SUNS: A NEW CLASS OF FACET DEFINING STRUCTURES FOR THE NODE PACKING POLYHEDRON

by

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ABSTRACT

Graph theory is a widely researched topic. A graph contains a set of nodes and a set of edges. The nodes often represent resources such as machines, employees, or plant locations. Each edge represents the relationship between a pair of nodes such as time, distance, or cost. Integer programs are frequently used to solve graphical problems. Unfortunately, IPs are NP-hard unless $P = NP$, which implies that it requires exponential effort to solve them. Much research has been focused on reducing the amount of time required to solve IPs through the use of valid inequalities or cutting planes. The theoretically strongest cutting planes are facet defining cutting planes.

This research focuses on the node packing problem or independent set problem, which is a combinatorial optimization problem. The node packing problem involves coloring the maximum number of nodes such that no two nodes are adjacent. Node packings have been applied to airline traffic and radio frequencies.

This thesis introduces a new class of graphical structures called suns. Suns produce previously undiscovered valid inequalities for the node packing polyhedron. Conditions are provided for when these valid inequalities are proven to be facet defining. Sun valid inequalities have the potential to more quickly solve node packing problems and could even be extended to general integer programs through conflict graphs.

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Acknowledgments

It is thanks to Dr. Todd Easton that this thesis is both tangible and logical. Were it not for his knowledge and exceptional teaching ability, I would not have had the success in my education that I have today. Additionally, I would like to recognize the efforts of Dr. Jessica Heier-Stamm and Dr. David Yetter in their work on my review committee. Lastly, my sincere thanks go to the faculty, staff, and students of the Kansas State University Department of Industrial and Manufacturing Systems Engineering for their support throughout my college career.

Dedication

My work is dedicated to my family and friends for their support and understanding.

Chapter 1

Introduction

An Integer Program (IP) is a mathematical optimization problem. IPs require integer values for their decision variables and take the form $Z^{IP} = max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n_+\}$. Unfortunately, IPs are known as NP-hard [22] which means that it requires exponential effort to solve them. For this reason, much research has been focused on reducing the amount of time required to solve IPs.

This thesis focuses on the integer programming formulation of the node packing problem, also known as the independent set problem. The node packing problem involves coloring the maximum number of nodes in a graph such that no two nodes are adjacent. This thesis introduces suns as a new graphic substructure that can generate useful cutting planes and may help to decrease the solution time required to find optimal node packings.

1.1 Integer Programs and the Node Packing Problem

Graph theory is a large portion of integer programming research. Research in this area has been applied to a number of unique instances such as truck routing [2, 23, 30, 32], facilities layouts [19], sports scheduling [14, 33], and even predicting protein flexibility [21].

Graphs are helpful visual representations of sometimes complicated relationships between variables. A graph $G = (V, E)$ is a set of vertices V, also referred to as nodes, and edges E such that $e = \{u, v\}$ where $u, v \in V$. The edges can also have weights; weights are values designating the cost or benefit of following a particular vertex and edge. The edges E are the relationships between parts of vertices such a time, distance, or cost and the vertices V are the entities such as machines, employees, or plant locations.

In order to optimize such a problem, it is often desirable to maximize the number of entities or resources that are used. This can be accomplished by what is known as node packing. For a graph $G = (V, E)$, a node packing contains a set of vertices $V' \subseteq V$ such that there is no edge $\{u, v\} \in E$ for any $u, v \in V'$.

In an integer program, each vertex $i \in V$ is assigned to an IP variable x_i. If $x_i = 1$, then x_i is in the node packing. If $x_i = 0$, it is not in the node packing. A weighted node packing is formulated as an IP: Maximize $\sum_{i=1}^{n} \omega_i x_i$ subject to $x_i + x_j \leq 1$ for all $\{i, j\} \in E$, $x_i \in \{0, 1\}$ for all $i \in V$.

Solving IPs is NP-Hard [22]. This means that they require exponential effort to solve and can either take a long time to solve or could require a greater amount of computer memory than is actually available. There are many methods/algorithms used to solve IPs.

Branch and Bound is the most commonly used method to solve IPs. This algorithm is initialized by solving the linear relaxation P_{LR} where the variables are not required to be integers. This initial solution is called the "root" of the branching tree. From the root, a series of child nodes, or child branches, are produced based upon the variable that is being branched on. For example, if $x_1 = 4.6$, the two child branches will be $x_1 \leq 4$ and $x_1 \geq 5$. This procedure continues until all child branched are fathomed. Fathoming occurs when the problem is infeasible, the linear relaxation solves to an integer value, or the objective value for the node is worse than the best known integer solution. In using branch and bound, exponentially many branches can be created before finding a solution. This is why a great deal of research has been focused on shortening IP solving time.

The most common method to decrease IP solve times is to implement cutting planes. Cutting planes are valid inequalities that eliminate non-integer space in the linear relaxation while leaving original integer solutions intact. These inequalities are often created by strengthening existing constraints in the original problem. Facet defining inequalities are theoretically the strongest form of cutting planes. Polyhedral Theory is the study of integer programs and their facets.

Lifting is one method used to strengthen cutting planes. Lifting can be accomplished by altering one or more coefficients of a known valid inequality. A valid cut that is stronger than the original inequality can be created by doing so. New lifting techniques are often being researched [3, 7, 8, 18, 28] and can be applied to myriad situations in graph theory.

Graph Theory involves studying graphs as visual representations of systems. Graphs consist of a set of vertices (also known as nodes) connected by edges. The edges are the relationships and the vertices are the entities. The edges can also have weights which are values designating the cost or benefit of following a particular edge between two vertices. For example, vertices could represent various locations while the edges represent the cost of flying between two cities and no edge represents a lack of flights between two particular cities.

Graph theory research includes various graph optimization problems. Graph Theory problems include the shortest route [5, 27], node colorings [12], and minimum cost flow problems [16]. Graph theory has been applied to a number of real-world applications including facilities layout planning [19], data clustering [34], and transportation [1, 2, 31]. Another interesting application of graph theory is in habitat dispersion. As Barahona describes in his paper [10], node packing can be used during forest planning to generate a cutting pattern to avoid destroying wildlife habitats. Because graph theory problems are discrete, they are often modeled as IPs.

This thesis focuses on the Node Packing problem, or the Independent Set problem, within Graph Theory. The goal of this problem is to select the maximum number of nodes such that no two selected nodes are adjacent. This problem is NP-hard [22].

Node packing can be applied to a number of situations. One example is discussed in a paper by Kuyumcu discusses the application of graph theory, more specifically the node packing problem, to revenue management in the airline industry [25]. The overall objective of this problem is to balance supply and demand in order to maximize revenue and profit. Factors that are considered in this problem include: competitiveness in the market, demand forecasts, fare classes and aircraft capacities. The polyhedral graph approach uses cutting planes and split graphs to reduce computer memory required to solve the IP. The author claims that this approach could be further applied to trucking, cruise lines, health care, etc. Other applications of node packing includes airline schedules [1], cellular frequencies [20], and habitat dispersion [10].

In certain instances, entities (vertices) and relationships (edges) fall such that certain graphical structures are subgraphs in the problem. Some such structures are cliques, wheels, and odd holes. These structures can generate valid or facet defining inequalities to make the IP easier to solve. This thesis introduces a new set of structures and their valid inequalities.

1.1.1 Motivation

The focus of this thesis is to explore a previously undiscovered class of graph structures which generate a set of valid inequalities. There has been a great deal of research at Kansas State University in the Industrial and Manufacturing Systems Engineering (IMSE) department on the node packing polytope and simultaneous lifting [11, 13, 15, 24, 29].

In 2009, Conley [13] expanded on research done on the node packing polytope. He interlaced two well-known graph structures: the clique and the odd hole. He called this new structure a cliqued hole. Conley concluded his research with a question as to whether another such structure could generate multiple valid inequalities by itself. This type of structure would be a version of synchronized simultaneous lifting as outlined in Boltons thesis [11]. The aim of this thesis is to discover such structures and to determine if they are facet-defining.

1.1.2 Contribution

During the research to find structures with implications to synchronized simultaneous lifting, a new class of structures called suns was discovered. Suns create valid inequalities of the node packing polyhedron. Certain classes of sun structures generate previously undiscovered classes of facet defining inequalities. Proofs are supplied to validate each of these claims and examples aid the reader identifying suns and their applications. The research also suggests that some of the inequalities produced by the structures are stronger than more commonly used inequalities.

Since suns generate new inequalities, they can be applied to help reduce the solution time required to solve node packing problems. Furthermore, suns are easily identifiable and a polynomial time algorithm is generated to identify instances where these cutting planes can be applied.

1.1.3 Outline

Chapter 2 gives the background information necessary to understand this thesis. First, basic Integer Programming and Node Packing is discussed. Polyhedral theory and cutting planes are explained. Finally, examples of several types of lifting are given.

Chapter 3 introduces the new class of structures: suns. Alternate classes of this structure and their valid inequalities are covered. Certain classes of suns are shown to produce certain facet defining inequalities. Proofs of validity and facet definition are provided. There is a polynomial time algorithm to identify a sun, and there are examples to demonstrate this algorithm.

Chapter 4 gives a conclusion of the major results and contributions to the field. In addition, areas for future research are provided along with suggested methods to approach these problems.

Chapter 2

Integer Programming Preliminaries

As stated earlier, an integer program (IP) is a maximization problem whose decision variables must be integer values and both the objective value and constraints are linear. An integer program takes the form: $Z^{IP} = max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n_+\}$ where $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^m$. Define P as the set of feasible integer points of an IP, where $P = \{x \in \mathbb{Z}^n_+ : Ax \leq b\}$ and let $N = \{1, ..., n\}$ be the indices of the variables.

IPs are typically solved by iteratively solving many linear programs. After this step, define the linear relaxation of an IP as $Z^{LR} = max\{c^T x : Ax \leq b, x \in \mathbb{R}^n_+\}$. Define the feasible region of a linear relaxation to be $P^{LR} = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$.

The branch and bound algorithm is one of the most commonly used methods to determine an optimal or feasible solution to an integer program. It is a search tree whose branches add constraints to the parents of the linear relaxation. To initialize this algorithm, create a search tree whose nodes are linear relaxation problems. The root node is the solution to the original IP relaxation. When this solution contains non-integers, branching is performed. Two branches are created from each node. The nodes stemming from these branches are called child nodes and the original node is called the parent node. One branch adds $x_i \leq \lfloor x_i \rfloor$ and the other branch is where $x_i \geq \lfloor x_i \rfloor + 1$. This process continues until all branches have been fathomed. Fathoming occurs when the problem is infeasible, the linear relaxation solves to an integer value, or the objective value for the node is worse than the best known integer solution.

Unfortunately, the branch and bound algorithm can require exponential effort. This is not surprising given the classification of integer programming problems. This is why many researchers have focused their research on decreasing the time it takes to solve an integer program.

2.1 Graphs and the Node Packing Problem

A large portion of integer programming research involves graph theory. Graphs are helpful visual representations of sometimes complicated relationships between entities. A graph $G = (V, E)$ is a set of vertices V, also referred to as nodes, and edges E such that $e = \{u, v\}$ where $u, v \in V$. The edges can also have weights; weights are values designating the cost or benefit of following a particular edge between two vertices. The edges E are the relationships and the vertices V are the entities.

Graph theory research includes various graph optimization problems such as: the shortest route [5, 27], node colorings [12], and minimum cost flow problems [16]. Graph theory has been applied to a number of real-world applications including facilities layout planning [19], data clustering [34], and transportation [1, 2, 31] and protein flexibility predictors [21]. As Jacob describes, in predicting protein flexibility the bond network is translated to a graph defined by covalent and hydrogen bonds. An algorithm is then used to count the degrees of freedom, or edges extending from each node, to determine the rigidity of the bonds. The number of excess edges past a certain limit quantifies the flexibility index of that bond. Jacobs claims that this graph theory method is one million times faster than molecular dynamics simulations.

