A METHOD FOR DETERMINING UPPER AND LOWER BOUNDS FOR TWO-DIMENSIONAL HEAT CONDUCTION PROBLEMS

by

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NOMENCLATURE

$t(x,y)$	temperature
$T(X,Y)$	transformed temperature
$t_U(x,y)$, $T_U(X,Y)$	upper bounding temperature
$t_L(x,y)$, $T_L(X,Y)$	lower bounding temperature
$t_m(x,y)$, $T_m(X,Y)$	mean value of upper and lower bounding temperature
z	complex variable
$R(f(z))$	real part of $f(z)$
$I(f(z))$	imaginary part of $f(z)$
q	internal heat generation per unit volume
k	thermal conductivity
δ	parameter
Er	percentage error relative to the maximum temperature
Er^*	percentage error relative to the local temperature
D	length of plate
B	height of plate
H	dimensionless height of plate
x,y	coordinates of plate
X,Y	dimensionless coordinates of plate
i,j,k,m,n	positive integers

INTRODUCTION

With the development of high-speed digital computers, we have been able to obtain approximate solutions for many boundary value problems for which the exact solutions are either impossible or too cumbersome. Unfortunately, when it is desired to determine the error associated with an approximate solution, the error analysis is frequently even more complicated than the solution itself. In this work, an attempt has been made to obtain exact solutions to modified problems such that these solutions provide upper and lower bounding functions for the original problems.

In general we seek solutions which will satisfy the differential equations and the boundary conditions. There are two ways to modify the problems i.e. either the differential equations or the boundary conditions are modified. In the present case we find an infinite series, every term of which satisfies the differential equations, and try to form the upper and lower bounds by certain techniques.

To demonstrate the technique, we will solve two two-dimensional, steady state, heat conduction problems.

SOLUTION OF A TWO-DIMENSIONAL HEAT CONDUCTION PROBLEM

Definition of the Problem

The differential equation for the temperature t in two-dimensional, steady state heat conduction, without internal heat generation, is shown by Schneider [2] to be

[ ] Numbers in brackets designate References at end of report.

$$\nabla^2 t = 0 \tag{1}$$

where ∇^2 indicates the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

The boundary conditions for a two-dimensional plate (see Fig. 1) are assumed in the following form.

1. $t(0,y) = g_1(y)$
2. $t(D,y) = g_2(y)$
3. $t(x,0) = f_1(x)$
4. $t(x,B) = f_2(x)$

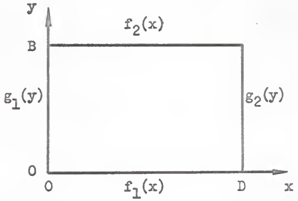

Fig. 1. Two-dimensional plate.

The problem is to determine the t which satisfies equation (1) and the prescribed boundary conditions.

Variational Procedure for Upper and Lower Bounds

Consider a complex variable

$z = x + iy$

and the polynomial

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots\cdots\cdots + a_n z^n + \cdots\cdots\cdots \tag{2}$$

Expanding each term into real and imaginary parts, we have

$$z = x + iy$$

$$z^2 = (x + iy)^2 = x^2 + 2ixy - y^2$$

$$R(z^2) = x^2 - y^2$$

$$I(z^2) = 2xyi$$

$$z^3 = (x + iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$$

$$R(z^3) = x^3 - 3xy^2$$

$$I(z^3) = (3x^2y - y^3)\, i$$

etc.

and the general term is

$$z^n = (x + iy)^n = x^n + nx^{n-1}iy + \binom{n}{2} x^{n-2}(iy)^2 + \binom{n}{3} x^{n-3}(iy)^3$$

$$+ \binom{n}{4} x^{n-4}(iy)^4 + \cdots\cdots + \binom{n}{k} x^{n-k}(iy)^k + \cdots\cdots + (iy)^n$$

$$R(z^n) = x^n - \binom{n}{2} x^{n-2}y^2 + \binom{n}{4} x^{n-4}y^4 - \cdots\cdots$$

$$\cdots + (-1)^{k-1} \binom{n}{2(k-1)} x^{n-2(k-1)}y^{2(k-1)} + \cdots\cdots \qquad (3)$$

$$I(z^n) = i\left[nx^{n-1}y - \binom{n}{3} x^{n-3}y^3 + \binom{n}{5} x^{n-5}y^5 - \cdots\cdots\right.$$

$$\left.\cdots + (-1)^{k-1} \binom{n}{2k-1} x^{n-2k+1}y^{2k-1} + \cdots\cdots\right] \qquad (4)$$

Utilizing these functions, an infinite series $f(x,y)$ is defined as

$$f(x,y) = b_0 + b_1x + b_2y + b_3(x^2-y^2) + b_4(2xy) + \cdots\cdots$$

$$\cdots + b_{2n-1}\left[x^n - \binom{n}{2} x^{n-2}y^2 + \binom{n}{4} x^{n-4}y^4 - \cdots\cdots + (-1)^{k-1}\cdot\right.$$

$$\left.\binom{n}{2(k-1)} x^{n-2(k-1)}\, y^{2(k-1)} + \cdots\cdots\right] + b_{2n}\left[nx^{n-1}y - \binom{n}{3} x^{n-3}y^3\right.$$

$$\left.+ \binom{n}{5} x^{n-5}y^5 - \cdots\cdots + (-1)^{k-1}\binom{n}{2k-1}x^{n-2k+1}y^{2k-1} + \cdots\right] + \cdots \qquad (5)$$

Now since each term of (3) and (4) satisfies equation (1), $f(x,y)$ is an infinite series every term of which satisfies the differential equation (1), see appendix A for the proof. We are going to choose a finite number of terms from (5) as an approximate solution. The coefficients will be determined such that the approximate temperature at every point on the boundary is in one case higher than the exact temperature. Then the approximate temperature will be higher than the exact temperature at every point in the region, see appendix B for the proof. This provides an upper bounding solution. Similarly, we get a lower bounding solution by choosing a new set of coefficients such that the approximate temperature at every point on the boundary is lower than the exact temperature.

The exact solution is somewhere between the upper and the lower bounds. If the mean value of the bounds is taken as the approximate solution, the associated maximum possible error is determined by half of the difference between the bounds. In order to obtain a good solution, an effort is made to find a set of coefficients such that the bounds are as close to each other as possible. The bounds thus obtained are defined as the least upper bound and the greatest lower bound, expressed as $t_U(x,y)$ and $t_L(x,y)$ respectively. The mean value is

$$t_m = \frac{t_U + t_L}{2}$$

The maximum possible error relative to the maximum temperature in the region will be

$$Er = \left(\frac{t_m - t_L}{t_{max}} \right) (100) \%$$

and that relative to the local temperature will be

$$Er^* = \left(\frac{t_m - t_L}{t_L} \right) (100) \, \%$$

EXAMPLE PROBLEM 1

Consider a two-dimensional problem without internal heat generation as shown in Fig. 2.

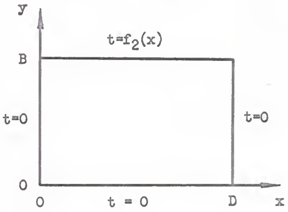

Fig. 2. Two-dimensional plate.

For this problem the general heat conduction equation is equation (1) and the boundary conditions are assumed to be

1. $t(0,y) = 0$
2. $t(D,y) = 0$
3. $t(x,0) = 0$
4. $t(x,B) = f_2(x) = 64 \, t_{max}(x/D)^3(1 - x/D)^3$

The equations are transformed into dimensionless form by letting $Y = y/D$, $X = x/D$, $H = B/D$ and $T = t/t_{max}$. The differential equation and the boundary conditions for the transformed problem become

$$\nabla^2 T = 0 \tag{6}$$

1. $T(0,Y) = 0$
2. $T(1,Y) = 0$
3. $T(X,0) = 0$
4. $T(X,H) = f_2(DX)/t_{max} = F_2(X) = 64\ X^3(1-X)^3$

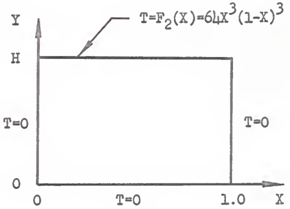

Fig. 3. Transformed two-dimensional plate.

