OPTIMUM DESIGNS OF PRESTRESSED CONCRETE PLATES

by 149

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TABLE OF CONTENTS

SYNOPSIS

The nonprestressed plate is first investigated by the conventional elastic method and the differential equations which are derived for the plate are solved by the finite difference method.

The load balancing method is then applied to the prestressed plate design. According to this method, prestressing balances a certain portion of the gravity loads so that the plate will not be subjected to bending stresses under a given loading condition. The method is illustrated by analysis of a plate having a combination of prestressed and reinforced concrete where varying amounts of the live load may be carried by nonprestressed reinforcement.

Finally, two economical methods are introduced. In the first method, the transformed membrane method, the equation $\Delta\Delta z=q/D$, which is derived for the nonprestressed plate, is replaced by the equation $\Delta z=q/S$ which is derived for the uniformly stretched membrane. The other method is the minimum tendon method. The tendon volume is a minimum if the plate is divided into regions in which the full load is carried by tendons running in one direction, and in the remaining regions, the full load is carried by the tendons running in the other direction.

Examples are worked out for each method.

INTRODUCTION

A plate may be regarded essentially as an indeterminate structure. The supporting conditions, whether simply supported along the edges or continuous over the side supports, make no fundamental difference. They affect only the degree of indeterminacy. As with any indeterminate structural system, a variety of analyses are possible, all more or less accurate, depending upon the degree to which their assumptions are fulfilled in the actual material.

Making a comparison between preatressed and nonprestressed plates, we find that in elastic analysis, stresses in a nonprestressed plate are fully determined by the boundary conditions and the external loading. The distribution of loading in two perpendicular directions, is invariable and can not be altered. An approximate method, such as one using finite differences equstions, is quite commonly used to solve these cases. In prestressed plates this distribution of the load is arbitrary, and a tendon arrangement can be calculated for any asaumed distribution. Consequently, the number of solutions for any particular loading and boundary condition is infinite. The most economical tendon arrangement would be of considerable importance to structural engineers.

In the following discussion, the elastic design of nonprestressed plates, the load balancing method and two techniques for the optimum design of prestressed plates are presented.

CONVENTIONAL DESIGN

The elastic theory of plate behavior is predicated upon the assumption that the material obeys Hooke's law and that it is homogeneous and isotropic. It is assumed further that all deflections will be very small compared with the thickness of the plate.

As a first step, it is necessary to consider the geometry involved when a slab deforms; i.e., becomes curved or warped under load. Assuming a plate in the x-y plane to be loaded perpendicular to that plane, it is obvious that the plate will deflect in the direction of the application of load. It follows that the top and bottom surfaces of the plate will trace curved lines on all planes cutting the plate. The curves of special interest are those drawn on planes perpendicular to the x-y plane. As is well known, from the theory of beams, bending stress is directly proportional to the curvature, assuming that the stress does not exceed the elastic limit of the material. Hence it is important to find the direction of maximum curvature.

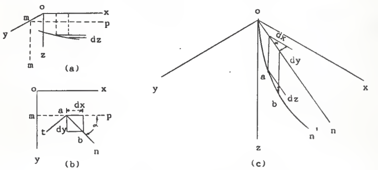


Fig. 1. Sections of the deflected plate.

Taking a normal section of the plate parallel to the x-z plane

(Fig. 1 a) we find that the slope of the surface in the x direction is $\partial z/\partial x$. In the same manner the slope in the y direction is $\partial z/\partial y$. Now Fig. 1-c is intended to show a transparent plate originally in the horizontal x-y plane which has deflected under load and which has been cut by a vertical plane making an angle α with the x-z plane. The intercept of this plane on the x-y plane is On, and its intercept with the surface of the plate is the curved line On'. From these figures, we find that the difference in the deflections of the two adjacent points a and b in the On direction is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy . \qquad (2\text{-}1)$$

The slope of the short segment then must be

$$\frac{\partial z}{\partial n} = \frac{\partial z}{\partial x}\frac{dx}{dn} + \frac{\partial z}{\partial y}\frac{dy}{dn} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha . \qquad (2\text{-}2)$$

We consider a curvature positive if it is convex downward. In determining the curvature of the surface of the plate we assume that the deflections of the plate are very small. In such a case the slope of the surface in any direction can be taken equal to the angle that the tangent to the surface in that direction makes with the x-y plane, and the square of the slope may be neglected compared to unity. The curvature in the x direction is approximately

$$\frac{1}{r_x} = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = -\frac{\partial^2 z}{\partial x^2} .$$

The curvature in the y direction then is

$$\frac{1}{r_y} = -\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial^2 z}{\partial y^2} .$$

Similarly, the curvature in the n direction may be taken to be

$$\frac{1}{r_n} = -\frac{\partial}{\partial n}\left(\frac{\partial z}{\partial n}\right) .$$

From Eq. (2-2), the operator $\partial/\partial n$ is seen to be defined by

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha \quad ,$$

so that

$$\frac{1}{r_n} = -\left(\frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha\right)\left(\frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha\right)$$

$$= -\left(\frac{\partial^2 z}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2 z}{\partial x\,\partial y}\sin\alpha\cos\alpha + \frac{\partial^2 z}{\partial y^2}\sin^2\alpha\right)$$

$$= \frac{1}{r_x}\cos^2\alpha - \frac{1}{r_{xy}}\sin 2\alpha + \frac{1}{r_y}\sin^2\alpha \quad , \qquad (2\text{-}3)$$

where $1/r_{xy} = \partial^2 z/\partial x\,\partial y$ is called the twist of the surfsce with respect to the x and y axes.

It is now possible to find the twist $1/r_{nt}$, where t is the coordinate perpendicular to n. By analogy with $1/r_{xy}$,

$$\frac{1}{r_{nt}} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial n}\right) .$$

The operator $\partial/\partial t$ can be derived from the operator $\partial/\partial n$ by replacing α by $\alpha + \pi/2$.

Then

$$\frac{1}{r_{nt}} = (-\frac{\partial}{\partial x}\sin\alpha + \frac{\partial}{\partial y}\cos\alpha)(\frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha)$$

$$= \frac{1}{2}\sin 2\alpha\,(-\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}) + \cos 2\alpha\frac{\partial^2 z}{\partial x\,\partial y}$$

$$= \frac{1}{2}\sin 2\alpha\,(\frac{1}{r_x} - \frac{1}{r_y}) + \cos 2\alpha\frac{1}{r_{xy}} \,. \qquad (2\text{-}4)$$

In further discussion we shall be interested in finding, in terms of α, the directions in which the curvature of the surface is a maximum or minimum and in finding the corresponding values of the curvature. We obtain the necessary equation for determining α by equating the derivative of expression (2-3) with respect to α to zero, which gives

$$\frac{1}{r_x}\sin 2\alpha + \frac{2}{r_{xy}}\cos 2\alpha - \frac{1}{r_y}\sin 2\alpha = 0 \,,$$

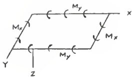
whence

$$\tan 2\alpha = -\frac{\frac{2}{r_{xy}}}{\frac{1}{r_x} - \frac{1}{r_y}} \,.$$

From this equation we find two values of α, differing by $\pi/2$. Substituting these in Eq. (2-3) we find two values of $1/r_n$, one representing the maximum and the other the minimum curvature at point a of the surface. These two curvatures are called the principal curvatures of the surface and the corresponding planes naz and taz the planes of principal curvature.

