SIMPLE CONTINUED FRACTIONS

by

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INTRODUCTION

The purpose of this paper is to develop the framework of simple continued fractions, without considering any applications. Simple continued fractions are the most important and most extensively studied continued fractions, and lie at the root of almost all arithmetic and very many analytic applications of continued fractions.[1]

A continued fraction is a function of the form

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4} \cdots}} ,$$

where the $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are any real numbers whatever. In particular, a simple continued fraction is a continued fraction of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}} ,$$

where the a_i, $i = 1, 2, 3, \ldots$ are positive integers with the exception that the integer a_1 may be positive, negative, or zero. If the number of

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[1]A. Ya. Khintchine, Continued Fractions, p. 3.

a_i is finite, the simple continued fraction is said to be a finite simple continued fraction. If the number of a_i is infinite, the simple continued fraction is said to be an infinite simple continued fraction.

The paper is divided into three sections. The first part is concerned with the defining and development of the finite simple continued fraction. All the theory and definitions are provided, and no previous knowledge of continued fractions is assumed.

Secondly, the convergents of a simple continued fraction are defined, and from this, recursion formulae are developed. These make possible the development of the properties of the convergents and the numerators and denominators of the convergents of finite simple continued fractions.

In the last section of the paper, infinite simple continued fractions are introduced and it is noted that while finite simple continued fractions represent rational numbers, infinite simple continued fractions represent irrational numbers.

FINITE SIMPLE CONTINUED FRACTIONS

Let r be any rational number. Then $r = \frac{u_1}{u_2}$, where u_1 and u_2 are unique relatively prime integers with $u_2 > 0$. By the division algorithm, there exist integers a_1 and u_3 such that

$$u_1 = u_2 a_1 + u_3, \qquad 0 \leq u_3 < u_2.$$

If $u_3 > 0$, then the division algorithm applied to the positive integers u_2 and u_3 yields unique integers a_2 and u_4 such that

$$u_2 = u_3 a_2 + u_4, \qquad 0 \leq u_4 < u_3.$$

The repeated application of the division algorithm gives unique integers $a_1, a_2, a_3, \ldots, a_j$ and $u_3, u_4, \ldots, u_{j+2}$ such that

$$u_i = u_{i+1} a_i + u_{i+2}, \quad (i = 1, 2, \ldots, j) \tag{1}$$

and $0 = u_{j+2} < u_{j+1} < \cdots < u_3 < u_2$.

Now let

$$B_i = \frac{u_i}{u_{i+1}}, \quad i = 1, 2, \ldots, j.$$

Then,

$$B_i = a_i + \frac{1}{B_{i+1}}, \quad i = 1, 2, \ldots, j-1,$$

$$B_j = a_j. \tag{2}$$

Thus the successive substitution of the value $a_i + \frac{1}{B_{i+1}}$ for B_i in B_{i-1} for $i = 2, 3, \ldots, j-1$ yields

$$B_1 = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots + \cfrac{1}{a_{j-1} + \cfrac{1}{B_j}}}}}\ .$$

Now, $B_1 = \frac{u_1}{u_2}$ and $B_j = a_j$, so

$$\frac{u_1}{u_2} = B_1 = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots + \cfrac{1}{a_{j-1} + \cfrac{1}{a_j}}}}}\ . \tag{3}$$

This expression for $\frac{u_1}{u_2}$ is called a continued fraction expansion, and the integers a_i are called the partial quotients, since they are the quotients obtained in the application of the division algorithm. By assumption, u_2 is positive, but u_1 may be positive, negative, or zero. Thus, a_1 may be positive, negative, or zéro; but since $0 < u_{i+1} < u_i$, $i = 2, 3, \ldots, j-1$, the succeeding partial quotients $a_2, a_3, \ldots, a_j$ are positive integers. In case $j \geqslant 2$, then $a_j = \frac{u_j}{u_{j+1}}$ and $0 < u_{j+1} < u_j$ imply that $a_j > 1$.