We chose H = 0.75 for the numerical study. Equation (5) can be written as

$$f(X,Y) = a_1 + a_2X + a_3Y + a_4(X^2-Y^2) + a_5(2XY) + \cdots\cdots$$

$$\cdots + a_{2n}\left[X^n - \binom{n}{2}X^{n-2}Y^2 + \binom{n}{4}X^{n-4}Y^4 - \cdots\cdots + (-1)^{k-1}\binom{n}{2(k-1)}\cdot X^{n-2(k-1)}Y^{2(k-1)} + \cdots\cdots\right] + a_{2n+1}\left[nX^{n-1}Y - \binom{n}{3}X^{n-3}Y^3 + \binom{n}{5}X^{n-5}Y^5 - \cdots\cdots + (-1)^{k-1}\binom{n}{2k-1}X^{n-2k+1}Y^{2k-1} + \cdots\cdots\right] + \cdots\cdots\cdots$$

$$f(X,Y) = a_1 + \sum_{i=1}^{\infty} a_{2i}\left[X^i - \binom{i}{2}X^{i-2}Y^2 + \binom{i}{4}X^{i-4}Y^4 - \cdots\cdots\right.$$

$$\cdots + (-1)^{k-1} \binom{i}{2(k-1)} X^{i-2(k-1)} Y^{2(k-1)} + \cdots\cdots$$

$$+ \sum_{i=1}^{\infty} a_{2i+1}\left[iX^{i-1} Y - \binom{i}{3}X^{i-3} Y^3 + \binom{i}{5}X^{i-5} Y^5 - \cdots\cdots\right.$$

$$\left.\cdots + (-1)^{k-1}\binom{i}{2k-1} X^{i-2k+1} Y^{2k-1} + \cdots\cdots\right] + \cdots\cdots \quad (7)$$

or expressed as

$$f(X,Y) = \sum_{i=0}^{\infty} a_i M_i(X,Y) \quad (8)$$

We know that $M_i(X,Y)$ $i=1,2,3,\cdots\cdots$ satisfies the differential equation (6). As stated before, if a set of coefficients a_1 through a_m can be determined such that the approximate solution, $f(X,Y)$, is higher than the exact temperature T on the boundary, then $f(X,Y)$ is an upper bound for $T(X,Y)$. To do this, we use the Method of Collocation. First, m collocation points are arbitrarily chosen along the boundary. Then at each point the following condition is imposed

$$f(X,Y) = T(X,Y) + \delta_1 \quad (9)$$

where δ_1 is an arbitrary parameter.

This results in m simultaneous equations in m unknowns a_1 through a_m. The simultaneous equations are solved by using the well-known Gauss Method [3]. The coefficients thus obtained are substituted into equation (8) to get the equation for the approximate solution.

This approximate temperature may now be compared with the exact temperature on the boundary. If δ_1 is big enough the approximate temperature will be higher than the exact temperature throughout the boundary, and will thus be an upper bound. The lowest value of δ_1 which satisfies the above condition will give us the least upper bound.

Similarly the greatest lower bound is obtained by starting with

$$f(X,Y) = T(X,Y) - \delta_2 \tag{10}$$

In order to get a good solution, i.e. the solution where the bounds are close together, a sufficient number of collocation points must be used. For the present case, the points and the exact temperatures are shown in Table 1.

Table 1. Collocation points and exact temperature.

i	X_i	Y_i	T_i	i	X_i	Y_i	T_i
1	0	0	0	15	.932	.750	.016291
2	0	.112	0	16	1	.750	0
3	0	.242	0	17	1	.663	0
4	0	.403	0	18	1	.540	0
5	0	.540	0	19	1	.403	0
6	0	.663	0	20	1	.242	0
7	0	.750	0	21	1	.112	0
8	.068	.750	.016291	22	1	0	0
9	.160	.750	.155374	23	.897	0	0
10	.270	.750	.490050	24	.760	0	0
11	.410	.750	.905905	25	.600	0	0
12	.590	.750	.905905	26	.400	0	0
13	.730	.750	.490050	27	.240	0	0
14	.840	.750	.155374	28	.103	0	0

Choosing $\delta_1 = 0.3977\ 10^{-3}$ and $\delta_2 = 0.5166\ 10^{-3}$, we obtain the least upper bound and the greatest lower bound. The difference between the bounding temperatures and the exact temperature on the boundary is shown in Fig. 4, 5 and 6. For interior points the mean value of the bounding temperatures, T_{mean}, is taken as the best approximation of the temperature distribution. The results are compared with the exact temperature, see appendix C, in Table 2. The difference between the bounding temperatures and the exact temperature is also shown in Fig. 7, 8 and 9 along the lines $Y = 0.1$, 0.3 and 0.5.

We see from Fig. 4 through 9 that the upper and the lower bounds are

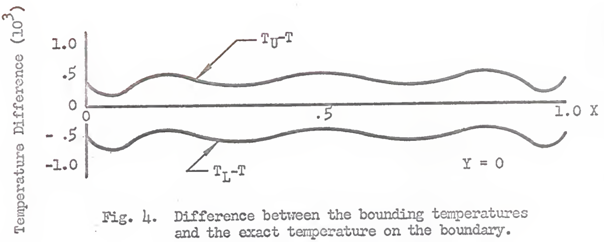

Fig. 4. Difference between the bounding temperatures and the exact temperature on the boundary.

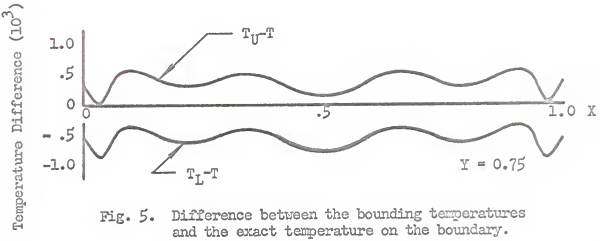

Fig. 5. Difference between the bounding temperatures and the exact temperature on the boundary.

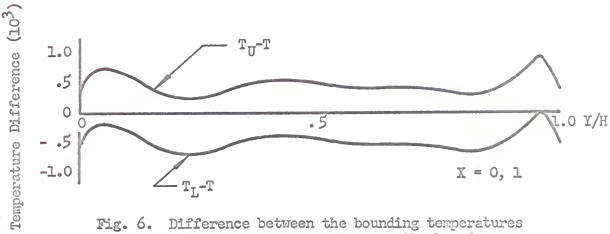

Fig. 6. Difference between the bounding temperatures and the exact temperature on the boundary.

Table 2. Values for the temperature distribution of example problem 1.

