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From Buildings to Point-line Geometries and Back Again*

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Abstract

A chamber system is a particular type of edge-labeled graph. We discuss when such chamber systems are or are not associated with a geometry, and when they are buildings. Buildings can give rise to point-line geometries under constraints imposed by how a line should behave with respect to the point-shadows of the other geometric objects (Pasini [24]). A recent theorem of Kasikova [21] shows that Pasini's choice is the right one. So, in a general way, one has a procedure for getting point-line geometries from buildings. In the other direction, we describe how a class of point-line geometries with elementary local axioms (certain parapolar spaces) successfully characterize many buildings and their homomorphic images. A recent result of K. Thas [32] makes this theory free of Tits' the classification of polar spaces of rank three [35]. One notes that parapolar spaces alone will not cover all of the point-line geometries arising from buildings by the Pasini-Kasikova construction, so the door is wide open for further research with points and lines.

1 Introduction

This paper represents an attempt to place in perspective the relation between the theory of buildings and characterizations of point-line geometries bearing

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simple local axioms.

2 Buildings

Buildings are really chamber systems rather than geometries. Often there is a class of geometries that goes with a chamber system, and one may want to think of these geometries as the buildings; but really they are not the buildings. The latter are simply nice geometries – some met by geometers a century ago, some met by Greek geometers more than two thouseand years ago – but they do not tell the real story. That role falls to chamber systems.

2.1 Chamber systems

A **chamber system** is a set of objects C, which we shall call "chambers", together with a mapping

$$\lambda: \text{ unordered pairs of distinct chambers} \rightarrow 2^I,$$

the set of all subsets of a set I called the **type set**; the mapping λ must satisfy this property: for any three-set of chambers $\{x, y, z\}$ one has

$$\lambda(x, y) \cap \lambda(y, z) \subseteq \lambda(x, z) \tag{1}$$

For any type $i \in I$, let us say that two distinct chambers x and y are i**-adjacent** if i is a member of the set $\lambda(x, y)$. Then equation (5) implies that when combined with the identity relationship, i-adjacency becomes an equivalence relation which we denote by i^*. Any i^*-equivalence class is called an i**-panel**.[1]

Of course one may let E be the collection of unordered pairs of chambers for which λ assumes a non-empty value. Then we may regard $C = (C, E)$ as a simple graph for which each edge e is assigned a non-empty set of types $\lambda(e)$ such that equation (5) holds. We say that the chamber system (C, E, λ) is **connected** if and only if the graph (C, E) is connected.]

The collection of all chamber systems over the set I forms a category when provided with morphisms f which are graph morphisms such that the typeset of any edge $\lambda(e)$ is mapped into the typeset $\lambda'(f(e))$, of any image

[1] This definition is equivalent to the one given in Tits' book as a system $\{\pi_i\}$ of (not-necessarily distinct) partitions of C indexed by elements of I.

of that edge: Precisely stated, if $e = (x, y)$ is an edge of C, and if $f(x)$ and $f(y)$ are distinct, then

$$\lambda(x, y) \subseteq \lambda'(f(x), f(y)).$$

This categorical view-point is useful, for it opens the door to the concepts of universal covers of various types and all sorts of functors.

Perhaps the most basic concept of chamber systems is that of a *residue*. Let J be any subset of the typeset I that you have selected. We define a new type-function λ_J whose value at any edge e is $\lambda(e) \cap J$. Suddenly, each label not in subset J is regarded as invisible. Now we have a new collection of edges E_J – those for which λ assumes values in set J – and now the graph $C_J = (C, E_J)$ may no longer be connected since we may have erased edges in E. The connected components of the graph C_J are called[2] the **residues of C of type J**. The cardinality of J is called the **rank** of the residue, the cardinality of $I - J$ is called its **corank**.

2.2 Chamber Systems and Geometries

A **geometry over typeset** I is a multipartite graph (V, E) with parts V_i indexed by the elements i belonging to the type-set I. The language takes a geometric shift: the "*objects of type i*" are simply the vertices of the coclique[3] V_i; an object of type i is said to be **incident** with an object of type j if and only if they are adjacent vertices of the multipartite graph (V, E). Obviously i must be distinct from j in order for this relationship to occur. We may also think of a geometry as a triple $\Gamma = (V, E, \tau)$, where (V, E) is the multipartite graph already referred to, and $\tau : V \to I$ is the **type function** which records the type indexing the unique component V_i that an object belongs to.

A morphism of one geometry into another is nothing more than a graph morphism of multipartite graphs which preserves the type of the object. In this way, the geometries over I form a category and once again we inherit the language of category theory – allowing one to discuss universal covers with respect to any desired composition-closed subclass of morphisms, and to discuss functors (for example truncations).

[2]In the language of Ronan and Brouwer/Cohen these would be called "$(I-J)$-*residues*".

[3]Now generally accepted even by graph-theorists, "coclique" is a term the author first learned form his friend Jaap Seidel

Do not worry; we do not carry this category-theory stuff any further than the basic language needed – no derived functors or unecessary homological algebra will appear here.

Suppose $\Gamma = (V, E, \tau)$ is a geometry over I. A **flag** is nothing more than a clique F in the multipartite graph (V, E) – so it is simply a set of pairwise adjacent vertices and can involve at most one vertex of each type. The subset $\tau(F)$ is called the **type of the flag** F. A flag F is called a **chamber flag of** Γ if and only if $\tau(F) = I$, that is, it contains one object of each type presented by the set I. Of course such a flag cannot exist unless all of the sets V_i are non-empty; but such flags might not exist in any event.[4] Two flags of geometry Γ are said to be **incident** if and only if they are distinct and their union is still a flag – that is, a clique of (V, E).

One last definition is needed for geometries. Let us select a flag F of type J in the geometry $\Gamma = (V, E, \tau)$. The collection $\mathrm{Res}_\Gamma(F)$ is the induced subgraph of all vertices $v \notin F$ such that $F \cup \{v\}$ is a clique – that is, the vertices whose type is disjoint from $\tau(F)$, but which are still incident with F. Such vertices form a geometry over $I - \tau(F)$, called the **the residue of the flag** F, denoted $\mathrm{Res}_\Gamma(F)$.

