Brolin's theorem for periodic points: speed of convergence for $z^2 + c$ with c in the main cardioid of the Mandelbrot set

by

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Abstract

Brolin's theorem states that for a monic polynomial f on the complex plane of degree d greater than or equal to 2, for a non-exceptional point a, the backwards orbit of a equidistributes on the Julia set of f [1]. Tortrat [9] proved a version of Brolin's theorem for periodic points. Drasin and Okuyama [3] proved a rate of convergence result for Brolin's theorem, and we use some of their work to prove a similar result for the periodic version of Brolin's theorem whenever f is a quadratic polynomial with parameter c in the main cardioid of the Mandelbrot set.

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1. Introduction

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ will be a rational function of degree $d \geq 2$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Define $f^{\circ 1} := f$ and $f^{\circ(n+1)} := f \circ f^{\circ n}$ for $n \geq 1$. We are interested in the distribution of points satisfying the equation $f^{\circ n}(z) = z$ for $n \geq 1$. We say a point z_0 has *period* n if $f^{\circ n}(z_0) = z_0$ and there is no smaller $k \geq 1$ such that $f^{\circ k}(z_0) = z_0$. Hence, the solutions to $f^{\circ n}(z) = z$ are the points of period n or period dividing n.

A point $z_0 \in \hat{\mathbb{C}}$ is a *fixed point* of f if its period is 1. A fixed point z_0 is classified according to its *multiplier* $|f'(z_0)|$. It is *attracting* if $|f'(z_0)| < 1$, *repelling* if $|f'(z_0)| > 1$, and *neutral* (or *indifferent*) if $|f'(z_0)| = 1$. For a neutral fixed point, $|f'(z_0)| = e^{2\pi i\theta}$, and we say z_0 is *rationally neutral* if θ is rational and *irrationally neutral* if θ is irrational.

Define the *Fatou set* F_f of f as

$$F_f := \{z \in \hat{\mathbb{C}} : (f^{\circ n})_{n\geq 1} \text{ is a normal family on some neighborhood of } z\},$$

and define the *Julia set* J_f of f as the complement $\hat{\mathbb{C}} \setminus F_f$. It is clear from the definition that F_f is open, and thence J_f is closed and hence compact (because $\hat{\mathbb{C}}$ is compact). It also follows easily from the definitions that all attracting fixed points are in F_f, and all repelling fixed points are in J_f. In fact, all rationally neutral fixed points are in J_f as well, but irrationally neutral fixed points may land in either set ([2], Theorem III.1.1). Both J_f and F_f are *completely invariant* in the sense that $f^{-1}(J_f) = J_f$ and $f^{-1}(F_f) = F_f$ ([2], Theorem III.1.3).

To obtain an alternative characterization for the Julia set of a polynomial f, we can define the *filled Julia set* K_f of f as

$$K_f := \hat{\mathbb{C}} \setminus \left\{z \in \hat{\mathbb{C}} : \lim_{z\to\infty} f^{\circ n}(z) = \infty\right\} = \left\{z \in \mathbb{C} : \{f^{\circ n}(z) : n \geq 1\} \text{ is bounded}\right\}.$$

For a polynomial f (but not for a general rational function), the Julia set J_f coincides with the boundary ∂K_f of the filled Julia set (see [2], Section III.4).

For a point $z \in J_f$ and a neighborhood U of z, consider the set $\bigcup_{n=1}^{\infty} f^{\circ n}(U)$. Since $(f^{\circ n})_{n\geq 1}$ is not a normal family on U, Montel's Theorem ([2] Theorem I.3.2) tells us that there are no more than 2 points in the set $E_z := \hat{\mathbb{C}} \setminus \bigcup_{n=1}^{\infty} f^{\circ n}(U)$. Indeed, E_z is actually independent of our choice of $z \in J_f$ (so we shall henceforth write E_f) and a point is in E_f if and only if it is a critical point of degree d ([2]Theorem III.1.5). When f is a polynomial, $\infty \in E_f$, and at most one finite point is in E_f. We call E_f the *exceptional set* for f, and we call a point in E_f an *exceptional point*. A point not in E_f is called a *non-exceptional point*.

Consider quadratic polynomials of the form $f_c(z) = z^2 + c$ for $c \in \mathbb{C}$. The *Mandelbrot set* $\mathcal{M}$ is defined by

$$\mathcal{M} := \{c \in \mathbb{C} : J_{f_c} \text{ is connected}\} = \{c \in \mathbb{C} : |f_c{}^{\circ n}(0)| \leq 2 \text{ for all } n \geq 1\}.$$

The *main cardioid* $\mathcal{C} \subseteq \mathcal{M}$ is defined by

$$\mathcal{C} := \{c \in \mathbb{C} : f_c \text{ has a finite attracting fixed point}\}.$$

To show that $\mathcal{C}$ is indeed a cardioid, one can solve the equation $z_c^2 + c = z_c$ for z_c and use the fact that $f_c'(z) = 2z$ to find that

$$|1 \pm \sqrt{1-4c}| = |2z_c| < 1 \quad (c \in \mathcal{C}).$$

From here, one finds that the boundary is of $\mathcal{C}$ is given by the parametric equation

$$c(\theta) = \frac{1}{4}[1 - (e^{i\theta} - 1)^2] \quad (\theta \in [0, 2\pi]),$$

which gives the boundary of a cardioid. See Figure 1.1.

Let $[z_1, z_2]$ denote the chordal distance between z_1 and z_2 in $\hat{\mathbb{C}}$, normalized so that

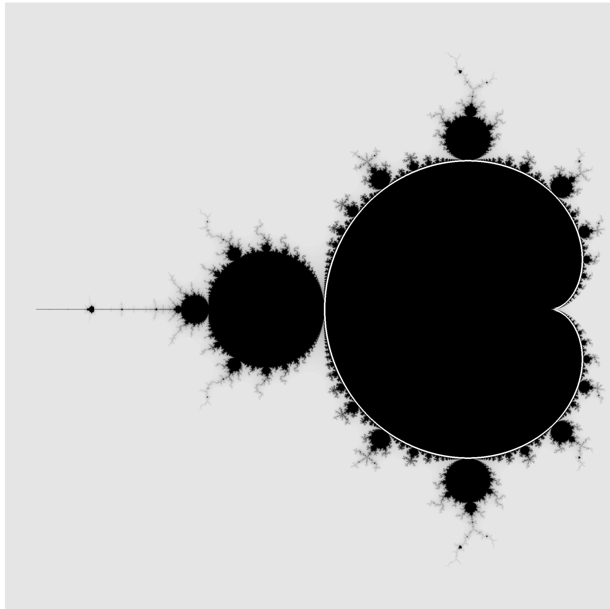


Figure 1.1: The boundary of $\mathcal{C}$ in white, and the rest of $\mathcal{M}$ in black.

$[0, \infty] = 1$:

$$[z_1, z_2] := \begin{cases} \dfrac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}} & z_1, z_2 \in \mathbb{C} \\ \dfrac{1}{\sqrt{1+|z_1|^2}} & z_2 = \infty \\ \dfrac{1}{\sqrt{1+|z_2|^2}} & z_1 = \infty. \end{cases}$$

Let δ_a denote the delta measure at the point $a \in \hat{\mathbb{C}}$, and let σ denote the spherical measure, normalized so that $\sigma(\hat{\mathbb{C}}) = 1$:

$$d\sigma(z) := \frac{1}{\pi} \frac{dxdy}{(1+|z|^2)^2},$$

where $z = x + iy$ and $dxdy$ indicates the standard Lebesgue measure on $\mathbb{C}$. Let $dd^c := \frac{1}{2\pi}\Delta dxdy$ denote the *normalized generalized Laplacian*, so that $dd^c(\log\sqrt{1+|z|^2}) = \sigma(z)$, $dd^c(\log|z-a|) = \delta_a$, and $dd^c\left(\log\frac{1}{[z,a]}\right) = \sigma(z) - \delta_a$.

For $k \in \mathbb{N}$, let

$$C_c^k(\mathbb{C}) := \{\phi : \mathbb{C} \to \mathbb{R} : \phi \text{ is } k \text{ times continuously differentiable and has compact support}\},$$

where $k = 0$ corresponds to continuous functions. Let $C^0(\hat{\mathbb{C}})$ be the Banach space of continuous functions $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with the sup norm. For a rational function f, the *pullback* $f^* : C^0(\hat{\mathbb{C}}) \to C^0(\hat{\mathbb{C}})$ is the continuous linear functional defined by $f^*\phi := \phi \circ f$. The *pushforward* $f_* : C^0(\hat{\mathbb{C}}) \to C^0(\hat{\mathbb{C}})$ is the continuous linear function defined by

$$(f_*\phi)(a) = \sum_{f(\xi)=a} \phi(\xi),$$

where the sum includes multiplicity (i.e., if ξ is a root of $f(z) - a$ of multiplicity p, then $\phi(\xi)$ will occur p times in the sum). All sums of this or a similar form will be counted with multiplicity in this paper.