X	Y	T(lower)	T(upper)	T(mean)	T(exact)	Er %	Er* %
.1	.075	.010454	.011369	.010912	.010953	.004130	.3770
.2	.075	.020510	.021423	.020967	.021003	.003645	.1735
.3	.075	.028659	.029575	.029116	.029196	.007995	.2738
.4	.075	.034086	.035000	.034543	.034595	.005160	.1492
.5	.075	.036015	.036927	.036472	.036484	.001194	.0327
.6	.075	.034086	.035000	.034543	.034594	.005124	.1481
.7	.075	.028659	.029575	.029116	.029195	.007926	.2715
.8	.075	.020510	.021423	.020967	.021002	.003550	.1690
.9	.075	.010454	.011369	.010912	.010952	.004020	.3670
.1	.150	.021885	.022800	.022342	.022409	.006620	.2954
.2	.150	.042540	.043455	.042998	.043051	.005339	.1240
.3	.150	.059462	.060377	.059919	.059980	.006083	.1014
.4	.150	.070688	.071603	.071146	.071197	.005180	.0727
.5	.150	.074638	.075552	.075096	.075136	.004038	.0537
.6	.150	.070688	.071603	.071146	.071197	.005104	.0717
.7	.150	.059462	.060377	.059919	.059979	.005940	.0990
.8	.150	.042540	.043455	.042998	.043049	.005145	.1195
.9	.150	.021885	.022800	.022342	.022406	.006394	.2854
.1	.225	.034318	.035233	.034776	.034839	.006369	.1828
.2	.225	.066666	.067581	.067124	.067179	.005584	.0831
.3	.225	.093493	.094407	.093950	.094007	.005716	.0608
.4	.225	.111457	.112372	.111915	.111969	.005428	.0485
.5	.225	.117805	.118720	.118263	.118314	.005072	.0429
.6	.225	.111457	.112372	.111915	.111968	.005308	.0474
.7	.225	.093493	.094407	.093950	.094005	.005491	.0584
.8	.225	.066666	.067581	.067124	.067176	.005280	.0786
.9	.225	.034318	.035233	.034775	.034835	.006015	.1727
.1	.300	.048129	.049043	.048586	.048626	.003974	.0817
.2	.300	.093863	.094778	.094320	.094374	.005393	.0571
.3	.300	.132549	.133464	.133006	.133064	.005746	.0432
.4	.300	.158891	.159806	.159349	.159406	.005750	.0361
.5	.300	.168284	.169199	.168742	.168798	.005620	.0333
.6	.300	.158891	.159806	.159349	.159405	.005578	.0350
.7	.300	.132549	.133464	.133006	.133061	.005424	.0408
.8	.300	.093863	.094778	.094320	.094370	.004960	.0526
.9	.300	.048129	.049043	.048586	.048621	.003474	.0715
.1	.375	.063413	.064328	.063871	.063915	.004374	.0684
.2	.375	.124925	.125840	.125383	.125438	.005482	.0437
.3	.375	.178595	.179509	.179052	.179112	.005970	.0333
.4	.375	.216077	.216992	.216535	.216597	.006172	.0285
.5	.375	.229623	.230537	.230080	.230141	.006093	.0265
.6	.375	.216077	.216992	.216535	.216594	.005934	.0274
.7	.375	.178595	.179509	.179052	.179107	.005528	.0309
.8	.375	.124925	.125840	.125383	.125432	.004895	.0390

Table 2. (cont.)

X	Y	T(lower)	T(upper)	T(mean)	T(exact)	Er %	Er^* %
.9	.375	.063413	.064328	.063871	.063908	.003703	.0579
.1	.450	.079757	.080672	.080214	.080276	.006122	.0763
.2	.450	.160147	.161062	.160605	.160661	.005616	.0350
.3	.450	.233783	.234698	.234241	.234304	.006306	.0269
.4	.450	.287094	.288009	.287551	.287620	.006861	.0239
.5	.450	.306704	.307619	.307161	.307228	.006675	.0217
.6	.450	.287094	.288009	.287551	.287617	.006536	.0227
.7	.450	.233783	.234698	.234241	.234298	.005710	.0244
.8	.450	.160147	.161062	.160605	.160653	.004838	.0301
.9	.450	.079757	.080672	.080215	.080267	.005248	.0654
.1	.525	.095428	.096343	.095886	.095957	.007126	.0743
.2	.525	.198592	.199507	.199050	.199115	.006533	.0328
.3	.525	.300415	.301329	.300872	.300938	.006575	.0218
.4	.525	.377699	.378614	.378157	.378229	.007189	.0190
.5	.525	.406733	.407648	.407191	.407267	.007595	.0186
.6	.525	.377699	.378614	.378157	.378224	.006744	.0178
.7	.525	.300415	.301329	.300872	.300930	.005775	.0192
.8	.525	.198592	.199507	.199050	.199105	.005515	.0277
.9	.525	.095428	.096343	.095886	.095946	.006011	.0627
.1	.600	.105654	.106569	.106111	.106184	.007232	.0681
.2	.600	.236559	.237474	.237017	.237085	.006770	.0286
.3	.600	.380819	.381734	.381277	.381341	.006422	.0168
.4	.600	.496572	.497487	.497029	.497108	.007821	.0157
.5	.600	.541002	.541917	.541459	.541554	.009504	.0175
.6	.600	.496572	.497487	.497029	.497101	.007202	.0145
.7	.600	.380819	.381734	.381277	.381330	.005336	.0140
.8	.600	.236559	.237474	.237017	.237071	.005439	.0229
.9	.600	.105654	.106569	.106111	.106170	.005837	.0550
.1	.675	.098712	.099627	.099170	.099211	.004109	.0414
.2	.675	.264524	.265439	.264984	.265061	.007924	.0299
.3	.675	.477260	.478175	.477717	.477765	.004766	.0100
.4	.675	.657616	.658531	.658073	.658152	.007850	.0119
.5	.675	.728050	.728964	.728507	.728655	.014843	.0204
.6	.675	.657616	.658531	.658073	.658143	.006966	.0106
.7	.675	.477260	.478175	.477717	.477750	.003258	.0068
.8	.675	.264524	.265439	.264982	.265043	.006175	.0233
.9	.675	.098712	.099627	.099170	.099194	.002403	.0242

exactly of the same shape, which makes the search for the parameters, δ_1 and δ_2, much easier than usual. Once a set of collocation points is decided, it is straightforward to let $\delta = 0$ and calculate the errors on the boundary and then look for the maximum and the minimum errors. To show how the number

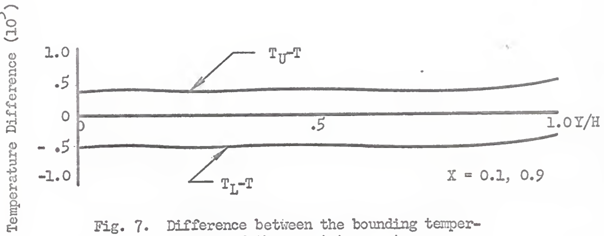

Fig. 7. Difference between the bounding temperatures and the exact temperature.

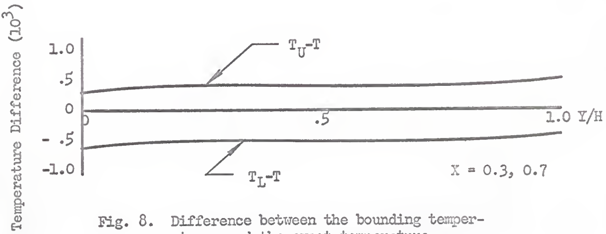

Fig. 8. Difference between the bounding temperatures and the exact temperature.

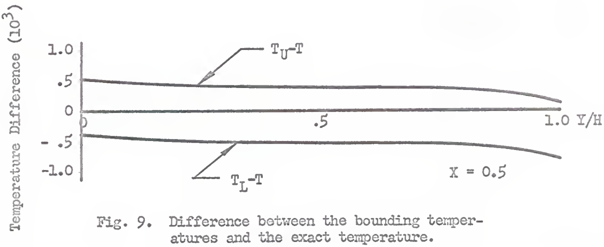

Fig. 9. Difference between the bounding temperatures and the exact temperature.

of collocation points affects the accuracy, we solve the same problem with 24 points and compare them as follows.

Table 3. Comparison of accuracy due to number of collocation points.

Number of points	δ_1	δ_2	Er %
24	.003304	.002394	.2849
28	.000517	.000398	.0458

EXAMPLE PROBLEM 2

A second problem will be considered with the following boundary conditions.

1. $t(0,y) = 0$
2. $t(D,y) = 0$
3. $t(x,0) = 0$
4. $t(x,B) = f_2(x) = t_{max}$

Note that the temperatures at (0,B) and (D,B) are undefined.