Of course the language itself reveals a suggested link between geometries over I and chamber systems over I. Here it is:

Starting with a geometry $\Gamma = (V, E, \tau)$, we consider the collection of chamber flags of Γ (if there are any) and declare two of them to be *i-adjacent* if and only if they differ only in their objects of type i. The definitions produce a chamber system $\mathbf{C}(\Gamma)$ with an extra property we had not insisted upon before. Two chambers of this structure can only be related by at most one value of I – that is λ assumes values only in the empty set and singleton subsets of i.

Now let us try it the other way round. We begin with a chamber system $C = (C, E, \lambda)$ and let V_i be the **residues of cotype** i – that is, the residues of type $I - \{i\}$. We say that a residue of cotype i is "incident" with a residue of cotype $\{j\}$, if and only if the two residues contain a common chamber. Clearly the result is a geometry over I which we call $\mathbf{\Gamma}(C)$.

[4]Recently the desire to have a property to ensure the existence of chamber flags – such as having each flag lie in a chamber flag – has been put forth as a revised definition of "geometry" – the geometries of this paper would then be labeled "*pregeometries*". Of course such a restiction seems to change the category and the definition of all the available universal covers without offering any advantage in proving general theorems. In this paper, we will stick to Tits' original definition of "geometry" as given above.

It is easy to see that the mappings

$$\mathbf{C} : \mathcal{G}^I \longrightarrow \mathcal{C}^I \tag{2}$$
$$\mathbf{\Gamma} : \mathcal{C}^I \longrightarrow \mathcal{G}^I \tag{3}$$

connecting the categories of geometries and chamber systems over I are actually functors. The problem is that the domains in either of the equations could be empty or otherwise miniscule. So, as it stands, the relationship between the two categories could be nothing more than a smoky vapor that would only interest politicians.

This is where the concept of residual connectedness comes in. It arrives in two versions; one for geometries and one for chamber systems.

A *geometry* Γ *over* I is said to be a **residually connected geometry** if and only if the residue of every corank one residue is non-empty and the residue of every flag of corank at least two is a connected non-empty geometry. [It is easy to prove that any truncation of a residually connected geometry to two or more types (that is, after throwing away all but at least two type-components V_i), the resulting geometry over the surviving type-set is still residually connected. In short, the truncation functor preserves residual connectedness.[5]

A chamber system over I is said to be **residually connected**, if and only if:

(CRC1) *For any family $\mathcal{F} = \{R_t\}$ of residues of C which pairwise intersect non-trivially, the global intersection $\cap\{R_t \in \mathcal{F}\}$ is non-empty and connected.*

(CRC2) *For any chamber c the intersection of all corank 1 residues of C which contain c is the set $\{c\}$ itself.*

Residual connectedness for chamber systems is a very strong condition. We record here two immediate consequences, which do not seem to be in the general literature.

Theorem 1 (Chapter 9 of [31]) *Assume $C = (C, E; \lambda)$ is a residually connected chamber system over I.*

[5]This is a slightly more general restatement of a result of Buekenhout (see [4], for example).

1. *Then any residue of type J, a proper subset of I, is the intersection of all the corank* 1 *residues which contain it.*

2. *There is more: Suppose $e = (x, y)$ is an edge bearing the label i – that is $e \in E$ and $i \in \lambda(e)$. Suppose $G = (x = x_0, x_1, \ldots, x_n = y)$ is any gallery connecting to x to y. Then for some integer j in the interval $[1, n]$, we have*
$$\lambda(x_{j-1}, x_j) = \{i\}.$$
In particular
   (a) *The type function λ never assumes multiple values – that is, for every edge $e \in E$, $\lambda(e)$ is a single-element subset I.*
   (b) *Each residue of cotype i is an induced subgraph of (C, E).*
   (c) *All residues are induced subgraphs.*

Theorem 2 [6] (Arjeh Cohen, in [2])

1. *If geometry G is residually connected of finite rank, then so is $\mathbf{C}(G)$, and there is a geometry isomorphism $\mathbf{\Gamma}(\mathbf{C}(G)) \simeq G$.*

2. *If C is a residually connected chamber system, then $\mathbf{\Gamma}(C)$ is residually connected, and $\mathbf{C}(\mathbf{\Gamma}(C)) \simeq C$.*

3. *There exists an isomorphism between the subcategory of residually connected geometries over a finite typeset I, and the subcategory of residually connected chamber systems over the same finite I.*

Upon first reading, it would seem that there is a slight asymmetry between the first two statements of the Theorem. Assertion 1. entails finite rank in its hypothesis while Assertion 2. does not. Does the second assertion really apply in the more general realm of chamber systems of infinite rank? The answer is no. Consider:

Theorem 3 (Kasikova and Shult.[22]) *If C is a chamber system over an infinite set I each of whose panels contain at least two chambers, then C is not residually connected. In particular, no building of infinite rank (definitions of these terms will appear below) is residually connected.*

[6] A neccessary and sufficient condition that a chamber system have the form $\mathbf{C}(\Gamma)$, is given in Proposition 12.34 of [24]. It does not necessarily imply the isomorphism of the second statement of this Theorem.

Put another way, if C is a residually connected chamber system with all panels having at least two chambers, then it has finite rank, thus restoring symmetry to the first two statements of Theorem 2.

But there is a larger meaning to be read from Theorem 3, for it reveals a basic rupture between geometries and chamber systems once one ventures into infinite rank. In fact the two categories seem to live seperate lives at infinite rank. On the one side, there are buildings (defined as chamber systems) at any conceivable rank; and on the other side, there are also classical geometries (such as projective spaces, polar spaces and certain Grassmannians of infinite singular rank) which exist and can be characterized, but cannot find a chamber system building to latch onto.

2.3 Buildings as Chamber Systems

2.3.1 Chamber systems of type M

Let M be a symmetric matrix whose whose rows and columns are indexed by I, and whose entries are positive integers or the symbol "∞". It is required that the diagonal entries are all equal to 1 and that the off-diagonal entries are integers greater than one or the infinity symbol. Then $M = (m_{ij})$ codifies the generators and relations of a group, $G(M)$, called the Coxeter group.[7]

A chamber system is said to be **type** M if and only its type set I indexes the rows of $M = (m_{ij})$ and if each residue of type $\{i, j\}$ is the chamber system of a generalized m_{ij}-gon. Note that in a chamber system $C = (C, E, \lambda)$ of type M each edge e is labeled by a single type $\lambda(e)$.