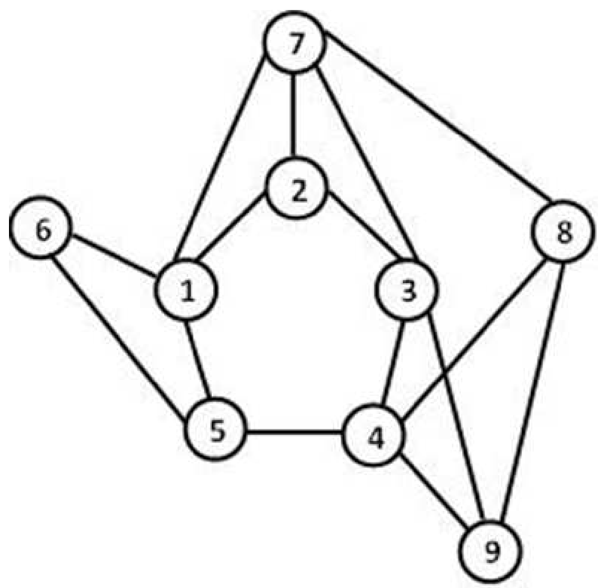

Figure 2.1: Sample Graph

For example, Figure 2.1 depicts a graph with nine nodes, nodes $1, ..., 9$ are the vertices in set V and each of the edges $\{1, 2\}, \{1, 5\}, ..., \{8, 9\}$ are the edges in set E. Many problems are solved by using graphs.

For a graph $G = (V, E)$, a node packing contains a set of vertices $V' \subseteq V$ such that there is no edge $\{u, v\} \in E$ for any $u, v \in V'$. In the nine-node example in Figure 2.2, a maximum of four nodes can be taken in the node packing such that no two chosen nodes are adjacent. This is called a node packing.

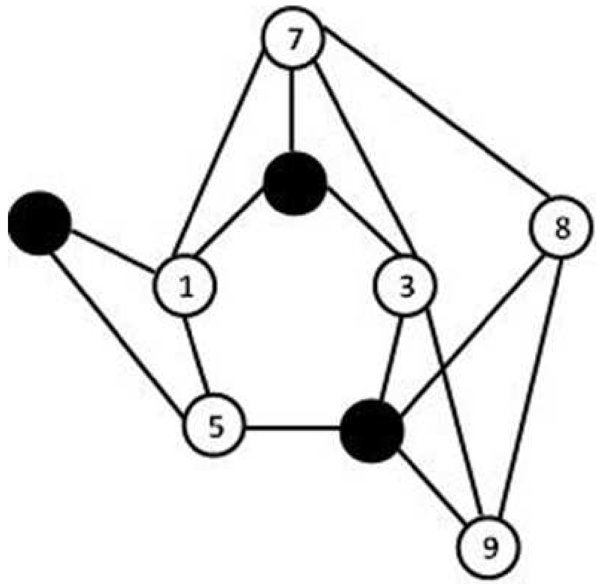

Figure 2.2: Sample Node Packing

The Node Packing problem is often solved as an IP. To model this problem, each vertex $i \in V$ is assigned to a binary variable x_i. If $x_i = 1$, then x_i is in the node packing, and if $x_i = 0$, it is not in the node packing. Then a weighted node packing is formulated as an IP: Maximize $\sum_{i=1}^{n} \omega_i x_i$ subject to $x_i + x_j \leq 1$ for all $\{i, j\} \in E$ $x_i \in \{0, 1\}$ for all $i \in V$ where ω_i is the weight of node i for all $i \in V$.

Node packing has been used to optimize a number of situations. Some examples of this include revenue management in the airline industry [25], train scheduling [35], and university course timetabling [6]. The last example is a case-study where a branch and bound algorithm yields the optimal solution of a timetabling problem of university courses. The problem is formulated as a set packing problem. Clique and lifted odd hole inequalities are used to improve the initial formulation. The combinatorial properties are then used to introduce new cutting planes. These cutting planes can yield the optimal solution for university timetabling instances with up to 69 courses, 59 teachers, and 15 rooms.

2.2 Polyhedral Theory and Cutting Planes

Polyhedral theory is an important area in mathematical programming research. It pertains to many of the definitions described in Section 2.1. This section further defines topics relevant to the studies of feasible space for both linear programs and integer programs.

The idea of convexity is critical to linear and integer programs. A set $S \subseteq \mathbb{R}^n$ is convex if and only if for all $x,\ y \in S$, $\lambda(x) + (1-\lambda)(y) \in S$ for all $\lambda \in [0,1]$. This means that a space is convex is and only if a line segment can be drawn from any point in the space to any other point in the space without touching any point outside of the space.

The feasible space of a linear relaxation P^{LR} is trivially convex. The integer points within a linear relaxation are the feasible points for an integer program. The set of integer points P^{LR} inside the linear relaxation are not convex. This is because the fractional points between the integer points are not part of the set P^{IP}. The minimum convex region containing a set of points P^{IP} is called the convex hull and is denoted as P^{ch}. The convex hull P^{ch} is the intersection of all convex sets that contain P^{IP}.

A half space is defined by $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$. A half space is convex. A finite intersection of half spaces is called a polyhedron. A bounded polyhedron is a polytope. In integer programming research, both P^{ch} and P^{LR} are critical polyhedrons. The integer corner points of the linear relaxation are critical points of the IP. The goal of polyhedral theory in relation to IP is to alter P^{LR} to become P^{ch}.

A cutting plane $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ is an inequality used to further constrain a linear relaxation. A cutting plane is considered a valid inequality if and only if $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ for all $x \in P$. Thus, every point in P must satisfy the inequality, ensuring that the cutting plane does not eliminate a valid point.

A face F created by the cutting plane on P^{LR} is the points in P^{ch} which meet the inequality at equality, $F = \{x \in P^{ch}) : \alpha^T x = \beta\}$. The inequality is said to be facet defining if and only if the dimension of its face is one less than the dimension of the polyhedron.

The points $x_1, ..., x_n$ are affinely independent if $\sum_{i=1}^{n} \lambda_i x_i = 0$ $\sum_{i=1}^{n} \lambda_i = 0$ and are uniquely solved by $\lambda_i = 0$. The affinely independent points on the face can be used to create one less linearly independent vector. Thus the dimension of a face F of the polyhedron is equal to the maximum number of linearly independent vectors or the maximum number of affinely independent points minus 1. To describe these concept consider the following example.

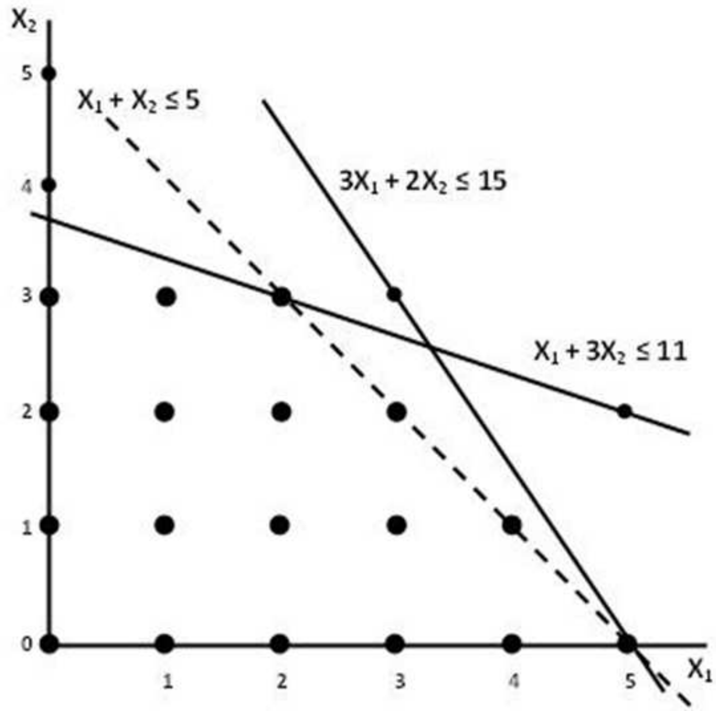

Figure 2.3: Cutting Planes

In this example, the objective function is maximize x_1+2x_2 subject to $3x_1+2x_2 \leq 15$ and $x_1 + 3x_2 \leq 11$ and $x_1, x_2 \in \mathbb{R}$. The optimal solution to the linear relaxation is $(x_1, x_2) = (\frac{23}{7}, \frac{18}{7})$. The cutting plane $x_1 + x_2 \leq 5$ eliminates this solution and results in a new linear relaxation solution $(2, 3)$.

To prove that this cut is facet defining, the dimension of the P^{ch} must first be bounded below; $dim(P^{ch}) \leq 2$ because it has two variables x_1 and x_2. Next it is bounded above; $dim(P^{ch}) \geq 2$ since the points $(0, 0)$, $(0, 1)$ and $(1, 0)$ are affinely independent and feasible. Thus $dim(P^{ch}) = 2$. It is trivial to see that the face $x_1 + x_2 \leq 5$ is valid since no points in P are eliminated. Since $dim(P^{ch}) = 2$ the $dim(F)$ is bounded below by the dimension of P^{ch} less one, $dim(F) \leq 1$. The points $(2, 3)$ and $(3, 2)$ meet the inequality

$x_1 + x_2 \leq 5$ at equality and are affinely independent. Thus $dim(F) = 1$ and it defines a facet.

This thesis focuses on the node packing problem, so let the set of feasible solutions for the integer programming node packing formulation be P^{NP}. The goal of this thesis is to more tightly describe (P^{ch}_{NP}).

2.2.1 Cutting planes and the Node Packing Problem

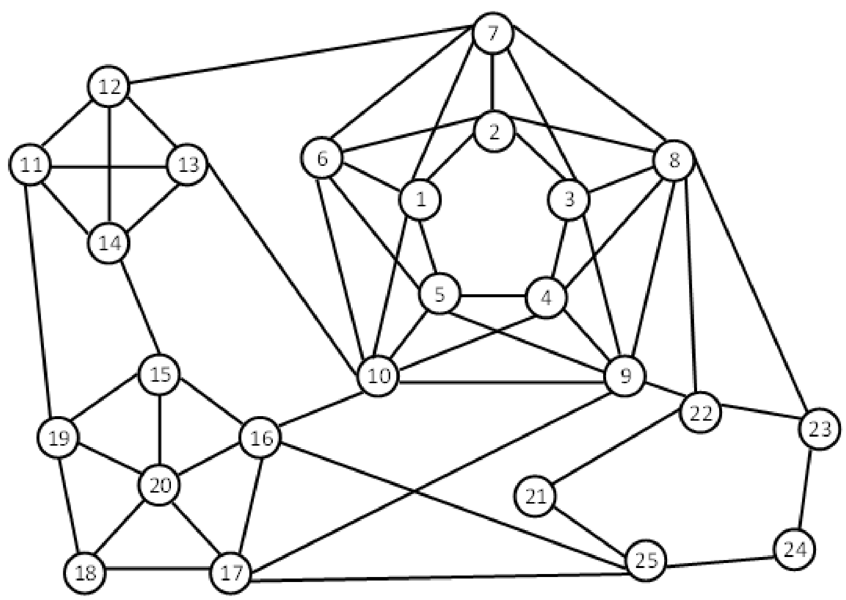

Figure 2.4: Graphical Structures

There are a series of structures that have been discovered within graphs that can generate valid, and at times, facet defining inequalities of P^{ch}_{NP}. Some such classic subgraphs are the clique, odd hole, and wheel.

A clique occurs when all vertices V within a subgraph are adjacent to every other vertex in the subgraph. A clique is denoted k_p, with p being the number of nodes in the clique. A k_4 clique can be seen in the example graph at vertices 11,12,13, and 14. Because each vertex is adjacent to every other vertex in the subgraph, only one vertex can be taken in a node packing. This creates what is known as a clique cut, $\sum_{i \in k_p} x_i \leq 1$. In the example, the inequality $x_{11} + x_{12} + x_{13} + x_{14} \leq 1$ is valid. Furthermore, since the clique is maximal [26, 32], this is a facet defining inequality.

An odd hole is denoted Hp and has an odd number of vertices in its subgraph. An odd hole is also called a chordless cycle. Each vertex is adjacent to only two other vertices and it is a connected graph. An odd hole can be identified in the example graph with vertices 21, 22, 23, 24, and 25. In an odd hole, at most one less than half of the vertices can be selected. This equates to the hole cut $\sum_{i \in Hp} x_i \leq \lfloor |Hp|/2 \rfloor$. More specifically, for the example subgraph, the cut $x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \leq 2$ is valid. Hole cuts can be facet defining but are frequently not.

A wheel w_p has a central node and a number of spokes branching from the center to outer nodes. A w_p can be identified in the example graph with vertices 15, 16, 17, 18, 29, and 20. The center node is denoted x_0 and the wheel had p outer nodes. The wheel cut inequality is $\lfloor \frac{p-1}{2} \rfloor x_0 + \sum x_i \leq \lfloor \frac{p-1}{2} \rfloor$.