By the same transformation, we arrive at equation (6) and the boundary conditions

1. $T(0,Y) = 0$
2. $T(1,Y) = 0$
3. $T(X,0) = 0$
4. $T(X,H) = F_2(X) = 1$

The procefure is the same except that T(0,H) and T(1,H) are 1 when they are used to determine the upper bound and zero when used to determine the lower bound. This is necessary if the condition of bounds is to be satisfied. The collocation points, which are different for the upper and the lower bounds, and the exact temperature are shown in Table 4 and 5.

Table 4. Collocation points and exact temperature for upper bound.

i	X_i	Y_i	T_i	i	X_i	Y_i	T_i
1	0	0	0	15	1	.75	1
2	0	.11	0	16	1	.62	0
3	0	.22	0	17	1	.54	0
4	0	.33	0	18	1	.44	0
5	0	.44	0	19	1	.33	0
6	0	.54	0	20	1	.22	0
7	0	.62	0	21	1	.11	0
8	0	.75	1	22	1	0	0
9	.20	.75	1	23	.88	0	0
10	.30	.75	1	24	.72	0	0
11	.42	.75	1	25	.58	0	0
12	.58	.75	1	26	.42	0	0
13	.70	.75	1	27	.27	0	0
14	.80	.75	1	28	.12	0	0

Table 5. Collocation points and exact temperature for lower bound.

i	X_i	Y_i	T_i	i	X_i	Y_i	T_i
1	0	0	0	15	1	.75	0
2	0	.11	0	16	1	.65	0
3	0	.22	0	17	1	.55	0
4	0	.33	0	18	1	.44	0
5	0	.44	0	19	1	.33	0
6	0	.55	0	20	1	.22	0
7	0	.65	0	21	1	.11	0
8	0	.75	0	22	1	0	0
9	.20	.75	1	23	.88	0	0
10	.29	.75	1	24	.73	0	0
11	.42	.75	1	25	.58	0	0
12	.58	.75	1	26	.42	0	0
13	.71	.75	1	27	.27	0	0
14	.80	.75	1	28	.12	0	0

The parameters for the least upper and the greatest lower bounds are respectively $\delta_1 = 0.01552$ $\delta_2 = 0.00912$, and the difference between the bounding temperatures and the exact temperature on the boundary is shown in Fig. 10, 11 and 12.

The results including the least upper bound, the greatest lower bound, the mean value which is taken as the approximate solution and the associated maximum possible errors relative to both the maximum temperature and the local temperature are shown in Table 6. The bounds are plotted along Y = 0.1, 0.3 and 0.5 in Fig. 13, 14 and 15.

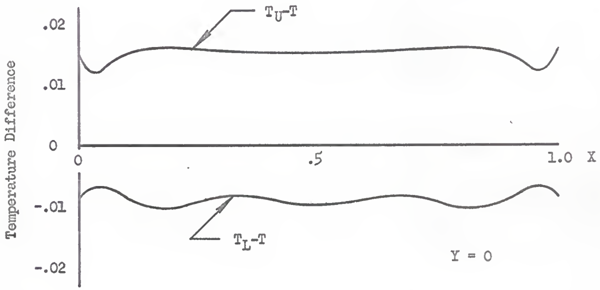

Fig. 10. Difference between the bounding temperatures and the exact temperature on the boundary.

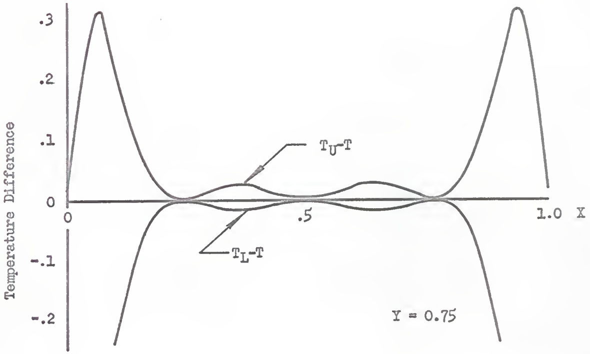

Fig. 11. Difference between the bounding temperatures and the exact temperature on the boundary.

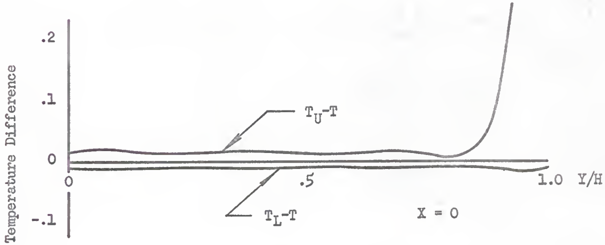

Fig. 12. Difference between the bounding temperatures and the exact temperature on the boundary.

Table 6. Values for the temperature distribution of example problem 2.

X	Y	T(lower)	T(upper)	T(mean)	Er %	Er* %
.1	.075	.008622	.034672	.021647	1.302	9.77
.2	.075	.023986	.051439	.037712	1.373	12.91
.3	.075	.036408	.064136	.050272	1.386	22.25
.4	.075	.044012	.071954	.057983	1.397	17.93
.5	.075	.046413	.074530	.060471	1.406	10.89
.6	.075	.044012	.071954	.057983	1.397	8.09
.7	.075	.036408	.064136	.050272	1.386	6.73
.8	.075	.023986	.051439	.037712	1.373	6.32
.9	.075	.008622	.034672	.021647	1.302	6.73
.1	.150	.027372	.054865	.041118	1.375	8.09
.2	.150	.059555	.089469	.074512	1.496	10.89
.3	.150	.084439	.115469	.099954	1.551	17.93
.4	.150	.099907	.131326	.115617	1.571	16.32
.5	.150	.105097	.136616	.120856	1.576	9.88
.6	.150	.099907	.131326	.115617	1.571	6.92
.7	.150	.084439	.115469	.099954	1.551	5.49
.8	.150	.059555	.089469	.074512	1.496	5.06
.9	.150	.027372	.054865	.041118	1.375	5.49
.1	.225	.048588	.078273	.063430	1.484	6.92
.2	.225	.099281	.132471	.115876	1.660	9.88
.3	.225	.137878	.172588	.155233	1.735	16.32
.4	.225	.161642	.196671	.179156	1.751	16.99
.5	.225	.169614	.204641	.187128	1.751	9.48
.6	.225	.161642	.196671	.179156	1.751	5.92
.7	.225	.137878	.172587	.155233	1.735	4.38
.8	.225	.099281	.132470	.115875	1.659	3.94
.9	.225	.048588	.078272	.063430	1.484	4.38
.1	.300	.073897	.106783	.090340	1.644	5.92
.2	.300	.145993	.183678	.164835	1.884	9.48
.3	.300	.199796	.238857	.219327	1.953	16.99
.4	.300	.232311	.271087	.251699	1.939	21.06
.5	.300	.243108	.281594	.262351	1.924	9.14
.6	.300	.232311	.271087	.251699	1.939	4.76
.7	.300	.199796	.238857	.219327	1.953	3.29
.8	.300	.145993	.183677	.164835	1.884	2.84
.9	.300	.073897	.106782	.090340	1.644	3.29
.1	.375	.105718	.143632	.124675	1.896	4.76
.2	.375	.203337	.247634	.225486	2.215	9.14
.3	.375	.273879	.318190	.296034	2.216	21.06
.4	.375	.315223	.357661	.336442	2.122	28.85
.5	.375	.328710	.370244	.349477	2.077	7.37
.6	.375	.315223	.357661	.336442	2.122	3.16
.7	.375	.273879	.318190	.296034	2.216	2.26
.8	.375	.203337	.247634	.225486	2.215	1.66

Table 6. (cont.)