2.3.2 Galleries

Suppose C is a chamber system of type M. A walk $w = (c_1, \ldots, c_n)$ in the graph (C, E) is called a *gallery* and its type $\lambda(w)$ is the word

$$\lambda(c_1, c_2)\lambda(c_2, c_3) \cdots \lambda(c_{n-1}, c_n)$$

in the free monoid I^* generated by the type set I. Now any word u in I^* corresponds to a product of the generating involutions t_i where the subscripts range over the letters of u, read from left to right. In turn, this product Πt_i

[7] Here $G(M) = \langle \{t_i | i \in I\}$ is generated by involutions t_i, and for distinct i, j, the product $t_i t_j$ has order m_{ij} with the understanding that if m_{ij} is the infinity symbol, then $t_i t_j$ has infinite order – i.e. t_i and t_j generate the infinite dihedral group.

is an element $\rho(u)$ of the Coxeter group $G(M)$. We say that the word u is **reduced** (with respect to M) if its corresponding expression Πt_i is a shortest such expression for $\rho(u)$.

A gallery is called a **geodesic** if it is a gallery of shortest possible length connecting its initial and terminal chambers. The type of any geodesic gallery is always a reduced word.

An **elementary M-homotopy** is the replacing of some subsegment of type $p(ij) := iji\cdots$ (of length m_{ij}) in a gallery, by a segment of type $p(ji) := jiji\cdots$ (also of length m_{ij}) (of course, this is possible only when m_{ij} is finite). We say two galleries G_1 and G_2 of $C(M)$ are **M-homotopic** if and only if one can be transformed into the other by a chain of elementary M-homotopies. Note that this type of homotopy is length-preserving.

2.3.3 Strong gated-ness

Let $H = (V', E')$ be a subgraph of a connected graph $G = (V, E)$ and choose a vertex $v \in V - V'$. Then H is said to be **strongly gated with respect to vertex** v if and only if there is a vertex $g \in V'$ such that for every vertex $h \in V'$ we have

$$d_G(v, h) = d_G(v, g) + d_H(g, h). \quad (4)$$

Here d_H and d_G are the distance metrics with respect to the graphs H and G respectively. We say H is **strongly gated** if and only if it is strongly gated with respect to every exterior vertex. [8]

Any strongly gated subgraph of G is a convex induced subgraph, and so is isometrically embedded in G.

2.3.4 Definition of building

Theorem 4 *Let $C = (C, E, \lambda)$ be a connected chamber system of type M. Then the following conditions are equivalent:*

(RG^1) Every residue of co-rank one (i.e. a residue of type $I - \{j\}$ for some $j \in I$) is strongly gated.

(RG_1) Every residue of rank one or two is strongly gated.

(RG) All residues are strongly gated.

[8] This is stronger than the condition of being "gated" introduced in [17].

(G) *Every gallery of reduced type is a geodesic.*

(P) *(Tits' condition) : Any two galleries of reduced type with the same initial and terminal chambers are M-homotopic.*

We call any chamber system obeying any of these equivalent conditions a **building**. This is justified since these conditions are also equivalent to the existence of a Tits sytem of apartments, the traditional definition of building. Note that none of the conditions require the type set I to be finite.

The conditions (RG_2) and (RG) allow one access to rather simple proofs of basic properties of buildings.[9] Thus one has:

Theorem 5 *Let C be a chamber system of finite rank satisfying these two conditions:*

(i) (RG): All residues are strongly gated.

(ii) (typ) The edges of C assume just one type-label.

Then C is residually connected.

In particular, any building B of finite rank is residually connected.

Theorem 6 [10] *Suppose C is a chamber system satisfying condition (typ) and condition condition (RG_2) which asserts that all residues of rank at most two are strongly gated in C.*

Then C is 2-simply connected — that is, all circuits of the graph $(C.E)$ are $\mathcal{C}_2$-contractible, where $\mathcal{C}_2$ is the class of circuits of (C, E), each of which lies in some rank 2 residue.

In particular, any building B of arbitrary rank is 2-simply connected.

3 Point-line geometries from buildings

3.1 Point-line geometries

Perhaps the simplest geometries to consider are the rank-two geometries. Of course these are just bipartite graphs describing the incidence relation between two classes of objects. We think of these as a point-line geometry

[9]The real idea behind (RG_2) is due to R. Scharlau in [25]. See also [27].

[10]The proofs of these two theorems are presented in Section 9.3 of [31].

$(\mathcal{P}, \mathcal{L})$ by designating one of the classes "points" ($\mathcal{P}$) and the other class "lines" ($\mathcal{L}$). Just introducing words doesnt change anything; two lines might have many incident points in common. Nonetheless, the idea is appealing for this is the sort of visually intuitive geometry which fascinated our Greek forbears.

3.1.1 A short glossary of concepts surrounding point-line geometries

Nonetheless, there is a shift in point of view when we declare one of the types to be "points": first there is the requirement that each line be incident with at least two points (while there is no such requirement about points. Secondly there is the asymmetric notion of "subspace". A **subspace** is a collection S of points such that the *point-shadow* of every line (that is, the collection of all points incident with the line) is either contained in S, or intersects S in at most one point. Clearly $\mathcal{P}$ and the empty set are subspaces. Since the intersection over any family of subspaces is also a subspace one may consider the intersection of all subspaces which contain a prescribed set of points X. This subspace is denoted $\langle X \rangle$ and is called the **subspace generated by** X.

Of course, with any point-line geometry Γ, there is a **point-collinearity graph** Δ whose vertex set is $\mathcal{P}$ and whose edges are pairs of distinct points incident with a common line (collinear points). The **distance** between points is simply their graph-theoretic distance as vertices of Δ. A subspace S of $(\mathcal{P}, \mathcal{L})$ is said to be **convex** if and only if any geodesic path connecting two points S has all its intermediate points in S. As is the custom with graphs, we let $p^\perp$ denote the vertex p together with all vertices that are adjacent to p — for the point collinearity graph, this would be point p together with all points which are collinear with p.