The notation $\langle a_1, a_2, \ldots, a_j \rangle$ is used to designate the continued fraction in (3). In general, if $X_1, X_2, \ldots, X_j$ are any real numbers, all positive except possibly for X_1, then

$$\langle X_1, X_2, \ldots, X_j \rangle = X_1 + \cfrac{1}{X_2 + \cdots + \cfrac{1}{X_{j-1} + \cfrac{1}{X_j}}} \quad . \tag{4}$$

If all the X_i are integers, then the continued fraction in (4) is said to be a <u>simple continued fraction</u>. By a <u>finite simple continued fraction</u> is meant a simple continued fraction with a finite number of partial quotients.

The result in equation (3) implies that every rational number can be expanded into a finite simple continued fraction. However, a finite simple continued fraction has an alternate form to that of (3); namely

$$\frac{u_1}{u_2} = \langle a_1, a_2, \ldots, a_{j-1}, a_j \rangle = \langle a_1, a_2, \ldots, a_{j-1}, a_j - 1, 1 \rangle \ .$$

The following theorem establishes that these are the only two simple continued fraction expansions of the rational number $\frac{u_1}{u_2}$.

THEOREM 1. If $\langle a_1, a_2, \ldots, a_j \rangle = \langle b_1, b_2, \ldots, b_n \rangle$, where these are finite simple continued fractions, and if $a_j > 1$ and $b_n > 1$, then $j = n$ and $a_i = b_i$ for $i = 1, 2, \ldots, n$.

Proof. Let $y_i = \langle b_i, b_{i+1}, \ldots, b_n \rangle$. Then

$$y_i = \langle b_i, b_{i+1}, \ldots, b_n \rangle = b_i + \frac{1}{\langle b_{i+1}, b_{i+2}, \ldots, b_n \rangle} .$$

Therefore,

$$y_i = b_i + \frac{1}{y_{i+1}} . \tag{5}$$

Thus, $y_i > b_i$ and $y_{i+1} > 1$ for $i = 1, 2, \ldots, n-1$ and $y_n = b_n > 1$. Hence, b_i is equal to the greatest integer in y_i, denoted by $[y_i]$, $i = 1, 2, \ldots, n$. From the hypothesis that the simple continued fractions are equal, it follows that $y_1 = B_1$. Now $B_i > 1$ for $i = 2, \ldots, j$, since $B_i = \frac{u_i}{u_{i+1}}$, and $0 < u_{i+1} < u_i$ for $i = 2, \ldots, j$. Thus $a_i = [B_i]$ for $i = 1, \ldots, j-1$ from equations (2). Now since $y_1 = B_1$, by taking integral parts, $b_1 = [y_1] = [B_1] = a_1$, and then from equations (2) and (5),

$$\frac{1}{B_2} = B_1 - a_1 = y_1 - b_1 = \frac{1}{y_2} .$$

Thus, $B_2 = y_2$, and hence $a_2 = [B_2] = [y_2] = b_2$.

Now assume that $B_k = y_k$ and $a_k = b_k$ for $k = 1, 2, \ldots, i$ where $i \leq j-1$, $i \leq n-1$. From equations (2) and (5), it follows that $\frac{1}{B_{i+1}} = B_i - a_i = y_i - b_i = \frac{1}{y_{i+1}}$, which implies that $B_{i+1} = y_{i+1}$. Hence $a_{i+1} = [B_{i+1}] = [y_{i+1}] = b_{i+1}$. Thus, by induction, $B_k = y_k$ and $a_k = b_k$ for $k = 1, 2, \ldots, i$ where $i \leq j$ and $i \leq n$. It remains to be shown that $j = n$.

Suppose that $j < n$. From the preceding argument, $a_k = b_k$ and $B_k = y_k$ for $k = 1, 2, \ldots, i$, where $i \leq j$, and $i \leq n$. From equation (2), $B_j = a_j$ and from equation (5), $y_j > b_j$. Thus, in particular, $a_j = b_j$ and $B_j = y_j$ and hence a contradiction. Now, assume that $j > n$. A similar contradiction arises, and it is concluded that $j = n$.

PROPERTIES OF THE CONVERGENTS OF A SIMPLE CONTINUED FRACTION

The part of the simple continued fraction up to and including a_k is called the k^{th} convergent and is denoted by C_k[2]. Thus, the convergents of the simple continued fraction $\langle a_1, a_2, \ldots, a_n \rangle$, are

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[2] Harry N. Wright, First Course in Theory of Numbers, p. 17.

$$C_1 = \langle a_1 \rangle ,$$

$$C_2 = \langle a_1, a_2 \rangle ,$$

$$. \quad . \quad .$$

$$C_n = \langle a_1, a_2, \ldots, a_n \rangle .$$

For a finite simple continued fraction $\langle a_1, a_2, \ldots, a_n \rangle$, the n^{th} convergent is, by definition, the simple continued fraction itself.

