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Uniqueness of the solution to inverse scattering problem with scattering data at a fixed direction of the incident wave

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Let $q(x)$ be real-valued compactly supported sufficiently smooth function. It is proved that the scattering data $A(\beta, \alpha_0, k)$ $\forall \beta \in S^2$, $\forall k > 0$, determine q uniquely. Here, $\alpha_0 \in S^2$ is a fixed direction of the incident plane wave. © *2011 American Institute of Physics*. [doi:10.1063/1.3666985]

I. INTRODUCTION

The scattering solution $u(x, \alpha, k)$ solves the scattering problem,

$$[\nabla^2 + k^2 - q(x)]u = 0 \quad in \quad \mathbb{R}^3, \tag{1}$$

$$u = e^{ik\alpha\cdot x} + A(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty,\ \beta := \frac{x}{r}. \tag{2}$$

Here, $\alpha, \beta \in S^2$ are the unit vectors, S^2 is the unit sphere, the coefficient $A(\beta, \alpha, k)$ is called the scattering amplitude, and $q(x)$ is a real-valued compactly supported sufficiently smooth function. The inverse scattering problem of interest is to determine $q(x)$ given the scattering data $A(\beta, \alpha_0, k)$ $\forall \beta \in S^2, \forall k > 0$. This problem is called *the inverse scattering problem with fixed direction of the incident plane wave data.*

The function $A(\beta, \alpha_0, k)$ depends on one unit vector β and on the scalar k, i.e., on three variables. The potential $q(x)$ depends also on three variables $x \in \mathbb{R}^3$. This inverse problem is, therefore, not over-determined in the sense that the data and the unknown $q(x)$ are the functions of the same number of variables.

Historical remark: In the beginning of the 1940s physicists raised the following question: is it possible to recover the Hamiltonian of a quantum-mechanical system from the observed quantities, such as S-matrix? In the non-relativistic quantum mechanics the simplest Hamiltonian $\mathbf{H} = -\nabla^2 + q(x)$ can be uniquely determined if one knows the potential $q(x)$. The S-matrix in this case is in one-to-one correspondence with the scattering amplitude A: $S = I - \frac{k}{2\pi i}A$, where I is the identity operator in $L^2(S^2)$, A is an integral operator in $L^2(S^2)$ with the kernel $A(\beta, \alpha, k)$, and $k^2 > 0$ is energy. Therefore, the question, raised by the physicists, is reduced to an inverse scattering problem: can one determine the potential $q(x)$ from the knowledge of the scattering amplitude. The inverse scattering problem with fixed direction α_0 of the incident plane wave scattering data $A(\beta, \alpha_0, k)$, known for all $\beta \in S^2$ and all $k > 0$ has been open from the 1940s . In this paper we prove uniqueness of the solution to this inverse problem under the *Assumption A* formulated below. Although there is a large literature on inverse scattering (see, e.g., references in Refs. 15 and 1), the above problem was not solved, and the references we give are only to the papers directly related to our presentation.

Let B_a be the ball centered at the origin and of radius a, and $H_0^\ell(B_a)$ be the closure of $C_0^\infty(B_a)$ in the norm of the Sobolev space $H^\ell(B_a)$ of functions whose derivatives up to the order ℓ belong to $L^2(B_a)$.

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Assumption A: We assume that q is compactly supported, i.e., $q(x) = 0$ for $|x| > a$, where $a > 0$ is an arbitrary large fixed number; $q(x)$ is real-valued, i.e., $q = \overline{q}$; and $q(x) \in H_0^{\ell}(B_a)$, $\ell > 3$.

It was proved in Refs. 9 and 10 (see also Ref. 17, Chap. 6), that if $q = \overline{q}$ and $q \in L^2(B_a)$ is compactly supported, then the resolvent kernel $G(x, y, k)$ of the Schrödinger operator $-\nabla^2 + q(x) - k^2$ is a meromorphic function of k on the whole complex plane k, analytic in Im$k \geq 0$, except, possibly, of a finitely many simple poles at the points ik_j, $k_j > 0$, $1 \leq j \leq n$, where $-k_j^2$ are negative eigenvalues of the self-adjoint operator $-\nabla^2 + q(x)$ in $L^2(\mathbb{R}^3)$. Consequently, the scattering amplitude $A(\beta, \alpha, k)$, corresponding to the above q, is a restriction to the positive semi-axis $k \in [0, \infty)$ of a meromorphic on the whole complex k-plane function.

It was proved by the author (Ref. 11), that the *fixed-energy scattering data*$A(\beta, \alpha) := A(\beta, \alpha, k_0)$, $k_0 = const > 0$, $\forall \beta \in S_1^2$, $\forall \alpha \in S_2^2$, determine real-valued compactly supported $q \in L^2(B_a)$ uniquely. Here, S_j^2, $j = 1, 2$, are arbitrary small open subsets of S^2 (solid angles). No uniqueness results for the potentials which decay at a power rate are known if the scattering data are known at a fixed energy. If the potentials decay faster than exponentially as $|x| \to \infty$, then a uniqueness result for this problem is obtained in Refs. 21 and 6. If the potential decays at a power rate but the scattering data are known for all $k > 0$, all $\alpha \in S^2$ and all $\beta \in S^2$ then a uniqueness results was obtained in Ref. 22.

In Ref. 14 (see also monograph,[15] Chap. 5, and Ref. 12), an analytical formula is derived for the reconstruction of the potential q from exact fixed-energy scattering data, and from noisy fixed-energy scattering data, and stability estimates and error estimates for the reconstruction method are obtained. To the author's knowledge, these are the only known theoretical error estimates until now for the recovery of the potential from noisy fixed-energy scattering data in the three-dimensional inverse scattering problem.