These cuts can eliminate non-integer space in the P_{NP}^{ch}. For example, the k_4 clique vertices can have coefficients equal to 2 such that $(x_{11}+x_{12}+x_{13}+x_{14}) = (0.5, 0.5, 0.5, 0.5)$ thus the $\sum_{i \in k_4} x_i = 2 > 1$.

A popular area of research is the modification of one or more of these structures, such as Conley's structures [13], the cliqued hole and the odd bipartite hole. A cliqued hole is denoted $CH_{m,P}$. It can be seen in the subgraph containing vertices 1,..,10. It contains an inner an outer hole. Each pair of inner vertices is "cliqued" with a pair of vertices in the outer hole. This structure creates the valid inequality $\sum_{i \in CH_{m,P}} x_i \leq \lfloor m/2 \rfloor$.

2.2.2 Lifting

Lifting is used to strengthen existing valid inequalities by introducing new variables. This technique was first introduced by Gomory [17]. There are many other methods of lifting such as exact sequential lifting, Balas' method [7], and simultaneous sequential lifting [4, 8, 9, 18, 28].

Restricted spaces are critical to lifting. Let $D \subset N$ and $K \subset \mathbf{Z}^{|D|}$ define the restricted space of the convex hull of the integer program on D and K as $P^{ch}_{D,K} = conv\{x \in P : x_j = k_j \text{ for all } j \in D\}$. Thus, each variable associated with D is assigned to a specific integer value k.

Lifting takes a valid inequality $\sum_{i \in D} \alpha_i x_i + \sum_{i \in N \setminus D} \alpha_i x_i \leq \beta$ for $P^{ch}_{D,K}$ and seeks to create a valid inequality of $\sum_{i \in D} \alpha'_i x_i + \sum_{i \in N \setminus D} \alpha_i x_i \leq \beta'$ of P^{ch}. There are four broad classes of lifting that are based upon the size of D, values selected for α' and β', the values of K and the number of new inequalities generated determine the particular class of lifting.

In up lifting, the elements in K are equal to zero in the restricted space and the right-hand side of the lifted inequality remains the same. In down lifting, the elements in K are set to their upper bounds and the value of the right-hand side often decreases. Middle lifting combines the two methods of up and down lifting.

In sequential lifting, one variable is lifted at a time so that $|D| = 1$ [8, 28]. Because one variable is lifted at a time, the inequality changes before lifting each remaining variable; this can have an effect on the coefficients of the later lifted variables. In simultaneous lifting, all variables in set D are lifted at once and $|D| \geq 2$.

Exact lifting produces the strongest possible inequality. That is, any increase in α' or decrease in β' makes the lifted inequality become invalid. Thus, an exact lifted inequality supports P^{ch}. This is ideal except that exact lifting can be very difficult to obtain the exact coefficients, which can require solving an optimization problem. Several approximate lifting heuristics [3, 8, 18] have been developed to create valid inequalities without the hurdle of exact lifting. These techniques are faster and can yield inequalities strong enough to be useful.

Synchronized simultaneous lifting was introduced by Bolton in 2009 [11] to find a new class of lifted inequalities. The first step of the synchronized simultaneous lifting algorithm (SSLA) is to create a set a table with two mutually exclusive sets or columns C and E. The values in C are the number of variables that can be picked up in the original set and values in E are the number of additional variables that can be picked up simultaneously. These points can then be graphed to find values of α_1 and α_2 for the extreme points. Let α_1 be the coefficient of the variables in the cover and α_2 be the coefficient of the variables outside the cover. The new facet defining inequality is of the form $\alpha_1 \sum_{i \in C} x_i + \alpha_2 \sum_{i \in E} x_i \leq \beta$ with β being the right-hand side of the original inequality.

The inequalities generated by synchronized simultaneous lifting can be used to more quickly solve integer programs. This thesis focuses on identifying induced subgraph structures that, if found within a graph, generates valid inequalities that follow SSL's technology. The hope is that these graph structures would help decrease the solving time of a node packing problem.

Chapter 3

Suns and the Node Packing Polyhedron

This chapter introduces a new graphical structure: The Sun. There are two classes of suns: Symmetric and Nonsymmetric. This structure generates valid inequalities for the node packing polyhedron and it can, at times, be facet defining.

A graph is a sun with parameters p, q, and r and is denoted $S(p,q,r)$ when the following conditions are met. The sun consists of two sets of nodes that create an inner hole with p vertices and one in an outer hole with qp vertices. The vertices in the outer hole are grouped into p clusters each of size q. The nodes in the inner hole are only adjacent to r clusters of q nodes in the outer hole with these clusters being consecutively ordered. Figures 3.1 and 3.2 demonstrate a $S(5,1,2)$ and a $S(7,3,3)$.

Formally, a graph $G = (V, E)$ is a sun with parameters p, q and r if and only if

$|V| = p(1+q)$ and V can be partitioned into V_I and V_O such that $V_I = \{v_1, v_2, ..., v_p\} = p$, $V_O = \{v_{p+1}, v_{p+2}, ..., v_{p+pq}\}$, the induced subgraphs of both V_I and V_O are holes and the only edges between V_I and V_O take the form $\{v_i, v_{(i-1)*q+j-1 \ mod \ p)+p+1}\}$ for all $j = 1, 2, ..., q*r$ and $i = 1, ..., p$.

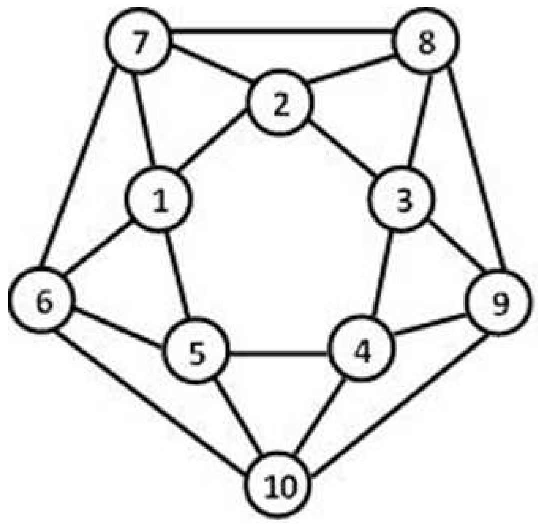

Figure 3.1: S(5,1,2) symmetric sun

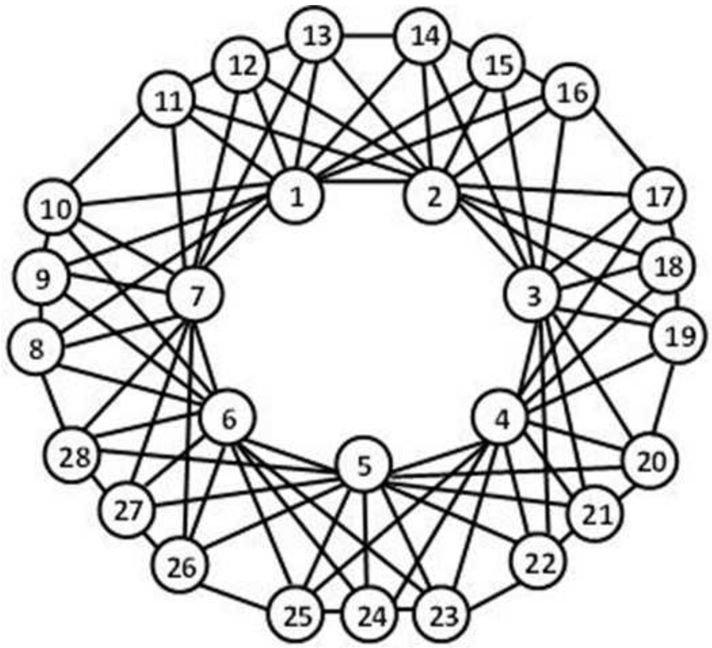

Figure 3.2: S(7,3,3) nonsymmetric sun

This chapter introduces sun cutting planes for the node packing polyhedron. In doing this, suns are divided into two classes, symmetric in section 3.1 and nonsymmetric in section 3.2. These sections provide valid inequalities and conditions for facet defining inequalities.

3.1 Symmetric Suns

A symmetric sun has the same number of nodes in its inner and outer holes, forcing $q = 1$. Because these suns are comparatively small to nonsymmetric suns with large q, they have some interesting properties. This section has three subclasses of suns: $r = 1$, $r = 2$, and $r \geq 3$. All symmetric suns produce valid inequalities.

3.1.1 Symmetric Suns with $r = 1$

Symmetric suns with $r = 1$ can be represented as $S(p, 1, 1)$. Each vertex in the inner or outer hole is connected to only the vertices in its hole and to one vertex in the opposing hole. Figure 3.3 depicts a $S(5, 1, 1)$. This type of sun produces a valid inequality that is stronger than the hole inequalities. However, this valid inequality is not facet defining for this type of sun.

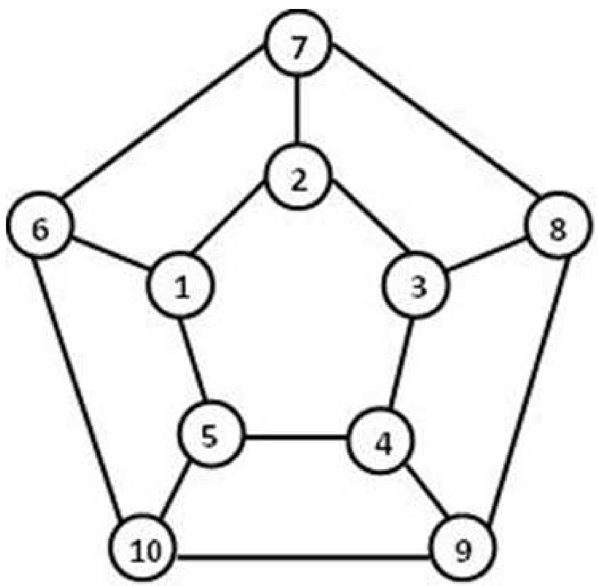

Figure 3.3: S(5,1,1)

Theorem 3.1.1 *Given a graph $G = (V, E)$ such that $S_{p,1,1}$ is an induced subgraph of G with $p \geq 5$ and odd, then $\sum_{i \in V_I} \frac{1}{p-1} x_i + \sum_{i \in V_0} \frac{1}{p-1} x_i \leq 1$ is a valid inequality of P_{NP}^{ch}.*

Proof:

Clearly, no more that $\frac{p-1}{2}$ nodes can be selected from either hole in any node packing due to p being odd. Thus, $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq p - 1$ must be satisfied by every node packing and the result follows.

□

Even though this inequality is valid, it is not a facet defining inequality. Unfortunately, the inequality has dimension one less than the dimension needed to prove facet defining. While this inequality may be useful, including both hole constraints would be far more useful as each of these inequalities are facet defining over $P^{ch}_{NP\ S_{p,1,1}}$

3.1.2 Symmetric Suns with $r = 2$

This subclass of suns is denoted $S(p, 1, 2)$ and Figure 3.4 shows a $S(5, 1, 2)$ sun. The symmetric suns with $r = 2$ create five interesting valid inequalities. Because of the symmetry of these suns, two of the five inequalities are found to be facet defining. Two inequalities are known not to be and one is suspected of being facet defining.

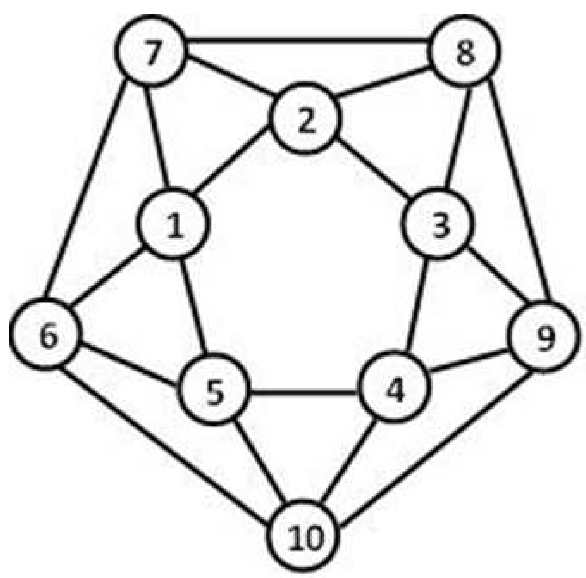

Figure 3.4: S(5,1,2)

If a symmetric sun has a sufficiently large p, it produces a set of five valid inequalities. Surprisingly the existence of the fifth inequality is dependant based on $p \; mod \; 3$. Formally,

Theorem 3.1.2 *Given a graph $G = (V, E)$ such that $S_{p,q,r}$ is an induced subgraph of G with $p \geq 5$ and odd, $q = 1$ and $r = 2$. Then the following inequalities are valid for P_{NP}^{ch}*

i) $\sum_{i \in V_I} x_i \leq \lfloor \frac{p}{2} \rfloor$,

ii) $\sum_{i \in V_I} 2 * x_i + \sum_{i \in V_0} x_i \leq p$,

iii) $\sum_{i \in V_I} x_i + \sum_{i \in V_0} 2 * x_i \leq p$,

iv) $\sum_{i \in V_O} x_i \leq \lfloor \frac{p}{2} \rfloor$.

