only if $\sigma(k) \in \{\sigma(i), \sigma(j)\}$ Therefore the entries of $\sigma(\mu_k(B))$ coincides with the entries of $\mu_{\sigma(k)}(\sigma(B))$.

For the second part; let $\mu_{i_k}\mu_{i_{k-1}}\dots\mu_{i_1}$ be a sequence of mutations, $p_{i_1 i_k} = \mu_{i_k}\mu_{i_{k-1}}\dots\mu_{i_1}(p)$, and $p'_{\sigma(i_1)\sigma(i_k)} = \mu_{\sigma(i_k)}\mu_{\sigma(i_{k-1})}\dots\mu_{\sigma(i_1)}(p')$, for $j \in [1, k]$. Part (1) tells us that $p_{i_1 i_{k-1}}$ and $p'_{\sigma(i_1)\sigma(i_{k-1})}$ are σ-similar. Then Lemma 2.1.7 implies that

$$T_{p_{i_1 i_k} p'_{\sigma(i_1)\sigma(i_k)},\sigma}(\mu_{i_k}(\mu_{i_{k-1}}\dots\mu_{i_1}(X))) = \mu_{i_{\sigma(k)}}(\mu_{i_{\sigma(k-1)}}\dots\mu_{\sigma(i_1)}(Y)). \tag{2.1.11}$$

So, it remains to show that

$$T_{p_{i_1 i_k} p'_{\sigma(i_1)\sigma(i_k)},\sigma}(\mu_{i_k}(\mu_{i_{k-1}}\dots\mu_{i_1}(X))) = T_{pp',\sigma}(\mu_{i_k}(\mu_{i_{k-1}}\dots\mu_{i_1}(X)). \tag{2.1.12}$$

To get to this, let $q = (Z, D)$, and $q' = (T, C)$ be any two σ-similar seeds, and let $q_1 = \mu_i(Z, D) = (Z', D')$, $q'_1 = \mu_{\sigma(i)}(T, C) = (T', C')$. Where $Z = (z_1, z_2, \dots, z_n)$, and $T = (t_1, t_2, \dots, t_n)$. Next we show that

$$T_{q_1 q'_1,\sigma}(\mu_k\mu_i(Z)) = T_{qq',\sigma}(\mu_k\mu_i(Z)). \tag{2.1.13}$$

Let z_j be a cluster variable in Z, then for $j \neq i$. Then, both of $T_{q_1 q'_1,\sigma}$, and $T_{qq',\sigma}$ leave z_j unchanged. Now, let $j = i$, and we have

$$\begin{aligned} T_{q_1 q'_1,\sigma}(\mu_i(z_i)) &= T_{q_1 q'_1,\sigma}\left(\frac{\prod_{d_{ij}>0} z_j^{d_{ij}} + \prod_{d_{ik}<0} z_k^{-d_{ik}}}{z_i}\right) \\ &= \frac{\prod_{d_{ij}>0} t_{\sigma(j)}^{d_{ij}} + \prod_{d_{ik}<0} t_{\sigma(k)}^{-d_{ik}}}{T_{p_1 p'_1,\sigma}(z_i)}. \end{aligned}$$

However,

$$\begin{aligned} T_{q_1q_1',\sigma}(\mu_i(z_i)) &= \mu_{\sigma(i)}(t_{\sigma(i)}) \\ &= \frac{\prod_{c_{ij}>0} t_{\sigma(j)}^{c_{ij}} + \prod_{c_{ik}<0} t_{\sigma(k)}^{-c_{ik}}}{t_{\sigma(i)}} \\ &= \frac{\prod_{d_{ij}>0} t_{\sigma(j)}^{ed_{ij}} + \prod_{d_{ik}<0} t_{\sigma(k)}^{-ed_{ik}}}{t_{\sigma(i)}}. \end{aligned}$$

Hence, $T_{p_1p_1',\sigma}(z_i) = t_{\sigma(i)}$. This shows that $T_{q_1q_1',\sigma}$, and $T_{qq',\sigma}$ have the same action on every cluster variable in Z, and since cluster variables from the $\mu_k\mu_i(Z)$ are integral laurent polynomials of cluster variables from Z, this gives (2.1.14).

For equation (2.1.13) we use induction on the length of the mutation sequence. Assume that equation (2.1.13) is true for any sequence of mutation of length less than or equal $k-1$. Now we have

$$\begin{aligned} T_{p_{i_1i_k}p'_{i_1i_k},\sigma}(\mu_{i_k}\mu_{i_{k-1}}\dots\mu_{i_1}(X)) &= T_{p_{i_1i_{k-2}}p'_{i_1i_{k-2}},\sigma}(\mu_{i_k}\mu_{i_{k-1}}\dots\mu_{i_1}(X)) \\ &= T_{pp',\sigma}(\mu_{i_k}\mu_{i_{k-1}}\dots\mu_{i_1}(X)), \end{aligned}$$

where the first equality is by identity (2.1.14), and the second is by the induction hypotheses.

□

Theorem 2.1.9. *Let $\mathcal{A}_n(S)$ be a cluster algebra, and (X, B) be a self σ-similar seed in S for some $\sigma \in \Sigma_n$. Then, σ_X is a cluster automorphism.*

Proof. Let $y \in \mathcal{X}_S$. Then there exists a sequence of mutations $\mu_{i_1}, \dots, \mu_{i_k}$ such that for some cluster variable x_i in X, we have $y = \mu_{i_1}\mu_{i_2}\dots\mu_{i_k}(x_i)$. So for some sequence of mutations $\mu_{i_1}\mu_{i_2}\dots\mu_{i_k}$, we apply this sequence of mutation to X.

We are left to show,

$$\sigma_X(y) = \mu_{\sigma(i_1)}\mu_{\sigma(i_2)}\cdots\mu_{\sigma(i_k)}(x_{\sigma(i)}). \tag{2.1.14}$$

From Theorem 2.1.7 (1), $\mu_{i_2}\mu_{i_3}\ldots\mu_{i_t}((X,B))$ and $\mu_{\sigma(i_2)}\mu_{\sigma(i_3)}\cdots\mu_{\sigma(i_t)}((X,B))$, are σ-similar $\forall t \in [1,k]$. Theorem 2.1.7 (2) implies that equation (2.1.15) is correct.

□

Example 2.1.10. Let $\mathcal{A}_n(S)$ be a cluster algebra of A_n-type. Then, $C_n(S)$ is a non trivial group.

To see that, fix an initial seed (X,B) such that $B=(b_{ij})$ where either $b_{ij}\geq 0, \forall i>j$ or $b_{ij}\leq 0, \forall i>j$, such seed always exists in any cluster algebra of A_n- type . The following permutation is a cluster automorphism with symmetric group action defined, with respect to the initial cluster X;

$$\tau_X = \begin{cases} (1 \quad 2k+1)_X(2 \quad 2k)_X\ldots(k \quad k+2)_X, & \text{if } n=2k+1, \\ (1 \quad 2k)_X(2 \quad 2k-1)_X\ldots(k \quad k+1)_X, & \text{if } n=2k. \end{cases} \tag{2.1.15}$$

Now one can see that (X,B) is self σ-similar.

2.2 Exchange groups

Every path in the cluster pattern defines a field automorphism, which we codify in the following definition. In this section, we study the intersection of the group generated by all such automorphisms and the cluster group.

Definitions 2.2.1. Let $p=(X,B)$ and $p'=(Y,B')$ be any two vertices in the cluster patern $\mathbb{T}_n(S)$ of $\mathcal{A}_n(S)$. For any $\sigma\in\mathfrak{S}_n$, the field automorphism $T_{pp',\sigma}:\mathcal{F}\to\mathcal{F}$ induced by $x_i\mapsto y_{\sigma(i)}$ is called an exchange automorphism.

The subgroup of $Aut._K(\mathcal{F})$ generated by the set of all exchange automorphisms is called the exchange group of $\mathcal{A}_n(S)$ and is denoted by $\widetilde{\mathbf{m}}_n(S)$.

Remark 2.2.2. Let $\mathcal{A}_n(S)$ be a simply laced cluster algebra, and fix an initial seed $p = (X, B)$. Then, every symmetric group element can be seen as a field automorphism, (as in the paragraph proceeding definition (2.1.3)), taking $T = X$. From Theorem 2.1.4, every symmetric group element (in the above sense) corresponds to a path in the cluster patern of $\mathcal{A}_n(S)$. So, the symmetric group elements can be seen as exchange automorphisms.

Before we state the main results of this section we sharpen the notations of the neighbors and monomials of the cluster variables. Let x_{i_0} be a cluster variable in $p = (X, B)$. The set of neighbors of x_{i_0} at the seed p, denoted by $N_p(x_{i_0})$ is defined to be, $N_p(x_{i_0}) := N_{p,+}(x_{i_0}) \cup N_{p,-}(x_{i_0})$, where $N_{p,+}(x_{i_0}) = \{x_i; b_{i_0 i} > 0\}$ and $N_{p,-}(x_{i_0}) = \{x_i; b_{i_0 i} < 0\}$. Define $m_{p,+}(x_{i_0})$, and $m_{p,-}(x_{i_0})$ respectively. Where, $m_{p,+}(x_{i_0}) = \prod_{x_i \in N_{p,+}(x_{i_0})} x_i^{b_{ii_0}}$, and $m_{p,-}(x_{i_0}) = \prod_{x_i \in N_{p,-}(x_{i_0})} x_i^{-b_{ii_0}}$ and call them the positive and negative monomials of the cluster variable x_{i_0}. We denote $f_{p,x_{i_0}} = m_{p,+}(x_{i_0}) + m_{p,-}(x_{i_0})$, one can see that $f_{p,x_{i_0}}$ is not divisible by $x_i, \forall i \in [1, n]$.

The following theorem provides a description for $C_n(S)$, through $\widetilde{\mathbf{m}}_n(S)$ and the equivalent classes of $\sim$. In the proof of the theorem, we assume that the positivity conjecture, Conjecture 1.1.13, is satisfied. However, a proof without the positivity conjecture, can be written in finite type cluster algebras of rank two.

Theorem 2.2.3. *Assume that $\mathcal{A}_n(S)$ satisfies the positivity conjecture. Let $p = (X, B)$ and $p' = (Y, B')$ be any two vertices in the cluster pattern of $\mathcal{A}_n(S)$. Then, for a fixed symmetric group element σ, the following are equivalent*

1. *$T_{pp',\sigma}$ is a cluster automorphism,*
2. *p and p' are σ- similar,*

3. $T_{pp',\sigma}$ permutes the clusters. Furthermore for any seed $q = (Z, D)$, we have;

$$T_{pp',\sigma}(Z) \in \{Y; \quad (Y, M) \in [q]\}. \tag{2.2.1}$$

Proof. $\Rightarrow$(2). Assume that $T_{pp',\sigma}$ is a cluster automorphism. From Lemma 2.1.7, to show p and p' are σ-similar, it is enough to show that, $T_{pp',\sigma}(\mu_i(x_i)) = \mu_{\sigma(i)}(y_{\sigma(i)}), \forall i \in [1, n]$.

Let $z = T_{pp',\sigma}(\mu_i(x_i))$, and $\xi = \mu_{\sigma(i)}(y_{\sigma(i)})$ where $X = (x_1, x_2, \ldots, x_n)$, and $Y = (y_1, y_2, \ldots, y_n)$. Then, we have

$$z = \frac{T_{pp',\sigma}(f_{p,x})}{y_{\sigma(i)}}, \qquad \text{and} \qquad \xi = \frac{f_{p',y_{\sigma(i)}}}{y_{\sigma(i)}}. \tag{2.2.2}$$

Both $T_{pp',\sigma}(f_{p,x})$ and $f_{p',y_{\sigma(i)}}$ are polynomials in $\mathbb{Z}[y_{\sigma(1)}, \cdots, y_{\sigma(i-1)}, y_{\sigma(i+1)}, \ldots, y_{\sigma(n)}]$, and are not divisible by $y_{\sigma(j)}$, for all j in $[1, n]$. Now, suppose that z is a cluster variable. Then, by Laurent phenomenon, z can be written uniquely as;

$$z = \frac{P(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(i-1)}, \xi, y_{\sigma(i+1)}, \ldots, y_{\sigma(n)})}{y_{\sigma(1)}^{\alpha_1} \cdots y_{\sigma(i-1)}^{\alpha_{i-1}} \xi^{\alpha_i} y_{\sigma(i+1)}^{\alpha_{i+1}} \cdots y_{\sigma(n)}^{\alpha_n}}, \tag{2.2.3}$$

where $P(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(i-1)}, \xi, y_{\sigma(i+1)}, \ldots, y_{\sigma(n)})$ is a polynomial with integers coefficients. Which is not divisible by any of the following $y_{\sigma(1)}$,$y_{\sigma(2)}$, …, $y_{\sigma(i-1)}$, ξ,$y_{\sigma(i+1)}, \ldots, y_{\sigma(n-1)}$ and $y_{\sigma(n)}$, and $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$. Comparing z from (2.2.2) and (2.2.3), we have

$$T_{pp',\sigma}(f_{p,x}) \ldots y_{\sigma(1)}^{\alpha_1} \cdots y_{\sigma(i-1)}^{\alpha_{i-1}} \cdots \xi^{\alpha_i} \cdots y_{\sigma(i+1)}^{\alpha_{i+1}} \cdots y_{\sigma(n)}^{\alpha_n} = P \cdot y_{\sigma(i)}. \tag{2.2.4}$$

Since, $f_{p,x}$ is not divisible by any cluster variable x_i, for any $i \in [1, n]$. Then $T_{pp',\sigma}(f_{p,x})$ is not divisible by $y_i, \forall i \in [1, n]$. More precisely $T_{pp',\sigma}(f_{p,x})$ is a sum of two monomials in variables from the cluster Y, with positive exponents. Therefore, $\alpha_j = 0$ for all $j \in [1, n] - \{i\}$, and $i = -1$. Hence, (2.2.4) can be simplified as

$$T_{pp',\sigma}(f_{p,x}) = P \cdot f_{p',y_{\sigma(i)}}. \tag{2.2.5}$$

Now we have that $f_{p',y_{\sigma(i)}}$ is also a sum of two monomials in variables from the cluster Y, with positive exponents. However, P is a polynomial with positive integers coefficients, and not divisible by any cluster variable from $Y' = \mu_{\sigma(i)}(Y)$. Then it must be either a sum of at least two monomials in variables from Y', or it is a positive integer. Equation (2.2.5) says that, the first option for P is impossible because $T_{pp',\sigma}(f_{p,x})$ and $f_{p',y_{\sigma(i)}}$ sums of exactly two monomials, so P must be an integer. Because the coefficients of $T_{pp',\sigma}(f_{p,x})$ and $f_{p',y_{\sigma(i)}}$ are all ones, which must be exactly 1.

Hence,

$$T_{pp',\sigma}(f_{p,x}) = f_{p',y_{\sigma(i)}}. \tag{2.2.6}$$

Therefore,

$$T_{pp',\sigma}(\mu_i(x_i)) = \mu_{\sigma(i)}(y_{\sigma(i)}), \forall i \in [1,n]. \tag{2.2.7}$$

Now, lemma 2.1.6 implies p and p' are σ-similar.

(2)$\Rightarrow$ (3). Let $Z = (z_1, z_2, \ldots, z_n)$ be the cluster of the seed (Z, D). Then there is a sequence of mutations $\mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}$ such that

$$(Z, D) = \mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}(X, B)$$

Since (X, B), and (Y, B') are σ-similar then, Theorem 2.1.7 part 1 implies that $\mu_{i_2}\ldots\mu_{i_k}(X, B)$ and $\mu_{\sigma(i_2)}\ldots\mu_{\sigma(i_k)}(Y, B')$ are σ-similar too.

But, from Theorem 2.1.7 part 2, we have,

$$T_{pp',\sigma}(\mu_{i_1}(\mu_{i_2}\ldots\mu_{i_k}(X))) = \mu_{\sigma(i_1)}\mu_{\sigma(i_2)}\ldots\mu_{\sigma(i_k)}(Y),$$

and since the right hand side is a cluster and the left hand side is only $T_{pp',\sigma}(Z)$, then $T_{pp',\sigma}$ sends Z to a cluster. So, $T_{pp',\sigma}(Z)$ permutes the clusters. For the belonging (3.1), is imme-

diate from the above argument.

(3)⇒(1) Permuting the clusters implies leaving χ_S invariant, because every cluster variable is contained in some cluster. □

The following is a corollary of the proof of Theorem 2.2.3, and is actually the generalization of the statement of the same theorem, to the level of cluster isomorphism.

Corollary 2.2.4. *Let $\mathcal{A}_n(S)$, and $\mathcal{A}_n(S')$ be any two cluster algebras over $\mathcal{F}$. If $p = (X, B) \in S$, $p' = (Y, B') \in S'$, and $\sigma \in \mathfrak{S}_n$. Then the following are equivalent*

1. *the field automorphism $\phi_{pp',\sigma} : F \to F$, given by $x_i \mapsto y_{\sigma(i)}$ is a cluster isomorphism from $\mathcal{A}_n(S)$ onto $\mathcal{A}_n(S')$,*
2. *p and p' are σ-similar,*
3. *$\phi_{pp',\sigma}$ sends every cluster in S_C onto a cluster in S'_C. In particular, two cluster algebras are cluster isomorphic if and only if they contain two σ-similar seeds for some permutation σ.*

Proof. Follow the proof of Theorem 2.2.3, mutatis mutandis. □

Corollary 2.2.5. *If $\mathcal{A}_n(S)$ is a cluster algebra of simply-laced type then $C_n(S) \neq 1$.*

Proof. This follow from Theorem 2.1.4 and Theorem 2.2.3. □

We remark that, the converse of Theorem 2.2.5 is not necessarily true. Consider the the cluster algebra given in Example 2.1.5, and let σ be the transpose (23). A routine check shows that, (23) corresponds to the sequence of mutations $\mu_2\mu_3\mu_2\mu_3\mu_2$. Then Theorem 2.1.4 implies that the transposition (23) is a cluster automorphism, while the cluster algebra is not simply-laced.

Conjecture 2.2.6. The set of all cluster variables χ_S can be complectly determined by $[p]$ as follows

$$\chi_S = \bigcup \{Y; \quad (Y, M) \in [p]\}, \tag{2.2.8}$$

In the following we calculate the cluster and exchange groups for some cluster algebras of low ranks.

Example 2.2.7 (Cluster and exchange groups of rank 1). In this case $\mathcal{F} = K(t)$, and $\mathcal{A}_1(S) = \mathbb{Z}[x^{\pm 1}]$. So the cluster group and the exchange groups are the same and they equal to the subgroup of $Aut._K(\mathcal{F})$ generated by the automorphism $T_1 : K(x) \to K(x)$ induced by $x \mapsto \frac{1}{x}$ and then

$$\widetilde{m}(S)) = C_1(S) \cong \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle < PGL_2(K).$$

Example 2.2.8 (Cluster groups of rank 2, $C_2(S)$). In this case $\mathcal{F} = K(x_1, x_2)$, and applying the mutations on the initial seed

$$p = \{(x_1, x_2), \begin{pmatrix} 0 & m \\ -n & 0 \end{pmatrix}\}$$

leads to the following recursive relation for the cluster variables of $\mathcal{A}(S)(m, n)$

$$x_{t-1}x_{t+1} = \begin{cases} x_t^m + 1, & \text{if t is odd ,} \\ x_t^n + 1, & \text{otherwise.} \end{cases} \tag{2.2.9}$$

Thus, the cluster algebra $\mathcal{A}(S)(m, n)$ corresponding to p is the subalgebra of $\mathcal{F} = K(x_1, x_2)$ generated by $\{x_t; t \in \mathbb{Z}\}$, however, since p is acyclic seed then $\mathcal{A}(S)(m.n) = \mathbb{Z}[x_0, x_1, x_2, x_3] \subset K(x_1, x_2)$.