A **gamma space** is a point-line geometry $(\mathcal{P}, \mathcal{L})$ for which $p^\perp$ is always a subspace. A **singular subspace** is a subspace S whose points are all pairwise collinear. In a gamma space, any clique in the collinearity graph generates a singular subspace. By some Zorn-like argument, maximal singular subspaces always exist in a gamma space.

The **point-shadow of a line** (or any other object) is just the set of points incident with it. In virtually all cases of interest, distinct lines possess distinct points shadows, and so may be regarded as sets of points subject to set-theoretic operations. A point-line geometry in which the point-shadows of any two distinct lines intersect in at most one point, is called a **partial**

linear space. When one thinks about it, a partial linear space is just a point-line geometry in which lines are subspaces. A **linear space** is a partial linear space which is singular — that is, any two distinct points are incident with a unique line.

3.2 Simple constructions

In describing a point-line geometry obtained from a building we need to consider certain flags defined by a *basic diagram.* For this purpose, let us suppose Γ is a residually connected geometry over a finite type set I for which the flag-chamber system $\mathbf{C}(\Gamma)$ is a chamber system of type M. Associated with the Coxeter matrix M is a **basic diagram graph** $D = (I, \sim)$ whose edges are pairs (i, j) for which $m_{ij} > 2$ (see [3]).

One simple way to form a point-line geometry $(\mathcal{P}, \mathcal{L})$ is to select a type k, let $\mathcal{P}$ be all objects of type k, and let $\mathcal{L}$ be the collection of all flags whose type is $D_1(k)$, the set of all vertices adjacent to k in the basic diagram graph D.

A classic example of this procedure would be the definition of the **Grassmann spaces** $A_{n,k}$ whose "points" are the k-dimensional vector subspaces of some $(n+1)$-dimensional vector space V, and whose lines are the $(k-1, k+1)$-dimensional subspace flags. (Incidence is inherited from incidence of flags in the projective geometry A_n.) In fact, for the geometries associated with the spherical buildings, one obtains a host of familiar geometries in this way. These are displayed in Figure 3.2.

But of course, from the original geometry Γ, one inherits certain further objects which are neither points or lines – just objects of Γ which are incident with their own collections of "points" and "lines" – let us call them *satellite objects*, for the sake of discussion. Thus for the Grassmannian $A_{n,k}$ mentioned above, the satellite objects include two classes of maximal singular subspaces, as well as a number of convex subspaces which are themselves Grassmannians.

3.3 More general constructions

Once again, we assume that we have a building B which, in the finite rank case, will be regarded as both geometry over I as well as a chamber system over I.[11] Our objective is to select a subset J of the typeset, and think

[11] In the infinite rank case, we must think of B as a chamber sytem. We need both points of view in order to render historical presentations in their original language.

Figure 1: Some of the more familiar Lie incidence geometries, excluding projective spaces and polar spaces. The points are the objects whose type is labeled by "$\mathcal{P}$" in the digram. The lines, $\mathcal{L}$. are those flags whose type is the collection of nodes which are neighbors of $\mathcal{P}$ in the diagram. This is a naive scheme. When points are to be flags of a fixed type in an arbitrary diagram, the recipe for defining "lines" is much more complicated.

of the flags of type J in the building *geometry* as the set of "points" of a geometry. There are two issues: (i) what are the other objects that we should be considering? (ii) How do we make a reasonable point-line geometry with the flags of type J as points?

3.3.1 The geometry of J-reduced objects

If F is any flag of the building geometry B, the **J-shadow of F** is simply the collection $sh_J(F)$ of all flags of type J which are *incident* with the flag F. (Recall that in a geometry, two flags are defined to be incident if and only if their union is also a flag.) The problem is that sometimes there are geometric objects which are members of a flag which are not essential in determining the shadow of that flag. Thus we could say that an object $x \in F$ is *innessential* relative to F if the J-shadow of $F - \{x\}$ is the same as that of F; and that x is *essential* relative to X otherwise. The point is that if Y is any sub-flag of X which contains x and if Y "still supports the J-shadow of X" – that is $sh_J(F) = sh_J(Y)$ – then, x is essential to Y as well. It follows that there is a a set X_J of elements of X which are essential to every subset of X which supports the shadow of X and moreover that any such supporting set Y must contain all of these essential elements. Thus for every flag F in the geometry B, there exists a subflag $r_J(F)$ consisting of only the J-essential objects. Such a flag is said to be **J-reduced**.

The next observation if that this reduction can be done universally in the poset of types. Thus, for any flag F of type K, the J-reduced object $r_J(F)$, always has the same type $r_J(K)$. Thus the idempotent operator on the poset of flags that takes each flag to its J-reduced subflag is actually induced by a similar idempotent ρ_J on the homomorphic image of the flag-poset under the *typ* homomorphism. Thus we have

$$typ \circ r_J = \rho_J \circ typ$$

as poset morphisms from the poset of flags of the building geometry B to the Boolean poset set of all subsets of I.

All of this is contained in Chapter 12 of Tits' book [35], an appendix entitled "Shadows". There, a numbered complex plays the role of the flag complex of a geometry, and the J-reduction is described in terms of operations in that complex. The result is a geometry with a distinguished set $\mathcal{P}$ of points (the flags of B of type J) and the set of all J-reduced objects, which are the flags of B of J-reduced type. The advantage is that *distinct*

J-reduced objects possess distinct J-shadows. I personally think that that was the whole point of the chapter. At this early stage, Tits was trying to open the door to future applications of his theory of buildings to geometries whose objects are describable as certain subsets of points.

And of course that is exactly what this talk is about.

3.3.2 J-Grassmannians

But which of these J-reduced objects should play the role of lines? In fact, to answer that question, we should be asking what properties lines should have. Looking at the classical examples, we might hope that

1. Any object (one of those J-reduced things) that is incident with a line is in fact incident with every point of the point-shadow of that line.

2. The lines, together with the points, should form a partial linear gamma space, if that is not asking too much.

3. Perhaps the J-reduced objects should be subspaces.

In fact such a proposal for lines was made in the book of Pasini ([24]). Here we follow the approach of Kasikova, which is stated in terms of a chamber system C of type M. As before. $D = D(M)$ is the *basic diagram graph*, whose vertices will be called "nodes". For each node, α, $D_{0,1}(\alpha)$ will denote the set consisting of the node α as well as all nodes which are adjacent to α in the graph D. From the chamber system point of view, our "points" are now the residues of C of type $S = I - J$, rather than flags of $\Gamma(C)$ of type J. Then, for any residue R, the "point shadow of R", $sh_S(R)$, is the collection of all residues of type S which intersect R non-trivially. Again, to form our geometry, we pay attention only to residues R which are of J-reduced type, as described above.