Furthermore if $p \; mod \; 3 = 1$ or 2, then

v)$\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{2p-2}{3}$ *and*

vi) $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{2p-1}{3}$

are valid for P_{NP}^{ch}, respectively.

Proof: Since the holes are symmetric, one only needs to consider the inequalities i), ii), v) and vi). Clearly, V_I is an odd hole on p nodes. Since an odd hole generates a valid inequality of the form i), i) is a valid inequality.

Assume ii) is not a valid inequality of P_{NP}^{ch}. Thus there exists an $x' \in P_{NP}$ such that $\sum_{i \in V_I} 2 * x'_i + \sum_{i \in V_0} x'_i > p$. At most $\frac{p-1}{2}$ x'_i can be selected from V_I due to the odd hole structure. In such a situation at most one x'_i can be selected from the outer hole, which satisfies this inequality. Due to the edges of a sun between the inner and outer hole, any removal of a vertex from the inner hole enables at most 2 vertices to be added to the outer hole. Thus, inequalities ii) and iii) are valid.

Now assume $p \ mod \ 3 = 1$ and let $x' \in P_{NP}$. It is evident that if node v_1 and v_k are in the node packing and no nodes are in the node packing between these two then the maximum number of nodes in any node packing between v_{p+1} and v_{p+k} occurs when v_{p+3}, v_{p+5}, ..., either v_{p+k-2} or v_{p+k-1} is in the node packing depending upon the value of k. The total number of nodes in the node packing is maximized when $k = 4$. Extending this pattern of skipping two nodes on the inner hole and also on the outer hole results in a total of $\frac{1}{3}(p-1) + \frac{1}{3}(p-1) = \frac{2}{3}(p-1)$ maximum nodes in the node packing for this sun structure. Thus, $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{2}{3}(p-1)$ is a valid inequality.

The case where $p \ mod \ 3 = 2$ follows similarly. Again the maximum number of nodes occur when the node packing contains v_1 and v_4 and no nodes are in the node packing between these two. In this situation. Extending this pattern of skipping two nodes on the inner hole and also on the outer hole results in a total of $\frac{1}{3}(p+1)$ nodes in the inner hole, which leaves at most $\frac{1}{3}(p-2)$ nodes in the outer hole. Observe that the additional node in the inner hole is always capable of being obtained since $p \ mod \ 3 = 2$. Thus, there are at most $\frac{1}{3}(p+1) + \frac{1}{3}(p-2) = \frac{2p-1}{3}$ nodes the node packing and the result follows.

□

These symmetric suns have some surprising properties. First the hole inequalities are facet defining, but the next inequalities are not facet defining. It is believed that the fifth and sixth inequalities are facet defining but a proof alludes me to this point. Fortunately, a lower boundis available on the faces for these inequalities as the following theorem shows.

Theorem 3.1.3 *Given a graph $G = (V, E)$ such that $S_{p,q,r}$ is an induced subgraph of G with $p \geq 5$ and odd, $q = 1$ and $r = 2$. Then the following statements are true and are shown as they relate to P_{NP}^{ch}:*

i) $\sum_{i \in V_I} x_i \leq \lfloor \frac{p}{2} \rfloor$ defines a face of at least $2p - 1$,

ii) $\sum_{i \in V_I} 2 * x_i + \sum_{i \in V_0} x_i \leq p$ defines a face with dimension of at least $p - 1$,

iii) $\sum_{i \in V_I} x_i + \sum_{i \in V_0} 2 * x_i \leq p$ defines a face with dimension of at least $p - 1$,

iv) $\sum_{i \in V_O} x_i \leq \lfloor \frac{p}{2} \rfloor$ defines a face with dimension of at least $2p - 1$.

Furthermore if p mod $3 = 1$ or 2, then

v) $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{2p-2}{3}$ defines a face with dimension of at least p, and

vi) $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{2p-1}{3}$ defines a face with dimension of at least p.

Proof: Since the point 0 is always feasible and never meets a sun inequality at equality, none of the faces of these inequalities is P_{NP}^{ch}. Therefore, it suffices to find the requisite number of feasible affinely independent points that satisfy each inequality at equality.

Consider the two hole inequalities i) and iv) of each set $\sum_{i \in V_I} x_i \leq \lfloor \frac{p}{2} \rfloor$. Then consider the following $2p$ points. The x values are $\sum_{i=0}^{\lfloor \frac{p}{2} \rfloor - 1} \xi_{((2*(i)-1+j) \ mod \ p)+1}$ for all $j = 1, ..., p$ and $\sum_{i=0}^{\lfloor \frac{p}{2} \rfloor - 1} \xi_{((2*(i)-1+j) \ mod \ p)+1} + \xi_{p+q-j+1}$ for all $j = 1, ..., p$. These points are clearly feasible, affinely independent and also meet $\sum_{i \in V_I} x_i \leq \lfloor \frac{p}{2} \rfloor$ at equality. Thus, this inequality is facet defining for P_{NP}^{ch}. When $q = 1$ the inner hole and outer hole are symmeteric. Thus, a similar set of points shows that the other hole is facet defining also.

For ii) consider the following points. $\xi_j + \sum_{i=0}^{\lfloor \frac{p}{2} \rfloor -1} \xi_{((2*(i)+1+j) \ mod \ p)+p+1}$ for all $j = 1, ..., p$. Clearly, no nodes are adjacent and these points meet ii) $\sum_{i \in V_I} 2*x_i + \sum_{i \in V_0} x_i \leq p$ at equality. Furthermore, when considered as a matrix, the upper right is the identity matrix and so these are p affinely independent points. Thus, its face is of dimension at least $p-1$ in P_{NP}^{ch}. Due to symmetry, inequality iii) follows similarly.

Assume $p \ mod \ 3 = 1$, then the inequality $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{2}{3}(p-1)$ is valid by Theorem 3.1.3 for P_{NP}^{ch}. Consider the following two sets of points. Set one contains the points $\sum_{i=0}^{\frac{p-4}{3}} \xi_{((3*(i)-1+j) \ mod \ p)+1} + \sum_{i=0}^{\frac{p-4}{3}} \xi_{((3*(i)+j+1) \ mod \ p)+p+1}$ for each $j = 1, ..., p$. The second set contains the points $\sum_{i=0}^{\frac{p-7}{3}} \xi_{((3*(i)-1+j) \ mod \ p)+1} + \sum_{i=0}^{\frac{p-7}{3}} \xi_{((3*(i)+j+1) \ mod \ p)+p+1} + \sum_{i=0}^{1} \xi_{((p-3+2*(i)+j-1) \ mod \ p)+p+1}$ for each $j = 1, ..., p$.

The first set of points contain $\frac{p-1}{3}$ $x_i = 1$ in the inner hole and $\frac{p-1}{3}$ $x_i = 1$ in the outer hole. Thus, there are $\frac{2p-2}{3}$ points in the sun with $x_i = 1$ and so these points meet the inequality at equality. The second set of points have $\frac{p-4}{3}$ $x_i = 1$ in the inner hole and $\frac{p-4}{3} + 2$ $x_i = 1$ in the outer hole. Thus, there are $\frac{2p-8+6}{3}$ points in the sun with $x_i = 1$ and so these points meet the inequality at equality. Thus these are $2p$ points that meet the sun inequality at equality.

It is simple to argue that these sets of points are in P_{NP}. Translating the x values into a set of vertices yields that no two nodes on either of the holes are adjacent to each other. Furthermore, due to the gap of size three between vertices in the node packing in the inner hole, there can exist exactly one vertex on the outer hole. Thus, there are no edges between vertices in the inner and outer hole. Consequently each of these points is in P_{NP}.

To show that these points are $p+1$ affinely independent, perform the following column operations. Replace the second set of points by subtracting the j^{th} point in set one from the j^{th} point in set two for $j = 1, ..., p$. Thus, the j^{th} column in the second set becomes the points $\sum_{i=0}^{\frac{p-4}{3}} \xi_{((3*(i)-1+j)\ mod\ p)+1} + \sum_{i=0}^{\frac{p-4}{3}} \xi_{((3*(i)+j+1)\ mod\ p)+p+1} - (\sum_{i=0}^{\frac{p-7}{3}} \xi_{((3*(i)-1+j)\ mod\ p)+1} + \sum_{i=0}^{\frac{p-7}{3}} \xi_{((3*(i)+j+1)\ mod\ p)+p+1} + \sum_{i=0}^{1} \xi_{((p-3+2*(i)+j-1)\ mod\ p)+p+1})$ for each $j = 1, ..., p$.

Observe that the upper right quadrant is now a permuted identity matrix and thus there are at least $p+1$ affinely independent points. The additional point comes from not having all zeros in the remainder of the matrix. Therefore, this face has dimension of at least p and this portion of the result holds.

Assume $p\ mod\ 3 = 2$, then the inequality $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{2p-1}{3}$ is valid by Theorem 3.1.3 for P_{NP}^{ch}. Consider the following two sets of points. Set one contains the points $\sum_{i=0}^{\frac{p-2}{3}} \xi_{((3*(i)-1+j)\ mod\ p)+1} + \sum_{i=0}^{\frac{p-5}{3}} \xi_{((3*(i)+j+1)\ mod\ p)+p+1}$ for each $j = 1, ..., p$. The second set contains the points $\sum_{i=0}^{\frac{p-5}{3}} \xi_{((3*(i)-1+j)\ mod\ p)+1} + \sum_{i=0}^{\frac{p-2}{3}} \xi_{((3*(i)+j+8)\ mod\ p)+p+1}$ for each $j = 1, ..., p$.

The first set of points contain $\frac{p+1}{3}$ $x_i = 1$ in the inner hole and $\frac{p-2}{3}$ $x_i = 1$ in the outer hole. Thus, there are $\frac{2p-1}{3}$ points in the sun with $x_i = 1$ and so these points meet the inequality at equality. The second set of points follow similarly and also meet the sun inequality at equality.

It is simple to argue that these sets of points are in P_{NP}. Translating the x values into a set of vertices yields that no two nodes on either of the holes are adjacent to each other. Furthermore, due to the gap of size three between vertices in the node packing in the inner hole, there can exist exactly one vertex on the outer hole. Thus, there are no edges between vertices in the inner and outer hole. Consequently each of these points is in P_{NP}.

To show that these points are $p+1$ affinely independent, perform the following column operations. Replace the second set of points by subtracting the j^{th} point in set one from the j^{th} point in set two for $j = 1, ..., p$. Thus, the j^{th} column in the second set becomes the points $\xi_{((3*(\frac{p-2}{3})-1+j) \ mod \ p)+1} + \sum_{i=0}^{\frac{p-2}{3}} \xi_{((3*(i)+j+8) \ mod \ p)+p+1} - \sum_{i=0}^{\frac{p-5}{3}} \xi_{((3*(i)+j+1) mod \ p)+p+1}$ for each $j = 1, ..., p$.

Next replace row $p+j$ by row $p+j+\sum_{i=0}^{\frac{p-2}{3}}$ row $j+1+ip+1-\sum_{i=0}^{\frac{p-5}{3}}$ row $j+1+ip+1$ for $j = 1, ..., p$. This creates an upper right matrix of 0s and a lower right permuted identity matrix. Thus, these are at least $p+1$ affinely independent points and its dimension is at least p and the result holds.

$\square$.

For example, reconsider the graph in Figure 3.4. Clearly this has a $S_{p,q,r}$ with $p = 5$ and odd for $q = 1$ and $r = 2$. Then the valid inequalities are

i) $\sum_{i \in V_I} x_i \leq 2$,

ii) $\sum_{i \in V_I} 2 * x_i + \sum_{i \in V_0} x_i \leq 5$,

iii) $\sum_{i \in V_I} x_i + \sum_{i \in V_0} 2 * x_i \leq 5$,

iv) $\sum_{i \in V_O} x_i \leq 2$,

v)$\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq 3$.