Theorem 2.2.9 ([34]). *The sequence (2.2.9) of the cluster variables $\{x_t\}_{t\in\mathbb{Z}}$ in $\mathcal{A}(S)(m.n)$ is periodic if and only if $mn \leq 3$, and is of period 5 (resp., 6, 8) if $mn = 1$ (resp., 2, 3).*

In the following, let $C_2(m,n)$ denote the cluster group associated to $\mathcal{A}(S)(m,n)$.

Lemma 2.2.10. *There is a cluster isomorphism between the cluster algebra $\mathcal{A}_2(S)(m,n)$ and $\mathcal{A}_2(S)(n,m)$.*

Proof. Let (x_1,x_2) and (y_1,y_2) be initial clusters for $\mathcal{A}_2(S)(m,n)$ and $\mathcal{A}_2(S)(n,m)$ respectively. Consider the following cluster isomorphism

$$\sigma_{12} : \mathcal{A}_2(S)(m,n) \to \mathcal{A}_2(S)(n,m)$$

given by:

$$x_1 \mapsto y_2 \quad \text{and} \quad x_2 \mapsto y_1,$$

one can see that this automorphism induces a one to one correspondence between the sets of all clusters of $\mathcal{A}_2(S)(m,n)$ and $\mathcal{A}_2(S)(n,m)$.

□

Corollary 2.2.11. $C_2(m,n) \cong C_2(n,m)$.

Proof. From the previous lemma, and Remarks 2.2.2(2). □

Example 2.2.12 (The cluster and exchange groups of $\mathcal{A}(\mathcal{S})(1,1)$). In this case, we have exactly 5 cluster variables which, in terms of the initial cluster variables (x_1,x_2) are

$$\{x_1, x_2, \frac{x_1+1}{x_2}, \frac{x_2+1}{x_1}, \frac{1+x_1+x_2}{x_1x_2}\},$$

and the following unordered pairs as clusters

$$(x_1,x_2), (x_1, \frac{x_1+1}{x_2}), (\frac{x_2+1}{x_1}, x_2), (\frac{x_2+1}{x_1}, \frac{1+x_1+x_2}{x_1x_2}), (\frac{1+x_1+x_2}{x_1x_2}, \frac{x_1+1}{x_2}).$$

So, $C_2(1,1)$ is the subgroup of $Aut._K\mathcal{F}(x_1,x_2)$ generated by the following involuting automorphisms T_1, and T_2 where, T_1 is induced by

$$x_1 \mapsto \frac{x_2+1}{x_1}, \quad \text{and} \quad x_2 \mapsto x_2, \tag{2.2.10}$$

and T_2

$$x_1 \mapsto x_1 \quad \text{and} \quad x_2 \mapsto \frac{x_1+1}{x_2}, \tag{2.2.11}$$

To show that these are generators. Consider the automorphism η induced by, $x_1 \mapsto \frac{x_1+1}{x_2}$ and $x_2 \mapsto x_2$ then we have

$$\eta((x_1, \frac{x_1+1}{x_2})) = (\eta(x_1), \eta(\frac{x_1+1}{x_2})) = (\frac{x_1+1}{x_2}, \frac{x_1+x_2+1}{x_2^2})$$

but $\frac{x_1+x_2+1}{x_2^2}$ is not a cluster variable, so the automorphism η cannot be a cluster automorphism. In a complete similar way we can argue all other possible choices. Therefore,

$$C_2(1,1) = \langle T_1, T_2 \rangle < Aut._K K(x_1, x_2).$$

Also, we can see that

$$C_2(1,1) = \widetilde{m}_2(S), \quad \forall \text{ seed } p \in S, \text{ and } \forall \ d \in [1,5].$$

Remark 2.2.13.

$$C_2(1,1) = \{T_1, T_2 | T_1^2 = T_2^2 = 1, (T_1T_2)^{10} = 1\}.$$

Example 2.2.14 (**The cluster group** $C_2(2,1)$**).** .

We have exactly 6 different cluster variables, which are

$$\{x_1, x_2, \frac{x_2+1}{x_1}, \frac{(x_2+1)^2+x_1^2}{x_1^2x_2}, \frac{x_1^2+x_2+1}{x_1x_2}, \frac{x_1^2+1}{x_1}\} \tag{2.2.12}$$

and the following unordered pairs as the set of clusters

$(x_1, x_2), (\frac{x_2+1}{x_1}, x_2), (\frac{x_2+1}{x_1}, \frac{(x_2+1)^2+x_1^2}{x_1^2x_2}), (\frac{x_1^2+x_2+1}{x_1x_2}, \frac{(x_2+1)^2+x_1^2}{x_1^2x_2}), (\frac{x_1^2+x_2+1}{x_1x_2}, \frac{x_1^2+1}{x_1})$.

Notice that, the cluster variables x_1 and x_2 are not symmetrical as in $\mathcal{A}(\mathcal{S})(1,1)$, which implies that the symmetric group element σ_{12} is not a cluster automorphism i.e. is not an element of $C_2(2,1)$, and hence the generators are only T_1 as defined in (2.2.10), together with automorphism $T_2 \in Aut_K K(x_1, x_2)$ which is induced by

$$x_1 \mapsto x_1 \quad \text{and} \quad x_2 \mapsto \frac{x_1^2+1}{x_2}. \tag{2.2.13}$$

Then we have

$$C_2(2,1) = \widetilde{m}_2(S) = \langle T_1, T_2 \rangle < Aut_K K(x_1, x_2), \quad \forall \text{ seed } p \in S, \quad \text{and } \forall \ d. \tag{2.2.14}$$

Remark 2.2.15. $C_2(2,1)$ is a Coexter group with the following presentation

$$C_2(2,1) = \{T_1, T_2 | \ \ T_1^2 = T_2^2 = 1, (T_1T_2)^3 = 1\}.$$

Example 2.2.16 (**The cluster group** $C_2(3,1)$**).** .

We have exactly 8 different cluster variables. In a similar way of calculating $C_2(2,1)$, one can see $C_2(3,1)$ is generated by T_1 as defined in (2.2.10) and T_2, induced by

$$x_1 \mapsto x_1 \quad \text{and} \quad x_2 \mapsto \frac{x_1^3+1}{x_2}, \tag{2.2.15}$$

and we have

$$C_2(3,1) = \widetilde{m}_2(S) = \langle T_1, T_2 \rangle < Aut_K K(x_1, x_2), \quad \forall \ p \in S, \quad \text{and } \forall \ d. \tag{2.2.16}$$

Remark 2.2.17. $C_2(3,1)$ is a Coexter group with the following presentation

$$C_2(2,1) = \{T_1, T_2 | \ \ T_1^2 = T_2^2 = 1, (T_1T_2)^4 = 1\}.$$

Chapter 3

Hyperbolic Cluster Algebras

In this chapter we set up for a class of non-commutative algebras with cluster structure which are generated by isomorphic copies of hyperbolic algebras. The first section is devoted to introducing the notion of the hyperbolic seeds and to generalize the cluster automorphisms to the hyperbolic seeds. In the second section we use the cluster structure to build indecomposable and irreducible representations for the associated hyperbolic algebra using combinatorial data called cluster strings. In the last section we introduce a ctategorification for the cluster structure in the Weyl algebra case.

3.1 Hyperbolic algebras

Definition 3.1.1 (Hyperbolic Algebra [30, 31]). Fix a commutative ring $\mathcal{R}$. Let $\theta = \{\theta_1, \ldots, \theta_n\}$ be a set of ring automorphisms of $\mathcal{R}$, and $\{\xi_1, \ldots, \xi_n\}$ be a fixed set of elements of $\mathcal{R}$. The hyperbolic algebra of rank n, denoted by $\mathcal{R}(\theta, \xi, n)$, is defined to be the ring generated by $\mathcal{R}$ and $x_1, \ldots, x_n, y_1, \ldots, y_n$ with the commutation relations:

$$x_i r = \theta_i(r) x_i \quad \text{and} \quad y_i r = \theta_i^{-1}(r) y_i, \quad \text{for any } i \in [1, n], \quad \text{and for any} \quad r \in \mathcal{R}, \tag{3.1.1}$$

$$x_i y_i = \xi_i, \forall i \in [1, n] \quad x_i y_j = y_j x_i, \quad x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i \quad \forall i \neq j. \tag{3.1.2}$$

We warn the reader that $x_i y_i \neq y_i x_i$ in general.

Example 3.1.2. Let A_n be the Weyl algebra generated by $2n$ variables $x_1, \ldots, x_n, y_1 \ldots, y_n$ over a field K and the relations

$$x_i y_i = y_i x_i + 1, \quad \forall i \in \{1, \ldots, n\} \text{ and } x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i \text{ for } i \neq j. \tag{3.1.3}$$

Let $\xi_i = y_i x_i$, $\mathcal{R} = K[\xi_1, \ldots, \xi_n]$, and $\theta_i : \mathcal{R} \to \mathcal{R}$, induced by $\xi_i \mapsto \xi_i + 1, \xi_j \mapsto \xi_j, j \neq i$. One can see that $A_n = \mathcal{R}(\theta, \xi, n)$ is a hyperbolic algebra of rank n.

Example 3.1.3 ([30, 31]). The coordinate algebra $A(SL_q(2,k))$ of algebraic quantum group $SL_q(2,k)$ is the K-algebra generated by x, y, u, and v subject to the following relations

$$qux = xu, \quad qvx = xv, \quad qyu = uy, \quad qyv = vy, \quad uv = vu, \quad q \in K^* \tag{3.1.4}$$

$$xy = quv + 1, \quad \text{and} \quad yx = q^{-1}uv + 1. \tag{3.1.5}$$

$A(SL_q(2,k)) = \mathcal{R}(\xi, \theta, 1)$ is a hyperbolic algebra of rank 1, with $\mathcal{R} = K[u, v]$ is the algebra of polynomials in u, v and $\theta \in Aut.(\mathcal{R})$ being given by $\theta(f(u,v)) = f(qu, qv)$ for any polynomial $f(u,v)$, and $\xi = 1 + q^{-1}uv$.

3.2 Hyperbolic cluster algebras

3.2.1 Generalized and hyperbolic seeds

Let $\mathbb{P}$ be a finitely generated free abelian group, written multiplicatively, with set of generators $F = \bigcup_{i=1}^{n} F_i$, where $F_i = \{f_{i1}, \ldots, f_{im_i}\}$ and $F_i \cap F_j = \emptyset$ for $i \neq j \in [1, n]$, and let $m = \sum_{i=1}^{n} m_i$. Denote the group ring of $\mathbb{P}$ over K by $R = K[\mathbb{P}]$. Let D be an Ore domain contains R such that there are $t_1, \ldots, t_n \in D$ so that $\{t_1^{\alpha_1}, \ldots, t_n^{\alpha_1}; (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n\}$ form a basis for D as a left R-module.

Let $\mathcal{D}$ denote the set of right fractions ab^{-1} with $a, b \in D$, and $b \neq 0$; two such fractions ab^{-1} and cd^{-1} are identified if $af = cg$ and $bf = dg$ for some non-zero $f, g \in D$. The ring D

is embedded into $\mathcal{D}$ via $d \mapsto d \cdot 1^{-1}$. The addition and multiplication in D extend to $\mathcal{D}$ so that $\mathcal{D}$ becomes a division ring. (Indeed, we can define $ab^{-1}+cd^{-1} = (ae+cf)g^{-1}$ where non-zero elements e, f , and g of D are chosen so that $be = df = g$; similarly, $ab^{-1} \cdot cd^{-1} = ae \cdot (df)^{-1}$, where non-zero $e, f \in D$ are chosen so that $cf = be$). In such case we say $\mathcal{D}$ is the division ring of fractions of D. More details about Ore domains are in [29] and [2].

Definition 3.2.1. Generalized Seeds. A generalized seed $\mathtt{i}$ of rank n in $\mathcal{D}$ is the triple (F, X, Γ), where

- F is as described above, which is called the set of frozen variables.
- $X = (x_1, \ldots, x_n) \in \mathcal{D}^n$ such that there is an $R-$linear automorphism on $\mathcal{D}$ that fixes the frozen variables and sends t_i to $x_{\sigma(i)}$, for each $i \in [1, n]$ and for some permutation σ. Elements of $\{x_1, \ldots, x_n\}$ are called cluster variables and

$$\widetilde{X} := \{f_{11}, \ldots, f_{1m_1}, \ldots, f_{n1}, \ldots, f_{nm_n}, x_1, \ldots, x_n\}$$

  is called a cluster.
- Γ is an oriented graph with set of vertices $I := I_F \cup I_X = [1, k]$, where $\mathtt{k} = m + n$, with no 2-cycles nor loops. Let $v : \widetilde{X} \to I$ be a one-to-one correspondence map, where $v(f) \in I_F$ for every frozen variable f, and $v(x_i) \in I_X$ for every cluster variable $x_i \in X$.

We need the following combinatorial data before introducing the generalized seed mutation. Let $\mathtt{i} = (F, X, \Gamma)$ be a generalized seed, and L be the lattice $\mathbb{Z}^{\mathtt{k}}$, where $\mathtt{k} = m+n$. For $k \in [1, n]$ we associate two vectors of L to the vertex $v(x_k)$ of Γ as follows, the first vector

$$\overrightarrow{r_{\mathtt{i}}}^k = (r_{11}, \ldots, r_{1m_1}, r_{21}, \ldots, r_{2m_2}, \ldots, r_{n1}, \ldots, r_{nm_n}, l_1, \ldots, l_{k-1}, -1, l_{k+1}, \ldots, l_n),$$

where r_{ij} is the number of arrows in Γ directed from the vertex $v(f_{ij})$ toward $v(x_k)$, and for $i \neq k$, l_i is the number of arrows from the vertex $v(x_i)$ toward $v(x_k)$, and we have -1 at the place of l_k. The second vector is

$$\overleftarrow{r_{\mathtt{i}}}^k = (r'_{11}, \ldots, r'_{1m_1}, r'_{21}, \ldots, r'_{2m_2}, \ldots, r'_{n1}, \ldots, r'_{nm_n}, l'_1, \ldots, l'_{k-1}, -1, l'_{k+1}, \ldots, l'_n),$$

which encodes the number of arrows targeting other vertices from $v(x_k)$, i.e each component of $\overleftarrow{r}_{\mathtt{i}}{}^k$ is the number of arrows with source as $v(x_k)$ and target as the corresponding vertex, except for the component corresponding to the vertex $v(x_k)$, we have -1. This defines a map, $r : X \to I \to \mathbb{Z}^{\mathtt{k}} \times \mathbb{Z}^{\mathtt{k}}$, given by $x_k \mapsto (\overrightarrow{r}_{\mathtt{i}}{}^k, \overleftarrow{r}_{\mathtt{i}}{}^k)$. Each of these two vectors defines a map from $\widetilde{X}$ to $\mathbb{Z}_{\geq -1}$, given by

$$\overrightarrow{r}_{\mathtt{i}}{}^k(t) = \begin{cases} r_{ij}, & \text{if } \ t = f_{ij}, \\ -1, & \text{if } t = x_k \\ l_j & \text{if } t = x_j, j \neq k. \end{cases}$$

For a cluster $\widetilde{X}$, we consider the following two maps $M_C^R, M_C^L : L \to \mathcal{D}$ given as follows, for a vector $a = (a_{11}, \ldots, a_{m_1 1}, a_{12}, \ldots, a_{m_2 2}, \ldots, a_{1n}, \ldots, a_{m_n n}, b_1, \ldots, \ldots, b_n) \in L$, we assign the following two monomials

$$M_C^R(a) = t_{11}^{a_{11}} \cdots t_{m_1 1}^{a_{m_1 1}} \cdot t_{12}^{a_{12}} \cdots t_{m_2 2}^{a_{m_2 2}} \cdots t_{1n}^{a_{1n}} \cdots t_{m_n n}^{a_{m_n n}} \cdot x_1^{b_1} \cdots x_n^{b_n}, \tag{3.2.1}$$

$$M_C^L(a) = x_1^{b_1} \cdots x_n^{b_n} \cdot t_{11}^{a_{11}} \cdots t_{m_1 1}^{a_{m_1 1}} \cdot t_{12}^{a_{12}} \cdots t_{m_2 2}^{a_{m_2 2}} \cdots t_{1n}^{a_{1n}} \cdots t_{m_n n}^{a_{m_n n}}. \tag{3.2.2}$$

In the following we skip the word generalized if there is no confusion.

Definition 3.2.2. Let $\mathtt{i} = (F, X, \Gamma)$ be a seed of rank n in $\mathcal{D}$. At each cluster variable (non-frozen) x_k we can obtain two new triples $\mu_k^R(\mathtt{i}) = (F', \mu_k^R(X), \Gamma')$ and $\mu_k^L(\mathtt{i}) = (F', \mu_k^L(X), \Gamma')$ from $\mathtt{i}$, by applying the following steps;

- $F' = F$,
- $\mu_k^R(X) = (x_1, \ldots, x_{k-1}, x_k', x_{k+1}, \ldots, x_n)$, where

$$x_k' = M_C^R(\overrightarrow{r}_{\mathtt{i}}{}^k) + M_C^R(\overleftarrow{r}_{\mathtt{i}}{}^k), \tag{3.2.3}$$

and $\mu_k^L(X) = (x_1, \ldots, x_{k-1}, \grave{x_k}, x_{k+1}, \ldots, x_n)$, where

$$\grave{x_k} = M_C^L(\overrightarrow{r}_{\mathtt{i}}{}^k) + M_C^L(\overleftarrow{r}_{\mathtt{i}}{}^k). \tag{3.2.4}$$

- Γ' is obtained from Γ by applying the same rules of quiver mutations given in Chapter 2 in the paragraph preceding Theorem 2.1.4.
- μ_k^R and μ_k^L are called right and left mutations in the k-direction respectively.

$\mu_k^R(\mathtt{i})$ and $\mu_k^L(\mathtt{i})$ obtained in the above way are said to be obtained from $\mathtt{i}$ by applying right and left mutations in the k-direction respectively.

Remark 3.2.3. This definition of mutations on oriented, no loos, and 2-cycles graphs is equivalent to the definition of mutations on skew-symmetric matrices corresponding to such graphs, definition 1.1.2.

In the following we will try to explore some of the commutation relations of the elements of $\widetilde{X}$ that are invariant under mutations for some particular seeds.

