ESTIMATION OF THE COMMON MEAN OF TWO NORMAL DISTRIBUTIONS

by

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CHAPTER 1.

INTRODUCTION AND LITERATURE REVIEW

The problem considered in this report is the estimation of the mean based on two independent samples from normal distributions with common mean and unknown variances. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be independent i.i.d. samples from the $N(\mu,\sigma_x^2)$ and $N(\mu,\sigma_y^2)$ distributions respectively. The parameter $(\mu,\sigma_x^2,\sigma_y^2) \in (-\infty,\infty)X(0,\infty)X(0,\infty)$ is unknown. We shall consider the estimation of the common mean μ.

Let $\rho = \sigma_y^2/\sigma_x^2$ denote the ratio of the variances. If ρ is known, then the minimal variance unbiased estimator of μ is given by

$\hat{\mu}_0 = \frac{\rho\bar{x} + \bar{y}}{1 + \rho}$, where $\bar{x} = n^{-1} \sum_{i=1}^{n} X_i$ and $\bar{y} = n^{-1} \sum_{i=1}^{n} Y_i$. $\hat{\mu}_0$ is also the maximum likelihood estimator of μ when ρ is known. In general, ρ is unknown.

When the ratio ρ of variances is unknown Graybill and Deal (1959), Seshadri (1963), Hogg (1960), Richter (1960) considered the estimation properties for medium-sized and large samples. Graybill and Deal (1959) suggested the use of the estimator

$$\hat{\mu}_{GD} = \frac{S_y^2\bar{x} + S_x^2\bar{y}}{S_x^2 + S_y^2}$$

where

$$S_x^2 = n^{-1} \sum_{i=1}^{n} (X_i - \bar{x})^2$$

$$S_y^2 = n^{-1} \sum_{i=1}^{n} (Y_i - \bar{y})^2 \quad .$$

They showed that the unbiased estimator $\hat{\mu}_{GD}$ has uniformly smaller variance than either sample mean, provided both sample sizes are greater than 10. The efficiency of $\hat{\mu}_{GD}$ has been studied by Graybill and Deal (1959).

S. Zacks (1966), Metha and Gurland (1969) presented estimators and considered small sample sizes. Zacks developed estimators for very small sample sizes. He considered two classes of randomized unbiased procedures. For both classes of estimators a two-sided F-test is performed. If $F = S_y^2/S_x^2$ falls in the interval $(1/\rho^*, \rho^*)$, where ρ^* is a constant , then the estimator $\hat{\mu}_z$ used is $\bar{\mu} = (\bar{x} + \bar{y})/2$. Otherwise, $\hat{\mu}_z$ is $\hat{\mu}_{GD} = \dfrac{S_y^2\bar{x} + S_x^2\bar{y}}{S_x^2 + S_y^2}$ for the first class, whereas for the second class estimator $\tilde{\mu}_z$ is estimated by $\bar{x}$ if $S_y^2/S_x^2 > \rho^*$ and by $\bar{y}$ if $S_y^2/S_x^2 < 1/\rho^*$. The value of ρ^*, in both classes, is the critical value of the F-test of significance according to which one decides whether to apply the estimators $\bar{\mu}$, $\hat{\mu}_{GD}$, $\bar{x}$ or $\bar{y}$.

The variances and the efficiency functions of these estimators are also studied in Zacks (1966) (2.3), (2.4), (3.4), (3.5), respectively. Explicit formulae for the efficiencies were given for the case of samples of size n=3 in Zacks (1966), (2.4) and (3.7). Further, as seen in Fig.(1), the efficiency of the estimator $\hat{\mu}_z$ is higher than that of $\tilde{\mu}_z$, over the range of $1/6 \leq \rho \leq 6$ for all values of ρ^*. Also Zacks concluded that the estimators $\hat{\mu}_z$ are superior to the estimators $\tilde{\mu}_z$.

Recently Cohen and Sackrowitz (1974) obtained a new unbiased estimator for the equal sample size case. They proved that the sample mean of the first population could be improved on provided the sample size is greater than 4; i.e., the estimator is uniformly better than the sample mean based on only one of the populations for $n \geq 5$. A particular estimator which results from Cohen and Sackrowitz (1974) is

$$\hat{\mu}_{CS} = [1 - C_n G(S_x^2, S_y^2)]\bar{x} + C_n G(S_x^2, S_y^2)\bar{y}$$

where

$$C_n = \begin{cases} (n-3)^2\ (n+1)^{-1}\ (n-1)^{-1} & \text{for n odd} \\ (n-4)\ (n+2)^{-1} & \text{for n even} \end{cases}$$

and

$$G(S_x^2, S_y^2) = \begin{cases} F(1, (3-n)/2,\ (n-1)/2,\ S_y^2/S_x^2 & 0 \le S_y^2/S_x^2 \le 1 \\ \frac{(n-3)}{(n-1)} \cdot \frac{S_x^2}{S_y^2} F(1,\ (5-n)/2,\ (n+1)/2,\ S_y^2/X_x^2) & S_y^2/S_x^2 \ge 1 \end{cases}$$

where F is the hypergeometric function. The estimator is unbiased and minimax for all $n \ge 5$. Although the given estimator is not based only on a sufficient statistic, it has sensible monotonicity properties and has desirable large sample properties.

Brown and Cohen (1974) showed that the sample mean of the first population can be improved on, provided the sample size in that population is greater than 2.

The estimator $\hat{\mu}_{BC} = \bar{x} + (\bar{y} - \bar{x})\{aS_{\bar{x}}^2/[S_{\bar{x}}^2 + (n-1)(S_{\bar{y}}^2/(n+2)) + (\bar{y} - \bar{x})^2/(n-2)]\}$ given by Brown and Cohen (1974) in Remark (2.1) is suggested when it is reasonable to feel that $\sigma_{\bar{y}}^2/\sigma_{\bar{x}}^2$ is large.

The plan of the present study is to derive the maximum likelihood estimator of the common mean μ, and to present some properties of this estimator. Further a Monte Carlo study is used to compare properties of the maximum likelihood estimator with properties of the other estimators presented.

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