In the papers,[4,5] the relation of the scattering data and the Dirichlet-to-Neumann map is used for proving the uniqueness theorems with the boundary data. Knowing these data is equivalent to knowing the Dirichlet-to-Neumann map. These data are overdetermined.

In paper.[2] inverse boundary problems with partial data are studied.

The scattering data $A(\beta, \alpha)$ depend on four variables (two unit vectors), while the unknown $q(x)$ depends on three variables. In this sense the inverse scattering problem, which consists of finding q from the fixed-energy scattering data $A(\beta, \alpha)$, is overdetermined.

In Ref. 13, stability results are obtained for the inverse scattering problem for obstacles.

The first uniqueness theorem for the three-dimensional inverse scattering problem with non-overdetermined data was announced by the author in Ref. 19, where the uniqueness of the solution to the three-dimensional inverse scattering problem with backscattering data was studied, and a proof of the uniqueness of its solution was outlined. In Ref. 8, the details of this proof were presented for the data $A(-\beta, \beta, k)$ $\forall \beta \in S^2$ $\forall k > 0$. The goal of this paper is to prove a uniqueness theorem for the three-dimensional inverse scattering problem with the scattering data $A(\beta, \alpha_0, k)$ $\forall \beta \in S^2 \forall k > 0$. These data are also non-overdetermined. Our work is based on the method developed in Ref. 18, but the presentation is self-contained. The technical details of our proof differ from these in Ref. 19, for instance, the derivation of the important relation (47).

Theorem 1.1: *If Assumption A holds, then the data $A(\beta, \alpha_0, k)$ $\forall \beta \in S^2$, $\forall k > 0$, and a fixed $\alpha_0 \in S^2$, determine q uniquely.*

Remark 1: The conclusion of Theorem 1.1 remains valid if the data $A(\beta, \alpha_0, k)$ are known $\forall \beta \in S_1^2$ and $k \in (k_0, k_1)$, where $(k_0, k_1) \subset [0, \infty)$ is an arbitrary small interval, $k_1 > k_0$, and S_1^2 is an arbitrary small open subset of S^2.

In Sec. II we formulate some auxiliary results. In Sec. III proof of Theorem 1.1 is given. In the Appendix some technical estimate are proved.

II. AUXILIARY RESULTS

Let

$$F(g) := \tilde{g}(\xi) = \int_{\mathbb{R}^3} g(x)e^{i\xi\cdot x}dx, \quad g(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i\xi\cdot x}\tilde{g}(\xi)d\xi. \tag{3}$$

If $f * g := \int_{\mathbb{R}^3} f(x-y)g(y)dy$, then

$$F(f * g) = \tilde{f}(\xi)\tilde{g}(\xi), \quad F(f(x)g(x)) = \frac{1}{(2\pi)^3}\tilde{f} * \tilde{g}. \tag{4}$$

If

$$G(x-y,k) := \frac{e^{ik[|x-y|-\beta\cdot(x-y)]}}{4\pi|x-y|}, \tag{5}$$

then

$$F(G(x,k)) = \frac{1}{\xi^2 - 2k\beta\cdot\xi}, \qquad \xi^2 := \xi\cdot\xi. \tag{6}$$

The scattering solution $u = u(x, \alpha, k)$ solves (uniquely) the integral equation,

$$u(x,\alpha,k) = e^{ik\alpha\cdot x} - \int_{B_a} g(x,y,k)q(y)u(y,\alpha,k)dy, \tag{7}$$

where

$$g(x,y,k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \tag{8}$$

If

$$v = e^{-ik\alpha\cdot x}u(x,\alpha,k), \tag{9}$$

then

$$v = 1 - \int_{B_a} G(x-y,k)q(y)v(y,\alpha,k)dy, \tag{10}$$

where G is defined in (5).

Define ϵ by the formula,

$$v = 1 + \epsilon. \tag{11}$$

Then, (10) can be rewritten as

$$\epsilon(x,\alpha,k) = -\int_{\mathbb{R}^3} G(x-y,k)q(y)dy - T\epsilon, \tag{12}$$

where

$$T\epsilon := \int_{B_a} G(x-y,k)q(y)\epsilon(y,\alpha,k)dy.$$

Fourier transform of (12) yields (see (4) and (6))

$$\tilde{\epsilon}(\xi,\alpha,k) = -\frac{\tilde{q}(\xi)}{\xi^2 - 2k\alpha\cdot\xi} - \frac{1}{(2\pi)^3}\frac{1}{\xi^2 - 2k\alpha\cdot\xi}\tilde{q} * \tilde{\epsilon}. \tag{13}$$

An essential ingredient of our proof in Sec. II is the following lemma, proved by the author in Ref. 15, p. 262, and in Ref. 14. For convenience of the reader a short proof of this lemma is given in the Appendix.

Lemma 2.1: If $A_j(\beta, \alpha, k)$ is the scattering amplitude corresponding to potential q_j, $j = 1, 2$, then

$$-4\pi[A_1(\beta,\alpha,k) - A_2(\beta,\alpha,k)] = \int_{B_a}[q_1(x) - q_2(x)]u_1(x,\alpha,k)u_2(x,-\beta,k)dx, \tag{14}$$

where u_j is the scattering solution corresponding to q_j.