Notations: For a seed $\mathtt{i} = (F, X, \Gamma)$ and $k \in [1, n]$. The following notations will be used in the rest of this chapter

- $\mathtt{i}' = \mu_k^R(\mathtt{i})$, (resp., $\grave{\mathtt{i}} = \mu_k^L(\mathtt{i})$).
- For $X' = (x'_1, \ldots, x'_n)$, where

$$x'_j = \begin{cases} x_j, & \text{if } \ j \neq k, \\ M_C^R(\overrightarrow{r_i}^k) + M_C^R(\overleftarrow{r_i}^k), & \text{if } j = k. \end{cases} \tag{3.2.5}$$

  (resp., $\grave{X} = (\grave{x_1}, \ldots, \grave{x_n})$).
- The mutation of $\mathtt{i}$ at k gives rise to two R-linear $\mathcal{D}$ automorphisms, $T^R_{\mathtt{ii}'}$ and $T^L_{\mathtt{i}\grave{\mathtt{i}}}$, where $T^R_{\mathtt{ii}'} : \mathcal{D} \to \mathcal{D}$ is induced by

$$T^R_{\mathtt{ii}'}(t) = t, \ \forall t \in R \ \text{ and } \ T^R_{\mathtt{ii}'}(x_j) := x'_j, \ \forall j \in [1, n] \tag{3.2.6}$$

  (resp., to $T^L_{\mathtt{i}\grave{\mathtt{i}}}(x_j) := \grave{x_j}, \forall j \in [1, n]$). $T^R_{\mathtt{ii}'}$ and $T^L_{\mathtt{i}\grave{\mathtt{i}}}$ are called right and left mutation automorphisms respectively.
- For a seed $\mathtt{i}$ *the neighborhood of a cluster variable* x_k is defined to be the subset of $\widetilde{X}$ corresponding to the non-zero components of $\overrightarrow{r_{\mathtt{i}}}^k$ or $\overleftarrow{r_{\mathtt{i}}}^k$ and is denoted by $N_{\mathtt{i}}(x_k)$. In

other words, $N_{\mathtt{i}}(x_k) := N_{\mathtt{i},+}(x_k) \cup N_{\mathtt{i},-}(x_k)$, where $N_{\mathtt{i},+}(x_k) = \{x \in \widetilde{X}; \overrightarrow{r_{\mathtt{i}}}^k(x) > 0\}$ and $N_{\mathtt{i},-}(x_k) = \{x \in \widetilde{X}; \overleftarrow{r_{\mathtt{i}}}^k(x) > 0\}$. In particular, Let $F^k_{\mathtt{i},+} = F_k \cap N_{\mathtt{i},+}(x_k)$, and $F^k_{\mathtt{i},-} = F_k \cap N_{\mathtt{i},-}(x_k)$.

Definition 3.2.4 (**Hyperbolic Seeds).** The quadruple (F, X, Γ, φ) is called a *Hyperbolic seed* of rank n in $\mathcal{D}$, if it satisfies the following conditions

1. the triple (F, X, Γ) is a generalized seed of rank n in $\mathcal{D}$ satisfying

$$N_{\mathtt{i}}(x_k) = F_k, \forall k \in [1, n]. \tag{3.2.7}$$

2. X consists of a commutative set of rational functions such that x_i commutes with elements of F_j for $j \neq i$.

3. φ is an R-linear automorphism of $\mathcal{D}$ satisfies the following equations

$$f^a x_i = \varphi^a(x_i) f^a, \forall f \in F_i, \forall i \in [1, n], a \in \mathbb{Z}_{\geq 0}. \tag{3.2.8}$$

The above equations induce the following equations

$$x_i f^a = f^a \varphi^{-a}(x_i), \forall f \in F_i, \forall i \in [1, n], a \in \mathbb{Z}_{\geq 0}. \tag{3.2.9}$$

Furthermore, (F, X, Γ, φ) is said to be *weak hyperbolic seed* if the condition (3.2.7) above is replaced by the condition

$$N_{\mathtt{i}}(x_k) \cap N_{\mathtt{i}}(x_i) \cap F_k = \emptyset, \forall i, \forall k \in [1, n]. \tag{3.2.10}$$

Lemma 3.2.5. *Let $\mathtt{i} = (F, X, \Gamma, \varphi)$ be a hyperbolic seed in $\mathcal{D}$. Then, the following are true*

1. *The ring D is an Ore domain.*

2. *For any sequence of right mutations (resp., to left) $\mu^R_{i_1}\mu^R_{i_2}\ldots\mu^R_{i_q}$, we have $\mu^R_{i_1}\mu^R_{i_2}\ldots\mu^R_{i_q}(\mathtt{i})$ (resp., to $\mu^L_{i_1}\mu^L_{i_2}\ldots\mu^L_{i_q}(\mathtt{i})$) is again a hyperbolic seed.*

3.

$$\mu_k^R \mu_k^L(\mathtt{i}) = \mu_k^L \mu_k^R(\mathtt{i}) = \mathtt{i}, \quad \forall k \in [1, n]. \tag{3.2.11}$$

More precisely $T_{\mathtt{i}\mathtt{i}'}(T_{\mathtt{i}'\mathtt{i}}(x_k)) = T_{\mathtt{i}'\mathtt{i}}(T_{\mathtt{i}\mathtt{i}'}(x_k)) = x_k$.

Proof. Condition 2 in definition 3.2.4 and the commutation relations (3.2.8) make part 1 immediate.

We prove part 2 for $\mu_k^R(\mathtt{i})$, (resp., $\mu_k^L(\mathtt{i})$) and the proof for an arbitrary sequence of right (resp., to left) mutations is by induction on the length of the sequence. First, the commutativity of the elements of $\{x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n\}$, is obvious by conditions (1) and (2) in the definition of the hyperbolic seeds. The same for the commutativity of x'_k and the elements of F_j for $j \neq k$. To finish the proof, it is enough to show that there is an R-linear automorphism φ' (resp., $\grave{\varphi}$) of $\mathcal{D}$ satisfies 3.2.8 (resp., $\mu_k^L(\mathtt{i})$). Let $\varphi' = T_{\mathtt{i}\mathtt{i}'}\varphi T_{\mathtt{i}'\mathtt{i}}$. We have

$$\begin{aligned} f_{ij}x'_j &= T_{\mathtt{i}\mathtt{i}'}(f_{ij}x_j) \\ &= T_{\mathtt{i}\mathtt{i}'}(\varphi(x_j f_{ij})) \\ &= T_{\mathtt{i}\mathtt{i}'}\varphi T_{\mathtt{i}'\mathtt{i}}(x'_j f_{ij}) \\ &= \varphi'(x'_j f_{ij}) \\ &= \varphi'(x'_j) f_{ij}. \end{aligned}$$

This finishes the proof of part (2).

For the part 3, one can see that $\mu_k^R \mu_k^L$ and $\mu_k^L \mu_k^R$ act like the identity on both of F and Γ, since the right and left mutations act like Fomin-Zelevinsky mutation on F and Γ, and Fomin-Zelevinsky mutation is involutive.

In the following we show that $\mu_k^R \mu_k^L(X) = X$, and $\mu_k^L \mu_k^R(X) = X$ is quite similar. Let $\grave{\mathtt{i}} = \mu_k^L(F, X, \Gamma)$, and $(\grave{\mathtt{i}})' = \mu_k^R \mu_k^L(F, X, \Gamma)$.

We have $\mu^L_{k,\mathtt{i}}(x_k) = \grave{x_k} = x_k^{-1}(\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) + \widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k))$, where $\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) = x_k M_C(\overrightarrow{r_{\mathtt{i}}}^k)$ and $\widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k) = x_k M_C(\overleftarrow{r_{\mathtt{i}}}^k)$. By definition of mutation on Γ, one has $\overrightarrow{r_{\mathtt{i}}}^k = \overleftarrow{r_{\grave{\mathtt{i}}}}^k$ and $\overleftarrow{r_{\mathtt{i}}}^k = \overrightarrow{r_{\grave{\mathtt{i}}}}^k$. Then condition 3.2.7, and commutation relations 3.2.8 guarantee $\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) = \widehat{M_C}(\overleftarrow{r_{\grave{\mathtt{i}}}}^k)$ and $\widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k) = \widehat{M_C}(\overrightarrow{r_{\grave{\mathtt{i}}}}^k)$. Hence

$$\begin{aligned}
\mu^R_{\grave{\mathtt{i}}}(\grave{x_k}) &= (\widehat{M_C}(\overrightarrow{r_{\grave{\mathtt{i}}}}^k) + \widehat{M_C}(\overleftarrow{r_{\grave{\mathtt{i}}}}^k)((\grave{x_k})^{-1}) \\
&= (\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) + \widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k)((\grave{x_k})^{-1}) \\
&= x_k.
\end{aligned}$$

Finally, for the automorphism $(\grave{\varphi})'$ of the seed $\mu^R_k\mu^L_k(\mathtt{i})$. From the proof of part (2), we have $(\grave{\varphi})' = T_{\grave{\mathtt{i}}\grave{(\mathtt{i})'}} T_{\mathtt{i}\grave{\mathtt{i}}} \varphi T_{\grave{\mathtt{i}}\mathtt{i}} T_{\grave{(\mathtt{i})'}\grave{\mathtt{i}}} = T_{\grave{\mathtt{i}}\mathtt{i}} T_{\mathtt{i}\grave{\mathtt{i}}} \varphi T_{\grave{\mathtt{i}}\mathtt{i}} T_{\mathtt{i}\grave{\mathtt{i}}} = id_{\mathfrak{F}} \varphi id_{\mathfrak{F}} = \varphi$.

To see that $T_{\mathtt{i}\mathtt{i}'}(T_{\mathtt{i}'\mathtt{i}}(x_k)) = x_k$.

One has $T_{\mathtt{i}\mathtt{i}'}(T_{\mathtt{i}'\mathtt{i}}(x_k)) = T_{\mathtt{i}\mathtt{i}'}(px_k^{-1}) = p(x'_k)^{-1} = p(px_k^{-1})^{-1} = x_k$, where p is the polynomial $M_C(\overrightarrow{r_{\grave{\mathtt{i}}}}^k) + M_C(\overleftarrow{r_{\grave{\mathtt{i}}}}^k) \in R$. □

Definition 3.2.6 (**The Cluster sets and (right and left) Cluster patterns).** 1. Let $\mathtt{i}$ be a seed (or a hyperbolic seed) an element $y \in \mathcal{D}$, is said to be a right cluster element (resp., to a left cluster element) of $\mathtt{i}$ if y is a cluster variable in some seed $\mathtt{j}$, where $\mathtt{j}$ can be obtained from $\mathtt{i}$ by applying some sequence of right mutations (resp., to left mutations). The set of all right cluster elements of $\mathtt{i}$ (resp., to cluster left) is called the *the right cluster set* (resp., to *the left cluster set*) of $\mathtt{i}$, and is denoted by $\chi^R(\mathtt{i})$ (resp., to $\chi^L(\mathtt{i})$). The set of all right and left cluster elements of $\mathtt{i}$ is called *the cluster set* of $\mathtt{i}$ and is denoted by $\chi(\mathtt{i})$. So $\chi(\mathtt{i}) = \chi^R(\mathtt{i}) \cup \chi^L(\mathtt{i})$. Elements of $\chi(\mathtt{i})$ are called cluster variables of $\mathtt{i}$ or simply cluster variables.

2. Let $\mathtt{i}$ be a hyperbolic seed. The cluster pattern $\mathbb{T}(\mathtt{i})$ of $\mathtt{i}$ is a directed graph built in the following way; label an initial vertex with $\mathtt{i}$, and from $\mathtt{i}$ we generate arrows as follows, every $k \in [1, n]$ corresponds to two arrows going out from $\mathtt{i}$ one for right mutation in k-direction and the other one is for the left mutation in same direction, each arrow is

targeting a new hyperbolic seed, which is generated by the indicated mutation applied to $\mathtt{i}$. Now repeat the process to the new vertices.

3. Right (resp., to left) cluster pattern is defined in same way with the restriction, every $k \in [1, n]$ corresponds to only one arrow which is the right mutation at k (resp., to left mutation) and is denoted by $\mathbb{T}^R(\mathtt{i})$ (resp., to $\mathbb{T}^L(\mathtt{i})$).

Open Problems 3.2.7. Given a (resp., to weak) hyperbolic seed $\mathtt{i} = (F, X, \Gamma, \varphi)$

- What are necessary and sufficient conditions on Γ and φ that guarantee $\chi(\mathtt{i})$ to be a finite set. Same question can be raised for right and left cluster sets.
- What are necessary and sufficient conditions on Γ and φ that guarantee $\mathbb{T}(\mathtt{i})$ (resp., to $\mathbb{T}^R(\mathtt{i})$ and $\mathbb{T}^L(\mathtt{i})$) to be a finite graph or a periodic graph.

In the following we will provide some sufficient conditions for cluster patterns to contain cycles.

Definition 3.2.8. A seed $\mathtt{i}$ is said to be a well-connected seed, if there are n nonnegative integers $a_1, \ldots, a_n$, such that

$$\sum_{f_{ik} \in F^k_{\mathtt{i},+}} \overrightarrow{r_{\mathtt{i}}}^k(f_{ik}) = \sum_{f_{ik} \in F^k_{\mathtt{i},-}} \overleftarrow{r_{\mathtt{i}}}^k(f_{ik}) = a_k, \quad \forall k \in [1, n]. \tag{3.2.12}$$

In this case the n-tuples $f_{\mathtt{i}} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ is called the frozen rank of $\mathtt{i}$.

Through the rest of the thesis, any statement contains μ_k without the superscript R or L, is true for right and left mutations μ_k^R and μ_k^L respectively. However, the proofs are written for right mutations and for left mutations the proofs are quite similar in most of the cases.

Proposition 3.2.9. *The mutation in any direction of a hyperbolic well-connected seed is again well-connected, and the frozen rank is invariant under the mutation.*

Proof. Immediate from the definitions of well-connected seeds and the definition of mutation.

□

Theorem 3.2.10. *Let* $\mathtt{i} = (F, X, \Gamma, \varphi)$ *be a well-connected weak hyperbolic seed with* φ *a finite order automorphism. Then* μ_k *is invertible on* $\mathtt{i}$ *for each* $k \in [1, n]$*. More precisely, there is a non negative integer* r *such that*

$$(\mu_k^R)^{2r}((F, X, \Gamma)) = (\mu_k^L)^{2r}((F, X, \Gamma)) = (F, X, \Gamma). \tag{3.2.13}$$

The proof of the theorem is a consequence of the following lemma.

Lemma 3.2.11. *Let* $\mathtt{i} = (F, X, \Gamma, \varphi)$ *be a well-connected weak hyperbolic seed. Then for every cluster variable* x_k*, we have*

$$(\mu_{\mathtt{i},k}^R)^2(x_k) = \varphi^{a_k}(x_k) \quad \textit{for some nonnegative integer} \quad a_k. \tag{3.2.14}$$

$$(\mu_{\mathtt{i},k}^L)^2(x_k) = \varphi^{-a_k}(x_k) \quad \textit{for some nonnegative integer} \quad a_k. \tag{3.2.15}$$

Proof. We start with proving 3.2.13. Let

$\overrightarrow{r_{\mathtt{i}}}^k = (r_{11}, \cdots, r_{m_1 1}, r_{12}, \cdots, r_{m_2 2}, \cdots, r_{1n}, \cdots, r_{m_n n}, l_1, \cdots, l_{k-1}, -1, l_{k+1}, \cdots, l_n)$.

and

$\overleftarrow{r_{\mathtt{i}}}^k = (r'_{11}, \cdots, r'_{m_1 1}, r'_{12}, \cdots, r'_{m_2 2}, \cdots, r'_{1n}, \cdots, r' m_n n, l'_1, \cdots, l'_{k-1}, -1, l'_{k+1}, \cdots, l'_n)$.

Then by definition of $M_C(\overrightarrow{r_{\mathtt{i}}}^k)$ and $M_C(\overleftarrow{r_{\mathtt{i}}}^k)$ and the commutativity of the elements of X one can see $M_C(\overrightarrow{r_{\mathtt{i}}}^k)$ and $M_C(\overrightarrow{r_{\mathtt{i}}}^k)$ can be written as follows

$$M_C(\overrightarrow{r_{\mathtt{i}}}^k) = \widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) x_k^{-1} \quad \text{and} \quad M_C(\overleftarrow{r_{\mathtt{i}}}^k) = \widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k) x_k^{-1}.$$

Where

$$\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) = t_{11}^{r_{11}} \cdots t_{m_1 1}^{r_{m_1 1}} \cdot t_{12}^{r_{12}} \cdots t_{m_2 2}^{r_{m_2 2}} \cdots t_{1n}^{r_{1n}} \cdots t_{m_n n}^{r_{m_n n}} \cdot x_1^{l_1} \cdots x_{k-1}^{l_{k-1}} \cdot x_{k+1}^{l_{k+1}} \cdots x_n^{l_n},$$

and

$$\widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k) = t_{11}^{r'_{11}} \cdots t_{m_1 1}^{r'_{m_1 1}} \cdot t_{12}^{r'_{12}} \cdots t_{m_2 2}^{r'_{m_2 2}} \cdots t_{1n}^{r'_{1n}} \cdots t_{m_n n}^{r'_{m_n n}} \cdot x_1^{l'_1} \cdots x_{k-1}^{l'_{k-1}} \cdot x_{k+1}^{l'_{k+1}} \cdots l'_n.$$

If C' is the cluster of the seed $\mathtt{i}' = \mu_k^R(\mathtt{i})$, then by the definition of mutation on Γ, we have $\overrightarrow{r_{\mathtt{i}}}^k = \overleftarrow{r_{\mathtt{i}'}}^k$, and $\overleftarrow{r_{\mathtt{i}}}^k = \overrightarrow{r_{\mathtt{i}'}}^k$. Hence, $\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) = \widehat{M_{C'}}(\overleftarrow{r_{\mathtt{i}'}}^k)$, and $\widehat{M_{C'}}(\overrightarrow{r_{\mathtt{i}'}}^k) = \widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k)$. Then one has

$$\begin{aligned}
(\mu_{\mathtt{i},k}^R)^2(x_k) &= \mu_{\mathtt{i}',k}^R((\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) + \widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k))x_k^{-1}) \\
&= (\widehat{M_{C'}}(\overrightarrow{r_{\mathtt{i}'}}^k) + \widehat{M_{C'}}(\overleftarrow{r_{\mathtt{i}'}}^k))x_k(\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) + \widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k))^{-1} \\
&= (\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) + \widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k))x_k(\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) + \widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k))^{-1}.
\end{aligned}$$

Since $\mathtt{i}$ is a well-connected seed, then there is a positive integer a_k such that

$$\sum_{f_{ik} \in F_{\mathtt{i},+}^k} \overrightarrow{r_{\mathtt{i}}}^k(f_{ik}) = \sum_{f_{ik} \in F_{\mathtt{i},-}^k} \overleftarrow{r_{\mathtt{i}}}^k(f_{ik}) = a_k.$$

In the case of $a_k = 0$, we have no thing to prove. Assume it is nonzero, then, using condition 2 of definition 3.2.4 one can see

$$(\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) + \widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k))x_k = \varphi^{a_k}(x_k)(\widehat{M_C}(\overrightarrow{r_{\mathtt{i}}}^k) + \widehat{M_C}(\overleftarrow{r_{\mathtt{i}}}^k)).$$

This finishes the proof of equations 3.2.13. The proof of 3.2.14 is quite similar except for using the commutation relations 3.2.9 instead of 3.2.8 in the step before the last one. □

Proof of theorem 3.2.10. We prove it for right mutations and the case of left mutation is quite similar. Assume that $\mathtt{i} = (F, X, \Gamma, \varphi)$ is as in the statement of theorem and $\varphi^r = id_{\mathcal{D}}$ for some non negative integer r. By definition of mutations on the cluster variables, the mutation in the k-direction leaves every cluster variable with no change except for x_k. Therefore the following sequence of repeated mutations in the k-direction $(\mu_k^R)^{2r}$ will leave every cluster variable, other than x_k unchanged, and for x_k, the lemma tells us

$$(\mu_k^R)^{2r}(x_k) = \varphi^{ra_k}(x_k) = x_k. \tag{3.2.16}$$

Also, one can see $(\mu_k^R)^2(\Gamma) = \Gamma$. Then, $(\mu_k^R)^{2r}(\Gamma) = \Gamma$.