Let $X_i = \langle a_i, a_{i+1}, \ldots, a_n \rangle$, for $1 \leq i \leq n$.

Then, X_i is called the i^{th} complete quotient.[3] The continued fraction itself may be called the first complete quotient.

Consider the first few convergents, say C_1, C_2, C_3, C_4 of the simple continued fraction $\langle a_1, a_2, \ldots, a_n \rangle$. When expressed as simple fractions, these become

$$C_1 = \frac{a_1}{1} ,$$

$$C_2 = \frac{a_2 a_1 + 1}{a_2}$$

$$C_3 = \frac{a_1(a_3 a_2 + 1) + a_3}{a_3 a_2 + 1}$$

$$C_4 = \frac{a_1(a_2(a_4 a_3 + 1) + a_4) + a_4 a_3 + 1}{a_2(a_4 a_3 + 1) + a_4} .$$

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[3] G. Chrystal, Textbook of Algebra, Part II, p. 431.

Now, rearrangement of terms gives

$$C_1 = \frac{a_1}{1} ,$$

$$C_2 = \frac{a_2a_1 + 1}{a_2}$$

$$C_3 = \frac{a_3(a_2a_1 + 1) + a_1}{a_3a_2 + 1}$$

and

$$C_4 = \frac{a_4(a_3(a_2a_1 + 1) + a_1) + (a_2a_1 + 1)}{a_4(a_3a_2 + 1) + a_2} .$$

A definite pattern appears to be developing in these simple fractions.

Let

$$p_1 = a_1 \text{ and } q_1 = 1,$$

so that

$$C_1 = \frac{p_1}{q_1} .$$

Then,

$$C_2 = \frac{a_2p_1 + 1}{a_2} .$$

Now let

$$p_2 = a_2p_1 + 1 \text{ and } q_2 = a_2.$$

so that

$$C_2 = \frac{p_2}{q_2} \quad .$$

Then,

$$C_3 = \frac{a_3 p_2 + p_1}{a_3 q_2 + q_1} \quad .$$

Finally, let

$$p_3 = a_3 p_2 + p_1 \quad ,$$

and

$$q_3 = a_3 q_2 + q_1 \quad ,$$

so that

$$C_3 = \frac{p_3}{q_3} \quad .$$

Then,

$$C_4 = \frac{a_4 p_3 + p_2}{a_4 q_3 + q_2} \quad .$$

This suggests that if p_i, q_i are defined by the recursion formulae

$$p_i = a_i p_{i-1} + p_{i-2} \ , \tag{6}$$

$$q_i = a_i q_{i-1} + q_{i-2} \ , \tag{7}$$

for $i \geq 3$, then

$$C_i = \frac{p_i}{q_i} \ .$$

To see that this is the case, suppose for $k < n$, and $i = 3, 4, \ldots, k$,

$$C_i = \frac{p_i}{q_i} \ ,$$

where p_i, q_i are defined as in equations (6) and (7), respectively. Then,

$$C_k = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}} \ .$$

Now from the recursion formulae defining the p_i and q_i, it follows that $p_{k-2}, p_{k-1}, q_{k-2}, q_{k-1}$ do not involve a_k. However, by definition the $(k+1)^{st}$ convergent is obtained from the k^{th} convergent by replacing a_k by $a_k + \dfrac{1}{a_{k+1}}$.