Consider an algebraic variety $\mathcal{M}$ in $\mathbb{C}^3$ defined by the equation,

$$\mathcal{M} := \{\theta \cdot \theta = 1, \quad \theta \cdot \theta := \theta_1^2 + \theta_2^2 + \theta_3^2, \quad \theta_j \in \mathbb{C}, \ 1 \le j \le 3.\}. \tag{15}$$

This is a non-compact variety, intersecting $\mathbb{R}^3$ over the unit sphere S^2.

Let $R_+ = [0, \infty)$. The following result is proved in Ref. 16, p. 62 (see also Refs. 9 and 15).

Lemma 2.2: If Assumption A holds, then the scattering amplitude $A(\beta, \alpha, k)$ is a restriction to $S^2 \times S^2 \times R_+$ of a function $A(\theta', \theta, k)$ on $\mathcal{M} \times \mathcal{M} \times \mathbb{C}$, analytic on $\mathcal{M} \times \mathcal{M}$ and meromorphic on $\mathbb{C}, \theta', \theta \in \mathcal{M}, k \in \mathbb{C}$.

The scattering solution $u(x, \alpha, k)$ is a meromorphic function of k in $\mathbb{C}$, analytic in Im$k \ge 0$, except, possibly, at the points $k = ik_j$, $1 \le j \le n$, $k_j > 0$, where $-k_j^2$ are negative eigenvalues of the self-adjoint Schrödinger operator, defined by the potential q in $L^2(\mathbb{R}^3)$. These eigenvalues can be absent, for example, if $q \ge 0$.

We need the notion of the Radon transform:

$$\hat{f}(\beta, \lambda) := \int_{\beta \cdot x = \lambda} f(x) d\sigma, \tag{16}$$

where $d\sigma$ is the element of the area of the plane $\beta \cdot x = \lambda$, $\beta \in S^2$, where λ is a real number. The following properties of the Radon transform will be used:

$$\int_{B_a} f(x) dx = \int_{-a}^{a} \hat{f}(\beta, \lambda) d\lambda, \tag{17}$$

$$\int_{B_a} e^{ik\beta \cdot x} f(x) dx = \int_{-a}^{a} e^{ik\lambda} \hat{f}(\beta, \lambda) d\lambda, \tag{18}$$

$$\hat{f}(\beta, \lambda) = \hat{f}(-\beta, -\lambda). \tag{19}$$

These properties are proved, e.g., in Ref. 20, pp. 12 and 15.

We also need the following Phragmen-Lindelöf lemma, which is proved in Ref. 3, p. 69, and in Ref. 7.

Lemma 2.3: Let f(z) be holomorphic inside an angle $\mathcal{A}$ of opening $<\pi$; $|f(z)| \le c_1 e^{c_2|z|}$, $\quad z \in \mathcal{A}, c_1, c_2 > 0$ are constants; $|f(z)| \le M$ on the boundary of $\mathcal{A}$; and f is continuous up to the boundary of $\mathcal{A}$. Then $|f(z)| \le M, \quad \forall z \in \mathcal{A}$.

III. PROOF OF THEOREM 1.1

The scattering data in Remark 1 determine uniquely the scattering data in Theorem 1.1 by Lemma 2.2.

Let us outline the ideas of the proof of Theorem 1.1.

Assume that potentials $q_j, j = 1, 2$, generate the same scattering data:

$$A_1(\beta, \alpha_0, k) = A_2(\beta, \alpha_0, k) \qquad \forall \beta \in S^2, \quad \forall k > 0,$$

and let

$$p(x) := q_1(x) - q_2(x).$$

Then by Lemma 2.1, see Eq. (14), one gets

$$0 = \int_{B_a} p(x) u_1(x, \alpha_0, k) u_2(x, -\beta, k) dx, \qquad \forall \beta \in S^2, \ \forall k > 0. \tag{20}$$

By (9) and (11), one can rewrite (20) as

$$\int_{B_a} e^{i\kappa\zeta\cdot x}[1+\epsilon(x,k)]p(x)dx = 0 \quad \forall\zeta\in S^2_+,\ \forall k>0, \tag{21}$$

where

$$\epsilon(x,k) := \epsilon := \epsilon_1(x,k)+\epsilon_2(x,k)+\epsilon_1(x,k)\epsilon_2(x,k),$$

and we have denoted $|\alpha_0 - \beta| := \tau$, $\zeta := (\alpha_0 - \beta)/\tau$, $\kappa := \tau k$. Without loss of generality one may assume that α_0 is the unit vector along x_3-axis. Then, τ runs through [0, 2] and the unit vector ζ runs through S^2_+, the upper half of the unit sphere S^2. Since $k\in[0,\infty)$ is arbitrary in (21), so is $\kappa = \tau k$. Because the left-hand side of (21) depends on ζ analytically on the variety $\mathcal{M}$, one concludes that relation (21) holds for any $\zeta\in S^2$ if it holds for $\zeta\in S^2_+$. So, from now on we will use formula (21) with $\zeta\in S^2$ being arbitrary.

By Lemma 2.2 the relations (20) and (21) hold for complex k,

$$\tau k = \kappa + i\eta, \quad \eta\ge 0. \tag{22}$$

Using formulas (3)–(4), one derives from (21) the relation,

$$\tilde{p}((\kappa+i\eta)\zeta) + \frac{1}{(2\pi)^3}(\tilde{\epsilon}*\tilde{p})((\kappa+i\eta)\zeta) = 0 \qquad \forall\zeta\in S^2,\ \forall\kappa\in\mathbb{R}, \tag{23}$$

where the notation $(f*g)(z)$ means that the convolution $f*g$ is calculated at the argument $z = (\kappa + i\eta)\zeta$.