X	Y	T(lower)	T(upper)	T(mean)	Er %	Er* %
.9	.375	.105718	.143631	.124675	1.896	17.93
.1	.450	.147782	.196018	.171900	2.412	16.32
.2	.450	.276550	.331204	.303877	2.733	9.88
.3	.450	.364622	.415068	.389845	2.522	6.92
.4	.450	.413738	.459147	.436442	2.270	5.49
.5	.450	.429330	.472781	.451055	2.173	5.06
.6	.450	.413738	.459147	.436442	2.270	5.49
.7	.450	.364622	.415068	.389845	2.522	6.92
.8	.450	.276550	.331204	.303877	2.733	9.88
.9	.450	.147782	.196017	.171900	2.412	16.32
.1	.525	.207136	.277517	.242326	3.519	16.99
.2	.525	.373799	.444705	.409252	3.545	9.48
.3	.525	.477474	.533969	.505721	2.825	5.92
.4	.525	.530820	.577294	.554057	2.324	4.38
.5	.525	.547127	.590189	.568658	2.153	3.94
.6	.525	.530820	.577294	.554057	2.324	4.38
.7	.525	.477474	.533969	.505721	2.825	5.92
.8	.525	.373799	.444706	.409253	3.545	9.48
.9	.525	.207136	.277518	.242327	3.519	16.99
.1	.600	.297855	.423313	.360584	6.273	21.06
.2	.600	.508499	.601480	.554990	4.649	9.14
.3	.600	.618407	.677290	.647848	2.944	4.76
.4	.600	.667985	.711889	.689937	2.195	3.29
.5	.600	.682679	.721405	.702042	1.936	2.84
.6	.600	.667985	.711890	.689937	2.195	3.29
.7	.600	.618407	.677290	.647848	2.944	4.76
.8	.600	.508499	.601482	.554990	4.649	9.14
.9	.600	.297855	.423315	.360585	6.273	21.06
.1	.675	.451809	.712503	.582156	13.035	28.85
.2	.675	.702938	.806601	.754770	5.183	7.37
.3	.675	.791335	.841269	.816302	2.497	3.16
.4	.675	.823202	.860362	.841782	1.858	2.26
.5	.675	.834321	.862089	.848205	1.388	1.66
.6	.675	.823202	.860362	.841782	1.858	2.26
.7	.675	.791335	.841269	.816302	2.497	3.16
.8	.675	.702938	.806602	.754770	5.183	7.37
.9	.675	.451809	.712508	.582158	13.035	28.85

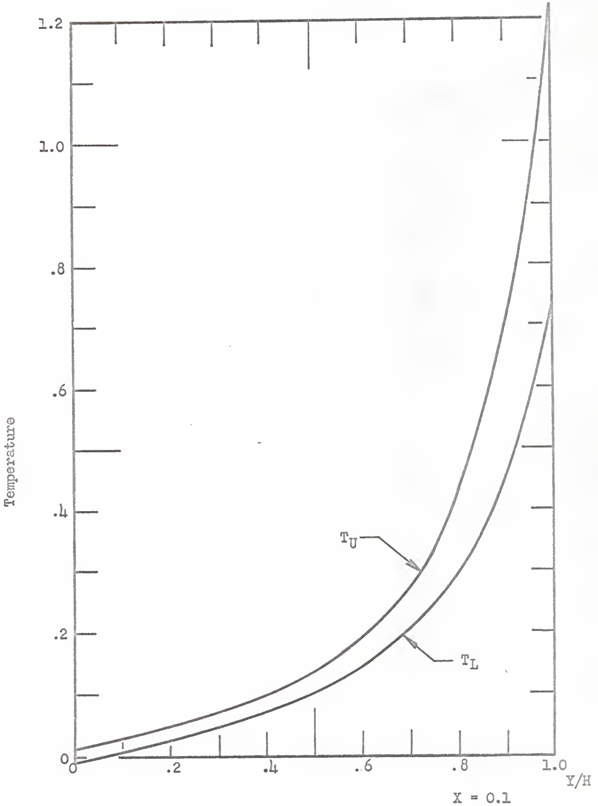

Fig. 13. Upper and lower bounds.

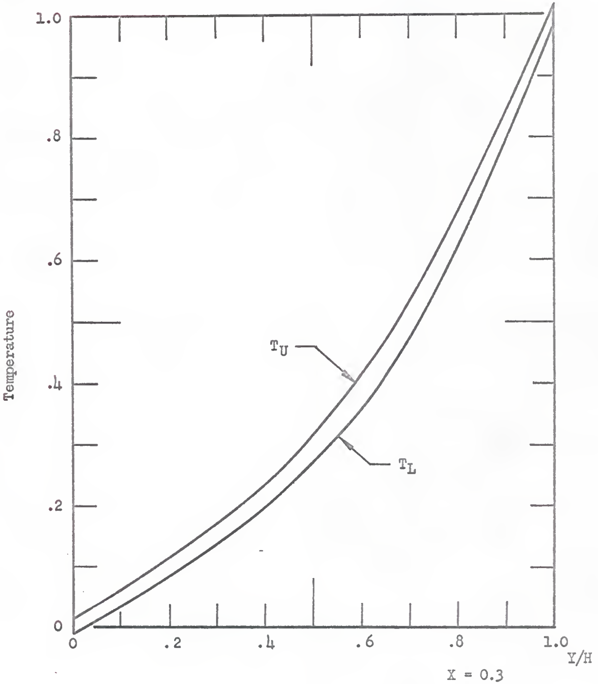

Fig. 14. Upper and lower bounds.

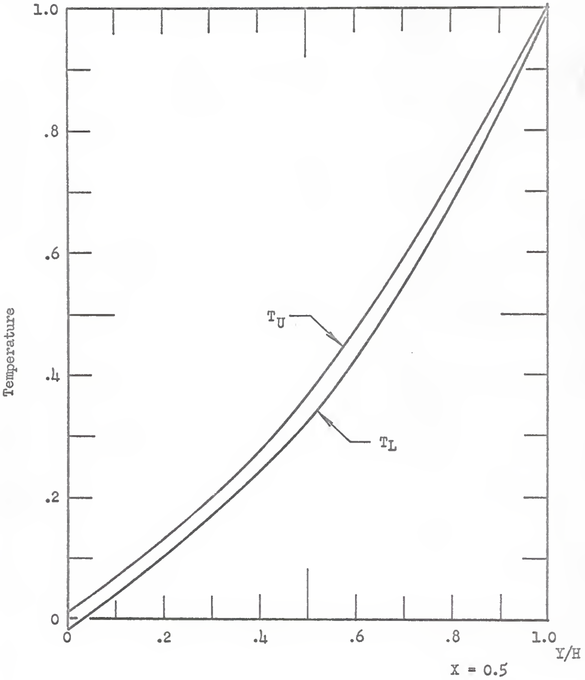

Fig. 15. Upper and lower bounds.

DISCUSSION AND CONCLUSIONS

As mentioned in the introduction, there are two ways to modify the problem to obtain upper and lower bounds for the original problem. It would then be desirable to compare the results obtained by applying each of the two. It is for this purpose that we first consider example problem 1, which has been solved by K. M. Hostetler. In the paper by Hostetler [5], the series was chosen as

$$T_{mn} = \frac{64}{H} YX^3(1-X)^3 + \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}\, X^i Y^j (X-1)(Y-H)$$

We see that every term of the series satisfies the boundary conditions, but not the differential equation. The coefficients a_{ij} were so determined as to provide upper and lower bounds from which the approximate solution and the associated error were obtained. Using 49 terms of the series, the error relative to the local temperature varied from 0.910 % to 29.942 %.

In the present work, we utilize equation (7), where every term satisfies the differential equation, and determine the coefficients to approximate the boundary condition. In comparison, the error relative to the local temperature varies from 0.010 % to 0.377 % using 28 terms of the series. In addition to the improved accuracy the present method has another advantage in that there is no need to vary δ in searching for better bounds.

In example problem 2, the boundary conditions are so defined that the temperature is discontinuous at the upper corners. As shown by Table 6, the solution is reasonably good using 28 terms of the series. The maximum possible percentage error with respect to the local temperature is considerably higher at the lower side of the plate. The actual error is not particularly large there, but the exact temperature is so near zero that the relative error becomes large. The maximum possible percentage error with respect to the maximum temperature seems the most reasonable criterion in this case.