Thus,

$$C_{k+1} = \frac{\left(a_k + \frac{1}{a_{k+1}}\right) p_{k-1} + p_{k-2}}{\left(a_k + \frac{1}{a_{k+1}}\right) q_{k-1} + q_{k-2}}$$

$$C_{k+1} = \frac{(a_{k+1}a_k + 1)p_{k-1} + a_{k+1}p_{k-2}}{(a_{k+1}a_k + 1)q_{k-1} + a_{k+1}q_{k-2}},$$

$$= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}}$$

$$= \frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}} .$$

Hence,

$$C_{k+1} = \frac{p_{k+1}}{q_{k+1}} ,$$

where p_{k+1}, q_{k+1} are defined by equations (6) and (7). Thus,

$$C_i = \frac{p_i}{q_i} ,$$

where

$$p_i = a_i p_{k-1} + p_{i-2} ,$$

and

$$q_i = a_i q_{i-1} + q_{i-2}$$

for i = 3, 4, . . . , n. Now define

$$p_{-1} = 0, \quad p_0 = 1,$$

$$q_{-1} = 1, \quad q_0 = 0.$$

Then,

$$p_1 = a_1 p_0 + p_{-1} ,$$

$$p_2 = a_2 p_1 + p_0 ;$$

and

$$q_1 = a_1 q_0 + q_{-1} ,$$

$$q_2 = a_2 q_1 + q_0 .$$

Thus, the recursion relation holds for $i = 1, 2, \ldots, n$.

The p_i and q_i in $C_i = \frac{p_i}{q_i}$ shall be referred to as the numerator and denominator of the i^{th} convergent, respectively.

THEOREM 2. The denominators of the successive convergents form an increasing sequence of integers.

Proof. The fact that $q_i > q_{i-1}$ follows immediately, since $q_i = a_i q_{i-1} + q_{i-2}$, where a_i, q_{i-1}, and q_{i-2} are non negative integers.

THEOREM 3. The ratios of successive numerators and denominators of the successive convergents of a simple continued fraction may be expressed as follows:

$$\frac{p_i}{p_{i-1}} = \langle a_i, a_{i-1}, \ldots, a_1 \rangle \qquad (8)$$

and

$$\frac{q_i}{q_{i-1}} = \langle a_i, a_{i-1}, \ldots, a_2 \rangle \;. \tag{9}$$

Proof. From equation (6), for i = 2,

$$\frac{p_2}{p_1} = a_2 + \frac{1}{a_1} \;.$$

Assume

$$\frac{p_k}{p_{k-1}} = \langle a_k, a_{k-1}, \ldots, a_1 \rangle \quad \text{for } k = 1, 2, \ldots, i-1.$$

To show that

$$\frac{p_i}{p_{i-1}} = \langle a_i, a_{i-1}, \ldots, a_1 \rangle \;,$$

equation (6) states

$$p_i = a_i p_{i-1} + p_{i-2} \;.$$

Therefore,

$$\frac{p_i}{p_{i-1}} = a_i + \frac{p_{i-2}}{p_{i-1}}$$

$$\frac{p_i}{p_{i-1}} = a_i + \frac{1}{\frac{p_{i-1}}{p_{i-2}}}$$

$$= a_i + \frac{1}{\langle a_{i-1}, \ldots, a_1 \rangle}$$

$$= \langle a_i, a_{i-1}, \ldots, a_1 \rangle .$$

Thus, $\frac{p_i}{p_{i-1}} = \langle a_i, a_{i-1}, \ldots, a_1 \rangle$, $i = 1, 2, \ldots, n$.

In a similar manner, equation (9) may be established.

THEOREM 4. Any simple continued fraction with a finite number of terms represents a rational number.

Proof. Since the simple continued fraction terminates, then by the definition of the last or i^{th} convergent, the simple continued fraction is equal to $C_i = \frac{p_i}{q_i}$. Since p_i and q_i are integers, the simple continued fraction represents a rational number.

THEOREM 5. For $i = 0, 1, 2, \ldots, n$,

$$p_i q_{i-1} - p_{i-1} q_i = (-1)^i \ . \tag{10}$$

Proof. By definition,

$$p_0 q_{-1} - p_{-1} q_0 = (1 \cdot 1 - 0 \cdot 0) = 1 = (-1)^0 .$$

Assume

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^k \text{ for } k = 0, 1, \ldots, i-1.$$

Now

$$\begin{aligned} p_i q_{i-1} - p_{i-1} q_i &= q_{i-1}(a_i p_{i-1} + p_{i-2}) - p_{i-1}(a_i q_{i-1} + q_{i-2}) \\ &= a_i p_{i-1} q_{i-1} + p_{i-2} q_{i-1} - a_i p_{i-1} q_{i-1} - p_{i-1} q_{i-2} \\ &= -(p_{i-1} q_{i-2} - p_{i-2} q_{i-1}) \\ &= -(-1)^{i-1} = (-1)^i. \end{aligned}$$

<u>COROLLARY 1.</u> The expression $\frac{p_i}{q_i}$ for the i^{th} convergent of a simple continued fraction is in its lowest terms.