One has

$$\sup_{\zeta\in S^2}|\tilde{\epsilon}*\tilde{p}| := \sup_{\zeta\in S^2}\left|\int_{\mathbb{R}^3}\tilde{\epsilon}((\kappa+i\eta)\zeta - s)\tilde{p}(s)ds\right| \le \nu(\kappa,\eta)\sup_{s\in\mathbb{R}^3}|\tilde{p}(s)|, \tag{24}$$

where

$$\nu(\kappa,\eta) := \sup_{\zeta\in S^2}\int_{\mathbb{R}^3}|\tilde{\epsilon}((\kappa+i\eta)\zeta - s)|ds.$$

We prove that if $\eta = \eta(\kappa) = O(\ln\kappa)$ is suitably chosen, namely as in (29) below, then the following inequality holds:

$$0 < \nu(\kappa,\eta(\kappa)) < 1, \qquad \kappa\to\infty. \tag{25}$$

We also prove that

$$\sup_{\zeta\in S^2}|\tilde{p}((\kappa+i\eta(\kappa))\zeta)| \ge \sup_{s\in\mathbb{R}^3}|\tilde{p}(s)|, \quad \kappa\to\infty, \tag{26}$$

and then it follows from (23)–(26) that $\tilde{p}(s)=0$, so $p(x)=0$, and Theorem 1.1 is proved. Indeed, it follows from (23) and (26) that, for sufficiently large κ and a suitable $\eta(k) = O(\ln k)$, one has

$$\sup_{s\in\mathbb{R}^3}|\tilde{p}(s)| \le \frac{1}{(2\pi)^3}\nu(\kappa,\eta(\kappa))\sup_{s\in\mathbb{R}^3}|\tilde{p}(s)|.$$

If (25) holds, then the above equation implies that $\tilde{p}=0$. This and the injectivity of the Fourier transform imply that $p=0$.

This completes the outline of the proof of Theorem 1.1.

Let us now give a detailed proof of estimates (25) and (26), that completes the proof of Theorem 1.1. We denote ζ by β in what follows, since both unit vectors run through all of S^2.

We assume that $p(x)\not\equiv 0$, because otherwise there is nothing to prove. Let

$$\max_{s\in\mathbb{R}^3}|\tilde{p}(s)| := \mathcal{P} \ne 0.$$

Lemma 3.1: If Assumption A holds and $\mathcal{P}\ne 0$, *then*

$$\limsup_{\eta\to\infty}\max_{\beta\in S^2}|\tilde{p}((\kappa+i\eta)\beta)| = \infty, \tag{27}$$

where $\kappa > 0$ is arbitrary but fixed. For any $\kappa > 0$, there is an $\eta = \eta(\kappa)$, such that

$$\max_{\beta \in S^2} |\tilde{p}((\kappa + i\eta(\kappa))\beta)| = \mathcal{P}, \tag{28}$$

where the number $\mathcal{P} := \max_{s \in \mathbb{R}^3} |\tilde{p}(s)|$, and

$$\eta(\kappa) = a^{-1} \ln \kappa + O(1) \quad as \quad \kappa \to +\infty. \tag{29}$$

Proof of Lemma 3.1: By formula (18), one gets

$$\tilde{p}((\kappa + i\eta)\beta) = \int_{B_a} p(x) e^{i(\kappa+i\eta)\beta\cdot x} dx = \int_{-a}^{a} e^{i\kappa\lambda - \eta\lambda} \hat{p}(\beta, \lambda) d\lambda. \tag{30}$$

The function $\hat{p}(\beta, \lambda)$ is compactly supported, real-valued, and satisfies relation (19). Therefore,

$$\max_{\beta \in S^2} |\tilde{p}((\kappa + i\eta(\kappa))\beta)| = \max_{\beta \in S^2} |\tilde{p}((\kappa - i\eta(\kappa))\beta)|. \tag{31}$$

Indeed,

$$\begin{aligned} \max_{\beta \in S^2} |\tilde{p}((\kappa + i\eta(\kappa))\beta)| &= \max_{\beta \in S^2} \left| \int_{-a}^{a} e^{i\kappa\lambda - \eta\lambda} \hat{p}(\beta, \lambda) d\lambda \right| \\ &= \max_{\beta \in S^2} \left| \int_{-a}^{a} e^{-i\kappa\mu + \eta\mu} \hat{p}(\beta, -\mu) d\mu \right| \\ &= \max_{\beta' \in S^2} \left| \int_{-a}^{a} e^{-i\kappa\mu + \eta\mu} \hat{p}(-\beta', -\mu) d\mu \right| \\ &= \max_{\beta' \in S^2} \left| \int_{-a}^{a} e^{-i\kappa\mu + \eta\mu} \hat{p}(\beta', \mu) d\mu \right| \\ &= \max_{\beta \in S^2} |\tilde{p}((\kappa - i\eta)\beta)|. \end{aligned} \tag{32}$$

At the last step we took into account that $\hat{p}(\beta, \lambda)$ is a real-valued function, so

$$\begin{aligned} \max_{\beta \in S^2} \left| \int_{-a}^{a} e^{-i\kappa\mu + \eta\mu} \hat{p}(\beta, \mu) d\mu \right| &= \max_{\beta \in S^2} \left| \int_{-a}^{a} e^{i\kappa\mu + \eta\mu} \hat{p}(\beta, \mu) d\mu \right| \\ &= \max_{\beta \in S^2} |\tilde{p}((\kappa - i\eta)\beta)|. \end{aligned} \tag{33}$$

If $p(x) \not\equiv 0$, then (30) and (31) imply (27), as follows from Lemma 2.3. Let us give a detailed proof of this statement.