□

Example 3.2.12. The simplest nontrivial well-connected hyperbolic seed. Consider the hyperbolic well-connected seed $\mathtt{i} = (F, X, \Gamma, \varphi)$ where $F = \{f_{11}, f_{12}\}$, $X = \{x\}$, and Γ is the following graph

$$\cdot_{1_1} \longleftarrow \cdot_1 \longleftarrow \cdot_{1_2} ,$$

and let φ be a R-linear automorphism of $\mathcal{D}$ satisfying the conditions 3.2.8. Here $I_f = \{1_1, 1_2\}$ and $I_a = \{1\}$, and the frozen rank is (1). This seed produces the following the cluster set

$$\chi(\mathtt{i}) = \{x, (f_{11} + f_{12})x^{-1}, x^{-1}(f_{11} + f_{12}), \varphi^k(x), (f_{11} + f_{12})\varphi^{-k}(x), \varphi^{-k}(x)(f_{11} + f_{12}); k \in \mathbb{Z}\}.$$

One can see that $\chi(\mathtt{i})$ is a finite set if and only if φ is of finite order.

Example 3.2.13. The simplest non well-connected hyperbolic seed. Consider the hyperbolic well-connected seed $\mathtt{i} = (F, X, \Gamma, \varphi)$ where $F = \{f\}$, $X = \{x\}$, Γ is the following graph

$$\cdot_{1_1} \longleftarrow \cdot_1 ,$$

and let φ be a R-linear automorphism of $\mathcal{D}$ satisfying the conditions 3.2.8. Here $I_f = \{1_1\}$ and $I_a = \{1\}$. We have the following infinite cluster set;
$\chi(\mathtt{i}) = \{x, (1+f)^{k+1}x^{-1}(1+f)^{-k}, (1+f)^k x(1+f)^{-k}, (1+f)^{-k}x^{-1}(1+f)^{k+1}, (1+f)^{-k}x^{-1}(1+f)^k, k \in \mathbb{Z}\}$. In this case, this seed has no frozen rank, and so condition 3.2.11 is not satisfied. So even if φ is of finite order we still have an infinite cluster sets. In the subsection 3.2.3, we will see how this seed is related to first Weyl algebra.

3.2.2 The groups of cluster automorphisms

Definition 3.2.14. An R-linear automorphism ϕ of $\mathcal{D}$ is called a cluster automorphism of a seed $\mathtt{i}$ if it leaves the cluster set $\chi(\mathtt{i})$ invariant as a whole set. The group of all such

automorphisms is called the group of Cluster automorphisms of $\mathtt{i}$ and is denoted by $H[\mathtt{i}]$. Right and left cluster automorphisms groups denoted by $H^R[\mathtt{i}]$ and $H^L[\mathtt{i}]$ respectively can be defined in the same way by replacing the cluster set of $\mathtt{i}$ by right and left cluster sets respectively.

Open Problems 3.2.15. For a given (hyperbolic) seed $\mathtt{i} = (F, X, \Gamma, \varphi)$ describe the group of all cluster automorphisms of $\mathtt{i}$ (resp., to right and left cluster automorphisms).

The following proposition and theorem provide a big class of cluster automorphisms of some seeds.

Proposition 3.2.16. *If $\mathtt{i} = (F, X, \Gamma, \varphi)$ is a well-connected weak hyperbolic seed then φ gives rise to an infinite set of cluster automorphisms of $\mathtt{i}$.*

Proof. Fix a well-connected hyperbolic seed $\mathtt{i} = (F, X, \Gamma, \varphi)$. For every $l \in \mathbb{Z}$ we define an R-linear ϕ_l automorphism on $\mathcal{D}$ induced by

$$\phi_l(t) = t, \quad \forall t \in R \quad \text{and} \quad \phi_l(x_k) = \varphi^{la_k}(x_k), \quad \forall k \in [1, n], \tag{3.2.17}$$

where $(a_1, \ldots, a_k, \ldots, a_n)$ is the frozen rank of $\mathtt{i}$.

In the following, we prove that ϕ_l is a cluster automorphism for every $l \in \mathbb{Z}$.

First, for nonnegative integers. Let $l = 1$. Lemma 3.2.12 tells us that, the action of this automorphism on the cluster variables of $\mathtt{i}$ is identified with the action of the sequence of the mutation automorphisms $\prod_{k=1}^{k=n}(\mu_k^R)^2$, which corresponds to the sequence of mutations $\prod_{k=1}^{k=n}(\mu_k^R)^2$. So, ϕ_1 sends every cluster variable in $\widetilde{X}$ to a cluster element of $\mathtt{i}$ which is a cluster variable in the seed $\prod_{k=1}^{k=n}(\mu_k^R)^2(\mathtt{i})$. By definition of ϕ_1, one can see it does depend only on the frozen rank of $\mathtt{i}$ which is invariant under mutation, thanks to proposition 3.2.9.

Now, let x be any cluster element of the seed $\mathtt{i}$, then it is a cluster variable in some seed $\mathtt{j}$, which can be obtain from $\mathtt{i}$ by applying some sequence of mutations say $\mu_{i_1}^R \ldots \mu_{i_d}^R$. Then

from lemma 3.2.12, we must have $\phi_1(x)$ is a cluster variable in the seed $\prod_{k=1}^{k=n}(\mu_k^R)^2(\mathtt{j}) = \prod_{k=1}^{k=n}(\mu^R)^2\mu_{i_1}^R\dots\mu_{i_d}^R(\mathtt{i})$, i.e., it is a cluster element of $\mathtt{i}$, and this proves it for $l=1$.

For $l \geq 2$, one can see that, the action of ϕ_l on the elements of $\widetilde{X}$ is the same as the action of $(\prod_{k=1}^{k=n}(\mu_k^R)^2)^l$. Then, following the same argument as above we can see that $\phi_l(x)$ sends every element in $\widetilde{X}$ to a cluster variable in the seed $(\prod_{k=1}^{k=n}(\mu_k^R)^2))^l(\mathtt{i})$, and the case of an arbitrary cluster element x is the same as $l=1$ with the obvious changes.

Second, the case of l is a negative integer is similar, with using the left mutations rather than the right mutation, i.e., the superscript R will be replaced by L when it makes sense, and use the equations 3.2.14 instead of equations 3.2.15. □

Remark 3.2.17. Let $\mathcal{S}_n$ be the set of all seeds of rank n in $\mathcal{D}$, which share the same set of frozen variables F and $\mathfrak{b}_n$ be the set of all the graphs of the elements of $\mathcal{S}_n$ with set of vertices $I = F \cup [1,n]$. Let $\mathfrak{S}_n$ be the symmetric group in n letters. We have, $\mathfrak{S}_n$ acts on $\mathfrak{b}_n$ as follows, for $\Gamma \in \mathfrak{b}_n$ and σ be a permutation in $\mathfrak{S}_n$, $\sigma(\Gamma)$ is obtained from Γ simply by permuting the vertices of Γ.

Lemma 3.2.18. *Let Γ be a graph as defined in 3.2.1. Then for any sequence of mutations $\mu_{i_k}, \mu_{i_{k-1}}, \dots, \mu_{i_1}$, we have*

$$\sigma(\mu_{i_k}\mu_{i_{k-1}}\dots\mu_{i_1}(\Gamma)) = \mu_{\sigma(i_k)}\mu_{\sigma(i_{k-1})}\dots\mu_{\sigma(i_1)}(\sigma(\Gamma)), \quad \forall\sigma \in \mathfrak{S}_n. \tag{3.2.18}$$

Proof. Part (1) of theorem 2.1.7. □

Theorem 3.2.19. *Let $\mathtt{i} = (F, X, \Gamma)$ and $\mathtt{i}' = (F, X', \Gamma')$ be two elements of $\mathcal{S}_n$, such that $\Gamma' = \sigma(\Gamma)$, for some permutation $\sigma \in \mathfrak{S}_n$. Then the automorphism $T_{\mathtt{ii}',\sigma}$, induced by $T_{\mathtt{ii}',\sigma}(t) = t, \forall t \in R$, and $T_{\mathtt{ii}',\sigma}(x_i) = x'_{\sigma(i)}$, is a cluster automorphism of $\mathtt{i}$. (The theorem can be phrased for left mutations as well)*

Proof. The proof will be broken into three steps.

Step one:

$$T_{\mathtt{ii}'}(\mu_{\mathtt{i}k}(x_k)) = \mu_{\mathtt{i}'\sigma(k)}(x'_{\sigma(k)}), \quad \forall k \in [1,n]. \tag{3.2.19}$$

Since $\Gamma' = \sigma(\Gamma)$, one can see $\overrightarrow{r_{\mathtt{i}}}^k = \overrightarrow{r_{\mathtt{i}'}}^{\sigma(k)}$ and $\overleftarrow{r_{\mathtt{i}}}^k = \overleftarrow{r_{\mathtt{i}'}}^{\sigma(k)}$.

Then

$$\begin{aligned} T_{\mathtt{i}\mathtt{i}',\sigma}(\mu_{\mathtt{i}k}(x_k)) &= T_{\mathtt{i}\mathtt{i}',\sigma}((M_C(\overrightarrow{r_{\mathtt{i}}}^k) + M_C(\overleftarrow{r_{\mathtt{i}}}^k)) \\ &= M_C(\overrightarrow{r_{\mathtt{i}'}}^{\sigma(k)}) + M_C(\overleftarrow{r_{\mathtt{i}'}}^{\sigma(k)}) \\ &= \mu_{\mathtt{i}'\sigma(k)}(x'_{\sigma(k)}). \end{aligned}$$

Second step: In this step we show,

$$T_{\mu_i(\mathtt{i})\mu_{\sigma(i)}(\mathtt{i}),\sigma}(\mu_{\mu_i(\mathtt{i})k}(\mu_{\mathtt{i}i}(x_i))) = T_{\mathtt{i}\mathtt{i}',\sigma}(\mu_{\mu_i(\mathtt{i})k}(\mu_{\mathtt{i}i}(x_i))), \quad \forall i,k \in [1,n]. \tag{3.2.20}$$

For $k \neq i$, we have $\mu_{\mu_i(\mathtt{i})k}(\mu_{\mathtt{i}i}(x_i)) = \mu_{\mathtt{i}i}(x_i)$, then

$$\begin{aligned} T_{\mu_i(\mathtt{i})\mu_{\sigma(i)}(\mathtt{i}),\sigma}(\mu_{\mu_i(\mathtt{i})k}(\mu_{\mathtt{i}i}(x_i))) &= T_{\mu_i(\mathtt{i})\mu_{\sigma(i)}(\mathtt{i}),\sigma}((M_C(\overrightarrow{r_{\mathtt{i}}}^i) + M_C(\overleftarrow{r_{\mathtt{i}}}^i)) \\ &= M_C(\overrightarrow{r_{\mathtt{i}}}^{\sigma(i)}) + M_C(\overleftarrow{r_{\mathtt{i}}}^{\sigma(i)}) \\ &= T_{\mathtt{i}\mathtt{i}',\sigma}(\mu_{\mu_i(\mathtt{i})k}((x_i))) \\ &= T_{\mathtt{i}\mathtt{i}',\sigma}(\mu_{\mu_i(\mathtt{i})k}(\mu_{\mathtt{i}i}(x_i))). \end{aligned}$$

Third step: For any cluster element $y \in \chi(\mathtt{i})$, $T_{\mathtt{i}\mathtt{i}',\sigma}(\mu_{\mathtt{i}k}(y)) \in \chi(\mathtt{i})$.
y must be a cluster variable in some seed. Then, $\mathtt{j} = \mu_{i_k}\mu_{i_{k-1}}\dots\mu_{i_1}(\mathtt{i}) = (F, Y, \Upsilon)$ for some sequence of mutations. Lemma 3.2.18 implies

$$T_{\mathtt{j}\mathtt{i}',\sigma}(Y) = \mu_{\sigma(i_k)}(\mu_{\sigma(i_{k-1})}\dots\mu_{\sigma(i_1)}(X')), \tag{3.2.21}$$

which means $T_{\mathtt{j}\mathtt{i}',\sigma}(y)$ is a cluster variable in $\mathtt{j}' = \mu_{\sigma(i_k)}\mu_{\sigma(i_{k-1})}\dots\mu_{\sigma(i_1)}(\mathtt{i}')$, where $\mathtt{j}' = \mu_{\sigma(i_k)}\mu_{\sigma(i_{k-1})}\dots\mu_{\sigma(i_1)}(\mathtt{i}')$. From 3.2.19, 3.2.20 and induction on the length of mutations sequence, one can deduce

$$T_{\mathtt{j}\mathtt{i}',\sigma}(Y) = T_{\mathtt{i}\mathtt{i}',\sigma}(Y). \tag{3.2.22}$$

□

Remark 3.2.20. The group of all Cluster automorphisms is invariant under mutation. More precisely, for any seed $\mathtt{i}$, the following equation is satisfied

$$H[\mathtt{i}] = H[\mu_{j_1}(\mathtt{i})] = \ldots = H[\mu_{j_1}\mu_{j_2}\ldots\mu_{j_n}(\mathtt{i})]\ldots = \ldots. \tag{3.2.23}$$

However, the equation is not satisfied for right and left cluster groups, and even the inclusion is not guaranteed.

3.2.3 Hyperbolic cluster algebra

Definition 3.2.21. A quadrable $\mathtt{h} = (\mathcal{R}, X, \Gamma, \theta)$ is said to be a hyperbolic feed (weak hyperbolic feed) in $\mathcal{D}$, if

- $\mathcal{R}$ is a commutative sub-ring of $\mathcal{D}$.
- $X = (x_1, \ldots, x_n) \in \mathcal{D}^n$ such that the division ring of fractions of the ring $\mathcal{R}[X]$ is an $\mathcal{R}$-automorphic copy of $\mathcal{D}$.
- Γ as in the definition 3.2.1. satisfying the condition (3.2.7) (resp., to (3.2.10)).
- $\theta = (\theta_1, \ldots, \theta_n)$ is an n-tuple of commutative ring automorphisms of $\mathcal{R}$ satisfies

$$x_i^{\pm 1} r = \theta_i^{\pm 1}(r) x_i^{\pm 1}, \quad \forall i \in [1, n], \quad \forall r \in \mathcal{R}. \tag{3.2.24}$$

Definition 3.2.22. Mutations in hyperbolic feeds is defined in same way as in the case of hyperbolic seeds, with the obvious change by leaving the commutative ring $\mathcal{R}$ invariant.

One can notice that, if $\mathtt{h} = (F, X, \Gamma, \varphi)$ is a hyperbolic seed then taking $\mathcal{R} = \mathbb{Z}[\mathbb{P}]$ as defined in 3.2.1, we may have a hyperbolic feed with the same data of $\mathtt{i}$ if there is an $\mathcal{R}$-automorphisms $\{\theta_i\}_{i=1}^n$ satisfies equations (3.2.24).

Definition 3.2.23. ***The Hyperbolic Cluster Algebra.*** For a hyperbolic feed $\mathtt{h} = (\mathcal{R}, X, \Gamma, \theta)$ (resp., to a hyperbolic seed $\mathtt{i} = (F, X, \Gamma, \varphi)$), the hyperbolic cluster algebra

$\mathcal{H}(\mathtt{h})$ (resp., $\mathcal{H}(\mathtt{i})$) is defined to be the $\mathcal{R}$-subalgebra (resp., R-subalgebra, $R = \mathbb{Z}[\mathbb{P}]$, $\mathbb{P}$ is the free abelian group generated by elements of F) of $\mathcal{D}$ generated by the cluster set $\chi(\mathtt{h})$ (resp., to $\chi(\mathtt{i})$).

Remark 3.2.24. Let $\mathtt{h} = (\mathcal{R}, X, \Gamma, \theta)$ is a hyperbolic feed (resp., to $\mathtt{i} = (F, X, \Gamma, \varphi)$ is a hyperbolic seed) with $\theta = id_{\mathcal{R}}$ (resp., to $\varphi = id_{\mathcal{D}}$), relaxing the conditions (3.2.7) and (3.2.10). Then the hyperbolic cluster algebra $\mathcal{H}(\mathtt{h})$ (resp., to $\mathcal{H}(\mathtt{i})$) coincides with the geometric Fomin-Zelevinsky (commutative) cluster algebra associated to the seed $\mathtt{p} = (\widetilde{X}, \Gamma)$, where $\widetilde{X} = F \cup X$ and in this case $\mathcal{D}$ is a (commutative) field.

Theorem 3.2.25. *Let $\mathtt{h} = (\mathcal{R}, X, \Gamma, \theta)$ be a hyperbolic feed of rank n with $X = (x_1, \ldots, x_n)$ and $\xi_k = x'_k x_k$ (resp., $\xi_k = x_k \grave{x_k}$) then the following are true:*

1. *$\mathcal{R}(\mathtt{i}) := \mathcal{R}(\xi, \theta^{-1}, n)$ is a hyperbolic algebra of rank n (resp., to $\mathcal{R}(\xi, \theta, n)$).*
2. *$\mu_k^R(\mathtt{i})$ (resp., $\mu_k^L(\mathtt{i})$) is again a hyperbolic feed.*
3. *Right and left mutations on feeds define isomorphisms between hyperbolic algebras from part (1).*
4. *There is an isomorphism $\psi : \mathcal{R}(\mathtt{i}) \longrightarrow \mathcal{R}[x_1^{\pm}, \ldots, x_n^{\pm}]$, where $\mathcal{R}[x_1^{\pm}, \ldots, x_n^{\pm}]$ is the ring of Laurent polynomials in $x_1, \ldots, x_n$, with coefficients from $\mathcal{R}$. More precisely, every element z of $\mathcal{R}(\mathtt{i})$ can be written uniquely as linear combinations of cluster monomials of the initial cluster X.*
5. *Let $\mathcal{R} = \mathbb{Z}[\mathbb{P}]$, then we have*

$$\mathcal{H}(\mathtt{i}) = \mathcal{R}(\mathtt{i}). \tag{3.2.25}$$

Proof. To prove first part, since $N_{\mathtt{i}}(x_k) \subset \mathcal{R}$ for all $k \in [1, n]$, then $x'_k x_k = \xi_k \in \mathcal{R}$ (resp., to $x_k \grave{x_k}$). For $r \in \mathcal{R}$. Then we have, $x_k r = \theta_k(r) x_k, \forall k \in [1, n]$ since $\mathtt{i}$ is a hyperbolic feed.