Proof. If p_i and q_i have a common factor, then by Theorem 5, it will divide $(-1)^i$. Hence, p_i is prime to q_i, and $\frac{p_i}{q_i}$ is in its lowest terms.

<u>COROLLARY 2.</u> The difference between two consecutive convergents is a fraction whose numerator is unity. In fact,

$$\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}} = \frac{(-1)^i}{q_i q_{i-1}}, \qquad \text{for } i = 2, 3, \ldots, n. \tag{11}$$

Proof. From Theorem 5,

$$p_i q_{i-1} - p_{i-1} q_i = (-1)^i.$$

Now, divide both sides of the previous equation by $q_i q_{i-1}$. Thus

$$\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}} = \frac{(-1)^i}{q_i q_{i-1}} .$$

<u>THEOREM 6.</u> The difference between two convergents of a simple continued fraction whose indices differ by two is

$$\frac{p_i}{q_i} - \frac{p_{i-2}}{q_{i-2}} = \frac{a_i(-1)^{i-1}}{q_i q_{i-2}} , \qquad \text{for } i = 3, 4, \ldots, n, \text{ or } i=1. \qquad (12)$$

Proof. Now,

$$\frac{p_i}{q_i} - \frac{p_{i-2}}{q_{i-2}} = \frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}} + \frac{p_{i-1}}{q_{i-1}} - \frac{p_{i-2}}{q_{i-2}} .$$

From Corollary 2,

$$\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}} = \frac{(-1)^i}{q_i q_{i-1}}$$

and

$$\frac{p_i}{q_i} - \frac{p_{i-2}}{q_{i-2}} = \frac{(-1)^{i-1}}{q_{i-1} q_{i-2}} .$$

Therefore,

$$\frac{p_i}{q_i} - \frac{p_{i-2}}{q_{i-2}} = \frac{(-1)^i}{q_i q_{i-1}} + \frac{(-1)^{i-1}}{q_{i-1} q_{i-2}} .$$

Thus,

$$\frac{p_i}{q_i} - \frac{p_{i-2}}{q_{i-2}} = \frac{(-1)^i q_{i-2} + (-1)^{i-1} q_i}{q_i q_{i-1} q_{i-2}}$$

$$= \frac{(-1)^{i-1}(-q_{i-2} + a_i q_{i-1} + q_{i-2})}{q_i q_{i-1} q_{i-2}}$$

$$= \frac{(-1)^{i-1} a_i q_{i-1}}{q_i q_{i-1} q_{i-2}} .$$

$$= \frac{a_i (-1)^{i-1}}{q_i q_{i-2}} .$$

THEOREM 7. The i^{th} convergent $\frac{p_i}{q_i}$ may be expressed by

$$\frac{p_i}{q_i} = a_1 + \frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \cdot \cdot \cdot \cdot + \frac{(-1)^i}{q_{i-1} q_i} , \tag{13}$$

where $i = 1, 2, \ldots, n$.

Proof. First,

$$\frac{p_i}{q_i} = \frac{p_1}{q_1} + \left(\frac{p_2}{q_2} - \frac{p_1}{q_1}\right) + \left(\frac{p_3}{q_3} - \frac{p_2}{q_2}\right) + \cdot \cdot \cdot + \left(\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}}\right).$$

Thus, by definition, $\frac{p_i}{q_i} = a_1$ and from Corollary 2, the term

$\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}}$ in parentheses is equal to $\frac{(-1)^i}{q_{i-1}q_i}$. Hence, the theorem

is proved.