We have already designated the set $\mathcal{P}$ of all residues of type S as "points''. Now we have a recipe for lines. A "line" is a residue of any one of the types

$$T := \{\alpha\} \cup (S - D_{0,1}(\alpha)) \tag{5}$$

as α ranges over $J = I - S$.[12] The set of all "lines" (as defined by equation (5)) is denoted $\mathcal{L}$. A line L, which is a residue of type T for one of the choices

[12] See Kasikova [21]. Of course one can write $D_1(\alpha)$ for $D_{0,1}(\alpha)$ in this formula (5). The reason for writing it this way is that in Pasini's theory of J-Grassmannians, one has

of T allowed in formula (5), is incident with a point p, itself a residue of type S, if and only if the two residues have a non-empty intersection – that is, they share a common chamber. The point-line geometry $(\mathcal{P}, \mathcal{L})$ is called the **J-Grassmannian** of the chamber system C of type M. Of course we will be interested in cases where $C = B$, a building.

Let's look at a classic example. Let B be a building of type A_n, so $I = \{1, \ldots, n\}$, and suppose we wish to consider the objects of type k as points, where $1 < k < n$. (This is the classic Grassmannian of k-spaces of an $n+1$-dimensional vector space.) Then $S = I - \{k\}$, and the formula for T gives the unique result $T = \{1, \ldots, k-2, k, k+2, \ldots, n\}$. Thus "lines", which are residues of type T correspond to flags of type $I - T = \{k-1, k+1\}$. This corresponds to our naive notion of line for the Grassmannians (see Figure 3.2).

Suppose now, $J = \{1, n\}$ in the building B of type A_n of the previous paragraph. Then our "points" are the point-hyperplane flags of the $PG(n)$-geometry. The reader can check that recipe of equation (5) produces two types of "lines": the residues corresponding to flags of types $\{1, n-2\}$ and flags of type $\{2, n\}$.

Now we come to the main theorem of this section.

Theorem 7 (Corollary 6.2 of Kasikova, [21]) *Let $(\mathcal{P}, \mathcal{L})$ be the J-Grassmannian of a building B (regarded as a chamber system over I) with basic diagram $D(M)$. Set $S = I - J$. Then for any residue R of B, the S-shadow $sh_S(R)$ is a convex subspace of the J-Grassmannian $(\mathcal{P}, \mathcal{L})$.*

Thus, the lines of a J-Grassmannian are doing exactly what they should.[13] The proof of this theorem utilizes the strongly gated property of all residues of a building (see Theorem 4).

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objects partitioned into sets $\mathcal{O}_k$ which are flags of one of the types

$$T = K \cup \{S - \cup_{\alpha \in K} D_{0,1}(\alpha)\}$$

where K ranges over $(k-1)-$subsets of I for which every connected component of K (as an induced subgraph of D) meets $I - S = J$ non-trivially. Then our points, the set $\mathcal{P}$ of residues of type S, is in fact the set $\mathcal{O}_1$. Our lines, as defined above, then form the set $\mathcal{O}_2$.

[13] Kasikova's paper includes theorems that under certain conditions on S allow one to recognize the S-shadows of an apartment of B. But that is beyond the scope of this section.

4 From point-line geometries to buildings

4.1 Introduction

Now we consider the opposite endeavor: beginning with a point-line geometry subject to certain simple axioms on points and lines, can we recognize it as a truncation of some well known geometry? Throughout we shall assume each line possesses at least three points.[14]

4.2 Two classic cases

Theorem 8 (Projective Spaces (Veblen-Young. ([38])) *If $(\mathcal{P}, \mathcal{L})$ is a linear space with at least two (thick) lines, and if $(\mathcal{P}, \mathcal{L})$ satisfies the Veblen axiom*[15]*, then it is either a projective plane or it is isomorphic to the geometry of* 1- *and* 2-*dimensional spaces of a (possibly infinite-dimensional) right vector space V.*

Theorem 9 (Polar Spaces (Veldkamp [39]/ Tits [35]/Buekenhout-Shult [5]/Johnson [20] /Johson-Pasini-Cuypers [14])) *In $\Gamma = (\mathcal{P}, \mathcal{L})$ suppose only*

(i) no point is collinear with all other points, and

(ii) for any non-incident point-line pair (p, L), p is either collinear with exactly one point of L or is collinear with all the points of L.

Then Γ is one of the following:

1. *a generalized quadrangle (rank 2 polar space),*
2. *a rank three polar space* (classified by J. Tits in [35])*, or*
3. *the geometry of* 1-*and* 2-*dimensional subspaces of a right vector space V which are either all such isotropic spaces with respect to a non-degenerate reflexive sesquilinear form f on V, or are all such subspaces which are totally singular with respect to a non-degenerate pseudoquadratic form q on V.*

[14]Some of the characterization theorems have versions which allow lines with two points, but we omit them in order to keep things simple.

[15]Sometimes called Pasch's axiom

[As the language of Theorem 9 suggests, a point-line geometry satisfying the hypotheses (i) and (ii) is called here a **polar space** (actually a *non-degenerate* polar space) in general contexts). It is not assumed in advance to be a partial linear space. Nor is it assumed that the singular subspaces are projective. Both of these statements can be proved using a theory of Teirlinck ([33]). The **(polar) rank** of a polar space is the rank of its geometry of singular subspaces when that number is finite, or is simply said to "infinite" otherwise.]

In theorems 8 and 9, the rank two cases (representing generalized 3-gons and 4-gons, respectively) have not been classified. [16] The classification of the rank three polar spaces exploits the Moufang property, and parameterizes the spaces by norms on Caley-Dickson algebras. I do not think it is an easy proof.