We can identify the topological dual space $C^0(\hat{\mathbb{C}})^*$ with the space of Borel regular (signed) measures on $\hat{\mathbb{C}}$, via the Riesz-Markov representation theorem, and for $\phi \in C^0(\hat{\mathbb{C}}), \nu \in C^0(\hat{\mathbb{C}})^*$, we write

$$\langle \phi, \nu \rangle := \int_{\hat{\mathbb{C}}} \phi d\nu.$$

Now we define the *pushforward* $f_* : C^0(\hat{\mathbb{C}})^* \to C^0(\hat{\mathbb{C}})^*$ to be the dual to the pullback $f^* : C^0(\hat{\mathbb{C}}) \to C^0(\hat{\mathbb{C}})$, and the *pullback* $f^* : C^0(\hat{\mathbb{C}})^* \to C^0(\hat{\mathbb{C}})^*$ to be the dual to the pushforward $f_* : C^0(\hat{\mathbb{C}}) \to C^0(\hat{\mathbb{C}})$. We can characterize f^* and f_* by the identities

$$\langle \phi, f^*\nu \rangle = \langle f_*\phi, \nu \rangle \quad \text{and} \quad \langle \phi, f_*\nu \rangle = \langle f^*\phi, \nu \rangle \quad (\phi \in C^0(\hat{\mathbb{C}}), \nu \in C^0(\hat{\mathbb{C}})^*).$$

Let K be a compact subset of $\mathbb{C}$, and let $\mathcal{P}(K)$ denote the space of Borel probability measures on $\mathbb{C}$ supported on K. For $\mu \in \mathcal{P}(K)$, we define its *energy* $I(\mu)$ to be

$$I(\mu) := \int_{\mathbb{C}} \int_{\mathbb{C}} \log |z - w| d\mu(z) d\mu(w).$$

If there is a measure $\mu_K \in \mathcal{P}(K)$ satisfying

$$I(\mu_K) = \sup_{\mu \in \mathcal{P}(K)} I(\mu),$$

then μ_K is called the *equilibrium measure* for K. All compact subsets of $\mathbb{C}$, and in particular, J_f, have an equilibrium measure ([7] Theorem 3.3.2). We call K a *polar* set if $I(\mu) = -\infty$ for all $\mu \in \mathcal{P}(K)$. If K is non-polar, then μ_K is unique, and $\text{supp}(\mu_K)$ is a subset of the exterior boundary of K, i.e., the boundary of the unbounded component of $\mathbb{C} \setminus K$ ([7] Theorem 3.7.6). The Julia set J_f is non-polar ([7] Theorem 6.5.1), so it has a unique equilibrium measure, which we will denote μ_f.

There is a connection between equilibrium measure and Brownian motion that gives a nice intuition for equilibrium measure. Suppose a Brownian motion process begins at ∞ and stops whenever it hits $K \subseteq \mathbb{C}$. Then for a Borel set $A \subseteq \mathbb{C}$, $\mu_K(A)$ is equal to the probability that the Brownian motion stops in A ([6] Chapter 3, Theorem 4.12). Therefore, we can intuitively think of μ_f as being more concentrated on the parts of J_f that "stick out" toward ∞.

Theorem 1.1. (Brolin's theorem, [1] Theorem 16.1) *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a monic polynomial of degree $d \geq 2$. For a non-exceptional point $a \in \hat{\mathbb{C}}$, let ν_n be the sequence of measures on $\hat{\mathbb{C}}$ defined by*

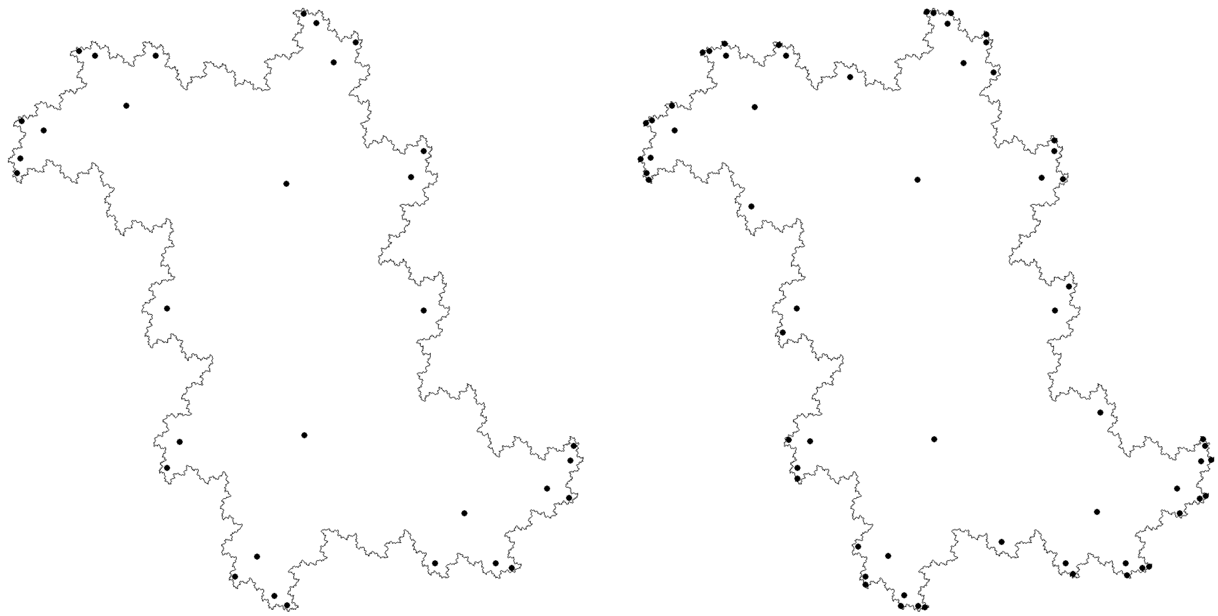
$$\nu_n := d^{-n}(f^{\circ n})^* \delta_a = \frac{1}{d^n} \sum_{f^{\circ n}(\xi)=a} \delta_\xi,$$

where the sum is again counted with multiplicity. Then $\nu_n \xrightarrow{w^*} \mu_f$.

Equivalently, for every $\phi \in C_c^0(\mathbb{C})$,

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{f^{\circ n}(\xi)=a} \phi(\xi) = \int_{\hat{\mathbb{C}}} \phi d\mu_f.$$

Intuitively, Brolin's theorem says that if we start a Brownian motion process at ∞ and stop it when it hits J_f, then all the points in $\{z \in \mathbb{C} : f^{\circ n}(z) = a\}$ (except for a bounded



(a) $n = 5$ (b) $n = 6$

Figure 1.2: The Julia set for $f(z) = z^2 + c$, with $c = 1/5 + i/2$, and solutions to $f^{\circ n}(z) = a$, where $a = -0.02 + 0.47i$.

number of points) are approximately equally likely to be closest to the stopping point, for sufficiently large n. So the pre-images of a under $f^{\circ n}$ concentrate on J_f, and they do so with greater density near parts of J_f that "stick out" toward ∞. See Figure 1.2.

Tortrat proved a related theorem for the periodic points of f (in fact, what he proved was more general: see the theorem in section IV of [9], and take $Q = -\,\mathrm{id}$ for the periodic version).