COROLLARY 3. For i = 1, 2, . . . , n,

$$p_i q_{i-2} - p_{i-2} q_i = (-1)^{i-1} a_i \,. \tag{14}$$

Proof. By substituting for p_i and q_i from equations (6) and (7),

$$\begin{aligned} p_i q_{i-2} - p_{i-2} q_i &= (a_i p_{i-1} + p_{i-2}) q_{i-2} - p_{i-2}(a_i q_{i-1} + q_{i-2}) \\ &= (p_{i-1} q_{i-2} - p_{i-2} q_{i-1}) a_i \,. \end{aligned}$$

Thus, by Theorem 5,

$$p_i q_{i-2} - p_{i-2} q_i = (-1)^{i-1} a_i \,.$$

C_i will be called an odd convergent if i is odd, and an even convergent if i is even.

THEOREM 8. The odd convergents of a finite simple continued fraction form an increasing finite sequence, the even convergents form a decreasing finite sequence, every odd convergent is less than any even convergent, every odd convergent is less than any following convergent, and every even convergent is greater than any following convergent.

Proof. From Corollary 2,

$$C_i - C_{i-1} = \frac{(-1)^i}{q_i q_{i-1}} \; ; \tag{15}$$

and from Theorem 6,

$$C_i - C_{i-2} = \frac{a_i(-1)^{i-1}}{q_i q_{i-2}} \; .$$

Now, $\frac{(-1)^i}{q_i q_{i-1}}$ and $\frac{a_i(-1)^{i-1}}{q_i q_{i-2}}$ are opposite in sign, which implies that C_i lies between C_{i-1} and C_{i-2}. Thus, from equation (15)

$$C_2 - C_1 = \frac{1}{q_2 q_1} \; .$$

Hence

$$C_1 < C_2 \text{ and } C_1 < C_3 < C_2 \, .$$

Similarly,

$$C_3 < C_5 < C_4, \; C_5 < C_6 < C_4, \cdots \; .$$

Combining these inequalities,

$$C_1 < C_3 < C_5 < \cdots < C_6 < C_4 < C_2 .$$

INFINITE SIMPLE CONTINUED FRACTIONS

An <u>infinite simple continued fraction</u> is an array of the following form,

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}} \quad ;$$

where the a_i are positive integers except for the fact that the integer a_1 may be positive, negative, or zero.

The part of the infinite simple continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_i + \cfrac{1}{a_{i+1} + \ddots}}}}$$

after a_i will be called the i^{th} remainder of the infinite simple continued fraction and is denoted by[4]

$$r_i = \langle a_{i+1}, a_{i+2}, \ldots \rangle \; .$$

Thus, an infinite simple continued fraction has the alternate form

$$\langle a_1, a_2, \ldots, a_i, r_i \rangle \; .$$

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[4]Khintchine, <u>op</u>. <u>cit</u>., p. 7.

If the process of writing the infinite simple continued fraction, is stopped at a_i, the resulting array is the finite simple continued fraction

$$\langle a_1, a_2, \ldots, a_i \rangle .$$

A <u>section</u>[5] of the infinite simple continued fraction $\langle a_1, a_2, \ldots \rangle$, denoted by S_i, is defined to be

$$S_i = \langle a_1, a_2, \ldots, a_i \rangle , \quad i \geq 1.$$

The i^{th} section of an infinite simple continued fraction is a finite simple continued fraction. Now the i^{th} convergent of an infinite simple continued fraction is defined to be the i^{th} convergent of the i^{th} section of the infinite simple continued fraction. Thus, with the convergents of an infinite simple continued fraction so defined, the same recursion formulae, and hence the same properties that were developed for the convergents of a finite simple continued fraction are valid for the convergents of an infinite simple continued fraction.

Now, the infinite simple continued fraction is said to converge to S, denoted by writing

$$S = \langle a_1, a_2, \ldots \rangle ,$$

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[5]Ibid.

if the infinite sequence of rational numbers $\{S_i\} = \{\langle a_1, a_2, \ldots, a_i \rangle\}$ converges to S. If the infinite sequence of rational numbers converges, then the infinite simple continued fraction is said to be convergent. Every simple continued fraction is convergent.

THEOREM 9. Every infinite simple continued fraction converges to a limit which is greater than any odd convergent and less than any even convergent.

Proof. The convergents of odd order, denoted by C_{2i+1} form a monotonically increasing sequence $\{C_{2i+1}\}$, each term of which is less than any even convergent C_{2i}. Thus, the convergents C_{2i+1} converge to a limit S_1 as i tends to infinity. Also, the even order convergents, $C_{2i,}$ form a monotonically decreasing sequence $\{C_{2i}\}$, each term of which is greater than any odd convergent C_{2i+1}. Thus, they converge to a limit S_2 as i tends to infinity.