A similar analysis is possible for $1/r_{nt}$, and

$$\tan 2\alpha = \frac{\frac{1}{r_x} - \frac{1}{r_y}}{\frac{2}{r_{xy}}} \,.$$



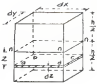


Fig. 2. The internal moments. Fig. 3. An element of the plate.

The next step is to relate the curvatures derived above to bending moments and thus to stresses. Let us begin with pure bending of a rectangular plate by moments that are uniformly distributed along the edges of the plate as shown in Fig. 2. We take the x-y plane to coincide with the middle plane of the plate before deflection and the x and y axes along the edges of the plate. The z axis, which is then perpendicular to the middle plane, is taken positive downward. We denote by Mx the bending moment per unit length acting on the edges parallel to the y axis and by My the moment per unit length acting on the edges parallel to the x axis. These moments we consider positive when they produce compression in the upper surface of the plate and tension in the lower.

Let us consider an element cut out of the plate by two pairs of planes, parallel to the x-z and y-z planes, as shown in Fig. 3. Assuming that during bending of the plate the lateral sides of the element remain plane and rotate about the neutral axis n-n so as to remain normal to the deflected middle surface of the plate, it can be concluded that the middle plane of the plate does not undergo any extension during this bending, and the middle surface is therefore the neutral surface. Then the unit elongations in the x and y directions of an elemental lamina a, b, c, d (Fig. 3), at a distance z from the neutral surface are

$$\epsilon_x = \frac{z}{r_x}, \qquad \epsilon_y = \frac{z}{r_y}.$$

From the theory of elasticity, the stresses and strains in the orthogonal directions are related by

$$\epsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y), \qquad \epsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x).$$

where σ_x, σ_y are stresses, μ is Poisson's ratio and E is the modulus of elasticity of the material.

Then the corresponding stresses in the lamina a, b, c, d are

$$\sigma_x = \frac{E\,Z}{1-\mu^2}\left(\frac{1}{r_x} + \mu\frac{1}{r_y}\right), \tag{2-5 a}$$

$$\sigma_y = \frac{E\,Z}{1-\mu^2}\left(\frac{1}{r_y} + \mu\frac{1}{r_x}\right). \tag{2-5 b}$$

Now we introduce a term

$$D = \frac{E\,h^3}{12\,(1-\mu^2)},$$

where D is called the flexural rigidity of the plate, u, Poisson's ratio, E, modulus of elasticity of the material, and h, full thickness of the plate. The moment in the x direction for a member of thickness h is

$$M_x = \int_{-h/2}^{+h/2} \sigma_x \, z \, dz .$$

Integrating,

$$M_x = D\left(\frac{1}{r_x} + \mu\frac{1}{r_y}\right) = -D\left(\frac{\partial^2 z}{\partial x^2} + \mu\frac{\partial^2 z}{\partial y^2}\right), \tag{2-6 a}$$

and similarly

$$M_y = -D\left(\frac{\partial^2 z}{\partial y^2} + \mu\frac{\partial^2 z}{\partial x^2}\right) . \qquad (2\text{-}6\ b)$$

Let us now consider the stresses acting on a section of the lamina a, b, c, d parallel to the z axis and inclined to the x and y axes. If acd (Fig. 4) represents a portion of the lamina cut by such a section, the stress acting on the side ac can be found by means of the equations of statics. Resolving this stress into a normal component σ_n and a shearing component τ_{nt}, the magnitudes of these components are obtained by projecting the forces acting on the element a c d n the n and t directions respectively, which gives:

$$\sigma_n = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha ,$$

$$\tau_{nt} = \frac{1}{2}(\sigma_y - \sigma_x)\sin 2\alpha .$$

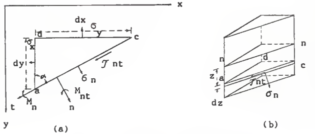


Fig. 4. A portion of the element.

Considering all laminae, such as acd in Fig. 4b, over the thickness of the plate, the normal stresses, σ_n, multiplied by lever arm, z, give the bending moment acting on the section a, c of the plate, the magnitude of which per unit length along a-c is

$$M_n = \int_{-h/2}^{+h/2} \sigma_n z dz = M_x \cos^2\alpha + M_y \sin^2\alpha \quad . \qquad (2-7)$$

The shearing stresses, τ_{nt}, multiplied by lever arm, z, give the twisting moment acting on the section a-c of the plate, the msgnitude of which per unit length a-c is

$$M_{nt} = -\int_{-h/2}^{+h/2} \tau_{nt} z dz = \frac{1}{2} \sin 2\alpha \; (M_x - M_y) \; . \qquad (2-8)$$

The third step is to derive the equationsof equilibrium of a small segment of plate in horizontal dimensions, but with the full thickness h of the plate. In addition to the bending moments and the twisting moment, there are vertical shearing forces on the sides of the element. The magnitudes of these shearing forces per unit length parallel to the y and x axes we denote by Q_x and Q_y respectively

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz \; , \qquad Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz \; .$$

Since the moments and the shearing forces are functions of the coordinates x and y, we must, in discussing the conditions of equilibrium of the element, take into consideration the smsll changes of these quantities when the coordinates x and y change by the small quantities dx and dy. The middle plane of the element is represented in Fig. 5 a, and b, and the directions in which the moments and forces are tsken as positive are indicated.

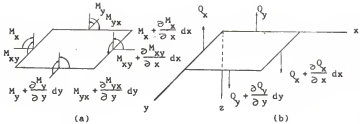


Fig. 5. Positiva moments and forces.

We must also consider the load distributed over the upper surface of the plate. The intensity of this load we denote by q, so that the load acting on the element is q dx dy.

Projecting all the forces acting on the element onto the z axis we obtain the following equation of equilibrium:

$$\frac{\partial Q_x}{\partial x} dx\, dy + \frac{\partial Q_y}{\partial y} dy\, dx + q\, dx\, dy = 0 ,$$

from which

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 . \tag{2-9}$$

Taking moments of all the forces acting on the element with respect to the x axis, we obtain the following equation:

$$\frac{\partial M_{xy}}{\partial x} dx\, dy - \frac{\partial M_y}{\partial y} dy\, dx + Q_y\, dx\, dy + \frac{\partial Q_y}{\partial y} (dy)^2 dx$$

$$+ \frac{\partial Q_x}{\partial x} dx \frac{(dy)^2}{2} + q\, dx \frac{(dy)^2}{2} = 0 . \tag{2-10 a}$$

The moment of the load q and the moment due to change in the force Q_x, Q_y are neglected in this equation, since they are small quantities of a higher order than those retained. After simplification, Eq. 2-10a becomes

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y = 0 . \qquad (2\text{-}10)$$

In the same manner, by taking moments with respect to the y axis, we obtain:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0 , \qquad (2\text{-}11)$$

then

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x \partial y} - \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = - q . \qquad (2\text{-}12)$$

From elementary statics,

$$\tau_{xy} = \tau_{yx} ,$$

so

$$M_{xy} = - M_{yx} = D (1 - \mu) \frac{\partial^2 z}{\partial x \partial y} .$$

Applying this fact and substituting Eqs. (2-6a) and (2-6b) into the Eq. (2-12) gives the differential equation of the plate:

$$\frac{\partial^4 z}{\partial x^4} + 2 \frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = \frac{q}{D} , \qquad (2\text{-}13)$$

where q is the load per unit area.

