

Essays in expectation driven business cycle and wage polarization

by

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B.S., North South University, Bangladesh, 2007

M.A., York University, Canada, 2010

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AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the  
requirements for the degree

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Department of Economics  
College of Arts and Sciences

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# Abstract

This dissertation investigates two essential features of the US economy. First, it explores how news about future productivity changes business cycle fluctuations. Using the a representative agent model, it shows that implementation labor in workplace organization could be an important channel through which news about the fundamentals can realistically generate US business cycle fluctuations. Further this idea is extended using the perspective of sunspot fluctuations. In particular, the model can lead to multiple equilibria under specific parameterizations. Second, a general equilibrium model has been developed with heterogeneous agents to explain the wage polarization feature of the US labor market, particularly how the price of an important technology is connected to lifetime earnings of agents and affects their college decisions. The following summarizes the three chapters of my dissertation.

The first chapter which I co-authored with Dr. Blankenau, argues that purchasing investment goods does not directly increase the productive capacity of a business. Changes in the business through the installation of capital, worker training, and workplace reorganization are often required. These changes themselves are not easily automated. Change requires workers. We build a model where investment requires a complementary labor input. This mechanism is embedded in a representative agent model with capacity utilization, adjustment costs, and separable preferences. We show that this environment can yield positive co-movement between consumption, investment, and labor hours when the economy experiences a news shock about future productivity, thus providing an additional channel through which news shocks can generate key business cycle features.

The second chapter is an extension of the first chapter. I investigate the indeterminacy in a representative agent model with implementation labor and increasing returns in production. First, my analysis shows that a representative agent with implementation labor can exhibit increasing returns to scale. Then I show that self-fulfilling beliefs of agents lead

to business cycle fluctuations in which multiple equilibria can arise under specific parameterizations. Specifically, implementation labor in the production of capital is the highly important, necessary condition for the self-fulfilling equilibrium outcome.

The third chapter, which is also a joint work with Dr. Blankenau, discusses the wage polarization feature of the US labor market. We build a general equilibrium model with heterogeneous agents, showing how wage polarization can emerge when the price of computer capital falls. Consequently, we find the share of the population with a college degree decreases. Our findings are consistent with recent empirical data that show a U-shaped wage growth pattern in the US as well as a slower growth rate of college-educated workers despite the high returns of investing in education. In the model, we assume that each agent is born with a portfolio of skills. Specifically, each agent can provide manual labor, routine labor, and abstract labor and must decide how much of each to provide. An agent can increase efficiency in all types of labor by attending college. All three types of labor are valued in the labor market at an endogenously determined wage rate. Computer capital is a substitute for routine labor. As its price falls and its quantity increases, agents with a relative aptitude for routine labor no longer find it advantageous to attend college. Since routinization of tasks harms middle-income agents, the model has government policy implications for observed wage polarization.

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# Abstract

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# Dedication

To my parents Quazi Abul Fida and Hossneara Ferooja.

# Chapter 1

## Investment brings change:

## Implications for news driven business cycles

### 1.1 Introduction

Capital investment at the firm level is often a component of broader change. A new computer system means that workers will have new tasks added to their workload while others are eliminated. New heavy equipment will not simply replace the old, but will change the production process in important ways. Investment installation itself is a sort of change as the firm's activity is a departure from routine. Even additional capital which simply scales capacity will cause change. Adding more trucks to a shipping fleet, for example, will change how the firm is optimally managed.

The notion that investment brings change has received considerable attention in the literature. So too has the complementary notion that change requires labor. Much of this literature focuses in particular on the change brought about by the revolution in information technology. A prominent example is the work of [Bresnahan et al. \(2002\)](#).<sup>1</sup> They find a close

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<sup>1</sup>Also see [Autor et al. \(1998\)](#), [Bartel and Lichtenberg \(1987\)](#), [Bresnahan and Greenstein \(1999\)](#), and [Bryn-](#)

relationship between improved information technology and workplace reorganization. They also show that this change itself cannot easily be automated. Labor is required to implement the reorganization.

In this paper we take a broad view of the investment/labor demand relationship, assuming that investment of any sort is more productive when ‘implementation labor’ is employed to accommodate the firm-level changes. We consider the implications of implementation labor for news driven business cycles. We build a model where news of future productivity improvements changes firms’ investment demand. Changes in investment demand cause changes in the demand for implementation labor. The productivity increase may be specific to investment. Investment-specific technology change is often associated with information technology. Following the way, we consider improvements in information technology as highlighted by [Bresnahan et al. \(2002\)](#). The productivity increase may instead affect all production symmetrically. In this case, we consider the effects of adding implementation labor in a more standard setting.

We show that adding implementation labor to a representative agent model can improve the model’s ability to generate current business cycles from news of future events. A common relationship resulting from optimal agent behavior in representative agent, and many other models, is that the marginal rate of substitution between consumption and labor is equal to the marginal productivity of labor. [Beaudry and Portier \(2004\)](#), [Beaudry and Portier \(2007\)](#), [Wang \(2012\)](#) and others point out that this relationship presents a challenge for modeling business cycles as resulting from news about future productivity.<sup>2</sup> If news of future productivity results in increased current consumption, the marginal rate of substitution will increase. To preserve the relationship, the marginal productivity of labor must increase; i.e. the labor input must fall. The relationship between consumption and labor hour violates a key feature of business cycles: consumption and labor hours are positively correlated. [Beaudry and Portier \(2007\)](#) refer to this as the ‘static problem’ and we adopt the terminology. The challenge is not present with contemporaneous productivity shocks since

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jolfsson and Mendelson (1993).

<sup>2</sup>See [Sims \(2013\)](#), [Schmitt-Grohé and Uribe \(2012\)](#).

such shocks increase the marginal product of labor even after accounting for the general equilibrium increase in employment.

The literature related to News-Driven Business Cycles (NDBC) explores several modifications of the baseline representative agent model that overcome the static problem. We show that implementation labor provides an additional useful modification of this sort. In essence, a news shock influences the supply of labor used to implement new investment capital as well as labor used in the production of a final good. Labor used for implementation enhances capital accumulation but has no impact on current production. Importantly, then, implementation labor has no direct effect on the marginal product of labor and allows more freedom of movement between the marginal rate of substitution and total labor employed.

Our model is most closely related to [Jaimovich and Rebelo \(2009\)](#). Their model has three key features which together allow consumption, investment, and labor hours to increase in response to a positive news shock. First they set the depreciation rate for capital equal to the endogenously determined rate of capacity utilization. An increase in capacity utilization has a similar effect to an increase in capital. With more capital employed, the marginal product of labor increases. The increase in capital with labor productivity help to overcome the static problem. Second they include adjustment costs which helps in assuring that current consumption is positively correlated with news of future productivity changes.

[Jaimovich and Rebelo \(2009\)](#) demonstrate that these two features of the law of motion for the capital stock fall short in generating the desired comovements in response to a news shock. A third feature, non-separable preferences, is essential. Their preference weakens the relationship between the marginal rate of substitution and the marginal product of labor. Our model does not include the third feature. We instead include an additional term in the law of motion for capital meant to capture the salient features the investment/labor demand relationship. Our new law of motion in the representative agent model able to generate news-driven business cycles even without the special utility function introduced in [Jaimovich and Rebelo \(2009\)](#).

We first consider a special case of our model with no adjustment costs which allows us to analytically examine conditions allowing positive comovement between consumption,



investment, and total labor hours. Importantly, we show that our new feature gives a boost to the effects of the capacity utilization rate. With the boost, capacity utilization can respond sufficiently to a news shock to allow a general equilibrium increase in the marginal product of labor at the same time that labor employed increases. Absent implementation labor, this cannot occur.

Moreover, we show that whether the increase in both labor hour and marginal productivity is closely related to returns to scale in the production of investment goods. In particular, we show that increasing returns to scale is a sufficient condition for comovement across these key variables. Because of increasing returns to scale, our results are related to those of [Guo et al. \(2015\)](#) in two ways. They show that a production externality can overcome the static problem. They focus on a production spillover that results in increasing returns to scale at the social level despite constant returns to scale at the firm level. There is no similar externality in our model. However, investment is produced using a final good and implementation labor. The final good is a standard Cobb-Douglas combination of capital and labor with constant returns to scale. Investment combines this final good with a second sort of labor input. Special type of investment allows the possibility of constant returns to scale in the production of the final good with increasing returns to scale for investment. Another commonality of our model with [Guo et al. \(2015\)](#) is that with increasing returns to scale, the model may be indeterminate. We characterize conditions which give rise to indeterminacy in the second chapter of this dissertation. Here, we restrict our parameter choices to cases where the model is determinate.

In our special case, the model can overcome the static problem. [Beaudry and Portier \(2007\)](#) also articulate the ‘dynamic problem’ of news driven business cycles. A future productivity increase is a positive lifetime income shock. The resulting consumption smoothing is a positive force affecting current consumption. At the same time, investment may increase as firms gear up for the anticipated productivity increase. The rise in productivity weighs in favor of decreased current consumption through the resources constraint. A news driven business cycle model must find a way for output to respond sufficiently, and its allocation to respond properly, such that consumption and investment both increase.

Our special case does not overcome the dynamic problem. As in [Jaimovich and Rebelo \(2009\)](#), we need adjustment costs for positive productivity shocks to yield both an increase in consumption and positive comovement between consumption, investment, labor hours, and output. The remainder of the paper, then, includes adjustment costs and considers the impact of implementation labor in our full model. We show that our model with implementation labor, variable capacity utilization, and investment adjustment cost can generate qualitatively realistic aggregate fluctuations driven by news shock to total factor productivity and investment specific technological change.

One caveat of the standard business cycle model is that it does not offer the realistic aggregate fluctuations of the key macroeconomic variables. Recent literature provides various modifications of the existing models to accommodate the realistic business cycles due to news shock. For example, [Beaudry and Portier \(2007\)](#) show a multi-sector model with internal cost complementarities between the production of different goods able to increase consumption, investment and output due to a positive news shock. They also argue that a two-sector model with no substitutability between consumption and investment is a potential candidate for expectation driven boom and bust cycles. Also, [Jaimovich and Rebelo \(2009\)](#); [Eusepi and Preston \(2011\)](#) introduce particular kinds of preferences in the neo-classical setup to achieve NDBC. Other examples that exhibit potential candidate for NDBC are models with imperfect competition and increasing returns to scale or externalities ([Khan and Tsoukalas, 2012](#); [Guo et al., 2015](#)). The idea of the NDBC is also extended using a New-Keynesian setup ([Christiano et al., 2010](#); [Eusepi and Preston, 2011](#)). They find that sticky prices combined with a particular type of monetary policy are needed to generate NDBC. In this case, the monetary policy must to sufficiently accommodative to the news. In addition, [Den Haan and Kaltenbrunner \(2009\)](#) show that convex adjustment costs of labor in the value function within a search and friction model creates a boom in response to good news.

In this chapter, we consider a neo-classical model with implementation labor that allows sufficient labor demand and capital utilization to exhibit news-driven business cycle. In particular, we examine the possibility of booms or busts with constant returns to scale in technology and without any market imperfection. We provide a novel mechanism which

explains why market economies exhibit business cycle fluctuations driven by changes in expectations.

In the next section we present our model. Our work is most closely related to [Jaimovich and Rebelo \(2009\)](#) so we pay particular attention to our point of departure with their work. In particular we highlight that a key feature of their model, non-separable preferences, is not required in our model. In the third section, we present a special case to provide intuition into the effects of implementation labor. In the fourth section, we show the model's results in the benchmark case, with a focus on impulse responses to news shocks. In the final section we summarize and discuss an extension to this work.

## 1.2 The Model

The economy is populated by a mass of identical agents who derive utility from consuming a final good in each period and disutility from providing a labor input. With  $c_t$  and  $n_t$  defined as consumption and the labor input in period  $t$ , the expected lifetime utility of a representative agent is

$$\max_{\{c_t, n_{f,t}, n_{i,t}, u_t, k_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} - \phi \frac{(n_{f,t} + n_{i,t})^{1+\gamma}}{1+\gamma}. \quad (1.1)$$

Here  $E_0$  is the expectations operator,  $\beta < 1$  is the discount rate,  $\phi > 0$  gauges the disutility of hours worked and  $\sigma, \gamma \geq 0$  govern elasticities.

A representative firm combines capital and labor to produce the final good subject to a Cobb-Douglas production function with share parameter  $\alpha \in [0, 1]$  and general productivity parameter  $a_t > 0$ . In general, capital employed by the firm will be some share,  $u_t$ , of the total capital stock available to the economy,  $k_t$ . Moreover, the labor input in the final good production,  $n_{f,t}$ , will be only part of the total labor input. Output of the final good,  $y_t$ , then, is

$$y_t = a_t (u_t k_t)^\alpha (n_{f,t})^{1-\alpha}. \quad (1.2)$$

The final good can be utilized as a consumption good or an investment good,  $i_t$ , with the resource constraint given by

$$y_t = c_t + \frac{i_t}{v_t}. \quad (1.3)$$

The rate at which a unit of the final good can be converted to a unit of the investment good depends on the technology parameter  $v_t$ .<sup>3</sup> An increase  $v_t$  reflects investment-specific technological progress and a decrease is technological regress. General and investment-specific technological progress are stochastic and governed by

$$(a_t) = \rho_a(a_{t-1}) + e_{1,t}^a + e_{2,t-j}^a \quad j > 0, \quad (1.4)$$

$$(v_t) = \rho_v(v_{t-1}) + e_{1,t}^v + e_{2,t-j}^v \quad j > 0, \quad (1.5)$$

where  $\rho_a, \rho_v \in (0, 1)$  and  $j > 0$ .

The law of motion for capital contains two of the key features driving business cycles in [Jaimovich and Rebelo \(2009\)](#). First, the rate of depreciation is positively related to the endogenous rate of capacity utilization. Specifically, we set depreciation equal to  $\frac{\varphi_2 u_t^{\varphi_1}}{\varphi_1}$  with  $\varphi_1 > 1$  and  $\varphi_2 > 0$ . Second, we include adjustment costs so that investment,  $i_t$ , is scaled by  $1 - \frac{\varphi_2}{2} \left( \frac{i_t}{i_{t-1}} - 1 \right)^2$  where  $\varphi_2 > 0$ .

[Jaimovich and Rebelo \(2009\)](#) demonstrate that these features of the law of motion fall short in generating the desired comovements in response to a news shock. Their third feature, non-separable preferences, is essential. Since preferences are separable in our model, an alternative feature is required. We assume that purchasing investment goods can more effectively add to the capital stock when combined with separate labor input,  $n_{i,t}$ . We refer to this separate labor input as implementation labor. Our law of motion is given by

$$k_{t+1} = \left( 1 - \frac{\varphi_2 u_t^{\varphi_1}}{\varphi_1} \right) k_t + i_t \left( \psi_1 - \frac{\psi_2}{2} \left( \frac{i_t}{i_{t-1}} - 1 \right)^2 \right) + \psi_3 (\theta_i i_t^\kappa + \theta_n n_{i,t}^\kappa)^{\frac{1}{\kappa}}. \quad (1.6)$$

The term  $\psi_3 (\theta_i i_t^\kappa + \theta_n n_{i,t}^\kappa)^{1/\kappa}$  where  $\theta_i, \theta_n \geq 0, \kappa \leq 1$ , represents our generalization of the law

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<sup>3</sup>This specification is similar to [Greenwood et al. \(1997, 2000, 1988\)](#).

of motion consistent with the discussion in the previous section. Through the expression, increments to the capital stock from investment depend upon how much labor is hired to implement the investment. The parameters  $\theta_i$  and  $\theta_n$  gauge the relative importance of investment and implementation services in producing physical capital and while  $\kappa$  governs their substitutability.

The above expression is meant to capture lessons from recent firm-level investigations of how technological change influences the structure of the firm. Absent the mechanism, the law of motion implicitly assumes that investment effortlessly increases productive capacity by having, for example, a new machine ready to go. However, consider the case of an investment in information technology (IT). [Bresnahan et al. \(2002\)](#) show that such investment leads to changes in organizational practices and even to changes in products and services. The purchase of information technology is the start, not the end, of the investment process. We argue that these remaining efforts are not easily mechanized and require a targeted labor input.

Like [Bresnahan et al. \(2002\)](#), much of the literature along these lines has focused in particular on IT investment, and for good reason. The quality-adjusted real price of computers has been declining at a compound rate of about 20% per year. Several studies suggest that the internal organization of the firm has been reshaped by the economics of information and communication.<sup>4</sup> To some extent we capture the organizational change by allowing a decrease in the price of investment through  $v_t$ . However, we take a broader view and assume that any sort of investment will require change in the firm and labor to implement it. Equipment of nearly any sort will be an improvement upon prior investment. A work environment optimized for one vintage of capital will not likely be optimized for the next. Training and restructuring will be required whether the investment is in IT or heavy equipment. Even investment in equipment which is just more of the same will require installation. Moreover, if investment reflects growth of the firm, the optimal structure may change for that reason alone. Our addition to the law of motion for capital is meant to be a general modeling of

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<sup>4</sup>Examples include [Milgrom and Roberts \(1990\)](#), [Brynjolfsson and Mendelson \(1993\)](#), [Radner \(1993\)](#), and [Black and Lynch \(2001\)](#).

the notion that investment brings firm-level change, and change requires labor.

Because not all investment expenditure is reflected in  $i_t$ , total output will not be reflected by production of the final good. The value of output in this economy is the sum of final goods production (our numeraire good) and the value of the services provided in putting investment goods into production. Looking ahead to an equilibrium, labor will have the same wage,  $w_t$ , whether employed in final goods production or investment implementation. The value of the implementation services, then will be equal to  $w_t n_{i,t}$  and total output is given by

$$Y_t = y_t + w_t n_{i,t}. \quad (1.7)$$

To solve the model, the social planner chooses  $c_t, n_{f,t}, n_{i,t}, i_t, k_{t+1}$  and  $u_t$  to maximize the equation (1.1) subject to equations (1.3), (1.6) and (1.2) and  $n_{f,t} + n_{i,t} = n_t$ . Substituting in for the last two constraints, and defining  $\lambda_t$  and  $\lambda_{k,t}$  as the Lagrangian multipliers on the first two constraints, first order conditions (F.O.Cs) are given by

$$c_t^{-\sigma} = \lambda_t \quad (1.8)$$

$$\phi(n_{f,t} + n_{i,t})^\gamma = \lambda_t (1 - \alpha) \frac{y_t}{n_{f,t}} \quad (1.9)$$

$$\phi(n_{f,t} + n_{i,t})^\gamma = \lambda_{k,t} \theta_n \psi_3 (\theta_i i_t^\kappa + \theta_n n_{i,t}^\kappa) \left( \frac{1}{\kappa} - 1 \right) n_{i,t}^{\kappa-1} \quad (1.10)$$

$$\lambda_{k,t} z(i_{t-1}, i_t, n_{i,t}) + E_t \left( \lambda_{k,t+1} \beta \psi_2 \frac{i_{t+1}^2}{i_t^2} \left( \frac{i_{t+1}}{i_t} - 1 \right) \right) = \frac{\lambda_t}{v_t} \quad (1.11)$$

$$E_t \left( \beta \lambda_{k,t+1} \frac{y_{t+1}}{k_{t+1}} \alpha \right) + E_t \left( \beta \lambda_{k,t+1} \left( 1 - \frac{u_{t+1}^{\varphi_1}}{\varphi_1} \right) \right) = \lambda_{k,t} \quad (1.12)$$

$$\lambda_t \alpha y_t = \lambda_{k,t} \psi_2 u_t^{\psi_1} k_t. \quad (1.13)$$

where

$$z(i_{t-1}, i_t, n_{i,t}) \equiv \psi_1 - \frac{\psi_2}{2} \left( \frac{i_t}{i_{t-1}} \right)^2 + \frac{\theta_i \psi_3 (\theta_i i_t^k + \theta_n n_{i,t}^k) \frac{1}{k}^{-1}}{i_t^{1-k}} - \frac{\psi_2 i_t}{i_{t-1}} \left( \frac{i_t}{i_{t-1}} - 1 \right). \quad (1.14)$$

Combining equations (1.8) and (1.9) gives

$$\phi n_t^\gamma c_t^\sigma = (1 - \alpha) \frac{y_t}{n_{f,t}} \quad (1.15)$$

which is the usual relationship equating the marginal rate of substitution between  $n_t$  and  $c_t$  to the marginal product of labor. As noted by [Beaudry and Portier \(2014\)](#), this fundamental relationship exposes two challenges to modeling new-driven business cycles. The first challenge is to have positive news cause an increase in  $c_t$  and the second is to preserve the equality given this increase.

To replicate key business cycle facts, consumption, investment and labor hours should all have positive comovement. Through the resource constraint, the above expression will assure positive comovement with output. News of a positive future productivity shock may lead to an increase in current consumption. A future productivity increase is a positive lifetime income shock. The resulting consumption smoothing is a positive force affecting current consumption. At the same time, investment may increase as firms gear up for the anticipated productivity increase. The relationship weighs in favor of decreased current consumption through the resources constraint. Absent an increase in output, increased current investment requires decreased current consumption. A news driven business cycle model must find a way for output to respond sufficiently, and its allocation to respond properly, such that consumption and investment both increase. The output increase cannot rely on technology changes or increased capital as these are fixed in the current period. The relationship is what [Beaudry and Portier](#) refer to as the ‘dynamic challenge’ of news driven business cycles.

A model that overcomes the dynamic challenge assures comovement between consump-

tion, investment, and output. It still faces a ‘static challenge’ to assure positive comovement with hours.<sup>5</sup> With  $c_t$  increased, other general equilibrium adjustments must preserve equality in equation (1.15). There might be a decrease in  $n_t$  on the left-hand-side. However, the relationship violates the required positive correlation between  $c_t$  and  $n_t$ . The other possibility is to increase the right-hand-side of the equality. That is, the model could generate an increase in the marginal product of labor when consumption increases. The increasing of right-hand-side, too, is problematic. In general, the marginal product of labor is decreasing in labor. To increase this would require less labor employed in producing the final good. We show later with numerical exercises that  $n_t$  and  $n_{f,t}$  are positively correlated over a wide range of parameters and make this assumption for now. In this case, increasing the marginal product of labor through a decrease in  $n_{f,t}$  means that  $n_t$  falls. The relationship again violates positive correlation between  $c_t$  and  $n_t$ .

One feature of [Jaimovich and Rebelo \(2009\)](#) helps toward overcoming both the dynamic and the static problem. Variable capacity utilization in their model allows the economy to respond to future productivity by utilizing, and hence depreciating, capital at a higher rate. The above expression allows output to increase with fixed capital and technology, creating the possibility that investment and consumption could both increase. A second feature of their model, properly calibrated adjustment costs, assures a proper allocation of this increased output and generates the appropriate relationship between  $c_t$  and  $i_t$ .