It is the belief of the author that both v) and vi) are facet defining. Numerous examples were applied and all had facet defining properties. However, the proof is still unanswered. The basis for a proof is implemented below and it appears to terminate in a p by p matrix that is most likely linearly independent, but the matrix is not structured enough for a proof.

The points used in Theorem 3.1.3 are depicted in the Matrix 3.1. The j^{th} column is subtracted from the "$(p + j)^{th}$" column which results in the Matrix 3.2. Next rows are combined to create a lower right matrix of 0s. This leaves the matrix in the lower left of Matrix 3.3. It appears as though that matrix is affinely independent, but no such proof has yet been obtained and is left as future research. In this particular instance, an inverse exists to this lower left matrix and thus this is a facet defining point.

$$\begin{vmatrix}
1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&1&0&0&1&0&0\\
0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&1&0&0&1&0\\
0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&1&0&0&1\\
1&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&1&0&0\\
0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&1&0\\
0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&1\\
1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0\\
0&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\
0&0&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0\\
1&0&0&1&0&0&1&0&0&1&0&0&0&0&1&0&0&1&0&0&1&0\\
0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&1&0&0&1&0&0&1\\
0&0&0&1&0&0&1&0&0&1&0&0&0&1&0&1&0&0&1&0&0&1\\
0&0&0&0&1&0&0&1&0&0&1&1&0&0&1&0&1&0&0&1&0&0\\
1&0&0&0&0&1&0&0&1&0&0&0&1&0&0&1&0&1&0&0&1&0\\
0&1&0&0&0&0&1&0&0&1&0&0&0&1&0&0&1&0&1&0&0&1\\
0&0&1&0&0&0&0&1&0&0&1&1&0&0&1&0&0&1&0&1&0&0\\
1&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0\\
0&1&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&0&1\\
0&0&1&0&0&1&0&0&0&0&1&1&0&0&1&0&0&1&0&0&1&0\\
1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1\\
0&1&0&0&1&0&0&1&0&0&0&1&0&1&0&0&1&0&0&1&0&0\\
0&0&1&0&0&1&0&0&1&0&0&0&1&0&1&0&0&1&0&0&1&0
\end{vmatrix}$$

(Matrix 3.1)

$$
\begin{vmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0
\end{vmatrix}
$$

(Matrix 3.2)

$$
\begin{vmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -3 & 4 & 0 & -1 & 3 & -1 & 1 & 2 & -2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 1 & -3 & 4 & 0 & -1 & 3 & -1 & 1 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 3 & 1 & -3 & 4 & 0 & -1 & 3 & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 3 & 1 & -3 & 4 & 0 & -1 & 3 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & -2 & 3 & 1 & -3 & 4 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 2 & -2 & 3 & 1 & -3 & 4 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & 1 & 2 & -2 & 3 & 1 & -3 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 1 & 2 & -2 & 3 & 1 & -3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 3 & -1 & 1 & 2 & -2 & 3 & 1 & -3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & -1 & 3 & -1 & 1 & 2 & -2 & 3 & 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 4 & 0 & -1 & 3 & -1 & 1 & 2 & -2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix}
$$

(Matrix 3.3)

3.1.3 Symmetric Suns with $r \geq 3$

This subclass of suns is denoted $S(p, 1, r)$ such that $r \geq 3$. As r increases in symmetric suns, less inequalities are generated. In fact, when $r \geq 3$, there is only a single inequality as the obvious hole inequalities are dominated by the sun inequality. Figure 2.7 shows a $S(7, 1, 4)$ sun.

The following theorem describes the valid inequalities generated by this class of suns.

Theorem 3.1.4 *Given a graph $G = (V, E)$ such that $S(p, 1, r)$ is an induced subgraph of G with $p \geq 5$ and odd and $r \geq 3$, then $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \lfloor \frac{p}{2} \rfloor$ is a valid inequality*

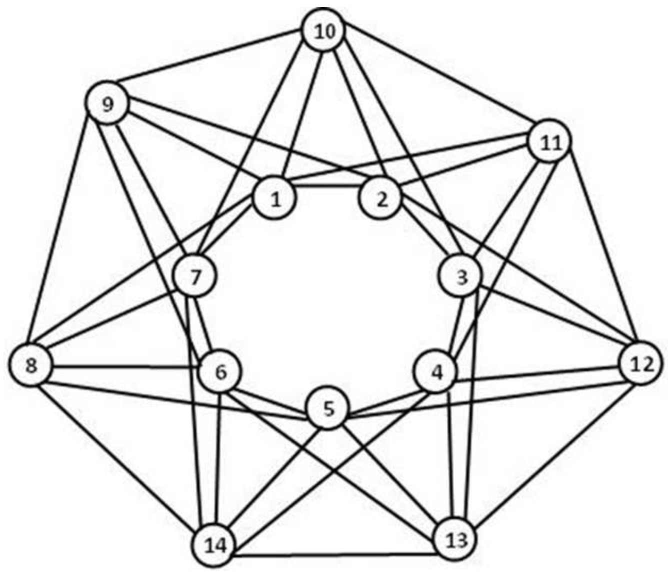

Figure 3.5: S(7,1,4)

of $P^{ch}_{NP\ S(p,1,r)}$.

Proof: Let x' be any point in $P^{ch}_{NP\ S(p,1,r)}$ and V' be the corresponding node packing. Clearly V' can have at most $\lfloor \frac{p}{2} \rfloor$ vertices in V_I and in such a situation there are no vertices in V_O. Similarly, V' can have at most $\lfloor \frac{p}{2} \rfloor$ vertices in V_O and in such a situation there are no vertices in V_I. Therefore, this inequality can only be invalid if there is at least one x_i set to one in both of the holes. Define $t_I = \sum_{i \in V_I} x'_i$ and $t_O = \sum_{i \in V_O} x'_i$. Observe that if $t \geq \lceil \frac{p-r}{2} \rceil + 1$, then x' has represents no vertices from V_O in V'. Thus, it suffices to consider t for $t = 1, ..., \lceil \frac{p-r}{2} \rceil$.

If $t = 1$, then at most $T_O \leq \lceil \frac{p-r}{2} \rceil$. Since $r \geq 3$, $1 + \lceil \frac{p-r}{2} \rceil \leq \lceil \frac{p-r+2}{2} \rceil \leq \lfloor \frac{p-1}{2} \rfloor$. Increasing t_I by one at least eliminates the ability to add two nodes in V_O. Thus, increasing t by one decreases t_O by at least one and the result follows.

□

With validity established, the focus turns toward determining the strength of the sun inequality in this case. It is not too surprising that this sun inequality is facet defining as the following result shows.

Theorem 3.1.5 *Given a graph $G = (V, E)$ such that $S(p, 1, r)$ is an induced subgraph of G with $p \geq 5$ and odd, and $r \geq 3$, then $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{p-1}{2}$ is a facet defining inequality over $P^{ch}_{NP\ S(p,1,r)}$. Furthermore, if every $v \in V \setminus S(p, 1, r)$ is incident to at most two vertices in either the inner or outer hole, then this inequality is facet defining for P_{NP}.*

Proof: By Theorem 3.1.5, this sun inequality is valid. Furthermore, 0 never meets this inequality at equality, and thus this inequality has dimension less than or equal to $2p - 1$ on $P^{ch}_{NP\ S(p,1,r)}$. It is therefore sufficient to find $2p$ affinely independent points that meet this inequality at equality.

Consider the points $\sum_{i=1}^{\frac{p-1}{2}} \xi_{((2i+j-3)\ mod\ p)+1}$ for each $j = 1, ..., p$. Additionally, include the points $\sum_{i=1}^{\frac{p-1}{2}} \xi_{((2i+j-3)\ mod\ p)+p+1}$ for each $j = 1, ..., p$. Each of these points is clearly feasible and meet this sun inequality at equality. Thus, it suffices to show that they are affinely independent.

Including these points into a matrix results in a block diagonal matrix. The first block is a cyclically permuted matrix of ones that corresponds to the inner hole. The second block is a cyclically permuted matrix of ones that corresponds to the outer hole. Each of these diagonals is linearly independent as long as p is odd. Thus these points are affinely independent. Consequently these are $2p$ affinely independent points that meet the inequality at equality and so $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{p-1}{2}$ is a facet defining inequality over $P^{ch}_{NP\ S(p,1,r)}$.

Now assume that every $v \in V \setminus S(p,1,r)$ is incident to at most two vertices in either the inner or outer hole. For each $v \in V \setminus S(p,1,r)$, there exists some $j \in \{1, ..., p\}$ and some $k \in \{1, ..., p\}$ such that $\xi_v + \sum_{i=1}^{\frac{p-1}{2}} \xi_{((2i+j-3)\ mod\ p)+1} \in P_{NP}$ or $\xi_v + \sum_{i=1}^{\frac{p-1}{2}} \xi_{((2i+k-3)\ mod\ p)+p+1} \xi_{((2i+k-3)\ mod\ qp)+1} \in P_{NP}$. Including either of these points to the previous matrix results in an additional affinely independent point that meets the inequality at equality and the result follows.

□

Reexamining the graph from Figure 3.5 provides a concrete example of this particular scenario of suns. Since this graph is a $S(7,1,4)$, the valid inequality is $\sum_{i=1}^{14} x_i \leq 3$. This inequality is facet defining follow Theorem 3.1.5 and the 14 affinely independent points that demonstrate this are in Matrix 3.4.

$$\left|\begin{array}{cccccccccccccc}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}\right|$$

(Matrix 3.4)

3.2 Nonsymmetric Suns

Suns can also be created where q does not equal 1 but is still odd. This means that there are more nodes in the outer hole than there are in the inner hole and thus they are not symmetric. The results here can easily be extended to even q; however, such situations provide an even outer hole and cyclically permuting this hole only results in two affinely independent points. Consequently, these inequalities would have a low dimensional face. As a result, q is restricted to odd for this section.

This section follows the outline of the Symmetric Suns section. First the case where $r = 1$ is explored and then larger values of $r = 2$ and $r \geq 3$ are considered. Unlike the case with symmetric suns, for nonsymmetric suns, $r = 1$ is facet defining while $r \geq 3$ is not.

3.2.1 Nonsymmetric Suns with $r = 1$

Nonsymmetric suns with $r = 1$ are denoted $S(p, q, 1)$. Unlike the case using symmetric suns, nonsymmetric suns such that $r = 1$ provide an inequality that is both valid and facet defining. Proofs of both of these properties are shown here. Figure 3.6 is a $S(5, 3, 1)$ is an example of this subclass of suns.

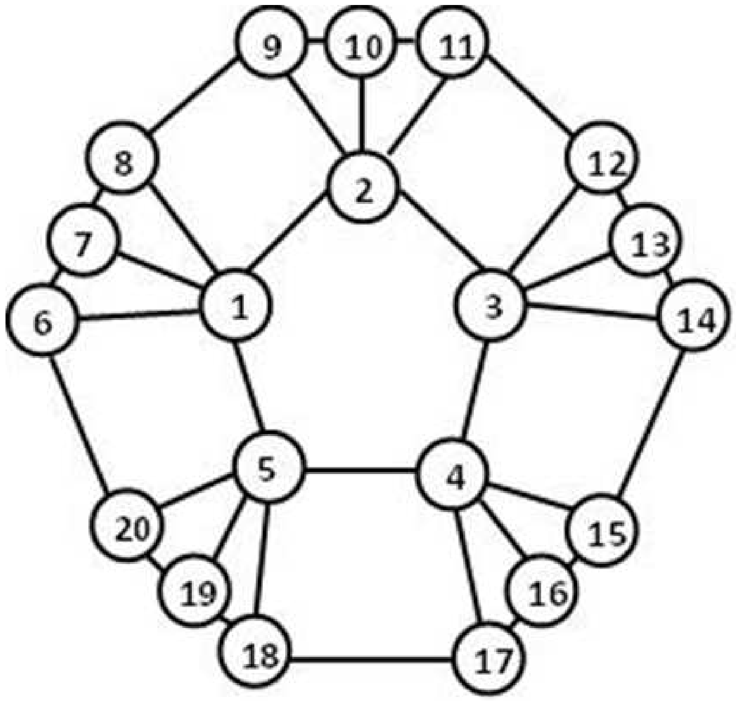

Figure 3.6: S(5,3,1)

Theorem 3.2.1 *Given a graph $G = (V, E)$ such that $S(p, q, 1)$ is an induced subgraph of G with $p \geq 5$ and odd and $q \geq 3$ and odd, then $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{qp-1}{2}$ is a valid inequality of P_{NP}^{ch}.*

Proof: Clearly, there can be no more than $\frac{qp-1}{2}$ nodes selected from the outer hole. Since $q \geq 3$, no nodes can be selected from the inner hole in this situation. Assume $\sum_{i \in V_I} x'_i = t$ for any $x' \in P_{NP}$. Since none of these nodes are adjacent, $\frac{qp-1}{2} - t$ nodes can be selected form the outer hole. Thus, the result follows.