This equation can also be written in the symbolic form

$$\Delta\Delta z = \frac{q}{D} ,$$

where

$$\Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} .$$

The basic differential equation of plates, Eq. (2-13), has no complete general solution. However, although numerical solutions are available for many specific problems, the finite difference method will be used here.

The finite difference operator used here is the central difference operator. Using this operator, the various derivatives may be expressed as follows:

$$\frac{\partial z}{\partial x} = \frac{z_{i+1} - z_{i-1}}{2\Delta x} ,$$

$$\frac{\partial z}{\partial y} = \frac{z_{j+1} - z_{j-1}}{2\Delta y} ,$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{z_{i+1} - 2z_i + z_{i-1}}{(\Delta x)^2} ,$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{z_{j+1} - 2z_j + z_{j-1}}{(\Delta y)^2} ,$$

$$\frac{\partial^3 z}{\partial x^3} = \frac{z_{i+2} - 2z_{i+1} + 2z_{i-1} - z_{i-2}}{2(\Delta x)^3} ,$$

$$\frac{\partial^3 z}{\partial y^3} = \frac{z_{j+2} - 2z_{j+1} + 2z_{j-1} - z_{j-2}}{2(\Delta y)^3} ,$$

$$\frac{\partial^4 z}{\partial x^4} = \frac{z_{i+2} - 4z_{i+1} + 6z_i - 4z_{i-1} + z_{i-2}}{(\Delta x)^4} ,$$

$$\frac{\partial^4 z}{\partial y^4} = \frac{z_{j+2} - 4z_{j+1} + 6z_j - 4z_{j-1} + z_{j-2}}{(\Delta y)^4} ,$$

$$\frac{\partial^4 z}{\partial^2 x \, \partial^2 y} = (z_{i+1,j+1} - 2z_{i+1,j} + z_{i+1,j-1} - 2z_{i,j+1} + 4z_{i,j}$$

$$- 2z_{i,j-1} + z_{i-1,j+1} - 2z_{i-1,j} + z_{i-1,j-1}) \frac{1}{(\Delta x)^2 (\Delta y)^2} .$$

Set $\Delta x = \Delta y = \Delta$, then Eq. (2-13) may be rewritten as

$$(z_{i+2,j} - 8z_{i+1,j} + 20z_{i,j} - 8z_{i-1,j} + z_{i-2,j} + z_{i,j+2}$$

$$- 8z_{i,j+1} - 8z_{i,j-1} + z_{i,j-2} + 2z_{i+1,j+1} + 2z_{i+1,j-1} + 2z_{i-1,j+1}$$

$$+ 2z_{i-1,j-1}) \frac{1}{\Delta^4} = \frac{q}{D} . \qquad (2-14)$$

This equation may best be expressed by the symbolic pattern as shown in Fig. 6.

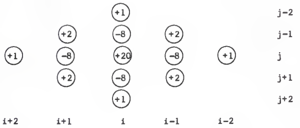


Fig. 6. Symbolic pattern of Eq. (2-14)

Concentrated loads may be handled very easily by this method. A concentrated load is turned into a uniformly distributed load by dividing it by Δ^2, if Δ^2 is greater than the area of application of the load. The resulting value of q is then used only at the net point where the concentrated load acts.

Example 1:

A rectangular plate, as shown in Fig. 7, is fixed against deflection and rotation along the sides y=0 and y=10' and is simply supported on rigid beams along x=0, x=15' and x=30'. The load is 80 psf on the left half of the plate and 40 psf on the right. Assumed thickness is 6", $E_c = 3 \times 10^6$ psi, poisson's ratio, μ, is zero. Find the deflections and moments of the plate.

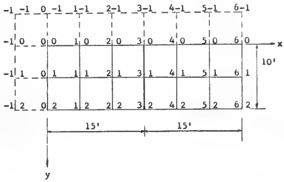


Fig. 7. Example 1.

Solution:

Set $\Delta x = \Delta y = \Delta = 5'$.

The boundary conditions may be written:

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_{x=0} = \frac{z_{1j} - 2z_{oj} + z_{-1j}}{\Delta^2} = 0 ,$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_{x=15} = \frac{z_{4j} - 2z_{3j} + z_{2j}}{\Delta^2} = 0 ,$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_{x=30} = \frac{z_{7j} - 2z_{6j} + z_{5j}}{\Delta^2} = 0 ,$$

$$\left(\frac{\partial z}{\partial y}\right)_{y=0} = \frac{z_{i1} - z_{i-1}}{2\Delta} = 0 ,$$

$$\left(\frac{\partial z}{\partial y}\right)_{y=10} = \frac{z_{i3} - z_{i1}}{2\Delta} = 0 ,$$

and $z_{oj} = z_{3j} = z_{6j} = z_{io} = z_{i2} = 0$.

Then $z_{-1j} = -z_{1j}$, $z_{2j} = -z_{4j}$,

$z_{5j} = -z_{7j}$, $z_{i1} = z_{i-1}$,

$z_{i1} = z_{i3}$.

For $i = 1, j = 1$, $21z_{11} - 8z_{21} = \Delta^4 \frac{q}{D}$.

For $i = 2, j = 1$, $-8z_{11} + 22z_{21} + z_{41} = \Delta^4 \frac{q}{D}$.

For $i = 4, j = 1$, $z_{21} + 22z_{41} - 8z_{51} = \Delta^4 \frac{q}{2D}$.

For $i = 5, j = 1$, $-8z_{41} + 21z_{51} = \Delta^4 \frac{q}{2D}$.

The solutions are

$z_{11} = 0.0754\, \Delta^4 \frac{q}{D}$, $z_{21} = 0.0704\, \Delta^4 \frac{q}{D}$,

$z_{41} = 0.0327\, \Delta^4 \frac{q}{D}$, $z_{51} = 0.0363\, \Delta^4 \frac{q}{D}$,

$\Delta = 5' = 60''$, $q = 80$ psf $= 0.55$ psi ,

$D = Eh^3/12(1 - \mu^2) = 3 \times 10^6 \times 6^3/12 = 54 \times 10^6$ #-"^{2}/" ,

then $z_{11} = 0.00994''$, $z_{21} = 0.00936''$,

$z_{41} = 0.00436''$, $z_{51} = 0.00484''$,

$$M_x = -D(\frac{\partial^2 z}{\partial x^2} + \mu \frac{\partial^2 z}{\partial y^2}) = -D\frac{\partial^2 z}{\partial x^2} = -D(\frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{\Delta^2}),$$

$$M_y = -D(\frac{z_{i,j+1} - 2z_{i,j} + z_{i,j-1}}{\Delta^2}) .$$

At point 1,1

$$M_{11x} = -D\ (\frac{z_{21} - 2z_{11} + z_{01}}{\Delta^2})$$

$$= -\Delta^2 q\ (0.0704 - 0.1490)$$

$$= 157.2^{\#'} ,$$

$$M_{11y} = -D\ (\frac{z_{12} - 2z_{11} + z_{10}}{\Delta^2})$$

$$= -\Delta^2 q\ (-0.1490)$$

$$= 298.0^{\#'} .$$

Similarly,

$M_{21x} = 132.6^{\#'}$, $M_{21y} = -298.0^{\#'}$,

$M_{31x} = -206.2^{\#'}$, $M_{31y} = 298.0^{\#'}$,

$M_{32x} = 0$, $M_{32y} = 281.6^{\#'}$.