Aside from its part in solving the dynamic problem, variable capacity utilization mitigates, but does not eliminate, the static problem. To see this, substitute the production function in for  $y_t$  in (1.15) and simplify to get

$$c_t^\sigma = \frac{(1 - \alpha)a_t (u_t k_t)^\alpha}{\phi \alpha n_{f,t}^\alpha n_t^\gamma}. \quad (1.16)$$

With  $a_t$  and  $k_t$  fixed at time  $t$ , an increase in the left-hand side through an increase in  $c_t$ , and an increase in the numerator through comovement between  $c_t$  and  $n_{f,t}$ , might be accommodated through an increase in the capacity utilization rate. Essentially, an increase

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<sup>5</sup>We are again using the terminology of [Beaudry and Portier \(2014\)](#).



in  $u_t$  increases the marginal product of labor at any level of  $n_{f,t}$ . The above expression allows for the possibility that both  $n_{f,t}$  and the marginal product of labor could increase. The increase in  $n_{f,t}$ , through its impact on  $y_t$ , amplifies the effect of increased capacity utilization in overcoming the dynamic problem.

### 1.3 A special case

In this subsection we show that in a special case, capacity utilization falls short of overcoming the static problem. We consider a simpler law of motion where  $\psi_1 = \psi_2 = \psi_3 = \kappa = 0$  and  $\theta_i = 1$  so that

$$k_{t+1} = \left(1 - \frac{\varphi_2 u_t^{\varphi_1}}{\varphi_1}\right) k_t + i_t n_{i,t}^{\theta_n}. \quad (1.17)$$

In this case there are no adjustment costs and implementation labor scales investment in creating physical capital. Then equations (1.2), (1.11), and (1.14) yield

$$u_t = \left( \frac{v_t \alpha a_t}{k_t^{1-\alpha}} n_{i,t}^{\theta_n} n_{f,t}^{1-\alpha} \right)^{\frac{1}{\varphi_1 - \alpha}}. \quad (1.18)$$

Substituting in for  $u_t$  in equation (1.16) gives

$$c_t^\sigma = \frac{\frac{\theta_n \alpha}{x_1 n_{i,t}^{\varphi_1 - \alpha}}}{\frac{\alpha(\varphi_1 - 1)}{n_t^\gamma n_{f,t}^{\varphi_1 - \alpha}}} \quad (1.19)$$

where  $x_1$  is a positive scalar. All exponents in equation (1.19) are positive. With  $\theta_n = 0$ , hours show up only in the denominator so it not possible for  $c_t$  and hours worked to both increase while preserving the equality. With  $\theta_n$  positive, labor hours are also in the numerator. The relationship makes comovement possible as the denominator may increase more than the numerator when hours increase. It is not clear whether the denominator

will increase sufficiently since  $n_{i,t}$  and  $n_{f,t}$  are related and  $n_t = n_{i,t} + n_{f,t}$ . However, when preferences are logarithmic in consumption and linear in the labor input ( $\sigma = 1, \gamma = 0$ ),  $n_{f,t}$  and  $n_{i,t}$  are linearly related to  $n_t$  such that

$$n_{i,t} = \frac{\theta_n n_t}{\theta_n + 1 - \alpha} - \frac{\theta_n(1 - \alpha)}{\phi(\theta_n + 1 - \alpha)} \quad (1.20)$$

$$n_{f,t} = \frac{n_t(1 - \alpha)}{\theta_n + 1 - \alpha} + \frac{\theta_n(1 - \alpha)}{\phi(\theta_n + 1 - \alpha)}. \quad (1.21)$$

We show in Appendix A that in this case  $\theta_n > (\varphi_1 - 1)$  is a sufficient condition for  $\frac{dc_t}{dn_t} > 0$  in equation (1.19).

Returning to equation (1.15), the static problem is that (i) the marginal rate of substitution is increasing in consumption and (ii) the marginal product of labor is decreasing in labor. Either (i) or (ii) must be overcome in some way to allow consumption and labor to simultaneously increase. Capacity utilization in [Jaimovich and Rebelo \(2009\)](#) is not enough to overcome (ii). They overcome (i) by introducing a more generalized set of preferences that allows the MRS to decrease in  $c_t$ . We instead allow a more general setting in the law of motion for capital to overcome (ii). The co-movement works through increasing the response of capacity utilization to a news shock.

It may appear that our model is highly susceptible to a particular concern with using capacity utilization as the channel through which news drives business cycles. With variable capacity utilization, a news shock increases capital through increased investment. However, it also decreases capital through increased depreciation. The mechanism can associate news shocks with decreased capital in the subsequent period. As our model gives increased scope to capacity utilization, the second effect is amplified. In our model, however, there is another feature favoring an increase in capital. As seen in equation (1.18) capacity utilization is associated with an increase in  $n_{i,t}$ . This, in turn, is associated with more productive implementation of investment through equation (1.6). The relationship makes it easier for our model to generate an increase in  $k_{t+1}$  resulting from a news shock.

A nice feature of the mechanism we introduce is that it makes it easier for news shocks to be associated with current increases in labor productivity. The mechanism is easiest to see in the final goods market. Using equations (1.2) and (1.18) we arrive at the following expression for labor productivity:

$$\frac{y_t}{n_{f,t}} = \left( \frac{a_t^{\varphi_1} v_t^\alpha \alpha^\alpha k_t^{\varphi_1-1} n_{i,t}^{\alpha\theta_n}}{n_{f,t}^{\alpha(\varphi_1-1)}} \right)^{\frac{1}{\varphi_1 - \alpha}}.$$

Each exponent here is positive. We see that with  $\theta = 0$ , hours appear only in the denominator so productivity and hours move in opposite directions. With  $\theta > 0$ , both can increase. In the special case expressed in equations (1.20) and (1.21), we can show that  $\theta > (\varphi_1 - 1)$  is a sufficient condition for positive comovement between hours and labor productivity.

### 1.3.1 The static problem with no adjustment costs

The argument above suggests that implementation effort can overcome the static challenge of news driven business cycles and shows this analytically for a special case. It does not suggest that this mechanism is helpful in overcoming the dynamic problem, and indeed we find that is not. As in [Jaimovich and Rebelo \(2009\)](#), we need adjustment costs in order for positive productivity shocks to yield both an increase in consumption and positive comovement between consumption, investment, labor hours, and output. In this subsection, we omit adjustment costs in order to focus on overcoming the static problem. Without adjustment costs, consumption decreases when the model generates the proper comovements. When we later add adjustment costs, the model generates these comovements along with an increase in consumption.

We first consider our special case above and then relax several of the assumptions in turn to show their impact on the comovement between consumption, hours, and investment. We show in the second chapter of this dissertation that in our special case the model can be indeterminate and find sufficient conditions where this can hold. For current purposes,

it is sufficient to have some intuition for why this can occur.<sup>6</sup> Let  $s_t$  be the endogenously determined share of the final good that is invested in period  $t$  so that from equation (1.3)  $i_t = s_t v_t y_t$ . Then with no adjustment costs, we can write equation (1.6) as

$$k_{t+1} - k_t + \left( \frac{\varphi_2 u_t^{\varphi_1}}{\varphi_1} \right) k_t = \psi_3 (\theta_i (s_t v_t y_t)^\kappa + \theta_n n_{i,t}^\kappa) \frac{1}{\kappa}. \quad (1.22)$$

The left-hand side of this is gross capital formation. From the right-hand side we see that gross capital formation is a function of  $y_t$  and  $n_{i,t}$ . The final good,  $y_t$ , is constant returns to scale in  $k_t$  and  $n_{f,t}$ . Given this, gross capital formation is constant returns to scale in all inputs if  $\theta_i + \theta_n = 1$ . However,  $\theta_i + \theta_n > 1$ , we have increasing returns to scale in this aggregate. Prior literature shows that indeterminacy can arise in models with increasing returns to scale in the production of the final good. Increasing returns to scale in gross capital formation gives rise to similar concerns in our model. While this is most clear in our special case, the issue of indeterminacy is robust. In particular we show in numerical exercises that the model can generally be indeterminate when  $\theta_i + \theta_n$  exceeds 1 and  $\beta$  or  $\varphi_1$  is sufficiently large.

Our goal is to generate news driven business cycles in a deterministic setting rather than consider sunspot equilibria. For this reason, the possibility of indeterminacy restricts our parameter choices. The model is determinate in our baseline calibration of the full model below. The determinacy holds also for the sensitivity analysis conducted around the baseline. However, the parameter restrictions in this special case are more severe. In particular, the next chapter shows that sufficient conditions for indeterminacy restrict us to relatively small values of  $\theta_n$ . With small values of  $\theta_n$ , the current response to future productivity changes is small. Nonetheless, the special case is useful for showing how implementation effort can yield the proper directional changes in the aggregates of interest.

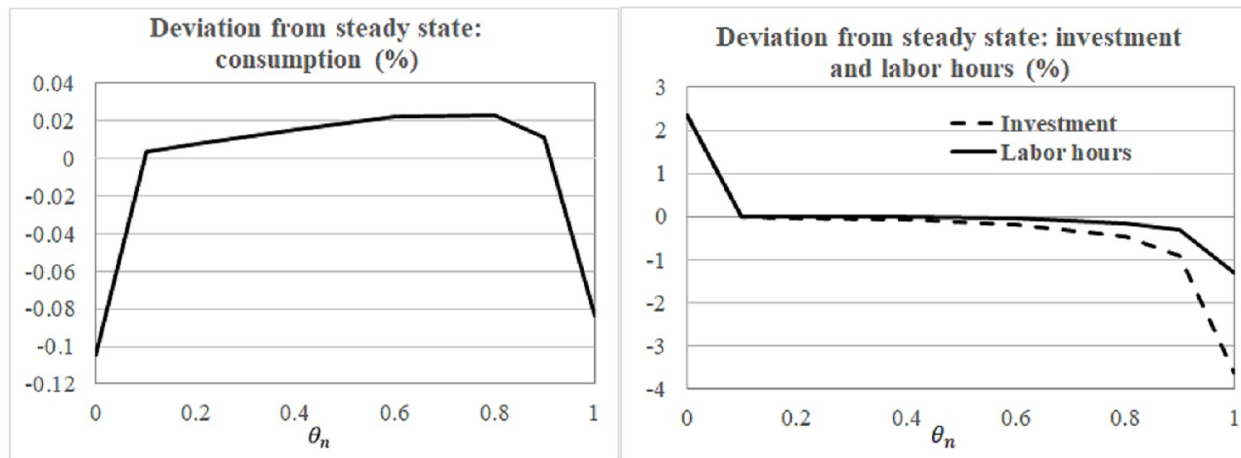
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<sup>6</sup>See [Benhabib and Farmer \(1994\)](#) for a further discussion of indeterminacy with increasing returns.

### 1.3.2 Results in the special case

Figure 1 below shows the current period response to news of a one-period-ahead total factor productivity increase equal to 1 of its mean value. It is common in the literature to consider news that pre-dates productivity increases by multiple periods. We choose a one-period-ahead shock for this discussion only to make the response larger for expositional purposes. To avoid indeterminacy we set  $\beta = .82$  which means that shocks further in the future have a small impact in the current period. Our qualitative findings are not sensitive to this, and we later consider a more standard time frame. We set parameters in the baseline for this exercise consistent with our special case:  $\psi_1 = \psi_2 = \kappa = \gamma = 0$  and  $\alpha = 1$ . We further set  $\alpha = .33$ ,  $\varphi_1 = 1.1$ ,  $v_t = \psi_2 = 1$  and  $\rho_a = .9$ . The parameter  $\varphi_2$  influences the capacity utilization rate. With this set to 1, we have a capacity utilization rate in excess of 1. While this does not cause mathematical problems in the model, it creates a challenge for interpreting the results. For this reason, we set  $\varphi_2 = 5$ . While the dynamics are qualitatively similar between  $\varphi_2 = 5$  and  $\varphi_2 = 1$ , the larger value allows for reasonable capacity utilization values.

Figure 1.1 shows the current percentage deviation of consumption, investment, and labor



**Figure 1.1:** *Comovement for different values of  $\theta_n$ .*

hours from their steady state values. We show the IRFs for a range of  $\theta_n$  values holding

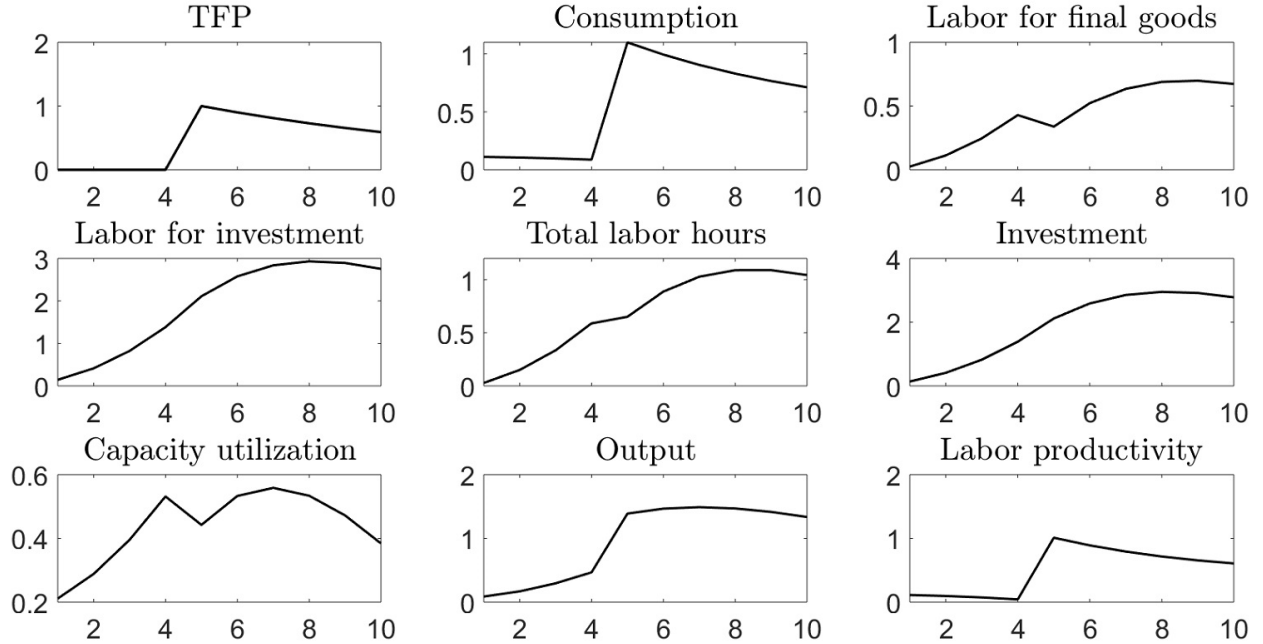
$\theta_i$  fixed at 1. The first panel shows that the effect of a news shock on consumption is non-monotonic in  $\theta_n$ . For small and large values of  $\theta_n$ , the productivity shock decreases consumption. For moderate values, it increases consumption. The second panel shows that the impact of the shock on investment and labor hours decreases with  $\theta_n$ . It shows that without our feature, i.e., when  $\theta_n = 0$ , consumption moves in the opposite direction of investment and labor hours. With  $\theta_n$  sufficiently large, the three items move in the same direction. Through the resource constraint, we know output also increases. Stated differently, we generate observed comovements across these aggregates and hence overcome the static problem with  $\theta_n$  large. With  $\theta_n$  small, we do not. While we show this only for the special case and only for the total factor productivity shock, we find in the subsequent section that this also holds with the investment specific shock and over a wide range of parameter values.

## 1.4 Results in the baseline model

We now turn to our baseline model. In this section, we consider a calibrated version of our model. We examine the economy's response as the representative agent learns about an upcoming change in productivity. We consider only parameters which maintain saddle path stability and a unique equilibrium. We assume that the economy is in its non-stochastic steady state at time zero. At period one, an unanticipated news shock arrives. The representative agent learns that total factor productivity (or investment specific productivity) will change three periods later, in period 4. Following a one standard deviation news shock, the level of  $a_t$  (or  $v_t$ ), does not increase immediately, by construction, but rises sharply at period 4 and returns to its steady state value over the horizon as expressed in equations (1.12) and (1.14).

We present our baseline model results into two parts. First, we consider the case of complementarity between investment and implementation labor in the production of capital,  $\kappa \leq 0$ . We consider this case of complementarity to be most consistent with the data. For example, [Bresnahan et al. \(2002\)](#) show that investment and workplace organization labor are complements. Next, we consider the case of high substitutability between investment

and implementation labor in production function of capital  $0 < \kappa \leq 1$ . We mainly show that comovements in macroeconomic variables can be achieved when investment and implementation labor are complementary in the production of capital. In case of substitutability between the two, our model does not generate comovements.



**Figure 1.2:** *Impulse responses from TFP news shocks in the baseline model.*

Figure 1.2 shows the impulse response results in our baseline model with complementarity between investment and implementation labor. In each panel, the line represents the impulse response of the respective variables due to the total factor productivity (TFP) news shock. We adopt the following parameterization that is commonly used in the real business cycle literature. The income share of capital,  $\alpha$ , is 0.33; the discount factor,  $\beta$ , is 0.985; labor supply elasticity,  $\gamma$ , is 0 (i.e., perfectly elastic), and,  $\sigma = 1$  (logarithmic utility in consumption). The preference parameter,  $\phi$ , is 1; the capital utilization parameter,  $\varphi$ , is 1.3; the adjustment cost parameters are  $\psi_1 = 1$ ,  $\psi_2 = 2$ , and  $\psi_3 = 3$ . We set the elasticity of substitution parameter,  $\kappa$ , to -8. The relative importance of investment,  $\theta_i$ , and implementation labor,  $\theta_n$ , in capital production are 0.5 and 0.5, respectively.

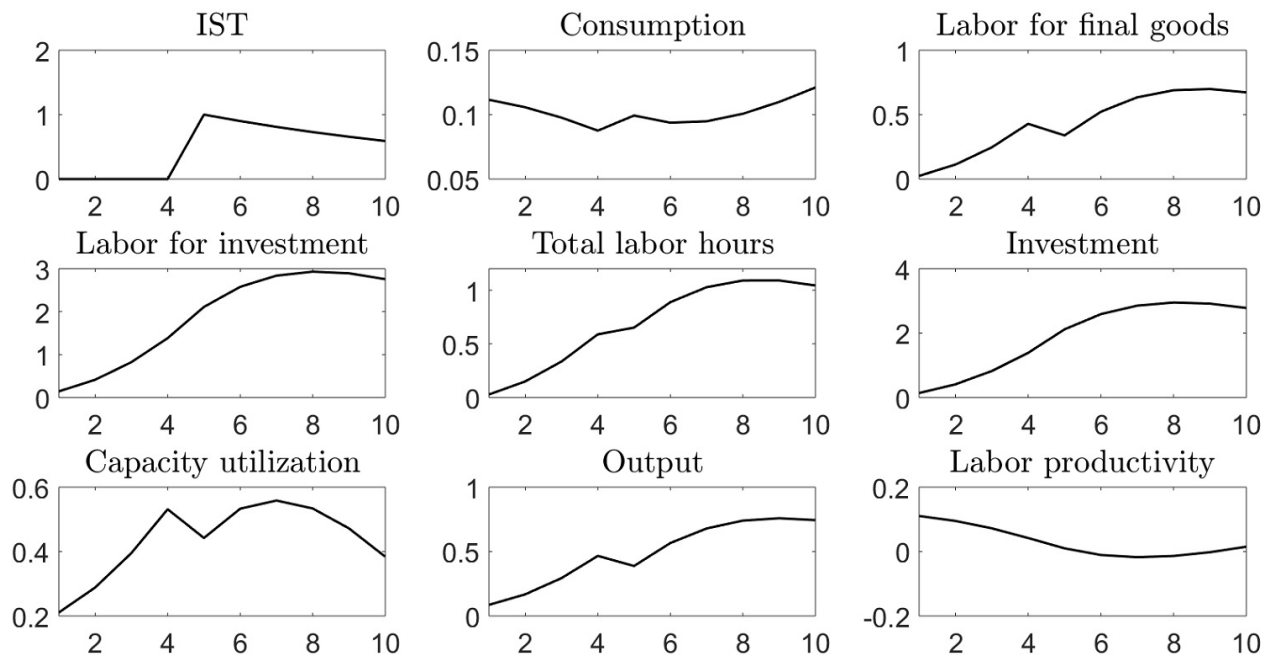
The first panel of Figure 1.2 shows the relationship between the news shock and total factor productivity. News arrives in period 1. Since this is news of a future increase in productivity, there is no immediate change in total factor productivity. Total factor productivity remains unchanged for four periods and then increases. Due to the AR(1) structure, the shock dies out through time, and total factor productivity returns eventually to its baseline.

The remaining panels show that the economy begins to adjust immediately to the news shock in anticipation of the eventual productivity increase. Recall in our special case that consumption decreased when the economy experienced a positive news shock. The reduction was due to a desire to build up capital in response to its future increased productivity. Here we have added adjustment costs. As a result, rapid increases in the capital stock are costly. The adjustment cost allows a countervailing effect to dominate. Improved future productivity increases lifetime income, resulting in increased consumption in all periods. That is, adjustment costs allow the model to avoid the dynamic problem.

As discussed above, from equation (1.15) an increase in consumption must be met with an increase in marginal productivity of labor. With a contemporaneous shock, the exogenous productivity improvement would generate increased marginal productivity of labor. With exogenous productivity and with the current capital stock fixed, this occurs through an increase in capacity utilization (panel 7). With implementation labor, the capacity utilization response is large enough to allow both increased labor in the production of the final good (panel 3) and increased productivity of labor (panel 9).

The initial effect on labor for investment is positive but small. It happens because two competing effects nearly offset with this parametrization. The increase in investment causes an increase in the demand for investment labor. However, as mentioned above (and suggested by equation (1.18)), implementation labor makes capacity utilization more responsive to productivity increases. The resulting large increase in the capacity utilization rate has a direct effect on productivity in the production of the final good, but has no direct effect on the productivity of implementation labor. Particularly, this relationship biases any increase in total labor toward increases in labor for final goods.