□

Unlike the symmetric case, this inequality happens to be facet defining as Theorem 3.2.2 shows. This is because the additional nodes in the outer hole's cluster allows some nodes adjacent to non-selected nodes in the inner hole to be selected as opposed to the symmetric suns where these nodes in the outer hole could not be selected due to the outer hole adjacency.

Theorem 3.2.2 *Given a graph $G = (V, E)$ such that $S(p, q, 1)$ is an induced subgraph of G with $p \geq 5$ and odd and $q \geq 3$ and odd, then $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{qp-1}{2}$ is a facet defining inequality $P^{ch}_{NP\ S(p,q,1)}$. Furthermore, if every $v \in V \setminus S(p, q, 1)$ is incident to at most two vertices in the outer hole or the vertex is adjacent to a single cluster and not adjacent to its corresponding vertex in the inner hole, then this inequality is facet defining for P^{ch}_{NP}.*

Proof: Given a $S(p, q, 1)$, the inequality $\sum_{i \in V_I} x_i + \sum_{i \in V_0} x_i \leq \frac{qp-1}{2}$ is valid for $P^{ch}_{NP\ S(p,q,1)}$ by Theorem 3.2.2. Clearly 0 does not meet this inequality at equality and so it suffices to find $p + pq$ affinely independent points in P_{NP} that meet this inequality at equality.

Consider the points $\xi_j + \sum_{i=1}^{\frac{qp-3}{2}} \xi_{((2i+qj-2)\ mod\ qp)+p+1}$ for each $j = 1, ..., p$ and $\sum_{i=1}^{\frac{qp-1}{2}} \xi_{((2i+j-3)\ mod\ qp)+p+1}$ for each $j = 1, ..., qp$. Each of these points are also feasible and meet this inequality at equality. Thus, it suffices to show that they are affinely independent.

Observe that the first p rows are the identity rows. Thus, one can easily change the first p columns to identity columns without changing the lower right values. Since the lower right qp rows and columns is a cyclically permuted set of ones based upon an odd hole, these points are clearly affinely independent and the result follows.

Now assume that every $v \in V \setminus S(p, q, r)$ is incident to at most two vertices in the outer hole or the vertex is adjacent to a single cluster and not adjacent to its corresponding vertex in the inner hole. For each $v \in V \setminus S(p,q,r)$, there exists some $j \in \{1,...,p\}$ and some $k \in \{1,...,qp\}$ such that $\xi_v + \sum_{i=1}^{\frac{qp-1}{2}} \xi_{((2i+k-3) \ mod \ p)p+1} \in P_{NP}$ or $\xi_v + \xi_j + \sum_{i=1}^{\frac{qp-3}{2}} \xi_{((jq+i) \ mod \ qp)+p+1} \in P_{NP}$. Including either of these points to the previous matrix results in an additional affinely independent point that meets the inequality at equality and the result follows.

□

Figure 3.6 is a sun inequality of the form $S(5,3,1)$. By Theorem 3.2.1 the inequality $\sum_{i \in V_I} x_i + \sum_{i \in V_O} x_i \leq 7$. By Theorem 3.2.2 this inequality is facet defining. The 20 affinely independent points that meet this inequality at equality are displayed in Matrix 3.5.

$$\begin{vmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0\\
0&1&0&1&0&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1\\
0&0&1&0&1&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0\\
1&0&0&1&0&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1\\
0&0&1&0&1&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0\\
1&0&0&1&0&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1\\
0&1&0&0&1&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0\\
1&0&0&1&0&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1\\
0&1&0&0&1&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0\\
1&0&1&0&0&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1\\
0&1&0&0&1&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0\\
1&0&1&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0\\
1&0&1&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
0&1&0&1&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1
\end{vmatrix}$$

(Matrix 3.5)

3.2.2 Nonsymmetric Suns with $r = 2$

Similar to the symmetric suns with $r = 2$, nonsymmetric suns such that $r = 2$ also provide five valid inequalities. In fact the five inequalities presented in Theorem 3.2.3 are applicable to $S(p,1,2)$ symmetric suns. In the subsection for symmetric suns such that $r = 2$, these inequalities have been specified because two of five of the valid inequalities were proven facet defining while the corresponding nonsymmetric suns only produce valid inequalities that may not be facet defining as it is difficult to define the appropriate number of affinely independent points due to the additional vertices in the outer hole.

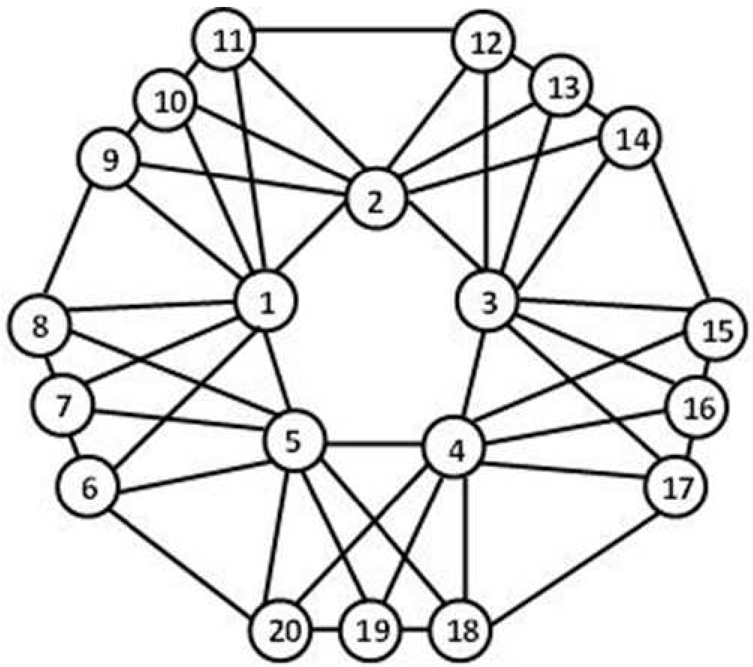

Figure 3.7: S(5,3,2)

Figure 3.7 is a $S(5,3,2)$ sun. There are $p = 5$ nodes in the inner hole, 5 clusters of $q = 3$ in the outer hole, and $r = 2$ connections between the inner nodes and outer clusters.

With more nodes, it becomes more difficult to define the necessary number of affinely independent points to prove facet definition. In fact, none of the examples attempted during this research induced facet defining inequalities. Determining this is a topic for additional research. However, a number of valid inequalities are applicable where q is large.

Theorem 3.2.3 *Given a graph $G = (V, E)$ such that $S(p, q, 2)$ is an induced subgraph of G with $p \geq 5$ and odd, $q \geq 3$ and odd. Then the following inequalities are valid for P^{ch}_{NP}*

i) $\sum_{i \in V_I} x_i \leq \lfloor \frac{p}{2} \rfloor$,

ii) $\sum_{i \in V_I} (2 * q - 1) * x_i + \sum_{i \in V_0} 2 * x_i \leq q * p$,

iii) $\sum_{i \in V_I} (q + 1) * x_i + \sum_{i \in V_0} x_i \leq \frac{q*p+p}{2}$,

iv) $\sum_{i \in V_O} x_i \leq \lfloor \frac{p*q}{2} \rfloor$.

Furthermore if p mod $3 = 1$ or 2, then

v) $\sum_{i \in V_I} q * x_i + \sum_{i \in V_O} x_i \leq \frac{\frac{3*q+1}{2} p - 2}{3}$ *or*

vi) $\sum_{i \in V_I} q * x_i + \sum_{i \in V_O} x_i \leq \frac{\frac{3*q+1}{2} p - 1}{3}$

are valid for $P^{ch}_{NP\ S(p,q,2)}$, respectively.

Proof: Clearly, inequalities i) and iv) are hole inequalities and are valid for $P^{ch}_{NP\ S(p,q,2)}$. Now let x' be any point in $P^{ch}_{NP\ S(p,q,2)}$ and V' be the corresponding node packing. Define $t_I = \sum_{i \in V_I} x'_i$ and $t_O = \sum_{i \in V_O} x'_i$. if $t_i \geq \lceil \frac{p-2}{2} \rceil + 1$, then x' has represents no vertices from V_O in V'. Thus, it suffices to consider $t_I = 1, ..., \lceil \frac{p-r}{2} \rceil$.

Consider ii) and assume $t_I = 1$, then $t_O \leq q\frac{p-3}{2} + \frac{q+1}{2}$. Checking this point on ii) results in $1 * (2 * q - 1) + q\frac{p-3}{2} + \frac{q+1}{2} = \frac{qp+5q-4}{2} \leq qp$ as long as $p \geq 5$. If t_I increases by 1, then at least q nodes must be removed from the outer hole in any node packing; equivalently, t_O decreases by q. Inputting this change into *ii*) results in an increase of $2q - 1$ and a decrease of $2q$. Thus, any increase in t_I, decreases the left hand value of the inequality and this inequality is valid.

Examine iii) and assume $t_I = 1$, then $t_O \leq q\frac{p-3}{2} + \frac{q+1}{2}$. Checking this point on iii) results in $(q+1) * t_i + t_O \leq q + 1 + q\frac{p-3}{2} + \frac{q+1}{2} = \frac{qp+3}{2} \leq \frac{q*p+p}{2}$ since $p \geq 5$. Again any increase in t_I decreases t_O by at least q. The impact of such a change on iii) is a decrease of $q+1$ and an increase of q. Again this decreases the left hand side of iii) and this inequality is valid.

Assume $p \; mod \; 3 = 1$ and $t_I = 1$. Again, $t_O \leq q\frac{p-3}{2} + \frac{q+1}{2}$. Checking this point on v) results in $(q) * t_i + t_O \leq q + q\frac{p-3}{2} + \frac{q+1}{2} = \frac{qp+1}{2} \leq \frac{q*p+p}{2}$ since $p \geq 5$. Again any increase in t_I decreases t_O by at least q. The impact of such a change on v) is a decrease of q and an increase of at most q. Thus this inequality is valid.

Finally, assume $p \; mod \; 3 = 2$ and $t_I = 1$. Again, $t_O \leq q\frac{p-3}{2} + \frac{q+1}{2}$. Checking this point on vi) results in $(q) * t_i + t_O \leq q + q\frac{p-3}{2} + \frac{q+1}{2} = \frac{qp+1}{2} \leq \frac{q*p+p}{2} = p\frac{q+1}{2} \leq p\frac{q+1}{2} - 1 + \frac{1}{2}pq = \frac{3*\frac{q+1}{2}p-1}{3}$ or the right hand side. Again any increase in t_I decreases t_O by at least q. The impact of such a change on vi) is a decrease of q and an increase of at most q. Thus this inequality is valid.

□

Now reconsider Figure 3.7 of the $S(5,3,2)$ sun. By Theorem 3.2.3, these are its valid inequalities:

i) $\sum_{i \in V_I} x_i \leq 2$,

ii) $\sum_{i \in V_I} 5 * x_i + \sum_{i \in V_0} 2 * x_i \leq 15$,

iii) $\sum_{i \in V_I} 4 * x_i + \sum_{i \in V_0} x_i \leq 10$,

iv) $\sum_{i \in V_O} x_i \leq 7$,

v) $\sum_{i \in V_I} 3 * x_i + \sum_{i \in V_O} x_i \leq 8$.

3.2.3 Nonsymmetric Suns with $r \geq 3$

Nonsymmetric suns such that $r \geq 3$ produce three valid inequalities. The middle inequality is facet defining, and because q is large, proof of facet definition requires $pq + p$ many affinely independent points. Figure 3.8 depicts a $S(7, 3, 3)$ sun as an example of this subclass of suns.