LOAD BALANCING METHOD

The load balancing concept is an attempt to balance the loads on a member by prestressing. In the design of a prestressed concrete structure, the effect of prestressing is viewed as the balancing of gravity loads so that structures will not be subjected to flexural stresses under a given loading condition. This enables the transformation of a flexural member into a member under direct stress. The application of this concept requires taking the concrete as a free body, and replacing the tendons with forces acting on the concrete.

The following assumptions are made:

1. The curvature of the tendon is small and the tendons exert vertical forces only on the plate. In reality, the forces are perpendicular to the tendons, but the horizontal components can be neglected, as they are extremely small and balance out to zero over the length of the tendon.
2. The friction between the tendons and ducts is negligible for the purpose of analysis.

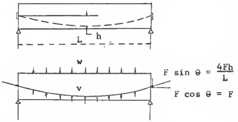


Fig. 8. Prestressed beam with parabolic tendon.

Figure 8 illustrates the balancing of a uniformly distributed load by means of a parabolic cable whose upward component v(lb/ft) is given by

$$v = \frac{8Fh}{L^2} .$$

If the externally applied load w (including the weight of the beam) is exactly balanced by the component v, there is no bending in the beam. The beam is under a uniform compression with stress

$$f = \frac{F}{A_c} ,$$

If the external load is different from w, it is only necessary to analyze the moment M produced by the load differential and compute the corresponding stresses by the formula

$$f = \frac{M_c C}{I} .$$

Example 2:

A prestressed-concrete rectangular beam 12 in. by 30 in. has a simple span of 32 ft. and is loaded by a uniform load of 2 k/ft including its own weight, Fig. 9. The prestressing tendon is located as shown and produces an effective prestress of 360k. Compute fiber stresses in the concrete at the midspan section.

Solution:

The upward uniform force from the tendon on the concrete is

$$w = \frac{8Fh}{L^2} = \frac{8 \times 360 \times 6/12}{32^2} = 1.4 \text{ k/ft} .$$

Hence the net downward load on the concrete beam is 2-1.4=0.6 k/ft, and the moment at midspan dua to that load is

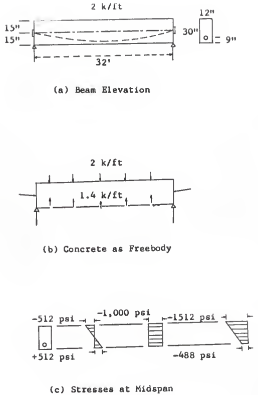


Fig. 9. Example 2.

$$M = \frac{wL^2}{8} = \frac{0.6 \times 32^2}{8} = 76.8 \text{ k-ft} .$$

The fiber stresses due to that moment are

$$f = \frac{Mc}{I} = \frac{6M}{bd^2} = \frac{6 \times 76.8 \times 1{,}000 \times 12}{12 \times 30^2} = \mp 512 \text{ psi} .$$

The fiber stress due to the direct load effect of the prestress is

$$\frac{F}{A} = \frac{-360{,}000}{12 \times 30} = -1{,}000 \text{ psi} .$$

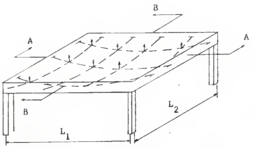
The resulting stresses are

$$-512 - 1{,}000 = -1{,}512 \text{ psi top fiber} ,$$

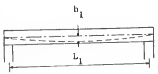
$$+512 - 1{,}000 = -488 \text{ psi bottom fiber} .$$

Two-dimensional load balancing differs from linear load balancing for beams in that the transverse component of the tendons in one direction either adds to or subtracts from that component in the other direction. Thus the prestress design in the two directions are closely related, one to the other. However, the basic principle of load balancing still holds, and the main aim of the design is to balance a given loading so that the entire structure will possess uniform stress distribution in each direction and will not have deflection or camber under this loading. Any deviation from this balanced loading will then be analyzed as loads acting on an elastic plate without further considering the transverse component of prestress.

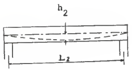
Let us consider a plate simply supported on four walls, Fig. 10. The cables in both directions exert an upward force on the plate, and if the sum of the upward components balances the downward load w, then we have a balanced design. Thus, if S_x and S_y are the prestressing forces in the two



(a) Isometric View of Slab and Supports



(b) Section A–A



(c) Section B–B

Fig. 10. Load balancing for two-way slabs.

directions per foot width of plate, we have

$$\frac{8S_x h_1}{L_1^2} + \frac{8S_y h_2}{L_2^2} = w .$$

Many combinations of S_x and S_y will satisfy the above equation. While the most economical design is to carry the load only in the short direction for the case of elongated rectangular plates, practical considerations might suggest different distributions. For example, if both directions are properly prestressed, it is possible to obtain a crack-free plate.

Under the action of S_x, S_y, and the load w, the entire plate has a uniform stress distribution in each direction equal to S_x/d and S_y/d, respectively. Any change in loading from the balanced amount of w can be analyzed by the conventional method discussed before.

Example 3:

An 8-in plate supported on four walls, Fig. 11, is to be post-tensioned in two directions. Design live load is 80 psf. Compute the amount of prestress, assuming that a minimum of 200 psi compression is desired in the concrete in each direction for the purpose of getting a watertight roof plate.

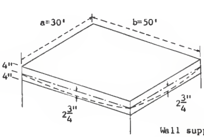


Fig. 11. Example 3.

Solution:

Since it is more economical to carry the load in the short direction, a minimum amount of prestress will be used in the long direction. At 200 psi compression in concrete, the prestress is,

$$200 \times 8 \times 12 \;=\; 19.2 \text{ k/ft of plate .}$$

This prestress will supply an upward force:

$$w \;=\; \frac{8\,Se}{L^2} \;=\; \frac{8 \times 19{,}200 \times 2.75}{50 \times 50 \times 12} \;=\; 16.8 \text{ psf.}$$

Since the weight of the plate is 100 psf. it will be necessary to supply another upward force of 100 - 16.8 = 83.2 psf. in order to balance the dead load. This will require a prestress in the 30-ft direction of

$$F \;=\; \frac{wL^2}{8e} \;=\; \frac{83.2 \times 30^2}{8 \times 2.75/12} \;=\; 40.9 \text{ k/ft ,}$$

which will give a uniform compression in the concrete of

$$\frac{40.9}{8 \times 12} \;=\; 426 \text{ psi .}$$

Thus, under the action of dead load alone, the plate will be under uniform stress of 200 psi in the 50-ft direction and 426 psi in the 30-ft direction.

The effect of live load can now be investigated. Referring to Timoshenko's treatise,[1] for span ratio 50/30=1.67, we have B=0.0894 B_1=0.0490. Thus, for w=80 psf. and a=30 ft,

$$(M_x)_{max} \;=\; Bwa^2 = 0.0894 \times 80 \times 30^2 = 6{,}430 \text{ ft-lb/ft of plate .}$$

---

[1] S. Timoshenko, Theory of Plates and Shells, McGraw-Hill, 1940, p. 133.

Hence, the concrete fiber stresses are

$$f = \frac{Se}{A} \pm \frac{Mc}{I}$$

$$= -426 \pm \frac{6,430 \times 12 \times 6}{12 \times 8^2} = -426 \pm 604 = -1,030 \text{ psi} \quad \text{top fiber}$$

$$+ 202 \text{ psi} \quad \text{bottom fiber} .$$

Stresses in the 50-ft direction can be similarly computed to be -530 psi for top fiber and +130 psi for bottom fiber.