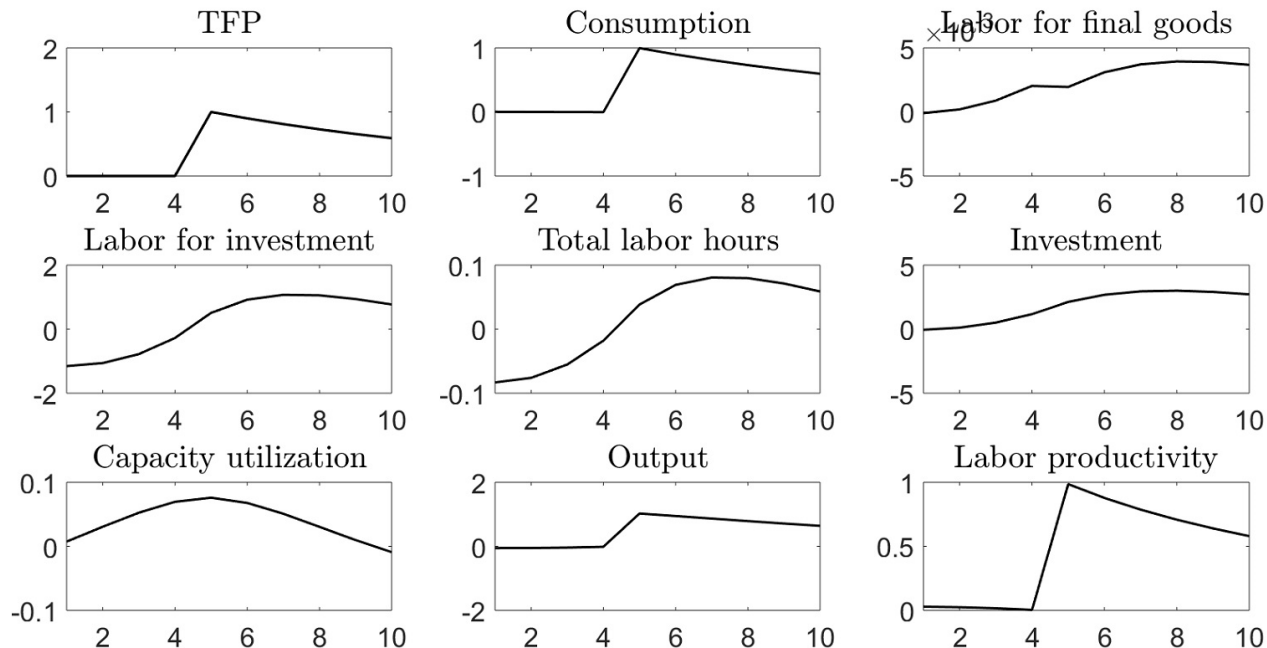




**Figure 1.3:** *Impulse responses from IST news shock in the baseline model.*

Each of these effects described for period one continue as time passes and the productivity shock becomes imminent. By periods 2 and 3, hours, output and investment have increased even more. The effect on consumption decreases over this time frame but remains positive. Thus our one sector model is able to generate qualitatively realistic business cycles driven solely by agents' changing expectations about future productivity.

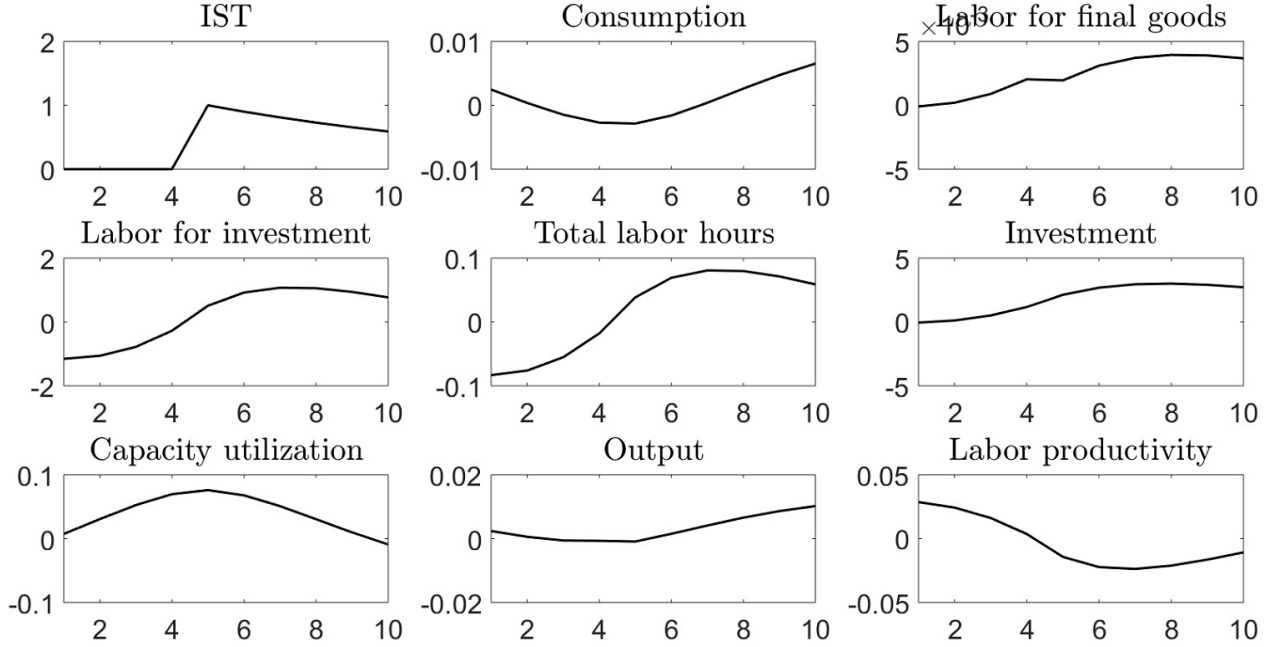
Results are similar to an investment specific technology shock (IST). In our model, this corresponds to an anticipated decrease in  $v_t$ . Figure 1.3 presents the impulse responses due to a favorable IST news shock. The organization is similar to that of Figure 1.2. At period 1, when the economy receives the news that more efficient and cheaper investment goods will be available in period 4, all the variables increase on impact. Consumption falls between period one and period four but starts rising again when the productivity increase arrives. The other impulse responses also follow similar patterns to the case with TFP shocks. There are some differences, but these do not stem principally from our mechanism. For example, labor productivity eventually decreases. The rise is due to an increase in labor employed with



**Figure 1.4:** *Impulse responses from TFP news shocks with high elasticity of substitution.*

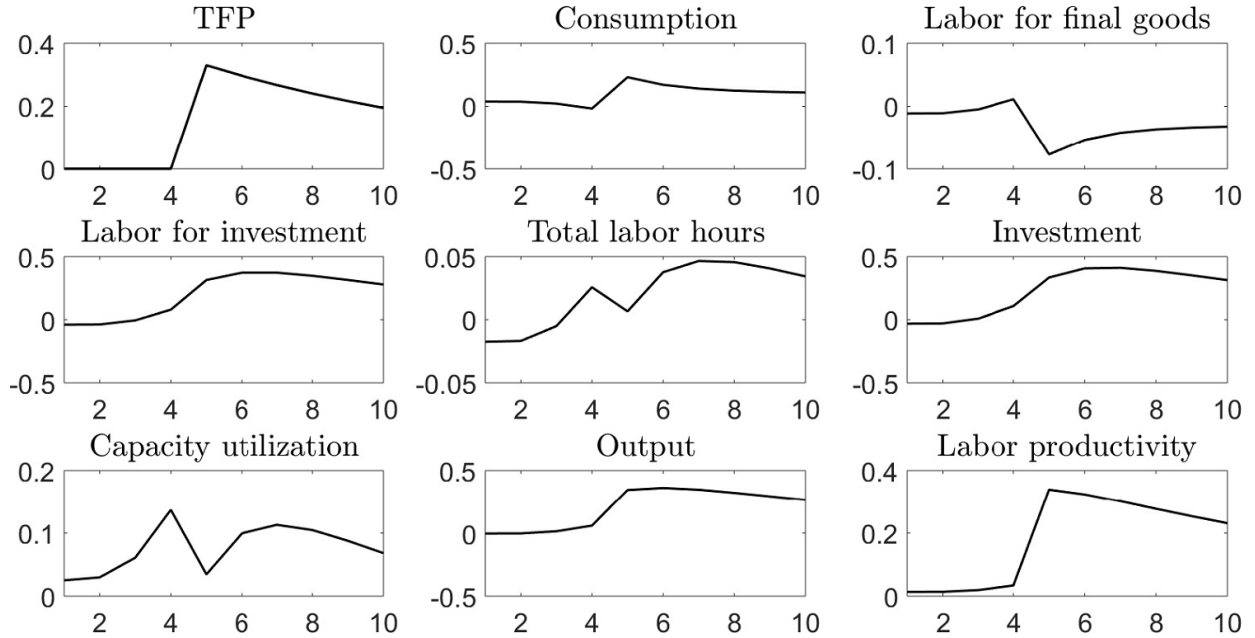
the same productivity parameter rather than stemming more deeply from implementation technology.

Figures 1.4 and 1.5 correspond to the impulse responses for TFP and IST news shocks when investment and labor are substitutes. We set  $\kappa = 0.1$  and otherwise use the baseline parameters. When investment and implementation labor are substitutes, our model does not generate positive comovements among the variables in response to anticipated technological changes. On impact, both types of labor and investment fall due to both TFP and IST news shocks as shown respectively in Figures 4 and 5. In this case, the largest immediate effect is a negative effect on labor for investment. With investment and labor substitutes, the effect of adjustment costs is muted. Rather than increasing the capital stock with investment alone, it is easier to substitute labor for this and avoid the rapidly increasing cost of adjusting. As a result, little occurs in terms of investment, capital utilization, or output until the productivity increase is closer to arriving. Work effort shifts away from the present and toward the future as a means of smoothing.



**Figure 1.5:** *Impulse responses from IST news shocks with high elasticity of substitution.*

Next, we focus on the relative importance of investment and implementation services in producing physical capital to generate news-driven business cycles. In particular, we change the corresponding values of  $\theta_i$  and  $\theta_n$  taking  $\theta_n = 0.001$  and  $\theta_i = 0.999$ . Figures 1.6 and 1.7 present impulse responses of our baseline model, respectively, for both TFP and IST news shocks. In both cases, a low share of implementation labor in capital production does not generate comovement. In particular, consumption increases while hours and investment fall in period 1. The result is consistent with our discussion in the special case. With a weak role for implementation labor, the mechanism is insufficient to overcome the ‘static problem’ expressed in equation (1.15). Intuitively, a sufficient boost of capacity utilization is required for comovements. The co-movement requires at least a moderate role for labor input in capital production to generate realistic comovements among variables.



**Figure 1.6:** *Impulse responses from TFP news shocks in the model with  $\theta_n = 0.001$ .*

In summary, we show that introducing implementation labor in capital production allows an otherwise standard general equilibrium model to produce realistic business cycles in response to an expectation of future technological changes. For our calibration of the model, this holds for either type of productivity improvement. However, a sufficient level of complementarity and a sufficient relative weight on implementation labor is required.

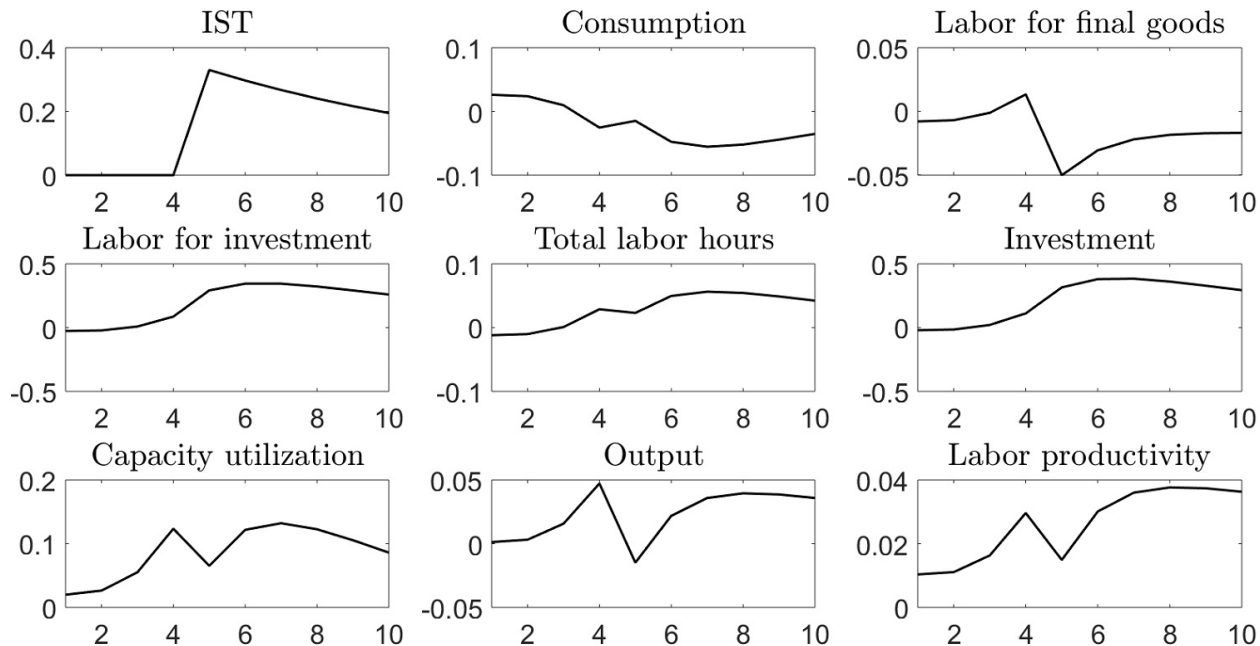


Figure 1.7: *Impulse responses from IST news shocks in the model with  $\theta_n = 0.001$ .*

## 1.5 Conclusion

The literature related to news-driven business cycles explores several modifications of the baseline real business cycle model that overcome the static and the dynamic problem. We show that implementation labor provides an additional useful modification of this sort. We begin by arguing that purchasing investment goods does not directly increase the productive capacity of the firm. Workplace reorganization, new management, training, and screening of new workers are often required with the changes in the firm. We motivate the organization changes from recent empirical findings that technological innovation is highly correlated with changes in firms' workplace organization and with the demand for labor to implement these changes [Bresnahan et al. \(2002\)](#).

We build a representative agent model that captures these changes in the firm through complementarity between investment and labor input. In essence, a news shock influences the demand for labor used to implement new investment capital as well as capital utilization. Labor used for implementation enhances capital utilization but has no impact on current production. Importantly, then, implementation labor has no direct effect on the marginal

product of labor. Implementation labor allows more freedom of movement between the marginal rate of substitution and total labor employed.

We first consider a special case of our model with no adjustment costs which allows us to analytically examine conditions allowing positive comovement between consumption, investment, and total labor hours. Importantly, we show that our new feature gives a boost to the effects of the capacity utilization rate. With this boost, capacity utilization can respond sufficiently to a news shock to allow a general equilibrium increase in the marginal product of labor at the same time that labor employed increases. Absent implementation labor, the co-movement cannot occur.

We then solve the dynamic problem for the general case. In particular, we include adjustment costs and consider the impact of implementation labor in our full model. We show that our model with implementation labor, variable capacity utilization, and investment adjustment cost can generate qualitatively realistic aggregate fluctuations driven by news shocks to total factor productivity and investment-specific technological change. Key parameters for achieving our results are the substitutability between investment and implementation labor, and the relative importance of implementation labor for increasing the capital stock.

This paper highlights the ability of implementation labor to deal with the static problem inherent in new-driven business cycles. The next step will be to place our mechanism in a somewhat fuller model along the lines of [Jaimovich and Rebelo \(2009\)](#) and explore the extent to which we can improve the fit of such models to observed data. We will examine in particular its effectiveness in explaining observed stochastic properties. We anticipate that the model will allow greater freedom in choosing appropriate utility functions and parameters since the static problem will not necessitate the use of non-separable preferences.

# Chapter 2

## Indeterminacy and increasing returns to scale with implementation labor

### 2.1 Introduction

The literature shows that a general equilibrium model with sufficient increasing returns in production results in an indeterminate steady state driven by agent self-fulfilling beliefs (See [Benhabib and Farmer \(1994\)](#); [Farmer and Guo \(1994\)](#), and others).<sup>1</sup> According to [Farmer \(2016\)](#), indeterminacy can be interpreted in two different ways. Using [Farmer \(2016\)](#)'s terminology, the first generation indeterminacy is introduced by [Benhabib and Farmer \(1994\)](#). They define indeterminacy as the possibility of having a continuum of equilibrium paths that converge to a unique steady state. The second generation of indeterminacy is developed up by Farmer and others displays multiple steady-state equilibrium unemployment rates or infinite equilibria close to each other. In both cases, non-fundamental shocks like sunspot or animal-spirit shocks may contribute to the economic fluctuations. In this paper, we focus the indeterminacy introduced by [Benhabib and Farmer \(1994\)](#).

Existing literature offers a few dynamic general equilibrium models that exhibit indeterminate equilibria. One of the keys to having indeterminacy of the first generation is large

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<sup>1</sup>In this paper, the terms “self-fulfilling beliefs” and “sunspots” are used interchangeably. The term refers to any randomness in the economy not related to uncertainties based on economic fundamentals.

increasing returns to scale in the labor input. Higher labor productivity means competitive firms will pay higher wages at equilibrium with more employment. Optimistic households decide to work more as the expectation of higher wages becomes self-fulfilling.<sup>2</sup> In this paper, we extend this idea by incorporating the implementation labor into the production process in a one sector representative agent model. In particular, we take a broad view of the investment-labor demand relationship, assuming that investment of any sort is more efficient when implementation labor accommodates firm-level changes.<sup>3</sup>

We pick the one-sector standard business cycle model augmented with implementation labor following [Blankenau and Farah \(2017\)](#).<sup>4</sup> For simplicity, we consider a special case of a model with no adjustment costs, which allows us to analyze the conditions for an indeterminate steady state. In this type of model, business demand for implementation labor coincides with future technological advances and the corresponding rise in investment demand. We show in our analysis that a representative agent model with implementation labor can exhibit increasing returns to scale. Existing literature shows that indeterminacy can arise in models with increasing returns to scale in the production of the end product. In contrast, we show that the source of indeterminacy in a model with implementation labor is due to increasing returns to scale in the production of capital goods. Then we show that self-fulfilling beliefs of agents lead to business cycle fluctuations, in which multiple equilibria can arise under specific parametrization.

Our model and the existence of indeterminacy of the first generation is comparable to [Benhabib and Farmer \(1994, 1996\)](#), [Wen \(1998\)](#), [Weder \(2005\)](#), and [Guo and Harrison \(2001\)](#) and can be explained as follows. Suppose the representative agent is optimistic about future

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<sup>2</sup> [Benhabib and Farmer \(1994\)](#) model provides dynamic arguments using non-standard slopes of the labor demand and supply curves, with strong increasing returns to labor input. In their framework, the labor demand curve is upward sloping and steeper than the labor supply curve.

<sup>3</sup>The notion that investment brings change has received considerable attention in the literature. Much of this literature focuses on the change brought about by the revolution in information technology. A prominent example is the work of [Bresnahan et al. \(2002\)](#). They find a close relationship between improved information technology and workplace reorganization. They also show that this change itself cannot easily be automated. Labor is required to implement reorganization. See also [Autor et al. \(1998\)](#), [Bartel and Lichtenberg \(1987\)](#), [Bresnahan and Greenstein \(1999\)](#), and [Brynjolfsson and Mendelson \(1993\)](#).

<sup>4</sup> [Blankenau and Farah \(2017\)](#) show that adding implementation labor to a representative agent can improve the model's ability to generate realistic business cycles from news about future events in the economy.



returns on investment and decides to invest more. The rise in demand for investment will increase the investment price. If increasing returns in business capital production are strong enough to yield higher labor productivity, the rate of return on today's investment will rise. As a result, the agent's initial optimistic expectations become self-fulfilling. Hence, indeterminacy is more likely to occur when increasing returns to scale in the production process is high enough. In our model, demand for labor increases through the implementation labor when investment rises. More labor input leads to higher labor productivity due to increasing returns to scale in labor input. As the marginal product of labor increases, so does the marginal product of capital. Moreover, variable capital utilization raises the equilibrium output as labor hours increase. As the demand for implementation labor increases, output increases, consumption and investment rise, and the economy moves to a new saddle path. The self-fulfilling beliefs of agents on the return on investment lead the economy to choose another trajectory to equilibrium path to converge to the unique steady states. Hence, we have indeterminacy in our model.

The literature on both the “sunspot” business cycle fluctuations and the “news-driven” business cycle provides an equilibrium outcome within the rational expectation framework, but a key difference between the two is that the first emphasizes random changes in expectations independent of fluctuations in the economy rather than shocks to fundamentals. These expectations are characterized as self-fulfilling. News-driven fluctuations, on the other hand, do not require self-fulfilling expectations equilibrium in the economy, requiring only an information set under which future shocks are anticipated. The causes of fluctuations are distinct, but we must understand situations where indeterminacy can arise in the model with implementation labor. We characterize conditions that give rise to indeterminacy in this paper, showing that mild increasing returns to scale are not enough to create indeterminate equilibria in our model. Instead, only a sufficiently large implementation labor share in the investment function can cause sunspot equilibria.

Our results are similar to those of [Benhabib and Farmer \(1994\)](#) and [Farmer and Guo \(1994\)](#) in two ways. Previous research shows that externality is a prerequisite for increasing returns to scale (IRTS) in the aggregate production function. Their focus on production

spillover that results in increasing returns to scale at the social level despite constant returns to scale at the individual business level is dissimilar to the externality in our model. However, in both their research and ours, investment is produced using a final good. The final good is a standard Cobb-Douglas combination of capital and labor with constant returns to scale. Investment combines this final good with a second type of labor input, allowing the possibility of constant returns to scale in producing the final good along with increasing returns to scale for the investment function. Another commonality between our model and theirs is that with sufficient increasing returns to scale, the model is indeterminate.<sup>5</sup> Our model, in using implementation labor, does provide a novel channel through which self-fulfilling expectations can propagate business cycle fluctuations.

This chapter is organized as follows. Section 2.2 intuitively describes how increasing returns to scale cause sunspot/indeterminate equilibria. Section 2.3 presents the model. Section 2.4 examines the plausibility of increasing returns to scale in our model. Section 2.5 explains the methodology for deriving the condition for self-fulfilling equilibria and the properties of indeterminacy. Section 2.6 describes the results, and Section 2.7 discusses conclusions.<sup>6</sup>

## 2.2 Indeterminacy in a Real Business Cycle model with increasing returns to scale

How does indeterminate equilibrium in a model connect to the increasing returns to scale in a Real Business Cycle (RBC) model? In this section, we provide an intuitive explanation. Benhabib and Farmer (1994) show that increasing returns to scale requires a sufficiently high value of external effects. In their simple RBC model, increasing returns to scale come from the social production function ( $Y_t$ ). The firm level output ( $y_t$ ) is produced by the

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<sup>5</sup>Also, Eusepi (2009) and Herrendorf and Valentinyi (2006), among others, show that a general equilibrium model with increasing returns to scale in aggregate production has great potential of sunspot driven equilibria.

<sup>6</sup>All the derivations and proofs are in the Appendix B.

Cobb-Douglas production function as

$$y_t = Y_t^{(\eta/(1+\eta))} k_t^\alpha n_t^{(1-\alpha)}.$$

Here,  $Y_t$  is the economy's social output taken as given by each individual firm, with  $\eta$  denoting the degree of productive externalities. In a symmetric equilibrium where  $Y_t = y_t$ , the social production function becomes

$$y_t = A_t k_t^{\alpha(1+\eta)} n_t^{(1-\alpha)(1+\eta)} \quad (2.1)$$

where  $k_t$  and  $n_t$  are the capital stocks and hours worked. The parameter  $\eta$  denotes the size of the externalities: with  $\eta = 0$ , the model becomes a standard RBC model with constant returns to scale in production, while  $\eta > 0$  indicates increasing returns to scale in the social production function. This type of production function is commonly used [Guo et al. \(2015\)](#); [Eusepi \(2009\)](#) to capture IRTS in an RBC framework.