Also, we have

$$x'_k r = \xi x_k^{-1} r = \xi \theta_k^{-1}(r) x_k^{-1} = \theta_k^{-1}(r) \xi x_i^{-1} = \theta_k^{-1}(r) x'_k.$$

Other commutation relations of the hyperbolic algebra structure are immediate from the commutation relations of the hyperbolic feeds and feed mutations, (left mutation case is quite similar).

Part (2), let $\mu_k^R(\mathtt{i}) = (R, X', \Gamma, \theta)$. By the definition of mutation on Γ it is easy to see that; since $N_{\mathtt{i}}(x_k) \subset \mathcal{R}$ for all $k \in [1, n]$ then $N_{\mu_k^R(\mathtt{i})}(x'_k) \subset \mathcal{R}$ for all $k \in [1, n]$, (resp., to left mutation on $\mathtt{i}$), which means $x'_k x_k = \xi_k \in \mathcal{R}$ (resp., to $x_k \grave{x_k}$). We have

$$x''_k r = \xi_k x_k \xi^{-1} r = \xi_k x_k r \xi^{-1} = \xi_k^{-1} \theta_k(r) x_k \xi = \theta_k(r) \xi_k x_k \xi^{-1} = \theta_k(r) x'',$$

and

$$x''^{-1}_k r = \xi_k x_k^{-1} \xi_k^{-1} r = \xi_k x_k^{-1} r \xi_k^{-1} = \theta_k^{-1}(r) \xi_k x_k^{-1} \xi_k^{-1} = \theta_k^{-1}(r) x''.$$

Also, since $N_{\mathtt{i}}(x_k) \cup N_{\mathtt{i}}(x_k) \cup \mathcal{R}$ is an empty set for every $i \in [1, n]$, then x_i commutes with ξ_k for $i \neq k$, and hence

$$x'_k x_i = \xi_k x_k^{-1} x_i = \xi_k x_i x_k^{-1} = x_i \xi_k x_k^{-1} = x_i x'_k, \forall i \in [1, n].$$

This finishes the proof of part (2).

To prove part (3), consider the $\mathcal{R}$-linear automorphism on $\mathcal{D}$, denoted by $T^R_{\mathtt{ii}',k} : \mathcal{D} \to \mathcal{D}$, and induced by, $x_k \mapsto x'_k$, and $x_i \mapsto x_i, \forall i \neq k \in [1, n]$. The restriction of this automorphism on $\mathcal{R}(\mathtt{i})$ induces the following algebra isomorphism $\widehat{T}^R_{\mathtt{ii}'} : \mathcal{R}(\mathtt{i}) \to \mathcal{R}(\mathtt{i}')$, given by $r \mapsto r, \forall r \in \mathcal{R}$, and $x_k \mapsto \xi x_k^{-1} = x'_k, \forall k \in [1, n]$. which implies $x'_k \mapsto \xi_k x_k \xi_k^{-1} = x''_k$. Finally, it is easy to see that the hyperbolic commutation relations (3.1.1) and (3.1.2) are invariant under $\widehat{T}^R_{\mathtt{ii}'}$.(the argument for $T^L_{\mathtt{ii}',k}$ and $\widehat{T}^L_{\mathtt{ii}'}$ is quite similar).

For part (4), By definition 3.1.1 and part (1) above, we have $\mathcal{R}(\mathtt{i})$ is generated by $\mathcal{R}$ and $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$, with relations (3.1.1) and (3.1.2) replacing y_i's with x'_i's. Let m

be any monomial in $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$, first part of relations (3.1.2) can be used to remove every possible sub monomials of the form $x_i x'_i$ and replace them with ξ_i. So, m can be written as a monomial from the following direct sum of non-commutative rings of polynomials

$$\mathcal{D} = \mathcal{R}^1[x_1] \oplus x_1^{-1}\mathcal{R}^1[x_1^{-1}] \oplus (\oplus_{i=2}^{n} x_i \mathcal{R}^i[x_i] \oplus x_i^{-1}\mathcal{R}^i[x_i^{-1}]), \tag{3.2.26}$$

where $\mathcal{R}^i = \mathcal{R}[x_1^{\pm} \ldots, x_{i-1}^{\pm}, x_{i+1}^{\pm} \ldots, x_n^{\pm}]$. The ring $\mathcal{D}$ inherits the multiplication and the relations of $\mathcal{R}(\mathtt{i})$. Finally, consider the map ψ that sends every monomial from $\mathcal{R}(\mathtt{i})$ to itself after applying the relations (3.1.1) whenever possible. One can see that ψ is an isomorphism.

For for part (5), we have $\mathcal{H}(\mathtt{i})$ is generated over $\mathcal{R}$ by the cluster elements of $\mathtt{i}$, and a random cluster element h can be written as $h = g_1 x_k^{\pm 1} g_2$, where g_1 and g_2 are elements of $\mathcal{R}$, thanks to the condition $N_{\mathtt{i}}(x_k) \subset \mathcal{R}$, and x_k is an initial cluster variable. Since, $g_2 \in \mathcal{R}$ then $g_2 = \sum_{j=1}^{j=t} n_j f_j^{\alpha_j}$, where $\alpha_j, n_j \in \mathbb{Z}$. Then

$$\begin{aligned} h &= g_1 x_k^{\pm 1} \sum_{j=1}^{j=t} n_j f_j^{\alpha_j} \\ &= g_1 \sum_{j=1}^{j=t} n_j (\theta(f_j))^{\pm \alpha_j} x_k^{\pm} \in \mathcal{R}(\mathtt{i}). \end{aligned}$$

Which means $\mathcal{H}(\mathtt{i}) \subseteq \mathcal{R}(\mathtt{i})$, and the other direction is obvious.

□

Corollary 3.2.26. *1. Every hyperbolic cluster algebra, comes from a hyperbolic feed (resp., to a hyperbolic seed) satisfies the conditions of theorem 3.2.26, is a hyperbolic algebra in the sense of definition 3.1.1.*

2. Deeper interpretations for for part (3) of theorem 3.2.26;

(a) every vertex of the cluster pattern of a hyperbolic feed $\mathtt{i}$ *gives raise to a hyperbolic*

algebra, the collection of all such hyperbolic algebras forms a scheme of hyperbolic algebras glued by the homomorphisms $\widehat{T}^R_{\mathtt{ii}'}$.

(b) From representation theory point of view, the homomorphism $\widehat{T}^R_{\mathtt{ii}'}$ *induces a functor on the category of representations of the hyperbolic algebra* $\mathcal{H}(\mathtt{i})$*, replacing the action of* x_i*'s by the action of* x'_i*'s. In the special case of Weyl algebra* A_n*,* $\widehat{T}^R_{\mathtt{ii}'}$ *as a functor on the category of representations of the weyl algebra* A_n*, exchanges the action of the deferential operators with the multiplication by the corresponding indeterminant.*

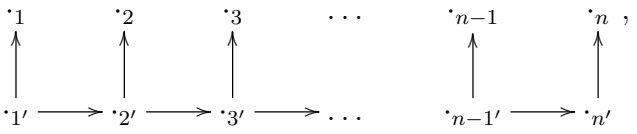
(c) $\widehat{T}^R_{\mathtt{ii}'}$ *coincides with the the canonical anti-automorphism mentioned in [30] page 62, which is plays an essential role in describing the representation theory of hyperbolic algebras.*

3. The hyperbolic algebra of rank 1, $\mathcal{R}(\mathtt{i})$ *is isomorphic to* $\mathcal{R}[x] \oplus x^{-1}\mathcal{R}[x^{-1}]$.

3.3 Examples

3.3.1 The Weyl algebra

Example 3.3.1. Let A_n be the Weyl algebra of $2n$ variables $x_1, \ldots, x_n, y_1 \ldots, y_n$ satisfying the relations (3.1.3). Consider the following seed $\mathtt{i} = (F, Y, \Gamma)$, where $Y = (y_1, \ldots, y_n)$ with $F = \{\xi_i |\ \xi_i = y_i x_i,\ 1 \leq i \leq n\}$ is the set of frozen variables, with $F_i = \{\xi_i\}$ and Γ is the graph



(3.3.1)

where $[1, n]$ corresponds to the elements of Y, and $[1', n']$ corresponds to the elements of F. Let $\mathbb{P}$ be the free abelian group generated by the elements of F and written multiplicatively. The hyperbolic cluster algebra $\mathcal{H}(\mathtt{i})$ corresponding to A_n is the $\mathbb{Z}[\mathbb{P}]$-subalgebra of the

division ring of fractions $\mathcal{D}$ of the ring $K[\mathbb{P}][Y]$, (we could take $\mathcal{D}$ to be the field of fractions of A_n) generated by the cluster set of the hyperbolic seed $\mathtt{i} = (F, Y, \Gamma, \varphi)$, where φ is a $\mathbb{Z}[\mathbb{P}]$-linear automorphism of $\mathcal{D}$ induced by $\varphi(x_i) = y_i x_i y_i^{-1}$ and $\varphi(y_i) = \xi_i y_i \xi_i^{-1}$. One can see that φ is an infinite order automorphism satisfying (3.2.8) and (3.2.9), and $\mathtt{i}$ is not a well-connected seed. In the following we will see that its cluster set is infinite.

A hyperbolic feed associated to A_n.

Consider the following assignments; $\mathcal{R} := K[\xi_1, \ldots, \xi_n]$ and $\theta = (\theta_1, \ldots, \theta_n)$ where

$$\theta_i : \mathcal{R} \to \mathcal{R} \text{ given by } \theta_i(\xi_j) = \begin{cases} \xi_i + 1, & \text{if } i = j, \\ \xi_j, & \text{if } i \neq j. \end{cases}.$$

These choices make $\mathtt{h} = (\mathcal{R}, Y, \Gamma, \theta)$ a hyperbolic feed in $\mathcal{D}$. This hyperbolic feed satisfies all the conditions of theorem 3.2.26, then the mutation on the feed $\mathtt{h}$ provides us with an infinite class of Weyl algebras connected (glued) by algebra homomorphisms induced by mutations.

The cluster set of the seed $\mathtt{i}$, $\chi(\mathtt{i})$: Since $\mu^R_{\mathtt{i}k}(y_k) = (\xi_k + 1)y_k^{-1} = x_k, k \in [1, n]$, we have

$$\begin{aligned} (y_1, \ldots, y_{k-1}, y_k, y_{k+1} \ldots, y_n) &\overset{\mu^R_k}{\Rightarrow} (y_1, \ldots, y_{k-1}, (\xi_k + 1)y_k^{-1}, y_{k+1} \ldots, y_n) \\ &= (y_1, \ldots, y_{k-1}, x_k, y_{k+1} \ldots, y_n) \\ &\overset{\mu^L_k}{\Rightarrow} (y_1, \ldots, y_{k-1}, y_k, y_{k+1} \ldots, y_n). \end{aligned}$$

Then, the right mutations in the directions $1, 2, \ldots, n$ cover all the generators of the Weyl algebra A_n, which means the fact

$$A_n \hookrightarrow \mathcal{H}(\mathtt{h}) = \mathcal{R}(\mathtt{h}). \tag{3.3.2}$$

Notice: The same phenomenon occurs if we start with $\mathtt{i} = (F, X, \Gamma)$ with $X = (x_1, \ldots, x_n)$ and apply left mutations rather than right mutations.

Yet, the mixed sequence mutations $\mu^L_k \mu^R_k$ and $\mu^R_k \mu^L_k$ act like identity on every seed (feed)

in the cluster pattern of $\mathtt{h}$, but the cluster set of $\mathtt{i}$ is an infinite set because the unmixed mutations sequences never reproduce the seed (feed) $\mathtt{i}$ again, as we will see in the following

$$
\begin{aligned}
(y_1,\ldots,y_{k-1},y_k,y_{k+1}\ldots,y_n) &\overset{\mu_k^L}{\Rightarrow} (y_1,\ldots,y_{k-1},y_k^{-1}(\xi_k+1),y_{k+1}\ldots,y_n)\\
&\overset{\mu_k^L}{\Rightarrow} (y_1,\ldots,y_{k-1},(\xi_k+1)^{-1}y_k(\xi_k+1),y_{k+1}\ldots,y_n)\\
&\overset{\mu_k^L}{\Rightarrow} (y_1,\ldots,y_{k-1},(\xi_k+1)^{-1}y_k^{-1}(\xi_k+1)^2,y_{k+1}\ldots,y_n)\\
&\overset{\mu_k^L}{\Rightarrow} (y_1,\ldots,y_{k-1},(\xi_k+1)^{-2}y_k(\xi_k+1)^{+2},y_{k+1}\ldots,y_n)\\
&\ldots\\
&\overset{\mu_k^L}{\Rightarrow} (y_1,\ldots,y_{k-1},(\xi_k+1)^{-j}y_k^{-1}(\xi_k+1)^{j+1},y_{k+1}\ldots,y_n)\\
&\overset{\mu_k^L}{\Rightarrow} (y_1,\ldots,y_{k-1},(\xi_k+1)^{-(j+1)}y_k(\xi_k+1)^{+j+1},y_{k+1}\ldots,y_n)\\
&\ldots\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
(x_1,\ldots,x_{k-1},x_k,x_{k+1}\ldots,x_n) &\overset{\mu_k^R}{\Rightarrow} (x_1,\ldots,x_{k-1},(\xi_k+1)x_k^{-1},x_{k+1}\ldots,x_n)\\
&\overset{\mu_k^R}{\Rightarrow} (x_1,\ldots,x_{k-1},(\xi_k+1)x_k(\xi_k+1)^{-1},x_{k+1}\ldots,x_n)\\
&\overset{\mu_k^R}{\Rightarrow} (x_1,\ldots,x_{k-1},(\xi_k+1)^2x_k^{-1}(\xi_k+1)^{-1},x_{k+1}\ldots,x_n)\\
&\overset{\mu_k^R}{\Rightarrow} (x_1,\ldots,x_{k-1},(\xi_k+1)^{+2}x_k(\xi_k+1)^{-2},x_{k+1}\ldots,x_n)\\
&\ldots\\
&\overset{\mu_k^R}{\Rightarrow} (x_1,\ldots,x_{k-1},(\xi_k+1)^{j+1}x_k^{-1}(\xi_k+1)^{-j},x_{k+1}\ldots,x_n)\\
&\overset{\mu_k^R}{\Rightarrow} (x_1,\ldots,x_{k-1},(\xi_k+1)^{j+1}x_k(\xi_k+1)^{-(j+1)},x_{k+1}\ldots,x_n)\\
&\ldots\ .
\end{aligned}
$$

Also, since $N_{\mathtt{i}}(x_k)$ does not contain any non frozen initial cluster variable, for every $k\in[1,n]$, one can see that $\mu_k^R\mu_i^R$ and $\mu_i^R\mu_k^R$ (resp., to $\mu_k^L\mu_i^L$ and $\mu_i^L\mu_k^L$) act in the same way on any seed (feed) in the cluster pattern of $\mathtt{i}$. Therefor the cluster set of $\mathtt{i}$ can be restricted

only to the following elements

$$\begin{aligned}
\chi(\mathtt{i}) &= \{y_1,\dots,y_n,(\xi_k+1)^{-j-1}y_k^{-1}(\xi_k+1)^{+j},(\xi_k+1)^{-j}y_k(\xi_k+1)^{+j};j\in\mathbb{N},k\in[1,n]\}\\
&\bigcup \{(\xi_k+1)^{j+1}y_k^{-1}(\xi_k+1)^{-j},(\xi_k+1)^{+j}y_k(\xi_k+1)^{-j};j\in\mathbb{N},k\in[1,n]\}\\
&= \{x_1,\dots,x_n,(\xi_k+1)^{j+1}x_k^{-1}(\xi_k+1)^{-j},(\xi_k+1)^{+j}x_k(\xi_k+1)^{-j};j\in\mathbb{N},k\in[1,n]\}\\
&\bigcup \{(\xi_k+1)^{-j}x_k^{-1}(\xi_k+1)^{j+1},(\xi_k+1)^{-j}x_k(\xi_k+1)^{+j};j\in\mathbb{N},k\in[1,n]\}
\end{aligned}$$

The cluster patterns of the first Weyl algebra A_1.

Let $A_1 = K < x,y > /(xy-\xi-1)$, where $\xi = yx$, consider the following seed $\mathtt{i} = (\mathcal{R}, y, \cdot_1 \longrightarrow \cdot_{1'})$, where $\mathcal{R} = \mathbb{Z}[\mathbb{P}]$, $\mathbb{P} = \{y^n; n\in\mathbb{Z}\}$ the free group generated by y, written multiplicatively . We have the following cluster patterns for this case

- $\mathbb{T}(\mathtt{i})$

$$\dots \underset{L}{\overset{R}{\longleftrightarrow}} \overset{y_{-3}}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \overset{y_{-2}}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \overset{y_{-1}}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \overset{y=y_0}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \overset{y_1}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \overset{y_2}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \overset{y_3}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \dots \tag{3.3.3}$$

(here $\underset{L}{\longleftarrow}\cdot$ is left mutation and $\cdot\overset{R}{\longrightarrow}$ is right mutation). Which can be encoded by the following equations

$$y_{k+1}y_k = y_ky_{k+1}+1, \quad \text{for } \ k\in 2\mathbb{Z}, \tag{3.3.4}$$

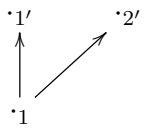
$$y_ky_{k+1} = y_{k+1}y_k+1, \quad \text{for } \ k\in 2\mathbb{Z}+1. \tag{3.3.5}$$

These equations are equivalent to say that, each arrow from the cluster pattern corresponds to a copy of the first Weyl algebra, denoted by $A_1^k = K\langle y_k, y_{k+1}\rangle, k\in\mathbb{Z}$ and mutations define algebra maps between these Weyl algebras, given by $T_k : A_1^k \to A_1^{k+1}$, $\ y_k \mapsto y_{k+1}$ for $k\in\mathbb{Z}_{\geq 0}$, and $T_k : A_1^k \to A_1^{k+1}$, $\ y_k \mapsto y_{k-1}$ for $k\in\mathbb{Z}_{<0}$.

Remark 3.3.2. Fomin-Zelevinsky finite type classification [16] does not work in this case. In the case of the first Weyl algebra $A_1 = K < x,y > /(xy-yx=1)$ with the seed $\mathtt{i} = (\xi = yx, y, \cdot_1 \longrightarrow \cdot_{1'})$ here $\mathtt{i}$ is of A_1-type as a cluster algebra based on Fomin-Zelevinsky finite type classification however $\chi(\mathtt{i})$ is an infinite set, which means Fomin-Zelevinsky finite type classification does not work in this case.

3.3.2 The coordinate algebra of $SL_q(2,k)$.

Recall definition 3.1.1. Consider the following hyperbolic feed $\mathtt{h} = (\mathcal{R}, x, \Gamma, \theta)$, where $\mathcal{R} = K[u,v]$ and $\theta : \mathcal{R} \to \mathcal{R}$ given by $\theta(f(u,v)) = f(qu, qv)$ and Γ is given by



(3.3.6)

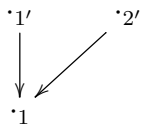
Consider the set of frozen variables F given by $F = \{qu, v\}$. In this case we deform the mutation (right and left) as follows, mutation of q is q^{-1}, i.e., for example $\mu_1(quv+1) = q^{-1}uv + 1$.