The problem which remains to be considered is that of cable layouts.

In Fig. 12, let AB represent the deflection curve of a string stretched by forces and uniformly loaded with a vertical load of intensity P_x. In deriving the equation of this curve we consider the equilibrium of an infinitesimal element mn. The tensile forces at pointa m and n have the directions of tangents to the deflection curve at these points; and, by projecting these forces and also the load $P_x dx$ on the z axis, we obtain

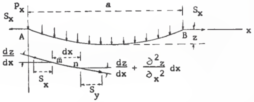


Fig. 12. A stretched string.

$$- S_x \frac{dz}{dx} + S_x \left(\frac{dz}{dx} + \frac{\partial^2 z}{\partial x^2} dx\right) + p_x dx = 0 ,$$

from which

$$\frac{\partial^2 z}{\partial x^2} = -\frac{p_x}{S_x} .$$

Similarly

$$\frac{\partial^2 z}{\partial y^2} = -\frac{p_y}{S_y} ,$$

and
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\left(\frac{p_x}{S_x} + \frac{p_y}{S_y}\right) ,$$

or

$$S_x \frac{\partial^2 z}{\partial x^2} + S_y \frac{\partial^2 z}{\partial y^2} = -(p_x + p_y) = -q .$$

If
$$S_x = S_y = S ,$$

then

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{q}{S} .$$

As compared with the differential equation (Eq. 2-13) derived for the plate, it can be seen that the degree of the differential equation has been reduced from four to two and the tendon ordinates are those of a funicular polygon for that portion of the external loading which is carried by the particular tendon.

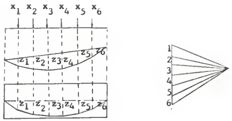


Fig. 13. Initial tendon curve.

As shown in Fig. 13, determine the funicular polygon for these forces, $x_1, \ldots, x_6$, for each tendon. Assuming unit force in the tendons at first, the funicular polygon will be identical to a bending moment diagram. This polygon will then also be the initial tendon curve.

OPTIMUM DESIGN METHODS

As mentioned in the load balancing method, for two-way prestressed plates a designer has an infinite choice of distribution of loading between the two directions of prestressing. One of the following criteria may be used to find an economical solution:

1. Assuming a depth of plate, the volume of the tendons should be a minimum.
2. The depth, D, of the plate should be a minimum.

Using the first criterion, we find that the distribution of loading is most economical when ΣSL is a minimum. S is the tendon force and L is the tendon length. Since we assumed a fixed depth, the final tendon force can be expressed as a function of the maximum tendon ordinate, z_{max}, in the initial tendon curve. Using a linear transformation, z_{max} can be reduced to the effective tendon depth d, otherwise tha tendon will not fit in the plate. Hence, all ordinates of the tendon curve are multiplied by d/z_{max} and the tendon force is multiplied by the reciprocal value, z_{max}/d. As we assumed unit force in the initial tendon, the final tendon force will be

$$S = z_{max}/d ,$$

so,

$$d\Sigma SL = d\Sigma L z_{max}/d = \Sigma z_{max} L .$$

When this expression is a minimum, the quantity of tendons is a minimum.

Using the second criterion, a load distribution has to be found where the largest z_{max} in the plate is a minimum. From the concept of linear transformation, this will also mean that S_{max} (the largest tendon force) is a minimum. The plate can be reduced in thickness provided the tendon forces are increased by the same ratio. It is evident that when S_{max} is a minimum, the tendon forces are evenly distributed in both directions, and these stresses will be less than in cases where the plate is stressed mainly in one direction, or where there is a much greater force in one tendon than in others.

From the above discussion, it can be seen that the two criteria for economy are:

(1) $\Sigma z_{max} L$ is a minimum.

(2) The largest z_{max} is a minimum.

The degree of economy of a particular tendon arrangement can be indicated by the following indices:

$$E_1 = \Sigma z_{max} L/(\Sigma z_{max} L)_{min}, \quad E_2 = z_{max}/(z_{max})_{min}.$$

Minimum Tendon Method

Before proceeding with the optimum design of prestressed plates, it is necessary to introduce the concepts of "panels" and "neutral lines."

Panels.

Type 1 panel (Fig. 14 a)--Type 1 panels are defined by using the following limitations:

1. The panel is a simply connected region.
2. The panel is symmetrical about not less than two axes.
3. The loading, q, is uniformly distributed.
4. All edges are either simply supported or continuous.

Type 2 panels (Fig. 16)--In this case, the following criteria differ from those of type 1 panels: (a) The panel has at least one axis of

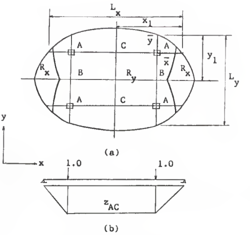


Fig. 14. Typical type 1 panel.

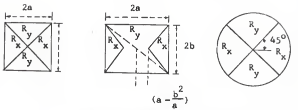


Fig. 15. Neutral lines of type 1 panels.

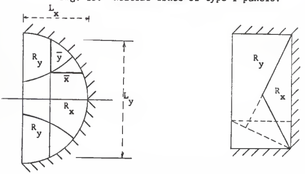


Fig. 16. Type 2 panels and neutral lines.

symmetry. (b) One edge of panel is unsupported and straight. This edge is parallel to the y-tendons. (c) Other edges are either fixed or continuous.

Type 3 panels (Fig. 17)--This type is defined by the following criteria: (a) Two edges are straight and unsupported. These are parallel to the x-tendons and y-tendons respectively. (b) Other edges are either fixed or continuous.

Neutral Lines.

For type 1 panel--The tendon volume of a type 1 prestressed plate is a minimum if the plate is divided into regions in which the full load is carried by tendons running in the x-direction, and in the remaining regions the full load is carried by the tendons running in the y-direction. The boundaries between these regions are called "Neutral lines" because along them the load distribution does not affect the tendon volume.

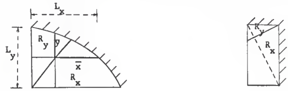


Fig. 17. Type 3 panels and neutral lines.

From Fig. 14 a, it will be proved that the neutral lines can be found from

$$\bar{x} L_x = \bar{y} L_y$$

where $\bar{x}$, $\bar{y}$ are the distances from the support to the neutral lines in x and y directions.

Areas where $\bar{x}\, L_x < \bar{y}\, L_y$ and $\bar{x}\, L_x > \bar{y}\, L_y$ will be referred to as x-region and y-region and will be denoted by R_x and R_y, respectively.

The following derivation shows that the minimum tendon volume is obtained if all of the load in the x-region or y-region is carried by tendons running in the x-direction or y-direction. In other words the tendon volume is a minimum, if in the x-region, load carried by x-tendon $P_x=q$ and load carried by y-tendon $P_y=0$; and in the y-region $P_x=0$, and $P_y=q$.

Assume a $\delta(x,y)$ that is (a) symmetrical about the axes of symmetry of the panel and (b) varies between 0 and q. If we vary P_x by $\delta(x,y)$ in the y-region, then the tendon loadings are

$$P_x = \delta ,$$

and $$P_y = q - \delta .$$

Because both loadings are positive (or zero) and symmetrical about the x-axis and y-axis, it follows that the maximum tendon ordinates occur along the same axes.