Consider the function h of the complex variable $z := \kappa + i\eta$:

$$h := h(z, \beta) := \int_{-a}^{a} e^{iz\lambda} \hat{p}(\beta, \lambda) d\lambda. \tag{34}$$

If (27) is false, then

$$|h(z, \beta)| \le c \quad \forall z = \kappa + i\eta, \quad \eta \ge 0, \quad \forall \beta \in S^2, \tag{35}$$

where $\kappa \ge 0$ is an arbitrary fixed number and the constant $c > 0$ does not depend on β and η.

Thus, $|h|$ is bounded on the ray $\{\kappa = 0, \eta \ge 0\}$, which is part of the boundary of the right angle $\mathcal{A}$, and the other part of its boundary is the ray $\{\kappa \ge 0, \eta = 0\}$. Let us check that $|h|$ is bounded on this ray also.

One has

$$|h(\kappa, \beta)| = |\int_{-a}^{a} e^{i\kappa\lambda} \hat{p}(\beta, \lambda) d\lambda| \le \int_{-a}^{a} |\hat{p}(\beta, \lambda)| d\lambda \le c, \tag{36}$$

where c stands in this paper for *various* constants. From (35)–(36), it follows that on the boundary of the right angle $\mathcal{A}$, namely, on the two rays $\{\kappa \ge 0, \eta = 0\}$ and $\{\kappa = 0, \eta \ge 0\}$ the entire function

$h(z, \beta)$ of the complex variable z is bounded, $|h(z, \beta)| \le c$, and inside $\mathcal{A}$ this function satisfies the estimate,

$$|h(z, \beta)| \le e^{a|\eta|} \int_{-a}^{a} |\hat{p}(\beta, \lambda)| d\lambda \le c e^{a|\eta|}, \tag{37}$$

where c does not depend on β. Therefore, by Lemma 2.3, $|h(z, \beta)| \le c$ in the whole angle $\mathcal{A}$.

By (31), the same argument is applicable to the remaining three right angles, the union of which is the whole complex z-plane $\mathbb{C}$. Therefore,

$$\sup_{z \in \mathbb{C}, \beta \in S^2} |h(z, \beta)| \le c. \tag{38}$$

This implies by the Liouville theorem that $h(z, \beta) = c \ \forall z \in \mathbb{C}$.

Since $\hat{p}(\beta, \lambda) \in L^1(-a, a)$, the relation

$$\int_{-a}^{a} e^{iz\lambda} \hat{p}(\beta, \lambda) d\lambda = c \quad \forall z \in \mathbb{C}, \tag{39}$$

and the Riemann-Lebesgue lemma imply that $c = 0$, so $\hat{p}(\beta, \lambda) = 0 \ \forall \beta \in S^2$ and $\forall \lambda \in \mathbb{R}$. Therefore $p(x) = 0$, contrary to our assumption. Consequently, relation (27) is proved.

Relation (28) follows from (27) because for large η the left-hand side of (28) is larger than $\mathcal{P}$ due to (27), while for $\eta = 0$ the left-hand side of (28) is not larger than $\mathcal{P}$ by the definition of the Fourier transform.

Let us derive estimate (29).

From the assumption $p(x) \in H_0^\ell(B_a)$, it follows that

$$|\tilde{p}((\kappa + i\eta)\beta)| \le c \frac{e^{a|\eta|}}{(1 + \kappa^2 + \eta^2)^{\ell/2}}. \tag{40}$$

This inequality is proved in Lemma 3.2, below.

The right-hand side of this inequality is of the order $O(1)$ as $\kappa \to \infty$ if $|\eta| = a^{-1} \ln \kappa + O(1)$ as $\kappa \to \infty$. This proves relation (29) and we specify $O(\ln \kappa)$ as in this relation.

Let us now prove inequality (40).

Lemma 3.2: If $p \in H_0^\ell(B_a)$ then estimate (40) holds.

Proof: Consider $\partial_j p := \frac{\partial p}{\partial x_j}$. One has

$$\begin{aligned} \left| \int_{B_a} \partial_j p e^{i(\kappa + i\eta)\beta \cdot x} dx \right| &= \left| -i(\kappa + i\eta)\beta_j \int_{B_a} p(x) e^{i(\kappa + i\eta)\beta \cdot x} dx \right| \\ &= (\kappa^2 + \eta^2)^{1/2} |\tilde{p}((\kappa + i\eta)\beta)|. \end{aligned} \tag{41}$$

The left-hand side of the above formula admits the following estimate:

$$\left| \int_{B_a} \partial_j p e^{i(\kappa + i\eta)\beta \cdot x} dx \right| \le c e^{a|\eta|},$$

where the constant $c > 0$ is proportional to $||\partial_j p||_{L^2(B_a)}$. Therefore,

$$|\tilde{p}((\kappa + i\eta)\beta)| \le c[1 + (\kappa^2 + \eta^2)]^{-1/2} e^{a|\eta|}. \tag{42}$$

Repeating this argument one gets estimate (40). Lemma 3.2 is proved. □

Estimate (42) implies that if relation (29) holds and $\kappa \to \infty$, then the quantity $\sup_{\beta \in S^2} |\tilde{p}((\kappa + i\eta)\beta)|$ remains bounded as $\kappa \to \infty$.