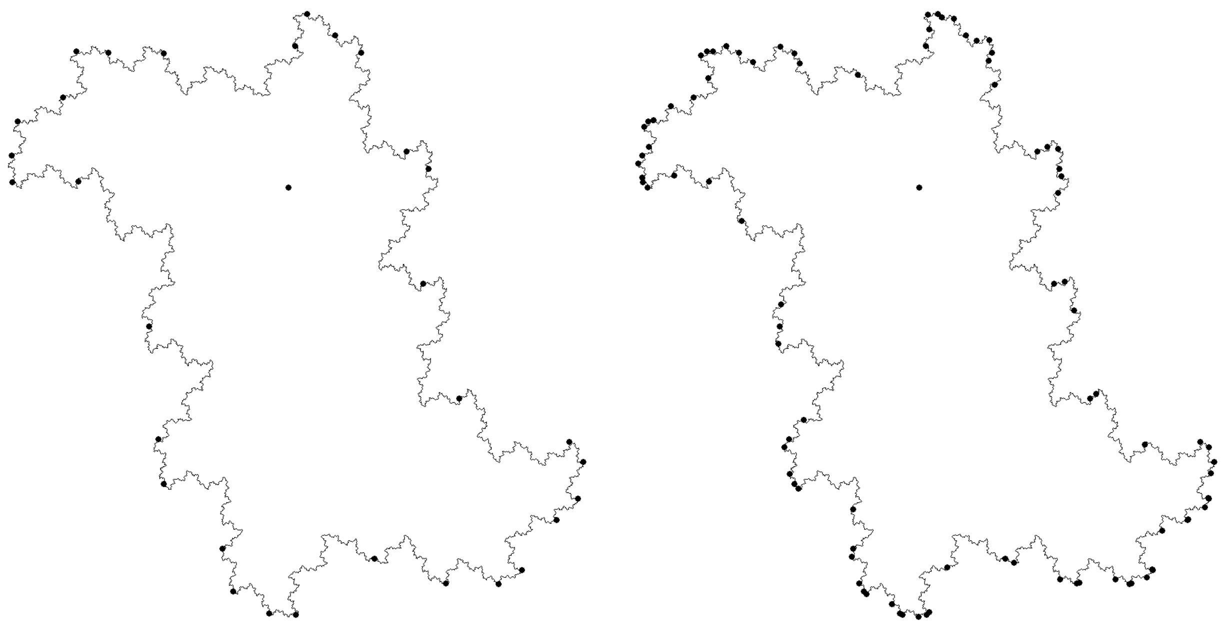
Theorem 1.2. (Brolin's theorem for periodic points, [9]) *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a monic polynomial of degree $d \geq 2$. Let μ_n be the sequence of measures on $\hat{\mathbb{C}}$ defined by*

$$\mu_n := d^{-n}(f^{\circ n} - \mathrm{id})^* \delta_0 = \frac{1}{d^n} \sum_{f^{\circ n}(\zeta) = \zeta} \delta_\zeta,$$

where the sum is counted with multiplicity (but we do not include ∞ in the sum even though $f^{\circ n}(\infty) = \infty$). Then $\mu_n \xrightarrow{w^*} \mu_f$.

Equivalently, for every $\phi \in C_c^0(\mathbb{C})$,

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{f^{\circ n}(\zeta)=\zeta} \phi(\zeta) = \int_{\hat{\mathbb{C}}} \phi d\mu_f.$$



(a) $n = 5$ (b) $n = 6$

Figure 1.3: The Julia set for $f(z) = z^2 + c$, with $c = 1/5 + i/2$, and solutions to $f^{\circ n}(z) = z$.

Lyubich proved Brolin's theorem and Brolin's theorem for periodic points more generally, replacing monic polynomials with rational functions and replacing equilibrium measure with the measure of maximal entropy [5].

The intuition here is similar to that of Brolin's original theorem. Brolin's theorem for periodic points says that if we start a Brownian motion process at ∞ and stop it when it hits J_f, then all the points in $\{z \in \mathbb{C} : f^{\circ n}(z) = z\}$ (except for a bounded number of points) are approximately equally likely to be closest to the stopping point, for sufficiently large n. So the points of period n or period dividing n are more concentrated on parts of J_f that "stick out" toward ∞. See Figure 1.3.

Note that "most" of the solutions to $f^{\circ n}(z) = z$ are in J_f (more precisely, all but a bounded number of solutions are; see the discussion on p. 23). On the other hand, if $a \notin J_f$, none of the solutions of $f^{\circ n}(z) = a$ are in J_f even though they accumulate on J_f. If $a \in J_f$, then all of the solutions of $f^{\circ n}(z) = a$ are in J_f, because J_f is completely invariant.

Drasin and Okuyama [3] proved the following rate of convergence result for $\nu_n \xrightarrow{w^*} \mu_f$.

Theorem 1.3. ([3], Theorem 3) *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a monic polynomial of degree $d \geq 2$. For every $\eta > 1$, there exists $C_\eta > 0$ and $N_\eta \in \mathbb{N}$ such that for every $n > N_\eta$ and every $\phi \in C_c^2(\hat{\mathbb{C}})$,*

$$|\langle \phi, \nu_n \rangle - \langle \phi, \mu_f \rangle| \leq C_\phi C_\eta \left(\frac{\eta}{d}\right)^n$$

for every $a \in J_f$, where $C_\phi \geq 0$ depends only on ϕ.

Equivalently,

$$\left| \frac{1}{d^n} \sum_{f^{\circ n}(\xi) = a} \phi(\xi) - \int_{\hat{\mathbb{C}}} \phi d\mu_f \right| \leq C_\phi C_\eta \left(\frac{\eta}{d}\right)^n$$

for every $a \in J_f$.

In fact, for any non-exceptional point $a \in \mathbb{C}$, rather than just $a \in J_f$, Drasin and Okuyama [3] found other bounds, but to state the various cases would require much background that is not needed elsewhere in this paper. They also proved the theorem more generally for rational functions and the measure of maximal entropy, but we state it only for monic polynomials and equilibrium measure. Our main result is to Brolin's theorem for periodic points as Drasin and Okuyama's result is to Brolin's theorem, but we have only proved it for a specific class of polynomials.

Theorem 1.4. *Let $f(z) = z^2 + c$, with $c \in \mathcal{C}$, the main cardioid of $\mathcal{M}$. Then for any $\phi \in C_c^2(\mathbb{C})$, there exist constants $\alpha_f \in (0, 1]$, dependent only on f, and $D_{\phi,f} > 0$, dependent on ϕ and f, such that for any $n \geq 1$,*

$$|\langle \phi, \mu_n \rangle - \langle \phi, \mu_f \rangle| \leq D_{\phi,f} \frac{(d^{1-\alpha_f})^n}{d^n},$$

or, equivalently,

$$\left| \frac{1}{d^n} \sum_{f^{\circ n}(\zeta)=\zeta} \phi(\zeta) - \int_{\hat{\mathbb{C}}} \phi d\mu_f \right| \leq D_{\phi,f} \frac{(d^{1-\alpha_f})^n}{d^n}.$$

2. Reducing the Problem to Proximity

For a point $a \in \hat{\mathbb{C}}$, define the *proximity* of $f^{\circ n}$ at a to be

$$m(a, f^{\circ n}) := \int_{\hat{\mathbb{C}}} \log \frac{1}{[a, f^{\circ n}(z)]} d\sigma(z).$$

Drasin and Okuyama showed that the rate of convergence for Brolin's theorem is related to proximity according to the following proposition.

Proposition 2.1. ([3]) *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree $d \geq 2$. For a non-exceptional point $a \in \hat{\mathbb{C}}$, let $\nu_n := d^{-n}(f^{\circ n})^*\delta_a$, as in Theorem 1.1. Then for any $\phi \in C_c^2(\mathbb{C})$, there exist constants $C_\phi > 0$, dependent only on ϕ, $C_f > 0$, dependent only on f, and $C_0 > 0$, with no dependencies, such that for every $n \geq 1$,*

$$|\langle \phi, \nu_n \rangle - \langle \phi, \mu_f \rangle| \leq C_\phi \frac{m(a, f^{\circ n}) + C_f C_0}{d^n},$$

or, equivalently,

$$\left| \frac{1}{d^n} \sum_{f^{\circ n}(\xi)=a} \phi(\xi) - \int_{\hat{\mathbb{C}}} \phi d\mu_f \right| \leq C_\phi \frac{m(a, f^{\circ n}) + C_f C_0}{d^n}.$$

They completed the rate of convergence result by finding the rate of growth of $m(a, f^{\circ n})$ for any $a \in \hat{\mathbb{C}}$. We will only state the rate of growth for $a \in J_f$, because that is the only one we will need.

Theorem 2.2. ([3], Theorem 2) *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a monic polynomial of degree $d \geq 2$. For every $\eta > 1$,*

$$m(a, f^{\circ n}) = o(\eta^n) \quad \text{as } n \to \infty,$$

uniformly for $a \in J_f$.

Now we define the *periodic proximity* of $f^{\circ n}$ to be

$$m(f^{\circ n}) := \int_{\hat{\mathbb{C}}} \log \frac{1}{[z, f^{\circ n}(z)]} d\sigma(z).$$

Perhaps not surprisingly, the relationship of the periodic version of Brolin's theorem to the periodic proximity is similar to that of Brolin's theorem to proximity.

Proposition 2.3. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a monic polynomial of degree $d \geq 2$. Let $\mu_n := d^{-n}(f^{\circ n} - \mathrm{id})^* \delta_0$, as in Theorem 1.2. Then for any $\phi \in C_c^2(\mathbb{C})$, there exist constants $C_\phi, C'_\phi \geq 0$, dependent only on ϕ, $C_f \geq 0$ dependent only on f, and C_0, with no dependencies, such that for every $n \geq 1$,*

$$|\langle \phi, \mu_n \rangle - \langle \phi, \mu_f \rangle| \leq \frac{C_\phi(m(f^{\circ n}) + C_f C_0) + C'_\phi}{d^n},$$

or, equivalently,

$$\left| \frac{1}{d^n} \sum_{f^{\circ n}(\zeta)=\zeta} \phi(\zeta) - \int_{\hat{\mathbb{C}}} \phi d\mu_f \right| \leq \frac{C_\phi(m(f^{\circ n}) + C_f C_0) + C'_\phi}{d^n}.$$

Actually, this theorem holds with rational functions in place of monic polynomials and with the measure of maximal entropy in place of equilibrium measure.