Remark that; $\mathtt{h}$ satisfies the conditions of theorem 3.2.26 and the conditions of the well-connected seeds.

One can see the right mutation on $\mathtt{h}$ will produce the following new feed $\mathtt{h}' = (\mathcal{R}, y, \Gamma', \theta)$, since

$$\mu_1^R(x) = (q^{-1}uv + 1)x^{-1} = \xi x^{-1} = y. \tag{3.3.7}$$

and Γ' is as follows



(3.3.8)

Applying left mutation on $\mathtt{h}'$ produces the original seed $\mathtt{h}$. Also, we have,

$$A(SL_q(2,k)) \hookrightarrow \mathcal{R}(\mathtt{h}) = \mathcal{H}(\mathtt{h}) \tag{3.3.9}$$

The cluster set of $\mathtt{h}$: Let $\zeta = quv + 1$. We have

$$\chi(\mathtt{h}) = \{x, \zeta^j x \xi^{-j}, \xi^{j+1} x^{-1} \zeta^{-j-1}, j \in \mathbb{N}\} \bigcup \{y, \xi^j y \zeta^{-j}, \zeta^{j+1} y^{-1} \xi^{-j-1}, j \in \mathbb{N}\}.$$

3.4 Irreducible representation arising from the *cluster graphs* of Weyl algebras.

In the following we introduce a family of indecomposable and irreducible representations for the Weyl algebra A_n arising from its cluster pattern. We speculate the same representations can be introduced for any hyperbolic algebra with a hyperbolic cluster structure.

Definition 3.4.1. Let $\mathtt{h} = (F, Y, \Gamma, \theta)$ be the hyperbolic feed of rank n, introduced in example 3.3.1. In addition to the n-th Weyl Algebra A_n, we have the following algebras related to $\mathtt{h}$.

- The Hyperbolic cluster algebra $\mathcal{H}(\mathtt{h})$.
- The algebra $\mathfrak{B} = K(y_1, \ldots, y_n)[y'_1, \ldots, y'_n]$, the algebra of polynomials in the first generation cluster variables $\mu_i^R(y_i) = y'_i, i \in [1, n]$, (resp., to $\mu_i^R(y_i) = y'_i, i \in [1, n]$), with coefficients from the field of fractions of the initial cluster variables.
- The algebra $B = K(\xi_1, \ldots, \xi_n)[\chi(\mathtt{h})]$ the algebra of polynomials in the cluster set of $\mathtt{h}$ with coefficients from the field of fractions of the frozen variables.

Remark 3.4.2. We have the following inclusions

$$\mathfrak{B} \hookleftarrow A_n \hookrightarrow \mathcal{H}(\mathtt{h}) \hookrightarrow B, \tag{3.4.1}$$

i.e., The hyperbolic cluster algebra is an intermediate algebra between the n-th Weyl Algebra and the algebra B.

Motivations: *The representation theory of the the three algebras A_n and the algebras $\mathfrak{B}$, and B are closely related, see for example [3].*

Definition 3.4.3. Space of Representations V_n. Let $\mathtt{h} = (F, Y, \Gamma, \theta)$ be a hyperbolic feed (resp., to seed) of rank n. A *cluster monomial* of $\mathtt{h}$ is a monomial formed from cluster

elements that are showing up (at least once) as cluster variables in some feed in the cluster pattern of $\mathtt{h}$. To visualize it, the monomial $m = z_1^{\beta_1} \cdots z_n^{\beta_n}, \beta_i \in \mathbb{Z}_{\geq 0}, i \in [1, n]$ is a cluster monomial if $(z_1, \ldots, z_n)$ is the cluster of some feed in the cluster pattern of $\mathtt{h}$. In case of $\beta_i \in \mathbb{Z}_{>0}, \forall i \in [1, n]$, m is called a *full cluster monomial*.

The space of representations V_n is defined to be the $K(\xi_1, \ldots, \xi_n)$-left span by the set of all cluster monomials of $\mathtt{h}$.

Lemma 3.4.4. *For any hyperbolic feed* $\mathtt{h}$ *(resp., to hyperbolic seed), the space of representations* V_n *is independent of* $\mathtt{h}$*, and depends only on* $\mathbb{T}(\mathtt{h})$ *the cluster pattern of* $\mathtt{h}$*.*

Proof. The statement of the lemma is equivalent to say that any two hyperbolic feeds (seeds) in the cluster pattern of $\mathtt{h}$ have the same cluster pattern. To see this fact; let $\mathtt{f}$ be any hyperbolic feed in $\mathbb{T}(\mathtt{h})$. Then, $\mathtt{f}$ can be obtained from $\mathtt{h}$ by applying some sequence of mutations, without lose of generality we may assume it is a sequence of right mutations only say $\mu_{i_1}^R \ldots \mu_{i_t}^R$. But part 3 of lemma 3.2.6 tells us that we can obtain $\mathtt{h}$ from $\mathtt{f}$ by applying the (same length) sequence of left mutations $\mu_{i_t}^L \ldots \mu_{i_1}^L$ which finishes the proof. However, we may realize this fact by recalling that any two vertices in the cluster pattern are connected by two oppositely directed pathes. □

Remark 3.4.5. In the case of $\mathtt{h}$ is the hyperbolic feed associated to the Weyl algebra or the coordinate algebra of $SL_q(2, K)$, the situation is easier since in this case a cluster monomial is any monomial formed from any set of cluster elements. In order to see this fact we need to recall the following two, easy to prove, combinatorial proposition.

Proposition 3.4.6. *If* $\mathtt{h}$ *is the hyperbolic feed associated to the Weyl algebra or the coordinate algebra of* $SL_q(2, K)$*, then the following are true*

1. *For any set of* n *(or less) different cluster elements, not including two elements produced from the same initial cluster variable, there is at least one seed in the cluster pattern of* $\mathtt{h}$ *which contains all of them.*

2. *For z_1 and z_2 any two cluster elements produced from the same initial cluster variable, then we have two cases for their product;*

   - *if z_2 can be obtained from z_1 by applying sequence of mutations of odd length, then $z_1 z_2 \in K(\xi_1, \ldots, \xi_n)$.*
   - *if z_2 can be obtained from z_1 by applying sequence of mutations of even length, then $z_1 z_2$ can be written as $g z_1^2$, for some $g \in K(\xi_1, \ldots, \xi_n)$.*

A left action of the algebras A_n, $\mathfrak{B}$, and B on V_n.

Consider the following notations; let $Y = (y_1, \ldots, y_n)$ be the initial cluster, for $t \in \mathbb{Z}_{\geq 0}$, $y_{i,t}$ denotes the cluster element obtained from the initial cluster variable y_i by applying one of the following sequence of mutations $(\mu_i^R)^t$ if $t \geq 0$ or $(\mu_i^L)^t$ if $t < 0$.

Using the above notation, a typical element of V_n can be written as a sum of monomials like the following monomial

$$v = f(\xi_1, \ldots, \xi_n) y_{1,m_1}^{\beta_1} \cdots y_{n,m_n}^{\beta_n}, \tag{3.4.2}$$

where $f(\xi_1, \ldots, \xi_n) \in R$, and $(\beta_1, \ldots, \beta_n) \in \mathbb{Z}_{\geq 0}^n$, and $(m_1, \ldots, m_n) \in \mathbb{Z}^n$.
A left action on the general term v is defined as follows

$$y_i(v) = f(\xi_1, \ldots, \xi_{i-1}, \theta_i^{-1}(\xi_i), \ldots, \xi_n) y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i-1}^{\beta_i} y_{i+1,m_{i+1}}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n}. \tag{3.4.3}$$

$$x_i(v) = \theta_i(\xi_i) f(\xi_1, \ldots, \xi_{i-1}, \theta_i(\xi_i), \ldots, \xi_n) y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i+1}^{\beta_i} y_{i+1,m_{i+1}}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n}. \tag{3.4.4}$$

Lemma 3.4.7. *1. The action of x_i and y_i is invertible. In particular, the action is compatible with mutations, and hence is defined for all cluster elements, and the action of x_i can be recovered from the action of y_i, for every i.*

2. *V_n is a left A_n, $\mathfrak{B}$, and B module with the action induced by the action of the initial clusters y_i, defined above.*

Proof. 1. If y_i^{-1} and x_i^{-1} denote the inverses of y_i and x_i respectively, then we have y_i^{-1} acts like $\theta^{-1}(\xi_i)x_i$, and x_i^{-1} acts like $\xi_i^{-1}y_i$. For the rest of the statement is an immediate calculations recalling that $\mu_i(y_i) = (1+\xi)y_i^{-1} = x_i$.

2. The action is consistent with relations (3.1.3). One can see $(x_iy_i - y_ix_i)(v)$ can be written as follows

$$\begin{aligned}
&= x_i(f(\xi_1,\ldots,\xi_{i-1},\theta_i^{-1}(\xi_i),\xi_{i+1},\ldots,\xi_n)y_{1,m_1}^{\beta_1}\cdots y_{i-1,m_{i-1}}^{\beta_{i-1}}y_{i,m_i+1}^{\beta_i}y_{i+1,m_i-1}^{\beta_{i+1}}\cdots y_{n,m_n}^{\beta_n})\\
&- y_i(\theta_i(\xi_i)f(\xi_1,\ldots,\xi_{i-1},\theta_i(\xi_i),\xi_{i+1},\ldots,\xi_n)y_{1,m_1}^{\beta_1}\cdots y_{i-1,m_{i-1}}^{\beta_{i-1}}y_{i,m_i+1}^{\beta_i}y_{i+1,m_i+1}^{\beta_{i+1}}\cdots y_{n,m_n}^{\beta_n})\\
&= \theta_i(\xi)f(\xi_1,\ldots,\xi_{i-1},\theta_i^{-1}(\theta(\xi_i)),\xi_{i+1},\ldots,\xi_n)y_{1,m_1}^{\beta_1}\cdots y_{i-1,m_{i-1}}^{\beta_{i-1}}y_{i,m_i}^{\beta_i}y_{i+1,m_i+1}^{\beta_{i+1}}\cdots y_{n,m_n}^{\beta_n})\\
&- \theta_i(\theta_i^{-1}(\xi_i))f(\xi_1,\ldots,\xi_{i-1},\theta_i(\theta^{-1}(\xi_i)),\ldots,\xi_n)y_{1,m_1}^{\beta_1}\cdots y_{i-1,m_{i-1}}^{\beta_{i-1}}y_{i,m_i}^{\beta_i}y_{i+1,m_i+1}^{\beta_{i+1}}\cdots y_{n,m_n}^{\beta_n})\\
&= (\theta_i(\xi_i)-\xi_i)v\\
&= v.
\end{aligned}$$

In a similar way, one gets $(x_iy_j - y_jx_i)(v) = 0$, for $i \neq j$.

□

Example 3.4.8. Consider the hyperbolic feed (seed) `i` introduced in example 3.1.1. The i-th branch of the cluster pattern $\mathbb{T}$(`i`) is as follows;

$$\cdot \longleftrightarrow \overset{(y_{1,m_1},\ldots,y_{i,m_i-1},\cdots,y_{n,m_n})}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \overset{(y_{1,m_1},\ldots,y_{i,m_i},\cdots,y_{n,m_n})}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \overset{(y_{1,m_1},\ldots,y_{i,m_i+1},\cdots,y_{n,m_n})}{\cdot} \longleftrightarrow \cdot. \tag{3.4.5}$$

Here, right mutations go to the right direction and left go to left. For sake of simplicity, we skipped labeling each vertex by the whole seed data and kept only the cluster variables, since the other data are all invariant under right and left mutations in the i-the direction. In this case, V_n is the left $K(\xi_1,\ldots,\xi_n)$-linear span generated by the following set

$$\{y_{1,m_1}^{\beta_1}\cdots y_{n,m_n}^{\beta_n} | \text{ for } m=(m_1,\ldots,m_n)\in\mathbb{Z}^n, \text{and } \beta=(\beta_1,\ldots,\beta_n)\in\mathbb{Z}_{\geq 0}^n\}. \tag{3.4.6}$$

3.4.1 Cluster strings and the string submodules of V_n.

Before introducing the cluster strings we need to develop some notations. For $b \in \mathbb{Z}$, we have

$$\theta^b(-) = \begin{cases} \overbrace{\theta(\theta(\dots\theta(-)))}^{b-times}, & \text{if } b > 0, \\ id_R, & \text{if } b = 0, \\ \overbrace{\theta^{-1}(\theta^{-1}(\dots\theta^{-1}(-)))}^{b-times}, & \text{if } b < 0. \end{cases}$$

Consider the following two sets of monomials in elements from the set $\{\theta^b(\xi); b \in \mathbb{Z}\}$

$$M^+(\xi) := \{1, \theta^t(\xi^{\pm 1})\theta^{t+1}(\xi^{\pm 1})\cdots\theta^{t+q}(\xi^{\pm 1}) | q, t \in \mathbb{Z}_{\geq 0}\},$$

$$M^-(\xi) := \{1, \theta^t(\xi^{\pm 1})\theta^{t+1}(\xi^{\pm 1})\cdots\theta^{t+q}(\xi^{\pm 1}) | q + t, t \in \mathbb{Z}_{\leq 0}\}.$$

Now we are ready to introduce one more set of monomials, $M(\xi)$

$$M(\xi) := \{m_1 m_2 | m_1 \in M^+(\xi) \text{ and } m_2 \in M^-(\xi)\}. \tag{3.4.7}$$

Let R be any one of the following rings $K[\xi_1, \dots, \xi_n], K(\xi_1, \dots, \xi_n)$ or $K[\mathbb{P}]$, and A be any of the algebras A_n, $\mathfrak{B}$ or B. Let $E = \{\xi_1^{\pm 1}, \dots, \xi_n^{\pm 1}\}$. The set of all monomials formed form the elements of E is denoted by $\mathcal{M}(E)$.

Fix a natural number $l \in \mathbb{N}$ and a 1-1 map $\sigma : [1, l] \to \mathbb{Z}_{\geq 0}^n \times \mathbb{Z}^n$. Let $\beta = (\beta_1, \dots, \beta_l) \in (\mathbb{Z}^n)^l$, and $m = (m_1, \dots, m_l) \in (\mathbb{Z}_{\geq 0}^n)^l$ be such that $\sigma(j) = (\sigma_1(j), \sigma_2(j)) = (\beta_j, m_j)$, where $\sigma_1(j) = \beta_j = (\beta_{j1}, \dots, \beta_{jn})$ and $\sigma_2(j) = m_j = (m_{j1}, \dots, m_{jn}), j \in [1, l]$.
For $t = (t_1, \dots, t_n) \in \mathbb{Z}^n$ and $h \in R$, we introduce one more important subset of R

$$\mathfrak{a}(t, h) := \{e\alpha_1 \cdots \alpha_n h(\theta_1^{t_1}(\xi_1), \dots, \theta_n^{t_n}(\xi_n)) | \ \alpha_i \in M(\xi_i), e \in \mathcal{M}(E), i \in [1, n]\}. \tag{3.4.8}$$

Definition 3.4.9. ***Cluster strings*** **of base** l**.** Every non-negative integer l, $f = (f_1, \dots, f_l) \in R^l$, and a $1-1$ map $\sigma : [1, l] \to \mathbb{Z}_{\geq 0}^n \times \mathbb{Z}^n$ corresponds to a *cluster string*, defined as follows

$$\mathrm{S}_l(\sigma, f) := \{\sum_{j=1}^{l} g_j y_{1,m_{j1}+t_{j1}}^{\beta_{1j}} \cdots y_{n,m_{jn}+t_{jn}}^{\beta_{nj}} | \ t_j = (t_{j1}, \dots, t_{jn}) \in \mathbb{Z}^n, g_j \in \mathfrak{a}(t_j, f_j), j \in [1, l]\}. \tag{3.4.9}$$

Example 3.4.10. A cluster string of base 2. Let $l = 2$, $\sigma_1(1) = (1,0), \sigma_1(2) = (1,2)$, $\sigma_2(1) = (1,1), \sigma_2(2) = (0,1)$, and $f = (\xi_1^2 + \xi_2, \xi_2\xi_1)$, we have for $t = (t_1, t_2)$

$$\mathfrak{a}(t, \xi_1^2 + \xi_2^{t_2}) = \{e\alpha_1\alpha_2(\theta_1^{2t_1}(\xi_1) + \theta(\xi_2))|\ \alpha_i \in M(\xi_i), e \in \mathcal{M}(E), i \in [1,2]\},$$

and

$$\mathfrak{a}(t, \xi_1\xi_2) = \{e\alpha_1\alpha_2\theta_1^{t_1}(\xi_1)\theta^{t_2}(\xi_2)|\ \alpha_i \in M(\xi_i), e \in \mathcal{M}(E), i \in [1,2]\}.$$

With the above data we have

$$S_2(\sigma, f) = \{g_1 y_{1,1+t_{11}} + g_2 y_{1,0+t_{21}} y_{2,1+t_{22}}^2 | g_1 \in \mathfrak{a}(t, \xi_1^2 + \xi_2^{t_2}), g_2 \in \mathfrak{a}(t, \xi_1\xi_2), t_{ij} \in \mathbb{Z}, i, j \in [1,2]\}.$$

Definition 3.4.11. Let $\mathrm{S}_l(\sigma, f)$ be a cluster string. The sub module of V_n generated by $\mathrm{S}_l(\sigma, f)$ is called a string submodule of base l associated to $\mathrm{S}_l(\sigma, f)$. This submodule is denoted by $W_l(\sigma, f)$ and called a string submodule if there is no possibility of confusion.

Remark 3.4.12. Each element of V_n gives rise to a cluster string and hence a submodule of V_n. To see that; every element v of V_n can be written as follows

$$f_1(\xi_1, \ldots, \xi_n) y_{1,m_{11}}^{\beta_{11}} \cdots y_{n,m_{1n}}^{\beta_{n1}} + \ldots + f_l(\xi_1, \ldots, \xi_n) y_{1,m_{l1}}^{\beta_{1l}} \cdots y_{n,m_{ln}}^{\beta_{nl}}.$$

Where $f_1, \ldots, f_l$ are elements of R, and a $1-1$ map $\sigma : [1,l] \to Z_{\geq 0}^n \times Z^n$ can be defined such that $\sigma(j) = (\sigma_1(j), \sigma_2(j))$, where $\sigma_1(j) = (\beta_{1j}, \ldots, \beta_{nj})$ and $\sigma_2(j) = (m_{1j}, \ldots, m_{nj}), j \in [1,l]$. Consider the cluster string $S_l(\sigma, f)$, with $f = (f_1, \ldots, f_l)$. This cluster string is denoted by $S(v)$ and the submodule of V_n generated by $S(v)$ denoted by $W(v)$.

The following lemma provides some basic properties of the cluster strings.