If η is fixed and $\kappa \to \infty$, then $\sup_{\beta \in S^2} |\tilde{p}((\kappa + i\eta)\beta)| \to 0$ by the Riemann-Lebesgue lemma. This, the continuity of $|\tilde{p}((\kappa + i\eta)\beta)|$ with respect to η, and relation (27) imply the existence of $\eta = \eta(\kappa)$, such that equality (28) holds, and, consequently, inequality (26) holds. This $\eta(\kappa)$ satisfies (29) because $\mathcal{P}$ is bounded.

Lemma 3.1 is proved □

To complete the proof of Theorem 1.1 one has to establish estimate (25).

This estimate will be established if one proves the following relation:

$$\lim_{\kappa\to\infty} \nu(\kappa) := \lim_{\kappa\to\infty} \nu(\kappa, \eta(\kappa)) = 0, \tag{43}$$

where $\eta(\kappa)$ satisfies (29) and

$$\nu(\kappa, \eta) = \sup_{\beta\in S^2} \int_{\mathbb{R}^3} |\tilde{\epsilon}((\kappa + i\eta)\beta - s)| ds. \tag{44}$$

Our argument is valid for ϵ_1, ϵ_2, and $\epsilon_1\epsilon_2$, so we will use the letter ϵ and Eq. (13) for $\tilde{\epsilon}$.

Below we denote $2k := \kappa + i\eta$ and we choose $\eta = \eta(\kappa) = a^{-1}\ln\kappa + O(1)$ as $\kappa \to \infty$.

We prove that Eq. (12) can be solved by iterations if $\eta \geq 0$ and $|\kappa + i\eta|$ is sufficiently large, because for such $\kappa + i\eta$ the operator T^2 has small norm in $C(B_a)$, the space of functions, continuous in the ball B_a, with the sup-norm. Since Eq. (12) can be solved by iterations and the norm of T^2 is small, the main term in the series, representing its solution, as $|\kappa + i\eta| \to \infty$, $\eta \geq 0$, is the free term of Eq. (12). The same is true for the Fourier transform of Eq. (12), i.e., for Eq. (13). Therefore, the main term of the solution $\tilde{\epsilon}$ to Eq. (13) as $|\kappa + i\eta| \to \infty$, $\eta \geq 0$, is obtained by using the estimate of the free term of this equation. Thus, it is sufficient to check estimate (43) for the function $\nu(\kappa, \eta(\kappa))$ using in place of $\tilde{\epsilon}$ the function $\tilde{q}(\xi)(\xi^2 - 2k\beta\cdot\xi)^{-1}$, with $2k$ replaced by $\kappa + i\eta$ and $\eta = a^{-1}\ln\kappa + O(1)$ as $\kappa \to \infty$.

For the above claim that Eq. (12) has the operator,

$$T\epsilon = \int_{B_a} G(x-y, k) q(y) \epsilon(y, \beta, k) dy,$$

with the norm $||T^2||$ in the space $C(B_a)$, which tends to zero as $|\kappa + i\eta| \to \infty$, $\eta \geq 0$, see the Appendix.

Thus, let us estimate the modulus of the factor $\nu(\kappa, \eta)$ in (24) with $\eta = \eta(\kappa)$ as in (29). Using inequality (40), and denoting $\xi = (\kappa + i\eta)\beta$, where $\beta \in S^2$ plays the role of α in (13), one obtains

$$\begin{aligned} I := &\sup_{\beta\in S^2} \int_{\mathbb{R}^3} \frac{|\tilde{q}((\kappa + i\eta)\beta - s)| ds}{|[(\kappa + i\eta)\beta - s)^2 - (\kappa + i\eta)\beta\cdot((\kappa + i\eta)\beta - s)]|} \\ &\leq c e^{a|\eta|} \sup_{\beta\in S^2} \int_{\mathbb{R}^3} \frac{ds}{|s^2 - (\kappa + i\eta)\beta\cdot s|[1 + (\kappa\beta - s)^2 + \eta^2]^{\ell/2}} \\ &:= c e^{a|\eta|} J. \end{aligned} \tag{45}$$

Let us prove that

$$J = o(\frac{1}{\kappa}), \ \ \kappa \to \infty.$$

If this estimate is proved and $\eta = a^{-1}\ln\kappa + O(1)$, then $I = o(1)$ as $\kappa \to \infty$, therefore relation (43) follows, and Theorem 1.1 is proved.

Let us write the integral J in the spherical coordinates with x_3-axis directed along vector β. We have

$$|s| = r, \qquad \beta\cdot s = r\cos\theta := rt, \quad -1 \leq t \leq 1.$$

Denote

$$\gamma := \kappa^2 + \eta^2.$$

Then

$$\begin{aligned} J &\leq 2\pi \int_0^\infty drr \int_{-1}^1 \frac{dt}{[(r - \kappa t)^2 + \eta^2 t^2]^{1/2}(1 + \gamma + r^2 - 2r\kappa t)^{\ell/2}} \\ &:= 2\pi \int_0^\infty drr B(r), \end{aligned} \tag{46}$$

where

$$B := B(r) = B(r,\kappa,\eta) := \int_{-1}^{1} \frac{dt}{[(r-\kappa t)^2+\eta^2 t^2]^{1/2}(1+\gamma+r^2-2r\kappa t)^{\ell/2}}.$$

Estimate of J we start with the observation

$$\tau := \min_{t\in[-1,1]}[(r-\kappa t)^2+\eta^2 t^2] = \min\{r^2\eta^2/\gamma,\ (r-\kappa)^2+\eta^2\}.$$

