ON GENERAL BURNSIDE PROBLEM

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B.S., National Taiwan University, 1964

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1970

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CONTENTS

I. Introduction

Is every torsion group locally finite? Burnside raised the question in 1902, which became one of the most famous in group theory. This is called the general Burnside problem. There is another restricted Burnside problem which states: Let G be a torsion group in which $x^N = 1$ for all $s \in G$, N a fixed positive integer. Is G then locally finite? Until 1964 almost all work was on the restricted Burnside problem - there was no real attack on the general Burnside problem. In 1964, Golod and Shafarevitch settled the general Burnside problem in the negative.

In 1941 Kurosh asked, for algebraic algebras, the analogue of the Burnside problem. In §IV the work of Golod and Shafarevitch is used to construct a finitely generated, infinite dimensional, nil algebra, thus setteling the Kurosh problem in the negative. Using this algebra, an infinite, finitely generated, torsion group is constructed which settles the general Burnside problem in the negative.

Many types of torsion groups are locally finite, including a class of groups which may be imbedded in certain rings [1]. That work is beyond the scope of this paper, but, because of its importance, we give the special case of Matrix Groups which was settled by Burnside himself. This result is presented in §II.

Except for §III, terminology and definitions are mostly from Herstein [3]. In §III definitions are used from [2].

II. The Burnside Problem For Matrix Groups.

2.1 Definition. A group G is said to be a torsion group if every element in G is of finite order.

2.2 Definition. A group G is said to be locally finite if every finitely generated subgroup of G is finite.

2.3 Lemma. Suppose that G is a group, N a normal subgroup of G such that both N and G/N are locally finite. Then G is locally finite.

Proof. Let $g_1, \ldots, g_n$ be a finite set of elements of G: we wish to show that they generate a finite subgroup of G. If $\bar{g}_1, \ldots, \bar{g}_n$ denote their images in G/N then, by assumption, these generate a finite subgroup of G/N. Let this subgroup be $\bar{g}_1, \ldots, \bar{g}_n, \ldots, \bar{g}_t$ and let $g_{n+1}, \ldots, g_t$ be any representative inverse images of $g_{n+1}, \ldots, g_t$ respectively in G. For any i, j, $g_i g_j = u_{ij} g_k$ for some k and some element u_{ij} in N. Let U be the subgroup of N generated by all the u_{ij}; the local finiteness of N implies that U is a finite group. Given any three g_i, g_j, g_m then $g_i g_j g_m = u_{ij} g_k g_m = u_{ij} u_{km} g_w$, so is of the form ug_w with $u \in U$. Similarly any word in the g_i's is of the form ug_w with $u \in U$, $1 \leq w \leq t$. Hence the $g_1, \ldots, g_t$ generate a group of order at most to $t \cdot o(U)$, that is, a finite group.

2.4 Definition. Let G be a group and suppose it has a series of subgroups $1 = G_i \subset \ldots \subset G_1 \subset G_0 = G$. If each $G_r \triangleleft G_{r-1}$ for $r = 1, \ldots, i-1$, then the series is called a subnormal series for G.

2.5 <u>Definition.</u> A group G is said to be solvable if it has a subnormal series with G_{r-1}/G_r an abelian group $(r = 0,1,..,i-1)$.

2.6 <u>Lemma.</u> A solvable torsion group is locally finite.

<u>Proof.</u> Let G be a solvable torsion group. By the solvability of G we can find subgroups G_i where G_i is normal in G_{i-1} and G_{i-1}/G_i is abelian and where $1 = G_i \subset \ldots \subset G_1 \subset G_0 = G$. An abelian torsion group is clearly locally finite; applying Lemma 2.3 we see that we can climb up this chain to get that G is locally finite.

2.7 <u>Lemma.</u> A group of triangular matrices over a field is solvable.

<u>Proof.</u> Because a subgroup of a solvable group is obviously solvable it is enough to show that the group of invertible triangular matrices is solvable. To see this let:

$$G = G_0 = \left\{ \begin{bmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \middle| a_i \neq 0 \right\}, \quad G_1 = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right\},$$

$$G_2 = \left\{ \begin{bmatrix} 1 & 0 & & * \\ & \ddots & & \\ & & \ddots & 0 \\ 0 & & & 1 \end{bmatrix} \right\}, \quad G_3 = \left\{ \begin{bmatrix} 1 & 0 & 0 & & * \\ & \ddots & & & \\ & & \ddots & & 0 \\ & & & \ddots & 0 \\ 0 & & & & 1 \end{bmatrix} \right\}$$

and so on. Each G_i is normal in G_{i-1}, G_{i-1}/G_i is abelian and $G_n = (1)$. Thus the group of triangular matrices is solvable.