Lemma 3.4.13. *1. The cluster strings are invariant under the action of every monomial formed from elements of the set $\mathcal{E} = E \cup \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, and we can recover*

any cluster string from any of its element. In particular for any cluster string $S_l(f,\sigma)$ *and for any* $v \in S_l(f,\sigma)$

$$\mathcal{M}(\mathcal{E})v = \{\sum_{j=1}^{l} g_j y_{1,m_{j1}+t_1}^{\beta_{1j}} \cdots y_{n,m_{jn}+t_n}^{\beta_{nj}} |\ t=(t_1,\ldots,t_n) \in \mathbb{Z}^n, g_j \in \mathfrak{a}(t,f_j)\} \subset S_l(f,\sigma). \tag{3.4.10}$$

Where $\mathcal{M}(\mathcal{E})$ *is the set of all monomials formed from the elements of the set* $\mathcal{E}$*. Hence, for any string submodule* $W_l(f,\sigma)$ *we have*

$$W_l(f,\sigma) = \sum \textit{copies of } S_l(f,\sigma). \tag{3.4.11}$$

2. $l \neq l'$*, then* $\mathrm{S}_l(\sigma,f) \neq \mathrm{S}_{l'}(\sigma',f)$*.*

3. *For* $\sigma = \sigma'$*, then* $\mathrm{S}_l(\sigma,g) = \mathrm{S}_l(\sigma,f)$*, if and only if* $g \in \mathfrak{a}_m(t,f)$*, for some* $t \in \mathbb{Z}^n$*.*

4. *Let* $g=(g_1,\ldots,g_n)$ *with* $g_i \in \mathfrak{a}_m(f,t_i)$ *for some* $t_i \in \mathbb{Z}^n$*. Then* $\mathrm{S}_l(\sigma',g) = \mathrm{S}_l(\sigma,f)$ *if and only if* $\sigma'(j) = \sigma(j) + (0,q_j) \forall j \in [1,l]$ *for some* $q_j \in \mathbb{Z}^l$*.*

Proof. 1. To see that the action of any element of $\mathcal{M}(\mathcal{E})$ on any element of $S_l(\sigma,f)$ is again an element of $S_l(\sigma,f)$.

We have x_i sends $f_j(\xi_1,\ldots,\xi_n)y_{1,m_{11}}^{\beta_{1j}} \cdots y_{n,m_{n1}}^{\beta_{nj}}$ to

$$\theta(\xi_i)(f_j(\xi_1,\ldots,\xi_{i-1},\theta_i(\xi),\xi_{i+1},\ldots,\xi_n)y_{1,m_{j1}}^{\beta_{j1}} \cdots y_{i,m_{ji-1}}^{\beta_{ji-1}} y_{i,(m_{ji})+1}^{\beta_{ji}} y_{i,m_{ji+1}}^{\beta_{ji+1}} \cdots y_{n,m_{jn}}^{\beta_{jn}}.$$

While y_i sends it to

$$f_j(\xi_1,\ldots,\xi_{i-1},\theta_i^{-1}(\xi),\xi_{i+1},\ldots,\xi_n)y_{1,m_{j1}}^{\beta_{j1}} \cdots y_{i,m_{ji-1}}^{\beta_{ji-1}} y_{i,(m_{ji})-1}^{\beta_{ji}} y_{i,m_{ji+1}}^{\beta_{ji+1}} \cdots y_{n,m_{jn}}^{\beta_{jn}}.$$

Which means non of the base l, f nor the map σ are changed under the action of $x_1,\ldots,x_n$ or $y_1,\ldots,y_n$. Then they keep every cluster string invariant, and hence same for every monomial formed from the set $\mathcal{E}$. The same change will occur in each term of the l-terms of every element of $S_l(f,\sigma)$, which justify (3.4.10).

Now, a random element of A is a linear combination of elements of $\mathcal{M}(\mathcal{E})$ with coefficients from the filed K. Remark that in the case of $A = \mathfrak{B}$, the inverses of x_i and y_i still keep the cluster strings invariant, we refer to part one of lemma 3.4.7. Therefore, any element of A will send any element of $S_l(\sigma, f)$ into a sum of elements each of them is an element of $S_l(f, \sigma)$, which proves that $W_l(\sigma, f)$ is entirely included in a sum of copies of $S_l(\sigma, f)$ and obviously any sum of copies of $S_l(\sigma, f)$ is included in $W_l(\sigma, f)$ which finishes the proof of (3.4.11).

2. This is immediate if we recall that the maps σ and σ' are $1 - 1$.

3. ($\Rightarrow$) Obvious.

   ($\Leftarrow$) if $g \in \mathfrak{a}(t, f)$ for some $t = (t_1, \ldots, t_n) \in \mathbb{Z}^n$, then there are $\alpha_i \in M(\xi_i)$, and $e \in \mathcal{M}(E)$ such that $g = e\alpha_1 \cdots \alpha_n f(\theta_1^{t_1}(\xi_1), \ldots, \theta_n^{t_n}(\xi_n))$. Remarking that elements of $M(\xi)$ are invertible for any choice of R, then $S_l(f, \sigma) \subseteq S_l(g, \sigma)$, and the other inclusion is direct.

4. ($\Rightarrow$). It is easy to see that, if $\sigma'_1(j) = \sigma_1(j), \forall j \in [1, l]$, then $\sigma' = \sigma + (0, q_j)$ for some $q_j \in \mathbb{Z}^n$.

   Assume that $\sigma'(j_0) \neq \sigma(j_0) + (0, q_j)$ for some $j_0 \in [1, l]$ and for every $q_j \in \mathbb{Z}^n$. Then $\sigma'_1(j) \neq \sigma_1(j)$. Then, the element $\sum_{j=1}^{l} g_j y_{1,m_{j1}+t_{j1}}^{\beta_{1j}} \cdots y_{n,m_{jn}+t_{jn}}^{\beta_{nj}}$, with $\sigma'_1(j) = (\beta_{1j}, \ldots, \beta_{nj})$, is an element of $\mathrm{S}_l(\sigma', g)$ but is not an element of $\mathrm{S}_l(\sigma, f)$.

   ($\Leftarrow$) Immediate.

□

Lemma 3.4.14. *For the cluster strings $S_l(\sigma, f)$ with $\sigma : [1, l] \to \mathbb{Z}_{>0}^n \times \mathbb{Z}^n$, i.e., all the cluster monomials are full. The following are true.*

1. *Every submodule of V_n is generated by a set of cluster strings*

2. *Any two proper submodules of a string submodule $W_l(\sigma, f)$ have non-zero intersection. In particular $W_l(\sigma, f)$ is indecomposable module, however it is not necessarily to be*

irreducible.

3. *If $W_l(\sigma, f)$ is a string module with base l, then for any $w \in W_l(\sigma, f)$ the string $S(w)$ is of base equals a multiple of l.*

4. *there is a bijection between the set of all cyclic submodules of V_n and the set of all string submodules.*

Proof. 1. We first notice that from parts 2 and 3 of lemma 3.4.13 and proof of part 8 of this lemma, we conclude that every two cluster strings are either identical or have zero intersection. So we can introduce the following equivalence relation

$$s \sim s' \text{ if and only if } s \text{ and } s' \text{ belong to the same string module.} \tag{3.4.12}$$

Let W be any submodule of V_n. Let $W^* = W/\sim$. Then we have the following identity

$$W = \oplus_{w \in W^*} W(w). \tag{3.4.13}$$

2. Let W_1 and W_2 be any two proper submodules of $W_l(\sigma, f)$. Then, there are two non zero elements $w_i \in W_i$ for $i = 1, 2$. The above arguments guarantee that $S(w_1)$ and $S(w_2)$ are of bases l_1 and l_2 respectively, such that they are multiples of l, and not equal to l, as we will see in the proof of part 7. WLOG assume $l_1 < l_2$, and $l_i = d_i l, i = 1, 2$, for some d_i and d_2 natural numbers. Let l' be the least common multiple of l_1 and l_2. So, $l' = n_i l_i$, for some $n_i \in \mathbb{N}, i = 1, 2$. Consider the element

$$w' = \sum_{i=1}^{l'} s_i, \text{where } \ s_i \in S(v).$$

Here we show $w' \in W(w_1) \cap W(w_2)$:

Write

$$w_1 = \sum_{b=1}^{l_1} \sum_{j=1}^{l} e_j^{(b)} \alpha_{1j}^{(b)} \cdots \alpha_{nj}^{(b)} f_j^{(b)} y_{1,m_{j1}+t_{j1}^{(b)}}^{\beta_{1j}} \cdots y_{n,m_{jn}+t_{jn}^{(b)}}^{\beta_{nj}}. \tag{3.4.14}$$

We have $e_j^{(b)} \in \mathcal{M}(E)$, $\alpha_{ij}^{(b)} \in M(\xi_i)$, and $f_j^{(b)} = f_j(\theta_1^{t_{j1}^{(b)}}(\xi_1), \ldots, \theta_n^{t_{jn}^{(b)}}(\xi_n))$, $\forall i \in [1, n]$.

Following remark 3.4.12, we can introduce the cluster string associated to w_1 as follows; Let $\sigma^{(b)}(j) = (\beta_j, t_j^b)$, where $\beta_j = (\beta_{1j}, \ldots, \beta_{nj})$, and $t_j^b = (t_{1j}^b, \ldots, t_{nj}^b)$. Let $\hat{\sigma} : [1, l_1] \to \mathbb{Z}_{>0}^n \times \mathbb{Z}_{>0}^n$, given by $\hat{\sigma}(j) = \sigma^b(j)$ for $j \in [(b-1)l, bl], b \in [1, d_1]$, and $\tilde{f} = (f_1^1, \ldots, f_n^1, \ldots, f_1^b, \ldots, f_n^b, \ldots, f_1^{d_1}, \ldots, f_n^{d_1}) \in R^{l_1}$.

Writing w' in the same way we can see that

$$w' \in \sum_{i=1}^{d_1 n_1} \text{copies of } S(w) \subset \sum_{i=1}^{n_1} \text{copies of } S(w_1).$$

In a quite similar way we can show $w' \in \sum_{i=1}^{n_1}$ copies of $S(w_2)$.

But we have $\sum_{i=1}^{n_1}$ copies of $S(w_i) \subset W(w_i), i = 1, 2$. Which means

$$w' \in W(w_1) \cap W(w_2). \tag{3.4.15}$$

3. The base of every cluster string contained in $W_l(\sigma, f)$ is a multiple of l.

   To see that; let w be an element of $W_l(\sigma, f)$. Then $w = av$ for some $a \in A$ and $v \in S_l(\sigma, f)$. Here, a can be written as $\sum_{i=1}^{d} k_i e_i$, where $k_i \in K^*$, and $e_i \in \mathcal{M}(\mathcal{E}), \forall i \in [1, n]$ ($e_1, \ldots, e_d$ are different monomials). Each of e_i, as we saw above, does not change the superscripts of the monomials of v, however it change the second subscripts simultaneously with the coefficients, such that the action's output is still an element of $S_l(\sigma, f)$. Also, since $\sigma : [1, l] \to Z_{>0}^n \times Z^n$ i.e., $\beta_{ij} >, \forall i \in [1, n], j \in [1, l]$. This condition guarantees that each term of v is a product of a coefficient from the ring R times a full cluster monomial. So, the action of any element of $\mathcal{M}(\mathcal{E})$ must change every term of v. Hence, in deed $w = \sum_{i=1}^{d} s_i$, where s_i is an element of $S_l(\sigma, f)$, for all $i \in [1, d]$. Therefore, the element $w = av$ is a sum of dl-different terms where each term belongs to a copy of $S_l(\sigma, f)$. Consider the cluster string $S(w)$ associated to w. One can see $W(w)$ is of base dl and every cluster string contained in W consists of elements of W i.e., every cluster string contained in W is of the form $W(w)$ which is of base equals a

multiple of l.

4. Let W be a cyclic module generated by w. Then $W(w) \subseteq W$. To see the other direction. We have every $v \in W$ is an element of a sum of copies of the cluster string $S(w)$ which is a subset of $W(w)$. So the bijection is defined to send W to $S(w)$.

   Now let $S_l(\sigma, f)$ be a cluster string. Fix a generic element w of $S_l(\sigma, f)$. One can see $S(w) = S_l(\sigma, f)$. So, $S_l(\sigma, f)$ is sent back to $W(w)$. Remark that, the above argument shows that, every element of $S_l(\sigma, f)$ can replace w, i.e., $S(w) = S(w')$ for every $w' \in S_l(\sigma, f)$.

□

Corollary 3.4.15. *1. For every two elements w and w' of the string module $W_l(\sigma, f)$. If $w = \sum_{i=1}^{d} s_i$ and $w' = \sum_{i=1}^{d} s'_i$, where s_i and s'_i are elements of $S_l(\sigma, f)$, for all $i \in [1, d]$, then $S(w) = S(w')$ and $W(w) = W(w')$, (immediate from the definition of cluster strings and the above arguments).*

2. *For any string module $W_l(\sigma, f)$, every cyclic module is of the form $W_{dl}(\hat{\sigma}, \hat{f})$, where $d \in \mathbb{Z}_{\geq 0}$, $\hat{\sigma} : [1, dl] \longrightarrow \mathbb{Z}_{>0}^n \times \mathbb{Z}^n$ with $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2)$, and*
   $\hat{\sigma}_1(j) = (\beta_{(j-il)1}, \ldots, \beta_{(j-il)n}), j \in [(il+1, (i+1)l], i \in [o, d-1]$,
   $\hat{\sigma}_2(j) = (t_{j1}^{(b)}, \ldots, t_{jn}^{(b)}) \in \mathbb{Z}^n, \forall j \in [1, dl]$
   and $\hat{f} = (f_1^1, \ldots, f_n^1, \ldots, f_1^b, \ldots, f_n^b, \ldots, f_1^{dl}, \ldots, f_n^{dl})$,
   where $f_j^{(b)} = f_j(\theta_1^{t_{j1}^{(b)}}(\xi_1), \ldots, \theta_n^{t_{jn}^{(b)}}(\xi_n))$, $\forall j \in [1, l], b \in [1, d]$.

(Horizontal) Infinite base *Cluster strings*. Let $\widetilde{V}_n$ be the ring of all infinite series formed by the set of all cluster monomials over the ring $\mathcal{R}$. The action defined in (3.4.3) and (3.4.4) can be extended to $\widetilde{V}_n$. Now fix $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_{>0}$. Consider the following element

$$w(\beta) = \sum_{t=(t_1,\ldots,t_n)\in\mathbb{Z}^n} y_{1,m_1+t_1}^{\beta_1} \cdots y_{n,m_n+t_n}^{\beta_n}. \tag{3.4.16}$$

Denote the cluster string of $w(\beta)$ by $S(\beta)$ and the string submodule by $W(\beta)$.

Theorem 3.4.16. *For every $\beta \in \mathbb{Z}^n_{>0}$, $W(\beta)$ is an irreducible $\mathfrak{B}$ module.*

Proof. One can see the following

$$y_i w(\beta) = w(\beta) \text{ and } x_i w(\beta) = \theta_i(\xi) w(\beta), \ \forall i \in [1, n]. \tag{3.4.17}$$

Therefore, for any $\mathfrak{b} \in \mathfrak{B}$, we have $\mathfrak{b} w(\beta) = \mathfrak{f}(x_1, \ldots, \xi_n) w(\beta)$, for some $\mathfrak{f} \in K(\xi_1, \ldots, \xi_n)$.

Then $\mathfrak{B} w(\beta) = K(\xi_1, \ldots, \xi_n) w(\beta)$. i.e., $W(\beta)$ is in fact a one dimensional vector space. Then it is an irreducible $\mathfrak{B}$ module.

□

3.5 Hyperbolic category

Definition 3.5.1. Hyperbolic category. Let $\mathcal{A}$ be an additive category with an n tuple of auto-equivalences $\theta = (\theta_1, \ldots, \theta_n)$ and an n-tuple of endomorphisms $\xi = (\xi_1, \ldots, \xi_n)$ of the identical functor of $\mathcal{A}$. *The hyperbolic category* of rank n on $\mathcal{A}$ is denoted by $\mathcal{A}\{\theta, \xi\}$ and is defined as follows:

The objects are the triples (γ, M, η) where M is an object of $\mathcal{A}$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$ are two n-tuples of $\mathcal{A}$-morphisms, given by

$$\gamma_i : M \longrightarrow \theta_i(M) \quad \text{and} \quad \eta_i : \theta_i(M) \longrightarrow M, \quad i \in [1, n],$$

where

$$\eta_i \circ \gamma_i = \xi_{iM} \quad \text{and} \quad \gamma_i \circ \eta_i = \xi_{i\theta_i(M)}, \quad \forall i \in [1, n]. \tag{3.5.1}$$

The morphisms from (γ, M, η) to (γ', M', η') are the n-tuples $f = (f_1, \ldots, f_n)$ of elements of $\mathfrak{Mor}._{\mathcal{A}}(M, M')$ which make the following diagrams commutative

$$\begin{array}{ccccc} M & \xrightarrow{\gamma_i} & \theta_i(M) & \xrightarrow{\eta_i} & M \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \theta_i(f_i)} & & \downarrow{\scriptstyle f_i} \\ M' & \xrightarrow{\gamma'_i} & \theta_i(M') & \xrightarrow{\eta'_i} & M' \end{array},$$

for $i \in [1, n]$.

Example 3.5.2. Let R be an associative ring with ϕ an R-automorphism, and z a central element of R, such that, R$\{\phi, z\}$ is a hyperbolic algebra of rank 1 in the two indeterminate x and y. Let $\mathcal{A} = \mathrm{R} - mod$ (the category of R-modules). The category $\mathcal{H}$ of R$\{\phi, z\}$-modules is equivalent to a hyperbolic category $\mathcal{A}\{\theta, \xi\}$ which is defined as; $\theta : \mathcal{A} \longrightarrow \mathcal{A}$ induced by the R-automorphism ϕ and for $M \in$ objets of $\mathcal{A}$, $\xi_M : M \longrightarrow M$ is given by $\xi(m) := zm$. Objects of $\mathcal{A}\{\theta, \xi\}$ are the triples $(\overline{x}, M, \overline{y})$ where M is an object in $\mathcal{A}$, $\overline{x} : M \longrightarrow \theta(M)$ given by $\overline{x}(m) = xm$ and $\overline{y} : \theta(M) \longrightarrow M$ given by $\overline{y}(m) = ym$.