[Benhabib and Farmer \(1994\)](#) explain the economic intuition of indeterminacy in a one sector IRTS model as follows. When agents suddenly start believing that the shadow price of investment increases over its present value and it is profitable to invest more now, they will reallocate consumption to investment. The falling marginal product of capital due to high investment will be offset by an increase in the shadow price of capital. The increasing shadow price will validate agent beliefs that higher rates of investment will yield a higher return. However, in the absence of externalities, a balanced path will over-accumulate capital and violate the transversality conditions. The agent will never consume enough to justify such a sacrifice for investment.

Suppose, now the model has sufficient externalities to permit multiple equilibria. If the agents believe there is an alternative saddle path where the price of investment is higher than the current value, then the higher price diverts consumption to investment. If, however, externalities are strong enough, agents will increase leisure as well. The increase in leisure will cause GDP to decline, and investment eventually falls. As the marginal product of capital decreases with the decline in labor supply, the capital stock begins to fall. Reducing

capital increases demand for labor. An alternative explanation of [Benhabib and Farmer \(1994\)](#) arguments uses the non-standard slope of the labor demand curve. If the degree of increasing returns to scale in the output production is high enough that the labor demand curve slopes upward and more steeply than labor supply, any rise in labor demand shifts the labor demand curve down and increases employment. Higher employment levels and less capital both increase the marginal product of capital. With the new price of investment, the economy will move to a new saddle path and a new equilibrium. In this case, the initial saddle path becomes indeterminate with the change in agent beliefs.

## 2.3 The model

The previous section shows how indeterminacy arises in an RBC model with increasing returns to scale in social production. In this section, we introduce our model with implementation labor, which can exhibit multiple equilibria with IRTS in the investment function. Consider the following environment. A mass of identical agents, infinitely lived, maximize discounted streams of utilities over the lifetime of the mass from consumption ( $c_t$ ) and labor supply ( $n_t$ ).

$$U_t = E_0 \sum_{t=0}^{\infty} \beta^t (\ln(c_t) - \phi(n_{f,t} + n_{i,t})).$$

In the utility function,  $E_0$  is the expectations operator,  $\beta < 1$  is the discount rate, and  $\phi > 0$  gauges the disutility of hours worked.

The representative agent also faces the following resource constraint:

$$y_t = c_t + \frac{i_t}{v_t} \tag{2.2}$$

where  $i_t$  is the investment,  $y_t$  is the final good production, and  $v_t$  represents the current state of investment specific technology for producing investment goods (machinery and equipment). The technology parameter  $v_t$  determines the rate at which a unit of the final good can

be converted to a unit of the investment good.<sup>7</sup> An increase in  $v_t$  reflects investment-specific technological progress, and a decrease represents technological regress.

The final good is produced by the following Cobb-Douglas production function

$$y_t = a_t(u_t k_t)^\alpha n_{f,t}^{1-\alpha} \quad (2.3)$$

where  $a_t$  is the TFP technology;  $u_t$  is the rate of capacity utilization (endogenously determined by the representative firm);  $k_t$  is the capital stock; and  $n_{f,t}$  is labor hours used for production. The parameter  $\alpha \in [0, 1]$  gauges the capital share of output, and  $1-\alpha$  is the labor share ( $n_{f,t}$ ) of output.

The accumulation of capital is expressed as

$$k_{t+1} = \left(1 - \frac{1}{\varphi} u_t^\varphi\right) k_t + i_t n_{i,t}^\theta. \quad (2.4)$$

The above law of motion for capital (2.4) is crucial to business cycle fluctuations. Here, the endogenous rate of capital utilization is positively related to the depreciation of capital. The depreciation rate is equal to  $\frac{1}{\varphi} u_t^\varphi$  with  $\varphi > 1$ . The specification  $\varphi > 1$  means that more intensive capital utilization accelerates the rate of depreciation. In addition, purchasing investment goods can more effectively add to capital stock when combined with separate labor input,  $n_{i,t}$ . [Blankenau and Farah \(2017\)](#) referred to this separate labor input as implementation labor. The term  $i_t n_{i,t}^\theta$  where  $\theta \geq 0$ , represents the generalization of the law of motion consistent with the discussion of implementation labor. Through this expression, increments to the capital stock from investment depend upon how much labor is hired to implement the investment. The parameter  $\theta$  gauges the share of implementation services in producing physical capital. Notice that when  $\theta \geq 0$ , capital production exhibits increasing returns to scale in investment  $i_t$  and implementation labor  $n_{i,t}$ . In our case, increasing returns to scale in capital production differs from increasing returns to scale in the aggregate production function in equation (2.1). Here, we attempt to discover whether and under what conditions

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<sup>7</sup>This specification is similar to what can be found in [Greenwood et al. \(1997\)](#), [Greenwood et al. \(2000\)](#) and [Greenwood et al. \(1988\)](#), and [Ben Zeev and Khan \(2015\)](#).

do increasing returns in capital production cause indeterminate equilibria.

The total labor hours is defined as the combination of hours used for final good production ( $n_{f,t}$ ) and implementing investment ( $n_{i,t}$ ).

$$n_{f,t} + n_{i,t} = n_t. \quad (2.5)$$

Finally, because not all investment expenditure is reflected in  $i_t$ , total output will not be reflected in production of the final good. The value of output in this economy is the sum of final goods production (our numeraire good) and the value of the services provided in putting investment goods into production. In our model, labor will have the same wage,  $w_t$ , whether employed in final goods production or investment implementation. The value of implementation services then will be equal to  $w_t n_{i,t}$ , and total output is given by

$$Y_t = y_t + w n_{i,t}. \quad (2.6)$$

The first order conditions for the representative agent are

$$c_t^{-1} = \lambda_t \quad (2.7)$$

$$\phi = \lambda_t (1 - \alpha) \frac{y_t}{n_{f,t}} \quad (2.8)$$

$$\phi = \lambda_{k,t} \theta i_t n_{i,t}^{\theta-1} \quad (2.9)$$

$$\frac{\lambda_t}{v_t} = \lambda_{k,t} n_{i,t}^\theta \quad (2.10)$$

$$E_t \left( \beta \lambda_{t+1} \frac{y_{t+1}}{k_{t+1}} \alpha \right) + E_t \left( \beta \lambda_{k,t+1} \left( 1 - \frac{1}{\varphi} u_{t+1}^\varphi \right) \right) = \lambda_{k,t} \quad (2.11)$$

$$\lambda_t \alpha y_t = \lambda_{k,t} u_t^\varphi k_t \quad (2.12)$$

where  $\lambda_t$  are the Lagrange multipliers associated with equations, and  $\lambda_{k,t}$  is (2.4). Equations (2.8) and (2.9) show the slope of household indifference curve to the marginal product of

labor, equation (2.10) is the first order condition with respect to  $i$ , and equation (2.11) is the euler equation for intertemporal consumption choices. Finally, equation (2.12) is the marginal gain and marginal loss of a change in the rate of capital utilization  $u_t$ . Equilibrium in our model is represented by equations (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.4), and (2.3).

## 2.4 Increasing returns to scale in the model

This section examines whether our model can generate increasing returns to scale in aggregate production from the investment function in our model. Benhabib and Farmer (1994) model shows strong increasing returns to scale in aggregate production and exhibiting indeterminacy. Here we illustrate that implementation labor in capital production eventually causes IRTS in final good production leading to multiple equilibria.

First, combining (2.8) and (2.9) gives

$$\frac{\lambda_{k,t}}{\lambda_t} = \frac{(1-\alpha)}{\theta} \frac{y_t}{n_{f,t}} \frac{1}{i_t n_{i,t}^{\theta-1}}. \quad (2.13)$$

Next, put the above equations into (2.10) to get

$$\frac{n_{f,t}}{n_{i,t} v_t} = \frac{1-\alpha}{\theta} \frac{y_t}{i_t}. \quad (2.14)$$

Equations (2.12) and (2.13) together yield the capacity utilization as

$$u_t = \left( \frac{\theta\alpha}{1-\alpha} \frac{n_{f,t}}{k_t} \frac{i_t}{n_{i,t}^{1-\theta}} \right)^{1/\varphi}. \quad (2.15)$$

Updating the equation above to the next period provides

$$u_{t+1} = \left( \frac{\theta\alpha}{1-\alpha} \frac{n_{f,t+1}}{k_{t+1}} \frac{i_{t+1}}{n_{i,t+1}^{1-\theta}} \right)^{1/\varphi}. \quad (2.16)$$

Substitute equation (2.15) for  $u_t$  into equation (2.3) to get

$$y = a_t \left( \frac{\theta \alpha}{1 - \alpha} \frac{i_t}{n_{i,t}^{1-\theta}} \right)^{\frac{\alpha}{\varphi}} \frac{\alpha}{n_{f,t}^{\varphi}} + (1-\alpha) \frac{\alpha}{k_t} \frac{\alpha}{\varphi}. \quad (2.17)$$

From (2.14), the value of  $i_t$  is

$$i_t = \frac{1 - \alpha}{\theta} \frac{n_{i,t}}{n_{f,t}} y_t \quad (2.18)$$

and

$$i_{t+1} = \frac{1 - \alpha}{\theta} \frac{n_{i,t+1}}{n_{f,t+1}} y_{t+1}. \quad (2.19)$$

Next, substituting the values of  $i_t$  and  $i_{t+1}$  into (2.17) provides

$$y_t = a_t^{\frac{\alpha}{\varphi - \alpha}} \frac{\alpha}{\alpha^{\varphi - \alpha}} \left( \frac{v_t}{n_{f,t}} \right)^{\frac{\alpha}{\varphi - \alpha}} \frac{\alpha}{k_t^{\varphi - \alpha}} \frac{(\varphi - 1)\alpha}{n_{f,t}^{\varphi - \alpha}} \frac{(1 - \alpha)\varphi + \alpha}{\varphi - \alpha} \frac{\theta \alpha}{n_{i,t}^{\varphi - \alpha}}.$$

In rewriting this expression, we get

$$y_t = a_t^{\frac{\alpha}{\varphi - \alpha}} \frac{\alpha}{\alpha^{\varphi - \alpha}} \frac{\alpha}{v_t^{\varphi - \alpha}} \frac{\alpha}{k_t^{\varphi - \alpha}} \frac{(\varphi - 1)\alpha}{n_{f,t}^{\varphi - \alpha}} \frac{(\varphi - 1)\alpha}{n_{f,t}^{\varphi - \alpha}} \frac{\theta \alpha}{n_{i,t}^{\varphi - \alpha}}. \quad (2.20)$$

The above equation is analogous to equation (2.1). In looking at the equation (2.20) carefully, we can see the shares of capital ( $k_t$ ) is  $\frac{(\varphi - 1)\alpha}{\varphi - \alpha}$ , and the two types of labor ( $n_{f,t}; n_{i,t}$ ) in

output are  $\frac{(\varphi - 1)\alpha}{\varphi - \alpha}$  and  $\frac{\theta \alpha}{\varphi - \alpha}$  respectively. The sum of the shares is

$$\frac{(\varphi - 1)\alpha}{\varphi - \alpha} + \frac{(\varphi - 1)\alpha}{\varphi - \alpha} + \frac{\theta \alpha}{\varphi - \alpha} = 1 + \frac{\theta \alpha}{\varphi - \alpha} > 1.$$

As long as  $\theta > 0$  and  $\varphi > \alpha$ , the aggregate output exhibits increasing returns to scale. In our model, the share of implementation labor parameter  $\theta$  must be greater than zero, and the capacity utilization must be a convex function, so  $\varphi > 1$ . Moreover, the capital share in final good production is  $\alpha \in [0, 1]$ , which is always less than capital utility parameter  $\varphi$ .



In other words, the condition  $\varphi > \alpha$  always holds. However, only IRTS in production does not guarantee indeterminacy. The degree of ITRS must be high enough to produce sunspot equilibria.

## 2.5 Conditions for indeterminacy

The RBC model in [Benhabib and Farmer \(1994, 1999\)](#) exhibits constant returns to scale at firm level production but increasing returns to scale at social aggregate production. Their model shows that business cycles driven by self-fulfillment are possible with a higher degree of IRTS. In the previous section, our model shows IRTS at the investment function eventually leading to IRTS in the production function at the firm level. In our model, we also find a threshold level of IRTS that can generate multiple equilibria. Our result is analogous to [Benhabib and Farmer \(1994, 1999\)](#).

### 2.5.1 Derivation of dynamic system of equations

In this section, the equilibrium conditions are reduced into two equations with one predetermined variable ( $k_t$ ) and one non-predetermined variable ( $n_{f,t}$ ).

Substitute equation (2.13) into equation (2.12)

$$\alpha \frac{1}{u_t^\varphi} \frac{y_t}{k_t} = \frac{\lambda_{k,t}}{\lambda_t}. \quad (2.21)$$

Further, use equations (2.15) and (2.16) in equations (2.11) and (2.4) to get

$$\beta \lambda_{t+1} \frac{y_{t+1}}{k_{t+1}} \alpha + \beta \lambda_{k,t+1} \left( 1 - \frac{1}{\varphi} \frac{\theta \alpha}{(1-\alpha)} \frac{n_{f,t+1}}{k_{t+1}} \frac{i_{t+1}}{n_{i,t+1}^{1-\theta}} \right) = \lambda_{k,t} \quad (2.22)$$

and

$$k_{t+1} = \left( 1 - \frac{1}{\varphi} \frac{\theta \alpha}{(1-\alpha)} \frac{n_{f,t}}{k_t} \frac{i_t}{n_{i,t}^{1-\theta}} \right) k_t + i_t n_{i,t}^\theta. \quad (2.23)$$

Next, by substituting equations (2.18) and (2.19) into equations (2.9), (2.13), (2.22),

(2.23), (2.2) and simplifying yields, we get

$$\phi = \theta \frac{(1-\alpha)}{\theta} v_t \frac{y_t}{n_{f,t}} n_{i,t}^\theta \lambda_{k,t}. \quad (2.24)$$

$$\frac{\lambda_{k,t}}{\lambda_t} = \frac{1}{n_{i,t} v_t}. \quad (2.25)$$

$$\beta \lambda_{t+1} \frac{y_{t+1}}{k_{t+1}} \alpha + \beta \lambda_{k,t+1} - \beta \lambda_{k,t+1} \frac{\alpha n_{f,t+1}}{\varphi} \frac{v_{t+1} y_{t+1}}{k_{t+1} n_{f,t+1}} n_{i,t+1}^\theta = \lambda_{k,t}. \quad (2.26)$$

$$c_t = y_t \left( 1 - \frac{(1-\alpha)}{\theta} \frac{n_{i,t}}{n_{f,t}} \right). \quad (2.27)$$

$$k_{t+1} = k_t - \frac{y_t v_t}{n_{f,t}} n_{i,t}^\theta \left( \frac{\alpha}{\varphi} n_{f,t} - \frac{(1-\alpha) n_{i,t}}{\theta} \right). \quad (2.28)$$

Then, by putting (2.7) into (2.25), we find the Lagrange multiplier

$$\lambda_{k,t} = \frac{1}{c_t} \frac{1}{n_{i,t}^\theta v_t} \quad (2.29)$$

and

$$\lambda_{k,t+1} = \frac{1}{c_{t+1}} \frac{1}{n_{i,t+1}^\theta v_{t+1}}. \quad (2.30)$$

Next, we substitute the values for  $\lambda_{k,t}$  and  $\lambda_{k,t+1}$  from these equations and  $\lambda_t$  from (2.7) into (2.24), (2.25), and (2.26) to get

$$\phi = \frac{\theta(1-\alpha)}{\theta n_{f,t} - (1-\alpha) n_{i,t}} \quad (2.31)$$

and

$$\beta \frac{1}{c_{t+1}} \frac{y_{t+1}}{k_{t+1}} \alpha + \beta \frac{1}{c_{t+1}} \frac{1}{n_{i,t+1}^\theta v_{t+1}} - \beta \frac{1}{c_{t+1}} \frac{1}{n_{i,t+1}^\theta v_{t+1}} \frac{\alpha n_{f,t+1}}{\varphi} \frac{v_{t+1}}{k_{t+1} y_{t+1}} n_{i,t+1}^\theta = \frac{1}{c_t} \frac{1}{n_{i,t}^\theta v_t}. \quad (2.32)$$

Update (2.27) for the next period as

$$c_{t+1} = y_{t+1} \left( 1 - \frac{(1-\alpha)}{\theta} \frac{n_{i,t+1}}{n_{f,t+1}} \right). \quad (2.33)$$

Substitute equations (2.27) and (2.33) into equation (2.32) and simplify it, which further yields

$$\beta \frac{y_{t+1}}{k_{t+1}} \alpha + \beta \frac{1}{n_{i,t+1}^\theta v_{t+1}} - \beta \frac{1}{c_{t+1}} \frac{1}{n_{i,t+1}^\theta v_{t+1}} \frac{\alpha}{\varphi} = \frac{y_{t+1} \left(1 - \frac{(1-\alpha) n_{i,t+1}}{\theta n_{f,t+1}}\right) \frac{1}{n_{i,t}^\theta v_t}}{y_t \left(1 - \frac{(1-\alpha) n_{i,t}}{\theta n_{f,t}}\right)}. \quad (2.34)$$

Next, rearranging the above equation gives

$$\beta \frac{y_{t+1}}{k_{t+1}} \alpha + \beta \frac{1}{n_{i,t+1}^\theta v_{t+1}} - \beta \frac{1}{c_{t+1}} \frac{1}{n_{i,t+1}^\theta v_{t+1}} \frac{\alpha}{\varphi} = \frac{y_{t+1} (\theta n_{f,t+1} - (1-\alpha) n_{i,t+1}) \frac{1}{n_{i,t}^\theta v_t}}{n_{f,t+1} (\theta n_{f,t} - (1-\alpha) n_{i,t})}. \quad (2.35)$$

The relationship between  $n_{i,t}$  and  $n_{f,t}$  from the equation (2.31) becomes

$$n_{i,t} = \frac{\theta n_{f,t}}{(1-\alpha)} - \frac{\theta}{\phi} \quad (2.36)$$

and

$$(\theta n_{f,t} - (1-\alpha) n_{i,t}) = \frac{\theta(1-\alpha)}{\phi}. \quad (2.37)$$

Now, by substituting these two equations into equation (2.28), we get

$$k_{t+1} = k_t - \frac{y_t v_t}{n_{f,t}} \left( \frac{\theta n_{f,t}}{(1-\alpha)} - \frac{\theta}{\phi} \right)^\theta \left( n_{f,t} \left( \frac{\alpha}{\varphi} - 1 \right) + \frac{(1-\alpha)}{\phi} \right). \quad (2.38)$$

Next, substitute equation (2.37) into (2.35) and simplify it as

$$n_{f,t+1} = \frac{\varphi}{\alpha \beta (\varphi - 1)} k_{t+1} \left( \frac{1}{n_{i,t}^\theta v_t} \frac{n_{f,t}}{y_t} - \beta \frac{n_{f,t+1}}{y_{t+1}} \frac{1}{n_{i,t+1}^\theta v_{t+1}} \right). \quad (2.39)$$

Solve for the ratio of  $n_{f,t}$  and  $y_t$  from equation (2.20):

$$\frac{n_{f,t}}{y_t} = \frac{1}{a_t \alpha^{(\alpha/(\varphi-\alpha))}} (v_t^\alpha k_t^{(\varphi-1)\alpha} n_{i,t}^{\theta\alpha} n_{f,t}^{\alpha(1-\varphi)}) \frac{1}{\alpha - \varphi} \quad (2.40)$$

and

$$\frac{n_{f,t+1}}{y_{t+1}} = \frac{1}{a_{t+1}\alpha^{(\alpha/(\varphi-\alpha))}} (v_{t+1}^\alpha k_{t+1}^{(\varphi-1)\alpha} n_{i,t+1}^{\theta\alpha} n_{f,t+1}^{\alpha(1-\varphi)})^{\frac{1}{\alpha-\varphi}}. \quad (2.41)$$

Now, by substituting equations (2.39) and (2.40) into equation (2.38), we get

$$k_{t+1} = k_t - \left(\frac{n_{f,t}}{k_t}\right)^{\frac{\alpha(\varphi-1)}{\alpha-\varphi}} \left(\frac{n_{f,t}}{(1-\alpha)} - \frac{1}{\phi}\right)^{\frac{\theta\varphi}{\varphi-\alpha}} \left(\frac{\frac{\theta\varphi}{\varphi-\alpha} \frac{\theta\varphi}{\alpha\varphi-\alpha}}{\varphi}\right) \left(n_{f,t} \frac{\alpha+\varphi}{\varphi} - \frac{(1-\alpha)}{\phi}\right). \quad (2.42)$$

Substituting equations (2.40) and (2.41) into equation (2.39) gives

$$n_{f,t+1} = \frac{\varphi}{\alpha\beta(\varphi-1)} \frac{1}{\alpha^{\varphi-\alpha}} k_{t+1} \left( (a_t v_t)^{\frac{\varphi}{\alpha-\varphi}} k_t^{\frac{(\varphi-1)\alpha}{\alpha-\varphi}} \frac{\alpha(1-\varphi)}{n_{f,t}^{\alpha-\varphi}} \frac{\theta\varphi}{n_{i,t}^{\alpha-\varphi}} - \beta(a_{t+1} v_{t+1})^{\frac{\varphi}{\alpha-\varphi}} k_{t+1}^{\frac{(\varphi-1)\alpha}{\alpha-\varphi}} \frac{\alpha(1-\varphi)}{n_{f,t+1}^{\alpha-\varphi}} \frac{\theta\varphi}{n_{i,t+1}^{\alpha-\varphi}} \right). \quad (2.43)$$

Substituting the values of  $n_{i,t}$  and  $n_{i,t+1}$  from equation (2.36) into equation (2.43) gives

$$\frac{n_{f,t+1}}{k_{t+1}} = \frac{\frac{\theta\varphi}{\varphi\theta^{\alpha-\varphi}}}{\beta\alpha^{\varphi-\alpha}(\varphi-1)} \left( \left(\frac{1}{a_t v_t}\right)^{\frac{\varphi}{\varphi-\alpha}} \left(\frac{n_{f,t}}{k_t}\right)^{\frac{(\varphi-1)\alpha}{\varphi-\alpha}} \left(\frac{n_{f,t}}{(1-\alpha)} - \frac{1}{\phi}\right)^{\frac{\theta\varphi}{\alpha-\varphi}} - \beta \left(\frac{1}{a_{t+1} v_{t+1}}\right)^{\frac{\varphi}{\varphi-\alpha}} \left(\frac{n_{f,t+1}}{k_{t+1}}\right)^{\frac{(\varphi-1)\alpha}{\varphi-\alpha}} \left(\frac{n_{f,t+1}}{(1-\alpha)} - \frac{1}{\phi}\right)^{\frac{\theta\varphi}{\alpha-\varphi}} \right). \quad (2.44)$$

Finally, equations (2.42) and (2.44) represent the reduced dynamic system of equations in our model.