When the rank is beyond 2 both cases give us big groups – even when the enriched geometry of subspaces has infinite rank. For projective spaces, this is ensured by the Jacobson density theorem; for rank three polar spaces one has the Moufang property, and for classical polar spaces this is ensured by the infinite version of Witt's theorem which tell us that isometries between finite dimensional subspaces always lift to an isometry of (V, f) or (V, q) as appropriate. Please note that isometries between infinite-dimensional subspaces of V need not lift. There are easy examples of sesquilinar forms (V, f) which possess maximal singular subspaces of two different infinite dimensions, and one cannot lift an isometry of the smaller into the larger.

In both theorems, the finite rank examples are residually connected and their associated chamber systems are indeed buildings belonging to diagrams A_n, B_n, C_n, D_n. But what happens when there are singular subspaces of infinite projective rank? Are they buildings?

In answering this question one has to ask what are the objects in the geometry? For the sake of discussion, consider a sesquilinear form (V, f) which has isotropic subspaces of infinite dimension. If one considers *all* isotropic subspaces to be objects of the geometry, we have a problem constructing the desired chamber system. True, unrefinable chains of subspaces exist (by a Zornification on the poset of chains) but how does one define i-adjacency when there is an ambiguity about assigning types by dimension? On the other hand, if one solves the problem of types by considering only isotropic

[16] A classification is not at all likely in the case of planes, but the tightness of the situation seems to increase for quadrangles.

subspaces of finite dimension to be the objects of the geometry, how does one prove residual connecteness, a definition that refers to flags of corank one? It is enough to give you a headache.

4.3 How point-line characterizations take place

There are many interesting point-line geometries. For some – such as generalized quadrangles – it is impossible to increase the rank of the geometry by adding new classes of subspaces. In these cases one hopes that postulating groups of automorphisms may help. Most spectacular in this direction is the theorem of Tits and Weiss (see [37]) classifying all Moufang generalized polygons. The Moufang condition is very natural here since rank two residues of higher rank buildings must possess this condition. But characterization theorems using smaller-than-Moufang groups exist for finite generalized quadrangles ([34]).

At other times, one is able to "enrich"[17] the geometry by creating certain classes of subspaces. For example, in a partial linear space, $\Gamma = (\mathcal{P}, \mathcal{L})$, exploiting a *diagonal axiom* frequently produces a class $\mathcal{C}$ of cliques of the point-collinearity graph. Characterizations of Grassmann spaces with one of the two classes of maximal singular spaces having finite projective rank can occur this way.

There seem to be two basic approaches.

1. Fischer-type theorems.

2. Theorems set in parapolar spaces.

In the Fischer approach, one has a point-line geometry $(\mathcal{P}, \mathcal{L})$ and then specifies the possible alternatives for the subspace generated by any two intersecting lines – say, a plane, an affine plane, a dual affine plane, or a c^*-geometry . Despite the fact that there is no visible diagram geometry, it is amazing how far such theorems can proceed. They are a perfect geometric analogue of theorems of Bernd Fischer which specify what is generated by three involutions of a "nearly simple" finite group. That theorem certainly amazed group-theorists of that time. Similarly there are surprising theorems characterising various classical point-line geometries minus a subspace.[18] In

[17]This useful term is due to Pasini, [24].

[18]The author was once priveledged to give a (now-outdated) survey of these geometric analogues of Fischer's theorems at a meeting in Bielefeld held in honor of Bernd Fischer.

recent years, the best work in this area has been due to Cuypers and his associates (see for example [12], [13], [9], and Cuypers and Passini, [15]. Here, we will follow the parapolar approach.

4.4 Introduction to parapolar spaces

In Cooperstein's early work on exceptional geometries [10], certain convex subgeometries called symplecta played a crucial role. When one looks at the way symplecta work in the other geometries of Figure 3.2, the definition of parapolar space seems to flow naturally.

In any point-line geometry, a **symplecton** is a convex subspace which happens to be a (non-degenerate) polar space, as that term was defined in Theorem 9 and the remark following. A **parapolar space** is a *connected* gamma space $\Gamma = (\mathcal{P}, \mathcal{L})$ with the property that for every pair of non-collinear points (x, y) either

1. $x^{\perp} \cap y^{\perp}$ is empty,
2. $x^{\perp} \cap y^{\perp}$ contains exactly one point (then (x, y) is called a **special pair**), or
3. $\{x, y\}$ is contained in some symplecton (in which case (x, y) is called a **polar pair**).

If special pairs do not occur, the space is called a **strong parapolar space**. The first four point-line geometries displayed in Figure 3.2 are strong parapolar spaces.

For any integer $k > 1$, a parapolar space is said to have **symplectic rank** k (**symplectic rank at least** k) if and only if every symplecton has polar rank k (at least k). For example Grassmann spaces have symplectic rank three while half-spin geometries have symplectic rank four. If a parapolar space has symplectic rank at least three, then every singular subspace is a projective space.

4.5 The beginnings

The idea was to use parapolar spaces as a stage on which to characterize geometries of each Lie type, using only purely local hypotheses that do not *prescribe point residues.* Thus one does not say that "each line lies in just two maximal singular subspaces" as in earlier characterizations of Grassmannians by Shult [26] and Bichara-Tallini [1]. If you are going to assume the parapolar space paradym, you must give up something, and according to this speaker's aesthetics, specified point-residues must be abandoned. Somehow one should be able to recover a point residue from even "more local" hypotheses.

The first great step along this line was taken by Arjeh Cohen in his magnificent paper "On a theorem of Cooperstein"[7]. He begins with a parapolar space of symplectic rank three; this is his hypothesis:

If L is a line and x is a point such that $x^{\perp} \cap L$ is empty, then $x^{\perp} \cap L^{\perp}$ is either empty or contains a line.

In this context, the hypothesis is equivalent to the following:

If x is a point not incident with symplecton S, then $x^{\perp} \cap S$ is either empty, or is a maximal singular subspace of S – in this case a plane.

In the course of the proof, one must consider (in a point-residue) a symplecton S (in this case a quadrangle) which is disjoint from a maximal singular space M; every point of S is collinear with a unique point of M, inducing a mapping $S \to M$ which one wishes to show is injective. If false, one acquires in S a very peculiar subquadrangle with a system of spread lines and grids which form a projective plane. In what I will refer to as the "technical lemma", Realizing that this must be a subquadrangle of a point-residue of a rank three polar space, Cohen uses Tits' classification of rank three polar spaces (see Theorem 9) to obtain its embedding in a quadrangle described in terms of norms on Cayley algebras. By a careful case-by case analysis, he shows that this is not an environment that can sustain such a bizarre quadrangle.