Now, since both sequences are monotonic and convergent,

$$S_1 = \text{lub } C_{2i+1} \text{ and } S_2 = \text{glb } C_{2i},$$

so that

$$C_{2i+1} \leq S_1 \leq S_2 \leq C_{2i} \, .$$

Thus,

$$0 \leq S_2 - S_1 \leq C_{2i} - C_{2i+1} ,$$

and

$$\lim_{i \to \infty} (C_{2i} - C_{2i+1}) = S_2 - S_1 ;$$

from Corollary 2,

$$C_{2i} - C_{2i+1} = \frac{1}{q_{2i} q_{2i+1}} ,$$

so

$$0 \leq S_2 - S_1 \leq C_{2i} - C_{2i+1} = \frac{1}{q_{2i} q_{2i+1}} ,$$

However,

$$\lim_{i \to \infty} \frac{1}{q_{2i} q_{2i+1}} = 0,$$

since $q_{2i} \to \infty$ and $q_{2i+1} \to \infty$ as $i \to \infty$.

Hence,

$$0 \leq S_2 - S_1 \leq 0 ,$$

which implies

$$S_2 = S_1 = S.$$

Thus,

$$\lim_{i \to \infty} C_i = S$$

and for each i,

$$C_{2i+1} \leqslant S \leqslant C_{2i} \quad .$$

<u>THEOREM 10.</u> The values of the infinite simple continued fraction $\langle a_1, a_2, \ldots \rangle$ satisfies the inequality

$$\left| S - \frac{p_i}{q_i} \right| < \frac{1}{q_i q_{i+1}} \quad \text{for all } i \geqslant 1.$$

Proof. Now $\frac{p_i}{q_i}$, S, $\frac{p_{i+1}}{q_{i+1}}$ are in order of magnitude either ascending or descending. Therefore

$$\left| S - \frac{p_i}{q_i} \right| < \left| \frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i} \right| = \frac{1}{q_i q_{i+1}} \quad \text{for all } i \geqslant 1.$$

<u>THEOREM 11.</u> The value of any infinite simple continued fraction

$$\langle a_1, a_2, \ldots \rangle$$

is irrational.

Proof. Let Q represent the infinite simple continued fraction

$$\langle a_1, a_2, \ldots \rangle .$$

From Theorem 9, Q lies between C_i and C_{i+1}, so that the following chain of inequalities holds:

$$0 < \left| Q - C_i \right| < \left| C_{i+1} - C_i \right| .$$

Multiplying by q_i, and making use of the result of Theorem 7 that

$$\left| C_{i+1} - C_i \right| = \frac{1}{q_i q_{i+1}} ,$$

gives

$$0 < \left| q_i \quad Q - p_i \right| < \frac{1}{q_{i+1}} .$$

Now suppose that Q is rational, say $Q = \frac{a}{b}$, where a and b are integers with b 0. The above inequality becomes upon multiplying by b,

$$0 < \left| q_i a - p_i b \right| < \frac{b}{q_{i+1}} .$$

The integers q_i increase with i, so choose i sufficiently large so that $b < q_{i+1}$. Then the integer $\left| q_i a - p_i b \right|$ would lie between 0 and 1 which is impossible. Hence, a contradiction and Q is irrational.

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SIMPLE CONTINUED FRACTIONS

by

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The purpose of this paper is to develop the framework of simple continued fractions, without considering any applications. Simple continued fractions are the most important and most extensively studied continued fractions, and lie at the root of almost all arithmetic and very many analytic applications of continued fractions.

A simple continued fraction is a function of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}$$

where the a_i, $i = 1, 2, 3, \ldots$ are positive integers with the exception that the integer a_1 may be positive, negative, or zero.

The paper is divided into three sections. The first part deals with the development of the finite simple continued fraction. All the theory and definitions are provided and no previous knowledge of continued fractions is assumed. Secondly, the properties of the convergents of a simple continued fraction are developed. Finally, the theory of infinite simple continued fractions is developed and it is shown that, while finite simple continued fractions represent rational numbers, infinite simple continued fractions represent irrational numbers.