For mathematical convenience, these two equations can be written as the following expressions:

$$z_{t+1} = \frac{x_1}{\beta(\varphi-1)} (s_t z_t^{x_2} q_t^{-x_3} - \beta s_{t+1} z_{t+1}^{x_2} q_{t+1}^{-x_3}). \quad (2.45)$$

$$q_{t+1} = \frac{z_{t+1}}{z_t} \left( q_t + \frac{1}{\phi} - z_t^{1-x_2} \left( q_t^{x_3} \frac{s_t}{x_1} \frac{\varphi-\alpha}{\varphi} \left( \frac{\alpha}{\phi(\varphi-\alpha)} - q_t \right) \right) \right) - \frac{1}{\phi} \quad (2.46)$$

where  $z_t = \frac{n_{f,t}}{k_t}$ ;  $q_t = \frac{n_{f,t}}{(1-\alpha)} - \frac{1}{\phi}$ ;  $s_t = \frac{1}{\frac{\varphi}{(a_t v_t)\varphi - \alpha}}$ ;  $x_1 = \frac{\varphi}{\frac{\theta\varphi}{\theta\varphi - \alpha} \frac{\varphi}{\alpha\varphi - \alpha}}$ ;  $x_2 = \frac{(\varphi - 1)\alpha}{\varphi - \alpha}$ ;  
and  $x_3 = \frac{\theta\varphi}{\varphi - \alpha}$ .

Finally, We use equations (2.45) and (2.46) to obtain the conditions for indeterminacy in our model.

## 2.5.2 The steady states

We can find the steady-state values in our model using this reduced form system of equations (2.45) and (2.46). First, we eliminate the time subscripts of all the variables because they remain same across time in steady state. Second, all technologies remain same, so  $s_t = 1$ . The previous two equations thus yield

$$z = \frac{x_1}{\beta(\varphi - 1)} (z^{x_2} q^{-x_3} - \beta z^{x_2} q^{-x_3}).$$

$$q = \frac{z}{z} \left( q + \frac{1}{\phi} - z^{1-x_2} \left( q^{x_3} \frac{1}{x_1} \frac{\varphi - \alpha}{\varphi} \left( \frac{\alpha}{\phi(\varphi - \alpha)} - q \right) \right) \right) - \frac{1}{\phi}.$$

Rearrange the above two equations to derive the steady state values of z and q:

$$q = \frac{\alpha}{\phi(\varphi - \alpha)}. \quad (2.47)$$

$$z = \left( \frac{x_1 s z (1 - \beta)}{q^{x_3} \beta (\varphi - 1)} \right) \frac{1}{1 - x_2}. \quad (2.48)$$

The steady state values will be used to derive the conditions for indeterminacy in section 2.5.3.

## 2.5.3 Log linearization of reduced system of equations

The system of non-linear difference equations (2.45) and (2.46) does not have a closed form solution. As such, the solution can be solved using approximation techniques. One particularly easy and very common approximation technique is log-linearization. First, we take

the natural logs of the system of nonlinear difference equations. Then, we linearize the log difference equations around a particular point (a steady state) and simplify until we have a system of linear differential equations where the variables of interests are percentage deviations around the steady-state values. Linearization is crucial because it allows us to use linear differential equations where the variables are percentage deviations from the steady state. After taking the log of the non-linear equations, we do the first order Taylor series expansion around the steady state and further simplify the equations to express percentage deviations from steady state. We use "tilde" to represent percentage deviations of the variables from their respective steady state values. The log-linearization of equations (2.45) and (2.46) are represented as follows:

$$\tilde{z}_{t+1} = \frac{\alpha}{\varphi} \tilde{q}_{t+1} - \frac{\alpha}{\varphi} \left( \frac{(1-\beta)\varphi - \alpha}{\varphi\beta(\varphi-1)} + 1 \right) \tilde{q}_t + \tilde{z}_t. \quad (2.49)$$

$$\tilde{q}_{t+1} = \frac{(\beta\alpha(\varphi-1) + (1-\beta)(\varphi-\alpha))}{\beta\theta\varphi} \tilde{z}_{t+1} + \frac{1}{\beta} \tilde{q}_t - \frac{(\varphi-1)\alpha}{\beta\theta\varphi} \tilde{z}_t. \quad (2.50)$$

#### 2.5.4 General conditions for indeterminacy

[Benhabib and Farmer \(1999\)](#) explain how the general conditions for indeterminacy can be obtained from a log-linearized system of equations. Technically, indeterminacy in a dynamic system comes from the explosive eigenvalues of a coefficient matrix. Following [Benhabib and Farmer \(1999\)](#), suppose a dynamic system of equations can be written in the following form:

$$\tilde{X}_t = \Omega \tilde{X}_{t+1} + \Gamma \tilde{e}_{t+1}. \quad (2.51)$$

Now, rearranging the above equation gives

$$\tilde{X}_{t+1} = \Omega^{-1} \tilde{X}_t - \Omega^{-1} \Gamma \tilde{e}_{t+1} \quad (2.52)$$

where  $X_t$  is a vector that contains endogenous variables found using Taylor series expansion (log-linearizing) around the balanced growth path. Because the condition of indeterminacy

is derived in the non-stochastic version of the model, here,  $e_{t+1}$  is a null vector.

When some of the roots of  $\Omega^{-1}$  lie outside the unit circle, equation (2.51) does not allow us to construct the stochastic process of  $\tilde{X}_t$  that gives rational expectation solutions. This happens because equation (2.51) is not bounded and violates the transversality conditions for the agent in the economy. As a result, equation (2.51) becomes explosive. To find a determinate rational expectation model, we can eliminate the effect of explosive roots of  $\Omega^{-1}$  by putting restrictions on  $X_t$ . Suppose, for instance, vector  $\tilde{X}_t$  is a combination of two disjoint sets  $\tilde{X}_t^1$  and  $\tilde{X}_t^2$ . The dimension of  $\tilde{X}_t^1$  is  $n_1$  that contains the predetermined variables in the model, and  $\tilde{X}_t^2$  is a dimension of  $n_2$  that contains the choice variables in the model. Let  $\lambda$  be the roots of  $\Omega$ , and it consists of two distinguished vectors:  $\lambda_1$  with a dimension of  $m_1$  and  $\lambda_2$ , with a dimension of  $m_2$ .  $\lambda_1$  consists of the roots of outside the unit circle, and  $\lambda_2$  consists of those inside the unit circle. The condition for a unique determinate solution is that there are exactly the same numbers of non- predetermined variables as non-explosive roots of  $\Omega$ ; in other words,  $n_2 = m_2$ . If there are fewer stable roots of  $\Omega$  than non-predetermined variables, we have the possibility of multiple indeterminate equilibria.

In a system of equations with one predetermined and one non-predetermined variable in the log-linearized form ( $\tilde{X}_t$  is a  $2 \times 2$  matrix), let  $\lambda_1$  and  $\lambda_2$  be the eigenvalues of  $\Omega^{-1}$ . In order to relate the indeterminate equilibria with the determinant and trace of a matrix, we consider the derived conditions of indeterminacy from Barsky and Sims (2011), Eusepi (2009), Wen (1998). According to Barsky and Sims (2011), there are two useful facts about eigenvalues and determinants of a matrix. First, the determinant of a matrix is the multiplication of the eigenvalues. Second, the trace of the matrix is the sum of the eigenvalues. Thus, we have

$$\lambda_1 \lambda_2 = \det|\Omega^{-1}|.$$

Now, rearranging the above equation gives

$$\lambda_1 + \lambda_2 = \text{trace}|\Omega^{-1}|.$$

For indeterminate equilibria, both eigenvalues must be outside the unit circle. Because both eigenvalues must be greater than one, then

$$(\lambda_1 - 1) + (\lambda_2 - 1) > 0.$$

$$(\lambda_1 - 1) + (\lambda_2 - 1) > 0.$$

Simplifying further, we have

$$\lambda_1\lambda_2 - \lambda_1 - \lambda_2 + 1 > 0.$$

$$\lambda_1\lambda_2 - (\lambda_1 + \lambda_2) > -1.$$

$$|\text{trace}|\Omega^{-1}| - \det|\Omega^{-1}|| < 1. \tag{2.53}$$

Equation (2.53) is the necessary and sufficient condition for the instability or indeterminate equilibrium of a model.

### 2.5.5 Conditions for indeterminacy in our model

We now turn to the model presented in this paper to show the conditions under which the model leads to indeterminacy. We show under what condition equation (2.53) is satisfied. The log-linearized system of equations from (2.49) and (2.50) can be written into the following matrix form:

$$M_1 E_t \begin{bmatrix} \tilde{z}_{t+1} \\ \tilde{q}_{t+1} \end{bmatrix} = M_2 \begin{bmatrix} \tilde{z}_t \\ \tilde{q}_t \end{bmatrix}$$

and rearranging it further gives

$$E_t \begin{bmatrix} \tilde{z}_{t+1} \\ \tilde{q}_{t+1} \end{bmatrix} = M_1^{-1} M_2 \begin{bmatrix} \tilde{z}_t \\ \tilde{q}_t \end{bmatrix}$$



which can be written as

$$E_t \begin{bmatrix} \tilde{z}_{t+1} \\ \tilde{q}_{t+1} \end{bmatrix} = M \begin{bmatrix} \tilde{z}_t \\ \tilde{q}_t \end{bmatrix}$$

Here we use the necessary and sufficient conditions for indeterminacy as stated in equation (2.53). The conditions for indeterminacy depend on the eigenvalues of matrix M. Indeterminacy occurs when both eigenvalues of M are outside the unit circle.

The necessary and sufficient conditions require

$$|\text{trace}(M) - \det(M)| < 1. \quad (2.54)$$

The matrix M in our model is

$$M = \begin{bmatrix} B \frac{\alpha}{(A\alpha - \theta\varphi)} - \frac{\theta\varphi}{A\alpha - \theta\varphi} & \frac{C\theta\varphi}{A\alpha - \theta\varphi} - \frac{\theta\alpha}{A\alpha\beta - \theta\varphi\beta} \\ B \frac{\varphi}{(A\alpha - \theta\varphi)} - A \frac{\varphi}{A\alpha - \theta\varphi} & \frac{AC\varphi}{A\alpha - \theta\varphi} - \frac{\theta\varphi}{A\alpha\beta - \theta\varphi\beta} \end{bmatrix}$$

where

$$A = \frac{(\beta\alpha(\varphi - 1) + (1 - \beta)(\varphi - \alpha))}{\beta\varphi}.$$

$$B = \frac{(\varphi - 1)\alpha}{\beta\varphi}.$$

$$C = \frac{\alpha}{\varphi} \left( \frac{(1 - \beta)(\varphi - \alpha)}{\varphi\beta(\varphi - 1)} + \frac{1}{\varphi - \alpha} \right).$$

The trace of matrix M is

$$B \frac{\alpha}{A\alpha - \theta\varphi} - \theta \frac{\varphi}{A\alpha\beta - \theta\beta\varphi} - \theta \frac{\varphi}{A\alpha - \theta\varphi} + AC \frac{\varphi}{A\alpha - \theta\varphi} \quad (2.55)$$

and the determinant of M is

$$\frac{BC\beta\varphi - \theta\varphi}{A\alpha\beta - \theta\beta\varphi}. \quad (2.56)$$

The condition for indeterminacy becomes,

$$\theta > \frac{1}{2\varphi}(A\alpha + B\alpha + C\frac{\varphi}{\beta}(1 - \alpha)(1 - \beta)). \quad (2.57)$$

The above equation is the necessary and sufficient condition for indeterminacy in our model.

## 2.6 Intuition of results

Using the preceding analysis, this section intuitively explains sunspot equilibrium in our model. The implementation labor share  $\theta$  must be between  $(0, \frac{1}{2\varphi}(A\alpha + B\alpha + C\frac{\varphi}{\beta}(1 - \alpha)(1 - \beta)))$  for the steady state to be saddle path stable or in unique equilibrium. When the implementation labor share of investment function is high enough ( $\theta > \frac{1}{2\varphi}(A\alpha + B\alpha + C\frac{\varphi}{\beta}(1 - \alpha)(1 - \beta))$ ), the economy exhibits an infinite number of equilibria.

Intuiting the existence of indeterminacy in our model is straightforward. Consider starting with an equilibrium trajectory of investment, output, and consumption. The agents become optimistic about the future development of the economy and inquire whether additional accumulation of capital can be justified as a new equilibrium. However, a new equilibrium path requires a higher return on investment. If the higher anticipated stock of capital increases marginal product of capital by raising the demand for labor, the expected higher rate of return on investment will be self-fulfilling.

In our model, when the agents become optimistic about future returns on investment, firms accumulate capital at a faster rate and the demand for implementation labor rises. The workers also become confident about market returns on labor because of higher demand and decide to work harder. To justify such optimism, a higher interest rate and higher wages are necessary. If increasing returns to scale induced by the investment function in the capital production result in higher capital and labor productivity, then the competitive firm will accumulate more capital and pay a higher wage. As a result, firm and household expectation of higher returns on investment and labor productivity become self-fulfilling. Because infinitely many saddle paths (that depend on the returns of investment and wage)

exist, through which the economy converges to a unique equilibrium, the initial equilibrium path becomes indeterminate. That is how a model with increasing returns to scale and with implementation labor leads to sunspot equilibria or becomes an indeterminate model.

Our analytical results indicate that the magnitude of returns to scale is critical for generating indeterminacy in our model. In fact, only when increasing returns to scale is not enough for self-fulfilling driven business cycles is a sufficiently strong return to scale necessary. In our model, firms internalize increasing returns to scale through increasing demand for implementation labor. Without implementation labor, the model cannot generate increasing returns to scale, and the economy will not be driven by self-fulfilling beliefs.

## 2.7 Conclusion

We have built a model [Blankenau and Farah \(2017\)](#) that captures news driven business cycles through complementarity between investment and labor input. In essence, a news shock influences the demand for labor used to implement new investment capital and capital utilization. Labor used for implementation enhances capital utilization but has no impact on current production. Our model, which includes implementation labor, capacity utilization, and adjustment cost of investment, can reproduce news-driven business cycles.

In a special case, our model exhibits increasing returns to scale in the investment function, which causes an indeterminate set of equilibria. Our non-stochastic version of the model with market imperfection (IRTS) is supported by stationary rational expectation that may be driven by sunspots. The following argument indicates that a representative agent allows self-fulfilling expectation with significant influence on the dynamics of macroeconomic variables. Our paper explores the conditions under which indeterminacy of equilibrium occurs in a representative agent model with implementation labor. Our main finding is that a certain higher level of the share of implementation labor in the production of capital may cause indeterminacy. We interpret this as follows: when an agent expects (purely driven by extrinsic belief) to have higher returns on investment and labor, a significant degree of increasing returns to scale of investment justifies the higher demand for investment, labor, and higher

wages. In this case, different beliefs lead to different equilibrium paths and converge to the steady state.

In the literature of expectation business cycles, local indeterminacy, which may drive the force of fluctuations, is the self-fulfilling belief of the agents. [Guo et al. \(2015\)](#) and [Jaimovich and Rebelo \(2009\)](#) published significant papers on the news-driven business cycle; these papers assume a unique saddle path and steady state equilibrium while mentioning the possibility of having indeterminate equilibrium. In this project, we first show that a representative agent model with implementation labor can exhibit increasing returns to scale. Increasing returns to scale cause the model to be indeterminate of first generation sort under some parametrizations. In this case, business cycle fluctuations can be caused by self-fulfilling beliefs. In other cases, news shocks can yield business cycle behavior based on a unique self-belief about future fundamentals. Because the root cause of business cycles is distinct in the two cases, we must understand situations where indeterminacy can arise. In this paper, we identify parameter values that lead to potential indeterminacy in a special case of the model we will introduce in the first paper.

# Chapter 3

## Declining price of computer capital, wage polarization, and college attendance<sup>1</sup>

### 3.1 Introduction

The U.S. labor market has seen strong job polarization since the late 1980s.<sup>2</sup> According to [David et al. \(2006\)](#), changes in technology, particularly in computerization, allows routine cognitive tasks like office, clerical, administration, to be done more efficiently, while complementing highly skilled labor like problem-solving and interactive tasks. So, we see an increase in the relative demand for jobs requiring highly skilled labor and a decline in the relative demand for routine jobs. [David and Dorn \(2013\)](#) argue that the increase in production of goods requiring highly skilled labor is connected to a rise in demand for services requiring low-skilled labor. Thus, highly skilled workers with higher earnings need other services

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<sup>1</sup>This chapter is a joint work with Dr. Blankenau. I developed the idea and wrote the paper. Dr. Blankenau helped me building the model and coding.

<sup>2</sup>Existing literature defines "job polarization" as the growth of employment of low-skilled workers who perform manual labor accompanied by growing employment of highly skilled workers (who perform specific tasks) and by declining employment of workers in the middle who perform routine tasks ([Goos and Manning, 2003](#)).

like assistance in caring for others, babysitting, food service, security, janitorial service, dry cleaning, hairdressers, etc. As a result, employment and wages have gone up in occupations requiring lower levels of skill. [David et al. \(2006\)](#) provide empirical evidence of such employment changes in the United States since the late 1980s that are strongly U-shaped in skill level. Simultaneously, wage growth is also U-shaped in the skill percentiles [David and Dorn \(2013\)](#).

Existing literature focuses on skilled-biased technological change and explicates the demand side of the labor market to show how job polarization is associated with the demand for skilled labor. A very few studies link the supply side to labor market polarization. In one important work, [Goldin and Katz \(2007\)](#) find that the demand for and supply of skilled labor have expanded disproportionately since the late 1980s. While demand for skilled labor has been increasing fairly consistently, the relative growth in the supply of skilled labor fell short of the growth in demand.

In this paper, we focus on two things. First, we show that wage polarization can be modeled as the result of the lower price of computer capital. In other words, wage growth is strong for relatively low and high-income groups but declines for middle-income workers.<sup>3</sup> Secondly, we look at both the demand and supply of the labor market and analyze how wage polarization affects time allocation in providing different types of skills among workers. Primarily, we consider whether the lower price of computer capital reduces growth in the earnings of workers with relative aptitude in routine labor and whether workers would benefit by attending college to gain skills. When skill accumulation is strongly associated with educational attainment, we hypothesize that the lack of demand and wage stagnation of routine labor discourages some agents from acquiring a college education. We base our research question on evidence of the slowing growth rate in college-educated workers despite high returns on education ([Goldin and Katz, 2007](#)).

We develop a general equilibrium model where an individual chooses when to provide three different types of skills: manual labor skill, routine labor skill, and abstract labor skill.

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<sup>3</sup>In our analysis wage growth means how much the wage increases/decrease due to reduced price of computer capital.

The choice should maximize the market value of their time allocation. The productivity of routine and abstract skills for each agent are drawn from a log-normal distribution; all workers provide each type of skill but in different proportions. Those with relative aptitude in abstract ability tend to supply more abstract skill, and this is true for the other skills as well. In our model, we assume that a business uses all types of skills but at different intensities. For example, manual labor skill is less productive, but a business uses it for low-skilled tasks. Also, this type of skill is a complement to computer capital. Routine labor skill is necessary for tasks like clerical, administrative, and mechanical tasks, which requires middle level skills. We assume computer capital as a substitute for routine labor skill. Consequently, these types of skills face the potential risk of devaluation as technology improves. However, abstract skill is more productive; it complements technology and thus used most rigorously in managerial and professional tasks that require highly skilled labor. In our model, agents are heterogeneous with a productivity portfolio by nature and with increased supplies of skills through education. Attaining education involves financial costs and time. The trade off faced by an individual is whether a portfolio of abilities assigned by nature and the cost of accumulating skills through education is repaid by higher labor productivity in the future.

In our model, technological advancement is captured by reducing the price of computers following [David and Dorn \(2013\)](#).<sup>4</sup> A technological improvement increases the efficiency of computer capital through lower prices. Computer capital takes the place of routine labor, which causes the business to reduce demand for routine labor skill. With lower demand, wages for routine skill labor falls. However, demand for abstract and manual labor rise because they complement computer capital. Thus it increases wages for these skills.

We argue that if an agent's lifetime income is higher with a college education, she will pursue a college degree. That is, if income from attending school is more than the cost associated with education and income tax, the agent will attend college. In our analysis, all human capital production exhibits diminishing marginal product of skill. In the face

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<sup>4</sup>This specification is also found in [Greenwood et al. \(1997\)](#), [Greenwood et al. \(2000\)](#), and [Ben Zeev and Khan \(2015\)](#).

of wage polarization and subsequently reduced wages for routine labor, if an individual's natural proclivities for manual and abstract skills are not high enough to cover fixed college costs, then the individual will not attend college. Otherwise, if the income for manual and abstract skills is high enough to cover the cost (even as routine labor income decreases), the agent will choose to attend college. Overall, we find that a 1% decrease in computer prices will reduce college attendance by 0.22%. In equilibrium, because of endogenous prices of different labor inputs, workers with high abstract skill ability tend to be at the highest level of income distribution, people with relatively higher manual skills are at the lower end of income distribution, and people who do routine intensive work are in the middle of income distribution.

[Goldin and Katz \(2007\)](#) note evidence that supply has expanded less smoothly than demand. However, the existing literature does not provide any structural interpretation of the dynamics of this change. To the best of our knowledge, our paper is the first to model the structural dynamics of technological change and include the supply of skilled labor that explains wage polarization and education attainment. Our analysis explains why the high skill labor supply has slowed over the last few decades. Our study provides some insight into information gathered in surveys of employers by the National at Federation of Independent Business ([Madigan, 2015](#)). In these surveys, many business owners and trade associations complain about the lack of skilled workers for hire, especially as the labor market recovers from the recent recession. Why might all of this happen? Our analysis indicates that labor market polarization changes the pattern of skill supplies in the labor market. Individuals with relative aptitude at routine skills are reluctant to go to college because wages for manual and abstract skill are relatively higher than wages for routine skill. As a result, the relative supply of routine skill workers has declined. Our model replicates the situation where technological change reduces college attendance and hence the supply of skilled labor.

In this chapter, we present the model in [Section 3.2](#). In [Section 3.3](#), we define the equilibrium in the model. [Section 3.4](#) provides our numerical analysis, and [Section 3.5](#) describes the results. In [Section 3.6](#), we provide our conclusions. The solutions in the model are in the [Appendix C](#).



## 3.2 The model

We use a general equilibrium model where agents live three periods and are heterogeneous in productivity assigned by nature. We assume all agents are high school graduates. In the first period, agents allocate time to supply different types of skill and end the period as young workers. In the second period, they work as adults, and, in the third period, they retire and then die. Those who go to college increase their productivity in all skills by a fixed efficiency unit. The opportunity cost of going to college is the time allocated for increasing the efficiency of human capital. Also, the financial cost is the tuition cost. Individuals decide on a college education and accumulate time to gain human capital. The first choice is whether or not to go to college. Second, given the option of educational attainment, an agent can choose how much of each type of skill to supply. The decision is based on comparing lifetime income with and without college.

The firms produces aggregate output that is determined by two types of capital stock: general capital ( $K_g$ ) and computer capital ( $K_r$ ) as well as the stock of three kinds of labor input: manual labor ( $M$ ), routine labor ( $R$ ), and abstract labor ( $A$ ). All workers supply manual labor, routine labor, and abstract labor. However, agents with the relative aptitude to manual productivity ( $z_M$ ) supply more manual skill ( $h_M$ ), agents with the corresponding proficiency to routine productivity ( $z_R$ ) provide more routine labor ( $h_R$ ), and agents with relative talent for abstract productivity ( $z_A$ ) supply more abstract labor ( $h_A$ ). Manual labor involves low skilled tasks like driving and cleaning, which require minimal creative and cognitive skill. Routine labor performs tasks that need clerical and the usual analytical skills and require medium skill. These types of employment involve repetitive work like calculating, coordinating, data analysis, and keeping records. Abstract labor uses creativity, generalized problem solving, and complex communication for tasks that demand cognitive and interpersonal skill. General capital is defined as infrastructure, machinery, and other physical capital. We assume that computer capital (information technology) only fulfills routine tasks.