Proof. Our proof of Proposition 2.3 is mostly a copy of the proof of Proposition 2.1 given in [3], but we give the details because of the differences.

Define

$$C_\phi := \sup_{\hat{\mathbb{C}}} \left| \frac{dd^c \phi}{\sigma} \right| = \sup_{z \in \hat{\mathbb{C}}} \left| (1/2)(1 + |z|^2)^2 \Delta \phi(z) \right| < \infty,$$

which is finite because ϕ has compact support in $\mathbb{C}$,

$$C_f := \sup_{\hat{\mathbb{C}}} \frac{f^* \sigma}{\sigma} = \sup_{z \in \hat{\mathbb{C}}} \frac{\Delta(\log \sqrt{1 + |f(z)|^2})}{\Delta(\log \sqrt{1 + |z|^2})} = \sup_{z \in \hat{\mathbb{C}}} \frac{|f'(z)|^2 (1 + |f(z)|^2)^{-2}}{(1 + |z|^2)^{-2}} < \infty,$$

$$C_0 := \langle \log(1/[w, a]), \sigma(w) \rangle,$$

which is finite and independent of $a \in \hat{\mathbb{C}}$, and

$$C'_\phi := \langle \phi, \sigma \rangle < \infty.$$

Then for any $a \in \hat{\mathbb{C}}$,

$$\begin{aligned}
\left|\left\langle \phi, (f^{\circ n})^* \sigma - (f^{\circ n})^* \delta_a \right\rangle\right| &= \left|\left\langle \phi, (f^{\circ n})^* dd^c \log(1/[\cdot, a]) \right\rangle\right| \\
&= \left|\left\langle \phi, dd^c \log(1/[f^{\circ n}(\cdot), a]) \right\rangle\right| \\
&= \left|\left\langle \log(1/[f^{\circ n}(\cdot), a]), dd^c \phi \right\rangle\right| \quad \text{by Green's identity} \\
&\leq C_\phi \left\langle \log(1/[f^{\circ n}(\cdot), a]), \sigma \right\rangle \\
&= C_\phi m(a, f^{\circ n}).
\end{aligned} \tag{2.4}$$

The use of Green's identity is the reason we require ϕ to be twice continuously differentiable.

Next, we calculate

$$\begin{aligned}
&\left|\left\langle \phi, (f^{\circ n})^* \sigma + \sigma - (f^{\circ n} - \mathrm{id})^* \delta_0 \right\rangle\right| \\
= &\left|\left\langle \phi, (f^{\circ n})^* dd^c (\log \sqrt{1 + |\cdot|^2}) + dd^c (\log \sqrt{1 + |\cdot|^2}) - (f^{\circ n} - \mathrm{id})^* dd^c \log|\cdot| \right\rangle\right| \\
= &\left|\left\langle \phi, dd^c (\log \sqrt{1 + |f^{\circ n}(\cdot)|^2}) + dd^c (\log \sqrt{1 + |\cdot|^2}) - dd^c \log|f^{\circ n}(\cdot) - \mathrm{id}(\cdot)| \right\rangle\right| \\
= &\left|\left\langle \phi, dd^c (\log(1/[f^{\circ n}(\cdot), \cdot])) \right\rangle\right| \\
= &\left|\left\langle \log(1/[f^{\circ n}(\cdot), \cdot]), dd^c \phi \right\rangle\right| \quad \text{by Green's identity} \\
\leq &\ C_\phi \left\langle \log(1/[f^{\circ n}(\cdot), \cdot]), \sigma \right\rangle \\
= &\ C_\phi m(f^{\circ n}).
\end{aligned} \tag{2.5}$$

Now, because $(f^{\circ n})^* \sigma = (f^*)^{\circ n} \sigma$, we have

$$\left\langle \phi, (f^{\circ (n+1)})^* \sigma \right\rangle = \left\langle (f^{\circ n})_* \phi, f^* \sigma \right\rangle = \left\langle \left\langle \phi, (f^{\circ n})^* \delta_w \right\rangle, (f^* \sigma)(w) \right\rangle. \tag{2.6}$$

Also,

$$(f^*\sigma)(\hat{\mathbb{C}}) := \langle \mathbb{1}_{\hat{\mathbb{C}}}, f^*\sigma\rangle = \langle f_*\mathbb{1}_{\hat{\mathbb{C}}}, \sigma\rangle = \left\langle \sum_{f(\zeta)=w} \mathbb{1}_{\hat{\mathbb{C}}}(\zeta), \sigma(w)\right\rangle = d\cdot\langle \mathbb{1}_{\hat{\mathbb{C}}}, \sigma\rangle = d\cdot\sigma(\hat{\mathbb{C}}) = d = \deg f, \tag{2.7}$$

where $\mathbb{1}_A$ is the indicator function of the set A.

Then

$$\begin{aligned}
& \left|\left\langle \phi, d\cdot (f^{\circ n})^*\sigma - (f^{\circ(n+1)})^*\sigma\right\rangle\right| \\
= \; & \left| d\left\langle \phi, (f^{\circ n})^*\sigma\right\rangle - \left\langle \phi, (f^{\circ(n+1)})^*\sigma\right\rangle\right| \\
= \; & \left|\left\langle \left\langle \phi, (f^{\circ n})^*\sigma\right\rangle f^*\sigma\right\rangle - \left\langle \left\langle \phi, (f^{\circ n})^*\delta_w\right\rangle, (f^*\sigma)(w)\right\rangle\right| && \text{by (2.7) and (2.6)} \\
= \; & \left|\left\langle \left\langle \phi, (f^{\circ n})^*\sigma - (f^{\circ n})^*\delta_w\right\rangle, (f^*\sigma)(w)\right\rangle\right| \\
\leq \; & \left\langle C_\phi m(w, f^{\circ n}), (f^*\sigma)(w)\right\rangle && \text{by (2.4)} \\
\leq \; & \left\langle C_\phi m(w, f^{\circ n}), C_f\sigma(w)\right\rangle \\
= \; & C_\phi C_f \left\langle \left\langle \log(1/[w, f^{\circ n}(\cdot)]), \sigma\right\rangle, \sigma(w)\right\rangle \\
= \; & C_\phi C_f \left\langle \left\langle \log(1/[w, f^{\circ n}(\cdot)]), \sigma(w)\right\rangle, \sigma\right\rangle && \text{by Fubini's Theorem} \\
= \; & C_\phi C_f \left\langle C_0, \sigma\right\rangle \\
= \; & C_\phi C_f C_0.
\end{aligned} \tag{2.8}$$

Therefore,

$$\left|\left\langle \phi, d^{-n}(f^{\circ n})^*\sigma\right\rangle - \left\langle \phi, d^{-n-1}(f^{\circ(n+1)})^*\sigma\right\rangle\right| \leq d^{-n-1}C_\phi C_f C_0,$$

so the sequence $d^{-n}(f^{\circ n})^*\sigma$ converges weak* to some measure, say μ_0, and

$$\left|\left\langle \phi, d^{-n}(f^{\circ n})^*\sigma - \mu_0\right\rangle\right| \leq \frac{C_\phi C_f C_0}{d^n}. \tag{2.9}$$

Hence,

$$\begin{aligned}
& \left| d^{-n} \left\langle \phi, (f^{\circ n} - \mathrm{id})_* \delta_0 \right\rangle - \langle \phi, \mu_0 \rangle \right| \\
= & \left| d^{-n} \left\langle \phi, (f^{\circ n} - \mathrm{id})_* \delta_0 - \sigma - (f^{\circ n})^* \sigma \right\rangle + \left\langle \phi, d^{-n} (f^{\circ n})^* \sigma - \mu_0 \right\rangle + d^{-n} \langle \phi, \sigma \rangle \right| \\
\leq & \ d^{-n} C_\phi m(f^{\circ n}) + d^{-n} C_\phi C_f C_0 + d^{-n} C'_\phi \quad \text{by (2.5) and (2.9)} \\
= & \ \frac{C_\phi (m(f^{\circ n}) + C_f C_0) + C'_\phi}{d^n}.
\end{aligned}$$

Now, all that remains is to show that $\mu_0 = \mu_f$. But we know by Theorem 1.2 that $d^{-n}(f^{\circ n} - \mathrm{id})_* \delta_0$ converges weak* to μ_f, and we just proved that it converges to μ_0, so we must have $\mu_0 = \mu_f$. □

3. Rate of Growth of Proximity

In the previous section, we reduced the problem to finding the rate of growth of $m(f^{\circ n})$. To do this, we will need the following proposition.