Let $\tau = r^2\eta^2/\gamma$, which is always the case if r is sufficiently small. In the case when $\tau = (r - \kappa)^2 + \eta^2$ the proof is considerably simpler and is left for the reader. If $\tau = r^2\eta^2/\gamma$, then

$$J \le 2\pi\gamma^{1/2}\eta^{-1}\int_0^\infty dr \int_{-1}^{1} dt[1+\gamma+r^2-2\kappa rt]^{-\ell/2}.$$

Integrating over t yields

$$J \le 2\pi\gamma^{1/2}\eta^{-1}[(\ell-2)\kappa]^{-1}\mathcal{J},$$

where

$$\mathcal{J} := \int_0^\infty drr^{-1}[(1+\gamma+r^2-2\kappa r)^{-b}-(1+\gamma+r^2+2\kappa r)^{-b}],$$

and $b := \ell/2 - 1$.

Since $\eta = O(\ln\kappa)$, one has $\frac{\eta}{\kappa} = o(1)$ as $\kappa\to\infty$. Therefore,

$$\gamma^{1/2}\eta^{-1}\kappa^{-1} = O(\eta^{-1}) \qquad \text{as} \quad \kappa\to\infty.$$

Since $\ell > 3$, one has $b > \frac{1}{2}$, and, as we prove below,

$$\mathcal{J} = o(\frac{1}{\kappa}) \quad \text{as} \quad \kappa\to\infty. \tag{47}$$

This relation implies the desired inequality,

$$J \le o(\frac{1}{\kappa}) \qquad \text{as} \quad \kappa\to\infty. \tag{48}$$

Let us derive relation (47). One has

$$\mathcal{J} = \int_0^1 + \int_1^\infty := J_1 + J_2,$$

$$J_1 \le \int_0^1 drr^{-1}\frac{(w^2+2r\kappa+r^2)^b-(w^2-2r\kappa+r^2)^b}{(w^2+2r\kappa+r^2)^b(w^2-2r\kappa+r^2)^b},$$

where

$$w^2 := 1+\gamma = 1+\eta^2+\kappa^2.$$

Furthermore,

$$(w^2+2r\kappa+r^2)^b-(w^2-2r\kappa+r^2)^b \le \frac{4br\kappa}{(w^2-2r\kappa+r^2)^{1-b}}.$$

Thus,

$$J_1 \le 4b\kappa\int_0^1 dr\frac{1}{(w^2+2r\kappa+r^2)^b(w^2-2r\kappa+r^2)}.$$

This implies the following estimate:

$$J_1 \le O(\kappa/w^{2+2b}) \le O(\kappa^{-(1+2b)}),$$

because $w = \kappa[1 + o(1)]$ as $\kappa \to \infty$. Furthermore,

$$J_2 \leq \int_1^\infty dr r^{-1}[(1 + \eta^2 + (r - \kappa)^2)^{-b} - (1 + \eta^2 + (r + \kappa)^2)^{-b}] := J_{21} - J_{22}.$$

One has $J_{22} \leq J_{21}$.

Let us estimate J_{21}. One obtains

$$J_{21} = \int_1^{\kappa/2} + \int_{\kappa/2}^\infty := j_1 + j_2,$$

and

$$j_1 \leq \frac{1}{[W^2 + \frac{\kappa^2}{4}]^b} \ln \kappa = o(\frac{1}{\kappa}), \qquad W^2 := 1 + \eta^2, \qquad b > \frac{1}{2}.$$

Furthermore,

$$j_2 \leq \frac{2}{\kappa} \int_{\kappa/2}^\infty \frac{dr}{[W^2 + (r - \kappa)^2]^b} \leq \frac{2}{\kappa} \int_{-\infty}^\infty \frac{dy}{[W^2 + y^2]^b} = o(\frac{1}{\kappa}).$$

Thus, if $b > \frac{1}{2}$, then $J_2 = o(\frac{1}{\kappa})$ and $\mathcal{J} = J_1 + J_2 = o(\frac{1}{\kappa})$. Thus, relation (47) is proved.

Relation (47) yields the desired estimate,

$$J = o(\frac{1}{\kappa}).$$

Thus, both estimates (47) and (48) are proved.

Estimate (45) implies

$$I \leq c e^{a|\eta|} o\left(\frac{1}{\sqrt{\kappa^2 + \eta^2}}\right), \quad \kappa \to \infty, \quad \eta = a^{-1} \ln \kappa + O(1). \tag{49}$$

The quantity $\eta = \eta(k) = a^{-1} \ln \kappa + O(1)$ was chosen so that if $\kappa \to \infty$, then the quantity $\frac{e^{|\eta|a}}{\sqrt{\kappa^2+\eta^2}}$ remains bounded as $\kappa \to \infty$. Therefore, estimate (49) implies

$$\lim_{\kappa \to \infty, \eta = a^{-1} \ln \kappa + O(1)} I = 0. \tag{50}$$

Consequently, estimate (43) holds.

Theorem 1.1 is proved. □

APPENDIX: ESTIMATE OF $||T^2||$ AND PROOF OF LEMMA 2.1

1. Estimate of the norm of the operator T^2

Let

$$Tf := \int_{B_a} G(x - y, \kappa + i\eta) q(y) f(y) dy. \tag{A1}$$

Assume $q \in H_0^\ell(B_a)$, $\ell > 2, f \in C(B_a)$. Our goal is to prove that Eq. (12) can be solved by iterations for all sufficiently large κ.