We write aggregate output as

$$Y = F(M, R, A; K_r, K_g). \quad (3.1)$$

Because we consider only steady state values in our model, no time subscript is included in the variables. We assume that  $K_r$ ,  $M$ ,  $R$ ,  $A$ , and  $K_g$  are endogenously determined by technological change. Technological innovation is measured using the falling price of computer capital and is the exogenous driving force in our model.

Computers take over jobs requiring routine skills, reducing demand for workers who supply relatively high routine labor and reducing wages for routine labor. On the other hand, computer capital complements abstract skills and manual skills. The productivity (wages) of abstract and manual labor rise because of high demand for them.

The government provides subsidies for those who enroll in college and balances its budget through income tax.

### 3.2.1 Firm decisions

The representative firm produces output using the following technology:

$$Y = K_g^{\theta_k} M^{\theta_m} A^{\theta_a} [\Phi R^\mu + (1 - \Phi) K_r^\mu]^{\frac{1 - \theta_k - \theta_m - \theta_a}{\mu}}.$$

where,  $\theta_k, \theta_m, \theta_a, \mu \in (0, 1)$ , and

$$\theta_k + \theta_m + \theta_a \leq 1.$$

$\Phi$  represents the weight on work activities of routine labor in the production process.

All three types of labor, manual, routine, and abstract, are imperfectly substituted. We assume computer capital ( $K_r$ ) is relatively complementary to general capital ( $K_g$ ), manual labor ( $M$ ), and abstract labor ( $A$ ) and a relative substitute for routine labor. The elasticity of substitution between routine labor ( $R$ ) and computer capital ( $K_r$ ) is constant, represented

by  $\frac{1}{1-\mu}$ . In our model we consider the value of  $\mu$  to be between 0 and 1 so routine labor ( $R$ ) and computer capital ( $K_r$ ) can substitute for each other. The elasticity of substitution between total routine task input and manual labor ( $M$ ), abstract labor ( $A$ ), or general capital ( $K_g$ ) is one.

Computer capital is supplied at market price  $P_t$ . Here, the time subscript of price indicates higher versus lower price of capital. Because our model captures no dynamics, we study the steady-state situation at different prices. Following [David and Dorn \(2013\)](#) The exogenous driving force in our model is the decline in the price of computer capital due to technological changes. The falling prices of computers replaces routine labor and reduces the wages of this skill group. On the other hand, computer capital increases utilization of abstract labor and general capital. The compensation for abstract ability also then rises, and this upsurge increases the demand for manual skill in the production process. No depreciation of capital is used in our model, so the net stock of capital is equal to total savings in the economy.

The firm maximizes profits and pays according to the marginal product of all inputs. The manual labor wage, routine labor wage, and abstract labor wage are represented as  $\omega_M$ ,  $\omega_R$ ,  $\omega_A$ , and the rental rates of general capital and computer capital are  $r_{k_g}$ ,  $r_{k_r}$ .

The firm's objective function is given below:

$$\max_{\{M,R,A,K_r,K_g\}} (Y - (\omega_M M + \omega_R R + \omega_A A + r_{k_r} K_r + r_{k_g} K_g))$$

subject to,

$$Y = K_g^{\theta_k} M^{\theta_m} A^{\theta_a} [\Phi R^\mu + (1 - \Phi) K_r^\mu] \frac{1 - \theta_k - \theta_m - \theta_a}{\mu}. \quad (3.2)$$

The profit-maximizing wages and interest rates are

$$\omega_M = \frac{\theta_m Y}{M} \quad (3.3)$$

$$\omega_A = \frac{\theta_a Y}{A} \quad (3.4)$$

$$\omega_R = \frac{\Phi Y(1 - \theta_k - \theta_m - \theta_a)}{R^{1-\mu}[\Phi R^\mu + (1 - \Phi)K_r^\mu]} \quad (3.5)$$

$$r_{k_g} = \frac{\theta_k Y}{K_g} \quad (3.6)$$

$$r_{k_r} = \frac{(1 - \Phi)(1 - \theta_k - \theta_m - \theta_a)Y}{(K_r^{1-\mu}[\Phi R^\mu + (1 - \Phi)K_r^\mu])}. \quad (3.7)$$

The arbitrage condition satisfies

$$r_{k_g} = \frac{r_{K_r}}{P_t}. \quad (3.8)$$

### 3.2.2 Agent problem

There are  $N$  heterogeneous agents in the economy. Agents are heterogeneous by nature on productivity endowments  $(z_{M,j}, z_{R,j}, z_{A,j})$  in all three types of skills (manual, routine, and abstract). We assume each agent has a draw for routine  $(z_{R,j})$  and abstract  $(z_{A,j})$  productivity assigned by nature from a log-normal distribution and inherits manual skill as  $z_{M,j} = 1$ . Agents live for three periods as a young, adult, and old agent and maximizes income over a lifetime. More specifically, agents, indexed by  $j$ , are born in the first period, acquire human capital, and end the period as a young worker. In the second period, agents work and retire at the end of the period. In the third period, agents live on their savings and exit the economy. Different inherent abilities, human capital accumulation costs, and wage structure in the economy motivate agents to allocate time for acquiring human capital portfolios <sup>5</sup>. Here portfolio means different compositions of three types of skill.

We assume that all agents are high school graduates and choose whether to attend college. The choice of supply depends on individual ability and the employment situation in the economy. However, the human capital portfolio can be improved by acquiring a college degree. A college degree raises human capital of each type by an efficiency unit  $(\eta)$ .

In the first period, an agent decides whether to attend college ( $H = 1$ ) or not ( $H = 0$ ), current consumption  $(C_1)$ , savings  $(S_1)$  and time allocation  $(t_M, t_R, t_c)$ . If they go to college, they spend  $t_c < 1$  unit of time for education and  $1 - t_c$  unit of time for work in the first period

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<sup>5</sup>The term is used in [Silos and Smith \(2015\)](#).

of life. In the second period of life, they only work and devote 1 unit of time to working. On the other hand, if they do not attend college, they have 1 unit of time to work in both of the periods of life. Also, They begin life with a portfolio of productivities  $z_{M,j}, z_{R,j}, z_{A,j}$ , wage structure of different skill types  $\omega_M, \omega_R, \omega_A$ , and a unit of time. For this presentation, we drop the individual specification  $j$ .

In the second period, agents choose consumption ( $C_2$ ), and savings ( $S_2$ ). In the third period, the agent only consumes ( $C_3$ ) and expires.

The agents' problem can be broken into three separate sub-problems. First is the income allocation, then the next two choices maximize income. First, they choose the time allocation to supply different types of skill and second, they decide whether to attend college or not to maximize lifetime income.

### 3.2.3 Utility maximization problem

We first consider the utility maximization problem of the agents.

The lifetime utility function is

$$U_t = \max_{\{C_1, C_2, C_3, h_M, h_R, h_A, H\}} \frac{C_1^\sigma}{\sigma} + \beta \frac{C_2^\sigma}{\sigma} + \beta^2 \frac{C_3^\sigma}{\sigma}$$

subject to

$$C_1 + S_1 = I_1.$$

$$C_2 + S_2 = I_2 + rS_1.$$

$$C_3 = rS_2.$$

where  $\beta \in (0, 1)$  is the discount factor, and  $\sigma > 0$  is the coefficient of risk aversion. Here,  $H$  is an indicator variable.  $H$  being 1 indicates somebody chooses college education otherwise  $H = 0$  indicates she will not.

### 3.2.4 Income allocation problem

Next we consider the agents income allocation problem. The solutions of the optimization problem are

$$C_1 = \frac{I_1 + \frac{I_2}{r}}{1 + \beta \frac{1}{1 - \sigma} \frac{1}{r} + \beta \frac{2}{1 - \sigma} \frac{\sigma}{r}}.$$

$$C_2 = C_1(r) \frac{1}{1 - \sigma}.$$

$$C_3 = C_1(r) \frac{2}{1 - \sigma}.$$

### 3.2.5 Time allocation problem

Now, agents decide how to allocate time providing skills to maximize lifetime income. We solve this separately for those who go to college and those who do not.

The agent's income in each period depends on the choice to attend college. The income in the first period  $I_1$  will be determined by this college decision, the acquired units of different human capital stocks, and the remuneration per unit.

If an agent does not attend college ( $H = 0$ ), his human capital accumulation follows

$$h_M = z_M t_M^\alpha \alpha \in (0, 1)$$

$$h_R = z_R t_R^\alpha$$

$$h_A = z_A t_A^\alpha$$

where  $t_M, t_R, t_A$  are time spent in acquiring manual, routine, abstract human capital and

$$1 = t_M + t_R + t_A.$$

Remuneration per unit of human capital will be  $\omega_M, \omega_R$ , and  $\omega_A$ , which are determined

by the equilibrium dynamics of the economy. The total income in the first period of life will be

$$I_1 = (\omega_M h_M + \omega_R h_R + \omega_A h_A)(1 - T). \quad (3.9)$$

In the second period of life, years of experience in work ( $\gamma > 1$ ) affects income. The present value of a high school graduate's income will be

$$I_2 = (\omega_M h_M + \omega_R h_R + \omega_A h_A) \frac{\gamma}{r_k} (1 - T). \quad (3.10)$$

If the agent chooses to go to college ( $H = 1$ ), he allocates  $t_c$  fraction of time to acquire a college degree. Also, each type of human capital accumulation will increase by an efficiency unit ( $\eta$ ) for attending college. The human capital will accumulate according to

$$h_M = z_M t_M^\alpha \eta$$

$$h_R = z_R t_R^\alpha \eta$$

$$h_A = z_A t_A^\alpha \eta$$

where  $\eta > 0$ . Thus

$$1 - t_c = t_M + t_R + t_A \eta.$$

Other than human capital stock and prevailing wage structure, agent income in the first period also depends on government subsidies for educational expenditures ( $G_s$ ), exogenous tuition cost ( $X$ ) to acquire college degree, and income tax ( $T$ ). Hence, the total income of a college-going agent in the first period ( $I_1$ ) will be

$$I_1 = (\omega_M h_M + \omega_R h_R + \omega_A h_A)(1 - t_c) \eta (1 - T) - (1 - G_s) X. \quad (3.11)$$

The present value of the second-period income of a college educated agent will be

$$I_2 = (\omega_M h_M + \omega_R h_R + \omega_A h_A) \frac{\gamma \eta}{r_k} (1 - T). \quad (3.12)$$

To find how much time to devote for supplying different types of kills, Agents solve

$$\max_{\{t_M, t_R, t_A\}} I_1 + \frac{I_2}{r} \quad (3.13)$$

subject to

$$h_M = z_M t_M^\alpha \quad (3.14)$$

$$h_R = z_R t_R^\alpha \quad (3.15)$$

$$h_A = z_A t_A^\alpha \quad (3.16)$$

$$1 = t_M + t_R + t_A$$

if  $H = 0$

Again the constraints are

$$h_M = z_M t_M^\alpha \eta \quad (3.17)$$

$$h_R = z_R t_R^\alpha \eta \quad (3.18)$$

$$h_A = z_A t_A^\alpha \eta \quad (3.19)$$

$$1 - t_c = t_M + t_R + t_A$$

if  $H = 1$ . Here the agent spends  $t_c$  units of time attending college.

By substituting the constraints into the objective function, we can rewrite the objective function as

$$\max_{\{t_M, t_R, t_A\}} (z_M t_M^\alpha + z_R t_R^\alpha + z_A t_A^\alpha)$$



If  $H=0$ , then the time constraint both in the first and second period of life is

$$1 = t_M + t_R + t_A.$$

If  $H=1$ , the time constraint for the first period of life is

$$1 - t_c = t_M + t_R + t_A.$$

Moreover, in the second period of life, the time constraint is

$$1 = t_M + t_R + t_A.$$

The optimization problem solves for non-college-educated workers for both periods as

$$t_M = \frac{(\omega_M z_M) \frac{1}{1-\alpha}}{\frac{1}{(\omega_M z_M) \frac{1}{1-\alpha}} + \frac{1}{(\omega_R z_R) \frac{1}{1-\alpha}} + \frac{1}{(\omega_A z_A) \frac{1}{1-\alpha}}}. \quad (3.20)$$

$$t_R = \frac{(\omega_R z_R) \frac{1}{1-\alpha}}{\frac{1}{(\omega_M z_M) \frac{1}{1-\alpha}} + \frac{1}{(\omega_R z_R) \frac{1}{1-\alpha}} + \frac{1}{(\omega_A z_A) \frac{1}{1-\alpha}}}. \quad (3.21)$$

$$t_A = \frac{(\omega_A z_A) \frac{1}{1-\alpha}}{\frac{1}{(\omega_M z_M) \frac{1}{1-\alpha}} + \frac{1}{(\omega_R z_R) \frac{1}{1-\alpha}} + \frac{1}{(\omega_A z_A) \frac{1}{1-\alpha}}}. \quad (3.22)$$

For college-educated workers in the first period, this solves as

$$t_M = \frac{(1 - t_c)(\omega_M z_M) \frac{1}{1-\alpha}}{\frac{1}{(\omega_M z_M) \frac{1}{1-\alpha}} + \frac{1}{(\omega_R z_R) \frac{1}{1-\alpha}} + \frac{1}{(\omega_A z_A) \frac{1}{1-\alpha}}}.$$

$$t_R = \frac{(1-t_c)(\omega_R z_R) \frac{1}{1-\alpha}}{(\omega_M z_M) \frac{1}{1-\alpha} + (\omega_R z_R) \frac{1}{1-\alpha} + (\omega_A z_A) \frac{1}{1-\alpha}}$$

$$t_A = \frac{(1-t_c)(\omega_A z_A) \frac{1}{1-\alpha}}{(\omega_M z_M) \frac{1}{1-\alpha} + (\omega_R z_R) \frac{1}{1-\alpha} + (\omega_A z_A) \frac{1}{1-\alpha}}$$

The second period solutions for college-educated workers are same as in equations (3.20), (3.21) and (3.20).

### 3.2.6 College decision problem

Finally agents decide whether go to college or not as follows.

The lifetime income for those attending college  $I_c$  is

$$I_c = (\omega_M h_M + \omega_R h_R + \omega_A h_A)(1-T)\eta \left( (1-t_c) + \frac{\gamma}{r} \right) - (1-G_s)X. \quad (3.23)$$

The lifetime income for those not attending college  $I_{nc}$  is

$$I_{nc} = (\omega_M h_M + \omega_R h_R + \omega_A h_A)(1-T) \left( 1 + \frac{\gamma}{r} \right). \quad (3.24)$$

An agent will choose to go to college if lifetime income after attending college is higher than lifetime income without attending college. We rewrite the college decision as

$$(\omega_M h_M + \omega_R h_R + \omega_A h_A)\eta > \frac{(1-G_s)X}{(1-T) \left( \left( (1-t_c) + \frac{\gamma}{r_k} \right) - \frac{(1+\frac{\gamma}{r})}{\eta} \right)}. \quad (3.25)$$

The left hand side (L.H.S) of the above equation is agents' income after attending college which depends on the endogenous wage of each skill and the human capital portfolio of every agent  $(h_M, h_R, h_M)$ . The portfolio is determined by an agent's time allocation and natural

endowment she inherits for each skill. Note that, any change in the price of computer capital affects the wage of each skill consequently, makes a difference to the L.H.S. of equation (3.25). Similarly, the right hand side (R.H.S) is the cost of attending college which is affected by government policies on subsidy and income tax. A subsidy reduces the cost of attending college where an income tax increases it. As a result, any government intervention change the threshold cost of attending college hence the R.H.S. of equation (3.25) is affected.

### 3.2.7 Government expenditures

The government provides subsidies to every individual going to college, financing the expenditure through the income tax collected from workers. The budget constraint satisfies

$$G_s X N_c = T Y. \quad (3.26)$$

where  $G_s$  is the share of tuition cost subsidized by the government,  $X$  is the tuition cost,  $N_c$  is the number of people attending college,  $T$  is the tax rate, and  $Y$  is total income in the economy.

## 3.3 Definition of equilibrium

A stationary competitive equilibrium in this environment is a set of quantities  $\{c_{1,j}, c_{2,j}, c_{3,j}, s_{1,j}, s_{2,j}\}$ , college choices, time allocations in each period of life, human capital accumulation  $\{h_{M,j}, h_{R,j}, h_{A,j}\}$  chosen by agents; the set of individual natural productivity  $\{z_{M,j}, z_{R,j}, z_{A,j}\}$  all taken as given by a set of outputs and inputs chosen by the representative firm in each period  $\{Y_t, K_{r,t}, K_{g,t}\}$ ; a set of prices  $\{\omega_M, \omega_R, \omega_A, r_r, r_g\}$ ; and government policy  $\{G_s, T\}$ . Each is such that it satisfies the following conditions:

1. the individual decides on college attendance based on maximizing expected lifetime income and also chooses  $\{c_t, c_{t+1}, c_{t+2}, s_{t+1}, s_{t+2}, t_M, t_R, t_A\}$ , taking prices and a set of natural productivity  $\{z_{M,j}, z_{R,j}, z_{A,j}\}$  as given;

2. the firm chooses  $Y_t, K_r, K_g, M, R$  and  $A$  in period  $t$  to maximize profits, taking prices as given;
3. the labor market clears so that  $M = \sum_{j=1}^N h_{m,j}$ ;  $R = \sum_{j=1}^N h_{R,j}$  and  $A = \sum_{j=1}^N h_{A,j}$ ;
4. the capital market clears so that  $K_g + K_r = \sum_{j=1}^N (s_{1,j} + s_{2,j})$ ;
5. the goods market clears; and
6. the government balances its budget constraint.

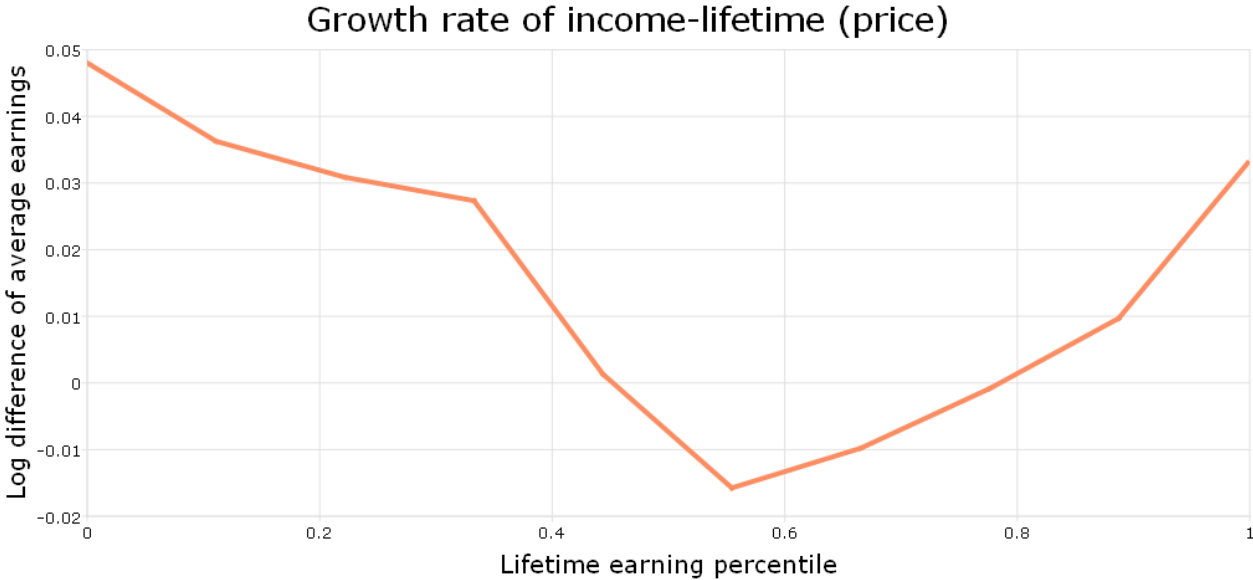
## 3.4 Results

In this section, we discuss the results in our model. First, we show that our model replicates wage polarization as find in [Acemoglu and Autor \(2011\)](#). Next, we discuss how wage polarization affects the supply of human capital and the agent decision about attending college in our model. Finally, government tax policy on college attendance is analyzed. We use the following parameter values in our model simulation. The share of manual skill ( $\theta_m$ ) is 0.1, abstract skill ( $\theta_a$ ) is 0.4, and general capital ( $\theta_k$ ) is 0.2. The elasticity of substitution parameter  $\mu$  is 0.5, the share of routine skill in production  $\phi$  is 0.2, risk aversion is 0.5, the discount rate  $\beta$  is 0.7, the efficiency parameter  $\eta$  is 1.2, the experience parameter  $\gamma$  is 1, the tuition cost  $X$  is 0.2, the time needed to attend college  $t_c$  is 0.15, tax is 0.05, and the human capital accumulation parameter  $\alpha$  is 0.5.

### 3.4.1 Wage polarization

Figure [3.1](#) illustrates the wage polarization of our model. First, we choose a price of computer capital and solve the model. In the process, we find the lifetime income of all the agents and sort lifetime incomes into percentiles. Next, we take the average income of all individuals in each decile and draw the income distribution. In this procedure we find the distribution of lifetime income when the price of computer capital is relatively high. Then holding all other

parameters the same, we repeat the procedure to find the income distribution when computer capital costs less. Further, we take the logarithm of each earning distribution and compare the change in income for each decile attributable to changes in computer capital price. We define this income change as log difference of average earning. The X-axis denotes income percentile ranked between 0 and 1 using the percentile of agent lifetime income, and the Y-axis is the log difference of average earning. The graph illustrates how the distribution of lifetime income changes due to changes in computer price. Our model replicates the pattern of wage polarization that [Acemoglu et al. \(2012\)](#) find empirically.<sup>6</sup>



**Figure 3.1:** *The growth rate of lifetime income for change in computer capital price.*

In looking at Figure 3.1 carefully, we can see the earning growth pattern is non-monotone in income percentile. Consistent with the conventional view of technological change and wage polarization, our model reflects that wage growth is U-shaped. The wage gain is relatively high at the lower deciles then gradually slows, so the wage growth is negative for a segment of the middle-income group. The wage growth picks up again in the upper tails. Intuitively, when technological change reduces computer prices, the representative firm

<sup>6</sup> [Acemoglu et al. \(2012\)](#) uses empirical data to define wage polarization. They consider income growth as changes in hourly wages relative to changes in median wages by earning percentile. In our model, we calculate the lifetime income of all the agents and define the income percentile by low, middle, and high income.

increases the stock of computer capital and reduces demand for routine skill. As computers replace labor for routine tasks, productivity of routine skill falls due to the increase in the substitute input. On the other hand, the technological change makes manual and abstract skill-intensive production more necessary. In turn, the demand for and productivity of both types of skills goes up.