Comparing the two problems, we see that the discontinuity in the boundary conditions makes it difficult to obtain good bounds in the vicinity of the corners. However with a bigger computer, we would be able to consider more terms in the series and thus improve the results. Accompanying a larger number of collocation points, a higher arithmetic precision would be required. Take the two problems for example, 8 place arithmetic precision is good for 16 collocation points, but 14 places are needed for 28 points. It is predicted that 18 places would be enough to consider 40 collocation points. This can easily by done by some digital computers in use.

ACKNOWLEDGEMENT

I wish to express my gratitude to Professor F. C. Appl of the Department of Mechanical Engineering, Kansas State University, for his suggestion of the topic and guidance throughout my graduate studies.

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APPENDIX A

INFINITE SERIES FOR LAPLACE EQUATION

We are going to prove that each term of the infinite series on the right side of equation (5) satisfies equation (1).

Rewrite (5) as

$$f(x,y) = b_0 + \sum_{n=1}^{\infty} b_{2n-1}\Big[x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 - \cdots\cdots$$
$$\cdots + (-1)^{k-1}\binom{n}{2(k-1)} x^{n-2(k-1)} y^{2(k-1)} + \cdots\cdots\Big] + \sum_{n=1}^{\infty} b_{2n}\Big[nx^{n-1} y$$
$$- \binom{n}{3} x^{n-3} y^3 + \binom{n}{5} x^{n-5} y^5 - \cdots\cdots + (-1)^{k-1}\binom{n}{2k-1} x^{n-2k+1} y^{2k-1}$$
$$+ \cdots\cdots\cdots\Big] + \cdots\cdots\cdots \tag{11}$$

First of all, the constant b_0 satisfies (1). Since (1) is a homogeneous equation, we need only to prove that each of the two general terms satisfies (1). Letting

$$t = b_{2n-1}\Big[x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 - \cdots\cdots$$
$$\cdots + (-1)^{k-1}\binom{n}{2(k-1)} x^{n-2(k-1)} y^{2(k-1)} + \cdots\cdots\Big]$$

and substituting into the left side of (1), we obtain

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = b_{2n-1}\Big[n(n-1)x^{n-2} - \binom{n}{2}(n-2)(n-3)x^{n-4} y^2 + \binom{n}{4}(n-4)(n-5)\cdot$$
$$x^{n-6} y^4 - \cdots\cdots + (-1)^{k-1}\binom{n}{2(k-1)}(n-2k+2)(n-2k+1)\, x^{n-2k} y^{2(k-1)}$$
$$+ \cdots\cdots - \binom{n}{2}\, 2\, x^{n-2} + \binom{n}{4}(4)(3)\, x^{n-4} y^2 - \cdots\cdots$$
$$\cdots + (-1)^{k-1}\binom{n}{2(k-1)}(2k-2)(2k-3)\, x^{n-2(k-1)} y^{2k-4} + \cdots\cdots\Big]$$

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = b_{2n-1}\Big[\binom{n}{2} 2\, x^{n-2} - \binom{n}{4}(3)(4)\, x^{n-4}\, y^2 + \binom{n}{6}(5)(6)\, x^{n-6}\, y^4$$

$$- \cdots\cdots + (-1)^{k-1} \binom{n}{2k} (2k-1)(2k)\, x^{n-2k}\, y^{2(k-1)} + \cdots\cdots\cdots$$

$$\cdots - \binom{n}{2} 2\, x^{n-2} + \binom{n}{4} (3)(4)\, x^{n-4}\, y^2 - \cdots\cdots\cdots$$

$$\cdots + (-1)^{k-1} \binom{n}{2(k-1)} (2k-3)(2k-2)\, x^{n-2(k-1)}\, y^{2k-4} + \cdots\cdots\Big]$$

Comparing the like-power terms, we see that the coefficients of the first part are exactly the same as those of the second part except that they have the opposite signs term by term and therefore

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0$$

The proof for the second general term is carried out in the same manner.

APPENDIX B

PROOF OF BOUNDS

Equation (5) is rewritten as

$$f(x,y) = b_0 + \sum_{n=1}^{\infty} b_{2n-1} \Big[x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 - \cdots\cdots$$

$$\cdots + (-1)^{k-1} \binom{n}{2(k-1)} x^{n-2(k-1)} y^{2(k-1)} + \cdots\cdots \Big] + \sum_{n=1}^{\infty} b_{2n} \Big[n x^{n-1} y$$

$$- \binom{n}{3} x^{n-3} y^3 + \binom{n}{5} x^{n-5} y^5 - \cdots\cdots + (-1)^{k-1} \binom{n}{2k-1} x^{n-2k+1} y^{2k-1}$$

$$+ \cdots\cdots \Big] + \cdots\cdots\cdots\cdots \tag{12}$$

For simplicity, we denote M_n and N_n for the two general terms in equation (12) respectively. The equation becomes

$$f(x,y) = b_0 + \sum_{n=1}^{\infty} b_{2n-1} M_n + \sum_{n=1}^{\infty} b_{2n} N_n \tag{13}$$

With the differential equation

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0 \tag{1}$$

and any prescribed boundary condition, we are going to prove that

$$t_U > t$$

if $$t_U = b_0 + \sum_{n=1}^{j} b_{2n-1} M_n + \sum_{n=1}^{m} b_{2n} N_n \tag{14}$$

where j and m are positive integers and $t_U > t$ throughout the prescribed boundary.

To prove this, we divide t_U into two parts

$$t_U(x,y) = t(x,y) + u(x,y)$$

where t_U, t and u are functions of x and y and t is the exact solution. Since t_U and t both satisfy equation (1), which is homogeneous, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (15)$$

And as $t_U > t$ throughout the prescribed boundary, we can say that u is the solution of equation (15) with positive boundary value. A corollary of Gauss' Theorem of the Arithmetic Mean [4] states: "Let R denote a closed bounded region (regular or not) of space, and let u be harmonic, but not constant, in R. Then u attains its maximum and minimum values only on the boundary of R." Now for our case u, satisfying equation (15), is harmonic and is greater than zero throughout the boundary; therefore u can not be negative in the region and therefore

$t_U(x,y) > t(x,y)$ throughout the region.

Thus we have completed the proof for the upper bound.

In a like manner, we can prove that

$t_L < t$

if $$t_L = b_0 + \sum_{n=1}^{j} b_{2n-1} M_n + \sum_{n=1}^{m} b_{2n} N_n \qquad (16)$$

where j and m are positive integers and $t_L < t$ throughout the prescribed boundary.

APPENDIX C

EXACT TEMPERATURE DISTRIBUTION

For the steady-state, two-dimensional, rectangular plate heat conduction problem shown in Fig. 2, Schneider [2] has shown the exact temperature distribution to be

$$t(x,y) = \frac{2}{D} \sum_{n=1}^{\infty} \frac{\mathrm{Sinh}\left(\frac{n\pi y}{D}\right)}{\mathrm{Sinh}\left(\frac{n\pi B}{D}\right)} \mathrm{Sin}\left(\frac{n\pi x}{D}\right) \int_0^D f(x)\, \mathrm{Sin}\left(\frac{n\pi x}{D}\right) dx \qquad (17)$$

By transforming the problem into the dimensionless problem shown in Fig. 3, the exact temperature distribution becomes

$$T(X,Y) = 2 \sum_{n=1}^{\infty} \frac{\mathrm{Sinh}(n\pi Y)}{\mathrm{Sinh}(n\pi H)} \mathrm{Sin}(n\pi X) \int_0^1 F(X)\, \mathrm{Sin}(n\pi X)\, dX \qquad (18)$$

Substituting $F(X) = 64\, T_{max}\, X^3\, (1-X)^3$ into equation (18), integrating and reducing, the exact temperature distribution becomes

$$T(X,Y) = \frac{18432\, T_{max}}{\pi^5} \sum_{n=1}^{\infty} \frac{\mathrm{Sinh}(n\pi Y)}{\mathrm{Sinh}(n\pi H)} \mathrm{Sin}(n\pi X) \frac{1}{n^5}\left[\frac{10}{n^2\pi^2} - 1\right] \qquad (19)$$

$$n = 1, 3, 5, 7 \cdots\cdots$$

In solving for the exact temperature with a computer, we sum the series to the point where the last term is less than 10^{-8}.