Proposition 3.1. *For any* $\zeta \in \mathbb{C}$,

$$\int_{\hat{\mathbb{C}}} \log|z-\zeta| d\sigma(z) = \frac{1}{2}\log(1+|\zeta|^2).$$

In order to prove Proposition 3.1, we introduce $h : [0,\infty) \to \mathbb{R}$ defined by

$$h(\rho) = \int_0^{2\pi} \log|1-\rho e^{i\alpha}| d\alpha,$$

which will arise naturally in the proof of Proposition 3.1.

Lemma 3.2.

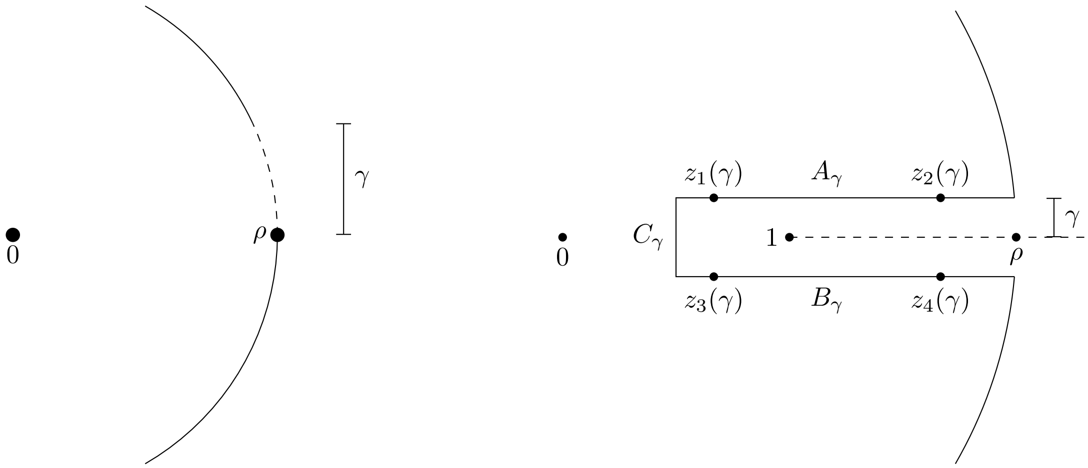
$$h(\rho) = 2\pi \log^+(\rho) := \begin{cases} 0, & \rho < 1 \\ 2\pi \log(\rho), & \rho \geq 1. \end{cases}$$

Proof of Lemma 3.2. The $\rho < 1$ case follows immediately from the fact that $\log|1-z|$ is harmonic except at $z = 1$. But for $\rho \geq 1$, we would be integrating over a circle that encloses the point $z = 1$, so this case is trickier. We will use $B(0,\rho)$ to denote the open ball of radius ρ centered at 0.

Now, we assume $\rho \geq 1$, and we note that

$$h(\rho) = \lim_{\gamma \to 0^+} \int_{\gamma}^{2\pi-\gamma} \log|1-\rho e^{i\alpha}| d\alpha = \lim_{\gamma \to 0^+} \int_{\gamma'}^{2\pi-\gamma'} \log|1-\rho e^{i\alpha}| d\alpha,$$

where γ' is the angular measure of the arc $\partial B(0,\rho)$ connecting ρ and the ray $\{x+i\gamma : x \in (0,\infty)\}$, so that $\gamma' \to 0$ and $\gamma \to 0$. See Figure 3.1a.



(a) With γ as shown, γ' is the arc length of the dashed part of the circle.

(b) With the branch cut and with $z_j(\gamma)$ as shown, $\lim_{\gamma\to 0^+}\arg(z_j(\gamma)) = 0$ for $j < 4$, and $\lim_{\gamma\to 0^+}\arg(z_4(\gamma)) = 2\pi$.

Figure 3.1: The contour of integration near ρ.

Then

$$\begin{aligned} h(\rho) &= \lim_{\gamma\to 0^+}\int_{\gamma'}^{2\pi-\gamma'} \log|1-\rho e^{i\alpha}|\,d\alpha \\ &= \lim_{\gamma\to 0^+}\text{Re}\left[\int_{\gamma'}^{2\pi-\gamma'} \left(\log|1-\rho e^{i\alpha}| + i\arg(1-\rho e^{i\alpha})\right) d\alpha\right] \\ &= \text{Re}\left[\lim_{\gamma\to 0^+}\int_{\gamma'}^{2\pi-\gamma'} \log(1-\rho e^{i\alpha})d\alpha\right] \\ &= \text{Re}\left[\lim_{\gamma\to 0^+}\oint_{\partial B(0,\rho)_\gamma} \frac{\log(1-z)}{iz}dz\right], \end{aligned}$$

where

$$\partial B(0,\rho)_\gamma = \partial B(0,\rho) \setminus \{x+iy : x > 0, |y| < \gamma\}$$

and $\frac{\log(1-z)}{iz}$ and $\arg(1-z)$ have $[1,+\infty)$ as a branch cut, so that

$$\lim_{\alpha\to 0^+}\arg(1-\rho e^{i\alpha}) = 0 \quad \text{and} \quad \lim_{\alpha\to 2\pi^-}\arg(1-\rho e^{i\alpha}) = 2\pi.$$

Now let $A_\gamma := \{x+i\gamma : x \in [1/2, \rho_\gamma]\}$, $B_\gamma := \{x-i\gamma : x \in [1/2, \rho_\gamma]\}$, and $C_\gamma := \{1/2+it : t \in [-\gamma, \gamma]\}$, where ρ_γ is the imaginary part of the point of intersection of $\partial B(0,\rho)$ and the ray $\{x+i\gamma : x \in (0,\infty)\}$. See Figure 3.1b, and note that the choice of $1/2$ is not particularly important; we could use any value strictly between 0 and 1.

Then

$$\begin{aligned}
\oint_{\partial B(0,\rho)_\gamma} \frac{\log(1-z)}{iz} dz &= \oint_{\partial B(0,\rho)_\gamma \cup A_\gamma \cup C_\gamma \cup B_\gamma} \frac{\log(1-z)}{iz} dz - \oint_{A_\gamma \cup C_\gamma \cup B_\gamma} \frac{\log(1-z)}{iz} dz \\
&= 0 + \oint_{B_\gamma \cup C_\gamma \cup A_\gamma} \frac{\log(1-z)}{iz} dz
\end{aligned}$$

(note the change to clockwise orientation in the last step), because $\frac{\log(1-z)}{iz}$ is holomorphic in the interior of (and a neighbrohood around) $\partial B(0,\rho)_\gamma \cup A_\gamma \cup C_\gamma \cup B_\gamma$: the only hiccups with holomorphicity could occur at 0 or the branch cut $[1, +\infty)$, but the contour avoids the branch cut, and in a neighborhood of 0,

$$\frac{\log(1-z)}{iz} = -\frac{\sum_{k=1}^{\infty} z^k/k}{iz} = i \sum_{k=1}^{\infty} \frac{z^{k-1}}{k},$$

which is holomorphic at 0.

Then

$$\begin{aligned}
h(\rho) &= \text{Re}\left[\lim_{\gamma \to 0^+} \oint_{B_\gamma \cup C_\gamma \cup A_\gamma} \frac{\log(1-z)}{iz} dz\right] \\
&= \text{Re}\left[\lim_{\gamma \to 0^+} \oint_{B_\gamma \cup A_\gamma} \frac{\log(1-z)}{iz} dz\right] + \text{Re}\left[\lim_{\gamma \to 0^+} \oint_{C_\gamma} \frac{\log(1-z)}{iz} dz\right].
\end{aligned}$$

Now, clearly

$$\text{Re}\left[\lim_{\gamma \to 0^+} \oint_{C_\gamma} \frac{\log(1-z)}{iz} dz\right] = 0,$$

(just use the trivial integral bound) so

$$\begin{aligned}
h(\rho) &= \mathrm{Re}\left[\lim_{\gamma\to 0^+}\oint_{B_\gamma\cup A_\gamma}\frac{\log(1-z)}{iz}dz\right]\\
&= \mathrm{Re}\left[\lim_{\gamma\to 0^+}\left(\oint_{B_\gamma}\frac{\log(1-z)}{iz}dz+\oint_{A_\gamma}\frac{\log(1-z)}{iz}dz\right)\right]\\
&= \mathrm{Re}\left[\lim_{\gamma\to 0^+}\left(\int_{1/2}^{\rho}\frac{\log(1-(x-i\gamma))}{i(x-i\gamma)}dx+\int_{\rho}^{1/2}\frac{\log(1-(x+i\gamma))}{i(x+i\gamma)}dx\right)\right]\\
&= \lim_{\gamma\to 0^+}\int_{1/2}^{\rho}\mathrm{Re}\left[\frac{\log(1-(x-i\gamma))}{i(x-i\gamma)}-\frac{\log(1-(x+i\gamma))}{i(x+i\gamma)}\right]dx.
\end{aligned}$$