The reason that I have gone into such detail is that Cohen's paper and his technical lemma are absolutely essential for virtually all the parapolar space characterizations that came afterward. If I may invoke a geographic metaphor, Cohen's paper and that vital technical lemma reside on the only isthmus from the mainland into the land of parapolar spaces. As a result, three decades of work on parapolar spaces still logically rested on the classification of rank three polar spaces.

Recently, Koen Thas simplified things. That strange quadrangle excluded by Cohen's Lemma can be excluded on the simpler ground that it cannot be

Moufang ([32]).[19]

Now that Cohen had opened the gates, it was only natural that Cohen and Cooperstein should collaborate on a series of more universal theorems.

These theorems and others that followed are listed below:

Theorem 10 (Cohen-Cooperstein-I [8], (updated in [31])) *Let Γ be a strong parapolar space, all of whose singular subspaces possess finite rank, and all of whose symplecta possess a constant symplectic rank $r \geq 3$. We assume the following conditions:*

1. *Γ is not itself a polar space.*
2. *For any non-incident point-symplecton pair (x, S), the intersection $x^{\perp} \cap S$ is never a hyperplane of a maximal singular subspace of S.*

Then one of the following conclusions must hold.

1. *If $r = 3$ then Γ is either*
   (a) *the Grassmannian $A_{n,k}(D)$ of k-spaces of a division ring D, or*
   (b) *the quotient $A_{2n-1,n}(D)/\langle\sigma\rangle$, where σ is a polarity of V of Witt index at most $n - 5$.*
2. *If $r = 4$, $\Gamma = (\mathcal{P}, \mathcal{L})$ is a homomorphic image of a half-spin geometry of type $D_{n,n}$ over a field F. This homomorphism is an isomorphism if $n \leq 9$.*
3. *If $r = 5$, then Γ is the Lie incidence geometry $E_{6,1}(F)$.*
4. *If $r = 6$, then Γ is the Lie incidence geometry $E_{7,7}(F)$ (in the Bourbaki node-numbering scheme).*

Under no circumstances can r exceed 6.

One can easily recognize the geometries of the conclusion in Figure 3.2.

A closely related theorem is proved in [31]:

Theorem 11 *Suppose Γ is a strong parapolar space of finite singular rank and symplectic rank at least three. Suppose Γ satisfies the hypothesis:*

[19] One does not have to *classify* rank 3 polar spaces in order to show that they and their residues – and even the subquadrangles of those residues – are Moufang.

(U) *Whenever* A *and* B *are two symplecta of the parapolar space* Γ *which intersect in a subspace properly containing a line – then, for each point* x *in* $A - (A \cap B)^{\perp}$*, the set* $x^{\perp} \cap B$ *is not contained in* $A \cap B$*.*

Then Γ *is one of the "***Cohen-Cooperstein geometries***" – that is, a polar space, a Grassmannian, a quotient of a Grassmannian* $A_{2n-1,n}(D)$ *by a polarity of index at most n-5, an appropriate homomorphic image of a half-spin geometry, or one of the exceptional Lie incidence geometries of types* $E_{6,1}$ *or* $E_{7,7}$*.*

Theorem 12 (Cohen-Cooperstein-II, [8] (updated in Kasikova-Shult [23])) *Suppose* Γ *is a parapolar space of symplectic rank at least three satisfying these axioms:*

(H1) *Given a point* x *not incident with a symplecton* S*, the space* $x^{\perp} \cap S$ *is never just a point.*

(H2) *Given a projective plane* π *and line* L *meeting* π *at point* p*, either (i) every line of* π *on* p *lies in a common symplecton with* L*, or else (ii) exactly one such line incident with* (p, π) *has this property.*

(H3) *Given a point-line flag* (p, L) *there exists a second line* N *such that* $L \cap N = \{p\}$ *and no symplecton contains* $L \cup N$ *— i.e.* (x, y) *is a special pair for each* $(x, y) \in (L - \{p\}) \times (N - \{p\})$*.*

(F) *If all symplecta have rank at least four, assume every maximal singular subspace has finite projective rank.*

Then Γ *is*

1. $E_{6,2}, E_{7,1},$ *or* $E_{8,8}$ *(in the Bourbaki numbering),*
2. *a metasymplectic space, or*
3. *a polar Grassmannian of lines of a non-degenerate polar space of (possibly infinite) rank at least four. In the case of finite polar rank, these would be classical Lie incidence geometries of type* $(B/C)_{n,2}$ *or* $D_{n,2}$*,* $n \geq 4$*.*

The first two geometry-classes in the conclusion of Theorem 12 are displayed in lines 5-8 of Figure 3.2.

So far, the general polar Grassmannians have not been characterized. The theorem which follows basically folds them in with metasymplectic spaces but requires point-residuals to possess the pentagon property which we now define:

(The Pentagon Property) *Suppose* $w = (x_o, x_1, x_2, x_3, x_4, x_5 = x_0)$, *is a 5-circuit in the point-collinearity graph of a parapolar space* $(\mathcal{P}, \mathcal{L})$. (The word "circuit" is understood here to mean w is a circular path and that there are no further collinearities to be found among the vertices of this path.) *Then there exists a symplecton containing* w.

For a parapolar space of symplectic rank at least 3, we say that the **pentagon property holds locally** if and only if it holds in the residual parapolar space $(\mathcal{L}_p, \pi_p)$ of all lines and projective planes on the point p.

Theorem 13 (Tits ([35]), Cohen ([6]), Shult ([28]), Ellard and Shult ([18])) *Let* $\Gamma = (\mathcal{P}, \mathcal{L})$ *be a parapolar space of symplectic rank at least three. (It is not assummed in advance that* Γ *is locally connected.) Assume the following hypotheses:*

(1) Every singular subspace of Γ *has finite projective rank.*

(2) The Pentagon Property holds locally (i.e. for each point p *the point-residual* $\Gamma_p = (\mathcal{L}_p, \Pi_p)$ *satisfies the Pentagon Property.)*

(3) If S *is a symplecton, and* $x \in \mathcal{P} - S$ *is such that* $x^\perp \cap S = \{p\}$*, a single point, then there exists a point* $y \in p^\perp - S$*, such that* $y^\perp \cap S$ *contains a plane.*

(4) There exists in Γ *at least one point-symplecton flag* (p, S) *such that for every line* L *on* p *which is not in* S*,* $L^\perp \cap S$ *is just the point shadow of a line* L' *(possibly depending on the choice of* L*).*

Then Γ *is one of the following:*

(i) A non-degenerate polar space of finite polar rank at least three.