APPENDIX D

LISTING OF FORTRAN PROGRAM TO SET
UP THE SIMULTANEOUS EQUATIONS.

```
MON$$     JOB  TEMPERATURE BOUNDS
MON$$     COMT 30,20,PAGES,6, F C APPL, T S CHOU, ME DEPT
MON$$     ASGN MJB,12
MON$$     ASGN MGO,16
MON$$     MODE GO,TEST
MON$$     EXEQ FORTRAN,,,14,,,,BOUNDS1
      DIMENSIONB(28,29),BINO(14,14),X(28),Y(28),XN(14),YN(14),CC(28)
    1 FORMAT(I5)
    2 FORMAT(E18.10,I5,I5)
    3 FORMAT(3E18.10,I5)
      READ(1,1) LN
      WRITE(3,1)LN
      READ(1,2) ZAC
      WRITE(3,2)ZAC
      DO 4 I=1,LN
      READ(1,3) X(I),Y(I),CC(I),I
    4 WRITE(3,3) X(I),Y(I),CC(I),I
      READ (1,3) DEL
      WRITE(3,3) DEL
      LNH=LN/2
      N1=LN+1
      DO 10 I=1,LNH
      DO 10 J=1,I
      BINO(I,J)=1.0
      DO 5 K=1,J
      GI=I
      GK=K
    5 BINO(I,J)=BINO(I,J)*(GI-GK+1.0)/GK
   10 CONTINUE
      DO 15 I=1,LN
   15 B(I,1)=1.0
      DO 70 I=1,LN
      XN(1)=X(I)
      YN(1)=Y(I)
      DO 20 JJ=2,LNH
      JJJ=JJ-1
      XN(JJ)=XN(JJJ)*X(I)
   20 YN(JJ)=YN(JJJ)*Y(I)
      DO 70 J=2,LN
      M=J/2
      IF ((J+1)/2-J/2) 30,30,55
   30 K=0
      B(I,J)=XN(M)
   35 K=K+2
      KK=M-K
      IF(KK) 70,36,40
   36 IF((K/2+1)/2-K/4) 37,37,38
   37 B(I,J)=B(I,J)+BINO(M,K)*YN(K)
      GO TO 70
   38 B(I,J)=B(I,J)-BINO(M,K)*YN(K)
      GO TO 70
   40 IF ((K/2+1)/2-K/4) 45,45,50
   45 B(I,J)=B(I,J)+BINO(M,K)*XN(KK)*YN(K)
      GO TO 35
   50 B(I,J)=B(I,J)-BINO(M,K)*XN(KK)*YN(K)
      GO TO 35
   55 K=1
      GM=M
      IF (M-1) 56,56,57
   56 B(I,J)=GM*Y(I)
      GO TO 60
   57 B(I,J)=GM*XN(M-1)*Y(I)
   60 K=K+2
      KK=M-K
```

```
      IF(KK) 70,61,64
   61 IF(((K-1)/2+1)/2-(K-1)/4) 62,62,63
   62 B(I,J)=B(I,J)+BINO(M,K)*YN(K)
      GO TO 70
   63 B(I,J)=B(I,J)-BINO(M,K)*YN(K)
      GO TO 70
   64 IF(((K-1)/2+1)/2-(K-1)/4) 65,65,66
   65 B(I,J)=B(I,J)+BINO(M,K)*XN(KK)*YN(K)
      GO TO 60
   66 B(I,J)=B(I,J)-BINO(M,K)*XN(KK)*YN(K)
      GO TO 60
   70 CONTINUE
      DO 72 I=1,LN
   72 B(I,N1)=DEL+CC(I)
      REWIND 6
      WRITE(6) LN,N1,LNH,ZAC
      WRITE(6) ((B(I,J),J=1,N1),I=1,LN)
      WRITE(6) ((BINO(I,J),J=1,I),I=1,LNH)
      REWIND 6
      STOP
      END
```

LISTING OF FORTRAN PROGRAM TO SOLVE
THE SIMULTANEOUS EQUATIONS.

```
MON$$     EXEQ FORTRAN,,,14,,,,BOUNDS2
      DIMENSION B(28,29),ABAXX(29),BSTAR(29),ABA(29)
 9090 FORMAT (I3, 17HTH COLUMN IS ZERO)
  501 FORMAT(2E18.10)
  505 FORMAT(2E18.10,I5)
      REWIND 6
      READ(6) LN,N1,LNH,ZAC
      READ(6) ((B(I,J),J=1,N1),I=1,LN)
      ZTRY=0.0
      DO 2000 J=1,LN
      ABA(J)=0.0
 2000 BSTAR(J)=0.0
 2511 DO 2200 J=1,LN
 2028 IF(J-LN) 2029,2059,2059
 2029 DO 2030 I=J,LN
      IF (B(I,J)) 2031,2032,2032
 2031 ABAXX(I)=-B(I,J)
      GO TO 2030
 2032 ABAXX(I)=B(I,J)
 2030 CONTINUE
      IC = J
      C=ABAXX(J)
      JJ = J+1
      DO 2040 I=JJ,LN
      IF(C-ABAXX(I))2039,2040,2040
 2039 C=ABAXX(I)
      IC = I
 2040  CONTINUE
      DO 2050 K=J,N1
      D = B(IC,K)
      B(IC,K) = B(J,K)
 2050 B(J,K) = D
 2059 DIV = B(J,J)
      IF (DIV) 2351,2350,2351
 2351 Z=1.0/DIV
      DO 2060 K=J,N1
 2060 B(J,K)=B(J,K)*Z
      DO 2080 I=1,LN
      IF (I-J) 2065,2080,2065
 2065 AIJ = -B(I,J)
      DO 2070 K= J,N1
 2070 B(I,K) = B(I,K) + AIJ*B(J,K)
 2080 CONTINUE
 2200 CONTINUE
      ZTRY=ZTRY+1.0
      DO 2300 I=1,LN
 2300 ABA(I)=B(I,N1)+ABA(I)
      REWIND 6
      READ(6) LN,N1,LNH,ZAC
      READ(6) ((B(I,J),J=1,N1),I=1,LN)
      DO2560 LI=1,LN
      DO2560 LJ=1,LN
 2560 BSTAR(LI)=ABA(LJ)*  B(LI,LJ)+BSTAR(LI)
      DO2561 LI=1,LN
      B(LI, N1)=  B(LI, N1)-BSTAR(LI)
 2561 BSTAR(LI)=0.0
      IF(ZTRY-ZAC)2562,2565,2565
 2562 ZZE=0.0
      DO2563 LI=1,LN
 2563 ZZE=ZZE+B(LI, N1)
      IF(ZZE)2564,2565,2564
 2564 GO TO2511
 2565 WRITE(3,501)  ZTRY,  ZAC
      DO2566 LI=1,LN
```

```
2566 WRITE(3,505)  ABA(LI),  B(LI, N1),  LI
     GO TO 2352
2350 WRITE (3,9090)  J
2352 CONTINUE
     READ(6) ((B(I,J),J=1,I),I=1,LNH)
     WRITE(6) (ABA(LI),LI=1,LN)
     REWIND 6
     STOP
     END
```