3.6 Hyperbolic cluster category

Definition 3.6.1. ***Categorical Seed***. Let $\mathcal{H}$ be a category. A categorical seed of rank n in $\mathcal{H}$ is the following data $\mathcal{S} = (\theta, \xi, \mathcal{C})$ where

1. $\theta = (\theta_1, \ldots, \theta_n)$ is an n-tuple of auto-equivalences in $\mathcal{H}$.
2. $\xi = (\xi_1, \ldots, \xi_n)$ is an n-tuples of endomorphisms of the identical functor of $\mathcal{H}$.
3. $\mathcal{C}$ is the following category;

   Objects are the pairs (M, η), where M is an object in $\mathcal{H}$ and $\eta = (\eta_1, \ldots, \eta_n)$ is an n-tuple of invertible elements of $\mathfrak{Mor}_{\cdot\mathcal{H}}(\theta(M), M)$ satisfy the following

$$\eta_i \circ \xi_{i,\theta_i(\theta_{j_k} \ldots \theta_{j_1}(M))} = \xi_{i,\theta_{j_k} \ldots \theta_{j_1}(M)} \circ \eta_i, \quad i, j_k, \ldots, j_1 \in [1, n]. \tag{3.6.1}$$

   Morphisms are $f \in \mathfrak{Mor}_{\cdot\mathcal{C}}, ((M, \eta), (M', \eta')) \subset \mathfrak{Mor}_{\cdot\mathcal{H}}(M, M')$ such that the following diagram is commutative for every $i \in [1, n]$

$$\begin{array}{ccc} \theta_i(M) & \xrightarrow{\eta_i} & M \\ \downarrow{\scriptstyle \theta_i(f_i)} & & \downarrow{\scriptstyle f_i} \\ \theta_i(M') & \xrightarrow{\eta'_i} & M' \end{array} . \tag{3.6.2}$$

   Furthermore, if $\{\theta_i; i \in [1, n]\}$ and $\{\xi_i; i \in [1, n]\}$ are two sets of commutative functors, then $\mathcal{S}$ is called *a categorical hyperbolic seed*.

Example 3.6.2. Let $\mathrm{R}\{\phi, z\}$ be a hyperbolic algebra of rank n in the the indeterminates $x = \{x_1, \ldots, x_n\}$ and $y = \{y_1, \ldots, y_n\}$, with $\phi = \{\phi_1, \ldots, \phi_n\} \subset Aut.(\mathrm{R})$ and $z = \{z_1, \ldots, z_n\} \subset Z(R)$ (center of R) and let $A = S^{-1}\mathrm{R}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$, where $S = \{z_i, \phi_i(z_i);\ i \in [1, n]\}$. Let $\mathfrak{A} = A - mod$, and $\mathfrak{R} = \varphi_*(\mathfrak{A})$, where $\varphi_* : \mathfrak{A} \longrightarrow \mathrm{R} - mod$, the functor that sends each object in $\mathfrak{A}$ to itself as an R-module forgetting the rest of the A-action. Let $\mathcal{S} = (\theta, \xi, \mathcal{C})$ where $\theta = (\theta_1, \ldots, \theta_n)$ is an n-tuple of $\mathfrak{R}$-auto equivalences where $\theta_i : \mathfrak{R} \longrightarrow \mathfrak{R}$ is induced by ϕ_i, and $\xi = (\xi_1, \ldots, \xi_n)$ is an n-tuple of endomorphisms of the identity functor of $\mathfrak{R}$, given by $\xi_{iW} : W \longrightarrow W$ where $\xi_{iW}(w) = z_i \cdot w, i \in [1, n]$, for any object W in $\mathfrak{R}$.

Objects of $\mathcal{C}$ are pairs $(M, \overline{y})$, where M is an object of $\mathfrak{R}$, and $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_n)$, where $\overline{y}_i : \theta_i(M) \longrightarrow M$, $\overline{y}_i(m) = y_i \cdot m$ for $i \in [1, n]$ and morphisms of $\mathcal{C}$ are given by $Mor._{\mathcal{C}}((M, \overline{y}), (M', \overline{y})) = Mor._{\mathfrak{A}}(M, M')$.

In the following we will show $\mathcal{S} = (\theta, \xi, \mathcal{C})$ is a categorical hyperbolic seed in $\mathfrak{R}$. The commutativity of $\{\theta_i\}_{i=1}^n$ is due to the commutativity of $\{\phi_i\}_{i=1}^n$, ad the commutativity of $\{\xi_i\}_1^n$ is because $\{z_1, \ldots, z_n\} \subset Z(R)$. To prove 3.2, for $t \in \theta_{j_k} \ldots \theta_{j_1}(M)$ we have

$$\begin{aligned}
\overline{y}_i \xi_{i, \theta_i(\theta_{j_k} \ldots \theta_{j_1}(M))}(t) &= \overline{y}_i(\phi_i(z_i)t) \\
&= y_i \phi_i(z_i)t \\
&= z_i y_i t \\
&= \xi_{i\theta_{j_k} \ldots \theta_{j_1}(M))}(\overline{y}_i(t)).
\end{aligned}$$

Equations 3.6.1 are consequences of the equations $y_i \theta_i(r) = r y_i$ for any $r \in \mathrm{R}$.

Before introducing the categorical mutations we need to introduce the following morphisms;

1.

$$\xi_{k,W} : W \longrightarrow W, \quad \xi_{k,W}(w) = z_k \cdot w.$$

2.

$$\xi_{\theta_k,W} : W \longrightarrow W, \quad \xi_{\theta_k,W}(w) = \theta_k(z_k) \cdot w.$$

3.

$$\xi_{\theta_k,W} : W \longrightarrow W, \quad \xi_{\theta_k,W}(w) = \theta_k(z_k) \cdot w.$$

4.

$$\xi_{\widehat{k},W} : W \longrightarrow W, \quad \xi_{\widehat{k},W}(w) = z_k^{-1} \cdot w.$$

One can see that the following identity is satisfied

$$\xi_{\widehat{k},W} \circ \xi_{k,W} = \xi_{k,W} \circ \xi_{\widehat{k},W} = id_W, \forall k \in [1,n], \forall\ W \in \ \textit{objects of } \mathfrak{R}.$$

Definition 3.6.3. Let $\mathcal{S} = (\theta, \xi, \mathcal{C})$ be a categorical seed of rank n in $\mathcal{H}$.

• **Right Categorical Mutations.** The right categorical mutation on $\mathcal{S}$ in the k-direction is defined as follows $\mu_k^R(\mathcal{S}) = (\widehat{\theta}_{(k)}^{-1}, \xi, \mathcal{C}_R^{(k,1)})$, where $\widehat{\theta}_{(k)}^{-1} = (\theta_1, \ldots, \theta_k^{-1}, \ldots, \theta_n)$ and $\mathcal{C}_R^{(k,1)}$ is a category with objects are pairs $(\theta_k(M), \xi_{k,\theta_k(M)} \circ \eta_k^{-1})$, and morphisms are given by

$$\mathrm{Mor}._{\mathcal{C}_R^{(k,1)}}((\theta_k(M), \xi_{\theta_k(M)} \circ \eta_k^{-1}), (\theta_k(M'), \xi_{\theta_k(M')} \circ \eta_k'^{-1})) := \theta_k(\mathrm{Mor}._{\mathcal{C}}((M,\eta),(M',\eta'))). \tag{3.6.3}$$

Second generation seeds Applying mutation on the same direction one more time gives us $\mu_k^R(\mu_k^R(\mathcal{S})) = (\theta, \xi, \mathcal{C}_R^{(k,2)})$, where the objects of $\mathcal{C}_R^{(k,2)}$ are the pairs $(M, \xi_{\theta_k,M} \circ \eta_k \circ \xi_{\widehat{k},\theta(M)})$.

So, the right mutation rules are the following:

1. *ξ is frozen.*
2. *θ is altered by replacing it by $\widehat{\theta}_{(k)}$.*
3. *$C^{(k,t)}$ with objects are the pairs $(W, \nu^{(k,t)})$, where $\nu^{(k,t)} = (\nu_1, \ldots, \nu_k)$, is replaced by $C^{(k,t+1)}$. where its objects are given by*

$$\textit{objects of } C_R^{(k,t+1)} = \begin{cases} (\theta_k(W), \nu^{(k,t+1)}), \textit{with} \\ \nu^{(k,t+1)} = (\nu_1, \ldots, \nu_{k-1}, \xi_{\theta_k(M)} \circ \nu_k^{-1}, \nu_{k+1}, \ldots, \nu_n), & \textit{if } t \textit{ is even,} \\ (W, \nu^{(k,t+1)}), \textit{with} \\ \nu^{(k,t+1)} = (\nu_1, \ldots, \nu_{k-1}, \xi_{\theta_k,M} \circ \nu_k^{-1}, \nu_{k+1}, \ldots, \nu_n), & \textit{if } t \textit{ is odd.} \end{cases} \tag{3.6.4}$$

• ***Left Categorical Mutations.*** *the left mutation rules for ξ and θ are the same as in the right mutations, and for $C_L^{(k,t)}$ with objects are the pairs $(W, \nu^{(k,t)})$, where $\nu^{(k,t)} = (\nu_1, \ldots, \nu_k)$, is replaced by $C_L^{(k,t+1)}$ where its objects are given by*

$$\textit{objects of } C_L^{(k,t+1)} = \begin{cases} (\theta_k(W), \nu^{(k,t+1)}), \textit{with} \\ \nu^{(k,t+1)} = (\nu_1, \ldots, \nu_{k-1}, \nu_k^{-1} \circ \xi_{k,M}, \nu_{k+1}, \ldots, \nu_n), & \textit{if } t \textit{ is even,} \\ (W, \nu^{(k,t+1)}), \textit{with} \\ \nu^{(k,t+1)} = (\nu_1, \ldots, \nu_{k-1}, \xi_{\widehat{k},M} \circ \nu_k \circ \xi_{k,\theta(M)}, \nu_{k+1}, \ldots, \nu_n), & \textit{if } t \textit{ is odd.} \end{cases} \tag{3.6.5}$$

The morphisms of $C_L^{(k,t+1)}$ is defined the same way as the the morphisms of $C_R^{(k,t+1)}$.

Lemma 3.6.4. *Let $\mathcal{S} = (\theta, \xi, \mathcal{C})$ be a categorical hyperbolic seed of rank n in $\mathfrak{A}$. Then the following are true*

1. *$\mu_{j_1}^R \ldots \mu_{j_t}^R(\mathcal{S})$ is a categorical hyperbolic seed for any sequence of right mutations $\mu_{j_1}^R \ldots \mu_{j_t}^R$ (resp., to $\mu_{j_1}^L \ldots \mu_{j_t}^L(\mathcal{S})$)*
2. *$\mu_k^R \mu_k^L(\mathcal{S}) = \mu_k^L \mu_k^R(\mathcal{S}) = \mathcal{S}$, for every $k \in [1, n]$*
3. *The categorical seed $\mathcal{S}$ together with the categorical seeds $\mu_1^R(\mathcal{S}), \ldots, \mu_n^R(\mathcal{S})$ give raise to a hyperbolic category (respect., to $\mu_1^L(\mathcal{S}), \ldots, \mu_n^L(\mathcal{S})$).*

Proof. For first part, we prove it for right mutations and for the left mutations is similar with the obvious changes. The proof is divided into into three steps; first notice that $\mu_1^R, \ldots, \mu_n^R$ are commutative on $\mathcal{S}$, this statement is a consequence of the identity $\mu_i^R \mu_j^R(\mathcal{S}) = \mu_j^R \mu_i^R(\mathcal{S})$, which is due to the commutativity of the following sets $\{\xi_i\}_{i=1}^n$, $\{\theta_i\}_{i=1}^n$, and $\{y_i\}_{i=1}^n$. This remark reduces the proof into proving it only for the case $j_i = \ldots = j_t = k$ for some $k \in [1, n]$. In the following we show part (1) for sequences of length one and two, and for the sequences of any odd or even length the proof is quite similar.

Secondly, we show that $\mu_k^R(\mathcal{S})$ is again a categorical seed. One can see that $\widehat{\theta}_{(k)}^{-1}$ still commutative.

Diagram (3.6.2) tells us that $(\theta_k(f) \circ \eta_k^{-1})(m) = (\eta_k^{'-1} \circ f)(m), \forall m \in M$. Then we have the following consecutive identities are satisfied

$$\begin{aligned}
\xi_k \cdot (\theta_k(f) \circ \eta_k^{-1})(m) &= \xi_k \cdot (\eta_k^{'-1} \circ f)(m) \\
\xi_k \cdot (\theta_k(f)(\eta_k^{-1}(m)) &= \xi_k \cdot (\eta_k^{'-1}(f(m)) \\
\theta_k(f)(\xi_{k,\theta_k(M)}\eta_k^{-1}(m) &= \xi_{k,\theta_k(M')}(\eta_k^{'-1}(f(m))) \\
\theta_k(f) \circ (\xi_{k,\theta_k(M)}\eta_k^{-1}) &= (\xi_{k,\theta_k(M')} \circ \eta_k^{'-1}) \circ f.
\end{aligned}$$

The last one says the following diagram is commutative

$$\begin{array}{ccc}
\theta_k(M) & \xleftarrow{\xi_{k\theta(M)} \circ \eta_k^{-1}} & M \\
\Big\downarrow{\scriptstyle \theta_k(f)} & & \Big\downarrow{\scriptstyle f} \\
\theta_k(M') & \xleftarrow[\xi_{k\theta(M')} \circ \eta_k^{'-1}]{} & M'
\end{array}, \tag{3.6.6}$$

which is equivalent to the commutativity of the following diagram

$$\begin{array}{ccc}
\theta_k(M) & \xleftarrow{\xi_{k,\theta(M)} \circ \eta_k^{-1}} & \theta_k^{-1}(\theta_k(M)) \\
\Big\downarrow{\scriptstyle \theta_k(f)} & & \Big\downarrow{\scriptstyle \theta^{-1}(\theta_k(f))} \\
\theta_k(M') & \xleftarrow[\xi_{k\theta_k(M')} \circ \eta_k^{-1}]{} & \theta_k^{-1}(\theta_k(M'))
\end{array}. \tag{3.6.7}$$

To prove (3.6.1) we have; for $i \neq k$ no thing to prove. Now, let $i = k$

$$\begin{aligned}(\xi_{k,\theta_k(\theta_{j_t}\ldots\theta_{j_1}(M))} \circ \eta_k^{-1}) \circ \xi_{k,\theta_k^{-1}(\theta_{j_t}\ldots\theta_{j_1}(\theta_k(M)))} &= (\xi_{k,\theta_k(\theta_{j_t}\ldots\theta_{j_1}(M))} \circ \eta_k^{-1}) \circ \xi_{k,\theta_{j_t}\ldots\theta_{j_1}(M)} \\ &= \xi_{k,\theta_k(\theta_{j_t}\ldots\theta_{j_1}(M))} \circ \xi_{k,\theta_k(\theta_{j_t}\ldots\theta_{j_1}(M))} \circ \eta_k^{-1} \\ &= \xi_{k,\theta_{j_t}\ldots\theta_{j_1}\theta_k(M)} \circ (\xi_{k,\theta_k(\theta_{j_t}\ldots\theta_{j_1}(M))} \circ \eta_k^{-1}).\end{aligned}$$

The commutativity of $\widehat{\theta}_{(k)}^{-1}$ is used for the first and the last equations and (3.6.2) for η_k^{-1} for the second. This finishes the proof for sequences of right mutations of length one.

Finally,the right mutations of length two, Altering $\mu_k^R(\mathcal{S})$ by mutation in K-direction, we get $\mu_k^R\mu_k^R(\mathcal{S}) = (\theta, \xi, \mathcal{C}^{(k,2)})$, where objects of $\mathcal{C}^{(k,2)}$ are given by the pairs $(M, \xi_{k,M} \circ (\eta_k \circ \xi_{\widehat{k},\theta_k(M)}))$. In the following we show that $(\theta, \xi, \mathcal{C}^{(k,2)})$ is a categorical hyperbolic seed.

First, we show (3.6.2) replacing η_k by $\xi_{\theta_k,M} \circ (\eta_k \circ \xi_{\widehat{k},\theta_k(M)})$

$$\begin{aligned}(\xi_{\theta_k,M'} \circ (\eta_k \circ \xi_{\widehat{k},\theta_k(M')})) \circ \xi_{k,\theta(M')} &= \xi_{\theta_k,M'} \circ \eta_k \\ &= \xi_{\theta_k,M'} \circ \eta_k \circ \xi_{k,\theta(M')} \circ \xi_{\widehat{k},\theta_k(M')} \\ &= \xi_{\theta_k,M'} \circ \xi_{k,M'} \circ \eta_k \circ \xi_{\widehat{k},\theta_k(M')} \\ &= \xi_{k,M'} \circ (\xi_{\theta_k,M'} \circ (\eta_k \circ \xi_{\widehat{k},\theta_k(M')})).\end{aligned}$$

For diagram (3.6.2), we have

$$\begin{aligned}f \circ (\xi_{\theta_k,M'} \circ (\eta_k \circ \xi_{\widehat{k},\theta_k(M')})) &= \xi_{\theta_k,M'} \circ f \circ (\eta_k \circ \xi_{\widehat{k},\theta_k(M')}) \\ &= \xi_{\theta_k,M'} \circ \eta_k' \circ \theta_k(f) \circ \xi_{\widehat{k},\theta_k(M')} \\ &= (\xi_{\theta_k,M'} \circ \eta_k' \circ \xi_{\widehat{k},\theta_k(M')}) \circ \theta_k(f).\end{aligned}$$

To prove second part, the only thing need to be checked here is that; the categories $\mu_k^R(\mu_k^L(\mathcal{C}))$ and $\mu_k^L(\mu_k^R(\mathcal{C}))$ are equivalent. Actually, one can see that the category $\mathcal{C}$ will be reproduced by applying $\mu_k^R\mu_k^L$ or $\mu_k^L\mu_k^R$. which is straightforward to prove.

Here we prove part three for $\mathcal{S}$ and $\mu_1^R(\mathcal{S}), \ldots, \mu_n^R(\mathcal{S})$. Let $\mathcal{S} = (\theta, \xi, \mathcal{C})$ be a categorical

seed and consider the categorical seeds $\mu_k^R(\mathcal{S}), k \in [1,n]$. We introduce the following full subcategory $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ with objects are the triples (γ, M, η), where $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_k = \xi_{k,\theta_k(M)} \circ \eta_k^{-1}$, and M is an object of C. The conditions of the hyperbolic category are immediate.

□

Theorem 3.6.5. *Every categorical seed in $\mathcal{H}$ is equivalent to one of the categories $\mathcal{R}[y_i, i \in [1,n]] - mod$ or $\mathcal{R}[x_i, i \in [1,n]] - mod$.*

Proof. Straightforward. □

Chapter 4

Conclusion

The thesis consists of two main parts, first part comes in the first two chapters, and the second part is chapter three. First part has to do with Fomin-Zelevinsky (commutative) cluster algebras. In the second part we introduce a noncommutative cluster structure, namely hyperbolic cluster algebra.

First chapter provides the reader with a short course on the basic notions and the original motivations of the theory.

In the second chapter we introduce the group of the cluster automorphisms. The main result of the chapter is giving three equivalent conditions describing the intersection of the group of automorphism with the group of the field automorphisms which are induced by mutations, (the exchange automorphisms). The theorem is proved for cluster algebras satisfying the Fomin-Zelevinsky positivity conjecture. An open question comes out of this chapter is; whether the positivity conjecture is a necessary condition for Theorem 2.2.3.

In Chapter 3 we introduce a non-commutative cluster structure on some hyperbolic algebras. This class of algebras is studied and we provide some results on the cluster automorphisms in this case. The cluster structure is been used to introduce representations for Weyl algebras, similar representations can be defined for other hyperbolic algebras as well.

Indecomposable and irreducible representations are described inside these representations. Fomin-Zelevinsky finite type classification fails in this case. An open problem is to find sufficient conditions for a hyperbolic cluster structure to be of finite type.

The last section of Chapter 3 is devoted to introduce a categorical version of the hyperbolic cluster structure for the Weyl algebra case.

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