And we have

$$\begin{aligned}
&\mathrm{Re}\left[\frac{\log(1-(x-i\gamma))}{i(x-i\gamma)}-\frac{\log(1-(x+i\gamma))}{i(x+i\gamma)}\right]\\
&= \mathrm{Re}\left[i\frac{(x-i\gamma)\log(1-(x+i\gamma))-(x+i\gamma)\log(1-(x-i\gamma))}{x^2+\gamma^2}\right]\\
&= \frac{1}{x^2+\gamma^2}\mathrm{Re}\Big[i\Big[(x-i\gamma)\left(\log|1-(x+i\gamma)|+i\arg(1-(x+i\gamma))\right)\\
&\quad -(x+i\gamma)\left(\log|1-(x-i\gamma)|+i\arg(1-(x-i\gamma))\right)\Big]\Big]\\
&= \frac{1}{x^2+\gamma^2}\mathrm{Re}\Big[(ix+\gamma)\left(\log|1-(x+i\gamma)|+i\arg(1-(x+i\gamma))\right)\\
&\quad -(ix-\gamma)\left(\log|1-(x-i\gamma)|+i\arg(1-(x-i\gamma))\right)\Big]\\
&= \frac{1}{x^2+\gamma^2}\Big[\gamma\Big(\log|1-(x+i\gamma)|+\log|1-(x-i\gamma)|\Big)\\
&\quad +x\Big(\arg(1-(x-i\gamma))-\arg(1-(x+i\gamma))\Big)\Big].
\end{aligned}$$

So

$$\begin{aligned} h(\rho) &= \lim_{\gamma\to 0^+}\int_{1/2}^{\rho}\left[\frac{\gamma\Big(\log|1-(x+i\gamma)|+\log|1-(x-i\gamma)|\Big)}{x^2+\gamma^2}\right]dx \\ &\quad + \lim_{\gamma\to 0^+}\int_{1/2}^{\rho}\left[\frac{x\Big(\arg(1-(x-i\gamma))-\arg(1-(x+i\gamma))\Big)}{x^2+\gamma^2}\right]dx \\ &=: E+F. \end{aligned}$$

We'll look at the second integral F first. We can use the dominated convergence theorem (with some constant function, for example, as our upper bound) to pass the limit inside the integral and obtain

$$\begin{aligned} F &= \lim_{\gamma\to 0^+}\int_{1/2}^{\rho}\left[\frac{x\Big(\arg(1-(x-i\gamma))-\arg(1-(x+i\gamma))\Big)}{x^2+\gamma^2}\right]dx \\ &= \int_{1/2}^{\rho}\left[\frac{x\Big(\lim_{\gamma\to 0^+}\arg(1-(x-i\gamma))-\lim_{\gamma\to 0^+}\arg(1-(x+i\gamma))\Big)}{x^2}\right]dx. \end{aligned}$$

Now

$$\lim_{\gamma\to 0^+}\arg(1-(x-i\gamma))-\lim_{\gamma\to 0^+}\arg(1-(x+i\gamma))=\begin{cases}0, & x<1\\ 2\pi, & x\geq 1\end{cases}$$

because of the branch cut of $[1,+\infty)$ (see Figure 3.1b), which gives us

$$F=2\pi\int_1^{\rho}\frac{x}{x^2}dx=2\pi\log(x)\Big|_1^{\rho}=2\pi\log(\rho).$$

Now it just remains to show $E=0$.

Since $|1-(x+i\gamma)| = |1-(x-i\gamma)|$, we have

$$\begin{aligned}\Big|\log|1-(x+i\gamma)| + \log|1-(x-i\gamma)|\Big| &= |\log|1-(x+i\gamma)|^2| \\ &= |\log[(1-x)^2+\gamma^2]| \\ &\leq \max\{|\log(\gamma^2)|, |\log(\rho^2)|\},\end{aligned}$$

which we may assume is equal to $|\log(\gamma^2)|$ by taking small enough γ. Then

$$\begin{aligned}|E| &= \lim_{\gamma\to 0^+}\left|\int_{1/2}^{\rho}\left[\frac{\gamma\Big(\log|1-(x+i\gamma)| + \log|1-(x-i\gamma)|\Big)}{x^2+\gamma^2}\right]dx\right| \\ &\leq \lim_{\gamma\to 0^+}\int_{1/2}^{\rho}\left[\frac{\gamma|\log(\gamma^2)|}{x^2+\gamma^2}\right]dx \\ &\leq \lim_{\gamma\to 0^+}\gamma|\log(\gamma^2)|\int_{1/2}^{\rho}\frac{1}{x^2}dx \\ &= 0.\end{aligned}$$

□

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Suppose $\zeta = |\zeta|e^{i\alpha}$. Then

$$\begin{aligned}\int_{\hat{\mathbb{C}}}\log|z-\zeta|d\sigma(z) &= \int_{\hat{\mathbb{C}}}\log|(\zeta)(z/\zeta-1)|d\sigma(z) \\ &= \log|\zeta| + \frac{1}{\pi}\int_{\mathbb{C}}\log|z/\zeta-1|\frac{dxdy}{(1+|z|^2)^2} \\ &= \log|\zeta| + \frac{1}{\pi}\int_0^\infty\left[\int_0^{2\pi}\log\left|\frac{re^{i\theta}}{|\zeta|e^{i\alpha}}-1\right|d\theta\right]\frac{rdr}{(1+r^2)^2} \\ &= \log|\zeta| + \frac{1}{\pi}\int_0^\infty\left[\int_0^{2\pi}\log\left|(r/|\zeta|)e^{i(\theta-\alpha)}-1\right|d\theta\right]\frac{rdr}{(1+r^2)^2}\end{aligned}$$

$$\begin{aligned}
&= \log|\zeta| + \frac{1}{\pi}\int_0^\infty \left[\int_{-\alpha}^{2\pi-\alpha} \log\left|(r/|\zeta|)e^{i\beta} - 1\right| d\beta\right] \frac{rdr}{(1+r^2)^2} \quad \beta = \theta - \alpha \\
&= \log|\zeta| + \frac{1}{\pi}\int_0^\infty \left[\int_0^{2\pi} \log\left|(r/|\zeta|)e^{i\beta} - 1\right| d\beta\right] \frac{rdr}{(1+r^2)^2} \\
&= \log|\zeta| + 2\int_0^\infty \log^+(r/|\zeta|)\frac{rdr}{(1+r^2)^2} \qquad \text{by Lemma 3.2} \\
&= \log|\zeta| + 2\int_{|\zeta|}^\infty \log(r/|\zeta|)\frac{rdr}{(1+r^2)^2} \\
&= \log|\zeta| + 2\int_{|\zeta|}^\infty \log(r)\frac{rdr}{(1+r^2)^2} - 2\int_{|\zeta|}^\infty \log|\zeta|\frac{rdr}{(1+r^2)^2} \\
&=: \log|\zeta| + 2A + 2B.
\end{aligned}$$

Now

$$\begin{aligned}
A &= \int_{|\zeta|}^\infty \log(r)\frac{rdr}{(1+r^2)^2} \\
&= \frac{1}{2}\int_{|\zeta|}^\infty \log(r^2)\frac{rdr}{(1+r^2)^2} \\
&= \frac{1}{4}\int_{|\zeta|^2}^\infty \log(u)\frac{du}{(1+u)^2} \qquad u = r^2, du = 2rdr \\
&= \frac{1}{4}\left[\log\left(\frac{u}{1+u}\right) - \frac{\log(u)}{1+u}\Big|_{|\zeta|^2}^\infty\right] \\
&= \frac{1}{4}\left[(0-0) - \log\left(\frac{|\zeta|^2}{1+|\zeta|^2}\right) + \frac{\log|\zeta|^2}{1+|\zeta|^2}\right] \\
&= \frac{1}{4}\left(\frac{\log|\zeta|^2}{1+|\zeta|^2}\right) - \frac{1}{4}\log\left(\frac{|\zeta|^2}{1+|\zeta|^2}\right),
\end{aligned}$$

and

$$\int_{|\zeta|}^\infty \frac{rdr}{(1+r^2)^2} = \frac{1}{2}\int_{1+|\zeta|^2}^\infty \frac{du}{u^2} = \frac{1}{2}\left[-\frac{1}{u}\Big|_{1+|\zeta|^2}^\infty\right] = \frac{1}{2}\left[0 + \frac{1}{1+|\zeta|^2}\right],$$

so

$$B = -\int_{|\zeta|}^\infty \log|\zeta|\frac{rdr}{(1+r^2)^2} = -\frac{1}{2}\left(\frac{\log|\zeta|}{1+|\zeta|^2}\right).$$