LISTING OF FORTRAN PROGRAM TO SEARCH FOR THE LEAST UPPER BOUND AND THE GREATEST LOWER BOUND.

```
MON$$      EXEQ FORTRAN,,,14,,,,BOUNDS3
      DIMENSION ABA(29),BINO(28,29),XXN(14),YYN(14)
    3 FORMAT(3E18.10,I5)
    4 FORMAT(5E18.10)
      REWIND 6
      READ(6) LN,N1,LNH,ZAC
      READ(6) ((BINO(I,J),J=1,N1),I=1,LN)
      READ(6) ((BINO(I,J),J=1,I),I=1,LNH)
      READ(6) (ABA(LI),LI=1,LN)
      XX=0.0
      YY=0.0
   80 T1=ABA(1)
      DO 165 J=2,LN
      M=J/2
      XXN(1)=XX
      YYN(1)=YY
      DO 100 II=2,LNH
      III=II-1
      XXN(II)=XXN(III)*XX
      YYN(II)=YYN(III)*YY
  100 CONTINUE
      IF((J+1)/2-J/2) 105,105,130
  105 K=0
      BX=XXN(M)
  110 K=K+2
      KK=M-K
      IF(KK) 160,111,115
  111 IF((K/2+1)/2-K/4) 112,112,113
  112 BX=BX+BINO(M,K)*YYN(K)
      GO TO 160
  113 BX=BX-BINO(M,K)*YYN(K)
      GO TO 160
  115 IF((K/2+1)/2-K/4) 120,120,125
  120 BX=BX+BINO(M,K)*XXN(KK)*YYN(K)
      GO TO 110
  125 BX=BX-BINO(M,K)*XXN(KK)*YYN(K)
      GO TO 110
  130 K=1
      GM=M
      IF(M-1) 135,135,140
  135 BX=GM*YY
      GO TO 145
  140 BX=GM*XXN(M-1)*YY
  145 K=K+2
      KK=M-K
      IF(KK) 160,146,149
  146 IF(((K-1)/2+1)/2-(K-1)/4) 147,147,148
  147 BX=BX+BINO(M,K)*YYN(K)
      GO TO 160
  148 BX=BX-BINO(M,K)*YYN(K)
      GO TO 160
  149 IF(((K-1)/2+1)/2-(K-1)/4) 150,150,151
  150 BX=BX+BINO(M,K)*XXN(KK)*YYN(K)
      GO TO 145
  151 BX=BX-BINO(M,K)*XXN(KK)*YYN(K)
      GO TO 145
  160 CONTINUE
  165 T1=T1+ABA(J)*BX
      IF(XX) 170,180,170
  170 IF(XX-1.0) 175,180,175
  175 IF(YY) 185,180,185
  180 T=0.0
      GO TO 200
  185 T=64.0*XX*XX*XX*(1.0-XX)*(1.0-XX)*(1.0-XX)
```

```
200 ER=T1-T
    WRITE(3,4) XX,YY,T1,T,ER
    IF(XX-1.0) 205,225,225
205 IF(YY) 210,215,210
210 IF(YY-0.75) 220,215,220
215 XX=XX+0.05
    GO TO 80
220 XX=1.0
    GO TO 80
225 IF(YY-0.75) 230,270,270
230 XX=0.0
    YY=YY+0.0375
    GO TO 80
270 CONTINUE
    STOP
    END
```

LISTING OF FORTRAN PROGRAM TO SOLVE FOR BOUNDS, EXACT SOLUTION AND ERRORS.

```
MON$$     EXEQ FORTRAN,,,14,,,,BOUNDS4
      DIMENSION ABA(29),BINO(28,29),XXN(14),YYN(14)
    3 FORMAT(3E18.10,I5)
    4 FORMAT(6E18.10)
      REWIND 6
      READ(6) LN,N1,LNH,ZAC
      READ(6) ((BINO(I,J),J=1,N1),I=1,LN)
      READ(6) ((BINO(I,J),J=1,I),I=1,LNH)
      READ(6) (ABA(LI),LI=1,LN)
      H=0.75
      E=0.0000000001
      DEL1=0.1
      DEL2=0.075
      XX=DEL1
      YY=DEL2
  405 AN=1.0
      ST=0.0
  410 X1=3.1415926*(AN*XX-300.)
      Y1=AN*3.1415926*YY
      H1=AN*3.1415926*H
      C1=10./(9.8696044*AN*AN)-1.
      SINH=(1.-EXP(-2.*Y1))/(EXP(H1-Y1)-EXP(-H1-Y1))
      STN=SINH*SIN(X1)*C1/(AN*AN*AN*AN*AN)
      ST=ST+STN
      RIS=ABS(STN)
      IF(RIS-E) 420,420,415
  415 AN=AN+2.0
      GO TO 410
  420 T=60.23142*ST
  430 T1=ABA(1)
      DO 515 J=2,LN
      M=J/2
      XXN(1)=XX
      YYN(1)=YY
      DO 450 II=2,LNH
      III=II-1
      XXN(II)=XXN(III)*XX
      YYN(II)=YYN(III)*YY
  450 CONTINUE
      IF((J+1)/2-J/2) 455,455,480
  455 K=0
      BX=XXN(M)
  460 K=K+2
      KK=M-K
      IF(KK) 510,461,465
  461 IF((K/2+1)/2-K/4) 462,462,463
  462 BX=BX+BINO(M,K)*YYN(K)
      GO TO 510
  463 BX=BX-BINO(M,K)*YYN(K)
      GO TO 510
  465 IF((K/2+1)/2-K/4) 470,470,475
  470 BX=BX+BINO(M,K)*XXN(KK)*YYN(K)
      GO TO 460
  475 BX=BX-BINO(M,K)*XXN(KK)*YYN(K)
      GO TO 460
  480 K=1
      GM=M
      IF(M-1) 485,485,490
  485 BX=GM*YY
      GO TO 495
  490 BX=GM*XXN(M-1)*YY
  495 K=K+2
      KK=M-K
      IF(KK) 510,496,499
```

```
496 IF(((K-1)/2+1)/2-(K-1)/4) 497,497,498
497 BX=BX+BINO(M,K)*YYN(K)
    GO TO 510
498 BX=BX-BINO(M,K)*YYN(K)
    GO TO 510
499 IF(((K-1)/2+1)/2-(K-1)/4) 500,500,501
500 BX=BX+BINO(M,K)*XXN(KK)*YYN(K)
    GO TO 495
501 BX=BX-BINO(M,K)*XXN(KK)*YYN(K)
    GO TO 495
510 CONTINUE
515 T1=T1+ABA(J)*BX
    ER=T1-T
    PERC=100.0*ER/T
    WRITE(3,4) XX,YY,T1,T,ER,PERC
    IF(XX+DEL1-1.0) 520,525,525
520 XX=XX+DEL1
    GO TO 405
525 IF(YY+DEL2-H) 530,535,535
530 XX=DEL1
    YY=YY+DEL2
    GO TO 405
535 CONTINUE
    STOP
    END
   MON$$       EXEQ LINKLOAD
               PHASEB1
               CALL BOUNDS1
               PHASEB2
               CALL BOUNDS2
               PHASEB3
               CALL BOUNDS3
               PHASEB4
               CALL BOUNDS4
   MON$$       EXEQ B1,MJB
   MON$$       EXEQ B2,MJB
   MON$$       EXEQ B3,MJB
   MON$$       EXEQ B4,MJB
```

A METHOD FOR DETERMINING UPPER AND LOWER
BOUNDS FOR TWO-DIMENSIONAL HEAT CONDUCTION PROBLEMS

by

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AN ABSTRACT OF A MASTER'S REPORT

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A method is presented for obtaining bounds for the solution of two-dimensional, steady-state heat conduction problems without internal heat generation. The method emphasizes the attainment of an approximate solution of known accuracy, is systematic, and is well adapted to analysis on high speed digital computers.

The method is applied to a rectangular plate with specified boundary temperature. The results of the examples are close to the known exact solution in the first case and reasonably good in the second.