In equilibrium, an agent who is naturally more productive in routine tasks tends to supply relatively more routine labor in the labor market, an agent with relative aptitude in abstract skill tends to provide more abstract labor, and a comparable worker using manual skills supplies more manual labor. The relative level of wages  $\frac{\omega_M}{\omega_R} < 1$  and  $\frac{\omega_A}{\omega_R} > 1$  shows falling wages for labor using routine skills mostly affects workers with relatively high productivity in routine tasks or the middle-income group. On the other hand, the rising productivity of labor using manual and abstract abilities means low income and high income groups are better off. In our simulated model, the wage growth rate is higher for the low-income group than the high-income group. The returns on skills are endogenized in our model through the dynamics of general equilibrium. Hence, the parameters  $\theta_m$  and  $\theta_a$  are vital in determining wage growth in our model.

### 3.4.2 Education choice

Our second key finding focuses on the decision to attend college. In particular, we show how non-monotonic earning growth or wage polarization affects the decision to attend college. We compute the expected earnings of agents who attend and agents who do not. The trade off between “attending” and “not attending” college depends on the expected return on education, financial costs, and individual productivity endowments.

An agent will choose to go to college if lifetime income after attending college is higher than lifetime income without attending college. That is if the L.H.S. of equation (3.25) is greater than the R.H.S. the agent will attend college, otherwise not.

In our model, all agents inherit and supply all types of skills, but their relative aptitudes varies. Also, an agent with relative abundance of routine skills tends to supply more routine

skill in the labor market; the same applies to manual and abstract skills.

Note that agent lifetime expected earnings and agent decisions about going to college are connected to the relative price of computer capital. A lower price for technology affects each individual agent differently because the effect depends on the individual portfolio. The price change on the lifetime earnings are mainly affected by two things. First, changes in the price of computer capital reduce routine labor wages ( $\omega_R$ ). Second, those changes increase the salaries for manual  $\omega_R$  labor and increases the wages of labor using manual ( $\omega_M$ ) and abstract ( $\omega_A$ ) skills. For some people, the downward impact dominates, so they choose not to attend college. These people are close to the cut off threshold for the benefits of going to college. On the other hand, for others, the upward impact dominates, so the agent will attend college.

To illustrate, when the wages for routine skill drops due to lower computer capital price, some agents whose lifetime incomes are at the break-even margin and thus are close to the cut off threshold for the benefits of going to college; these agents likely will not choose to attend college if they are relatively efficient at providing routine labor. So people who have a relative abundance of routine skills and low manual and abstract skills do not find it advantageous to go to the college, making it reasonable to choose not to attend college because their return from routine skill trends down, especially if, in addition, the income from manual and routine labor is not high enough to cover the tax, tuition cost, and time needed to pursue a college education. On the other hand, some of the agents at the margin may have low  $R$  but high  $A$  and  $M$ . These people are more likely to pursue college as their income after going college exceeds the cost.

The dynamics can be better understood by looking at equation (3.25), where the LHS reflects the benefit of attending college and the RHS is the cost. If RHS is greater than LHS, agents opt out of attending college. However, if agent income from manual and abstract skill is high enough to cover the fixed costs of going to college, agents with relative aptitude in routine skill may still attend college. In that case, their income after attending college is higher than their income with no college degree even though the return on routine skill falls. The net benefit of attending college depends on which factor works more strongly. Overall,

we find that a 1% decrease in the price of computer capital reduces college attendance by 0.22% in our model.

Our analysis reveals that the recent wage structure in the U.S. economy inherently affects human capital accumulation. Technological change affects wage growth of workers with different relative incomes, accentuating income inequality<sup>7</sup>. Thus, in turn, affects the decision to attend college. Our analysis is similar to [Goldin and Katz \(2007\)](#), who state "... changes in the rate of expansion in the supply of skills explain more of the variance in inequality over time than demand does". To the best of our knowledge, our paper is the first to show how wage polarization and technological advancement affect the supply side of the labor market and individual decisions about attending college.

### 3.5 Government subsidy policy

This section illustrates the policy implications of our model. In particular, we want to examine how subsidies change lifetime income across income distribution. In addition, we analyze the effects of government subsidies on college education. Against the backdrop of wage polarization and the subsequent decline in rates of college attendance, we want to investigate whether a subsidy could encourage college enrollment and how the distribution of income is affected. Agents college decision after subsidy depends on the dynamics between reduced education cost, the increase in income tax and the general equilibrium adjustment of wages. Our analysis indicates that a 0.5% increase in income tax rises college attendance by 8%. In particular, we show that 60% of people attend college when no tax or subsidy is available. The attendance rate increases to 68% in the presence of 0.5% income tax.

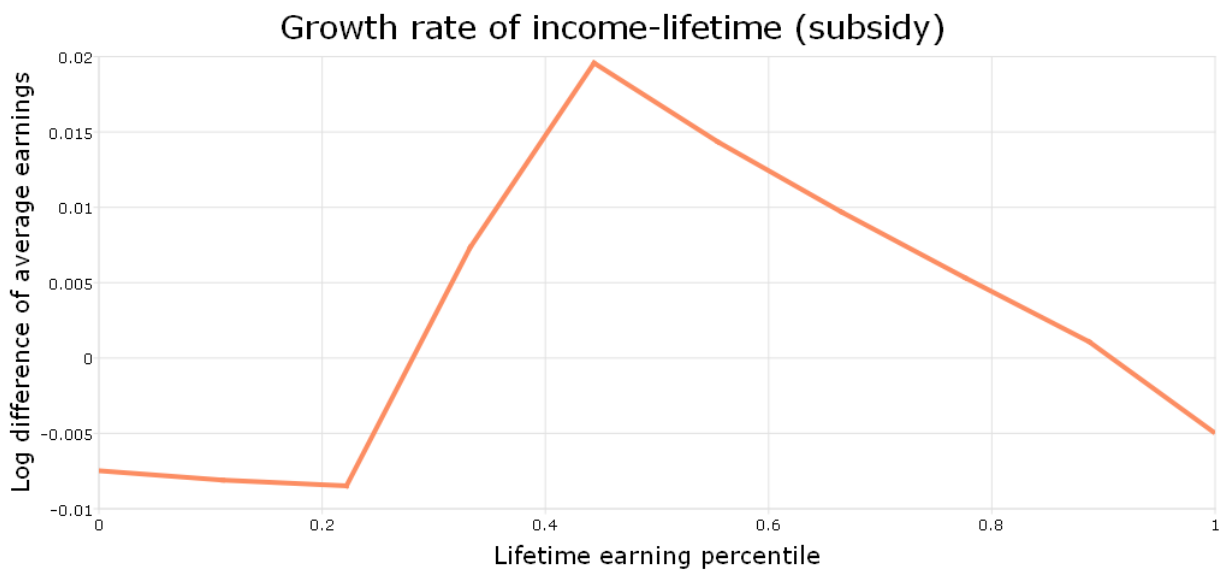
First, we set the price of technology as one ( $P = 1$ ), with no tax or subsidy provided by the government ( $T=0$ ). After solving the model, we find the lifetime income of all agents and sort lifetime income by percentile. Next, we take the average income of all individuals in each percentile and draw an income distribution to find the distribution of lifetime income when the government provides no subsidy. Then holding all the other parameters the same,

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<sup>7</sup>Similar to [Autor \(2014\)](#).



we repeat the procedure to find the income distribution for a positive tax level ( $T = 0.05$ ). Further, we take the logarithm of each earning distribution and compare changes in income for each decile due to changes in government compensation. In Figure 3.2, the X-axis denotes income percentile ranked between 0 and 1 by taking the percentile of agent lifetime incomes, and the Y-axis is the change in lifetime income due to government subsidy. The inverse U shape curve shows the income difference in each situation. We use the parameter values from the previous section, keeping the price constant at  $P_t = 1$ .



**Figure 3.2:** *The growth rate of lifetime income for change in government subsidy.*

To understand the result we again refer to equation (3.25). The RHS of equation (3.25) is the threshold cost of attending the college, and the LHS is the income of an individual attending college. Now, suppose the government provides an education subsidy ( $G_s$ ) consisting of a fraction of total tuition cost ( $X$ ). The government policy affects the economy in three ways: i) increase in tax for all; ii) provide a subsidy for some; iii) changes in wages. Note that the government policy influences the relative supply of labor inputs. Consequently, the equilibrium wages for different skill groups change due to general equilibrium adjustments. People who are college goers tend to supply more abstract and routine skills compared to the manual skill. The equilibrium wage change reflects these dynamics. In our analysis, we find

that an increase in subsidy reduces the equilibrium wage of abstract skill ( $\omega_A$ ) and routine skill ( $\omega_R$ ) and increases manual skill wage ( $\omega_M$ ).

An agent will choose college if lifetime income after government subsidy and wage changes is higher than the income tax and tuition cost. That is the R.H.S. of equation (3.25) is greater than the L.H.S. of (3.25). The subsidy reduces the threshold cost of attending college (R.H.S. of equation (3.25) ) and thus encourages more people to attend college. Boosting education subsidies is, however, a double-edged sword for agent lifetime earnings. The income of agents depends on what kind of ability individuals have and the market rate of those skills. The subsidies increase lifetime earnings for some agents but also increases tax burdens. The lower cost of education due to subsidy has both direct and indirect positive effects on agent lifetime earnings. The lower cost itself increases the income directly, and, at the same time, agents earn higher wages with a college degree. However, because tax revenue finances subsidies, agents will incur higher taxes on their income. Note that the government provides a fixed subsidy only to those people who attend college, whereas it imposes a proportional tax across the board. Thus, the net effect on the RHS is lower because the percentage increase in tax is lower than the percentage increase in subsidy. In equilibrium, taxes and subsidies level off. Only a segment of the people, those who attend college, benefits because of the lower cost of education.

So far we see the growth in earning percentile for low and high-income groups whereas the middle-income group witnesses a growth in earning. The economic intuition that increasing education subsidies increases college enrollment can be argued as follows. Assume three different characteristics among individuals: (i) individuals who always go to college regardless of government policy; (ii) individuals who opt to go to college when there is a subsidy for college education; and (iii) individuals who never go to college. Further, we can loosely assume that most of the individuals who have always intended to go to college regardless of government policy belong to a higher income group and most of the individuals who never intend to go to college belong to low-income groups. The college decision of individuals with these characteristics are not affected by government policy on education subsidies. We cannot expect much increase or decrease in college enrollment among these two groups of

people.

To understand how government policy affects different income groups, let us examine Figure 3.2 more closely. Figure 3.2 shows the log difference of average earnings before and after a subsidy. People in the low and high-income group show negative wage growth due to increased tax because the subsidy is financed by the income tax collected from the agents regardless of income level. To get a more in-depth understanding, let us follow the income group from left to right on the horizontal axis. The income change is negative between 0 to approximately 30 percentile. People belong to this income group hardly go to the college. They do not get any benefit from the subsidy. Though their equilibrium wage changes, the proportionate tax burden causes relatively a sizeable negative impact on their lifetime income.

The middle-income group experience a positive income growth over the lifetime. It mainly because people on the margin, who was indecisive about going to the college, now go to college due to the subsidy. These people are mostly at the margin of 30 to 45 percentile of the income distribution. As the threshold of cost decreases because of government subsidies, some people at the cut-off point will choose to attend college. The higher enrollment from the margin increases the overall lifetime earnings of the middle-income growth. After college, their efficiency gain will give them a higher return on income than the taxes they pay. Also, the net effect of the tax on the lifetime earnings is small because this group is directly benefited from the subsidy. Finally, we do not see any significant changes in college enrollment decision from the high-income group (roughly ten percentile). However, their lifetime income growth becomes negative mainly because of a high tax burden.

Overall, most of the individuals who always choose college are better off due to government subsidy. However, individuals at the 90 percentile of the earnings distribution are worse off as they pay higher tax and receive lower wages from abstract and routine skill. Similarly, individuals at the 30 percentile of the earnings receive lower income due to the income tax. Note that most of the people in this income group do not attend college so they never receive any subsidies. However, some individuals within this group may find their earnings to go up with the changes in wages. Also, we can reasonably assume that individuals whose college

decisions are affected by government support mostly come from the middle-income group.

Next, we consider how changes in tax structure affect the economy's gross and net income. We find that if the government increases the tax by 0.05%, in aggregate the net income of all the agents falls by 0.18%. Though all the individuals are required to pay income tax regardless of attending college or not, the government policy does not affect all the individuals equally. As argued previously the lower and upper-income group are negatively affected by the policy while the middle-income people are the primary beneficiary. Overall, the net income goes down. The lower income leads the total consumption to decrease. The overall net activities of the economy see a downward trend. However, this downtrend is offset by the increase in government expenditure. We find that the government policy increases the gross income by 0.71%. However, the increase in output is offset by increase in the education expenditure so that the consumption goes down. In summary, the government subsidy helps to redistribute income towards the middle-income group at the cost of lower and higher income groups. In the process, the total output in the economy increases while the net income falls due to income tax.

The policy implication from the above exercise is that the government can help the middle-income group, those who are hurt by the reduction of computer capital price and pull out from the college, by providing a subsidy for education, thus, increasing the rate of college attendance in this group. Also, government subsidy causes an increase in the gross income of the economy.

## 3.6 Conclusion

The primary objective of our paper is to develop a model that displays wage polarization using empirical data. We built a general equilibrium model with heterogeneous agents characterized by different productivity endowments. We considered two types of capital, namely, general capital and computer capital, and three types of labor inputs. Manual labor is used for low productivity tasks; routine labor involved assisting with repetitive tasks, and abstract labor used creativity in tasks that demanded high cognitive and interpersonal skill. Gen-

eral capital is infrastructure, machinery, and other physical capital, and computer capital is computer equipment or information technology that performs only routine tasks. Computer capital replaces routine labor and complements abstract labor, manual labor, and general capital. In our model, the reduction of the price of computer capital is the exogenous factor that changes returns for different kinds of labor. Notably, low computer prices reduced wages for routine labor, increased returns for manual and abstract labor. The agents in our model are heterogeneous in their ability to supply different types of labor: manual, routine, and abstract. All individuals are high school graduates who decided whether to attend college. In addition, every individual supplied all types of labor at varying capacities. Those who are relatively more talented in routine tasks tended to supply more routine labor, with the same applying to manual and abstract labor.

Our model produces U-shaped wage growth for income distribution in the lifetime earnings of agents. That is, when the price of computer capital decline, our model show that wages grew for the lower and upper-income groups and decreased for the middle-income group. Then we show how wage polarization affects human capital accumulation, especially in the decision of heterogeneous agents to attend college . The college decision depends on individual ability; the wages prevailing in the economy, and tuition and time costs associated with college education. If the income of those attending college is higher over an agents lifetime than the income of agents without a college education, an agent should choose to go to college. The lower wages for routine labor because of changes in computer capital price discourage agents who have more routine skills as well as fewer manual and abstract skills from attending college. Not all agents with high talent for routine labor will refrain from pursuing college. Some may have enough manual and abstract talent to have higher lifetime income if they attend college that if they do not. In net, we find that a 1% decrease in the price of computer capital reduces college attendance by 0.22%. Moreover, our model can analyze the implications of government policies on college enrollment. For example, if the government provides subsidies for college education, more agents will choose to attend college. In this way, although some individuals in the lower and higher income group may be worse off, the middle-income group is better off.

In the literature, the effect of technological change on demand for labor has been emphasized. However, very few studies focus on the supply side of the labor market, especially due to technological improvements. In our paper, we analyzed how wage polarization affects the supply of labor in the economy and the decision to obtain higher education. Our results are consistent with [Goldin and Katz \(2007\)](#), who empirically show the relative growth of college graduates differs from the relative growth in the demand for college graduates. Our study emphasizes that wage polarization is one reason for the slowing growth rate of college-educated workers despite high returns on education.

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# Appendix A

## Appendix A

### A.0.1 The relationship between $n_{f,t}$ and $n_{i,t}$

This section derives the relationship between the labor input in the final good production ( $n_{f,t}$ ) and implementation labor ( $n_{i,t}$ ). We solve the representative agent's problem in the special case using the market clearing conditions. The first order conditions are

$$\begin{aligned}c_t^{-\sigma} &= \lambda_t \\ \phi(n_{f,t} + n_{i,t})^\gamma &= \lambda_t(1 - \alpha)\frac{y_t}{n_{f,t}} \\ \phi(n_{f,t} + n_{i,t})^\gamma &= \lambda_{k,t}\theta_n i_t n_{i,t}^{\theta_n - 1} \\ \lambda_{k,t} n_{i,t}^{\theta_n} &= \frac{\lambda_t}{v_t} \\ \beta \lambda_{k,t+1} \frac{y_{t+1}}{k_{t+1}} \alpha + \beta \lambda_{k,t+1} \left(1 - \varphi_2 \frac{u_{t+1}^{\varphi_1}}{\varphi_1}\right) &= \lambda_{k,t} \\ \lambda_t \alpha y_t &= \lambda_{k,t} \psi_2 u_t^{\psi_1} k_t.\end{aligned}$$

Further assuming that  $\sigma = 1$ ,  $\gamma = 1$ , and  $\varphi_2 = 1$ , we simplify the above as

$$c_t^{-1} = \lambda_t \tag{A.1}$$

$$\begin{aligned}
\phi &= \lambda_t(1 - \alpha) \frac{y_t}{n_{f,t}} \\
\phi &= \lambda_{k,t} \theta i_t n_{i,t}^{\theta-1} \\
\frac{\lambda_t}{v_t} &= \lambda_{k,t} n_{i,t}^\theta \\
\beta \lambda_{t+1} \frac{y_{t+1}}{k_{t+1}} \alpha + \beta \lambda_{k,t+1} \left(1 - \frac{1}{\varphi} u_{t+1}^\varphi\right) &= \lambda_{k,t} \\
\lambda_t \alpha y_t &= \lambda_{k,t} u_t^\varphi k_t.
\end{aligned} \tag{A.2}$$

Substituting (A.1) and (A.2), we get

$$\phi n_{f,t} c_t = (1 - \alpha) y_t. \tag{A.3}$$

$$c_t k_t u_t^\varphi n_{i,t}^{(1-\theta)} \phi = \alpha \theta_n y_t i_t. \tag{A.4}$$

$$\frac{\alpha y_t}{u_t^\varphi k_t} n_{i,t}^\theta = \frac{1}{v_t}. \tag{A.5}$$

We further substitute the resource constraint into (A.4),

$$c_t k_t u_t^\varphi n_{i,t}^{(1-\theta)} \phi = \alpha \theta_n y_t (v_t (y_t - c_t)). \tag{A.6}$$

We substitute (A.3) into (A.6) to get,

$$c_t k_t u_t^\varphi n_{i,t}^{(1-\theta)} \phi = \alpha \theta_n y_t \left( v_t \left( c_t \frac{n_{f,t} \phi}{(1 - \alpha)} - c_t \right) \right).$$

Further simplifications yields,

$$\frac{1}{v_t} k_t u_t^\varphi n_{i,t}^{(1-\theta)} \phi = \alpha \theta_n y_t \left( \frac{n_{f,t} \phi}{(1 - \alpha)} - 1 \right). \tag{A.7}$$

Substituting (A.5) into (A.8), we get

$$\frac{\alpha y_t}{u_t^\varphi k_t} n_{i,t}^\theta k_t u_t^\varphi n_{i,t}^{(1-\theta)} \phi = \alpha \theta_n y_t \left( \frac{n_{f,t} \phi}{(1 - \alpha)} - 1 \right). \tag{A.8}$$

Further simplifying,

$$n_{f,t} = \frac{n_{i,t}(1-\alpha)}{\theta_n} + \frac{(1-\alpha)}{\phi}. \quad (\text{A.9})$$

We know,

$$n_t = n_{i,t} + n_{f,t}.$$

Substituting value of  $n_{f,t}$  into the above expression to get,

$$n_{i,t} = \frac{\theta_n n_t}{\theta_n + 1 - \alpha} - \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)}. \quad (\text{A.10})$$

We find  $n_{i,t}$  in terms of  $n_t$  as

$$n_{f,t} = \frac{(1-\alpha)n_t}{\theta_n + 1 - \alpha} + \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)}. \quad (\text{A.11})$$

## A.0.2 The sufficient condition for consumption and labor comovement

In this section we show that sufficient condition for  $\frac{dc_t}{dn_t} > 0$  is  $\theta_n > \varphi_1 - 1$ .

We know,

$$c_t^\sigma = \frac{\frac{\theta_n \alpha}{x_1 n_{i,t}^{\varphi_1 - \alpha}}}{\frac{\alpha(\varphi_1 - 1)}{n_t^\gamma n_{f,t}^{\varphi_1 - \alpha}}}.$$

Substituting (A.10) and (A.11) into the above expression we find,

$$\frac{dc_t}{dn_t} = \frac{R - S}{\left( \frac{(1-\alpha)n_t}{\theta_n + 1 - \alpha} + \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)} \right) \frac{\alpha(\varphi_1 - 1)}{\varphi_1 - \alpha}} \quad (\text{A.12})$$

where,

$$\begin{aligned}
R &= \left( \frac{(1-\alpha)n_t}{\theta_n + 1 - \alpha} + \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)} \right) \frac{\alpha(\varphi_1 - 1)}{\varphi_1 - \alpha} x_1 \frac{\theta_n \alpha}{\varphi_1 - \alpha} \\
&\quad \left( \frac{\theta_n n_t}{\theta_n + 1 - \alpha} - \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)} \right) \frac{\theta_n \alpha}{\varphi_1 - \alpha}^{-1} \frac{(1-\alpha)}{\theta_n + 1 - \alpha} \\
Q &= x_1 \left( \frac{\theta_n n_t}{\theta_n + 1 - \alpha} - \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)} \right) \frac{\theta_n \alpha}{\varphi_1 - \alpha} \frac{\alpha(\varphi_1 - 1)}{\varphi_1 - \alpha} \\
&\quad \left( \frac{(1-\alpha)n_t}{\theta_n + 1 - \alpha} + \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)} \right) \frac{\alpha(\varphi_1 - 1)}{\varphi_1 - \alpha}^{-1} \frac{(1-\alpha)}{\phi(\theta_n + 1 - \alpha)}.
\end{aligned}$$

In order to have  $\frac{dc_t}{dn_t} > 0$  it requires  $R > S$  as

$$\begin{aligned}
&\left( \frac{(1-\alpha)n_t}{\theta_n + 1 - \alpha} + \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)} \right) \frac{\alpha(\varphi_1 - 1)}{\varphi_1 - \alpha} x_1 \frac{\theta_n \alpha}{\varphi_1 - \alpha} \left( \frac{\theta_n n_t}{\theta_n + 1 - \alpha} - \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)} \right) \frac{\theta_n \alpha}{\varphi_1 - \alpha}^{-1} \\
&\quad \frac{(1-\alpha)}{\theta_n + 1 - \alpha} > x_1 \left( \frac{\theta_n n_t}{\theta_n + 1 - \alpha} - \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)} \right) \frac{\theta_n \alpha}{\varphi_1 - \alpha} \frac{\alpha(\varphi_1 - 1)}{\varphi_1 - \alpha} \\
&\quad \left( \frac{(1-\alpha)n_t}{\theta_n + 1 - \alpha} + \frac{\theta_n(1-\alpha)}{\phi(\theta_n + 1 - \alpha)} \right) \frac{\alpha(\varphi_1 - 1)}{\varphi_1 - \alpha}^{-1} \frac{(1-\alpha)}{\phi(\theta_n + 1 - \alpha)}.
\end{aligned}$$

Further simplification yields

$$\theta_n > (\varphi_1 - 1). \quad (\text{A.13})$$

If  $\theta_n > (\varphi_1 - 1) > 0$ , a positive comovement occurs between  $c_t$  and  $n_t$ .