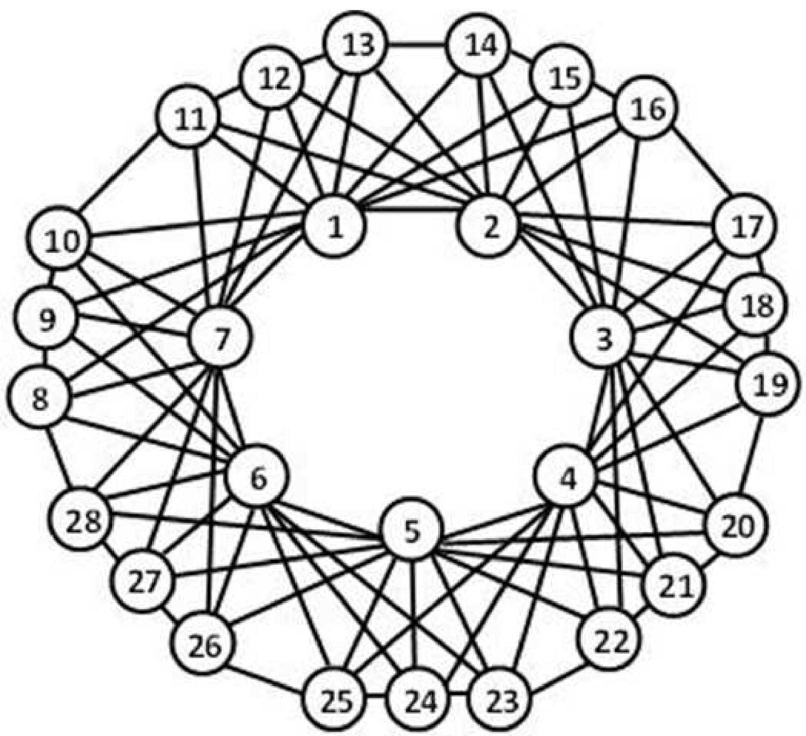

Figure 3.8: S(7,3,3)

Theorem 3.2.4 *Given a graph $G = (V, E)$ such that $S(p, q, r)$ is an induced subgraph of G with $p \geq 5$ and odd, $q \geq 3$ and odd and $r \geq 3$, then the following inequalities are valid.*

i) $\sum_{i \in V_I} x_i \leq \lfloor \frac{p}{2} \rfloor$,

ii) $\sum_{i \in V_I} \frac{1}{\frac{p-1}{2}} x_i + \sum_{i \in V_O} \frac{1}{\frac{q*p-1}{2}} x_i \leq 1$,

iii) $\sum_{i \in V_O} x_i \leq \lfloor \frac{p*q}{2} \rfloor$.

Proof: Clearly, inequalities i) and iii) are valid since they are hole inequalities. Furthermore, inequality ii) can only be invalid if there is at least one x_i set to one in both of the holes or the inequality becomes a hole inequality. Let $x' \in P^{NP}$ and define $t_I = \sum_{i \in V_I} x'_i$. Observe that if $t_I \geq \lceil \frac{p-r}{2} \rceil + 1$, then x' has no vertices from V_O in its corresponding node packing. Thus, it suffices to consider t_I for $t_I = 1, ..., \lceil \frac{p-r}{2} \rceil$.

If $t = 1$, then at most there can be $\frac{qp-1}{2} - r\frac{q-1}{2} - \lfloor \frac{r}{2} \rfloor$. Each time t escalates by one, it eliminates at least two clusters from consideration in the node packing. In eliminating these two clusters, the value of $\sum_{i \in V_O} x'_i$ decreases by at least q. Inputing this change into ii), results in $\frac{1}{\frac{p-1}{2}} t + \frac{1}{\frac{q*p-1}{2}} (\frac{qp-1}{2} - r\frac{q-1}{2} - \lfloor \frac{r}{2} \rfloor - q(t-1))$. Simplifying leads to $\frac{1}{\frac{p-1}{2}} t + 1 - \frac{1}{\frac{q*p-1}{2}} (r\frac{q-1}{2} + \lfloor \frac{r}{2} \rfloor + q(t-1))$. This is less than or equal to one whenever $\frac{1}{\frac{p-1}{2}} t \leq \frac{1}{\frac{q*p-1}{2}} (r\frac{q-1}{2} + \lfloor \frac{r}{2} \rfloor + q(t-1))$. Simplifying yields $\frac{q*p-1}{p-1} t \leq r\frac{q-1}{2} + \lfloor \frac{r}{2} \rfloor + q(t-1)$. Observe that $\frac{q-1}{2} + \lfloor \frac{r}{2} \rfloor + q(t-1) \leq r\frac{q-1}{2} + \frac{r}{2} + q(t-1) = \frac{rq-r+r+2qt-2q}{2} = q\frac{r+2t-2}{2} = \frac{qr}{2} + qt - q$. Thus, it is sufficient to consider the case when $\frac{q*p-1}{p-1} t \leq \frac{qr}{2} + qt - q$. Cross multiplying leads to $2t(p*q-1) \leq qr*(p-1) + 2qt(p-1) - 2q(p-1)$. Simplifying leads to $2t(q-1) \leq qr(p-1) - 2q(p-1)$, which implies $2t(q-1) \leq (qr-2q)(p-1)$. Since

$2t(q-1) \leq 2tq$, it implies that $2t \leq (r-2)(p-1)$. Since t is at most $\lceil \frac{p-r}{2} \rceil$ $2\lceil \frac{p-r}{2} \rceil \leq p-r+1$. Forcing $p-r+1-(r-2)(p-1) \leq 0$ simplifies to $p-r+1-(pr-2p-r+2) \leq 0$. This can be reduced to $3p - pr \leq 0$ whenever $r \geq 3$. Thus, the result follows.

□

Unlike the symmetric case where the two hole inequalities were facet defining, in this case neither of the hole inequalities are facet defining. However, the sun inequality ii) is facet defining as the following theorem shows.

Theorem 3.2.5 *Given a graph $G = (V, E)$ such that $S(p, q, r)$ is an induced subgraph of G with $p \geq 5$ and odd, $q \geq 3$ and odd and $r \geq 3$, then $\sum_{i \in V_I} \frac{1}{\frac{p-1}{2}} x_i + \sum_{i \in V_O} \frac{1}{\frac{q*p-1}{2}} x_i \leq 1$ is a facet defining inequality over $P^{ch}_{NP\ S(p,q,r)}$. Furthermore, if every $v \in V \setminus S(p, q, r)$ is incident to at most two vertices in either the inner or outer hole, then this inequality is facet defining for P^{ch}_{NP}.*

Proof: By Theorem 3.2.5, the sun inequality is valid. Furthermore, 0 never meets this inequality at equality, and thus this inequality has dimension less than $p(q + 1)$ on $P^{ch}_{NP\ S(p,q,r)}$. It is therefore sufficient to find $p(q + 1)$ affinely independent points that meet this inequality at equality.

Consider the points $\sum_{i=1}^{\frac{p-1}{2}} \xi_{((2i+j-3)\ mod\ p)+1}$ for each $j = 1, ..., p$. Additionally, include the points $\sum_{i=1}^{\frac{qp-1}{2}} \xi_{((2i+j-3)\ mod\ qp)+p+1}$ for each $j = 1, ..., qp$. Each of these points is clearly feasible and meet the sun inequality at equality. Thus, it suffices to show that they are affinely independent.

Including these points into a matrix results in a block diagonal matrix. The first block is a cyclically permuted matrix of ones that corresponds to the inner hole. The second block is a cyclically permuted matrix of ones that corresponds to the outer hole. Each of these diagonals is linearly independent as long as p and q are odd. Thus these points are affinely independent. Consequently these are $p + qp$ affinely independent points that meet the inequality at equality and so $\sum_{i \in V_I} \frac{1}{\frac{p-1}{2}} x_i + \sum_{i \in V_O} \frac{1}{\frac{q*p-1}{2}} x_i \leq 1$ is a facet defining inequality over $P^{ch}_{NP\ S(p,q,r)}$.

Now assume that every $v \in V \setminus S(p,q,r)$ is incident to at most two vertices in either the inner or outer hole. For each $v \in V \setminus S(p,q,r)$, there exists some $j \in \{1,...,p\}$ and some $k \in \{1,...,qp\}$ such that $\xi_v + \sum_{i=1}^{\frac{p-1}{2}} \xi_{((2i+j-3)\ mod\ p)+1} \in P_{NP}$ or $\xi_v + \sum_{i=1}^{\frac{qp-1}{2}} \xi_{((2i+k-3)\ mod\ qp)+p+1} \in P_{NP}$. Including either of these points to the previous matrix results in an additional affinely independent point that meets the inequality at equality and the result follows.

$\square$

Reconsider Figure 3.8 which shows a $S(7,3,3)$. The inequality $\sum_{i \in V_I} \frac{1}{3} * x_i + \sum_{i \in V_O} \frac{1}{10} * x_i \leq 1$, has 28 affinely independent points. Matrix 3.6 shows these points.

Observe that $\sum_{i \in V_I} \frac{1}{3} * x_i + \sum_{i \in V_O} \frac{1}{10} * x_i \leq 1$ is precisely the inequality that is generated by Conley [13]. However for Conley to generate this inequality, the structure would have needed a complete graph between both holes. In this case it would have required nearly twice as many edges as in the sun inequality. Thus, the sun inequalities strengthen his result and are previously undiscovered inequalities. As an asside, similar comments could have been made regarding the other cases with $r \geq 2$.

$$
\begin{vmatrix}
1&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1\\
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0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1\\
0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1\\
0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1\\
0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1
\end{vmatrix}
$$

(Matrix 3.6)

3.3 Identifying Sun Structures in Arbitrary Graphs

In identifying sun structures for cutting planes, observe that if a sun is a subgraph of a graph on $p(q+1)$ nodes, then the cutting planes are valid. These inequalities may no longer be facet defining with the additional edges, but the cutting planes are still valid. Thus, they may help solve a node packing instance.

An exponential algorithm was developed that could exactly identify a sun structure from a hole. However, the applicability of such an algorithm is essentially useless. Instead a polynomial time algorithm is generated that can identify super graphs with suns as a subgraph. Thus, the cutting planes contained in sections 3.1 and 3.2 can be identified and applied in polynomial time.

Given an odd hole, the algorithm determines if there exists a graph with a $S(p, q, r)$

as a subgraph. Without loss of generality, assume there is an odd hole $H_p = \{v_1, ..., v_p\}$ where there is an edge between v_i and $v_{i \ mod \ p+1}$. for $i = 1, ..., p$.

The algorithm begins by identifying potential candidates to be in a $S(p, 1, r)$. If a node $v_j \in V \setminus V(H_p)$ is in an $S(p, 1, r)$, then by necessity the edges to $V(H_P)$ must be exactly of the form $\{v_i, v_j\}$, $\{v_{i \ mod \ p+1}, v_j\}$, $\{v_{i+1 \ mod \ p+1}, v_j\}$,..., $\{v_{i+r-2 \ mod \ p+1, v_j}\}$ for some $i \in \{1, ..., p\}$. If this is the case, the v_j is a candidate to be in this sun that would correspond to node v_i. This list of candidate nodes is stored in $list_{v_i,r}$.

With this list of candidate adjacent nodes, for each possible r, a multilayer graph is created. The vertices in layer i correspond to vertex $v_i \in H_p$ and are the vertices in $list_{v_i,r}$. The induced subgraph of these vertices is then considered and a modified breadth first search is utilized to identify whether or not a path in this multilayer graph exists. Observe that chords are allowed and is why the algorithm is identifying supergraphs and not just suns.

The modified breadth first search seeks a path from a vertex in the first layer back to itself of length p. If such a path exists, then there is a cycle, but there may be chords. Due to the structure of the breadth first search, the algorithm can identify all suns of form $S(p, 1, r)$. Clearly, it does not necessarily report these as there may be exponentially many such suns. More on this is discussed after the example along with a modification of moving from $q = 1$ to larger q.

Super Symmetric Sun Identification Algorithm (SSSIA)

Initialization:

Identify an odd hole $H_p = (v_1, ..., v_p)$ in G

Main Step:

For $r = 1$ to p

For every vertex $v_j \in V \setminus V(H_p)$ add v_i to $list_{v_j,r}$ if and only if the edges in G.

between v_j and $V(H_p)$ include $\{v_i, v_j\}$, $\{v_{i \ mod \ p+1}, v_j\}$, $\{v_{i+1 \ mod \ p+1}, v_j\}$,...,

$\{v_{(i+r-2) \ mod \ p+1, v_j}\}$.