Consider T as an operator in $C(B_a)$. One has

$$\begin{aligned} T^2 f &= \int_{B_a} dz G(x - z, \kappa + i\eta) q(z) \int_{B_a} G(z - y, \kappa + i\eta) q(y) f(y) dy \\ &= \int_{B_a} dy f(y) q(y) \int_{B_a} dz q(z) G(x - z, \kappa + i\eta) G(z - y, \kappa + i\eta). \end{aligned} \tag{A2}$$

Let us estimate the integral

$$\begin{aligned}
I(x,y) &:= \int_{B_a} G(x-z,\kappa+i\eta)G(z-y,\kappa+i\eta)q(z)dz \\
&= \int_{B_a} \frac{e^{i(\kappa+i\eta)[|x-z|-\beta\cdot(x-z)+|z-y|-\beta\cdot(z-y)]}}{16\pi^2|x-z||z-y|}q(z)dz \\
&= \frac{1}{16\pi^2}\int_{B_a} \frac{e^{i(\kappa+i\eta)[|x-z|+|z-y|-\beta\cdot(x-y)]}}{|x-z||z-y|}q(z)dz \\
&:= \frac{e^{-i(\kappa+i\eta)\beta\cdot(x-y)}}{16\pi^2}I_1(x,y).
\end{aligned} \tag{A3}$$

Let us use the following coordinates (see Ref. 16, p. 391):

$$z_1 = \ell st + \frac{x_1+y_1}{2}, \quad z_2 = \ell\sqrt{(s^2-1)(1-t^2)}\cos\psi + \frac{x_2+y_2}{2}, \tag{A4}$$

$$z_3 = \ell\sqrt{(s^2-1)(1-t^2)}\sin\psi + \frac{x_3+y_3}{2}. \tag{A5}$$

The Jacobian J of the transformation $(z_1, z_2, z_3) \to (\ell, t, \psi)$ is

$$J = \ell^3(s^2-t^2), \tag{A6}$$

where

$$\ell = \frac{|x-y|}{2}, \quad |x-z|+|z-y| = 2\ell s, \quad |x-z|-|z-y| = 2\ell t, \tag{A7}$$

$$|x-z||z-y| = 4\ell^2(s^2-t^2), \quad 0 \le \psi < 2\pi, \quad t \in [-1,1], \; s \in [1,\infty). \tag{A8}$$

One has

$$I_1 = \ell \int_a^\infty e^{2i(\kappa+i\eta)\ell s} Q(s)ds, \tag{A9}$$

where

$$Q(s) := Q(s,\ell,\frac{x+y}{2}) = \int_0^{2\pi} d\psi \int_{-1}^{1} dt q(z(s,t,\psi;\ell,\frac{x+y}{2})), \tag{A10}$$

and the function $Q(s) \in H_0^2(\mathbb{R}^3)$ for any fixed x, y. Therefore, an integration by parts in (A9) yields the following estimate:

$$|I_1| = O\left(\frac{1}{|\kappa+i\eta|}\right), \quad |\kappa+i\eta| \to \infty. \tag{A11}$$

From (A2), (A3), and (A11), one gets

$$\|T^2\| = O\left(\frac{1}{\sqrt{\gamma}}\right), \qquad \gamma := \kappa^2+\eta^2 \to \infty. \tag{A12}$$

Therefore, integral Eq. (12), with k replaced by $\frac{\kappa+i\eta}{2}$, can be solved by iterations if γ is sufficiently large and $\eta \ge 0$. Consequently, integral Eq. (13) can be solved by iterations. Thus, estimate (43) holds if such an estimate holds for the free term in Eq. (13), that is, for the function $\frac{\tilde{q}}{\xi^2-(\kappa+i\eta)\beta\cdot\xi}$, namely, if estimate (50) holds.

2. Proof of Lemma 2.1

Let $L_j G_j := [\nabla^2 + k^2 - q_j(x)]G_j(x, y, k) = -\delta(x - y)$ in $\mathbb{R}^3$, $j = 1, 2$. Applying Green's formula, one gets

$$G_1(x, y, k) - G_2(x, y, k) = \int_{B_a} [q_2(z) - q_1(z)]G_1(x, z, k)G_2(z, y.k)dz. \tag{A13}$$

In Ref. 16, p. 46, the following formula is proved:

$$G_j(x, y, k) = \frac{e^{ik|y|}}{4\pi|y|}u_j(x, \alpha, k) + o(\frac{1}{|y|}), \qquad |y| \to \infty, \alpha := -\frac{y}{|y|}, \tag{A14}$$

where $u_j(x, \alpha, k)$ is the scattering solution, $j = 1, 2$. Applying formula (A14) to (A13), one obtains

$$u_1(x, \alpha, k) - u_2(x, \alpha, k) = \int_{B_a} [q_2(z) - q_1(z)]G_1(x, z, k)u_2(z, \alpha, k)dz \tag{A15}$$

using the definition (2) of the scattering amplitude $A(\beta, \alpha, k)$, one derives from (A15) the relation

$$4\pi[A_1(\beta, \alpha, k) - A_2(\beta, \alpha, k)] = \int_{B_a} [q_2(z) - q_1(z)]u_1(z, -\beta, k)u_2(z, \alpha, k)dz. \tag{A16}$$

This formula is equivalent to (14) because of the well-known reciprocity relation $A(\beta, \alpha, k) = A(-\alpha, -\beta, k)$.

Lemma 2.1 is proved. □

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