(ii) One of the following three types of metasymplectic spaces classified by Tits:

(a) *The Lie incidence geometry of a (non-weak) building of type $F_{4,1}$.*

(b) *The Polar Grassmannian of lines of a non-degenerate non-oriflame polar space of polar rank four – a Lie incidence geometry of type $C_{4,2}$.*

(c) *The polar Grassmannian of lines of a non-degenerqate oriflame polar space of polar rank four – type $D_{4,2}$.*

(iii) *A polar Grassmannian of singular $PG(k)'s$, $k > 1$, in a non-degenerate polar space of finite polar rank at least $k + 2 > 4$.*

This Theorem has its origin in a paper of Cohen, [6], characterizing metasymplectic spaces which first introduced the pentagon property. (Of course, when speaking of "origins", all the theorems just listed must be played against the background of Tits characterizations of these geometries (sometimes as point-line geometries rather than buildings, as in the case of polar spaces and metasymplectic spaces). The polar Grassmanian conclusion requires the use of "Hanssens principle" (see Chapter 13 of [31]) and Tits' "local approach theorem" [36].)

4.6 Characterizations by singular subspaces

Of course we have not covered all the spherical Lie-incidence geometries whose points are the objects whose type is represented by an end-node of the spherical diagram. We are missing $E_{7,2}$, $E_{8,1}$ and $E_{8,2}$. These and many non-shperical geometries of type M, whose points are represented by a single node, can be characterized as parapolar spaces with special conditions regarding the relation of points and a class of maximal singular subspaces (not necessarily all maximal singular subspaces).

One begins with a class $\mathcal{M}$ of maximal singular spaces of a of a parapolar space of symplectic rank at least 3, and one supposes that there exists an positive integer d such that for every pair $(x, M) \in \mathcal{P} \times \mathcal{M}$, $x^{\perp} \cap M$ is either empty or a $PG(d)$. Then $d = 1$ or 2. In the case that $d = 1$, one must assume that there exists a line incident with at least two members of M. The conclusions are polar spaces, Grassmannians, Grassmanians mod a polarity, and half-spin geometries. Next, taking such a space to represent the point-residuals of a parapolar space of symplectic rank at least 4, one can show that there is a uniform outcome for residuals, thus yielding a polar space, or a geometry which is locally a Grassmannian, or a twisted Grassmannian

.

Figure 2: The diagrams and polarities which belong to the building geometries or building geometries mod a diagram polarity, whose point-line truncationes are preimages of the geometries of Theorem 14.

modulo a polarity, or locally a homorphic image of a half-spin geometry. All of these cases yield geometries that are homomorphic images of building geometries or a buiding geometry modulo a diagram polarity. (In the latter case, one must use Tits' local approach theorem on certain covers that admit the diagram polarity.) In this way, we obtain

Theorem 14 (Chapter 16 of [31] and [29] and [30]) *Suppose* $\Gamma = (\mathcal{P}, \mathcal{L})$ *is a parapolar space of symplectic rank at least four having a class of maximal singular subspaces* $\mathcal{M}$ *such that every plane is contained in a member of* $\mathcal{M}$*, and for which there exists an integer* $d \geq 2$ *such that for any point* x *not in* $M \in \mathcal{M}$*, one has* $x^{\perp} \cap M$ *empty, a single point, or a* $PG(d)$*. If* $d = 2$ *assume some plane lies in at least two members of* $\mathcal{M}$*.*

Then $d = 2$ *or* 3 *and* Γ *is a point-line truncation of a homomorphic image of building geometry (or buildings geometry modulo a diagram polarity) belonging to the diagrams displayed in Figure 4.6 below.*

Note that the heretofore un-characterized spherical end-node geometries mentioned at the beginning of this subsection are now covered. Details can be found in Chapter16 of [31].

4.7 Two open questions

4.7.1 Question 1.

One may notice that at the beginning the previous paragraph, there was an extra condition needed in the case that $d = 1$. If it fails, then the spaces in $\mathcal{M}$ pairwise intersect in at most one point. This means that in a point-residue of such a geometry, symplecta are quadrangles and the subspaces $\mathcal{M}_p$ pairwise intersect at point p – that is, they partition the "points" of the residual. Moreover in this residual geometry, $\mathrm{Res}_\Gamma(p)$, each point not in a maximal singular subspace M_p derived from an element of $\mathcal{M}_p$, is collinear to exactly one point of M_p. One now certainly has the situation that set up Cohen's "Technical Lemma. The argument forces convex non-grid quadranges to exist in this geometry. But there is no end-result showing that this picture of a point-residual cannot exist. In fact, using the notion of an *admissable triple* (introduced by Bart DeBruyn [16]), one can construct examples which fulfill all the requirements – suggeting there was a good reason to place that condition in the theorems described above. But the residual we are speaking of is the point-residual of a parapolar space of symplectic rank 3, and each non-oriflame symplecton S (recall that they must exist) now possesses a collection of maximal singular $PG(2)$'s which pair-wise meet in at most one point. In other words, they form an *ovoidal hyperplane* of the dual polar space DS associated with S. we should mention that since the elements of $\mathcal{M}$ are not planes, the planes of S are Desarguesian, so S (being non-oriflame) are classic embeddable rank 3 polar spaces. As far as the writer knows the non-existence of such ovoidal hyperplanes has been shown only for the finite dual-polar spaces of type $W(3, q)$ (Cooperstein and Pasini [11], a difficult proof using hard theorems of Woldar and Hemmeter [19].)

My point here is simply to pin-point the connection of the ovoidal hyperplane problem with the singular characterizations of the previous subsection when $d = 1$.

4.7.2 Question 2

The conclusions of Theorems 12 and 13 overlap: They both contain the metasymplectic spaces. This suggests that it may be possible to prove Theorem 13 without invoking the local pentagon property. The reader is invited to unravel this mystery.

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