Create a multilayer graph $G_M = (V, E) = (V_1 \cup V_2 \cup ... \cup V_p \cup V_{p+1}, E')$ where

$V_k = list_{v_j,r}$ and $V_{p+1} = V_1$. The edges between V_i and V_{i+1} $\{u, v\}$ such that

$u \in V_i$, $v \in V_{i+1}$ and $\{u, v\} \in E(G)$ for $i = 1$ to p.

For $layer = 2$ to $p + 1$

For each node $v_j \in V$ set $predlist_{v_j,layer} \leftarrow \emptyset$.

For each $layer = 1$ to p

For each $v_l \in V_{layer}$.

if v_l is adjacent to $v_k \in V_{layer+1}$, then $pred_{v_k,layer} \leftarrow pred_{v_k,layer} \cup \{v_l\}$.

For each $v_{l \in V_{p+1}}$

Follow the predecessors until the vertex v_l is reached in V_1. In following predecessors, it is important not to revisit a node visited, besides v_l which is repeated exactly once. Thus, once a node is used a predecessor, it is marked and only unmarked nodes are considered in each successive predecessor. Report the nodes on this path and the original hole as being a sun.

It is straightforward to show that SSSIA creates a supergraph that contains a $S(p, 1, r)$. First, the inner odd hole is generated or could be provided as input. The vertices on the outer hole are only considered if they have at least the r requisite adjacencies to the inner hole. Thus, the edges between the two holes are guaranteed to be in the reported graph. Finally, the multilayer graph is used to find the outer hole among these candidate nodes. Due to the multilayer graph and the marking of vertices in traversing this graph, enables the algorithm to find a cycle with p nodes. Consequently, the algorithm does identify a supergraph of a $S(p, 1, r)$.

The runtime of this algorithm is straightforward. The initialization can be performed using breadth first search, $O(n + m)$ where $|V| = n$ and $|E| = m$. The mainstep is repeated p times and the first step is done in $O(np)$. Creating the multilayer graph requires $O(p(n + m))$. Assigning the predecessors to the empty set. The last steps are just a modified version of breadth first search and require $O(pn + pm)$ effort. Thus, the algorithm requires $O(p^2n)$ effort and since $p \leq n$, this algorithm requires $O(n^3)$ effort.

The following example depicts some of this algorithm.

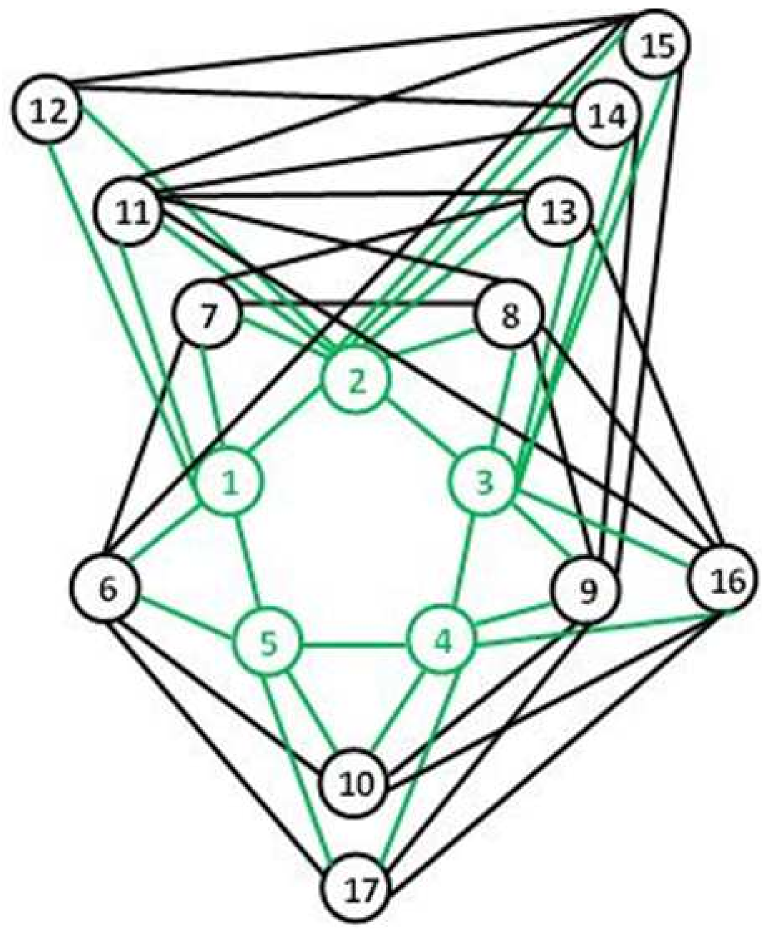

Figure 3.9: Algorithm Example

In Figure 3.9, the initial hole can be seen in nodes $1, 2, 3, 4, and/5$. The remaining nodes in the figure are candidates for the outer hole of the sun. In this particular instance, the loop when $r = 2$ is examined. Thus, any sun generated would be a $S(5, 1, 2)$.

Next the lists for each layers are created as follows. The vertices in $List_{(1,2)}$ are precisely those vertices that are adjacent to both vertex 1 and 2 since $r = 2$. Thus, $List_{(1,2)} = \{7, 11, 12\}$. The final values of these lists are as follows.

$List_{(1,2)} = \{7, 11, 12\}$

$List_{(2,2)} = \{8, 13, 14, 15\}$

$List_{(3,2)} = \{9, 16\}$

$List_{(4,2)} = \{10, 17\}$

$List_{(5,2)} = \{6\}$

Next the multilayer graph is created as shown in Figure 3.10.

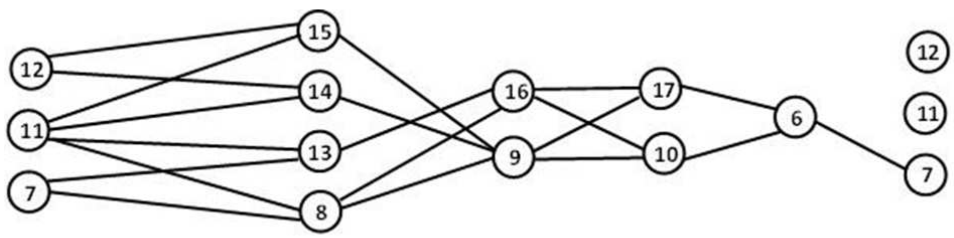

Figure 3.10: Layers

The predecessors are found in this multilayer graph. For instance, the predecessor of node 16 is $\{8, 13\}$. Starting with node 7 in layer 6, predecessors are followed in an effort to obtain a path to 7 in layer 1. Any such path generates a sun. In this example, there are six paths and thus, six suns with $r = 2$. The outer holes for each different sun are:

$7 - 8 - 9 - 10 - 6$

$7 - 8 - 9 - 17 - 6$

$7 - 8 - 16 - 10 - 6$

$7 - 8 - 16 - 17 - 6$

$7 - 13 - 16 - 10 - 6$

$7-13-16-17-6$

Since these suns have $p=2$, $q=1$, and $r=2$, one can apply the $S(5,1,2)$ sun inequalities. Take the first discovered sun for instance $7-8-9-10-6$ and the valid inequalities are as follows.

i) $x_1+x_2+x_3+x_4+x_5 \leq 2$,

ii) $2*(x_1+x_2+x_3+x_4+x_5)+x_6+x_7+x_8+x_9+x_{10} \leq 5$,

iii) $x_1+x_2+x_3+x_4+x_5+2*(x_6+x_7+x_8+x_9+x_{10}) \leq 5$,

iv) $x_6+x_7+x_8+x_9+x_{10} \leq 2$,

v) $x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10} \leq 3$.

To identify suns with $q \geq 3$, groups of nodes must be considered at each layer. Thus if a layer had l nodes, then l choose q sets could be examine at each layer. Furthermore, these nodes must have at least a path connecting them. In performing the path searching algorithm between the layers, only one (the end of the path) is adjacent to the starting set of q in the next layer. Obviously, the algorithm could trivially be extended to include these sets of q candidates at each layer. For arbitrary q, this is an exponential algorithm, but for fixed q it is a polynomial time algorithm.

The only change is in the creation of the multilayer graph. It begins with *List*. If $q=3$, then $List_{(2,2)}=\{8,13,14,15\}$ describes this process. First twenty four chains (4*3*2) are examined. For instance, one possiblility would be for the hole to have (8,13,14) in that order in the multigraph. Another would be (8,13,15), etc. These new "super" nodes only are added to the multilayer graph if there exists a path from the starting node to the ending node. The connections between layers now are in the multilayer graph if the end of layer i's chain has an edge to the beginning of layer $i+1$'s chain for $i = 1$ to p. Once the multilayer graph is changed, the remainder of the algorithm is identical.

Chapter 4

Conclusions and Future Research

This research presented a new series of structures called Suns $S(p, q, r)$. There are two classes of suns. The first is symmetric suns which have an equal number of nodes in the inner and outer holes. The second is nonsymmetric suns which have more nodes in the outer hole than the inner hole. Within the two classes of suns, there are also three subclasses: $r = 1$, $r = 2$, and $r \geq 3$, where r represents the number of connections between nodes in the inner and outer holes.

All classes of suns produce different valid inequalities known as sun inequalities. Symmetric suns with $r = 2$ and $r \geq 3$ and nonsymmetric suns with $r = 1$ and $r \geq 3$ have been proven to have facet defining inequalities. These inequalities are theoretically stronger than the existing hole inequalities that could be applied to the the odd hole subgraphs within each sun. Thus, in integer programs whose graphs contain sun subgraphs, these valid and facet defining inequalities can be used to more quickly solve the problem.

Additionally, a polynomial time algorithm has been provided to aid in identifying whether or not a sun exists in the IP. This algorithm can be used for various classes and sizes of suns. Hence the generation of sun inequalities can be completed in polynomial time.

4.1 Future Research

The work done for this thesis provides several areas for future research. In particular, for symmetric suns such that $r = 2$, research suggests that the middle sun inequality is likely facet defining. The matrix of the suspected affinely independent points has an inverse. However, the proof that this valid inequality is in fact facet defining still remains.

During the research for this thesis, in both symmetric and nonsymmetric cases, five valid inequalities were created for suns with $r = 2$. Out of these five inequalities, the two hole inequalities have been proven and the middle sun inequality is suspected to be facet defining for symmetric suns. The remaining two inequalities each share end points with a hole inequality and the middle sun inequality. However, we were unable to prove that these inequalities are facet defining. The question that remains in this instance is: what facets exist between the three facet defining inequalities?

To this point, every sun structure that has been studied is comprised of two holes:

an inner and outer hole. It would be possible to create a multilayer sun with n holes. These multilayer suns could also be either symmetrical or nonsymmetrical and would have a similar set of valid inequalities to the two-hole suns. Although the number of nodes would be exponential, some of these inequalities could still be facet defining due to the cyclical nature of the node packings. These suns may be harder to identify in real-world examples since they are large and require a great deal of intricacy.

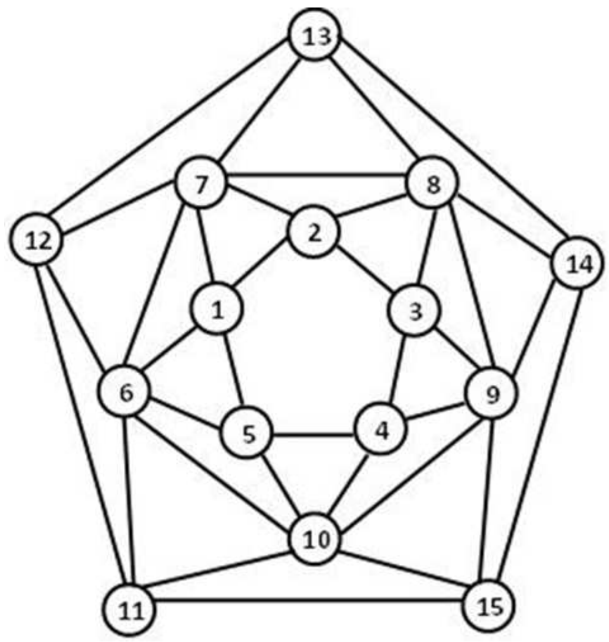

Figure 4.1: Multilayer Sun

Finally, additional graphical structures can be created to generate different valid inequalities. These could be intersections of odd holes or they could be intersections of other structures such as cliques or wheels. Using varying numbers of edges between the intersected structures may generate new, stronger inequalities. This line of reasoning was a primary motivation for this this research and it is believe that many more previously undiscovered facet defining inequalities remain for P_{NP}^{ch}.

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