Considering an arbitrary set of symmetrical elements in the y-region, the criterion of the volume of tendons is

$$\Sigma\, L\, z_{max} .$$

Considering the symmetrical crossing points A of tendons ACA and ABA (Fig. 14 a), the load distribution at these points affects two maximum values of z only—at points B and C. A unit force at A on Tendon ABA will cause a deflection of z_{AB} in tendon ABA at point B, and similarly, a unit force at points A on tendon ACA will cause a deflection of z_{AC} in tendon ACA at point C.

From p. 28, we know that the tendon ordinates are those of a funicular polygon for the portion of the external loading which is carried by the particular tendon, and the funicular polygon is identical to the bending moment diagram if the force in the tendon is unity.

Then

$$\text{Momant at } A = 1 \times \bar{x} = 1 \times z_{AC} ,$$

and $$z_{max} = z_{AC} = \bar{x} .$$

Similarly, z_{AB} is equal to $\bar{y}$ the distance of the point A from the support (in the direction of tendon ABA, Fig. 14 a). Now if a q dx dy load is placed at point A, let δ dx dy be carried by tendon ACA and let $(q-\delta)$dx dy ba carried by tendon ACA. Then for this loading

$$z_{max} L = \delta \, dx \, dy \, \bar{x} \, L_x \qquad \text{for tendon ACA} ,$$

$$z_{max} L = (q-\delta) \, dx \, dy \, \bar{y} \, L_y \qquad \text{for tendon ABA} .$$

Then for the four tendons,

$$\theta = \Sigma z_{max} L = 2 \delta \bar{x} L_x \, dx \, dy + 2(q - \delta) \bar{y} L_y \, dx \, dy$$

$$= 2 q \bar{y} L_y \, dx \, dy + 2 \delta (\bar{x} L_x - \bar{y} L_y) \, dx \, dy .$$

So, the variation of θ is

$$\delta dx \, dy \, (2\bar{x} L_x - 2\bar{y} L_y) ,$$

if $\bar{x}$ and $\bar{y}$ are the distances of the elements from the boundary.

Hence, the variation for the whole region is

$$\Delta\theta = \int_{Ry}\int 2\ \delta(\bar{x}\ L_x - \bar{y}\ L_y)\ dx\ dy\ ,$$

in which $\delta \geq 0$, and $\bar{x}\ L_x - \bar{y}\ L_y > 0$ in the y-region. Consequently, $\Delta\theta \geq 0$ and the value of θ is not a minimum. In other words, a variation of $0 \leq \delta \leq q$ of P_x would result in a greater tendon than the optimum solution, provided that δ has at least one value that is different from zero.

From this conclusion, it is known that the neutral lines can be found from

$$\bar{x}\ L_x - \bar{y}\ L_y = 0\ ,$$

$$\bar{x}\ L_x = \bar{y}\ L_y\ . \tag{4-1}$$

Assume that the coordinate axes are the axes of symmetry and that x_1 and y_1 are the coordinates of the boundary points. Then, it follows from Eq. (4-1) that

$$2y_1(y_1 - y) = 2x_1(x_1 - x)\ ,$$

and

$$y_1^2 - yy_1 = x_1^2 - xx_1\ . \tag{4-2}$$

Because y_1 and x_1 can always be represented as a function of x and y respectively, the equation of the neutral lines can readily be obtained from Eq. (4-2).

The neutral lines of some panels are given in Fig. 15.

For type 2 panel--The neutral lines of these panels can be found from Fig. 16 by the equation

$$\bar{x}\, L_x = \frac{\bar{y}\, L_y}{4} .$$

$\bar{y}\, L_y$ is divided by 4 for the following reasons:

(1) In a symmetrical set of tendons there are twice as many x-tendons as y-tendons.

(2) The effective tendon depth of the y-tendons is twice that of the x-tendons.

For type 3 panel--The neutral lines of these panels can be found from Fig. 17 by the equation

$$\bar{x}\, L_x = \bar{y}\, L_y .$$

Example 4:

A 30' by 40' rectangular plate (Fig. 18 a), the effective tendon depth is given, d=3". The uniformly distributed load over the plate is q=100 lbs per sq. ft. including its own weight. It is required to obtain a minimum tendon design of the plate.

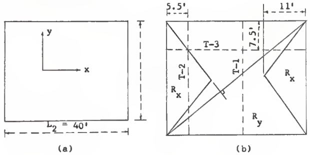


Fig. 18. Example 4.

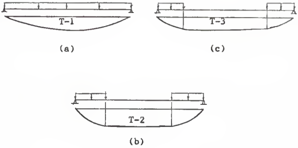


Fig. 19. Tendon curves.

Solution:

The neutral lines are shown in Fig. 18 b. In R_y, the load will be carried in the y direction, while in R_x, the load will be transferred in the x direction.

Here we will discuss the design of three tendons only, as the design of the other tendons is similar.

Tendon T-1 (Fig. 19 a)--Initial tendon curves for the central 18 ft of R_y in the y direction are parabolic, and assume one tendon per foot.

$$z_{max} = 100 \times 30^2/8 = 11250' .$$

By linear transformation

$$d = 3'' ,$$

$$S = \frac{11250 \times 12}{3} = 45{,}000^{\#} = 45 \text{ kips} .$$

Use 7 a strand tendon composed of 0.276" diameter wires

Nominal steel area = 0.06 sq. in ,

Maximum design load = 8.4 kips ,

Effective stress $= \frac{8.4}{0.06} = 140$ kips per sq. in ,

Effective prestressing force = 140 x 7 x 0.06 = 59 kips ,

Spacing of the cables $= \frac{59 \times 12}{45} = 15.7''$.

Tendon T-2--The initial tendon curves for two 11 ft wide end strips in the y direction will have a straight section at the middle. The cable at 5.5 ft from the edge, the loading and the initial tendon curve is shown in Fig. 19 b.

$$z_{max} = 100 \times 7.5^2/2 = 2{,}810' ,$$

$$S = \frac{2810 \times 12}{3} = 11{,}240^{\#} = 11.24 \text{ kips} ,$$

Spacing of the cables $= \frac{59 \times 12}{11.24} = 63''$.

Tendon T-3--The initial tendon curves in the x direction will be of similar shape to that of tendon T-2. The cable at 7.5 ft from the edge is shown in Fig. 19 c.

$$z_{max} = 100 \times 5.5^2/2 = 1{,}512' ,$$

$$S = \frac{1512 \times 12}{3} = 6{,}048^{\#} = 6.05 \text{ kips} ,$$

Spacing of the cables $= \frac{59 \times 12}{6.05} = 116''$.

The shape of the final tendon curves is the same as that of the initial tendon curves, but the ordinates are scaled down to provide a maximum ordinate of 3 in. for each tendon.

Assuming 5" total concrete depth, the maximum concrete stress R_y (neglecting the effect of the compression in the other direction) will be

$$45/12x5 = 0.75 \text{ kips per sq. in.}$$

Transformed Membrane Method.

In this method, it is assumed that at any tendon crossing, the initial tendon ordinates are identical for the two intersecting tendons

$$z_x(x, y) = z_y(x, y)$$

where $z_x(x, y)$ and $z_y(x, y)$ are the deflections of the tendons in the x and y directions, respectively.