# Appendix B

# Appendix B

## B.0.1 The reduced form system of equations

The reduced system of equations is

$$\frac{n_{f,t+1}}{k_{t+1}} = \frac{\frac{\theta\varphi}{\varphi\theta^{\alpha-\varphi}}}{\beta\alpha^{\varphi-\alpha}(\varphi-1)} \left( \left( \frac{1}{a_t v_t} \right)^{\frac{\varphi}{\varphi-\alpha}} \left( \frac{n_{f,t}}{k_t} \right)^{\frac{(\varphi-1)\alpha}{\varphi-\alpha}} \left( \frac{n_{f,t}}{(1-\alpha)} - \frac{1}{\phi} \right)^{\frac{\theta\varphi}{\alpha-\varphi}} \right. \\ \left. - \beta \left( \frac{1}{a_{t+1} v_{t+1}} \right)^{\frac{\varphi}{\varphi-\alpha}} \left( \frac{n_{f,t+1}}{k_{t+1}} \right)^{\frac{(\varphi-1)\alpha}{\varphi-\alpha}} \left( \frac{n_{f,t+1}}{(1-\alpha)} - \frac{1}{\phi} \right)^{\frac{\theta\varphi}{\alpha-\varphi}} \right). \quad (\text{B.1})$$

$$k_{t+1} = k_t - \left( \frac{n_{f,t}}{k_t} \right)^{\frac{\alpha(\varphi-1)}{\alpha-\varphi}} \left( \frac{n_{f,t}}{(1-\alpha)} - \frac{1}{\phi} \right)^{\frac{\theta\varphi}{\varphi-\alpha}} \left( \frac{\frac{\theta\varphi}{\varphi-\alpha} \frac{\theta\varphi}{\alpha\varphi-\alpha}}{\varphi} \right) \left( n_{f,t} \frac{\alpha+\varphi}{\varphi} - \frac{(1-\alpha)}{\phi} \right). \quad (\text{B.2})$$

Let's assume,

$$z_t = \frac{n_{f,t}}{k_t}. \quad (\text{B.3})$$

$$q_t = \frac{n_{f,t}}{(1-\alpha)} - \frac{1}{\phi}. \quad (\text{B.4})$$

Now rewriting equation (B.5)



$$z_{t+1} = \frac{\frac{\theta\varphi}{\varphi\theta^{\alpha-\varphi}}}{\beta\alpha\varphi - \alpha(\varphi - 1)} \left( \left( \frac{1}{a_t v_t} \right)^{\frac{\varphi}{\varphi - \alpha}} z_t^{\frac{(\varphi - 1)\alpha}{\varphi - \alpha}} \left( \frac{n_{f,t}}{(1 - \alpha)} - \frac{1}{\phi} \right)^{\frac{\theta\varphi}{\alpha - \varphi}} \right. \\ \left. - \beta \left( \frac{1}{a_{t+1} v_{t+1}} \right)^{\frac{\varphi}{\varphi - \alpha}} z_{t+1}^{\frac{(\varphi - 1)\alpha}{\varphi - \alpha}} \left( \frac{n_{f,t+1}}{(1 - \alpha)} - \frac{1}{\phi} \right)^{\frac{\theta\varphi}{\alpha - \varphi}} \right). \quad (\text{B.5})$$

Premultiplying and dividing  $n_{f,t+1}$  into the L.H.S and  $n_f$  into the R.H.S of equation (B.2) to get

$$k_{t+1} \frac{n_{f,t+1}}{n_{f,t+1}} = k_t \frac{n_{f,t}}{n_{f,t}} - z_t^{\frac{\alpha(\varphi - 1)}{\alpha - \varphi}} \left( \frac{n_{f,t}}{(1 - \alpha)} - \frac{1}{\phi} \right)^{\frac{\theta\varphi}{\varphi - \alpha}} \left( \frac{\frac{\theta\varphi}{\theta\varphi - \alpha} \frac{\theta\varphi}{\alpha\varphi - \alpha}}{\varphi} \right) \left( n_{f,t} \frac{\alpha - \varphi}{\varphi} - \frac{(1 - \alpha)}{\phi} \right).$$

Further simplification gives,

$$n_{f,t+1} = \frac{z_{f,t+1}}{z_t} \left( n_{f,t} - z_t^{\frac{\alpha(\varphi - 1)}{\alpha - \varphi}} \left( \frac{n_{f,t}}{(1 - \alpha)} - \frac{1}{\phi} \right)^{\frac{\theta\varphi}{\varphi - \alpha}} \left( \frac{\frac{\theta\varphi}{\theta\varphi - \alpha} \frac{\theta\varphi}{\alpha\varphi - \alpha}}{\varphi} \right) \left( n_{f,t} \frac{\alpha - \varphi}{\varphi} - \frac{(1 - \alpha)}{\phi} \right) \right).$$

Now, assume,

$$s_t = \frac{1}{\frac{\varphi}{(a_t v_t)^{\varphi - \alpha}}}.$$

$$x_1 = \frac{\varphi}{\frac{\theta\varphi}{\theta\varphi - \alpha} \frac{\varphi}{\alpha\varphi - \alpha}}.$$

$$x_2 = \frac{(\varphi - 1)\alpha}{\varphi - \alpha}.$$

$$x_3 = \frac{\theta\varphi}{\varphi - \alpha}.$$

Substituting the above expressions into the system of equations gives,

$$z_{t+1} = \frac{x_1}{\beta(\varphi - 1)} \left( s_t z_t^{x_2} \left( \frac{n_{f,t}}{(1-\alpha)} - \frac{1}{\phi} \right)^{-x_3} - \beta s_{t+1} z_{t+1}^{x_2} \frac{n_{f,t+1}}{(1-\alpha)} - \frac{1^{-x_3}}{\phi} \right).$$

$$n_{f,t+1} = \frac{z_{t+1}}{z_t} \left( n_{f,t} - z_t^{1-x_2} \left( \frac{n_{f,t}}{(1-\alpha)} \right)^{x_3} \frac{s_t}{x_1} \frac{\alpha - \varphi}{\varphi} \left( \frac{n_{f,t}(\alpha - \varphi)}{\varphi} - \frac{(1-\alpha)}{\phi} \right) \right).$$

Use the expression  $q_t = \frac{n_{f,t}}{(1-\alpha)} - \frac{1}{\phi}$ , the first equation becomes

$$z_{t+1} = \frac{x_1}{\beta(\varphi - 1)} (s_t z_t^{x_2} q_t^{-x_3} - \beta s_{t+1} z_{t+1}^{x_2} q_{t+1}^{-x_3}). \quad (\text{B.6})$$

Further add and subtract  $\frac{1}{\phi}$  in both side of the second equations gives

$$n_{f,t+1} - \frac{1}{\phi} + \frac{1}{\phi} = \frac{z_{t+1}}{z_t} \left( n_{f,t} - \frac{1}{\phi} + \frac{1}{\phi} - z_t^{1-x_2} \left( \frac{n_{f,t}}{(1-\alpha)} \right)^{x_3} \frac{s_t}{x_1} \left( \frac{n_{f,t}}{(1-\alpha)} - \frac{1}{\phi} + \frac{1}{\phi} \frac{(\varphi)}{\phi(\alpha - \phi)} \right) \right).$$

Rewriting

$$q_{t+1} = \frac{z_{t+1}}{z_t} \left( q_t + \frac{1}{\phi} - z_t^{1-x_2} \left( q_t^{x_3} \frac{s_t}{x_1} \frac{\varphi - \alpha}{\varphi} \left( \frac{\alpha}{\phi(\varphi - \alpha)} - q_t \right) \right) \right) - \frac{1}{\phi}. \quad (\text{B.7})$$

Equations (B.7) and (B.6) are the final form of the reduced system of equilibrium conditions in our model.

## B.0.2 Deriving conditions for indeterminacy

The Taylor series expansion of equations (B.7) and (B.6) yields,

$$\tilde{z}_{t+1} z \left( \frac{z^{x_2-1} x_1 x_2}{q^{x_3} (\varphi - 1)} + 1 \right) - \tilde{q}_{t+1} \left( \frac{z^{x_2-1} x_1 x_3}{q^{x_3} (\varphi - 1)} \right) + \tilde{q}_t \left( \frac{z^{x_2-1} x_1 x_3}{q^{x_3} \beta (\varphi - 1)} \right) - \tilde{z}_t \left( \frac{z^{x_2} x_1 x_2}{q^{x_3} \beta (\varphi - 1)} \right). \quad (\text{B.8})$$

$$\begin{aligned}
& -\tilde{z}_{t+1} \left( q + \frac{1}{\phi} - \frac{(q^{x_3} z^{1-x_2} (\alpha + q\alpha\phi - q\phi\varphi))}{\varphi\phi x_1} \right) + q\tilde{q}_{t+1} + q\tilde{q}_t \left( \frac{q^{x_3} z^{1-x_2} (\alpha - \varphi)}{\varphi x_1} + \right. \\
& \left. \frac{q^{x_3} z^{1-x_2} x_3 (\alpha + q\alpha\phi - q\phi\varphi)}{\varphi\phi x_1} - 1 \right) + \tilde{z}_t \left( \frac{z^{x_2} \varphi x_1 - q^{x_3} \alpha x_2 + qz^{x_2} \varphi\phi x_1 - qq^{x_3} z \alpha \phi x_2 + qq^{x_3} z \varphi\phi x_2}{z^{x_2} \varphi\phi x_2} \right).
\end{aligned} \tag{B.9}$$

Next simplify both of the equations by using the steady state values of  $q$  and  $z$  and the values of  $x_2$  and  $x_3$ . The first equation

$$\tilde{z}_{t+1} = \frac{\alpha}{\varphi} \tilde{q}_{t+1} - \frac{\alpha}{\varphi} \left( \frac{(1-\beta)\varphi - \alpha}{\varphi\beta(\varphi-1)} + 1 \right) \tilde{q}_t + \tilde{z}_t$$

and the second equation

$$\tilde{q}_{t+1} = \frac{(\beta\alpha(\varphi-1) + (1-\beta)(\varphi-\alpha))}{\beta\theta\varphi} \tilde{z}_{t+1} + \frac{1}{\beta} \tilde{q}_t - \frac{(\varphi-1)\alpha}{\beta\theta\varphi} \tilde{z}_t.$$

Put the above equations in the matrix form,

$$\begin{bmatrix} \frac{(\beta\alpha(\varphi-1) + (1-\beta)(\varphi-\alpha))}{\beta\theta\varphi} & 1 \\ 1 & -\frac{\alpha}{\varphi} \end{bmatrix} \begin{bmatrix} \tilde{z}_{t+1} \\ \tilde{q}_{t+1} \end{bmatrix} = \begin{bmatrix} -\frac{(\varphi-1)\alpha}{\beta\theta\varphi} & \frac{1}{\beta} \\ 1 & -\frac{\alpha}{\varphi} \left( \frac{(1-\beta)(\varphi-\alpha)}{\varphi\beta(\varphi-1)} + \frac{1}{\varphi-\alpha} \right) \end{bmatrix} \begin{bmatrix} \tilde{z}_t \\ \tilde{q}_t \end{bmatrix}$$

Let

$$A = \frac{(\beta\alpha(\varphi-1) + (1-\beta)(\varphi-\alpha))}{\beta\theta\varphi}.$$

$$B = \frac{(\varphi-1)\alpha}{\beta\theta\varphi}.$$

$$C = \frac{\alpha}{\varphi} \left( \frac{(1-\beta)(\varphi-\alpha)}{\varphi\beta(\varphi-1)} + \frac{1}{\varphi-\alpha} \right).$$

Rewriting the expression,

$$\begin{bmatrix} -\frac{A}{\theta} & 1 \\ 1 & -\frac{\alpha}{\varphi} \end{bmatrix} \begin{bmatrix} \tilde{z}_{t+1} \\ \tilde{q}_{t+1} \end{bmatrix} = \begin{bmatrix} -\frac{B}{\theta} & \frac{1}{\beta} \\ 1 & -C \end{bmatrix} \begin{bmatrix} \tilde{z}_t \\ \tilde{q}_t \end{bmatrix}$$

Let,  $M_1 =$

$$\begin{bmatrix} -\frac{A}{\theta} & 1 \\ 1 & -\frac{\alpha}{\varphi} \end{bmatrix}$$

and  $M_2 =$

$$\begin{bmatrix} -\frac{B}{\theta} & \frac{1}{\beta} \\ 1 & -C \end{bmatrix}$$

Taking inverse of  $M_1$ :

$$\begin{bmatrix} -\theta \frac{\alpha}{A\alpha - \theta\varphi} & -\theta \frac{\varphi}{A\alpha - \theta\varphi} \\ -\theta \frac{\varphi}{A\alpha - \theta\varphi} & -A \frac{\varphi}{A\alpha - \theta\varphi} \end{bmatrix}$$

Now, premultiply  $M_1^{-1}M_2$

$$\begin{bmatrix} -\theta \frac{\alpha}{A\alpha - \theta\varphi} & -\theta \frac{\varphi}{A\alpha - \theta\varphi} \\ -\theta \frac{\varphi}{A\alpha - \theta\varphi} & -A \frac{\varphi}{A\alpha - \theta\varphi} \end{bmatrix} \begin{bmatrix} -\frac{B}{\theta} & \frac{1}{\beta} \\ 1 & -C \end{bmatrix}$$

$$M = \begin{bmatrix} -B \frac{\alpha}{A\alpha - \theta\varphi} - \theta \frac{\alpha}{A\alpha - \theta\varphi} & C\theta \frac{\varphi}{A\alpha - \theta\varphi} - \theta \frac{\alpha}{A\alpha\beta - \theta\varphi\beta} \\ -B \frac{\varphi}{A\alpha - \theta\varphi} - A \frac{\varphi}{A\alpha - \theta\varphi} & AC \frac{\varphi}{A\alpha - \theta\varphi} - \theta \frac{\alpha}{A\alpha\beta - \theta\varphi\beta} \end{bmatrix}$$

Now, the trace of  $M$

$$-B \frac{\alpha}{A\alpha - \theta\varphi} - \theta \frac{\alpha}{A\alpha - \theta\varphi} + AC \frac{\varphi}{A\alpha - \theta\varphi} - \theta \frac{\alpha}{A\alpha\beta - \theta\varphi\beta}. \quad (\text{B.10})$$

and the determinant of  $M$

$$\frac{BC\beta\varphi - \theta\varphi}{A\alpha\beta - \theta\beta\varphi}. \quad (\text{B.11})$$

Now, the indeterminacy condition,  $|\text{trace}(M) - \det(M)| < 1$  yields,

$$\left| -B \frac{\alpha}{A\alpha - \theta\varphi} - \theta \frac{\alpha}{A\alpha - \theta\varphi} + AC \frac{\varphi}{A\alpha - \theta\varphi} - \theta \frac{\alpha}{A\alpha\beta - \theta\varphi\beta} - \frac{BC\beta\varphi - \theta\varphi}{A\alpha\beta - \theta\beta\varphi} \right| < 1.$$

rearranging,

$$\left| \frac{1}{A\alpha - \theta\varphi} \left( B\alpha - \theta\varphi + C\varphi \frac{1}{\beta} (1 - \alpha)(1 - \beta) \right) \right| < 1.$$

When  $\left( B\alpha - \theta\varphi + C\varphi \frac{1}{\beta} (1 - \alpha)(1 - \beta) \right) > 0$ ,  $\theta$  drops out from the condition and the indeterminacy condition becomes

$$B\alpha + C\varphi \frac{1}{\beta} (1 - \alpha)(1 - \beta) < A\alpha.$$

But let

$$\frac{1}{A\alpha - \theta\varphi} \left( B\alpha - \theta\varphi + C\varphi \frac{1}{\beta} (1 - \alpha)(1 - \beta) \right) < 0.$$

Then since the condition on this is the absolute value, the opposite of this need to be less than one. The condition becomes

$$\frac{1}{A\alpha - \theta\varphi} \left( B\alpha - \theta\varphi + C\varphi \frac{1}{\beta} (1 - \alpha)(1 - \beta) \right) < 1.$$

Now, rearranging the above expression, the indeterminacy condition is

$$\theta > \frac{1}{2\varphi} (A\alpha + B\alpha + C\varphi \frac{1}{\beta} (1 - \alpha)(1 - \beta)).$$

# Appendix C

# Appendix C

In this section we show the details derivations of the solutions of  $t_M, t_R, t_A$  mentioned in Section 2.2. First, the income for the non-college workers in the first period is

$$\max_{\{t_M, t_R, t_A\}} (\omega_M h_M + \omega_R h_R + \omega_A h_A)(1 - T) \quad (\text{C.1})$$

subject to

$$h_M = z_M t_M^\alpha$$

$$h_R = z_R t_R^\alpha$$

$$h_A = z_A t_A^\alpha$$

$$1 - t_c = t_M + t_R + t_A.$$

FOCs for the first-period of life are

$$\omega_M \alpha z_M t_M^{\alpha-1} (1 - T) = \lambda_1 \quad (\text{C.2})$$

$$\omega_R \alpha z_R t_R^{\alpha-1} (1 - T) = \lambda_1 \quad (\text{C.3})$$

$$\omega_A \alpha z_A t_A^{\alpha-1} (1 - T) = \lambda_1 \quad (\text{C.4})$$

where  $\lambda_1$  is the Lagrange multiplier of the time constraint.

Now substitute (C.2), (C.3) into (C.4). the above expressions to get,

$$\omega_R \alpha z_R t_R^{\alpha-1} (1-T) = \omega_M \alpha z_M t_M^{\alpha-1} (1-T).$$

$$\omega_A \alpha z_A t_A^{\alpha-1} (1-T) = \omega_M \alpha z_M t_M^{\alpha-1} (1-T).$$

Simplifying the above expression gives,

$$\omega_R \alpha z_R t_R^{\alpha-1} = \omega_M \alpha z_M t_M^{\alpha-1}.$$

$$\omega_A \alpha z_A t_A^{\alpha-1} = \omega_M \alpha z_M t_M^{\alpha-1}. \quad (\text{C.5})$$

Rewriting the above expression and find the expressions for  $t_R$  and  $t_A$  in terms of  $t_M$  as

$$t_R = \left( \frac{\omega_R z_R}{\omega_M z_M} \right)^{\frac{1}{1-\alpha}} t_M.$$

$$t_A = \left( \frac{\omega_A z_A}{\omega_M z_M} \right)^{\frac{1}{1-\alpha}} t_M.$$

The second-period problem for non-college workers can be written as

$$\max_{\{t_M, t_R, t_A\}} (\omega_M h_M + \omega_R h_R + \omega_A h_A) \frac{\gamma \eta}{r_k} (1-T)$$

subject to

$$1 = t_M + t_R + t_A.$$

FOCs for the second period of life are

$$\omega_M \alpha z_M t_M^{\alpha-1} (1-T) \left(1 + \frac{\gamma}{r_{kr}}\right) = \lambda_2 \quad (\text{C.6})$$

$$\omega_R \alpha z_R t_R^{\alpha-1} (1-T) \left(1 + \frac{\gamma}{r_{kr}}\right) = \lambda_2 \quad (\text{C.7})$$

$$\omega_A \alpha z_A t_A^{\alpha-1} (1-T) \left(1 + \frac{\gamma}{r_{k_r}}\right) = \lambda_2 \quad (\text{C.8})$$

where  $\lambda_2$  is the Lagrange multiplier of the time constraint.

Now substitute (C.6), (C.7) into (C.8). the above expressions to get,

$$\omega_R \alpha z_R t_R^{\alpha-1} (1-T) \left(1 + \frac{\gamma}{r_{k_r}}\right) = \omega_M \alpha z_M t_M^{\alpha-1} (1-T)$$

$$\omega_A \alpha z_A t_A^{\alpha-1} (1-T) \left(1 + \frac{\gamma}{r_{k_r}}\right) = \omega_M \alpha z_M t_M^{\alpha-1} (1-T).$$

Following the same procedure above, we get the solutions as

$$t_M = \frac{\frac{1}{(\omega_M z_M)^{\frac{1}{1-\alpha}}}}{\frac{1}{(\omega_M z_M)^{\frac{1}{1-\alpha}}} + \frac{1}{(\omega_R z_R)^{\frac{1}{1-\alpha}}} + \frac{1}{(\omega_A z_A)^{\frac{1}{1-\alpha}}}}.$$

$$t_R = \frac{\frac{1}{(\omega_R z_R)^{\frac{1}{1-\alpha}}}}{\frac{1}{(\omega_M z_M)^{\frac{1}{1-\alpha}}} + \frac{1}{(\omega_R z_R)^{\frac{1}{1-\alpha}}} + \frac{1}{(\omega_A z_A)^{\frac{1}{1-\alpha}}}}.$$

$$t_A = \frac{\frac{1}{(\omega_A z_A)^{\frac{1}{1-\alpha}}}}{\frac{1}{(\omega_M z_M)^{\frac{1}{1-\alpha}}} + \frac{1}{(\omega_R z_R)^{\frac{1}{1-\alpha}}} + \frac{1}{(\omega_A z_A)^{\frac{1}{1-\alpha}}}}.$$

Now consider the agent who goes to college. The agent spends  $t_c$  unit of time in college and so has  $(1 - t_c)$  units of time for work in the first period. The first-period problem is

$$\max_{\{t_M, t_R, t_A\}} (\omega_M h_M + \omega_R h_R + \omega_A h_A) \frac{\gamma \eta}{r_k} (1-T) - (1 - G_s) X$$

subject to

$$1 - t_c = t_M + t_R + t_A.$$



Following the same steps from the previous problem the solutions are

$$t_M = \frac{\frac{1}{(1-t_c)(\omega_M z_M)^{1-\alpha}}}{\frac{1}{(\omega_M z_M)^{1-\alpha}} + \frac{1}{(\omega_R z_R)^{1-\alpha}} + \frac{1}{(\omega_A z_A)^{1-\alpha}}}.$$

$$t_R = \frac{\frac{1}{(1-t_c)(\omega_R z_R)^{1-\alpha}}}{\frac{1}{(\omega_M z_M)^{1-\alpha}} + \frac{1}{(\omega_R z_R)^{1-\alpha}} + \frac{1}{(\omega_A z_A)^{1-\alpha}}}.$$

$$t_A = \frac{\frac{1}{(1-t_c)(\omega_A z_A)^{1-\alpha}}}{\frac{1}{(\omega_M z_M)^{1-\alpha}} + \frac{1}{(\omega_R z_R)^{1-\alpha}} + \frac{1}{(\omega_A z_A)^{1-\alpha}}}.$$

Also, the solutions in the second period of life, where the agents have one unit of time to work are same as the non-college individuals.