Therefore,

$$\begin{aligned}
\int_{\hat{\mathbb{C}}} \log|z-\zeta| d\sigma(z) &= \log|\zeta| + 2A + 2B \\
&= \log|\zeta| + \frac{1}{2}\left(\frac{\log|\zeta|^2}{1+|\zeta|^2}\right) - \frac{1}{2}\log\left(\frac{|\zeta|^2}{1+|\zeta|^2}\right) - \left(\frac{\log|\zeta|}{1+|\zeta|^2}\right) \\
&= \log|\zeta| - \frac{1}{2}\log\left(\frac{|\zeta|^2}{1+|\zeta|^2}\right) \\
&= \frac{1}{2}\log(1+|\zeta|^2).
\end{aligned}$$

□

Theorem 3.3. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a monic polynomial of degree $d \geq 2$. For $n \geq 1$ and $a \in \mathbb{C}$,*

$$m(f^{\circ n}) = m(a, f^{\circ n}) + C_a + \frac{1}{2}\log\left(\frac{\prod_{f^{\circ n}(\xi)=a}(1+|\xi|^2)}{\prod_{f^{\circ n}(\zeta)=\zeta}(1+|\zeta|^2)}\right),$$

where

$$C_a = \int_{\hat{\mathbb{C}}} \log\left[\frac{\sqrt{1+|z|^2}}{\sqrt{1+|a|^2}}\right] d\sigma(z).$$

Proof. First, by Proposition 3.1,

$$\begin{aligned}
\int_{\hat{\mathbb{C}}} \log|f^{\circ n}(z) - z| d\sigma(z) &= \sum_{f^{\circ n}(\zeta)=\zeta} \int_{\hat{\mathbb{C}}} \log|z-\zeta| d\sigma(z) \\
&= \frac{1}{2} \sum_{f^{\circ n}(\zeta)=\zeta} \log(1+|\zeta|^2) \\
&= \frac{1}{2}\log\left(\prod_{f^{\circ n}(\zeta)=\zeta} (1+|\zeta|^2)\right).
\end{aligned}$$

Similarly,

$$\int_{\hat{\mathbb{C}}} \log|f^{\circ n}(z) - a| d\sigma(z) = \frac{1}{2}\log\left(\prod_{f^{\circ n}(\xi)=a} (1+|\xi|^2)\right).$$

Then

$$
\begin{aligned}
m(f^{\circ n}) &:= \int_{\hat{\mathbb{C}}} \log \frac{1}{[f^{\circ n}(z), z]} d\sigma(z) \\
&= \int_{\hat{\mathbb{C}}} \log \left[\frac{\sqrt{1+|f^{\circ n}(z)|^2}\sqrt{1+|z|^2|}}{|f^{\circ n}(z) - z|} \right] d\sigma(z) \\
&= \int_{\hat{\mathbb{C}}} \log \left[\frac{\sqrt{1+|f^{\circ n}(z)|^2}\sqrt{1+|a|^2|}}{|f^{\circ n}(z) - a|} \right] d\sigma(z) + \int_{\hat{\mathbb{C}}} \log \left[\frac{\sqrt{1+|z|^2}}{\sqrt{1+|a|^2}} \right] d\sigma(z) \\
&\quad + \int_{\hat{\mathbb{C}}} \log |f^{\circ n}(z) - a| d\sigma(z) - \int_{\hat{\mathbb{C}}} \log |f^{\circ n}(z) - z| d\sigma(z) \\
&= \int_{\hat{\mathbb{C}}} \log \frac{1}{[f^{\circ n}(z), a]} d\sigma(z) + C_a + \frac{1}{2} \log \left(\prod_{f^{\circ n}(\xi)=a} (1+|\xi|^2) \right) - \frac{1}{2} \log \left(\prod_{f^{\circ n}(\zeta)=\zeta} (1+|\zeta|^2) \right) \\
&= m(a, f^{\circ n}) + C_a + \frac{1}{2} \log \left(\frac{\prod_{f^{\circ n}(\xi)=a}(1+|\xi|^2)}{\prod_{f^{\circ n}(\zeta)=\zeta}(1+|\zeta|^2)} \right).
\end{aligned}
$$

□

Now assume $a \in J_f$, so that $f^{\circ n}(\xi) = a$ implies $\xi \in J_f$, and let β_n be the number of $\zeta \notin J_f$ satisfying $f^{\circ n}(\zeta) = \zeta$, counting multiplicity. We want an upper bound for β_n. If ζ_0 is a root of $f^{\circ n}(z) - z$ of multiplicity $p > 1$, then $(f^{\circ n})'(\zeta_0) - 1 = 0$, so ζ_0 is in a rationally neutral cycle. By Theorem 1.1 in Chapter III of [2], J_f contains all repelling cycles and all rationally neutral cycles. Therefore, β_n will be unaffected by multiplicity. And by Corollary 1 of [8], the number of attracting cycles plus the number of neutral cycles is at most $2d-2$. Therefore, there are at most $2d-2$ cycles not in the J_f, and all points in these cycles have multiplicity 1 as solutions to $f^{\circ n}(z) = z$. So for all $n \geq 1$,

$$\beta_n \leq \beta_f := N_f(2d-2),$$

where N_f is the number of points in the largest of the cycles not in J_f.

Next, order the ξs as $\{\xi_1, \xi_2, \ldots, \xi_{d^n}\}$ and the ζs as $\{\zeta_1, \zeta_2, \ldots, \zeta_{d^n}\}$, where $\zeta_j \in J_f$ for $j \in \{1, \ldots, d^n - \beta_n\}$ and $\zeta_j \notin J_f$ for $j \in \{d^n - \beta_n + 1, d^n\}$. Let $M_f := \sup_{z \in J_f}(1 + |z|^2)$. Then

$$\begin{aligned}
\log\left(\frac{\prod_{f^{\circ n}(\xi)=a}(1+|\xi|^2)}{\prod_{f^{\circ n}(\zeta)=\zeta}(1+|\zeta|^2)}\right) &= \log\left(\prod_{j=1}^{d^n}\frac{1+|\xi_j|^2}{1+|\zeta_j|^2}\right) \\
&= \log\left(\prod_{j=1}^{d^n-\beta_n}\frac{1+|\xi_j|^2}{1+|\zeta_j|^2}\right) + \log\left(\prod_{j=d^n-\beta_n+1}^{d^n}\frac{1+|\xi_j|^2}{1+|\zeta_j|^2}\right) \\
&\leq \log\left(\prod_{j=1}^{d^n-\beta_n}\frac{1+|\xi_j|^2}{1+|\zeta_j|^2}\right) + \log M_f^{\beta_n} \\
&\leq \log\left(\prod_{j=1}^{d^n-\beta_n}\frac{1+|\xi_j|^2}{1+|\zeta_j|^2}\right) + \beta_f \log M_f.
\end{aligned}$$

Now suppose we have matched up the ξ_js and the ζ_js in such a way that $|\xi_j - \zeta_j| \leq \epsilon_n$ for $j \in \{1, \ldots, d^n - \beta_n\}$, for some $\epsilon_n > 0$. We can certainly do this in such a way that

$\epsilon_n \leq \mathrm{diam}(J_f)$ for all n, but we will find a better value ϵ_n later. Then

$$\begin{aligned}
\frac{1+|\xi_j|^2}{1+|\zeta_j|^2} &\leq \frac{1+(|\xi_j-\zeta_j|+|\zeta_j|)^2}{1+|\zeta_j|^2} \\
&\leq \frac{1+(\epsilon_n+|\zeta_j|)^2}{1+|\zeta_j|^2} \\
&= \frac{1+|\zeta_j|^2+2\epsilon_n|\zeta_j|+\epsilon_n^2}{1+|\zeta_j|^2} \\
&= 1+\frac{2\epsilon_n|\zeta_j|+\epsilon_n^2}{1+|\zeta_j|^2} \\
&= 1+\epsilon_n\left(\frac{2|\zeta_j|+\epsilon_n}{1+|\zeta_j|^2}\right) \\
&\leq 1+\epsilon_n(2M_f+\mathrm{diam}(J_f)) \\
&= 1+\epsilon_n C_f' \\
&\leq e^{\epsilon_n C_f'},
\end{aligned}$$

since, in general $1+x \leq e^x$ for $x \in \mathbb{R}$.