The following differential equation previously developed for plates,

$$\frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = \frac{q}{D},$$

can be written in the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = \frac{q}{D}.$$

The following moment expressions have also been previously derived:

$$M_x = -D\left(\frac{\partial^2 z}{\partial x^2} + \mu\frac{\partial^2 z}{\partial y^2}\right),$$

$$M_y = -D\left(\frac{\partial^2 z}{\partial y^2} + \mu\frac{\partial^2 z}{\partial x^2}\right),$$

$$M_x + M_y = -D(1 + \mu)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right).$$

Introducing a new notation,

$$M = \frac{M_x + M_y}{1 + \mu} = -D\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right),$$

then

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = -q,$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{M}{D}.$$

These equations are of the same type as those obtained for a uniformly stretched and laterally loaded membrane.

Next we consider a mesh of tendons and denote the external loading at point (x, y) by q(x, y), then the loading carried by tendons in the x and y directions will be $p_x(x, y)$ and $p_y(x, y) = q(x, y) - p_x(x, y)$, respectively. We can consider the total load q(x, y) as taken by the tendons in the y direction if we put an internal redundant force $p_x(x, y)$ at the crossing points, acting upwards on the y tendons and downwards on the x tendons.

Castigliano's second theorem states that the strain energy is a minimum when the relative displacement in the direction of the redundant forces is zero. The strain energy is equal to the external work,

$$W_E = \int\int_R [q(x, y) - p_x(x, y)]\, z_y(x, y) + p_x(x, y)\, z_x(x, y)\, d_x\, d_y .$$

The external work is a minimum when

$$\frac{\partial W_E}{\partial p_x(x, y)} = \text{displacement} = 0,$$

for any value of x and y.

Hence, this method, where the relative displacement at the crossing points is zero, provides a tendon arrangement where the sum of the products of the tendon deflections and the distributed loads in each tendon direction at the crossing points is a minimum. This is not identical to Criterion 1 ($\sum Z_{max}$ L is minimum) but the difference is usually not great.

If the tendon spacing approaches zero, the tendon system becomes a uniformly stretched membrane, provided that the horizontal component of the tendon force is the same in each tendon. For this case, it is known from p. 27 that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{q}{S}, \qquad (4\text{-}2)$$

or $\Delta z = \frac{q}{S}$.

This equation can be solved by the finite difference method (Fig. 20).

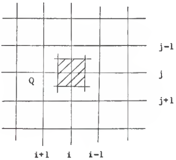


Fig. 20. Tendon grid in transformed membrane method.

$$\frac{\partial^2 z}{\partial x^2} = \frac{z_{i+1} - 2z_i + z_{i-1}}{(\Delta x)^2} ,$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{z_{j+1} - 2z_j + z_{j-1}}{(\Delta y)^2} .$$

Setting $\Delta x = \Delta y = \Delta$, Eq. (4-2) may be rewritten as

$$(z_{i+1,j} - 2z_{i,j} + z_{i-1,j} + z_{i,j+1} - 2z_{i,j} + z_{i,j-1}) \frac{1}{\Delta^2} = - \frac{q}{S} ,$$

$$4z_{i,j} - (z_{i+1,j} + z_{i-1,j} + z_{i,j+1} + z_{i,j-1}) = \frac{Q}{S} , \qquad (4\text{-}3)$$

where Q = total loading at the point (i,j).

This may be expressed by the symbolic pattern as shown in Fig. 21.

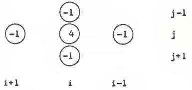


Fig. 21. Symbolic pattern of Eq. (4-3).

If the tendon forces are different in the x and y directions, then Eq. (4-2') becomes

$$S_x \frac{\partial^2 z}{\partial x^2} + S_y \frac{\partial^2 z}{\partial y^2} = - q .$$

This can also be written as

$$S_x\left[2z_{i,j}-(z_{i+1,j}+z_{i-1,j})\right]+S_y\left[2z_{i,j}-(z_{i,j+1}+z_{i,j-1})\right]=Q\ .$$

The initial tendon curves can, then, be determined using these equations, and the following design procedure is the same as for the load balancing method.

Example 5:

The layout of the plate is indicated in Fig. 22 b. It is assumed that there is no deflection in the column capitals which are represented by shaded squares on the diagram. The loading is 100 psf, and the effective tendon depth of the plate is 6". Obtain all initial tendon ordinates and the necessary tendon forces.

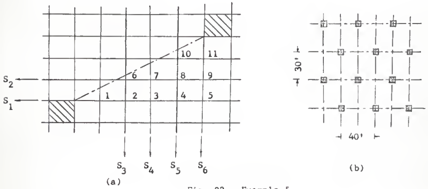


Fig. 22. Example 5.

Solution:

Figure 22 a shows a typical panel with equivalent tendons at 5' centers. Assume unit forces at first in the tendons. There are eleven typical crossing points.

From Eq. (4-3) which is

$$4z_{i,j} - (z_{i+1,j} + z_{i-1,j} + z_{i,j+1} + z_{i,j-1}) = \frac{Q}{S},$$

we can obtain eleven simultaneous equations.

For example, for point 1,

$$4z_1 - (z_1 + z_2 + z_{10} + 0) = \frac{Q}{S},$$

where z_1 and z_{10} are in another panel on Fig. 22 a, but can be determined from symmetry. To simplify the equations, we assume at first that the spacing of the equivalent tendons is 1 ft (instead of 5 ft) and the intensity of the loading is also unity. Then $\frac{Q}{S} = 1$, and the equation for point 1 is

$$3z_1 - z_2 - z_{10} = 1 .$$

In a similar manner, we can obtain all eleven equations and express them in matrix form as:

point	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	Q/S
1	+3	-1								-1		+1
2	-1	+3	-1			-1						+1
3		-1	+3	-1			-1					+1
4			-1	+3	-1			-1				+1
5				-1	+2				-1			+1
6		-1				+4	-2			-1		+1
7			-1			-2	+4	-1				+1
8				-1			-1	+4	-1	-1		+1

point	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	Q/S
9					-1			-1	+3		-1	+1
10	-1					-1		-1		+4	-1	+1
11									-1	-1	+3	+1

By computer,[2] the solution of these equations gives:

$$z_1 = 5.4863 , \quad z_5 = 9.3048 , \quad z_9 = 8.1911 ,$$

$$z_2 = 8.1090 , \quad z_6 = 8.6448 , \quad z_{10} = 7.3499 ,$$

$$z_3 = 9.1959 , \quad z_7 = 9.0601 , \quad z_{11} = 5.5137 .$$

$$z_4 = 9.4185 , \quad z_8 = 8.7549 ,$$

Q was assumed to be 1, but it is actually $5^2 \times 0.1 = 2.5$. Consequently, all initial ordinates have to be multiplied by 2.5.

$z_{max.}$: the difference between the smallest and greatest tendon ordinates of a tendon.

For Tendon 1.

$$z_{max.} = 2.5(z_4 - 0) = 2.5 \times 9.4185 = 23.546 .$$

For Tendon 2.

$$z_{max.} = 2.5(z_7 - z_{11}) = 2.5 \times 2.5464 = 6.366 .$$

[2]Refer to Appendix II.