Hence,

$$\begin{aligned}
\log\left(\prod_{j=1}^{d^n-\beta_n}\frac{1+|\xi_j|^2}{1+|\zeta_j|^2}\right) &\leq \log\left(\prod_{j=1}^{d^n-\beta_n} e^{C_f'\epsilon_n}\right) \\
&\leq \log(e^{C_f'\epsilon_n d^n}) \\
&= C_f'\epsilon_n d^n.
\end{aligned}$$

So now we just need to show that we can obtain a suitable sequence ϵ_n, preferably equal to a constant times d^{-n}. In the case that $f(z)=z^2+c$, with c in the main cardioid of the Mandelbrot set, we can use a constant times $d^{-\alpha n}$ for some $\alpha \in (0,1]$, which is the final step to prove our main theorem, which we re-state here for convenience.

Theorem 1.4 *Let $f(z)=z^2+c$, with c in the main cardioid of the Mandelbrot set. Then for any $\phi \in C_c^2(\mathbb{C})$, there exist constants $\alpha_f \in (0,1]$, dependent only on f, and $D_{\phi,f} > 0$,* *dependent on ϕ and f, such that for any $n \geq 1$,*

$$|\langle \phi, \mu_n \rangle - \langle \phi, \mu_f \rangle| \leq D_{\phi,f} \frac{(d^{1-\alpha_f})^n}{d^n},$$

or, equivalently,

$$\left| \frac{1}{d^n} \sum_{f^{\circ n}(\zeta)=\zeta} \phi(\zeta) - \int_{\hat{\mathbb{C}}} \phi d\mu_f \right| \leq D_{\phi,f} \frac{(d^{1-\alpha_f})^n}{d^n}.$$

Proof. If c is in the main cardioid of the Mandelbrot set, then there is a quasiconformal map $\tau : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ conjugating f to $z \mapsto z^d$ (with $d = 2$) in a neighborhood of N of $\partial\mathbb{D}$, i.e.,

$$\tau^{-1} \circ f^{\circ n} \circ \tau(z) = z^{d^n} \quad (z \in N);$$

in [2], see Theorem 2.1 and the following example in Chapter VI, along with Theorem 1.3 in Chapter VIII. In fact, this proof holds more generally when the Fatou set of f has exactly two components and f is hyperbolic on J_f, thanks to the aforementioned Theorem 2.1 in [2]. In particular, since $\partial\mathbb{D}$ is the Julia set of $z \mapsto z^d$, we have $\tau(\partial\mathbb{D}) = J_f$.

By Theorem 11.14 in [4], τ is quasisymmetric. Therefore, by Corollary 11.5 in [4], τ is Hölder continuous on $\partial\mathbb{D}$, say with constant $\alpha \in (0, 1]$, i.e., there exists $L_f > 0$ such that

$$|\tau(z) - \tau(w)| \leq L_f |z - w|^\alpha \quad (z, w \in \partial\mathbb{D}).$$

Now, choose $a = \tau(1) \in J_f$, and let $\Xi_n := \{\xi_1, \xi_2, \ldots, \xi_{d^n}\}$ and U_k be the set of the kth roots of unity. We claim $\Xi_n = \tau(U_{d^n})$.

($\subseteq$) Suppose $\xi \in \Xi_n$. Then $\xi \in J_f$, so $\xi = \tau(\omega)$ for some $\omega \in \partial\mathbb{D}$. Then we have

$$\tau(1) = a = f^{\circ n}(\xi) = f^{\circ n}(\tau(\omega)),$$

so

$$1 = \tau^{-1} \circ f^{\circ n} \circ \tau(\omega) = \omega^{d^n},$$

so $\omega \in U_{d^n}$. Hence, $\xi \in \tau(U_{d^n})$.

($\supseteq$) Suppose $\xi \in \tau(U_{d^n})$. Then $\xi = \tau(\omega)$ for some $\omega \in U_{d^n}$. Then

$$f^{\circ n}(\xi) = f^{\circ n} \circ \tau(\omega) = \tau(\omega^{d^n}) = \tau(1) = a,$$

so $\xi \in \Xi_n$.

Next, define $Z_n := \{\zeta_1, \ldots, \zeta_{d^n - \beta_n}\} := \{\zeta \in J_f : f^{\circ n}(\zeta) = \zeta\}$, and we claim $Z_n = \tau(U_{d^n - 1})$.

($\subseteq$) Suppose $\zeta \in Z_n$. Then $\zeta = \tau(\omega)$ for some $\omega \in \partial\mathbb{D}$, and

$$\tau(\omega) = \zeta = f^{\circ n}(\zeta) = f^{\circ n} \circ \tau(\omega) = \tau(\omega^{d^n}),$$

so $\omega = \omega^{d^n}$, so $\omega \in U_{d^n - 1}$ or $\omega = 0$, but $\omega \in \partial\mathbb{D}$, so $\omega \in U_{d^n - 1}$. Hence, $\zeta \in \tau(U_{d^n - 1})$.

($\supseteq$) Suppose $\zeta \in \tau(U_{d^n - 1})$ Then $\zeta = \tau(\omega)$, with $\omega^{d^n - 1} = 1$, or $\omega^{d^n} = \omega$. Then

$$f^{\circ n}(\zeta) = f^{\circ n} \circ \tau(\omega) = \tau(\omega^{d^n}) = \tau(\omega) = \zeta.$$

Also, since $\omega \in \partial\mathbb{D}$, $\zeta \in J_f$, so $\zeta \in Z_n$.

Now that we know $\Xi_n = \tau(U_{d^n})$ and $Z_n = \tau(U_{d^n - 1})$, we can pair up the points in Ξ_n and Z_n properly. Let $\omega_k := e^{2\pi i/k}$ be the principal kth root of unity. Then we let $\zeta_j = \tau(\omega_{d^n - 1}^{j-1})$ for $j = 1, \ldots, d^n - 1$ and $\xi_j = \tau(\omega_{d^n}^{j-1})$ for $j = 1, \ldots, d^n - 1$ (so the points we are "throwing out" are $\xi_{d^n} = \tau(\omega_{d^n}^{d^n - 1})$ and ζ_{d^n}, which is the attracting fixed point, meaning $\beta_n = 1$ in this case).

Then for $j \in \{1, \ldots, d^n - 1\}$, we have

$$\begin{aligned}
|\zeta_j - \xi_j| &= |\tau(\omega_{d^n-1}^{j-1}) - \tau(\omega_{d^n}^{j-1})| \\
&\leq L_f |\omega_{d^n-1}^{j-1} - \omega_{d^n}^{j-1}|^\alpha \qquad \text{since } \tau \text{ is } \alpha\text{-Hölder} \\
&\leq L_f |\omega_{d^n-1}^{j-1} - \omega_{d^n-1}^{j-2}|^\alpha \qquad \text{since } \frac{j-2}{d^n-1} \leq \frac{j-1}{d^n} \leq \frac{j-1}{d^n-1} \text{ for } j \in \{1, \ldots, d^n - 1\} \\
&\leq L_f \left(\frac{2\pi}{d^n - 1} \right)^\alpha \qquad \text{since chordal distance is less than arc length} \\
&\leq C''_f d^{-\alpha n},
\end{aligned}$$

so we define

$$\epsilon_n := 2C''_f d^{-\alpha n},$$

and we obtain

$$\begin{aligned}
m(f^{\circ n}) &= m(a, f^{\circ n}) + C_a + \frac{1}{2} \log \left(\frac{\prod_{f^{\circ n}(\xi)=a} (1 + |\xi|^2)}{\prod_{f^{\circ n}(\zeta)=\zeta} (1 + |\zeta|^2)} \right) \\
&\leq m(a, f^{\circ n}) + C_a + \beta_f \log M_f + \frac{1}{2} C'_f \epsilon_n d^n \\
&= m(a, f^{\circ n}) + C_a + \beta_f \log M_f + C'_f C''_f d^{-\alpha n} d^n \\
&= m(a, f^{\circ n}) + C_a + \beta_f \log M_f + C'_f C''_f (d^{1-\alpha})^n.
\end{aligned}$$

Now, combining this with Proposition 2.3, we obtain the desired result:

$$\begin{aligned}
& \left| \frac{1}{d^n} \sum_{\zeta = f^{\circ n}(\zeta)} \phi(\zeta) - \int_{J_f} \phi d\mu_f \right| \\
\leq\ & \frac{C_\phi(m(f^{\circ n}) + C_f C_\sigma) + C'_\phi}{d^n} \\
\leq\ & \frac{C_\phi(m(a, f^{\circ n}) + C_a + \beta_f \log M_f + C'_f C''_f (d^{1-\alpha})^n + C_f C_\sigma) + C'_\phi}{d^n} \\
\leq\ & \frac{C_\phi(C'''_f (d^{1-\alpha})^n + C_{\tau(1)} + \beta_f \log M_f + C'_f C''_f (d^{1-\alpha})^n + C_f C_\sigma) + C'_\phi}{d^n} \quad \text{Theorem 2.2 with } \eta = d^{1-\alpha} \\
\leq\ & D_{\phi,f} \frac{(d^{1-\alpha})^n}{d^n}.
\end{aligned}$$

□

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