For Tendon 3.

$$z_{max.} = 2.5(z_3 - z_2) = 2.5 \times 1.0869 = 2.717 .$$

For Tendon 4.

$$z_{max.} = 2.5(z_3 - z_2) = 2.5 \times 1.0869 = 2.717 .$$

For Tendon 5.

$$z_{max.} = 2.5(z_4 - z_1) = 2.5 \times 3.9322 = 9.831 .$$

For Tendon 6.

$$z_{max.} = 2.5(z_5 - 0) = 2.5 \times 9.3048 = 23.262 .$$

The effective tendon depth is 6". The necessary tendon force will be $z_{max.}/0.5$.

S_1 = 23.546/0.5 = 47.092 kips per ft.

S_2 = 6.366/0.5 = 12.732 kips per ft.

S_3 = 2.717/0.5 = 5.434 kips per ft.

S_4 = 2.717/0.5 = 5.434 kips per ft.

S_5 = 9.831/0.5 = 19.662 kips per ft.

S_6 = 23.262/0.5 = 46.524 kips per ft.

CONCLUSION

The methods presented in this paper enable the designer to quickly select economical solutions for the design of prestressed plates.

Being based on the load balancing method, the methods presented have the following advantages:

a. There are no compatibility equations to satisfy along the continuous edges, as the deformation throughout the structure is zero.

b. Each member can be analyzed separately and continuity provided merely by matching the eccentricity of the tendons on either side of a continuous edge.

c. For the particular design load, the bending moment, shear, and deflection are zero in all sections of the structure, thus avoiding the effects of deflection and creep.

The transformed membrane method is applicable to any loading and boundary condition and is suitable for use in an electronic digital computer solution. With the minimum tendon method, for each type of boundary condition, a different approach is required, and therefore, it is not applicable for complicated cases. For the above reasons, the author favors the transformed membrane method for most cases.

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APPENDIX I -- NOTATION

σx, σy, σn normal components of stress parallel to x, y and n directions.

τxy, τxz, τyz shearing stress components in Cartesian coordinates respectively.

τnt shearing stress components in t direction.

A_c area of concrete.

b width of member.

D flexural rigidity of a plate.

d effective depth of member.

ϵx, ϵy unit elongations in x, y directions respectively.

E modulus of elasticity of the material.

f stress in outermost fiber.

F total force in concrete.

h full thickness of the plate.

I moment of inertia.

L tendon length.

M_x, M_y, M_n bending moments per unit length of plate perpendicular to x, y and n directions respectively.

M_{xy}, M_{nt} twisting moments per unit length of a plate perpendicular to the x and n directions respectively.

P_x, P_y intensities of load in the x, y directions respectively.

q load per unit area.

Q_x, Q_y shearing forces parallel to the z axis per unit length of a plate perpendicular to the x and y axes respectively.

r_x, r_y, r_n	radii of curvature in the x, y and n directions respectively.
R_x, R_y	x- and y-regions respectively.
S	uniform tendon force per unit width.
S_x, S_y	tendon forces in the x and y directions respectively.
u	Poisson's ratio.
x, y, z	cartesian coordinates.
z	deflections, tendon ordinates.
θ	Z_{max} *L for four tendons.

APPENDIX II -- COMPUTER PROGRAM FOR SOLVING SIMULTANEOUS EQUATIONS

```
C  C  SOLUTION OF N LINEAR EQUATIONS OF N UNKNOWNS BY MATRIX INVERSION
      DIMENSION A (24, 25)
   1  FORMAT (E11.5)
   2  FORMAT (I2)
      READ 2, N
      N1=N+1
      DO 4 J=I, N1
      DO 4 I=1, N
      READ 1,A(I,J)
   4  CONTINUE
      DO 8 I=I,N
      DEL = A(I,I)
      DO 5 J=I, NI
   5  A(I,J) = A(I,J)/DEL
      DO 8 K=1, N
      IF(K-I)6,8,6
   6  ELS=A(K,I)
      DO 7 J=1, NI
   7  A(K,J)=A(K,J)-ELS*A(I,J)
   8  CONTINUE
      DO 9 I=1, N
   9  PUNCH 1,A(I,NI)
      STOP
      END
```

```
C  C  SOLUTION OF N LINEAR EQUATIONS OF N UNKNOWNS BY MATRIX INVERSION
  .5486E+01
  .81090E+01
  .91959E+01
  .94185E+01
  .93048E+01
  .86448E+01
  .90601E+01
  .87549E+01
  .81911E+01
  .73499E+01
  .55137E+01
      STOP END OF PROGRAM AT STATEMENT 0009 + 01 LINES
```

APPENDIX III -- BIBLIOGRAPHY

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OPTIMUM DESIGNS OF PRESTRESSED CONCRETE PLATES

by

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B. S., National Taiwan University, 1960

AN ABSTRACT OF A MASTER'S REPORT

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1967

The purpose of this paper was to present three methods for the analysis of nonprestressed and prestressed concrete plates:

1. Conventional Method: the analysis is predicated upon perfectly elastic homogenous materials for nonprestressed plates.
2. Load balancing method for prestressed plate design.
3. Optimum design methods for prestressed plate design.
   (a) Minimum tendon method.
   (b) Transformed membrane method.

The number of conventional analyses available for plates is small compared with the many possible and practical shapes which could be used in construction. In many practical designs, the formulation of the boundary conditions and the solutions of the differential equations are impossible, and even tha approximations which may be made often involve formidable numerical work. The method of finite differences offers one tool of practical importance in plate design.

As a first step, it is necessary to consider the geometry involved when a plate deforms; i.e., becomes curved or warped, under load.

The next step is to relate the curvature derived above to bending moments and thus to stresses.

The third step is to derive the equations of equilibrium of a segment of plate small in horizontal dimensions, but with the full thickness of the plate. Thus the differential equation of the plate is:

$$\frac{\partial^4 z}{\partial x^4} + z\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = \frac{q}{D}$$

where

x, y, z = Cartesian coordinates.

q = The load per unit area.

D = $Eh^3/12\ (1-u^2)$, flexural rigidity of plate.

and

h = Full thickness of the plate.

u = Poisson's ratio.

E = Modulus of elasticity of the material.

The last step is to solve the differential equation by the finite difference method.

The "balanced-load-concept" sees prestressing in the concrete as primarily an attempt to balance a portion of the load on the structure. Two-dimensional load balancing differs from linear load balancing for beams and columns in that the transverse component of the tendons in one direction either adds to or subtracts from the component in the other direction. However, the basic principle of load balancing still holds and the main aim of the design is to balance a given loading so that the entire structure will possess uniform stress distribution in each direction and will not have deflection or camber under this loading. Any deviation from this balanced loading will then be analyzed as loads acting on an elastic plate without further considering the transverse component of prestressing.

For a prestressed plate, we have an infinite choice of distribution of loading between the two directions of prestressing. One of the following criteria may be used to find an economical solution.

1. Assuming a depth, minimize the volume of the tendons.
2. Minimize the depth of the slab.

Two economical methods are discussed:

1. Minimum tendon method

   (a) Minimum volume: In this case, the prestressed method provides minimum tendon volume.

   (b) Minimum force: This method requires a minimum tendon force.

2. Transformed membrane method

If the tendon spacing approaches zero the tendon system becomes a uniformly stretched membrane. For this case:

$$4z_a - \sum_{b}^{a} z = \frac{Q_a}{S}$$

where

S = Uniform tendon force per unit width.

z_a = The initial tendon ordinate at the crossing point a.

a = Load at point a.

z = The initial tendon ordinate.

Once the ordinates of the initial tendon curves are calculated from this equation, the design procedure is the same as for the balanced load method.

Examples are given to illustrate each method.